Sample 4-6

線形シフト不変システム

循環畳み込み行列

画像処理特論

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動作確認: MATLAB R2020a

Linear shift-invariant systems

Circular convolution matrix

Advanced Topics in Image Processing

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Verified: MATLAB R2020a

準備

(Preparation)

```
close all
```

単変量循環畳み込み

(Univariate circular convolution)

有限インパルス応答(FIR)$\{h[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{h}\subset\mathbb{Z}}$を有する線形シフト不変システム $T(\cdot)$を仮定する． (Let us assume a linear shift-invariant system $T(\cdot)$ with a finite impulse response (FIR) $\{h[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{h}\subset\mathbb{Z}}$.)

配列 $\{u[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{u}\subset\mathbb{Z}}$に対する周期拡張後の$T(\cdot)$ の応答 $\{v[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{v}\subset\mathbb{Z}}$は周期 $Q\in\mathbb{N}$ を法とする単変量循環畳み込み (The response $\{v[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{v}\subset\mathbb{Z}}$ of $T(\cdot)$ to a sequence $\{u[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{u}\subset\mathbb{Z}}$ after periodic extension can be represented by univariate circular convolution with period Q as)

$$\{v[n]\}_n = T(\{u[n]\}_n) = \sum_{k\in\Omega_\mathrm{h}} h[k]\{u[((n-k))_Q]\}_n.$$

により表現できる．ただし，(where)

$$((n))_Q = n - Q\lfloor Q^{-1}n\rfloor.$$

は， Qを法とする n の剰余である．(denotes the n modulo Q.)

信号の生成

(Signal generation)

```
% Generating an input sequence u[n] of finite support region
Q = 6;
ugen = "(1:Q)";
u = eval(ugen)
```

```
u = 1×6
     1     2     3     4     5     6
```

インパルス応答の設定

(Setting the impulse response)

```
% Setting the shift amount
h = [1 0 -1];
```

写像の定義

(Definition of a map)

```
% Definition of map T as a modulo-Q circular convolution with h[n]
mapT = @(x) cconv(x,h,Q);
```

写像の結果

(Result of mapping)

```
% Mapping with the circular convolution T(.)
v = mapT(u)
```

```
v = 1×6
    -4    -4     2     2     2     2
```

単変量循環畳み込みの行列表現

(Matrix representation of the univariate circular convolution)

単変量循環畳み込み演算も (The univariate circular convolution can also be represented as a matrix as)

$\mathbf{v} = \mathbf{T}\mathbf{u}$,

のように行列表現できる.

インパルス応答 $\{h[n]\}_n$ のサポート領域 $\Omega_\mathrm{h} = \{0, 1, 2\}$ が，入力信号 $\{u[n]\}_n$のサポート領域$\Omega_\mathrm{u} = \{0, 1, 2, 3, 4, 5\}$ よりも短く周期 $Q = |\Omega_\mathrm{u}|$ と設定されているとき，出力信号 $\{v[n]\}_n$のサポート領域も$\Omega_\mathrm{v} = \Omega_\mathrm{u}$となり，(If the support region $\Omega_\mathrm{h} = \{0, 1, 2\}$ of the impulse response $\{h[n]\}_n$ is shorter than the support region $\Omega_\mathrm{u} = \{0, 1, 2, 3, 4, 5\}$ of the input signal $\{u[n]\}_n$ and the period is set as $Q = |\Omega_\mathrm{u}|$, then the support region of the output signal $\{v[n]\}_n$ becomes also $\Omega_\mathrm{v} = \Omega_\mathrm{u}$, and we have)

$$\mathbf{v} = \mathrm{vec}(\{v[n]\}_n) = \begin{pmatrix} v[0] \\ v[1] \\ v[2] \\ v[3] \\ v[4] \\ v[5] \end{pmatrix}, \mathbf{u} = \mathrm{vec}(\{u[n]\}_n) = \begin{pmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \\ u[4] \\ u[5] \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} h[0] & 0 & 0 & 0 & h[2] & h[1] \\ h[1] & h[0] & 0 & 0 & 0 & h[2] \\ h[2] & h[1] & h[0] & 0 & 0 & 0 \\ 0 & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & 0 & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & 0 & h[2] & h[1] & h[0] \end{pmatrix}.$$

と表現できる.

単変量循環畳み込みの行列生成

(Matrix generation of univariate circular convolution)

```
% Find the matrix representation of the univariate circular convolution
T = zeros(length(u));
for idx = 1:length(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = mapT(e);
end
% Matrix representation of the univariate circular convolution
T
```

```
T = 6×6
    1.0000         0   -0.0000         0   -1.0000         0
         0    1.0000         0   -0.0000         0   -1.0000
   -1.0000         0    1.0000         0         0         0
         0   -1.0000         0    1.0000         0   -0.0000
   -0.0000         0   -1.0000         0    1.0000         0
         0         0         0   -1.0000         0    1.0000
```

行列演算による単変量循環畳み込み

(Univariate circular convolution by matrix operation)

循環畳み込みも可換図に沿って (Circular convolution can also be computed as)

$\{v[n]\}_n = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1}(\mathbf{v}) = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1}(\mathbf{Tu}) = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}(\{u[n]\}_n)$

のように行列演算が可能である．すなわち，(along the commutative diagram. That is, we have)

$T = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}.$

と表現できる．

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the row sequnece
recv = reshape(vecv,[1 Q])
```

```
recv = 1×6
   -4.0000   -4.0000    2.0000    2.0000    2.0000    2.0000
```

行列演算による単変量循環畳み込みの評価

(Evaluation of univariate circular convolution by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse = @(x,y) mean((double(x)-double(y)).^2,'all');
mymse(v,recv)
```

```
ans = 3.6156e-31
```

2 変量循環畳み込み

(Bivariate circular convolution)

有限インパルス応答(FIR)インパルス応答 $\{h[\mathbf{n}]\}_\mathbf{n}$ を有する線形シフト不変システム $T(\cdot)$ を仮定する．(Assume a linear shift-invariant system T with a bivariate finite impulse response (FIR) $\{h[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n}\in\Omega_\mathrm{h}\subset\mathbb{Z}^2}$.)

配列 $\{u[\mathbf{n}]\}_\mathbf{n}$ に対する周期拡張後の $T(\cdot)$ の応答 $\{v[\mathbf{n}]\}_\mathbf{n}$ は周期 $\mathbf{Q} \in \mathbb{Z}^{2\times 2}(\det\mathbf{Q} \neq 0)$ を法とする 2 変量循環畳み込み

$$\{v[\mathbf{n}]\}_\mathbf{n} = T(\{u[\mathbf{n}]\}_\mathbf{n}) = \sum_{\mathbf{k}\in\Omega_\mathrm{h}} h[\mathbf{k}]\{u[((\mathbf{n}-\mathbf{k}))_\mathbf{Q}]\}_\mathbf{n},$$

により表現できる．(The response $\{v[\mathbf{n}]\}_\mathbf{n}$ of $T(\cdot)$ after periodic extension to array $\{u[\mathbf{n}]\}_\mathbf{n}$ can be represented by bivariate circular convolution with period $\mathbf{Q}$,) ただし，$\mathbf{Q} \in \mathbb{Z}^{2\times 2}(\det\mathbf{Q} \neq 0)$ は周期行列，$\{v[\mathbf{n}] \in \mathbb{K}\}_{\mathbf{n}\in\Omega\subset\mathbb{Z}^2}$ は出力配列，(where $\mathbf{Q} \in \mathbb{Z}^{2\times 2}(\det\mathbf{Q} \neq 0)$ is the period matrix, $\{v[\mathbf{n}] \in \mathbb{K}\}_{\mathbf{n}\in\Omega\subset\mathbb{Z}^2}$ is the destination array and)

$((\mathbf{n}))_{\mathbf{Q}} = \mathbf{n} - \mathbf{Q}\lfloor \mathbf{Q}^{-1}\mathbf{n} \rfloor$

は，$\mathbf{Q}$を法とする $\mathbf{n}$ の剰余である．(denotes the $\mathbf{n}$ modulo $\mathbf{Q}$.)

信号の生成

(Signal generation)

```
% Generating an input array u[n1,n2] of finite support region
N1 = 2; % # of rows
N2 = 3; % # of columns
ugen = "reshape((0:N1*N2-1),[N1 N2])";
u = eval(ugen)
```

```
u = 2×3
     0     2     4
     1     3     5
```

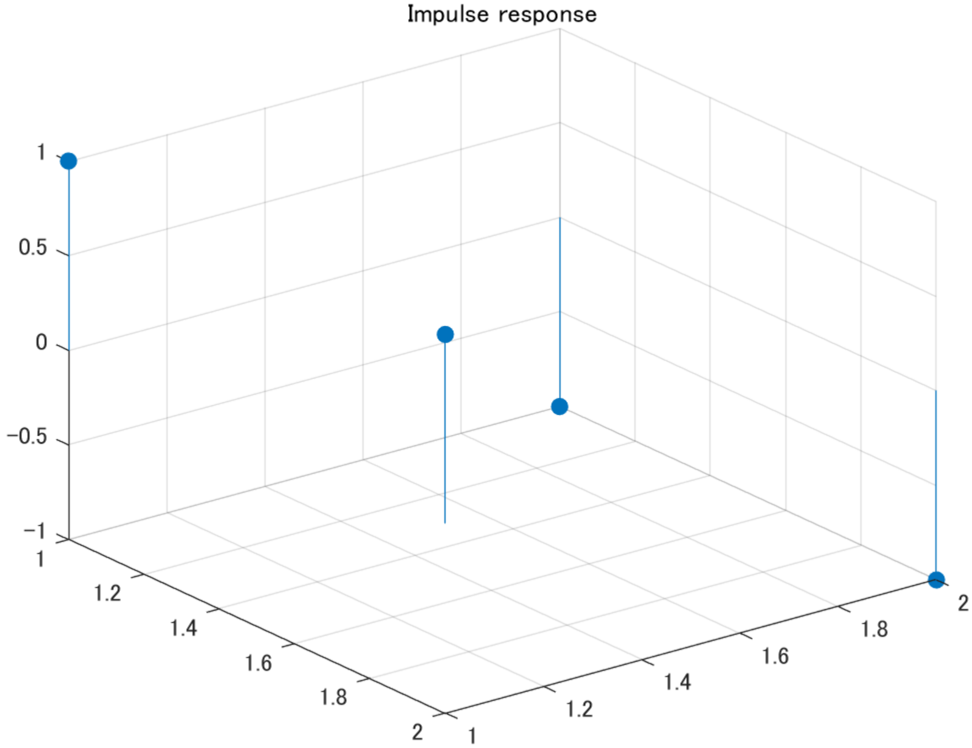
インパルス応答の設定

(Setting the impulse response)

```
h = [1 -1 ; 1 -1]
```

```
h = 2×2
     1    -1
     1    -1
```

```
figure(1)
stem3(h,'filled')
axis ij
title('Impulse response')
```



写像の定義

(Definition of a map)

```
% Definition of map T as a circular convolution with h[n]
mapT = @(x) imfilter(x,h,'conv','circ');
```

写像の結果

(Result of mapping)

```
% Mapping with the circular shift T(.)
v = mapT(u)
```

```
v = 2×3
     4     4    -8
     4     4    -8
```

2 変量循環畳み込みの行列表現

(Matrix representation of the bivariate circular convolution)

2 変量循環畳み込み演算も (The bivariate circular convolution can also be represented as a matrix as)

$\mathbf{v} = \mathbf{T}\mathbf{u},$

のように行列表現できる.

インパルス応答 $\{h[\mathbf{n}]\}_\mathbf{n}$ のサポート領域 $\Omega_\mathrm{h}=\{0,1\}\times\{0,1\}$ が，入力信号 $\{u[\mathbf{n}]\}_\mathbf{n}$ のサポート領域 $\Omega_\mathrm{u}=\{0,1\}\times\{0,1,2\}$ よりも狭く周期行列が $\mathbf{Q}=\begin{pmatrix}N_1 & 0\\ 0 & N_2\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$ と設定されているとき，出力信号 $\{v[\mathbf{n}]\}_\mathbf{n}$ のサポート領域も $\Omega_\mathrm{v}=\Omega_\mathrm{u}$ となり，(If the support region $\Omega_\mathrm{h}=\{0,1\}\times\{0,1\}$ of the impulse response $\{h[\mathbf{n}]\}_\mathbf{n}$ is narrower than the support region $\Omega_\mathrm{u}=\{0,1\}\times\{0,1,2\}$ of the input signal $\{u[\mathbf{n}]\}_\mathbf{n}$ and the period is set as $\mathbf{Q}=\begin{pmatrix}N_1 & 0\\ 0 & N_2\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$, then the support region of the output signal $\{v[\mathbf{n}]\}_\mathbf{n}$ becomes also $\Omega_\mathrm{v}=\Omega_\mathrm{u}$, and we have)

$$\mathbf{v}=\mathrm{vec}(\{v[\mathbf{n}]\}_\mathbf{n})=\begin{pmatrix}v[0,0]\\ v[1,0]\\ v[0,1]\\ v[1,1]\\ \vdots\\ v[1,2]\end{pmatrix},\mathbf{u}=\mathrm{vec}(\{u[\mathbf{n}]\}_\mathbf{n})=\begin{pmatrix}u[0,0]\\ u[1,0]\\ u[0,1]\\ u[1,1]\\ \vdots\\ u[1,2]\end{pmatrix}$$

$$\mathbf{T}=\begin{pmatrix}h[1,1] & h[0,1] & h[1,0] & h[0,0] & 0 & 0\\ h[0,1] & h[1,1] & h[0,0] & h[1,0] & 0 & 0\\ 0 & 0 & h[1,1] & h[0,1] & h[1,0] & h[0,0]\\ 0 & 0 & h[0,1] & h[1,1] & h[0,0] & h[1,0]\\ h[1,0] & h[0,0] & 0 & 0 & h[1,1] & h[0,1]\\ h[0,0] & h[1,0] & 0 & 0 & h[0,1] & h[1,1]\end{pmatrix}.$$

と表現できる．

2 変量循環畳み込みの行列生成

(Bivariate circular convolution matrix generation)

```
% Find the matrix representation of the circular convolution
T = zeros(numel(u));
for idx = 1:numel(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = reshape(mapT(e),[size(T,1) 1]);
end
% Matrix representation of the convolution
T
```

```
T = 6×6
    -1    -1     1     1     0     0
    -1    -1     1     1     0     0
     0     0    -1    -1     1     1
```

```
     0     0    -1    -1     1     1
     1     1     0     0    -1    -1
     1     1     0     0    -1    -1
```

行列演算による 2 変量循環畳み込み

(Bivariate circular convolution by matrix operation)

2 変量循環畳み込みも可換図に沿って (Bivariate circular convolution can also be computed as)

$$\{v[\mathbf{n}]\}_{\mathbf{n}} = \mathrm{vec}^{-1}_{\Omega_\mathrm{v}}(\mathbf{v}) = \mathrm{vec}^{-1}_{\Omega_\mathrm{v}}(\mathbf{Tu}) = \mathrm{vec}^{-1}_{\Omega_\mathrm{v}} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}(\{u[\mathbf{n}]\}_{\mathbf{n}})$$

のように行列演算が可能である．すなわち，(along the commutative diagram. That is, we have)

$$T = \mathrm{vec}^{-1}_{\Omega_\mathrm{v}} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}.$$

と表現できる.

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the row sequnece
recv = reshape(vecv,[N1 N2])
```

```
recv = 2×3
     4     4    -8
     4     4    -8
```

行列演算による 2 変量循環畳み込みの評価

(Evaluation of bivariate circular convolution by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse(v,recv)
```

```
ans = 0
```