

Sample 4-6

線形シフト不変システム

循環畳み込み行列

画像処理特論

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動作確認: MATLAB R2023a

Linear shift-invariant systems

Circular convolution matrix

Advanced Topics in Image Processing

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Verified: MATLAB R2023a

準備

(Preparation)

```
close all
```

単変量循環畳み込み

(Univariate circular convolution)

有限インパルス応答(FIR) $\{h[n] \in \mathbb{R}\}_{n \in \Omega_h \subset \mathbb{Z}}$ を有する線形シフト不変システム $T(\cdot)$ を仮定する. (Let us assume a linear shift-invariant system $T(\cdot)$ with a finite impulse response (FIR) $\{h[n] \in \mathbb{R}\}_{n \in \Omega_h \subset \mathbb{Z}}$.)

配列 $\{u[n] \in \mathbb{R}\}_{n \in \Omega_u \subset \mathbb{Z}}$ に対する周期拡張後の $T(\cdot)$ の応答 $\{v[n] \in \mathbb{R}\}_{n \in \Omega_v \subset \mathbb{Z}}$ は周期 $Q \in \mathbb{N}$ を法とする単変量循環畳み込み (The response $\{v[n] \in \mathbb{R}\}_{n \in \Omega_v \subset \mathbb{Z}}$ of $T(\cdot)$ to a sequence $\{u[n] \in \mathbb{R}\}_{n \in \Omega_u \subset \mathbb{Z}}$ after periodic extension can be represented by univariate circular convolution with period Q as)

$$\{v[n]\}_n = T(\{u[n]\}_n) = \sum_{k \in \Omega_h} h[k] \{u[(n-k)_Q]\}_n.$$

により表現できる. ただし, (where)

$$(n)_Q = n - Q \lfloor Q^{-1}n \rfloor.$$

は, Q を法とする n の剰余である. (denotes the n modulo Q .)

信号の生成

(Signal generation)

```
% Generating an input sequence u[n] of finite support region
Q = 6;
ugen = "(1:Q)";
u = eval(ugen)
```

```
u = 1×6
    1     2     3     4     5     6
```

インパルス応答の設定

(Setting the impulse response)

```
% Setting the shift amount
h = [1 0 -1];
```

写像の定義

(Definition of a map)

```
% Definition of map T as a modulo-Q circular convolution with h[n]
mapT = @(x) cconv(x,h,Q);
```

写像の結果

(Result of mapping)

```
% Mapping with the circular convolution T(.)
v = mapT(u)
```

```
v = 1×6
   -4    -4     2     2     2     2
```

単変量循環畳み込みの行列表現

(Matrix representation of the univariate circular convolution)

単変量循環畳み込み演算も (The univariate circular convolution can also be represented as a matrix as)

$$\mathbf{v} = \mathbf{T}\mathbf{u},$$

のように行列表現できる.

インパルス応答 $\{h[n]\}_n$ のサポート領域 $\Omega_h = \{0, 1, 2\}$ が, 入力信号 $\{u[n]\}_n$ のサポート領域 $\Omega_u = \{0, 1, 2, 3, 4, 5\}$ よりも短く周期 $Q = |\Omega_u|$ と設定されているとき, 出力信号 $\{v[n]\}_n$ のサポート領域も $\Omega_v = \Omega_u$ となり, (If the support region $\Omega_h = \{0, 1, 2\}$ of the impulse response $\{h[n]\}_n$ is shorter than the support region $\Omega_u = \{0, 1, 2, 3, 4, 5\}$ of the input signal $\{u[n]\}_n$ and the period is set as $Q = |\Omega_u|$, then the support region of the output signal $\{v[n]\}_n$ becomes also $\Omega_v = \Omega_u$, and we have)

$$\mathbf{v} = \text{vec}(\{v[n]\}_n) = \begin{pmatrix} v[0] \\ v[1] \\ v[2] \\ v[3] \\ v[4] \\ v[5] \end{pmatrix}, \mathbf{u} = \text{vec}(\{u[n]\}_n) = \begin{pmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \\ u[4] \\ u[5] \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} h[0] & 0 & 0 & 0 & h[2] & h[1] \\ h[1] & h[0] & 0 & 0 & 0 & h[2] \\ h[2] & h[1] & h[0] & 0 & 0 & 0 \\ 0 & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & 0 & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & 0 & h[2] & h[1] & h[0] \end{pmatrix}.$$

と表現できる.

単変量循環畳み込みの行列生成

(Matrix generation of univariate circular convolution)

```
% Find the matrix representation of the univariate circular convolution
T = zeros(length(u));
for idx = 1:length(u)
    % Generating a standard basis vector
    e = zeros(size(u),'like',u);
    e(idx) = 1;
    % Response to the standard basis vector
    T(:,idx) = mapT(e);
end
% Matrix representation of the univariate circular convolution
T
```

```
T = 6x6
    1.0000         0    -0.0000         0    -1.0000         0
         0    1.0000         0    -0.0000         0    -1.0000
   -1.0000         0    1.0000         0         0         0
         0   -1.0000         0    1.0000         0   -0.0000
   -0.0000         0   -1.0000         0    1.0000         0
         0         0         0   -1.0000         0    1.0000
```

行列演算による単変量循環畳み込み

(Univariate circular convolution by matrix operation)

循環畳み込みも可換図に沿って (Circular convolution can also be computed as)

$$\{v[n]\}_n = \text{vec}_{\Omega_v}^{-1}(\mathbf{v}) = \text{vec}_{\Omega_v}^{-1}(\mathbf{T}\mathbf{u}) = \text{vec}_{\Omega_v}^{-1} \circ \mathbf{T} \text{vec}_{\Omega_u}(\{u[n]\}_n)$$

のように行列演算が可能である。すなわち, (along the commutative diagram. That is, we have)

$$T = \text{vec}_{\Omega_v}^{-1} \circ \mathbf{T} \text{vec}_{\Omega_u}.$$

と表現できる。

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the row sequence
recv = reshape(vecv,[1 Q])
```

```
recv = 1x6
    -4.0000    -4.0000     2.0000     2.0000     2.0000     2.0000
```

行列演算による単変量循環畳み込みの評価

(Evaluation of univariate circular convolution by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse = @(x,y) mean((double(x)-double(y)).^2,'all');
mymse(v,recv)
```

```
ans = 3.6156e-31
```

2 変量循環畳み込み

(Bivariate circular convolution)

有限インパルス応答(FIR)インパルス応答 $\{h[\mathbf{n}]\}_n$ を有する線形シフト不変システム $T(\cdot)$ を仮定する. (Assume a linear shift-invariant system T with a bivariate finite impulse response (FIR) $\{h[\mathbf{n}] \in \mathbb{R}\}_{\mathbf{n} \in \Omega_h \subset \mathbb{Z}^2}$.)

配列 $\{u[\mathbf{n}]\}_n$ に対する周期拡張後の $T(\cdot)$ の応答 $\{v[\mathbf{n}]\}_n$ は周期 $\mathbf{Q} \in \mathbb{Z}^{2 \times 2} (\det \mathbf{Q} \neq 0)$ を法とする 2 変量循環畳み込み

$$\{v[\mathbf{n}]\}_n = T(\{u[\mathbf{n}]\}_n) = \sum_{\mathbf{k} \in \Omega_h} h[\mathbf{k}] \{u[(\mathbf{n} - \mathbf{k})_{\mathbf{Q}}]\}_n,$$

により表現できる. (The response $\{v[\mathbf{n}]\}_n$ of $T(\cdot)$ after periodic extension to array $\{u[\mathbf{n}]\}_n$ can be represented by bivariate circular convolution with period \mathbf{Q} .) ただし, $\mathbf{Q} \in \mathbb{Z}^{2 \times 2} (\det \mathbf{Q} \neq 0)$ は周期行列, $\{v[\mathbf{n}] \in \mathbb{K}\}_{\mathbf{n} \in \Omega_C \mathbb{Z}^2}$ は出力配列, (where $\mathbf{Q} \in \mathbb{Z}^{2 \times 2} (\det \mathbf{Q} \neq 0)$ is the period matrix, $\{v[\mathbf{n}] \in \mathbb{K}\}_{\mathbf{n} \in \Omega_C \mathbb{Z}^2}$ is the destination array and)

$$((n))_Q = n - Q \lfloor Q^{-1}n \rfloor$$

は、 Q を法とする n の剰余である. (denotes the n modulo Q .)

信号の生成

(Signal generation)

```
% Generating an input array u[n1,n2] of finite support region
N1 = 2; % # of rows
N2 = 3; % # of columns
ugen = "reshape((0:N1*N2-1),[N1 N2])";
u = eval(ugen)
```

```
u = 2x3
    0     2     4
    1     3     5
```

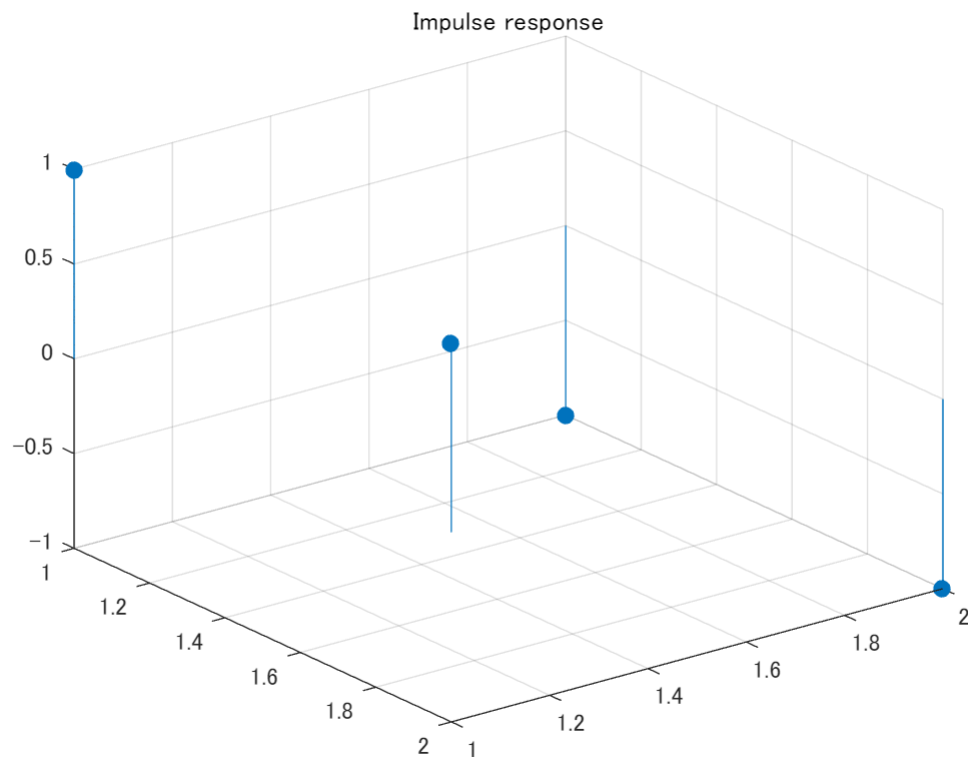
インパルス応答の設定

(Setting the impulse response)

```
h = [1 -1 ; 1 -1]
```

```
h = 2x2
    1    -1
    1    -1
```

```
figure(1)
stem3(h,'filled')
axis ij
title('Impulse response')
```



写像の定義

(Definition of a map)

```
% Definition of map T as a circular convolution with h[n]
mapT = @(x) imfilter(x,h,'conv','circ');
```

写像の結果

(Result of mapping)

```
% Mapping with the circular shift T(.)
v = mapT(u)
```

```
v = 2x3
    4    4   -8
    4    4   -8
```

2 変量循環畳み込みの行列表現

(Matrix representation of the bivariate circular convolution)

2 変量循環畳み込み演算も (The bivariate circular convolution can also be represented as a matrix as)

$$v = Tu,$$

のように行列表現できる.

インパルス応答 $\{h[\mathbf{n}]\}_n$ のサポート領域 $\Omega_h = \{0, 1\} \times \{0, 1\}$ が, 入力信号 $\{u[\mathbf{n}]\}_n$ のサポート領域

$\Omega_u = \{0, 1\} \times \{0, 1, 2\}$ よりも狭く周期行列が $\mathbf{Q} = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ と設定されているとき, 出力信号

$\{v[\mathbf{n}]\}_n$ のサポート領域も $\Omega_v = \Omega_u$ となり, (If the support region $\Omega_h = \{0, 1\} \times \{0, 1\}$ of the impulse response $\{h[\mathbf{n}]\}_n$ is narrower than the support region $\Omega_u = \{0, 1\} \times \{0, 1, 2\}$ of the input signal $\{u[\mathbf{n}]\}_n$ and the period

is set as $\mathbf{Q} = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, then the support region of the output signal $\{v[\mathbf{n}]\}_n$ becomes also $\Omega_v = \Omega_u$, and we have)

$$\mathbf{v} = \text{vec}(\{v[\mathbf{n}]\}_n) = \begin{pmatrix} v[0, 0] \\ v[1, 0] \\ v[0, 1] \\ v[1, 1] \\ \vdots \\ v[1, 2] \end{pmatrix}, \mathbf{u} = \text{vec}(\{u[\mathbf{n}]\}_n) = \begin{pmatrix} u[0, 0] \\ u[1, 0] \\ u[0, 1] \\ u[1, 1] \\ \vdots \\ u[1, 2] \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} h[1, 1] & h[0, 1] & h[1, 0] & h[0, 0] & 0 & 0 \\ h[0, 1] & h[1, 1] & h[0, 0] & h[1, 0] & 0 & 0 \\ 0 & 0 & h[1, 1] & h[0, 1] & h[1, 0] & h[0, 0] \\ 0 & 0 & h[0, 1] & h[1, 1] & h[0, 0] & h[1, 0] \\ h[1, 0] & h[0, 0] & 0 & 0 & h[1, 1] & h[0, 1] \\ h[0, 0] & h[1, 0] & 0 & 0 & h[0, 1] & h[1, 1] \end{pmatrix}.$$

と表現できる.

2 変量循環畳み込みの行列生成

(Bivariate circular convolution matrix generation)

```
% Find the matrix representation of the circular convolution
```

```
T = zeros(numel(u));
```

```
for idx = 1:numel(u)
```

```
    % Generating a standard basis vector
```

```
    e = zeros(size(u), 'like', u);
```

```
    e(idx) = 1;
```

```
    % Response to the standard basis vector
```

```
    T(:, idx) = reshape(mapT(e), [size(T, 1) 1]);
```

```
end
```

```
% Matrix representation of the convolution
```

```
T
```

```
T = 6x6
```

```
    -1    -1     1     1     0     0
    -1    -1     1     1     0     0
     0     0    -1    -1     1     1
```

0	0	-1	-1	1	1
1	1	0	0	-1	-1
1	1	0	0	-1	-1

行列演算による 2 変量循環畳み込み

(Bivariate circular convolution by matrix operation)

2 変量循環畳み込みも可換図に沿って (Bivariate circular convolution can also be computed as)

$$\{v[\mathbf{n}]\}_{\mathbf{n}} = \text{vec}_{\Omega_v}^{-1}(\mathbf{v}) = \text{vec}_{\Omega_v}^{-1}(\mathbf{T}\mathbf{u}) = \text{vec}_{\Omega_v}^{-1} \circ \mathbf{T} \text{vec}_{\Omega_u}(\{u[\mathbf{n}]\}_{\mathbf{n}})$$

のように行列演算が可能である。すなわち, (along the commutative diagram. That is, we have)

$$T = \text{vec}_{\Omega_v}^{-1} \circ \mathbf{T} \text{vec}_{\Omega_u}.$$

と表現できる。

```
% Column vectorization of sequence u[n]
vecu = u(:);

% Matrix operation
vecv = T*vecu;

% Reshaping the result into the row sequence
recv = reshape(vecv,[N1 N2])
```

```
recv = 2x3
     4     4    -8
     4     4    -8
```

行列演算による 2 変量循環畳み込みの評価

(Evaluation of bivariate circular convolution by matrix operation)

```
% Comparizon between mapping and matrix operation
mymse(v,recv)
```

```
ans = 0
```

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