MSJ Math Club

Week 12: Power of a Point and Radical Hatchets

February 21, 2013

Overall, this handout is oriented in the Olympiad direction, and thus is expected to be harder than previous handouts. We encourage you to spend as much time on each problem before consulting help, as each one is very instructive in its own way.

1 Tips and Tricks

- The **power** of a point P with respect to a circle ω (radius r)is defined as follows. A line through P intersects the circle at two points A and B. The power is the value of the directed product $PA \cdot PB$. (If P is outside of the circle, the power is positive. If it is inside, the power is negative.) It turns out that the power of a point P is constant regardless of your choices of A and B. If P is x units away from the center of the circle, the power turns out to be $x^2 - r^2$.
- The **radical axis** of two circles O_1 and O_2 is the locus of points P that have the same power with respect to both circles. The radical axis of the two circles turns out to be a line perpendicular to O_1O_2. If the two circles intersect, then the axis is the line connecting the two intersection points. This is a very powerful result, but it will not be proved here.
- The **Radical Axis Theorem** states that the pairwise radical axes of three circles concur at a single point, called the radical center. This can be proved by considering the intersection point of two radical axes, and showing that it has equal power with respect to all three circles.
- The Radical Axis Theorem is very useful proving collinearity. If two circles intersect at two points P and Q, you can try finding four points (let's say, A_1, A_2 on one circle, and B_1, B_2 on the other). If these four points are collinear, then lines A_1A_2, B_1B_2, and PQ concur.
- A useful configuration is as follows. Points A, B, C lie on a line in that order. A point P not on the line satisfies that $\angle APB = \angle ACP$. Then $\Delta ABP \sim \Delta APC$ and $AP^2 = AB \cdot AC$. Notice that if any of these relationships are given to you, the other two must follow.

2 Hints for Problems Below

Don't read these until you have tried the problems below. These hints are out of order.

1. Look for cyclic quadrilaterals. If you could show something is cyclic, then how can you get the the conclusion?
2. Examine the angle conditions. Can you find any similar triangles?

3. Construct the A-excircle of triangle ABC. Let D, E, and F be the tangency points to BC, AC, and AB.
4. First, make a guess as to which point MN passes through. Since MN is the radical axis of two circles, it might help to find another cyclic quadrilateral...
5. If P, Q, R, S are on a circle, can you immediately say anything about concurrency?
6. After following the previous hint, notice that $AF = 2$, so X is the midpoint of AF. Thus, XY is the radical axis of the excircle and the circle at A with radius of 0. What else is on this radical axis?
7. Find two similar triangles.

3 Practice Problems

I tried ordering the problems in increasing order of difficulty. There are hints above for the harder problems. It is encouraged that you spend as much time on a problem before looking at the hint or ask for a solution.

1. ($\sim$Mandlebrot 2011-12) Two circles intersecting at the points $A(0,1)$ and $B(x,3)$ are tangent to the x-axis at $(0,0)$ and $(10,0)$. Find x.
2. (NIMO) Circle $\odot O$ with diameter $\overline{AB}$ has chord $\overline{CD}$ drawn such that $\overline{AB}$ is perpendicular to $\overline{CD}$ at P. Another circle $\odot A$ is drawn, sharing chord $\overline{CD}$. A point Q on minor arc $\overline{CD}$ of $\odot A$ is chosen so that $\mathrm{m}\angle AQP + \mathrm{m}\angle QPB = 60^\circ$. Line l is tangent to $\odot A$ through Q and a point X on l is chosen such that $PX = BX$. If $PQ = 13$ and $BQ = 35$, find QX. **Hints: 7**
3. Angle XOY is given, with points A and B chosen on rays OX and OY such that the length of $OA+OB$ is fixed. A circle centered at point A with radius of length OB and a circle centered at point B with radius of length OA intersect at points M and N. Show that line MN passes through a fixed point. **Hints: 4**
4. (HMMT 2008) Let C_1 and C_2 be externally tangent circles with radius 2 and 3, respectively. Let C_3 be a circle internally tangent to both C_1 and C_2 at points A and B, respectively. The tangents to C_3 at A and B meet at T, and $TA = 4$. Determine the radius of C_3.
5. (USAJMO 2012) Given a triangle ABC, let P and Q be points on segments AB and AC, respectively, such that $AP = AQ$. Let R and S be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle). **Hints: 2**
6. (IMO 1995) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent. **Hints: 1**
7. (USAMO 2009) Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY. **Hints: 5**
8. (MOP) The perimeter of triangle ABC is 4. On rays AB and AC, points X and Y are chosen, respectively, so that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of one of the triangles ABM and ACM is 2. **Hints: 3, 6**