Sample 4-4

線形シフト不変システム

循環シフト

画像処理特論

村松 正吾

動作確認: MATLAB R2020a

Linear shift-invariant systems

Circular shift

Advanced Topics in Image Processing

Shogo MURAMATSU

Verified: MATLAB R2020a

準備

(Preparation)

```
close all
```

単変量循環シフト

(Univariate circular shift)

単変量の有限なサポート領域をもつ配列 $\{u[n]\in\mathbb{K}\}_{n\in\Omega_\mathrm{u}\subset\mathbb{Z}}$ の循環シフトは, (The circular shift of sequences with univariate finite support region can be represented as)

$$\{v[n]\}_n = T(\{u[n]\}_n) = \{u[((n-k))_Q]\ \}_n,$$

のように表現できる．ここで， $Q\in\mathbb{N}$ は周期, $\{v[n]\in\mathbb{K}\}_{n\in\Omega\subset\mathbb{Z}}$ は出力配列, (where $Q\in\mathbb{N}$ is the period, $\{v[n]\in\mathbb{K}\}_{n\in\Omega\subset\mathbb{Z}}$ is the destination sequence and)

$$((n))_Q = n - Q\lfloor Q^{-1}n\rfloor.$$

は, Q を法とする n の剰余である. (denotes the n modulo Q.)

【Example】(In case of) $Q=6,\ k=1$ の場合：

$$\begin{aligned}&(v[0]\quad v[1]\quad v[2]\quad v[3]\quad v[4]\quad v[5])\\&=T(u[0]\quad u[1]\quad u[2]\quad u[3]\quad u[4]\quad u[5])=(u[5]\quad u[0]\quad u[1]\quad u[2]\quad u[3]\quad u[4])\end{aligned}$$

信号の生成

(Signal generation)

```
% Generating an input sequence u[n] of finite support region
Q = 6;
ugen = "(0:Q-1)";
u = eval(ugen)
```

```
u = 1×6
     0     1     2     3     4     5
```

シフト量の設定

(Setting the shift amount)

```
% Setting the shift amount
k = 1;
```

写像の定義

(Definition of a map)

```
% Definition of map T as a circular shift
mapT = @(x) circshift(x,k);
```

写像

(Mapping)

```
% Mapping with the circular shift T(.)
v = mapT(u)
```

```
v = 1×6
     5     0     1     2     3     4
```

循環シフトの行列表現

(Matrix representation of the circular shift)

$Q=6,\ k=1$ の循環シフトは (The circular shift of $Q=6,\ k=1$ can be represented as a matrix as)

$\mathbf{v}=\mathbf{T}\mathbf{u},$

のように行列表現できる．ただし，(where)

$$\mathbf{v} = \mathrm{vec}(\{v[n]\}_n) = \begin{pmatrix} v[0] \\ v[1] \\ v[2] \\ v[3] \\ v[4] \\ v[5] \end{pmatrix}, \mathbf{u} = \mathrm{vec}(\{u[n]\}_n) = \begin{pmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \\ u[4] \\ u[5] \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

循環シフトの行列生成

(Circular shift matrix generation)

```
% Find the matrix representation of the circular shift
T = zeros(length(u));
for idx = 1:length(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = mapT(e);
end
% Matrix representation of the circular shift
T
```

```
T = 6×6
     0     0     0     0     0     1
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
```

行列演算による単変量循環シフト

(Univariate circular shift by matrix operation)

循環シフトは可換図に沿って (Circular shifts can be computed as)

$$\{v[n]\}_n = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1}(\mathbf{v}) = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1}(\mathbf{T}\mathbf{u}) = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}(\{u[n]\}_n)$$

のように行列演算が可能である．すなわち，(along the commutative diagram. That is, we have)

$$T = \mathrm{vec}_{\Omega_\mathrm{v}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_\mathrm{u}}.$$

と表現できる．

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the original
recv = reshape(vecv,[1 Q])
```

```
recv = 1×6
     5     0     1     2     3     4
```

行列演算による単変量循環シフトの評価

(Evaluation of univariate circular shift by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse = @(x,y) mean((double(x)-double(y)).^2,'all');
mymse(v,recv)
```

```
ans = 0
```

2変量循環シフト

(Bivariate circular shift)

2変量の有限なサポート領域をもつ配列 $\{u[\mathbf{n}] \in \mathbb{K}\}_{\mathbf{n}\in\Omega_{\mathrm{u}}\subset\mathbb{Z}^2}$ の循環シフトは， (The circular shift of arrays with bivariate finite support region can be represented as)

$$\{v[\mathbf{n}]\}_{\mathbf{n}} = T(\{u[\mathbf{n}]\}_{\mathbf{n}}) = \{u[((\mathbf{n}-\mathbf{k}))_{\mathbf{Q}}]\ \}_{\mathbf{n}},$$

のように表現できる．ただし， $\mathbf{Q} \in \mathbb{Z}^{2\times 2}(\det \mathbf{Q} \neq 0)$ は周期行列， $\{v[\mathbf{n}] \in \mathbb{K}\}_{\mathbf{n}\in\Omega\subset\mathbb{Z}^2}$ は出力配列， (where $\mathbf{Q} \in \mathbb{Z}^{2\times 2}(\det \mathbf{Q} \neq 0)$ is the period matrix, $\{v[\mathbf{n}] \in \mathbb{K}\}_{\mathbf{n}\in\Omega\subset\mathbb{Z}^2}$ is the destination array and)

$$((\mathbf{n}))_{\mathbf{Q}} = \mathbf{n} - \mathbf{Q}\lfloor \mathbf{Q}^{-1}\mathbf{n} \rfloor$$

は， $\mathbf{Q}$ を法とする $\mathbf{n}$ の剰余である． (denotes the $\mathbf{n}$ modulo $\mathbf{Q}$.)

【Example】 (In case of) $\mathbf{Q} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\mathbf{k} = (k_1 \quad k_2)^T = (1 \quad 2)^T$ の場合：

$$\begin{pmatrix} v[0,0] & v[0,1] & v[0,2] \\ v[1,0] & v[1,1] & v[1,2] \end{pmatrix}$$
$$= T\begin{pmatrix} u[0,0] & u[0,1] & u[0,2] \\ u[1,0] & u[1,1] & u[1,2] \end{pmatrix} = \begin{pmatrix} u[1,1] & u[1,2] & u[1,0] \\ u[0,1] & u[0,2] & u[0,0] \end{pmatrix}$$

信号の生成

(Signal generation)

```
% Generating an input array u[n1,n2] of finite support region
N1 = 2; % # of rows
N2 = 3; % # of columns
ugen = "reshape((0:N1*N2-1),[N1 N2])";
u = eval(ugen)
```

```
u = 2×3
     0     2     4
     1     3     5
```

シフト量の設定

(Setting the shift amount)

```
% Settings of the shift amount
k1 = 1; % # of shifts in the vertical direction
k2 = 2; % # of shifts in the horizontal direction
```

写像の定義

(Definition of a map)

```
% Definition of map T as a circular shift
mapT = @(x) circshift(x,[k1,k2]);
```

写像

(Mapping)

```
% Mapping with the circular shift T(.)
v = mapT(u)
```

```
v = 2×3
     3     5     1
     2     4     0
```

循環シフトの行列表現

(Matrix representation of the circular shift)

$\mathbf{Q} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\mathbf{k} = \begin{pmatrix} k_1 & k_2 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$ の循環シフトは (The circular shift of $\mathbf{Q} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\mathbf{k} = \begin{pmatrix} k_1 & k_2 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$ can be represented as a matrix as)

$$\mathbf{v} = \mathbf{T}\mathbf{u},$$

のように行列表現できる．ただし，(where)

$$\mathbf{v} = \mathrm{vec}(\{v[\mathbf{n}]\}_{\mathbf{n}}) = \begin{pmatrix} v[0,0] \\ v[1,0] \\ v[0,1] \\ v[1,1] \\ v[0,2] \\ v[1,2] \end{pmatrix}, \mathbf{u} = \mathrm{vec}(\{u[\mathbf{n}]\}_{\mathbf{n}}) = \begin{pmatrix} u[0,0] \\ u[1,0] \\ u[0,1] \\ u[1,1] \\ u[0,2] \\ u[1,2] \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

循環シフトの行列生成

(Circular shift matrix generation)

```
% Find the matrix representation of the circular shift
T = zeros(numel(u));
for idx = 1:numel(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = reshape(mapT(e),[size(T,1) 1]);
end
% Matrix representation of the circular shift
T
```

```
T = 6×6
     0     0     0     1     0     0
     0     0     1     0     0     0
     0     0     0     0     0     1
     0     0     0     0     1     0
     0     1     0     0     0     0
     1     0     0     0     0     0
```

行列演算による2変量循環シフト

(Bivariate circular shift by matrix operation)

循環シフトは可換図に沿って (Circular shifts can be computed as)

$$\{v[\mathbf{n}]\}_{\mathbf{n}} = \mathrm{vec}_{\Omega_{\mathrm{v}}}^{-1}(\mathbf{v}) = \mathrm{vec}_{\Omega_{\mathrm{v}}}^{-1}(\mathbf{T}\mathbf{u}) = \mathrm{vec}_{\Omega_{\mathrm{v}}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_{\mathrm{u}}}(\{u[\mathbf{n}]\}_{n})$$

のように行列演算が可能である．すなわち，(along the commutative diagram. That is, we have)

$$T = \mathrm{vec}_{\Omega_{\mathrm{v}}}^{-1} \circ \mathbf{T}\mathrm{vec}_{\Omega_{\mathrm{u}}}.$$

と表現できる.

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the original
recv = reshape(vecv,[N1 N2])
```

```
recv = 2×3
     3     5     1
     2     4     0
```

行列演算による単変量循環シフトの評価

(Evaluation of univariate circular shift by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse(v,recv)
```

```
ans = 0
```