

Sample 4-5

線形シフト不変システム

畳み込み行列

画像処理特論

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動作確認: MATLAB R2023a

Linear shift-invariant systems

Convolution matrix

Advanced Topics in Image Processing

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Verified: MATLAB R2023a

準備

(Preparation)

```
close all
```

単変量畳み込み

(Univariate convolution)

有限インパルス応答(FIR) $\{h[n] \in \mathbb{R}\}_{n \in \Omega_h \subset \mathbb{Z}}$ を有する線形シフト不変システム $T(\cdot)$ を仮定する. (Let us assume a linear shift-invariant system $T(\cdot)$ with a finite impulse response (FIR) $\{h[n] \in \mathbb{R}\}_{n \in \Omega_h \subset \mathbb{Z}}$.)

$T(\cdot)$ の配列 $\{u[n] \in \mathbb{R}\}_{n \in \Omega_u \subset \mathbb{Z}}$ に対する応答 $\{v[n] \in \mathbb{R}\}_{n \in \Omega_v \subset \mathbb{Z}}$ は畳み込み (The response $\{v[n] \in \mathbb{R}\}_{n \in \Omega_v \subset \mathbb{Z}}$ to a sequence $\{u[n] \in \mathbb{R}\}_{n \in \Omega_u \subset \mathbb{Z}}$ of $T(\cdot)$ can be represented by convolution)

$$\{v[n]\}_n = T(\{u[n]\}_n) = \sum_{k \in \Omega_h} h[k] \{u[n-k]\}_n.$$

により表現できる.

信号の生成

(Signal generation)

```
% Generating an input sequence u[n] of finite support region  
Q = 6;
```

```
ugen = "(1:Q)";
u = eval(ugen)
```

```
u = 1×6
    1    2    3    4    5    6
```

インパルス応答の設定

(Setting the impulse response)

```
% Setting the shift amount
h = [1 0 -1];
```

写像の定義

(Definition of a map)

```
% Definition of map T as a convolution with h[n]
mapT = @(x) conv(x,h);
```

写像の結果

(Result of mapping)

```
% Mapping with the circular shift T(.)
v = mapT(u)
```

```
v = 1×8
    1    2    2    2    2    2   -5   -6
```

単変量畳み込みの行列表現

(Matrix representation of the univariate convolution)

FIR システムの畳み込み演算は (The convolution of an FIR system can be represented as a matrix as)

$$\mathbf{v} = \mathbf{T}\mathbf{u},$$

のように行列表現できる.

インパルス応答 $\{h[n]\}_n$ のサポート領域が $\Omega_h = \{0, 1, 2\}$, 入力信号 $\{u[n]\}_n$ のサポート領域が $\Omega_u = \{0, 1, 2, 3, 4, 5\}$ のとき, 出力信号 $\{v[n]\}_n$ のサポート領域は $\Omega_v = \{0, 1, 2, 3, 4, 5, 6, 7\}$ となり, (When the support region of the impulse response $\{h[n]\}_n$ is $\Omega_h = \{0, 1, 2\}$ and the support region of the input signal $\{u[n]\}_n$ is $\Omega_u = \{0, 1, 2, 3, 4, 5\}$, the support region of the output signal $\{v[n]\}_n$ is $\Omega_v = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and we have)

$$\mathbf{v} = \text{vec}(\{v[n]\}_n) = \begin{pmatrix} v[0] \\ v[1] \\ v[2] \\ v[3] \\ v[4] \\ v[5] \\ v[6] \\ v[7] \end{pmatrix}, \mathbf{u} = \text{vec}(\{u[n]\}_n) = \begin{pmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \\ u[4] \\ u[5] \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} h[0] & 0 & 0 & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 & 0 & 0 \\ h[2] & h[1] & h[0] & 0 & 0 & 0 \\ 0 & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & 0 & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & 0 & h[2] & h[1] & h[0] \\ 0 & 0 & 0 & 0 & h[2] & h[1] \\ 0 & 0 & 0 & 0 & 0 & h[2] \end{pmatrix}.$$

と表現できる. なお, 出力のサポート領域が $\Omega_v = \{(n+k) \in \mathbb{Z} | n \in \Omega_u, k \in \Omega_h\}$ となり \mathbf{T} の行数が⁸ (Note that the support area of the output is $\Omega_v = \{(n+k) \in \mathbb{Z} | n \in \Omega_u, k \in \Omega_h\}$ and the number of rows in \mathbf{T} is

$$|\Omega_v| = |\Omega_u| + |\Omega_h| - 1 = 6 + 3 - 1 = 8.$$

となることに注意する.

単変量畳み込みの行列生成

(Matrix generation of univariate convolution)

```
% Find the matrix representation of the univariate convolution
T = zeros(length(u)+length(h)-1,length(u));
for idx = 1:length(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = mapT(e);
end
% Matrix representation of the univariate convolution
T
```

```
T = 8x6
    1     0     0     0     0     0
    0     1     0     0     0     0
   -1     0     1     0     0     0
    0    -1     0     1     0     0
    0     0    -1     0     1     0
```

0	0	0	-1	0	1
0	0	0	0	-1	0
0	0	0	0	0	-1

関数 CONVMTX の利用

(Using the CONVMTX function)

単変量畳み込み行列の生成に関数 CONVMTX も利用できる.

(The function CONVMTX can also be used to generate univariate convolutional matrices.)

```
% Generating the matrix representation of the univariate convolution by CONVMTX
H = convmtx(h(:),length(u)) %#ok
```

```
H = 8x6
     1     0     0     0     0     0
     0     1     0     0     0     0
    -1     0     1     0     0     0
     0    -1     0     1     0     0
     0     0    -1     0     1     0
     0     0     0    -1     0     1
     0     0     0     0    -1     0
     0     0     0     0     0    -1
```

行列演算による単変量畳み込み

(Univariate convolution by matrix operation)

畳み込みは可換図に沿って (Convolution can be computed as)

$$\{v[n]\}_n = \text{vec}_{\Omega_v}^{-1}(\mathbf{v}) = \text{vec}_{\Omega_v}^{-1}(\mathbf{T}\mathbf{u}) = \text{vec}_{\Omega_v}^{-1} \circ \mathbf{T} \text{vec}_{\Omega_u}(\{u[n]\}_n)$$

のように行列演算が可能である. すなわち, (along the commutative diagram. That is, we have)

$$T = \text{vec}_{\Omega_v}^{-1} \circ \mathbf{T} \text{vec}_{\Omega_u}.$$

と表現できる.

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the row sequence
recv = reshape(vecv,[1 (length(u)+length(h)-1)])
```

```
recv = 1x8
     1     2     2     2     2     2    -5    -6
```

行列演算による単変量畳み込みの評価

(Evaluation of univariate convolution by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse = @(x,y) mean((double(x)-double(y)).^2,'all');
mymse(v,recv)
```

```
ans = 0
```

2 変量畳み込み

(Bivariate convolution)

2 変量の有限インパルス応答(FIR) $\{h[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_h \subset \mathbb{Z}^2}$ を有する線形シフト不変システム $T(\cdot)$ を仮定する.

(Assume a linear shift-invariant system T with a bivariate finite impulse response (FIR) $\{h[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_h \subset \mathbb{Z}^2}$.)

$T(\cdot)$ の配列 $\{u[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_u \subset \mathbb{Z}^2}$ に対する応答 $\{v[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_v \subset \mathbb{Z}^2}$ は畳み込み (The response $\{v[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_v \subset \mathbb{Z}^2}$ to the array $\{u[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_u \subset \mathbb{Z}^2}$ of $T(\cdot)$ can be represented by convolution)

$$\{v[\mathbf{n}]\}_{\mathbf{n}} = T(\{u[\mathbf{n}]\}_{\mathbf{n}}) = \sum_{\mathbf{k} \in \Omega_h} h[\mathbf{k}] \{u[\mathbf{n} - \mathbf{k}]\}_{\mathbf{n}}.$$

により表現できる.

信号の生成

(Signal generation)

```
% Generating an input array u[n1,n2] of finite support region
N1 = 2; % # of rows
N2 = 3; % # of columns
ugen = "reshape((0:N1*N2-1),[N1 N2])";
u = eval(ugen)
```

```
u = 2x3
    0     2     4
    1     3     5
```

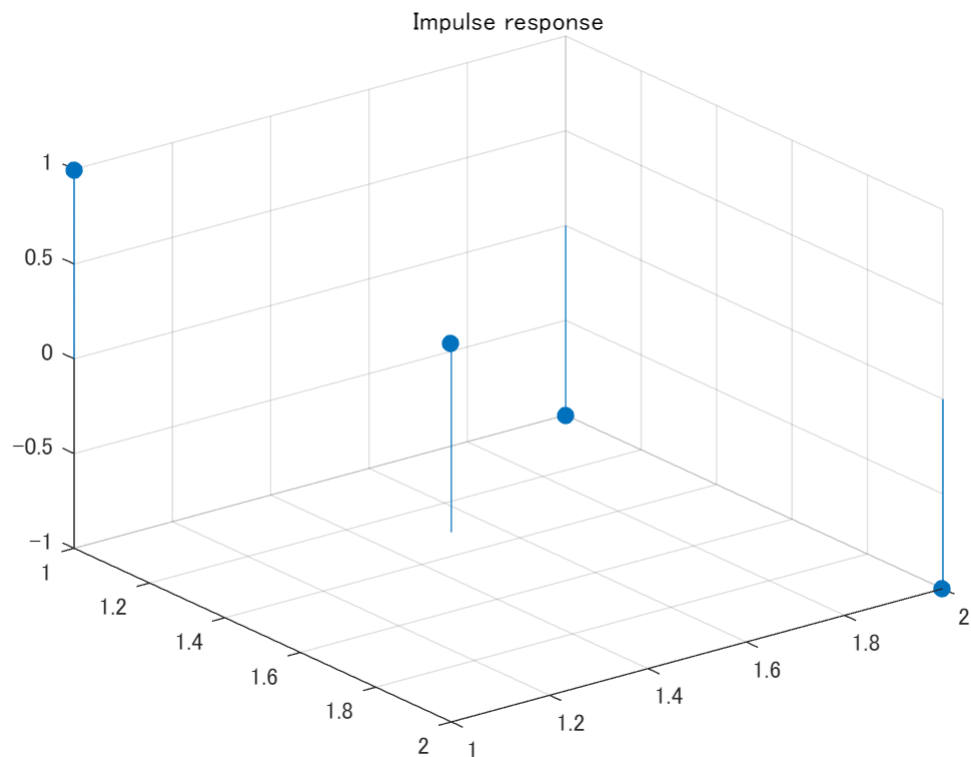
インパルス応答の設定

(Setting the impulse response)

```
h = [1 -1 ; 1 -1]
```

```
h = 2x2
    1    -1
    1    -1
```

```
figure(1)
stem3(h,'filled')
axis ij
title('Impulse response')
```



写像の定義

(Definition of a map)

```
% Definition of map T as a convolution with h[n]
mapT = @(x) imfilter(x,h,'conv','full'); % or conv2(x,h);
```

写像の結果

(Result of mapping)

```
% Mapping with the circular shift T(.)
v = mapT(u)
```

```
v = 3x4
    0     2     2    -4
    1     4     4    -9
    1     2     2    -5
```

2 変量畳み込みの行列表現

(Matrix representation of the bivariate convolution)

2 変量 FIR システムの畳み込み演算は (The convolution of a bivariate FIR system can be represented as a matrix as)

$$\mathbf{v} = \mathbf{T}\mathbf{u},$$

のように行列表現できる.

インパルス応答 $\{h[\mathbf{n}]\}_{\mathbf{n}}$ のサポート領域が $\Omega_h = \{0, 1\} \times \{0, 1\}$, 入力信号 $\{u[\mathbf{n}]\}_{\mathbf{n}}$ のサポート領域が $\Omega_u = \{0, 1\} \times \{0, 1, 2\}$ のとき, 出力信号 $\{v[\mathbf{n}]\}_{\mathbf{n}}$ のサポート領域は $\Omega_v = \{0, 1, 2\} \times \{0, 1, 2, 3\}$ となり, (If the support region of the impulse response $\{h[\mathbf{n}]\}_{\mathbf{n}}$ is $\Omega_h = \{0, 1\} \times \{0, 1\}$, and the support region of the input signal $\{u[\mathbf{n}]\}_{\mathbf{n}}$ is $\Omega_u = \{0, 1\} \times \{0, 1, 2\}$, then the support region of the output signal $\{v[\mathbf{n}]\}_{\mathbf{n}}$ is $\Omega_v = \{0, 1, 2\} \times \{0, 1, 2, 3\}$ and can be expressed as)

$$\mathbf{v} = \text{vec}(\{v[\mathbf{n}]\}_{\mathbf{n}}) = \begin{pmatrix} v[0, 0] \\ v[1, 0] \\ v[2, 0] \\ v[0, 1] \\ \vdots \\ v[2, 3] \end{pmatrix}, \mathbf{u} = \text{vec}(\{u[\mathbf{n}]\}_{\mathbf{n}}) = \begin{pmatrix} u[0, 0] \\ u[1, 0] \\ u[0, 1] \\ u[1, 1] \\ \vdots \\ u[1, 2] \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} h[0, 0] & 0 & 0 & 0 & \cdots & 0 \\ h[1, 0] & h[0, 0] & 0 & 0 & \cdots & 0 \\ 0 & h[1, 0] & h[0, 0] & 0 & \cdots & 0 \\ h[1, 0] & 0 & h[1, 0] & h[0, 0] & \cdots & 0 \\ h[1, 1] & h[1, 0] & 0 & h[1, 0] & \cdots & 0 \\ 0 & h[1, 1] & h[1, 0] & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & h[1, 1] \end{pmatrix}.$$

と表現できる. なお, 出力のサポート領域が $\Omega_v = \{(n+k) \in \mathbb{Z} | n \in \Omega_u, k \in \Omega_h\}$ となり \mathbf{T} の行数が (Note that the support region of the output is $\Omega_v = \{(n+k) \in \mathbb{Z} | n \in \Omega_u, k \in \Omega_h\}$ and the number of rows in \mathbf{T} is)

$$|\Omega_v| = (2+2-1) \times (3+2-1) = 2 \times 4 = 12.$$

となることに注意する.

2 変量畳み込みの行列生成

(Matrix generation of bivariate matrix)

```
% Find the matrix representation of the convolution
T = zeros(prod(size(u)+size(h)-1),numel(u));
for idx = 1:numel(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = reshape(mapT(e),[size(T,1) 1]);
end
% Matrix representation of the convolution
```

T

```
T = 12x6
    1     0     0     0     0     0
    1     1     0     0     0     0
    0     1     0     0     0     0
   -1     0     1     0     0     0
   -1    -1     1     1     0     0
    0    -1     0     1     0     0
    0     0    -1     0     1     0
    0     0    -1    -1     1     1
    0     0     0    -1     0     1
    0     0     0     0    -1     0
    ⋮
    ⋮
```

関数 CONVMTX2 の利用

(Using the CONVMTX2 function)

2 変量畳み込み行列の生成に関数 CONVMTX2 も利用できる.

(The function CONVMTX2 can also be used to generate bivariate convolutional matrices.)

```
% Generating the matrix representation of the bivariate convolution by CONVMTX2
H = convmtx2(h,size(u)) %#ok
```

```
H =
(1,1)      1
(2,1)      1
(4,1)     -1
(5,1)     -1
(2,2)      1
(3,2)      1
(5,2)     -1
(6,2)     -1
(4,3)      1
(5,3)      1
(7,3)     -1
(8,3)     -1
(5,4)      1
(6,4)      1
(8,4)     -1
(9,4)     -1
(7,5)      1
(8,5)      1
(10,5)     -1
(11,5)     -1
(8,6)      1
(9,6)      1
(11,6)     -1
(12,6)     -1
```

行列演算による 2 変量畳み込み

(Bivariate convolution by matrix operation)

畳み込みは可換図に沿って (Convolution can be computed as)

$$\{v[n]\}_n = \text{vec}_{\Omega_v}^{-1}(\mathbf{v}) = \text{vec}_{\Omega_v}^{-1}(\mathbf{T}\mathbf{u}) = \text{vec}_{\Omega_v}^{-1} \circ \mathbf{T} \text{vec}_{\Omega_u}(\{u[n]\}_n)$$

のように行列演算が可能である。すなわち, (along the commutative diagram. That is, we have)

$$T = \text{vec}_{\Omega_v}^{-1} \circ \mathbf{T} \text{vec}_{\Omega_u}.$$

と表現できる.

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the row sequence
recv = reshape(vecv,(size(u)+size(h)-1))
```

```
recv = 3x4
      0      2      2     -4
      1      4      4     -9
      1      2      2     -5
```

行列演算による単変量畳み込みの評価

(Evaluation of univariate convolution by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse(v,recv)
```

```
ans = 0
```

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