# Sample 10-6

## 冗長変換

$\ell_1$ -ノルム最小化

画像処理特論

村松 正吾

動作確認: MATLAB R2023a

## Redundant transforms

$\ell_1$ -norm minimization

Advanced Topics in Image Processing

Shogo MURAMATSU

Verified: MATLAB R2023a

### 準備

(Preparation)

```
close all
```

### 非線形近似の設定

(Settings of non-linear approximation)

```
% # of Coefs.
K = 32;
```

### 入力信号の生成

(Generation of input sequence)

```
% # of input samples
nSamples = 128;

% Random process in AR(1) model
rng('default');
w = 0.1*randn(nSamples,1);
w(floor(end/2)) = 1;
u = filter(1,[1 -0.95],w);
```

### 合成辞書

(Synthesis dictionary)

```
% Synthesis filters
f0 = [  1 1 ]/2;
f1 = [ -1 1 ]/2;

% (Circular) convolution matrix
nF = max(length(f0),length(f1));
X = [zeros(nF-1,nSamples-nF+1) eye(nF-1); eye(nSamples)]; % Circular extension
matrix
C = [zeros(nSamples,nF-1) eye(nSamples) zeros(nSamples,nF-1)]; % Clipping matrix

% Atoms in (circular) convolution matrix
d0 = C*convmtx(f0.',nSamples+nF-1)*X;
d1 = C*convmtx(f1.',nSamples+1)*X;
```

辞書 (Dictionary) $\mathbf{D}$

```
% Dictionary D (Global matrix representation of synthesis filter bank)
D = zeros(nSamples,2*nSamples);
D(:,1:2:end) = d0;
D(:,2:2:end) = d1;
disp(D)
```

```
    0.5000   -0.5000         0         0         0         0         0         0         0         0         0
    0.5000    0.5000    0.5000   -0.5000         0         0         0         0         0         0         0
         0         0    0.5000    0.5000    0.5000   -0.5000         0         0         0         0         0
         0         0         0         0    0.5000    0.5000    0.5000   -0.5000         0         0         0
         0         0         0         0         0         0    0.5000    0.5000    0.5000   -0.5000         0
         0         0         0         0         0         0         0         0    0.5000    0.5000    0.5000   -0
         0         0         0         0         0         0         0         0         0         0    0.5000    0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0         0         0
```

## $\ell_1$ -ノルム最小化による非線形近似

(Non-linear approximation with $\ell_1$ -norm minimization)

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}\in\mathbb{R}^L} \|\mathbf{s}\|_1 \ \text{s.t.} \ \mathbf{v} = \mathbf{D}\mathbf{s}$$

凸緩和法による分析処理と係数選択 (Analysis process and coefficient selection by a convex-relaxation)

- 基底追跡法 (Basis Pursuit; BP)

線形計画問題に帰着させる．(Reduced to a linear programming problem)

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathbb{R}^{2L}} \mathbf{1}^T \mathbf{x} \ \text{ s.t. } \ \mathbf{v} = \mathbf{D}(\mathbf{I} \quad -\mathbf{I})\mathbf{x} \cap \mathbf{x} \in [0, \infty)^{2L}$$

ただし，(where)

$$\mathbf{s} = \mathbf{s}_+ - \mathbf{s}_- \in \mathbb{R}^L$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{s}_+ \\ \mathbf{s}_- \end{pmatrix} \in \mathbb{R}^{2L}$$

```
% Initialization
M  = size(D,2);
f  = ones(2*M,1);
lu = zeros(2*M,1);
% Linear programming
z  = linprog(f,[],[],[D -D],u,lu,[]);
```

最適解が見つかりました。

```
s  = z(1:M) - z(M+1:end);
```

係数選択 (Coefficient selection)

```
s = s(:);
[~,ix] = sort(abs(s),'descend');
s(ix(K+1:end)) = 0;
```

近似結果 (Approximation result)

```
v = D*s;
```

近似誤差 (Residual)

```
r = u - v;
```

## グラフ描画

(Graph plot)

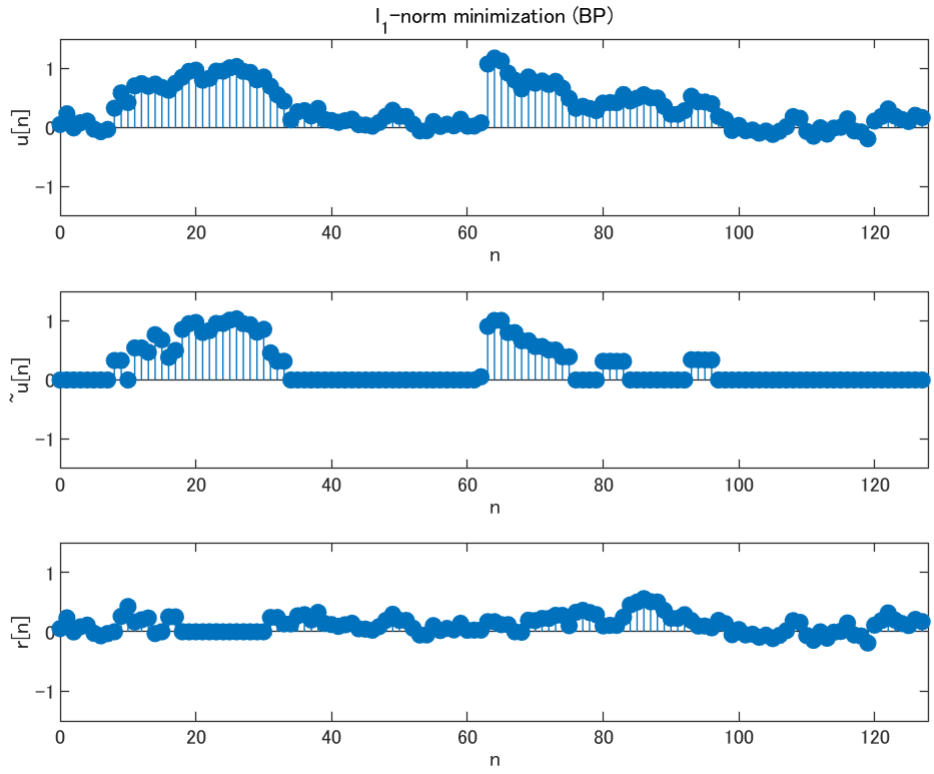
```
figure(1)

% Input
subplot(3,1,1)
stem(0:nSamples-1,u,'filled')
axis([0 nSamples -1.5 1.5])
xlabel('n')
ylabel('u[n]')
```

```
title('l_1-norm minimization (BP)')

% NLA
subplot(3,1,2)
stem(0:nSamples-1,v,'filled')
axis([0 nSamples -1.5 1.5])
xlabel('n')
ylabel('~u[n]')

% Residual
subplot(3,1,3)
stem(0:nSamples-1,r,'filled')
axis([0 nSamples -1.5 1.5])
xlabel('n')
ylabel('r[n]')
```




MSE 評価 (MSE evaluation)

```
mymse = @(x,y) mean((x(:)-y(:)).^2);
fprintf('mse = %f\n',mymse(u,v));
```

```
mse = 0.035184
```