Blitz Round Solutions

MagMaR 2014

January 26, 2014

A1. [**2**] A factorial, or $n!$, is equal to $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. Find the sum of the last eight digits of $(((4!)!)!)!$.

A1. **0**

Solution: Let $N = 4!!!$, and observe that N is very large (precisely, greater than 10^8). Then $N!$ has at least 8 factors of 5 and 8 factors of two, so all eight trailing digits are zero, hence the final answer of $\boxed{0}$.

A2. [**3**] A date is written on the board in the form $MM - DD - YY$, where MM represents the month, DD represents the date, and YY represents the last two digits of the year. However, Alvin misreads this as the expression $(MM - D) \cdot (D - YY)$. List all dates in 2014 for which this expression is a positive perfect cube in $MM - DD - YY$ format.

A2. **01-26-14**

Solution: Here, $YY = 14$. So we want $1 \le m \le 12$ and digits d_1, d_2, with $(m - d_1)(d_2 - 14)$. Clearly $d_2 - 14 < 0$, so $m - d_1 < 0$ follows. But $d_1 \in \{1, 2, 3\}$ and hence $m \in \{1, 2\}$. Checking the cases, we see that the only possibility is January 26, 2014.

A3. [**3**] In a high school curriculum, one must learn Pre-Algebra before learning either Algebra or Probability. Additionally, Geometry and Calculus each require an Algebra background, while Statistics and Discrete Math each require Probability as a prerequisite. If Arthur must learn all seven subjects (Pre-Algebra, Algebra, Probability, Geometry, Calculus, Statistics, and Discrete Math), in how many orders can he learn the subjects?

A3. **80**

Solution: Clearly Arthur must learn pre-algebra first. Let A, A_1, A_2 denote algebra, geometry, calculus; let B, B_1, B_2 denote probability, statics, discrete math. Then the orderings are any permutation of the above six characters such that A precedes A_1, A_2 and likewise for B.

If we ignore the subscripts then there are $\binom{6}{3} = 20$ possible ways. Then for each of A and B, we have two ways to assign the subscripts (either AA_1A_2 or AA_2A_1). Hence we obtain $20 \cdot 2^2 = 80$.

Note: this is a very specific case of a more general problem, counting the number of linear extensions of a partially ordered set.

A4. [**4**] A convex n-gon $P_1P_2\dots P_n$ is given. The sum of the interior angles at P_1, P_2, $\dots$, P_{n-1} and the supplement of the interior angle at P_n is 2000°. Compute the largest possible value of n.

A4. **14**

Solution: Let x be the angle at P_n. We obtain

$$180(n-2) - x + (180 - x) = 2000$$

or

$$180n - 2x = 2180.$$

We know $0 \le x \le 180$. From here it is straightforward to deduce that the largest n is 14, achieved at $x = 170$.

B1. [**2**] Jared is shooting arrows at a target. So far, he has hit the target 13 times out of a total of 37 shots. How many more consecutive hits must he make in order to have a 75% success rate?

B1. **59**

Solution: Solving $\frac{13+x}{37+x} = \frac{3}{4}$ gives $x = 59$.

B2. [**3**] Four cubes, each with 5 green sides and 1 red side, are arranged together on the table to form a larger square prism. What is the probability that no red sides are visible from the sides or top?

B2. **1/16**

Solution: We simply view each cube independently (as the configuration is symmetric). The red face can either be turned face down or face one of the two interior sides, so each cube has a $\frac{1}{2}$ chance of being good, so the answer is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

B3. [**3**] Let m be the result when $\sqrt{41}$ is rounded to the nearest tenth. Find $10m$.

B3. **64**

Solution: The trick is to approximate $10\sqrt{41} \approx \sqrt{4096} = \sqrt{2^{12}} = 64$.

B4. [**4**] You walk into a random store and buy a random item priced uniformly at random between \$0.01 and \$99.99. (The cost is an integer number of cents.) Since you do not know how much the item costs, you hand the cashier a \$100 bill. The cashier tries to give you change in as few coins as possible. If he is out of quarters and does not have half-dollars, what is the expected number of coins that you will receive? (Remember that for change that exceeds \$0.99, the cashier will pay you the whole dollars with paper bills.)

B4. **7**

Solution: The number of dimes given is equal the tens digit, which has an expected value of $\frac{1}{10}(0+1+\cdots+9) = 4.5$. The number of pennies given is equal to the remainder when the change is divided by 5, so the expected number of pennies is $\frac{1}{5}(0+1+\cdots+4) = 2$. We will either need 0 or 1 nickels depending on whether the units digit of the change is greater than 4 or not, so the expected number of nickels is 0.5. Summing (valid by linearity of expectation), we get 7.

C1. [**2**] In a weird variation of Rock-Paper-Scissors (members' edition), each of two players can choose from 101 different signs to play on his or her turn, such that each symbol trumps 50 symbols and loses to 50 other symbols. What is the probability that in a single round of Rock-Paper-Scissors (members' edition), the two players tie?

C1. **1/101**

Solution: This is just the probability the throws are the same, which is $1 - 2 \cdot \frac{50}{101} = \frac{1}{101}$.

C2. [**3**] The positive difference between the largest two angles of triangle ABC is equal to the positive difference between the smallest two angles. Suppose that $BC < CA < AB = 10$, and let D be the foot of the altitude from A to $\overline{BC}$. Find BD.

C2. **5**

Solution: Evidently $\angle A, \angle B, \angle C$ form an increasing arithmetic progression in that order. Thus, $\angle A + \angle B + \angle C = (\angle B - x) + \angle B + (\angle B + x) = 180°$, which implies that $\angle B = 60°$. As a result, $BD = 10 \cdot \frac{1}{2} = 5$.

C3. [**3**] Compute the number of ways to tile a 3×12 grid with L-trominoes (**not** tetrominos). (An L-tromino is a shape composed of three squares in an L-shape, as depicted below.)

C3. **64**

Solution: Dissect the grid into six 3×2 grids. Each can be tiled in two ways. It is easy to verify that any L-tiling must correspond to tiling each of these six sub-grids, giving an answer of $2^6 = 64$.

C4. [**4**] Square $ABCD$ and $AEFG$ have the same side length and are oriented in the same direction. (In other words, $ABCD$ and $AEFG$ are either both clockwise or counterclockwise.) If the length of $ED = 20\sqrt{13}$ and $BG = 20\sqrt{14}$. Compute the side length of each square.

C4. $30\sqrt{3}$

Solution:

We rotate the original diagram by 90 degrees clockwise about point A, mapping point E to point G, C to C', and D to D'. Thus, $DE = D'G$ because rotations preserve lengths. Furthermore, since $AD' = AB = AG = x$ and A is on the side length BD' of triangle $BD'G$, triangle $BD'G$ is right. Thus, by the Pythagorean Theorem, we have that $BG^2 + D'G^2 = D'B^2$, or $(20\sqrt{14})^2 + (20\sqrt{13})^2 = (2x)^2$. Solving gives that $x = 30\sqrt{3}$.

D1. [**2**] A circular piece of cardboard weighs 20 mg and has circumference 1 meter. Mr. Recycle Bin comes and eats part of the paper. The remaining piece of paper weighs 5mg and is also circular. How many centimeters are in its circumference now?

D1. **50**

Solution: The area changed by a factor of $\frac{1}{4}$, so the circumference changes by a factor of $\sqrt{\frac{1}{4}} = \frac{1}{2}$. That means the new circumference is $\frac{1}{2} \cdot 100 = 50$ centimeters. (Watch units!)

D2. [**3**] What is the smallest positive integer that can be written as the sum of two positive squares in two distinct ways?

D2. **50**

Solution: $50 = 7^2 + 1^2 = 5^2 + 5^2$, and one can check this is minimal. The "elegant" proof that this is minimal involves factoring Gaussian integers.

D3. [**3**] You start on the center square of a 3×3 grid. Every minute, you move to an adjacent square (up, down, left, or right). What is the probability that you are on the center square after 7 minutes?

D3. **0**

Solution: Color the squares alternatively red and blue (like a chessboard/checkerboard). Every minute, we must change the color of the square we are on. If the center is red, we end at a blue square after seven minutes (as seven is odd), so the probability is zero.

D4. [**4**] Gary lives in a triangular house BRN with $BR = BN = 3\sqrt{10}$ meters and $RN = 6$ meters, with one exit at each corner of the house. One day, while burning CDs, his computer overheats and starts a fire in his house. Immediately, he runs to the closest exit. What is the probability that he exits through corner B?

D4. **5/18**

Solution:

Let M, X, and Y be the midpoints of sides RN, BR, and BN respectively. Since $BR = BN$, we know that $BM \perp RN$.

Next we split up his house into regions to determine when he exits through a certain door in the house. Since he exits through the closest door, we consider the perpendicular bisectors of the sides, OX, OY, and OM. Any point on the perpendicular bisector of a line segment is equidistant from each of the endpoints of the line segment. Thus, the three regions in the triangle formed by cutting along the perpendicular bisectors dictate the door Gary chooses to escape through.

To compute the area of the shaded region $BXOY$, we first compute the length of $ON = OB = OR = x$. Notice that since BMN is a right triangle, $BM = \sqrt{BN^2 - MN^2} = \sqrt{(3\sqrt{10})^2 - (6/2)^2} = 9$. Thus, $OM = BM - OB = 9 - x$, so with Pythagorean Theorem on $\triangle OMN$, we can write $3^2 + (9 - x)^2 = x^2$. This equation expands to $9 + 81 - 18x + x^2 = x^2$, so $x = 5$.

To compute the area, notice that each of the triangles BXO, RXO, BYO, and NYO are congruent, and thus have the same area A. We have that:

$$[BRN] = BM \cdot RN/2 = 4A + [ORN] = 4A + OM \cdot RN/2$$
$$4A = (BM - OM)(RN/2) = (BO)(RN/2) = (5)(6/2) = 15$$

Thus, the area of the desired region is $2A = 15/2$. Since the area of the entire triangle $BRN = 9 \cdot 6/2 = 27$, the probability is $(15/2)/27 = 5/18$.

Note: The three perpendicular bisectors of the three sides of a triangle always intersect at a single point called the circumcenter of a triangle.

E1. [**2**] Let ABC be a triangle with area 800 and let P be a point inside it. Let D, E, F be the midpoints of $\overline{AP}$, $\overline{BP}$, $\overline{CP}$. What is the area of DEF?

E1. **200**

Solution: It's easy to see (say, by homothety or similar triangles) that $\triangle DEF \sim \triangle ABC$ with a factor of two. Hence the answer is $\frac{1}{4} \cdot 800 = 200$.

E2. [**3**] Each of the four basic operators $+, -, \times$, and $\div$ is filled into one of the blanks in the expression $1 \underline{\ \ \ } 2 \underline{\ \ \ } 3 \underline{\ \ \ } 4 \underline{\ \ \ } 5$. What is the largest value attainable from evaluating the expression?

E2. **61/3**

Solution: To try and maximize the expression, we try placing the $\times$ symbol between 4 and 5 first, because multiplication will ultimately boost the final number the most. We do not want to place a $\div$ or $-$ symbol between the 3 and the 4, or else it will negate the 4×5 at the end, so we are left with $1 \underline{\ \ \ } 2 \underline{\ \ \ } 3 + 4 \times 5$. Checking the possibilities for the remaining two blanks, we have that the maximum value is $1 - 2 \div 3 + 4 \times 5 = 61/3$.

One can check that no bigger value exists by noting that the next largest product possible (3×4) added to the remaining three numerals (1, 2, 5) is still not large enough to overtake 61/3.

E3. [**3**] Given a regular hexagon, what is the ratio of the area of the largest equilateral triangle that can fit inside the hexagon to the area of the smallest equilateral triangle that the hexagon can fit into?

E3. **3**

Solution:

The largest and smallest equilateral triangles are depicted above on the left in blue and red respectively. To compute the area of the red triangle relative to the hexagon, notice that we can fold three vertices of the hexagon to form the triangle, so the small triangle has half the area of the hexagon. For the large triangle, we divide the hexagon into equilateral triangles, and notice that the area of the large triangle is 3/2 times the that of the hexagon. Thus, the ratio is $(3/2)/(1/2) = 3$.

E4. [**4**] Mrs. Phair is generating a sequence of numbers $a_1, a_2, a_3, \cdots$ on her whiteboard. In the first minute, she writes $a_1 = 1$. During the n^{th} minute for $n \geq 2$, Mr. Unfair randomly orders the numbers $\{1, 2, 3, \cdots, n-1\}$ and lets them equal to $b_1, b_2, \cdots, b_{n-1}$ respectively. Mrs. Phair then lets $a_n = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_{n-1} b_{n-1}$. Let k be the smallest positive integer such that the maximal and minimal possible values for a_k differ by more than 2013. Compute this difference.

E4. **2376**

Solution: It's easy to see $a_1 < a_2 < \ldots$, and so the maxima and minima are obtained when the b_i are always increasing or always decreasing, respectively. (One can see this by, say, the rearrangement inequality.) Explicitly, let C_i and c_i denote the maximal and minimal values of a_i respectively. Then $C_i = c_i = 1$ and

$$C_n = C_1 + 2C_2 + \cdots + (n-1)C_{n-1}, \quad c_n = c_{n-1} + 2c_{n-2} + \cdots + (n-1)c_1$$

We compute

$$C_n : 1, 1, 3, 12, 60, 360, 2520,$$

$$c_n : 1, 1, 3, 8, 21, 55, 144, \ldots$$

We can see that the answer is $2520 - 144 = 2376$.

Note: For a bonus challenge, try to prove that $C_n = n!/2$ and that $c_n = F_{2n-3}$ for $n > 1$. Here, $k! = k \times (k-1) \times \cdots \times 3 \times 2 \times 1$ and F_k denotes the k^{th} Fibonacci number, with $F_0 = F_1 = 1$.

F1. [**2**] What is the sum of the distinct positive factors of 72?

F1. **195**

Solution: Since $72 = 2^3 \cdot 3^2$, every factor of 72 can be distinctly written as $2^a \cdot 3^b$ for some $0 \le a \le 3$ and $0 \le b \le 2$. The answer is just $(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2) = 195$.

Note: The above factorization expands to $2^0 3^0 + 2^0 3^1 + 2^0 3^2 + 2^1 3^0 + 2^1 3^1 + 2^1 3^2 + 2^2 3^0 + 2^2 3^1 + 2^2 3^2 + 2^3 3^0 + 2^3 3^1 + 2^3 3^2 = 1 + 3 + 9 + 2 + 6 + 18 + 4 + 12 + 36 + 8 + 24 + 72$, which is indeed the sum of the distinct positive factors of 72.

F2. [**3**] A cone has a height of 12 units and a circular base of radius 4. A cut is made through the middle of the cone's height, parallel to its base, forming a mini cone and another solid. What is the volume of this other solid?

F2. 56π

Solution: The original cone has volume $\frac{1}{3} \cdot 12 \cdot 4^2\pi = 64\pi$. By 3D similarity, the mini-cone has $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ volume of the original, and so the desired solid has $\frac{7}{8}$ of the volume. This is $\frac{7}{8} \cdot 64\pi = 56\pi$.

F3. [**3**] One card is missing from a pack of 52 standard playing cards. (Standard playing cards have four suits – spades, hearts, diamonds, clubs – each with one of 13 possible face values.) From the remaining cards in the pack, one card is drawn and is found to be spades. Find the probability the missing card is also a spade.

F3. **12/51**

Solution: Suppose that whatever spade card you drew is the missing card instead, and that the missing unknown card is actually a card randomly drawn from the remaining deck of cards. There are 12 spades remaining and 51 cards, so the answer is 12/51.

Note: One can also use Bayes' Theorem to solve this problem.

F4. [**4**] Let $\Pi(n)$ denote the product of the digits of a positive integer n. If n only has one digit, then $\Pi(n) = n$. How many prime numbers $p \leq 100$ have the property that $\Pi(p)$ is also a prime number?

F4. **8**

Solution: All 1-digit positive primes satisfy that $\Pi(n) = n$, which include 2, 3, 5, and 7. Additionally, for a two-digit prime number to satisfy the equation, one of the digits must be 1 and the other must be a prime number. The primes that satisfy this are 13 and 17 if the 1 comes in the tens digit, and 31 and 71 if the 1 comes in the units digit. Thus, the answer is $4 + 2 + 2 = 8$.

G1. [**2**] Compute $1 - 3 + 5 - 7 + \cdots + 101$.

G1. **51**

Solution: This sum is equal to $(1-3)+(5-7)+\cdots+(97-99)+101 = 25(-2)+101 = 51$.

G2. [**3**] A five-digit positive integer $n = 123AB$ with A and B as digits is chosen. For what value of A is n never divisible by 11?

G2. **1**

Solution: By the divisibility rule for 11, we know that $1+3+B$ must leave the same remainder as $2 + A$ when each is divided by 11. In order for no single-digit values of B to exist, $2 + A$ cannot leave the same remainder as any of $1 + 3 + 0$, $1 + 3 + 1$, $\cdots$, $1 + 3 + 8$, $1 + 3 + 9$, or that $2 + A$ and $1 + 3 + 10$ must leave the same remainder upon division by 11. Since A is a digit, it must be equal to 1.

G3. [**3**] How many positive integers n less than 9000 can be written in the form 2014_b, for some base $b \geq 5$? (The subscript denotes a number base. When we write a numeral, such as 421, the value of it is $4 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0$ because we use base 10.)

G3. **12**

Solution: We want to find the largest n such that $2014_n = 2n^3+n+4 < 9000$. Careful testing reveals that $2014_{16} = 2(16)^3 + 16 + 4 = 2^{13} + 20 = 8212$ and $2014_{17} = 2(17)^3 + 17 + 4 = 9847$, and since $n \geq 5$, the answer is $16 - 5 + 1 = 12$.

G4. [**4**] We say $(A, B; X, Y)$ is a *harmonic bundle* if A, B, X, Y are four points satisfying the following conditions: (i) the points are distinct and collinear, (ii) $\frac{XA}{XB} = \frac{YA}{YB}$ and (iii) the segments $\overline{AB}$ and $\overline{XY}$ overlap. For example, $A = (-5, 0)$, $B = (5, 0)$, $X = (1, 0)$ and $Y = (25, 0)$ form a harmonic bundle.

Suppose X is selected on segment $\overline{AB}$ such that it is impossible to find Y for which $(A, B; X, Y)$ is a harmonic bundle. If $AB = 100$, what is AX?

G4. **50**

Solution: Let $A = (-50, 0)$, $B = (50, 0)$, $X = (x, 0)$, and $Y = (y, 0)$ with $-50 < x < 50$ and $y > 50$. Using the definition of a harmonic bundle, we have that $XA \cdot YB = XB \cdot YA$, or $(x-(-50))(y-50) = (50-x)(y-(-50))$. This equation expands to $xy+50y-50x-2500 = -xy - 50x + 50y + 2500$, which is $xy = 5000$ or $y = 5000/x$. If there are no possible values for y, then the right-hand side of the last equation must be undefined, so $x = 0$ or $AX = 50$.

Note: Harmonic bundles come up in a branch of geometry called projective geometry.

H1. [**2**] Choose a point O on a plane. Let $\mathcal{A}$ be the set of all points of distance at most 2 units from O. Let $\mathcal{B}$ be the set of all points of distance at most 3 from some point in $\mathcal{A}$. Let $\mathcal{C}$ be the set of all points of distance at most 5 from some point in $\mathcal{B}$. To the nearest integer, what is the area of $\mathcal{C}$?

H1. **314**

Solution: $\mathcal{A}$ is a circle with a radius of 2 centered at point O. We can see that $\mathcal{B}$ is a circle with a radius of $2 + 3 = 5$ centered at point O, and $\mathcal{C}$ is a circle with a radius of $5 + 5 = 10$ centered at point O. Thus, the area of $\mathcal{C}$ is $10^2 \cdot \pi \approx 314$.

H2. [**3**] Let M be the midpoint of side BC of triangle ABC with $AB = AC = 13$ and $BC = 10$. A semicircle centered at point M is tangent to sides AB and AC. What is the radius of the semicircle?

H2. **60/13**

Solution: By the Pythagorean Theorem, we have that $AM = \sqrt{AB^2 - BM^2} = \sqrt{13^2 - (10/2)^2} = 12$. If P is the tangency point of the semicircle with side AB, then $MP \perp AB$ by the definition of tangency. We can express the area of triangle ABM is two ways: $K = AM \cdot MB/2 = AB \cdot MP/2$, or that $r = MP = AM \cdot MB/AB = 12 \cdot 5/13 = 60/13$.

H3. [**3**] The answer to this question is a two-digit number which can be written in the form $10a+b$, where a and b are positive integers such that $0 < a, b < 10$. Find $2ab$.

H3. **36**

Solution: The trick to this problem is to use the two-digit number in the form $10a + b$, and write the equation $10a + b = 2ab$ or $2ab - 10a - b = 0$. Adding 5 to either side, we see that $2ab - 10a - b + 5 = 5$, or $(2a - 1)(b - 5) = 5$. Thus, $2a - 1$ and $b - 5$ must be equal to 1 and 5 in some order. Because a and b must both be digits, we have that $b - 5 = 5$ is impossible, so $b - 5 = 1$ and $2a - 1 = 5$, or $(a, b) = (3, 6)$. The answer is 36.

H4. [**4**] A man is choosing among three different entrees for his dinner. He first considers plate A, eating it with a 1/3 chance and moving on to plate B with 2/3 chance. In general, from whatever plate he is considering consuming, he has a 1/3 chance of eating it and 2/3 of considering the next plate alphabetically (C after B after A after C). What is the probability that he ends up eating plate A?

H4. **9/19**

Solution: We proceed with casework. The probability that he eats it on his first consideration is 1/3. If he eats it on his second consideration, he must reject the first three plates (A, B, C), and accept the fourth plate (A), leaving a probability of $(2/3)^2 \cdot (1/3)$. If he eats A on his third consideration, he must reject the first six plates (A, B, C, A, B, C) and accept the seventh plate, which gives a probability of $(2/3)^6 \cdot (1/3)$. This casework continues forever. The final probability is:

$$\begin{aligned}
&\frac{1}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3}\left(\frac{2}{3}\right)^6 + \frac{1}{3}\left(\frac{2}{3}\right)^9 + \cdots \\
&= \frac{1}{3}\left(1 + \left(\frac{8}{27}\right) + \left(\frac{8}{27}\right)^2 + \left(\frac{8}{27}\right)^3 + \cdots\right) \\
&= \frac{1}{3} \cdot \frac{1}{1-(8/27)} \\
&= \frac{9}{19}
\end{aligned}$$

Note: In this solution, we used that the sum of an infinite geometric sequence $1+p+p^2+p^3+\cdots$ with $|p|<1$ is $1/(1-p)$. To prove this let $S = 1+p+p^2+p^3+\cdots$. Then $p \cdot S = p+p^2+p^3+\cdots$, and subtracting gives $(1-p)\cdot S = 1$.

I1. [**2**] A random day in January 2014 is chosen. What is the probability that it is a Saturday or Sunday? (Today, January 26, 2014, is a Sunday.)

I1. **8/31**

Solution: The Sundays in January are on January $5, 12, 19, 26$ and the Saturdays are on January $4, 11, 18, 25$, so the answer is 8/31.

I2. [**3**] In triangle ABC, side $AB = 5$, $AC = 13$, and $\angle B = 96^\circ$. How many points P on segment BC (distinct from points B and C) are there such that the length of AP is an integer?

I2. **7**

Solution: If P can be any point on the segment including points B and C, then the minimum possible length is $AB = 5$ (since $\angle B$ is obtuse, there cannot be anything shorter) and the maximum is $AC = 13$. Thus, the possible values for the length of AP are $6, 7, \cdots, 11, 12$, so the answer is 7.

I3. [**3**] Two congruent equilateral triangles overlap as shown below. If the side length of each triangle is 12 and the height h marked is $7\sqrt{3}$, compute the area of the region of overlap.

I3. $\frac{45}{2}\sqrt{3}$

Solution: The height of each equilateral triangle is $(12/2) \cdot \sqrt{3} = 6\sqrt{3}$, so the side length of each of the small equilateral triangles on the top and bottom are 2 and the side length of the four equilateral triangles on the side are 5.

To compute the area of the overlap, we split the region of overlap into two congruent trapezoids by drawing the horizontal line in the middle. The area of each trapezoid is the area of an equilateral triangle with side length of 7 minus the area of an equilateral triangle of side length 2. Thus, the total area is $2 \cdot \left(\frac{7^2\sqrt{3}}{4} - \frac{2^2\sqrt{3}}{4}\right) = 45\sqrt{3}/2$.

I4. [**4**] How many integers $m < 100$ of the form pq for distinct prime numbers p, q cannot be written in the form $\dbinom{n}{r}$, for positive integers $n \geq r \geq 0$?

I4. **0**

Solution: Any positive integer m can be expressed as $\binom{m}{1}$, so the answer is 0.

J1. [**2**] John and his brother are running around a 400 meter track. John runs at 8 m/s while his brother runs at 3 m/s. If they start running from the same spot at the same time in the same direction, how many seconds will have passed when John laps his brother the second time?

J1. **160**

Solution: John moves at $8 - 3 = 5$ meters per second relative to his brother. Thus, John passes his brother for the second time after $2 \times (400/5) = 160$ seconds.

J2. [**3**] What is the slope of the line connecting the two points of intersection of $20x + 14y = 24$ and $x^2 + y^2 = 42$?

J2. $-10/7$

Solution: The two points of intersection lie on the line $20x + 14y = 24$, which is equivalent to $y = (-10/7)x + 12/7$, so the slope is $-10/7$.

J3. [**3**] If $\dfrac{x-2}{3}, \dfrac{x-4}{5}, \dfrac{x-6}{7}$ are all positive integers, what is the smallest possible value of x?

J3. **104**

Solution: Let $y = x + 1$. Then $\frac{y-3}{3} = \frac{y}{3} - 1$, $\frac{y-5}{5} = \frac{y}{5} - 1$, and $\frac{y-7}{7} = \frac{y}{7} - 1$ must all be integers. Hence, y must be divisible by 3, 5, and 7, so the smallest positive value of y that satisfies this is 105. Thus, $x = 104$.

J4. [**4**] What is the area of a triangle with the coordinates $(0, 0, 0)$, $(3, 4, 4)$, and $(-6, -8, 12)$?

J4. **50**

Solution: Consider the plane $\mathcal{P}$ that passes through the three points. Since $(0, 0)$, $(3, 4)$, and $(-6, -8)$ lie on a line in the xy plane, then the plane passing through the three points is perpendicular to the xy plane. Let the a-axis of $\mathcal{P}$ be the line $3y = 4x$ and the b-axis of $\mathcal{P}$ be the z-axis of the original space.

Notice that $(0, 0, 0)$ is the origin of $\mathcal{P}$. To find the coordinates of the other two points, we know that the a-coordinate of each point becomes the perpendicular distance to the b-axis and that the b-coordinate is the z coordinate of the original triple. Thus, $(x, y, z) = (3, 4, 4)$ becomes $(a, b) = (\sqrt{3^2 + 4^2}, 4) = (5, 4)$, and $(x, y, z) = (-6, -8, 12)$ becomes $(a, b) = (-\sqrt{(-6)^2 + (-8)^2}, 12) = (-10, 12)$. (Since the two points are on opposite sides of the origin, one of the points must have a negative a-coordinate.) We now want to find the area of the following triangle:

To find the area of the triangle, we inscribe it in a rectangle, and subtract out the areas of three right triangles. The area of the right triangle on top is $15 \cdot 8/2 = 60$, the area of the right triangle on the left is $12 \cdot 10/2 = 60$, and the area of the triangle in the bottom right is $5 \cdot 4/2 = 10$. Since the area of the entire rectangle is $12 \cdot 15 = 180$, the area of the triangle formed by the three points is $180 - 60 - 60 - 10 = 50$.

Note: One can take the cross-product of two vectors to compute the area of a triangle with any three points in space.

K1. [**2**] On a 20-question True-False quiz, Brenda guesses on all of the questions and gets 16 of the questions correct. If she flips the answers to each of the first 10 questions, she will have 12 questions correct. How many questions will she answer right if she changes the answer to each of the last 10 problems as well?

K1. **4**

Solution: In the end, she will have flipped all of the answers, so she will have $20 - 16 = 4$ correct.

K2. [**3**] Forty 2×3 rectangles are arranged together into a figure with no gaps inside. What is the minimum possible perimeter of this figure?

K2. **62**

Solution: The area of the final figure will always be $40 \times 2 \times 3 = 240$. In order to minimize the perimeter of the figure, we take a rectangle with dimensions as close as possible. Since $240 = 15 \times 16$, the minimum possible perimeter is $2(15 + 16) = 62$.

K3. [**3**] The median and mean of the five numbers in $\{5, 9, x, 14, 6\}$ are both equal. What is the sum of all possible values of x?

K3. **15.5**

Solution: We do casework on the median of the set. Notice that $5 < 6 < 9 < 14$, so neither 5 nor 14 can be the median.

Case 1: 6 is the median. The mean is $6 = (5 + 9 + x + 14 + 6)/5$, or $x = 30 - 34 = -4$. Checking indeed puts 6 as the median.

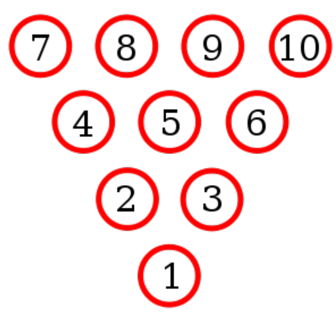
Case 2: x is the median. The mean is $x = (5 + 9 + x + 14 + 6)/5$, or $5x - x = 34$. Thus, $x = 8.5$, which indeed makes x the median.

Case 3: 9 is the median. The mean is $9 = (5 + 9 + x + 14 + 6)/5$, or $x = 45 - 34 = 11$. Checking indeed confirms that 9 is the median in this case.

Thus, the sum is $(-4) + 8.5 + 11 = 15.5$.

K4. [**4**] Bob is bowling, only knocking down the first pin (marked as 1 in the diagram below). When a pin is toppled, each of the one or two pins has a $\frac{1}{2}$ chance of being knocked over as well (independent of the other pin's results). What is the probability that Bob bowls a strike (knocks all 10 pins down)?

K4. **27/4096**



Solution: All pins must be knocked over in the end. Each of the pins on the edge (2, 4, 7, 3, 6,10) can only be knocked over by one possible pin, each with probability $1/2$. Each of the pins in the center (5, 8, 9) must be knocked over by one of two pins in front of it. The probability that a given pin in the front does not knock down a given pin behind it is $1-1/2=1/2$, so the probability that each of the center pins are knocked is:

$$1-P(\text{neither of the two pins in front knock it down}) = 1-\left(\frac{1}{2}\cdot\frac{1}{2}\right)=\frac{3}{4}.$$

Thus, the probability is $\left(\frac{1}{2}\right)^6\cdot\left(\frac{3}{4}\right)^3=\frac{27}{1024}$.

L1. [**2**] Farmer Don wants to build a closed fence around his (rectangular) house farm to keep neighbors out and buys a bunch of posts. If he puts them each 1 foot apart, he will still need 150 more posts to fully surround his plot of land, but if he puts them 1 yard apart, he will have 90 poles too many. How many fenceposts does he have?

L1. **210**

Solution: Let x be the number of posts he has and p be the perimeter of his land, in feet. From the first piece of information, we have that $x+150=p$, and from the second piece of information, we have that $3(x-90)=p$. Thus, $x+150=3x-270$, or $x=210$.

L2. [**3**] What is the sum of the decimal digits of $11\times 101\times 10,001\times 100,000,001$?

L2. **16**

Solution: We have:

$$\begin{aligned} 11\times 101\times 10,001\times 100,000,001 & \\ &= 1111\times 10,001\times 100,000,001\\ &= 11,111,111\times 100,000,001\\ &= 1,111,111,111,111,111 \end{aligned}$$

There are 16 digits, so the answer is 16.

L3. [**3**] For how many values of n does a regular n-gon have integer interior angle measures?

L3. **22**

Solution: The sum of the interior angles of a regular n-gon is $180(n-2)$, so $(180n-360)/n = 180-360/n$ must be an integer. Since 180 is an integer, n must be a factor of $360=2^3\cdot 3^2\cdot 5^1$, which has $(3+1)(2+1)(1+1)=24$ factors. However, n cannot be either 1 or 2, so the answer is $24-2=22$.

L4. [**4**] Find the sum of the decimal digits of $999{,}999{,}999^2 - 899{,}999{,}999^2$.

L4. **73**

Solution: We observe that by the difference of squares:

$$d = (999,999,999 - 899,999,999)(999,999,999 + 899,999,999) = 1{,}888{,}888{,}888 \cdot 10^8$$

Hence, the answer is $\boxed{73}$.

M1. [**2**] The sequence $a_0, a_1, \ldots, a_9$ satisfies $a_0 = 0$ and $a_n = n + a_{n-1}$ for each integer n. Find a_9.

M1. **45**

Solution: We have that $a_n = n + (n-1) + \cdots + 2 + 1$, so $a_9 = 9 + 8 + \cdots + 2 + 1 = 45$.

M2. [**3**] A total of 73 students registered online for MagMaR 2014. Henry, Evan, and Aaron handled the registrations, and the average number of students they each checked in was 20. Only one student checked in who had not registered online. How many students who registered online did not show up?

M2. **14**

Figure 1: 1

Solution: The number of people who registered online is $73 - 3(20) + 1 = 14$.

M3. [**3**] In the diagram below, three circles with radii of 4, 8, and 12 have the same center. Compute the area of the shaded region.

M3. 64π

Solution: If we fold the bottom half to overlap the top half of the dart board, the shaded region is a semicircle of radius 12 missing a semicircle of radius 4. The area of this region is $(12^2 - 4^2) \cdot \pi/2 = 64\pi$.

M4. [**4**] Compute the area of the region bounded by the graph of

$$y = ||x + 20| + |x + 13|| - 14$$

and the x-axis.

M4. **147/2**

Solution: We try to determine what the function's graph looks like. First, notice that since $|x + 20| + |x + 13| \geq 0$ for all x, then the outer absolute value signs do not change the function. For $x < -20$, the function is equal to $-(x+20)-(x+13)-14 = -2x-47$. For $-20 \leq x \leq -13$, the function is evaluates to $x + 20 - (x + 13) - 14 = -7$. For $x > 13$, the function becomes $(x + 20) + (x + 13) - 14 = 2x + 19$. Thus, the figure is a trapezoid with bases along the lines $y = 0$ and $y = -7$.

To find the intersection points of the graph with the x-axis, we solve $-2x - 47 = 0$, and $2x + 19 = 0$, which lie at $-47/2$ and $-19/2$. These values, respectively, satisfy the domains of x assumed for the piecewise function. The length of the base along the x-axis is $-19/2 - (-47/2) = 14$, and the length of the base along the line $y = -7$ is $-13 - (-20) = 7$. Thus, the area of the trapezoid is $(14 + 7) \cdot 7/2 = 147/2$.

N1. [**2**] Ernesto is gluing 27 unit cubes together to form a $3 \times 3 \times 3$ cube. What is the fewest pairs of faces he must glue together in order for the large cube to not fall apart?

N1. **25**

Solution: To join 2 pieces, Ernesto needs to glue 1 pair of faces. To join 3 pieces, Ernesto must glue another pair of faces, or 2 in total. In general, Ernesto needs to glue $(n - 1)$ pairs of faces to connect n pieces. However, the center piece of the cube does not need to be glued, so the answer is $27 - 1 - 1 = 25$.

N2. [**3**] Without using calculus, one can intuitively prove that the area of a circle is πr^2 by splitting a circle into many equally sized sectors and arranging them into a parallelogram, as shown in the left diagram in the figure below. Since the two bases have a total length equal to the circumference of the circle, $2\pi r$, and the height is the radius of the circle, then the area of the parallelogram and the area of the circle is $r \cdot (2\pi r)/2$.

A circle of radius 2 is split into 6 sectors and arranged similarly into a parallelogram. What is the area of the analogous parallelogram, as marked in red in the right diagram in the figure below?

N2. $6\sqrt{3}$

Solution: The parallelogram is the union of six equilateral triangles, each with side length 2. Thus, the area of the parallelogram is $6 \cdot (2^2\sqrt{3}/4) = 6\sqrt{3}$.

Note: One can prove that the area of an equilateral triangle with side length of s is $s^2\sqrt{3}/4$ by dropping an altitude. Since the base has length s and the height has length of $s\sqrt{3}/2$, the area formula follows.

N3. [**3**] A line segment has endpoints at $(3, 7)$ and $(38, 72)$. Through the interiors of how many unit squares does this line segment cut?

N3. **95**

Solution:

One can count the number of unit squares that the line segment passes through by drawing a good diagram if so desired.

To properly compute the number of squares that the line segment passes through, we imagine that the diagonal is physically being drawn from $(3, 7)$ to $(38, 72)$. Notice that if the line segment crosses a single gridline (i.e. not at a lattice point), the line segment enters a new unit square, and thus increases the number of squares that it passes through by 1. If the line segment crosses two gridlines simultaneously (i.e. at a lattice point), then the count of the number of unit squares passed through still increments by only 1. To simplify the counting process, we can split the 35×65 rectangle into 25 7×13 rectangles so that the diagonal in each of these mini-rectangles does not pass through any lattice points. The answer we seek is the number of unit squares that the diagonal of a 7×13 rectangle passes through multiplied by 5.

From $(3, 7)$ to $(3+7, 7+13) = (10, 20)$, the line segment crosses the vertical lines $x = 4$, $x = 5$, $\cdots$, $x = 9$ (6 lines)and the horizontal lines $y = 8$, $y = 9$, $\cdots$, $y = 19$ (12 lines). Thus, the diagonal of this 7×13 rectangle passes through $12+7+1 = 19$ unit squares. (The reason that 1 must be added is that the line segment automatically starts in a unit square before crossing any gridlines.) Thus, the entire line segment passes through $19 \times 5 = 95$ unit squares.

N4. [**4**] Necklaces can have beads of two colors: red and gold. Call a necklace *Au-rich* if the majority of its beads are gold. How many distinct necklaces with 11 beads are *Au-rich*? (Two necklaces are the same if one can be reached by a rotation of the other. However, flipping a necklace counts as a different configuration.)

N4. **94**

Solution: We temporarily break the necklace at a certain point, creating a string of beads rather than a loop of beads. Notice that a fixed necklace with both color beads can be broken to represent 11 distinct broken necklaces. For example, a necklace with 10 red beads and 1 gold bead only constitutes 1 necklace pattern, but it can be broken into any of the following:

$$GRR\cdots RRR$$
$$RGR\cdots RRR$$
$$RRG\cdots RRR$$
$$\vdots$$
$$RRR\cdots GRR$$
$$RRR\cdots RGR$$
$$RRR\cdots RRG$$

There are 11 sequences listed above. The key here is that 11 is prime. If a necklace had 12 beads instead, then a necklace with alternating red and gold beads would only have 2 broken configurations, rather than 12. Namely:

$$RGRGRGRGRGRG$$
$$GRGRGRGRGRGR$$

Thus, for any necklace with at least one bead of each color (total of 11 beads), we can count the total number of broken necklaces that have more gold than red beads, divide by 11 to account for rotations in a fixed necklace, and add 1 to include the case in which all of the beads are gold.

There are 2 possible colors of each bead in a broken necklace, so there are a total of 2^{11} broken necklaces. Two of these have either all red or all gold beads. Since half of these necklaces have more red beads than gold beads and the other half have more gold beads than red beads (there cannot be an equal number), the total number of broken necklaces that have more gold than red beads (excluding ones with only gold) is $(2^{11} - 2)/2 = 2^{10} - 1$. Next, we account for rotation by dividing this number by 11, and adding 1 finally. Thus, the final answer is $(2^{10} - 1)/11 + 1 = 1023/11 + 1 = 94$.

Note: The end of this solution uses the physical interpretation of the binomial/choose functions, denoted by $\binom{n}{r}$. The reader may be interested in learning more about this to help clarify the motivation in this solution. Additionally, we used the fact that 11 was prime to justify for the division by 11 to nicely account for rotations. This necklace problem closely resembles the combinatorial proof for Fermat's Little Theorem, showing that p must divide $a^{p-1} - 1$ if a is relatively prime to p. (Here $p = 11$ and $a = 2$.)

O1. [**2**] A spinner is split into eight congruent sections, labelled 1 through 8 clockwise, as shown below. Sean flicks the spinner. Using the number that he lands on, he moves the arrow that many spaces counter-clockwise. At the end of this process, what is the probability that the arrow is pointing to the number '8'?

O1. **1**

Solution: Let n be the number that the spinner lands on. Because the spinner moves back n spaces, he ends up on space $n - n = 0$, which corresponds to 8. Thus, the probability is 1.

O2. [**3**] How many positive integers less than 100 have exactly 4 factors?

O2. **32**

Solution: A positive integer can have exactly 4 factors only if it is a cube of a prime number or the product of two distinct prime numbers. We can see that there are only 2 such perfect cubes less than 100 (namely, 8 and 27). To compute the number of integers that are the products of two distinct primes, we can list the primes less than 50 and carefully examine the multiples of 2, 3, 5 and 7. There are 30 such numbers.

Thus the answer is $30 + 2 = 32$.

O3. [**3**] Aurick chooses a point P in space such that a $90°$ rotation about point P sends $(0, 0, 0)$ to $(20, 14, 2)$. What is the minimum possible distance from the origin to point P?

O3. $10\sqrt{3}$

Solution: Let O be the origin and A be the point $(20, 14, 2)$. Then, we have: $\angle OPA = 90°$ and $OP = AP$ which implies that triangle OAP is a 45-45-90 triangle and $OA = OP\sqrt{2}$. By the Pythagorean Theorem, $OA = \sqrt{20^2 + 14^2 + 2^2} = 10\sqrt{6}$, so $OP = \frac{OA}{\sqrt{2}} = 10\sqrt{3}$.

O4. [**4**] Your mean friend stole your items and is charging you money in return for the items. If you must pay \$14 to get your laptop, 1 folder, and 1 pencil back and \$20 to receive your laptop, 3 folders, and 5 pencils, how much must you pay your friend to get your laptop, all 4 folders, and all 7 pencils back?

O4. **\$23**

Solution: Let the cost of the laptop be A, a folder be B, and a pencil be C. Then we can write the two equations:

$$a + b + c = 14$$

$$a + 3b + 5c = 20$$

Adding these two equations gives $2a + 4b + 6c = 34$, or $a + 2b + 3c = 17$. We subtract this from twice the latter equation, $2a+6b+10c = 40$, which gives $(2a+6b+10c)-(a+2b+3c) = a + 4b + 7c = 23$.