# Sample 12-5

## 画像復元

主-双対近接分離法 (PDS)

画像処理特論

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動作確認: MATLAB R2023a

## Image restoration

Primal-dual splitting (PDS)

Advanced Topics in Image Processing

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Verified: MATLAB R2023a

### 準備

(Preparation)

```
clear
close all
import msip.download_img
msip.download_img
```

```
kodim01.png already exists in ./data/
kodim02.png already exists in ./data/
kodim03.png already exists in ./data/
kodim04.png already exists in ./data/
kodim05.png already exists in ./data/
kodim06.png already exists in ./data/
kodim07.png already exists in ./data/
kodim08.png already exists in ./data/
kodim09.png already exists in ./data/
kodim10.png already exists in ./data/
kodim11.png already exists in ./data/
kodim12.png already exists in ./data/
kodim13.png already exists in ./data/
kodim14.png already exists in ./data/
kodim15.png already exists in ./data/
kodim16.png already exists in ./data/
kodim17.png already exists in ./data/
kodim18.png already exists in ./data/
kodim19.png already exists in ./data/
kodim20.png already exists in ./data/
kodim21.png already exists in ./data/
kodim22.png already exists in ./data/
kodim23.png already exists in ./data/
kodim24.png already exists in ./data/
See Kodak Lossless True Color Image Suite
```

## 問題設定

(Problem settings)

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}} \frac{1}{2}\|\mathbf{v} - \mathbf{D}\mathbf{s}\|_2^2 + \lambda\|\mathbf{s}\|_1, \quad \text{s.t.} \ \ \mathbf{D}\mathbf{s} \in [a, b]^2$$

- $\mathbf{D} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix}: \quad \mathbb{R}^2 \to \mathbb{R}^1$
- $\mathbf{v} = \frac{1}{2} \in \mathbb{R}^1$
- $\lambda \in [0, \infty)$
- $\mathbf{s} \in \mathbb{R}^2$
- $a, b \in \mathbb{R}$

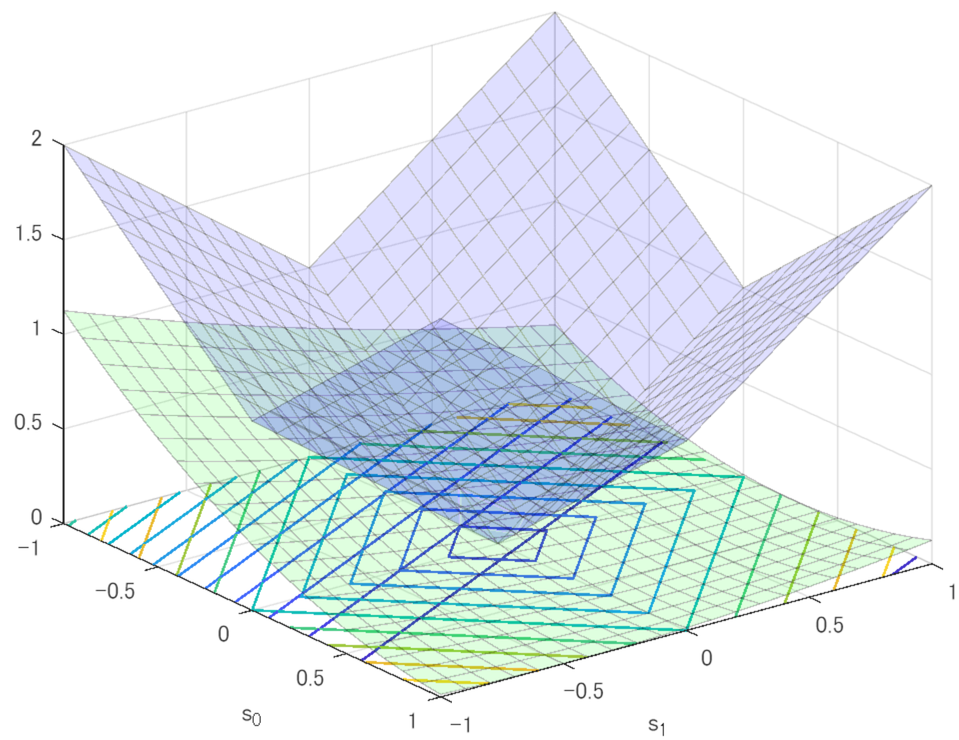
```
D = [2 1]/3;
v = 0.5;
a =  -0.5;
b =   0.0;
```

## 関数プロット

(Function plot)

```
% Function settings
f = @(s0,s1) 0.5*(v-(D(1)*s0+D(2)*s1)).^2;
g = @(s0,s1) (abs(s0)+abs(s1));
% Variable settins
s0 = linspace(-1,1,21);
s1 = linspace(-1,1,21);
[S0,S1] = ndgrid(s0,s1);
F = f(S0,S1);
G = g(S0,S1);
% Surfc plot of the fidelity
figure(1)
hf = surfc(s0,s1,F);
hf(1).FaceAlpha = 0.125;
hf(1).FaceColor = 'green';
hf(1).EdgeAlpha = 0.25;
hf(2).LineWidth = 1;
set(gca,'YDir','reverse');
hold on
% Surfc plot of the regularizer
hg = surfc(s0,s1,G);
hg(1).FaceAlpha = 0.125;
hg(1).FaceColor = 'blue';
hg(1).EdgeAlpha = 0.25;
hg(2).LineWidth = 1;
xlabel('s_1')
```

```
ylabel('s_0')
hold off
```




## パラメータ設定

(Parameter settings)

```
lambda = 0.2;
gamma1 = 0.4;
niters = 20;
```

# 制約付き$\ell_1$ -ノルム正則化最小自乗法による近似

($\ell_1$ -norm-regularized least square method with constraint)

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}} \frac{1}{2}\|\mathbf{v}-\mathbf{D}\mathbf{s}\|_2^2 + \lambda\|\mathbf{s}\|_1, \quad \mathbf{D}\mathbf{s} \in [a,b]^2$$

主-双対近接分離法に帰着させる．(Reduced to a primal-dual splitting method)

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}\in V} f(\mathbf{x}) + g(\mathbf{x}) + h(\mathbf{L}\mathbf{x})$$

- $\mathbf{L} \in \mathbb{R}^{K\times L}$
- $f(\cdot), g(\cdot) \in \Gamma_0(\mathbb{R}^L)$, $h(\cdot) \in \Gamma_0(\mathbb{R}^K)$: Convex functions

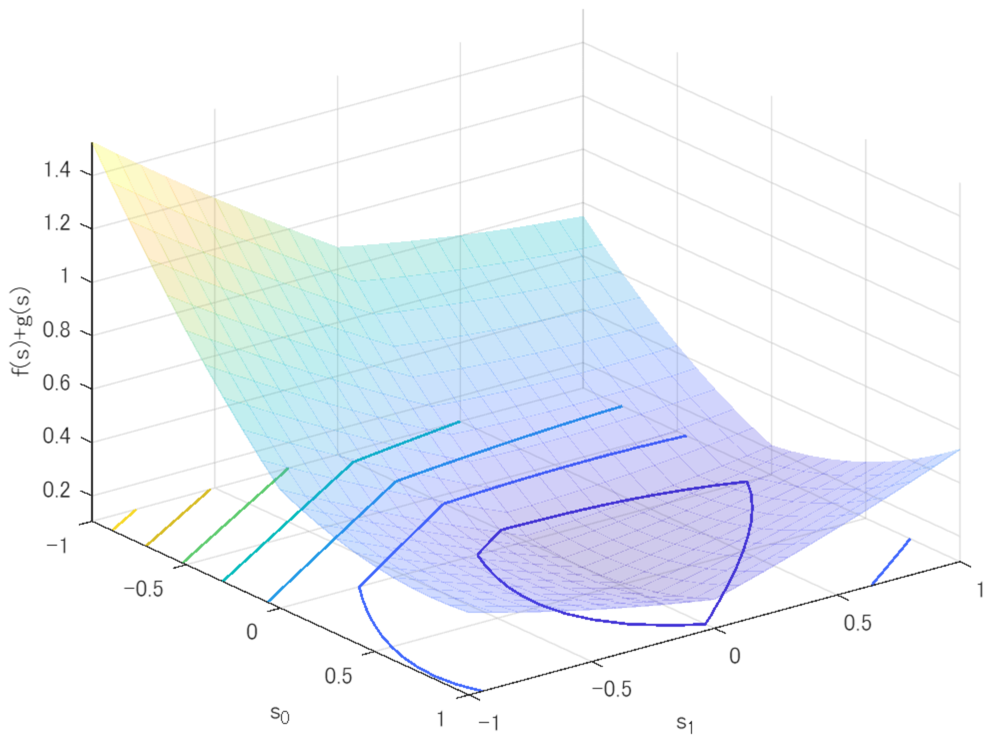
- $f(\cdot)$ is differentiable ( $\beta$-Lipschitz continuous)
- $\Gamma_0(\mathbb{R}^L)$ : Set of proper semi-lower-continuous convex functions

【Example】

- $f(\mathbf{s}) = \frac{1}{2}\|\mathbf{v} - \mathbf{D}\mathbf{s}\|_2^2$
- $g(\mathbf{s}) = \lambda\|\mathbf{s}\|_1$
- $h(\mathbf{L}\mathbf{s}) = \iota_{[a,b]^2}(\mathbf{D}\mathbf{s})$

関数プロット (Function plot)

```
% Function setting
fg = @(s0,s1) 0.5*(v-(D(1)*s0+D(2)*s1)).^2 + lambda*(abs(s0)+abs(s1));
% Surfc plot of cost function f+g
figure(2)
J = fg(S0,S1);
hf = surfc(s0,s1,J);
hf(1).FaceAlpha = 0.25;
hf(1).EdgeAlpha = 0.25;
hf(1).EdgeColor = 'interp';
hf(2).LineWidth = 1;
set(gca,'YDir','reverse')
ylabel('s_0')
xlabel('s_1')
zlabel('f(s)+g(s)')
hold on
```




## 主-双対近接分離法

(Primal-dual splitting method)

1. Initialization: $\mathbf{x}^{(0)}$, $t \leftarrow 0$
2. Primal: $\mathbf{x}^{(t+1)} \leftarrow \mathrm{prox}_{\gamma_1 g}\left(\mathbf{x}^{(t)} - \gamma_1\left(\nabla_x f(\mathbf{x}^{(t)}) + \mathbf{L}^T\mathbf{y}^{(t)}\right)\right)$
3. Dual: $\mathbf{y}^{(t+1)} \leftarrow \mathrm{prox}_{\gamma_2 h^*}\left(\mathbf{y}^{(t)} + \gamma_2\mathbf{L}(2\mathbf{x}^{(t+1)} - \mathbf{x}^{(t)})\right)$
4. If a stopping critera is satisfied then finish, otherwise $t \rightarrow t+1$ and go to Step 2.

【Example】

- $\nabla_{\mathbf{x}} f(\mathbf{x}) = \mathbf{D}^T(\mathbf{D}\mathbf{x} - \mathbf{v})$
- $\mathrm{prox}_{\gamma\lambda\|\cdot\|_1}(\mathbf{x}) = \mathscr{T}_{\gamma\lambda}(\mathbf{x}) = \mathrm{sign}(\mathbf{x}) \odot \max(\mathrm{abs}(\mathbf{x}) - \gamma\lambda\mathbf{1}, \mathbf{0})$
- $\mathrm{prox}_{\gamma\iota_C^*}(\mathbf{y}) = \mathbf{y} - \gamma\mathscr{P}_C(\gamma^{-1}\mathbf{y})$

ただし，(where) $C = [a, b]^N$

$\mathscr{P}_{[a,b]^N}(\mathbf{x}) = \min\{\max\{\mathbf{x}, a\}, b\}$

ソフト閾値処理 (Soft-thresholding)

```
softthresh = @(x,t) sign(x).*max(abs(x)-t,0);
```

初期化 (Initialization)

```
sp = 2*rand(2,1)-1; % in [-1,1]^2
```

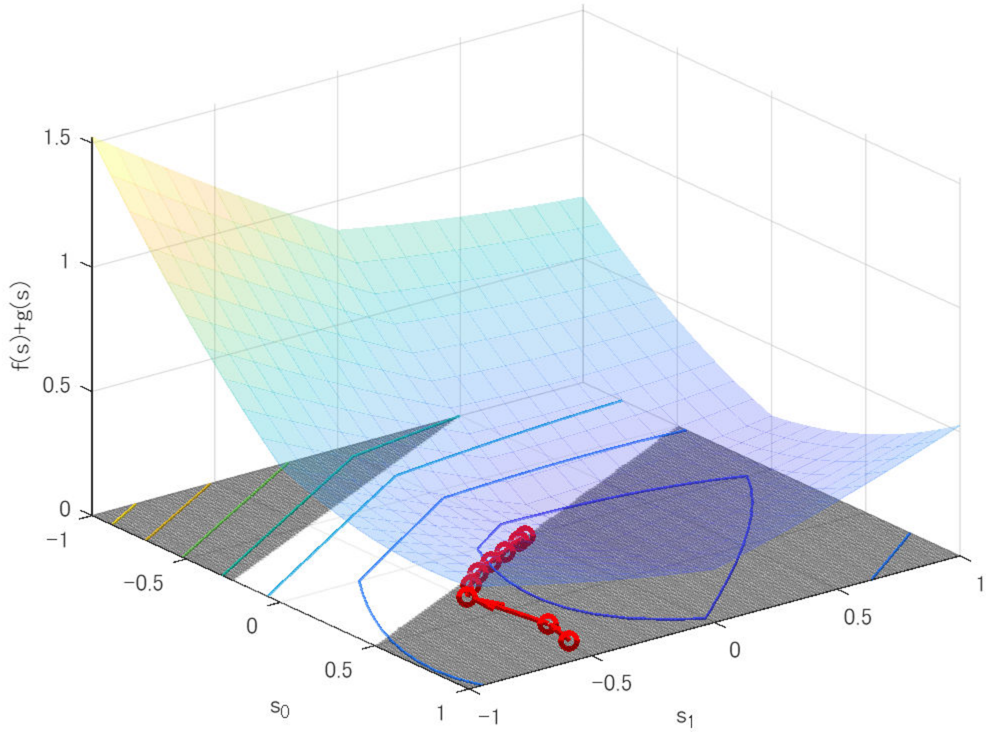
ステップサイズ $\gamma_2$ の設定 (Settings for stepsize $\gamma_2$)

```
beta = D*D';
tau2 = D*D';
gamma2 = 1/(1.05*tau2)*(1/gamma1-beta/2);
assert((1/gamma1 - gamma2*tau2) > beta/2,'Step size condition is violated.')
```

主双対近接分離法 (Prima-dual splitting method)

```
sf0 = -1:.01:1;
sf1 = -1:.01:1;
[Sf0,Sf1] = ndgrid(sf0,sf1);
ic = @(x1,x2) (D(1)*x1+D(2)*x2)>=a & (D(1)*x1+D(2)*x2)<=b;
C = repmat(ic(Sf0,Sf1),[1 1 3]);
hc = surf(sf0,sf1,zeros(size(C,1),size(C,2)),double(C));
hc.EdgeColor = 'interp';
hc.EdgeAlpha = 0.25;
hc.FaceAlpha = 0.25;
y = D*sp;
for idx=0:niters-1
    % Preious state
    s(1,1) = sp(1); % s0
    s(2,1) = sp(2); % s1
    % Primal
    sg = sp-gamma1*D'*((D*sp-v)+y);
    sc = softthresh(sg,gamma1*lambda);
    % Dual
    u = y + gamma2*D*(2*sc-sp);
    y = u - gamma2*min(max(u/gamma2,a),b);
    % Current state
    s(1,2) = sc(1); % s0
    s(2,2) = sc(2); % s1
    % Quiver plot
    xp = s(2,1);
    yp = s(1,1);
    xn = s(2,2);
    yn = s(1,2);
    hp = quiver(xp,yp,xn-xp,yn-yp);
    hp.Marker = 'o';
    hp.ShowArrowHead = 'on';
    hp.MaxHeadSize = 120;
    hp.MarkerSize = 6;
    hp.MarkerEdgeColor = 'r';
    hp.Color = 'r';
```

```
    hp.LineWidth = 2;
    % Update
    sp = sc;
end
hold off
```




## パラメータ設定

(Parameter settings)

- sgm: ノイズ標準偏差 $\sigma_w$ (Standard deviation of noise)

```
% Parameter settings
lambda = 10^-2.4
```

```
lambda = 0.0040
```

```
gamma1 = 10^-2.4
```

```
gamma1 = 0.0040
```

```
sgmuint8 = 10;
sgm = sgmuint8/255;
niters = 80;
```

## 画像の読込

(Read image)

```
u = rgb2gray(im2double(imread('./data/kodim23.png')));
```

## 観測画像

(Observation image)

- $\mathbf{v} = \mathbf{P}\mathbf{u} + \mathbf{w}$
- $\mathbf{w} \sim \mathrm{Norm}\left(\mathbf{w}|\boldsymbol{\mu}_w = \mathbf{0}, \sigma_w^2\mathbf{I}\right)$

```
% Definition of measurment process
psf = fspecial('motion',21,11);
measureproc = @(x) imfilter(x,psf,'conv','circular');
% Adjoint process of the measurment process
measureadjp = @(x) imfilter(x,psf,'corr','circular');
% Simulation of AWGN
v = imnoise(measureproc(u),'gaussian',0,sgm^2);
```

## 全変動正則化

(Total-variation regularization)

PDS の応用例として全変動正則化による画像復元を行う．(As an application of PDS, total variation regularization is applied to image restoration.)

### 問題設定 (Problem settings)

$$\hat{\mathbf{u}} = \arg\min_{\mathbf{u}\in[0,1]^N} \frac{1}{2}\|\mathbf{v} - \mathbf{P}\mathbf{u}\|_2^2 + \lambda\|\mathbf{u}\|_{\mathrm{TV}}$$

$$\Downarrow$$

$$\hat{\mathbf{u}} = \arg\min_{u\in\mathbb{R}^N} \frac{1}{2\lambda}\|\mathbf{v} - \mathbf{P}\mathbf{u}\|_2^2 + \iota_{[0,1]^N}(\mathbf{u}) + \|\nabla\mathbf{u}\|_{1,2}$$

### アルゴリズム (Algorithm)

1. Initialization: $\mathbf{u}^{(0)}, \mathbf{y}^{(0)}, t \leftarrow 0$
2. $\mathbf{u}^{(t+1)} \leftarrow \mathscr{P}_{[0,1]^N}\left(\mathbf{u}^{(t)} - \gamma_1(\lambda^{-1}\mathbf{P}^T(\mathbf{P}\mathbf{u}^{(t)} - \mathbf{v}) + \nabla^T\mathbf{y}^{(t)})\right)$
3. $\mathbf{y}^{(t+1)} \leftarrow \mathrm{prox}_{\gamma_2(\|\cdot\|_{1,2})^*}\left(\mathbf{y}^{(t)} + \gamma_2\nabla(2\mathbf{u}^{(t+1)} - \mathbf{u}^{(t)})\right)$
4. If a stopping critera is satisfied then finish, otherwise $t \rightarrow t+1$ and go to Step 2.

ただし，(where)

- $\mathrm{prox}_{\gamma\|\cdot\|_{1,2}^{*}}(\mathbf{x}) = \mathbf{x} - \gamma\mathrm{prox}_{\gamma^{-1}\|\cdot\|_{1,2}}(\gamma^{-1}\mathbf{x})$
- $\left[\gamma\mathrm{prox}_{\gamma^{-1}\|\cdot\|_{1,2}}(\gamma^{-1}\mathbf{x})\right]_{\mathcal{I}_n} = \mathbf{x}_{\mathcal{I}_n} \odot \max\left\{\mathbf{1} - \frac{1}{\|\mathbf{x}_{\mathcal{I}_n}\|_2}\mathbf{1}, 0\right\}$ (Soft-thresholding for magnitude of $\nabla\mathbf{x}$)
- $\gamma_1^{-1} - \gamma_2(\sigma_{\max}(\nabla))^2 \geq (2\lambda)^{-1}(\sigma_{\max}(\mathbf{P}))^2$

勾配フィルタ (Gradient filter)

```
g0 = fspecial('sobel'); % Vetical difference
g1 = rot90(g0);  % Horizontal difference
gradproc = @(x) cat(3,imfilter(x,g0,'conv','circ'),imfilter(x,g1,'conv','circ'));
gradadjp = @(x) imfilter(x(:,:,1),g0,'corr','circ') +
imfilter(x(:,:,2),g1,'corr','circ');
```

初期化 (Initialization)

```
up = v;
yp = gradproc(v);
```

ステップサイズ $\gamma_2$ の設定 (Settings for stepsize $\gamma_2$)

```
nPoints = 2.^nextpow2(size(v));
beta = max(abs(fftn(psf,nPoints)).^2,[],'all')/lambda; % (σmax(P))^2/λ
tau2 = max(abs(fftn(g0,nPoints)).^2,[],'all');  % (σmax(G))^2
gamma2 = 1/(1.05*tau2)*(1/gamma1-beta/2);
assert((1/gamma1 - gamma2*tau2) > beta/2,...
    ['Step size condition is violated. γ1 must be less than ' num2str(2/beta)])
```

主-双対近接分離法 (Primal-dual splitting method)

```
for idx=0:niters-1
    % Primal step
    ug = measureadjp(measureproc(up)-v);
    uc = min(max(up-gamma1*(ug/lambda + gradadjp(yp)),0),1);
    % Dual step
    yt = yp + gamma2*gradproc(2*uc - up);
    yc = yt - magshrink(yt); % MAGSHRINK is defined at the end of this script
    % Update
    up = uc;
    yp = yc;
end
```

## 復元画像

(Restored image)

```
r = uc;
```

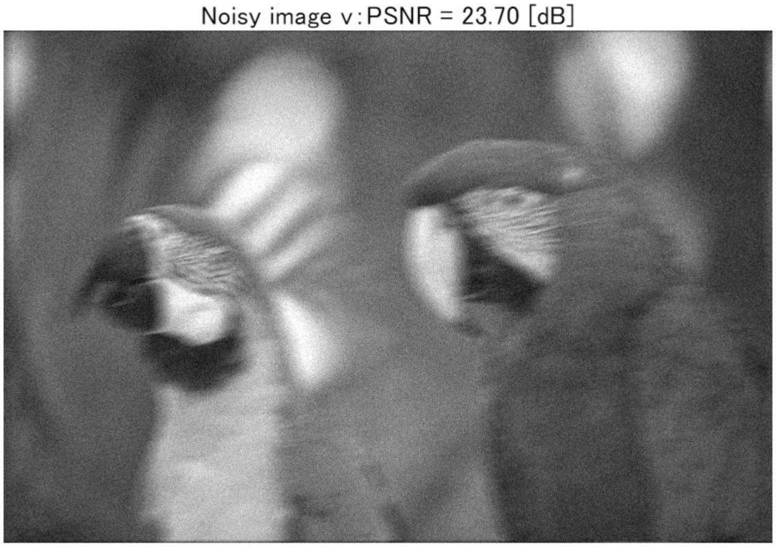
## 画像表示

(Image show)

```
figure(1)
imshow(u);
title('Original image u')
```

```
figure(2)
imshow(v)
title(sprintf('Noisy image v：PSNR = %5.2f [dB]',psnr(u,v)))
```

Noisy image v：PSNR = 23.70 [dB]



```
figure(3)
imshow(r)
```

```
title(sprintf('Restored image r /w TV-regularization : PSNR = %5.2f [dB]',psnr(u,r)))
```


### 振幅ソフト閾値関数

(Magnitude soft-thresholding)

```
function y = magshrink(x)
    Gy = x(:,:,1);
    Gx = x(:,:,2);
    Gmag = imgradient(Gx,Gy);
    map = max(1 - 1./Gmag,0);
    y = x.*map;
end
```