Week 9: Special Sauce Problem Set

MSJ Math Club

December 6, 2012

For those of you who are not taking or have not taken calculus, we have a special problem set for you this week. The answers and links to solutions will be available after this meeting.

Good luck!

Part I: Special Sauce Medley

1. *(Classic)* Calculate $2 + 2$.

2. *(MOEMS)* In an election, Ethan got 5 fewer votes than Christopher, who got 3 votes more than Olivia, who got 4 fewer votes than Ava. How many more votes did Ava get than Ethan?

3. *(Mathcounts)* Mike wrote a list of six positive integers on his paper. He chose the first and second integers randomly, but the third integer was the sum of the first and second, and each of the remaining integers was the sum of the two previous integers in the list. He then found the sum of all six integers. What is the ratio of the fifth integer in his list to this sum? Express your answer as a common fraction.

4. *(2009 AMC 8 #6)* Steve's empty swimming pool will hold $24,000$ gallons of water when full. It will be filled by 4 hoses, each of which supplies 2.5 gallons of water per minute. How many hours will it take to fill Steve's pool?

5. *(2008 AMC 10A #22)* Jacob uses the following procedure to write down a sequence of numbers. First he chooses the first term to be 6. To generate each succeeding term, he flips a fair coin. If it comes up heads, he doubles the previous term and subtracts 1. If it comes up tails, he takes half of the previous term and subtracts 1. What is the probability that the fourth term in Jacob's sequence is an integer?

6. *(2005 AMC 12A #17)* How many distinct four-tuples (a, b, c, d) of rational numbers are there with $a \log_{10} 2 + b \log_{10} 3 + c \log_{10} 5 + d \log_{10} 7 = 2005$?

7. *(2007 AIME I #7)* Let
$$N = \sum_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor).$$
Find the remainder when N is divided by 1000.

8. *(2002 AIME I #13)* In triangle ABC the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

9. *(2008 USAMO #2)* Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

10. *(1988 IMO #6)* Let a and b be two positive integers such that $a \cdot b + 1$ divides $a^2 + b^2$. Show that $\frac{a^2+b^2}{a \cdot b+1}$ is a perfect square.

Part II: Special Sauce AIME

1. *(1998 AIME #1)* For how many values of k is 12^{12} the least common multiple of the positive integers 6^6, 8^8, and k?

2. *(1999 AIME #2)* Consider the parallelogram with vertices $(10, 45), (10, 114), (28, 153)$, and $(28, 84)$. A line through the origin cuts this figure into two congruent polygons. The slope of the line is m/n, where m and n are relatively prime positive integers. Find $m + n$.

3. *(2000 AIME I #3)* In the expansion of $(ax+b)^{2000}$, where a and b are relatively prime positive integers, the coefficients of x^2 and x^3 are equal. Find $a + b$.

4. *(2001 AIME I #4)* In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

5. *(2002 AIME I #5)* Let $A_1, A_2, A_3, \cdots, A_{12}$ be the vertices of a regular dodecagon. How many distinct squares in the plane of the dodecagon have at least two vertices in the set $A_1, A_2, A_3, \cdots, A_{12}$?

6. *(2003 AIME I #6)* The sum of the areas of all triangles whose vertices are also vertices of a 1 by 1 by 1 cube is $m + \sqrt{n} + \sqrt{p}$, where m, n, and p are integers. Find $m + n + p$.

7. *(2004 AIME I #7)* Let C be the coefficient of x^2 in the expansion of the product $(1 - x)(1 + 2x)(1 - 3x) \cdots (1 + 14x)(1 - 15x)$. Find $|C|$.

8. *(2005 AIME I #8)* The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

9. *(2006 AIME I #9)* The sequence $a_1, a_2, \cdots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

10. *(2007 AIME I #10)* In a 6×4 grid (6 rows, 4 columns), 12 of the 24 squares are to be shaded so that there are two shaded squares in each row and three shaded squares in each column. Let N be the number of shadings with this property. Find the remainder when N is divided by 1000.

11. *(2008 AIME I #11)* Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

12. *(2009 AIME I #12)* In right $\triangle ABC$ with hypotenuse $\overline{AB}$, $AC = 12$, $BC = 35$, and $\overline{CD}$ is the altitude to $\overline{AB}$. Let ω be the circle having $\overline{CD}$ as a diameter. Let I be a point outside $\triangle ABC$ such that $\overline{AI}$ and $\overline{BI}$ are both tangent to circle ω. The ratio of the perimeter of $\triangle ABI$ to the length AB can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. *(2010 AIME I #13)* Rectangle $ABCD$ and a semicircle with diameter AB are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment AB, and segment CD at distinct points N, U, and T, respectively. Line ℓ divides region $\mathcal{R}$ into two regions with areas in the ratio $1 : 2$. Suppose that $AU = 84, AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

14. *(2011 AIME I #14)* Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let M_1, M_3, M_5, and M_7 be the midpoints of sides $\overline{A_1A_2}, \overline{A_3A_4}, \overline{A_5, A_6}$, and $\overline{A_7, A_8}$ respectively. For $i = 1, 3, 5, 7$, ray R_i is constructed from M_i towards the interior of the octagon such that $R_1 \perp R_3, R_3 \perp R_5, R_5 \perp R_7$, and $R_7 \perp R_1$. Pairs of rays R_1 and R_3, R_3 and R_5, R_5 and R_7, and R_7 and R_1 meet at B_1, B_3, B_5, B_7 respectively. If $B_1B_3 = A_1A_2$, then $\cos 2\angle A_3M_3B_1$ can be written in the form $m - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

15. *(2012 AIME I #15)* There are n mathematicians seated around a circular table with n seats numbered $1, 2, 3, \cdots, n$ in clockwise order. After a break they again sit around the table. The mathematicians note that there is a positive integer a such that

    (a) for each k, the mathematician who was seated in seat k before the break is seated in seat ka after the break (where seat $i + n$ is seat i);

    (b) for every pair of mathematicians, the number of mathematicians sitting between them after the break, counting in both the clockwise and the counterclockwise directions, is different from either of the number of mathematicians sitting between them before the break.

    Find the number of possible values of n with $1 < n < 1000$.