# Week 4: Mods and Digits

MSJ Math Club

October 11, 2012

## 1 Tips and Tricks

- Two integers $a$ and $b$ are said to be congruent in a **modulus** (abbreviated mod) $m$ if $a-b$ is divisible by $m$. We would write this as $a \equiv b \pmod{m}$. Note that addition and multiplication by the same number on both sides of the mod is legal, but you have to be careful with division.
- If you are working with a modulus that is not prime, sometimes it helps to break it down into moduli that are prime. For example, a common technique for finding the last three digits of a number (such as for AIMEs) is computing the remainder upon division by $2^3 = 8$ and $5^3 = 125$. Sometimes this might not be easy though.
- For the last whatever digits of big exponents or large numbers, if you don't know what to do, sometimes it helps to look at the first 4 or 5 cases, and see if something repeats.
- Think about divisibility rules. In particular, because the divisibility test for 9 preserves the remainder upon division by 9, sometimes it can be quite helpful. Also, the divisibility test for 11 is quite useful.
- A common way to take care of digit-type problems is to split a 2+ digit number into the form $10a + b$, where $b$ is the units digit and $a$ is everything else.

## 2 Practice Problems

1. *(AMC)* What is the remainder when $3^0 + 3^1 + 3^2 + \ldots + 3^{2009}$ is divided by 8?
2. *(AIME)* The integer $n$ is the smallest positive multiple of 15 with only the digits 0 and 8. What is $\frac{n}{15}$?
3. *(AMC)* What are the last two nonzero digits of 90! (the number)?
4. Jeffrey writes the number 23! on the board. However, Aaron decides to make life difficult and erases three digits. The number remaining on the board is: $25,852,016,738,\_\_\_,976,640,000$. What are the three missing digits?
5. When the first digit of a 6-digit integer $n$ is moved to the last digit to form different 6-digit integer $m$, Jerry notices that $n$ is a multiple of $m$. How many such $n$ satisfy this?
6. *(AMC)* Given that $2^{2004}$ is a 604-digit number whose first digit is 1, how many elements of the set $S = \{2^0, 2^1, 2^2, \ldots, 2^{2003}\}$ have a first digit of 4?
7. *(AIME)* Let $R$ be the set of all possible remainders when a number of the form $2^n$, $n$ a nonnegative integer, is divided by 1000. Let $S$ be the sum of all elements in $R$. Find the remainder when $S$ is divided by 1000.
8. *(OMO)* The numbers $1, 2, \ldots, 2012$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers $x$ and $y$, erases them, and writes the number $2x + 2y$ on the board. This continues until only one number $N$ remains. Find the remainder when the maximum possible value of $N$ is divided by 1000.
9. Let $a$ be the sum of the digits of $4444^{4444}$. Let $b$ be the sum of the digits of $a$. Find the sum of the digits of $b$.