MSJ Math Club

Complex Numbers and Complex Trig

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1 Introduction

The *imaginary unit* i is defined such that $i^2 = -1$. Notice that $i \neq \sqrt{-1}$, which is still undefined for our purposes (due to the lack of distinction between positive and negative when it comes to imaginary numbers.

A complex number $x \in \mathbb{C}$ is defined such that $x = a + bi$, where $a, b \in \mathbb{R}$. We'll give a proper definition when we get to constructions of the reals and other advanced topics.

To apply arithmetic operations to complex numbers, one treats i as a variable, and replaces instances of i^2 with -1 accordingly.

2 Motive

Suppose you had two complex numbers $z_1 = \cos(\theta_1) + i\sin(\theta_1)$, and $z_2 = \cos(\theta_2) + i\sin(\theta_2)$. If you multiply these two numbers, you get:

$$\begin{aligned} z_1 z_2 &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \end{aligned}$$

Or, if you define a function $\operatorname{cis}(x) = \cos(x) + i\sin(x)$, then $\operatorname{cis}(\theta_1)\operatorname{cis}(\theta_2) = \operatorname{cis}(\theta_1 + \theta_2)$

Also, $e^{i\theta} = \operatorname{cis}(\theta)$. We won't prove this in this handout because the proof requires calculus.

3 Basics

With the definition of $\operatorname{cis}(x)$, we can derive these two equations:

$$\sin(x) = \frac{(\cos(x) + i\sin(x)) - (\cos(x) - i\sin(x))}{2i} = \frac{\operatorname{cis}(x) - \operatorname{cis}(-x)}{2i}$$

$$\cos(x) = \frac{(\cos(x) + i\sin(x)) + (\cos(x) - i\sin(x))}{2} = \frac{\operatorname{cis}(x) + \operatorname{cis}(-x)}{2}$$

We can use these equations to derive various trig identities. For example if we want to express $\cos(x)^3$ in terms of only first powers of trig functions, we can do this:

$$\begin{aligned} \cos(x)^3 &= \left(\frac{\operatorname{cis}(x) + \operatorname{cis}(-x)}{2}\right)^3 = \frac{\operatorname{cis}(x)^3 + 3\operatorname{cis}(x)^2\operatorname{cis}(-x) + 3\operatorname{cis}(-x)^2\operatorname{cis}(x) + \operatorname{cis}(-x)^3}{8} \\ &= \frac{1}{4}\frac{\operatorname{cis}(x)^3 + \operatorname{cis}(-x)^3}{2} + \frac{3}{4}\frac{\operatorname{cis}(x) + \operatorname{cis}(-x)}{2} = \frac{\cos(3x) + 3\cos(x)}{4} \end{aligned}$$

4 Tips and Tricks

- The idea of complex numbers can seem really theoretical at first, but once you get used to it, they can begin to feel really natural to use.
- The complex conjugate of a complex number $\overline{a+bi} = a - bi$.
- Complex roots of unity are complex solutions to polynomials of the form $x^n - 1 = 0$.

5 Examples

1. (2014 AIME I) Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = arg(\frac{w-z}{z})$. Find the maximum possible value of $\tan^2(\theta)$.
2. (Classic) Evaluate
$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n}$$

6 Practice Problems

1. (Classic) Show that $\sin^2(x) + \cos^2(x) = 1$ using complex numbers.
2. (Classic) Try proving some of the harder trig identities that you had in your math classes with complex numbers. Smile at how easy they are now.
3. (Classic Variant) Show that $\sin(20^\circ)\sin(40^\circ)\sin(80^\circ) = \sin(30^\circ)\sin(30^\circ)\sin(60^\circ)$
4. (2008 AIME II) A particle is located on the coordinate plane at $(5, 0)$. Define a move for the particle as a counterclockwise rotation of $\pi/4$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p, q), find the greatest integer less than or equal to $|p| + |q|$.
5. (1990 AIME) The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C?
6. (2005 AIME II) For how many positive integers n less than or equal to 1000 is $(\sin t + i\cos t)^n = \sin nt + i\cos nt$ true for all real t?
7. (2008 AMC 12A) A sequence $(a_1, b_1), (a_2, b_2), (a_3, b_3), \cdots$ of points in the coordinate plane satisfies
$$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n).$$
Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?
8. (2000 AIME II) Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.
9. **Puzzle of the Week:** Suppose, like in the previous handout, there are n pirates and 1000 gold coins in a chest. The pirates line up in order, and starting from the first pirate, he proposes a way to split up the gold coins. If and only if a strict majority of the pirates agree with his scheme, the pirates split up according to his plan. Otherwise, the pirate is forced to walk the plank. Then, control goes over to the next pirate in line. We keep doing this until an agreement is reached by a majority of the pirates remaining. Considering that each pirate is a perfect mathematician, what is the smallest value of n such that the first pirate gets less than half the coins? In order, the pirates have these priorities: living, getting the most gold coins, and finally killing off as many other pirates as possible.