Sample 5-6

周波数解析

多変量循環畳み込み

画像処理特論

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動作確認: MATLAB R2023a

Fourier analysis

Multivariate circular convolution

Advanced Topics in Image Processing

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Verified: MATLAB R2023a

準備

(Preparation)

```
close all
```

サンプル画像 $\{u[\mathbf{n}]\}_\mathbf{n}$ の準備

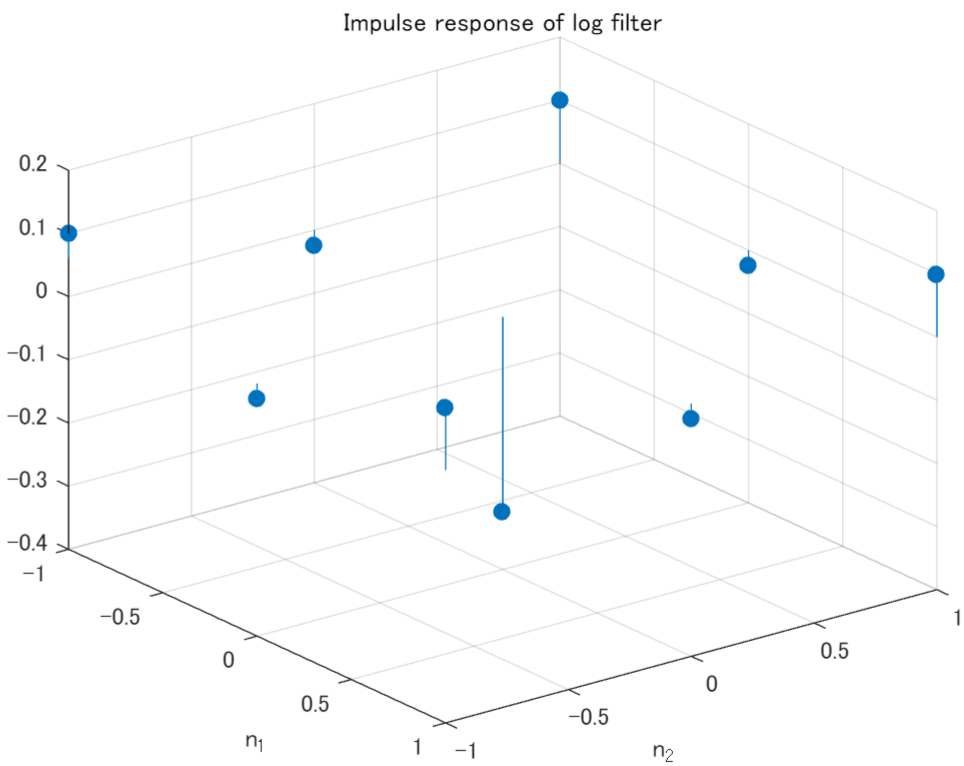
(Preparation of sample image $\{u[\mathbf{n}]\}_\mathbf{n}$)

```
% Reading original image
u = im2double(imread('cameraman.tif'));
figure(1)
imshow(u)
title('Original')
```

線形シフト不変システムのインパルス応答 $\{h[\mathbf{n}]\}_\mathbf{n}$

(Impulse response of a linear shift-invariant system $\{h[\mathbf{n}]\}_{\mathbf{n}}$

```
% Impulse response h[n]
hsize1 = 3;
hsize2 = 3;
sigma = 1;
ftype = "log";
h = rot90(fspecial(ftype,[hsize1 hsize2],sigma),2);
figure(2)
[n1,n2] = meshgrid(-floor((hsize2-1)/2):ceil((hsize2-1)/2),-floor((hsize1-1)/
2):ceil((hsize1-1)/2));
stem3(n1,n2,h,'filled')
xlabel('n_2')
ylabel('n_1')
axis ij
title(['Impulse response of ' char(ftype) ' filter'])
```



周期行列 $\mathbf{Q}$ の循環畳み込みの出力応答 $\{v[\mathbf{n}]\}_{\mathbf{n}}$

(Output response $\{v[\mathbf{n}]\}_{\mathbf{n}}$ of circular convolution with period $\mathbf{Q}$)

循環畳み込み演算 (Circular convolution)

$$\{v[\mathbf{n}]\}_{\mathbf{n}} = \{h[\mathbf{n}]\}_{\mathbf{n}} \bigcirc \{u[\mathbf{n}]\}_{\mathbf{n}} = \sum_{\mathbf{k}\in\Omega\subset\mathbb{Z}^2} u[\mathbf{k}]\{h[((\mathbf{n}-\mathbf{k}))_{\mathbf{Q}}]\}_{\mathbf{n}}$$

```
% Setting the period N
nPeriod1 =258;
nPeriod2 =258;
nPeriod = [nPeriod1 nPeriod2];
nZeroPadding = [nPeriod1 nPeriod2] - size(u);

% Zero padding
uzpd = padarray(u,nZeroPadding,0,'post');
figure(3)
imshow(uzpd)
```

```

% Output v[n]
v = imfilter(uzpd,h,'conv','circ');
```

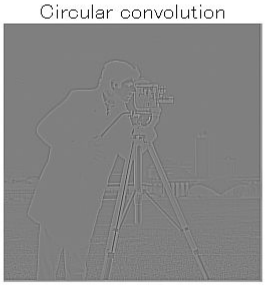
畳み込み演算との比較

(Comparison with convolution)

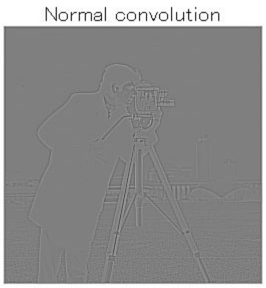
```
% Normal convolution
w = imfilter(u,h,'conv','full');

% v[n]
figure(4)
imshow(v+(min(v(:))<0)/2)
title('Circular convolution')
```

Circular convolution



```
% w[n]
figure(5)
imshow(w+(min(w(:))<0)/2)
title('Normal convolution')
```



通常の畳み込みと循環畳み込みが一致する条件

(The condition that normal convolution and circular convolution match)

$\exists \mathbf{m} \in \mathbb{Z}^D$ s.t. $\{\mathbf{n}+\mathbf{m} | \mathbf{n} \in \Omega_{\mathrm{v}}\} \subseteq \mathcal{N}(\mathbf{Q})$

$\Omega_{\mathrm{v}} = \{\mathbf{n}+\mathbf{k} | \mathbf{n} \in \Omega_{\mathrm{u}}, \mathbf{k} \in \Omega_{\mathrm{h}}\}$

ただし、(where)

- Ω_{v}: 出力のサポート領域 (Output support region)
- Ω_{h}: インパルス応答のサポート領域 (Support region of impulse response)
- Ω_{u}: 入力のサポート領域 (Input support region)

以下では周期行列 $\mathbf{Q}$ を対角行列 (In the following, the periodic matrix $\mathbf{Q}$ is set to a diagonal matrix)

$$\mathbf{Q} = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

に設定する．すなわち，(That is,)

$\mathcal{N}(\mathbf{Q}) = \mathcal{N}(\mathbf{Q}^T) = \{0, 1, 2, \cdots, N_1 - 1\} \times \{0, 1, 2, \cdots, N_2 - 1\}$

$N = |\mathcal{N}(\mathbf{Q})| = |\det(\mathbf{Q})| = N_1 N_2$

ただし，$\Omega_{\mathrm{u}} \subseteq \mathcal{N}(\mathbf{Q})$ を仮定する．(and $\Omega_{\mathrm{u}} \subseteq \mathcal{N}(\mathbf{Q})$ is assumed.)

【Example】もし，(If)

$\Omega_{\mathrm{u}} = \{0, 1, 2, \cdots, L_{\mathrm{u}1} - 1\} \times \{0, 1, 2, \cdots, L_{\mathrm{u}2} - 1\}$

$\Omega_{\mathrm{h}} = \{-1, 0, 1\} \times \{-1, 0, 1\}$

ならば，(then,)

$\Omega_{\mathrm{v}} = \{-1, 0, 1, 2, \cdots, L_{\mathrm{u}1}\} \times \{-1, 0, 1, 2, \cdots, L_{\mathrm{u}2}\}.$

よって，(Therefore, from)

$\{\mathbf{n} + (1, 1)^T | \mathbf{n} \in \Omega_{\mathrm{v}}\} = \{0, 1, 2, \cdots, L_{\mathrm{u}1} + 1\} \times \{0, 1, 2, \cdots, L_{\mathrm{u}2} + 1\},$

より,

$N_1 \geq L_{\mathrm{u}1} + 2,\ N_2 \geq L_{\mathrm{u}2} + 2,$

ならば，通常と畳み込みと循環畳み込みの結果が一致する．(then, the results of normal, convolution and circular convolution are consistent.)

```
% Adjusting the sizes for evaluation
dsz = size(v)-size(w);
if dsz(1) > 0
    vc = v;
    wc = padarray(w,[dsz(1) 0],0,'post');
else
    wc = w;
    vc = padarray(v,[-dsz(1) 0],0,'post');
end
if dsz(2) > 0
    wc = padarray(wc,[0 dsz(2)],0,'post');
else
    vc = padarray(vc,[0 -dsz(2)],0,'post');
end
% Compensate the circular shift
wc = circshift(wc,-ceil((size(h)-1)/2));

% Sizes and MSE
mymse = @(x,y) mean((double(x)-double(y)).^2,'all');
fprintf('Period:         N1  = %d, N2  = %d',nPeriod1,nPeriod2);
```

```
Period:         N1  = 258, N2  = 258
```

```
fprintf('Size of image:  Lu1 = %d, Lu2 = %d',size(u,1),size(u,2));
```

```
Size of image:  Lu1 = 256, Lu2 = 256
```

```
fprintf('Size of filter: Lh1 = %d, Lh2 = %d',size(h,1),size(h,2));
```

```
Size of filter: Lh1 = 3, Lh2 = 3
```

```
fprintf('MSE: %f', mymse(vc,wc))
```

```
MSE: 0.000000
```

入力信号 $\{u[\mathbf{n}]\}_{\mathbf{n}}$ の DFT

(DFT of input signal $\{u[\mathbf{n}]\}_{\mathbf{n}}$)

$$U[\mathbf{k}] = \sum_{\mathbf{n}\in\mathcal{N}(\mathbf{Q})} u[\mathbf{n}] e^{-\jmath 2\pi \mathbf{k}^T \mathbf{Q}^{-1}\mathbf{n}},\ \ \mathbf{k} \in \mathcal{N}(\mathbf{Q}^T)$$

```
% DFT of u[n]
U = fftn(u,nPeriod);
```

フィルタ $\{h[\mathbf{n}]\}_{\mathbf{n}}$ の DFT

(DFT of impulse response $\{h[\mathbf{n}]\}_{\mathbf{n}}$)

$$H[\mathbf{k}] = \sum_{\mathbf{n}\in\mathcal{N}(\mathbf{Q})} h[\mathbf{n}] e^{-\jmath 2\pi \mathbf{k}^T \mathbf{Q}^{-1}\mathbf{n}},\ \ \mathbf{k} \in \mathcal{N}(\mathbf{Q}^T)$$

```
% DFT of h[n]
H = fftn(h,nPeriod);
```

出力信号 $\{v[\mathbf{n}]\}_{\mathbf{n}}$ の DFT

(DFT of output signal $\{v[\mathbf{n}]\}_{\mathbf{n}}$)

$$V[\mathbf{k}] = \sum_{\mathbf{n}\in\mathcal{N}(\mathbf{Q})} v[\mathbf{n}] e^{-\jmath 2\pi \mathbf{k}^T \mathbf{Q}^{-1}\mathbf{n}},\ \ \mathbf{k} \in \mathcal{N}(\mathbf{Q}^T)$$

```
% Frequency response of v[n]
V = fftn(v,nPeriod);
```

DFT 積

(DFT product)

$$V[\mathbf{k}] = H[\mathbf{k}]U[\mathbf{k}],\ \ \mathbf{k} \in \mathcal{N}(\mathbf{Q})$$

循環畳み込みとの比較 (Comparison with circular convolution)

```
% IDFT of DFT product
y = ifftn(H.*U);
% Compensate the circular shift
y = circshift(y,-ceil((size(h)-1)/2));
% MSE with the cconv result 'v'
fprintf('MSE: %f', mymse(v,y))
```

```
MSE: 0.000000
```

循環畳み込みのスペクトルノルム

(Spectral norm of the circular convolution)

$$\|\mathbf{T}\|_2 = \sigma_1(\mathbf{T}) = \max_{\mathbf{k}\in\mathcal{N}(\mathbf{Q}^T)} |H[\mathbf{k}]|$$

ただし，(where)

- $\sigma_1(\mathbf{T})$: $\mathbf{T}$の最大特異値．(Maximum singular value of $\mathbf{T}$)

```
% Definition of map T as a circular convolution with h[n]
mapT = @(x) imfilter(x,h,'conv','circ');
```

2 変量循環畳み込みの行列表現

(Matrix representation of the bivariate circular convolution)

```
% Redefining the period
N1 =8;
N2 =8;

% Find the matrix representation of the circular convolution
N  = N1*N2;
T = zeros(N);
for idx = 1:N
    % Generating a standard basis vector
    e = zeros(N1,N2);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = reshape(mapT(e),[N 1]);
end
% Matrix representation of the circular convolution
T
```

```
T = 64×64
   -0.3079   -0.0234         0         0         0         0         0   -0.0234 ···
   -0.0234   -0.3079   -0.0234         0         0         0         0         0
         0   -0.0234   -0.3079   -0.0234         0         0         0         0
         0         0   -0.0234   -0.3079   -0.0234         0         0         0
         0         0         0   -0.0234   -0.3079   -0.0234         0         0
         0         0         0         0   -0.0234   -0.3079   -0.0234         0
         0         0         0         0         0   -0.0234   -0.3079   -0.0234
   -0.0234         0         0         0         0         0   -0.0234   -0.3079
   -0.0234    0.1004         0         0         0         0         0    0.1004
    0.1004   -0.0234    0.1004         0         0         0         0         0
       ⋮
```

スペクトルノルム

(Spectral norm)

```
% Function NORM evaluates the operator norm for a matrix
opnorm = norm(T,2)
```

```
opnorm = 0.7096
```

最大特異値

(Maximum singular value)

```
sigma1 = max(svd(T))
```

```
sigma1 = 0.7096
```

最大振幅応答

(Maximum magnitude response)

```
H = fftn(h,[N1 N2]);
maxmgn = max(abs(H(:)))
```

```
maxmgn = 0.7096
```

関数 NORM に関する注意

(Notes on the Function NORM)

行列に関するノルムを評価する際には引数の渡し方、オプションの指定に注意すること．(When evaluating the entorywise norm of a matrix, pay attention to the way of passing the arguments and options.)

```
% Froubenius norm
norm(T,'fro')
```

```
ans = 2.9649
```

```

% Entrywise 2-norm, which is identical to the Frobenius norm
norm(T(:),2)
```

```
ans = 2.9649
```

```

% Operator 1-norm
norm(T,1)
```

```
ans = 0.8033
```

```

% Entrywise 1-norm
norm(T(:),1)
```

```
ans = 51.4119
```