MSJ Math Club

Mass Points

15 October 2015

1 Introduction to Mass Points

Let us begin with a practical problem. Suppose you had a 98% in your AP Calculus class, and the final exam was worth 20% of your grade. Given that you want at least a 90% as your final grade, what is the lowest grade you get on the exam so that you can maintain your A?

We can visualize this as a line with two weights on opposite sides. On one side is your current grade of 98% with a weight of 0.8, and on the other side is the minimum exam grade, with a weight of 0.2, such that you can balance the two weights with a fulcrum at exactly 90%.

Since your original grade has a weight of 0.8, it makes sense that the fulcrum is nearer to that. In fact, the basic property of **mass points** is that

$$(\text{weight at A} * \overline{AC}) = (\text{weight at B} * \overline{BC})$$

Therefore we have

$$\frac{\overline{BC}}{\overline{AC}} = \frac{\text{weight at A}}{\text{weight at B}} = \frac{0.8}{0.2} = 4$$

And since $\overline{AC} = 8\%$, we get that $\overline{BC} = 32\%$, so B is at 58%. Pretty easy with mass points, right?

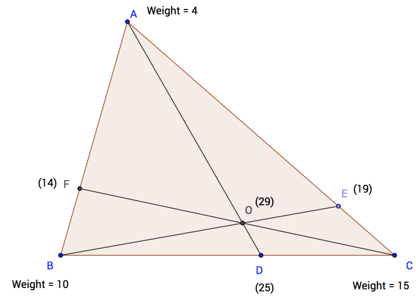
2 Geometry

Mass points is a easy way to solve certain types of geometry problems as well.

Suppose you had a triangle ABC with points such that D is on the side of the triangle opposite A, E is opposite B, and F is opposite C. AD, BE, and CF intersect at a point O. Let $2BD = 3CD$, and $2AF = 5BF$. Find the ratio of the length between AE and CE.

How would you solve this?

If you place a mass of $10g$ on the point B, $4g$ on point A, and $15g$ on point C, you can balance the triangle on a needle at point O. How do we know that? Well, we know that if you put a rigid edge on the line CF, it balances on the line CF because the mass on point B multiplied by its distance from that line is equal to the mass on point A multiplied by its distance, thus the torques are equal and they balance out. Similarly, the masses balance on the line AD. Therefore, since both CF and AD pass through the center of mass of the triangle, we see that O **must be the center of mass**.



And since the line BE passes through then center of mass O, the triangle would also balance on the line BE. Because the point B doesn't contribute to any torque on the line, we know that, by balancing torques, $AE * 4g = CE * 15g$, and thus we have our desired ratio.

What if we wanted to find the ratio of BO to OE? Well, we know that the points A and C have a center of mass at E; we can replace both points with a mass of $4g + 15g = 19g$ on E. Again, using torques, we get that $BO * 10g = OE * 19g \implies BO : OE = 19 : 10$

3 Examples

1. What assigned weights on the vertices of a triangle would make the centroid the center of mass?
2. (AIME 1989) P is inside $\triangle$ABC. $\overline{APD}$, $\overline{BPE}$, and $\overline{CPF}$ are drawn with D, E, and F on $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$. Given that $\overline{AP} = 6$, $\overline{BP} = 9$, $\overline{PD} = 6$, $\overline{PE} = 3$, and $\overline{CF} = 20$, find the area of $\triangle$ABC.

4 Practice Problems

1. (Varignon's Theorem) If the midpoints of a quadrilateral are connected, the resulting quadrilateral is a parallelogram.
2. In triangle ABC, points D and E are on sides BC and CA, respectively, and points F and G are on side AB with G between F and B. BE intersects CF at point O_1 and BE intersects DG at point O_2. If $FG = 1$, $AE = AF = DB = DC = 2$, and $BG = CE = 3$, compute $\frac{O_1O_2}{BE}$.
3. (AMC 10B 2013) In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?
4. In triangle ABC, D, E, and F are on BC, CA, and AB, respectively, so that $AE = AF = CD = 2$, $BD = CE = 3$, and $BF = 5$. If DE and CF intersect at O, compute $\frac{OD}{OE}$ and $\frac{OC}{OF}$.
5. (AIME 1985) In triangle $\triangle$ABC, cevians $\overline{AD}$,$\overline{BE}$, and $\overline{CF}$ intersect at point P. The areas of triangles $\triangle$PAF, $\triangle$PFB, $\triangle$PBD, and $\triangle$PCE are 40, 30, 35, and 84, respectively. Find the area of $\triangle$ABC.
6. **Puzzle of the Week:** Fill in the spaces of the grid to the right with positive integers so that in each 2×2 square with top left number a, top right b, bottom left c, and bottom right d, either $a + d = b + c$ or $ad = bc$. *This is #1 on this year's USA Math Talent Search (usamts.org). You guys should all try USAMTS! In each of its 3 rounds, you are given a month to solve 5 very interesting problems.*

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