Wild Round Solutions

Mission Math Regional

January 19, 2013

1. How many arithmetic sequences with only integer terms start with 1 and end with 2013?

   Solution: We wish to find the number of possible common differences. Because the difference between the first and last terms is $2013 - 1 = 2012 = 2^2 \cdot 503$, the common difference can be any factor of 2012, of which there are $\boxed{6}$.

2. Compute the sum $21 + 23 + \cdots + 99$.

   Solution: There are $99 - 21 + 1 = 40$ terms and the average of the terms is $\frac{21+99}{2} = 60$, so the sum of the terms is $40 \times 60 = \boxed{2400}$.

3. In a sequence of numbers, the n^{th} term is 3 less than the square of the previous term. If $a_1 = 1$, what is a_9?

   Solution: Testing the first few terms gives that $a_2 = 1^2 - 3 = -2$ and $a_3 = (-2)^2 - 3 = 1$. Thus, the sequence oscillates between 1 and -2, so $\boxed{a_9 = 1}$.

4. A right triangle has two sides of length 5 and 13. What is the largest possible length of the third side of the triangle?

   Solution: We let the legs of the triangle be 5 and 13, so the third side, the hypotenuse, has length $\sqrt{5^2 + 13^2} = \boxed{\sqrt{194}}$.

5. A point P is chosen inside of a square of side length 2. What is the probability that P is not within 1 unit of the center of the square?

   Solution: The desired region is the entire square minus a circle of radius 1 with center coinciding with the center of the square. Hence, the probability is $\dfrac{4-\pi}{4} = \boxed{1 - \dfrac{\pi}{4}}$.

6. If the block is 7 units wide and 9 units tall, what is the perimeter of the figure?

   Solution: The shorter vertical segments add up to the length of the longest one. Similarly, the shorter horizontal segments add up to the length of the longest horizontal segment. Therefore, the answer is $2(7 + 9) = \boxed{32}$.

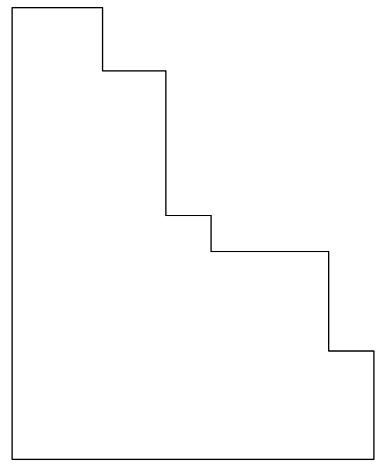

Figure 1: Problem 6

7. Pokeblocks and Stabblocks are sold in cylindrical cans. A can of Pokeblocks has a radius of 2 and height of 5, while a can of Stabblocks has a radius of 3 and height of 3. What is the ratio of the volume of a can of Pokeblocks to the volume of a can of Stabblocks?

   Solution: Because the volume of a cylinder is $hr^2\pi$, the volume of a Pokéblock is $5 \times 2^2 \times \pi$ while the volume of a Stabblock is $3 \times 3^2 \times \pi$. Dividing the two quantities and cancelling the π gives the correct answer of $\boxed{20/27}$.

8. Triangle ABC has a perimeter of 42. Points M and N are on sides AB and AC respectively such that $MN \parallel BC$. If $AB = 18$ and $AM = 8$, then what is the perimeter of triangle AMN?

   Solution: Notice that $\triangle AMN \sim \triangle ABC$ by AA similarity. Thus,

$$\frac{p}{42} = \frac{\text{perimeter}(AMN)}{\text{perimeter}(ABC)} = \frac{AM}{AB} = \frac{8}{18} = \frac{4}{9}$$

   The scaled perimeter is $p = 42 \cdot 4/9 = \boxed{56/3}$.

9. The angles of a polygon form an arithmetic sequence. If the largest angle has a measure of 168° and the smallest has a measure of 120°, what is the sum of all of the angles in the polygon, in degrees?

   Solution: Suppose the polygon has n sides and angles. Because the angles form an arithmetic sequence, we see that the sum of the angles must be $n \cdot \frac{120+168}{2} = 144n$. The sum of the angles in a polygon must also be$180(n-2)$. Hence $144n = 180n - 360$, so $\boxed{n = 10}$.

10. Right triangle ABC with $\angle B = 90°$ has $AB = 3$ and $BC = 4$. Let M be the midpoint of side AC. What is the length of MB?

Solution: It is well-known that M is also the circumcenter of ABC (the circumcenter of a triangle is the point that is equidistant from all three vertices), so $MB = AC/2 = (MA + MC)/2 = \boxed{5/2}$.

11. Two children can finish six cakes in three hours. How many minutes does it take four children to finish two cakes if each child eats at the same rate?

Solution: We have twice as many children and $\frac{1}{3}$ the number of cakes. Because increasing children and decreasing the number of cakes both decrease the time needed, the answer is 3 hours $\times \frac{1}{2} \times \frac{1}{3}$, which is $\boxed{\text{30 minutes}}$.

12. In a triathlon, Peter runs 10 miles in 2 hours, bikes 7 miles in 40 minutes, and swims 9 miles in 100 minutes. What is Peter's average rate throughout the race in miles per hour?

Solution: Peter spends $120+40+100 = 260$ minutes, or $4+\frac{1}{3} = \frac{13}{3}$ hours. The total distance covered is $10 + 7 + 9 = 26$ miles. Dividing gives the average rate of $\boxed{\text{6 miles per hour}}$.

13. Alan leaves Boston at 12:00 P.M. and drives to D.C. at an average of 50 miles per hour. Barry leaves D.C. later at 1:00 P.M. and drives toward Boston along the same road at an average of 30 miles per hour. If the distance between Boston and D.C. is 440 miles, at what time do they meet?

Solution: At 1:00 P.M., Alan will have traveled 50 miles towards D.C. At that point, Alan's relative speed to Barry will be $50+30 = 80$ miles per hour. Since they have $440-50 = 390$ miles left to travel, they meet $390/80 = 4\frac{7}{8}$ hours after 1:00. Thus, the answer is $\boxed{\text{5:52:30 P.M.}}$. (Answers of 5:52 and 5:53 were both accepted.)

14. Express the value of $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}$ in simplest form.

Solution: Let N be the answer. We can write that $N = \sqrt{2 + N}$. Hence, we conclude $N^2 = N + 2$. Solving this quadratic equation yields that $N = -1$ or $N = 2$. However, since the expression is positive the answer must be $\boxed{2}$.

15. Compute the product $77 \times 91 \times 143$.

Solution: Note that $7 \cdot 11 \cdot 13 = 1001$, and

$$\begin{aligned} 77 &= 7 \cdot 11 \\ 91 &= 13 \cdot 7 \\ 143 &= 11 \cdot 13 \end{aligned}$$

Hence, the answer is $1001^2 = \boxed{1002001}$.

16. A recipe that makes 24 cookies calls for 4 cups of flour and 3 cups of water. How many cookies can be made with 3 cups of flour and 2 cups of water?

Solution: We first figure out that water is the limiting ingredient in the recipe. Thus, the answer is $24 \cdot \frac{2}{3} = \boxed{16}$.

17. The average age of 18 boys is 12 years old, and the average age of 12 girls is 17 years old. After two 28 year-olds leave the group, what is the average age of the remaining 28 people?

Solution: Before, the sum of the ages was $18 \cdot 12 + 12 \cdot 17 = 420$. After the departure, the new average is $(420 - 2 \cdot 28)(18 + 12 - 2) = \boxed{13}$.

18. Compute 111^3.

Solution: The value $111^2 = 12321$, so $111^3 = (100 + 10 + 1)(111^2) = 1232100 + 123210 + 12321 = \boxed{1367631}$.

19. If 14 callops are worth 9 jints, 12 spanters are worth 7 callops, and 3 gogs are worth 5 spanters, how many gogs is 20 jints worth?

Solution: We can use dimensional analysis to solve the problem:

$$20 \text{ jints} \cdot \frac{14 \text{ callops}}{9 \text{ jints}} \cdot \frac{12 \text{ spanters}}{7 \text{ callops}} \cdot \frac{3 \text{ gogs}}{5 \text{ spanters}} = \boxed{32 \text{ gogs}}.$$

Notice that each of the fractions above has a value of 1 from the givens.

20. Express the product $20\frac{5}{6} \times 18\frac{3}{5}$ as mixed number.

Solution: We have

$$20\frac{5}{6} \times 18\frac{3}{5} = \frac{125}{6} \cdot \frac{93}{5} = \frac{25 \cdot 31}{2} = \boxed{387\frac{1}{2}}.$$

21. How many prime numbers less than 50 satisfy that the sum of their digits is equal to another prime number?

Solution: The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$. Checking shows that the sum of the digits of each of the numbers $13, 17, 19, 31, 37$ is composite, so the answer is $15 - 5 = \boxed{10}$.

22. Let $a \star b = \frac{1}{a} + \frac{1}{b}$. Compute $(1 \star 2) \star (3 \star 4)$.

Solution: We evaluate $1 \star 2$ to be $\frac{1}{1} + \frac{1}{2} = \frac{3}{2}$, and $3 \star 4$ to be $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. Then, $(1 \star 2) \star (3 \star 4)$ $= \frac{3}{2} \star \frac{7}{12} = \frac{2}{3} + \frac{12}{7} = \boxed{50/21}$.

23. In a school, 40% of the boys and 50% of girls participate in math contests. If 42% of the entire school participates in math contests, what percent of the school population is female?

Solution: Let there be b boys and g girls. Then, we know that:

$$\begin{aligned} 0.4b + 0.5g &= 0.42(b+g) \\ 0.4b + 0.5g &= 0.42b + 0.42g \\ 0.08g &= 0.02b \\ 4g &= b \end{aligned}$$

Since there are four times as many boys as girls, so the percent of the school population that is female is $\frac{1}{1+4} = \boxed{20\%}$.

24. Compute the sum of reciprocals of the positive factors of 28.

Solution: We know that the sum of the reciprocals is:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{28}$$

By multiplying the top and bottom of the fractions so that they all have the same denominator of 28, we get:

$$\frac{28}{28} + \frac{4}{84} + \frac{2}{28} + \dots \frac{1}{28}$$

Notice that the numerators are the factors of 28 listed exactly once. Thus the sum of the reciprocals of the factors of 28 is:

$$\frac{\text{sum of factors of 28}}{28}$$

The sum of the factors of 28, which is $2^2 \times 7$, is $(1+2+4)(1+7)$, or 56. Thus the desired number is $\frac{56}{28} = \boxed{2}$.

25. Find the remainder when 7^{2013} is divided by 11.

Solution: By Fermat's Little Theorem we may remark that $7^{10} \equiv 1 \pmod{11}$. (Alternatively, you may check the first few cases, because you know that the remainder must repeat after a certain point.) In that case, the problem reduces to

$$7^{2013} \equiv 7^3 \equiv 343 \equiv \boxed{2} \pmod{11}$$

26. The value of $11 \times 10 \times 9 \times \cdots \times 2 \times 1$ is in the form 3_9_6800. What is the sum of the two blank digits?

    Solution: Let us denote the first blank to be x and the second blank to be y. We know that this number is divisible by 11, so the difference between the sum of alternating digits should be divisible by 11. The sum of every other digit starting from the first is 3+9+6+0=18. The sum of every other digit starting from the second is $x + y + 8 + 0 = x + y + 8$. The difference, $(x + y + 8) - (18) = x + y - 10$ must be divisible by 11. Then $x + y$=10, 21, 32, or so on. But we know that the maximum possible sum for two digits is 9+9=18, so $x + y = \boxed{10}$.

27. Ronald is buying pencils in packs of 10 and 13. If he buys at least one pack of each, what is the largest number of pencils he cannot purchase exactly?

    Solution: Based on the Chicken McNugget Theorem, the largest number of pencils he can not buy, with the restriction, is $13 \times 10 = \boxed{130}$.

28. What is the smallest positive integer n such that the product $n \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot (n+4)$ ends with two 0's?

    Solution: We know that in order for a number to end with two 0's, it must be divisible by 100, or in other words, divisible by both 4 and 25. The number is always divisible by 4, because we have the product of at least 4 consecutive integers, and one of them must be divisible by 4. Among these 5 consecutive integers, we can have exactly one that is divisible by 5. Thus, the only way for the number to be divisible by 25, is for the multiple of 5 to be a multiple of 25 as well. Since 25 is the smallest multiple of 25, we can set $n + 4 = 25$. Then, the smallest positive integer n such that the product ends with two 0's is $\boxed{21}$.

29. What is the smallest positive integer that leaves a remainder of 5 upon division by 7 and 7 upon division by 9?

    Solution: Another way to think about this problem is to find the smallest positive integer that, when added by 2, would be divisible by both 7 and 9. Thus, the answer is $63 - 2 = \boxed{61}$.

30. A pineapple under the sea has four doors. How many ways can Patrick enter the house through one door and exit through a different door?

    Solution: Patrick can choose from 4 doors to enter, and from 3 doors to exit. Thus, there are a total of $4 \times 3 = \boxed{12}$ ways to do so.

31. There are 24 people in math club, 17 people in chess club, and 8 in both clubs. How many people are in at least one club?

Solution: The number of people in math club but not chess club is $24 - 8 = 16$, and the number of people in chess club but not math club is $17 - 8 = 9$. Then, the people who are in at least one club is $16 + 9 + 8 = \boxed{23}$.

32. Two dice are rolled. What is the probability that the total number of dots on the top faces add up to a multiple of 6?

Solution: We note that no matter what we roll on the first die, there will be exactly one face on the second die such that the total number of dots on the top faces add up to a multiple of 6. For example, if we roll a 1 on the first die, then we must roll a 5 on the second die. Thus, the probability is just $\boxed{1/6}$.

33. A fair coin is flipped 7 times. What is the probability of flipping at most 3 heads?

Solution: We know that the probability of flipping n heads is the same as the probability of flipping $7 - n$ heads by symmetry. Thus, the probability of flipping at most 3 heads is the same as flipping 4 or more heads. Since there must be either at most 3 heads or at least 4 heads, the answer is $\boxed{1/2}$.

34. A burger can have possible condiments of lettuce, tomatoes, onions, and pickles. How many combinations of condiments are possible? A burger can have no condiments on it.

Solution: There are two options for each condiment (that is, to either include or exclude the condiment). Thus, the total number of combinations is $2 \times 2 \times 2 \times 2 = \boxed{16}$.

35. One spinner is divided into three equal sections labeled 1, 4, and 5. The other spinner is divided into 100 equal sections labeled $1, 2, 3, \ldots, 100$. Both spinners are spun. What is the probability that the sum of the two numbers is even?

Solution: For the sum of two numbers to be even, they must be both even or both odd. The probability of getting two odd numbers is $\frac{2}{3} \times \frac{50}{100} = \frac{1}{3}$. The probability of getting two even numbers is $\frac{1}{3} \times \frac{50}{100} = \frac{1}{6}$. Then, the probability that the sum of the two numbers is even is $\frac{1}{3} + \frac{1}{6} = \boxed{\dfrac{1}{2}}$.

36. What is the sum of the distinct prime factors of 1599?

Solution: The prime factorization of 1599 is $3 \times 13 \times 41$. Thus, the sum of the prime factors are $3 + 13 + 41 = \boxed{57}$.

37. If 2 pens and 1 notebook is worth \$3.10, and 1 pen and 2 notebooks is worth \$3.50, how much is 1 pen and 1 notebook worth, in dollars?

Solution: We know that 2 pens and 1 notebook and 1 pen and 2 notebooks, or 3 pens and 3 notebooks, is worth $\$3.10 + \$3.50 = \$6.60$. Thus, 1 pen and 1 notebook is worth $\$6.60/3 = \boxed{\$2.20}$.

38. In how many ways can four students Albert, Berta, Candice, and Denise line up for lunch?

Solution: There are 4 ways to choose who is first in line. After choosing this student, there are 3 options for the second student, 2 options for the third student, and 1 option for the final student. Thus, there are $4 \times 3 \times 2 \times 1 = \boxed{24}$ ways for the students to line up for lunch.

39. How many integers satisfy that $|x - 3| \leq 6$ and $|2x + 5| \leq 8$?

Solution: Looking at the first inequality, we know that

$$\begin{aligned} -6 \leq x - 3 &\leq 6 \\ -3 \leq x \quad &\leq 9 \end{aligned}$$

We can do the same thing for the second inequality.

$$\begin{aligned} -8 \leq 2x + 5 &\leq 8 \\ -13 \leq 2x \quad &\leq 3 \\ -\frac{13}{2} \leq x \quad &\leq \frac{3}{2} \end{aligned}$$

By looking at both inequalities, we see that $-3 \leq x \leq \frac{3}{2}$. Thus, the only integers that satisfy both inequalities are $-3, -2, -1, 0$, and 1, for a total of $\boxed{5}$ integers.

40. The Youtube video Gangnam Style had 10,000,000 views in the beginning of August 2012 and 100,000,000 views in the beginning of September 2012. On average, how many views did the video get each day in the month of August 2012, to the nearest million?

Solution: There are 31 days in August. During August, there were $100,000,000 - 10,000,000 = 90,000,000$ views. This means that on average there were $\frac{90{,}000{,}000}{31}$ views or about $\boxed{3,000,000}$ views each day.