Sample 4-5

線形シフト不変システム

畳み込み行列

画像処理特論

村松 正吾

動作確認: MATLAB R2020a

Linear shift-invariant systems

Convolution matrix

Advanced Topics in Image Processing

Shogo MURAMATSU

Verified: MATLAB R2020a

準備

(Preparation)

```
close all
```

単変量畳み込み

(Univariate convolution)

有限インパルス応答(FIR)$\{h[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{h}\subset\mathbb{Z}}$を有する線形シフト不変システム $T(\cdot)$ を仮定する． (Let us assume a linear shift-invariant system $T(\cdot)$ with a finite impulse response (FIR) $\{h[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{h}\subset\mathbb{Z}}$.)

$T(\cdot)$ の配列 $\{u[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{u}\subset\mathbb{Z}}$に対する応答 $\{v[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{v}\subset\mathbb{Z}}$は畳み込み (The response $\{v[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{v}\subset\mathbb{Z}}$ to a sequence $\{u[n]\in\mathbb{R}\}_{n\in\Omega_\mathrm{u}\subset\mathbb{Z}}$ of $T(\cdot)$ can be represented by convolution)

$$\{v[n]\}_n = T(\{u[n]\}_n) = \sum_{k\in\Omega_\mathrm{h}} h[k]\{u[n-k]\}_n.$$

により表現できる．

信号の生成

(Signal generation)

```
% Generating an input sequence u[n] of finite support region
Q = 6;
```

```
ugen = "(1:Q)";
u = eval(ugen)
```

```
u = 1×6
     1     2     3     4     5     6
```

インパルス応答の設定

(Setting the impulse response)

```
% Setting the shift amount
h = [1 0 -1];
```

写像の定義

(Definition of a map)

```
% Definition of map T as a convolution with h[n]
mapT = @(x) conv(x,h);
```

写像の結果

(Result of mapping)

```
% Mapping with the circular shift T(.)
v = mapT(u)
```

```
v = 1×8
     1     2     2     2     2     2    -5    -6
```

単変量畳み込みの行列表現

(Matrix representation of the univariate convolution)

FIRシステムの畳み込み演算は (The convolution of an FIR system can be represented as a matrix as)

$\mathbf{v} = \mathbf{T}\mathbf{u}$,

のように行列表現できる.

インパルス応答 $\{h[n]\}_n$ のサポート領域が $\Omega_{\mathrm{h}} = \{0,1,2\}$, 入力信号 $\{u[n]\}_n$ のサポート領域が $\Omega_{\mathrm{u}} = \{0,1,2,3,4,5\}$ のとき, 出力信号 $\{v[n]\}_n$ のサポート領域は $\Omega_{\mathrm{v}} = \{0,1,2,3,4,5,6,7\}$ となり, (When the support region of the impulse response $\{h[n]\}_n$ is $\Omega_{\mathrm{h}} = \{0,1,2\}$ and the support region of the input signal $\{u[n]\}_n$ is $\Omega_{\mathrm{u}} = \{0,1,2,3,4,5\}$, the support region of the output signal $\{v[n]\}_n$ is $\Omega_{\mathrm{v}} = \{0,1,2,3,4,5,6,7\}$, and we have)

$$\mathbf{v} = \mathrm{vec}(\{v[n]\}_n) = \begin{pmatrix} v[0] \\ v[1] \\ v[2] \\ v[3] \\ v[4] \\ v[5] \\ v[6] \\ v[7] \end{pmatrix}, \mathbf{u} = \mathrm{vec}(\{u[n]\}_n) = \begin{pmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \\ u[4] \\ u[5] \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} h[0] & 0 & 0 & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 & 0 & 0 \\ h[2] & h[1] & h[0] & 0 & 0 & 0 \\ 0 & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & 0 & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & 0 & h[2] & h[1] & h[0] \\ 0 & 0 & 0 & 0 & h[2] & h[1] \\ 0 & 0 & 0 & 0 & 0 & h[2] \end{pmatrix}.$$

と表現できる．なお，出力のサポート領域が$\Omega_{\mathrm{v}} = \{(n+k) \in \mathbb{Z} | n \in \Omega_{\mathrm{u}}, k \in \Omega_{\mathrm{h}}\}$となり $\mathbf{T}$ の行数が (Note that the support area of the output is $\Omega_{\mathrm{v}} = \{(n+k) \in \mathbb{Z} | n \in \Omega_{\mathrm{u}}, k \in \Omega_{\mathrm{h}}\}$ and the number of rows in $\mathbf{T}$ is

$$|\Omega_{\mathrm{v}}| = |\Omega_{\mathrm{u}}| + |\Omega_{\mathrm{h}}| - 1 = 6 + 3 - 1 = 8.$$

となることに注意する．

単変量畳み込みの行列生成

(Matrix generation of univariate convolution)

```
% Find the matrix representation of the univariate convolution
T = zeros(length(u)+length(h)-1,length(u));
for idx = 1:length(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = mapT(e);
end
% Matrix representation of the univariate convolution
T
```

```
T = 8×6
     1     0     0     0     0     0
     0     1     0     0     0     0
    -1     0     1     0     0     0
     0    -1     0     1     0     0
     0     0    -1     0     1     0
     0     0     0    -1     0     1
```

```
     0     0     0     0    -1     0
     0     0     0     0     0    -1
```

関数CONVMTXの利用

(Using the CONVMTX function)

単変量畳み込み行列の生成に関数CONVMTXも利用できる.

(The function CONVMTX can also be used to generate univariate convolutional matrices.)

```
% Generating the matrix representation of the univariate convolution by CONVMTX
H = convmtx(h(:),length(u)) %#ok
```

```
H = 8×6
     1     0     0     0     0     0
     0     1     0     0     0     0
    -1     0     1     0     0     0
     0    -1     0     1     0     0
     0     0    -1     0     1     0
     0     0     0    -1     0     1
     0     0     0     0    -1     0
     0     0     0     0     0    -1
```

行列演算による単変量畳み込み

(Univariate convolution by matrix operation)

畳み込みは可換図に沿って (Convolution can be computed as)

$$\{v[n]\}_n = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1}(\mathbf{v}) = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1}(\mathbf{Tu}) = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}(\{u[n]\}_n)$$

のように行列演算が可能である．すなわち，(along the commutative diagram. That is, we have)

$$T = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}.$$

と表現できる.

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the row sequnece
recv = reshape(vecv,[1 (length(u)+length(h)-1)])
```

```
recv = 1×8
     1     2     2     2     2     2    -5    -6
```

行列演算による単変量畳み込みの評価

(Evaluation of univariate convolution by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse = @(x,y) mean((double(x)-double(y)).^2,'all');
```

```
mymse(v,recv)
```

```
ans = 0
```

2変量畳み込み
(Bivariate convolution)

2変量の有限インパルス応答(FIR) $\{h[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_h \subset \mathbb{Z}^2}$ を有する線形シフト不変システム $T(\cdot)$ を仮定する.

(Assume a linear shift-invariant system T with a bivariate finite impulse response (FIR) $\{h[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_h \subset \mathbb{Z}^2}$.)

$T(\cdot)$ の配列 $\{u[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_u \subset \mathbb{Z}^2}$ に対する応答 $\{v[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_u \subset \mathbb{Z}^2}$ は畳み込み (The response $\{v[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_u \subset \mathbb{Z}^2}$ to the array $\{u[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_u \subset \mathbb{Z}^2}$ of $T(\cdot)$ can be represented by convolution)

$$\{v[\mathbf{n}]\}_{\mathbf{n}} = T(\{u[\mathbf{n}]\}_{\mathbf{n}}) = \sum_{k \in \Omega_h} h[\mathbf{k}]\{u[\mathbf{n}-\mathbf{k}]\}_{\mathbf{n}}.$$

により表現できる.

信号の生成
(Signal generation)

```
% Generating an input array u[n1,n2] of finite support region
N1 = 2; % # of rows
N2 = 3; % # of columns
ugen = "reshape((0:N1*N2-1),[N1 N2])";
u = eval(ugen)
```

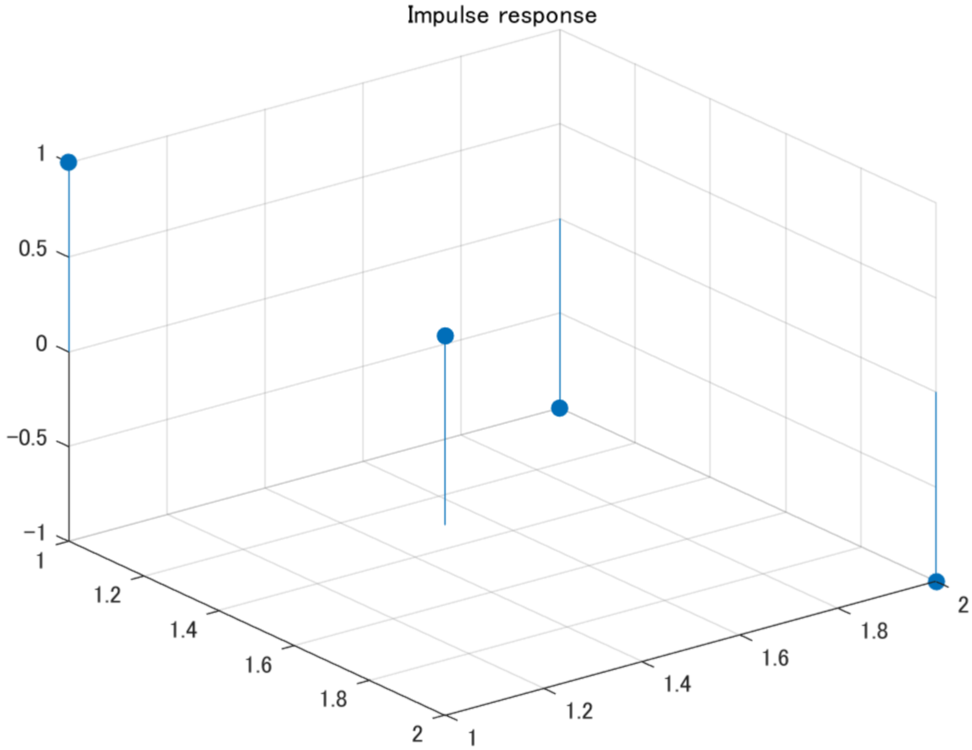
```
u = 2×3
     0     2     4
     1     3     5
```

インパルス応答の設定
(Setting the impulse response)

```
h = [1 -1 ; 1 -1]
```

```
h = 2×2
     1    -1
     1    -1
```

```
figure(1)
stem3(h,'filled')
axis ij
title('Impulse response')
```



写像の定義

(Definition of a map)

```
% Definition of map T as a convolution with h[n]
mapT = @(x) imfilter(x,h,'conv','full'); % or conv2(x,h);
```

写像の結果

(Result of mapping)

```
% Mapping with the circular shift T(.)
v = mapT(u)
```

```
v = 3×4
     0     2     2    -4
     1     4     4    -9
     1     2     2    -5
```

2変量畳み込みの行列表現

(Matrix representation of the bivariate convolution)

2変量FIRシステムの畳み込み演算は (The convolution of a bivariate FIR system can be represented as a matrix as)

$\mathbf{v} = \mathbf{T}\mathbf{u},$

のように行列表現できる．

インパルス応答 $\{h[\mathbf{n}]\}_{\mathbf{n}}$ のサポート領域が $\Omega_{\mathrm{h}}=\{0,1\}\times\{0,1\}$，入力信号 $\{u[\mathbf{n}]\}_{\mathbf{n}}$ のサポート領域が $\Omega_{\mathrm{u}}=\{0,1\}\times\{0,1,2\}$ のとき，出力信号 $\{v[\mathbf{n}]\}_{\mathbf{n}}$ のサポート領域は $\Omega_{\mathrm{v}}=\{0,1,2\}\times\{0,1,2,3\}$ となり，(If the support region of the impulse response $\{h[\mathbf{n}]\}_{\mathbf{n}}$ is $\Omega_{\mathrm{h}}=\{0,1\}\times\{0,1\}$, and the support region of the input signal $\{u[\mathbf{n}]\}_{\mathbf{n}}$ is $\Omega_{\mathrm{u}}=\{0,1\}\times\{0,1,2\}$, then the support region of the output signal $\{v[\mathbf{n}]\}_{\mathbf{n}}$ is $\Omega_{\mathrm{v}}=\{0,1,2\}\times\{0,1,2,3\}$ and can be expressed as)

$$\mathbf{v}=\mathrm{vec}(\{v[\mathbf{n}]\}_{\mathbf{n}})=\begin{pmatrix} v[0,0]\\ v[1,0]\\ v[2,0]\\ v[0,1]\\ \vdots\\ v[2,3]\end{pmatrix},\mathbf{u}=\mathrm{vec}(\{u[\mathbf{n}]\}_{\mathbf{n}})=\begin{pmatrix} u[0,0]\\ u[1,0]\\ u[0,1]\\ u[1,1]\\ \vdots\\ u[1,2]\end{pmatrix}$$

$$\mathbf{T}=\begin{pmatrix} h[0,0] & 0 & 0 & 0 & \cdots & 0\\ h[1,0] & h[0,0] & 0 & 0 & \cdots & 0\\ 0 & h[1,0] & h[0,0] & 0 & \cdots & 0\\ h[1,0] & 0 & h[1,0] & h[0,0] & \cdots & 0\\ h[1,1] & h[1,0] & 0 & h[1,0] & \cdots & 0\\ 0 & h[1,1] & h[1,0] & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & h[1,1]\end{pmatrix}.$$

と表現できる．なお，出力のサポート領域が $\Omega_{\mathrm{v}}=\{(n+k)\in\mathbb{Z}|n\in\Omega_{\mathrm{u}},k\in\Omega_{\mathrm{h}}\}$ となり $\mathbf{T}$ の行数が (Note that the support region of the output is $\Omega_{\mathrm{v}}=\{(n+k)\in\mathbb{Z}|n\in\Omega_{\mathrm{u}},k\in\Omega_{\mathrm{h}}\}$ and the number of rows in $\mathbf{T}$is)

$$|\Omega_{\mathrm{v}}|=(2+2-1)\times(3+2-1)=2\times4=12.$$

となることに注意する．

2変量畳み込みの行列生成

(Matrix generation of bivariate matrix)

```
% Find the matrix representation of the convolution
T = zeros(prod(size(u)+size(h)-1),numel(u));
for idx = 1:numel(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = reshape(mapT(e),[size(T,1) 1]);
end
% Matrix representation of the convolution
```

```
T
```

```
T = 12×6
     1     0     0     0     0     0
     1     1     0     0     0     0
     0     1     0     0     0     0
    -1     0     1     0     0     0
    -1    -1     1     1     0     0
     0    -1     0     1     0     0
     0     0    -1     0     1     0
     0     0    -1    -1     1     1
     0     0     0    -1     0     1
     0     0     0     0    -1     0
     ⋮
```

関数**CONVMTX2**の利用

(Using the CONVMTX2 function)

2変量畳み込み行列の生成に関数CONVMTX2も利用できる．

(The function CONVMTX2 can also be used to generate bivariate convolutional matrices.)

```
% Generating the matrix representation of the bivariate convolution by CONVMTX2
H = convmtx2(h,size(u)) %#ok
```

```
H =
   (1,1)        1
   (2,1)        1
   (4,1)       -1
   (5,1)       -1
   (2,2)        1
   (3,2)        1
   (5,2)       -1
   (6,2)       -1
   (4,3)        1
   (5,3)        1
   (7,3)       -1
   (8,3)       -1
   (5,4)        1
   (6,4)        1
   (8,4)       -1
   (9,4)       -1
   (7,5)        1
   (8,5)        1
  (10,5)       -1
  (11,5)       -1
   (8,6)        1
   (9,6)        1
  (11,6)       -1
  (12,6)       -1
```

行列演算による**2**変量畳み込み

(Bivariate convolution by matrix operation)

畳み込みは可換図に沿って (Convolution can be computed as)

$$\{v[n]\}_n = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1}(\mathbf{v}) = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1}(\mathbf{Tu}) = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}(\{u[n]\}_n)$$

のように行列演算が可能である．すなわち，(along the commutative diagram. That is, we have)

$$T = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}.$$

と表現できる．

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the row sequnece
recv = reshape(vecv,(size(u)+size(h)-1))
```

```
recv = 3×4
     0     2     2    -4
     1     4     4    -9
     1     2     2    -5
```

行列演算による単変量畳み込みの評価

(Evaluation of univariate convolution by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse(v,recv)
```

```
ans = 0
```