MSJ Math Club

Combo 1: Counting

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1 Introduction

Problems involving counting the number of ways to choose something come up frequently on math competitions like the AMC's and AIME's.

You should probably be familiar with the following notation:

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

which counts the number of ways to choose m items from a total of n items, irrespective of order (which is the same thing as choosing $n-m$ items from a total of n items, since we are just choosing the complement.) Thus for all all n and m, we have

$$\binom{n}{m} = \binom{n}{n-m}$$

One consequence of this is its appearance in binomial expansions. When we expand a binomial raised to the nth power, we get

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots \binom{n}{n-1}ab^{n-1} + b^n$$

Although this expansion isn't really useful by itself, learning to manipulate it and adapt it to pertinent problems is a crucial skill.

Vandermonde's Identity (A consequence of combinations):

$$\sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r}\binom{m}{0} = \binom{n+m}{r}$$

2 Tips and Tricks

- Having a solid understanding of combinations and ways of choosing something is extremely beneficial for the AMC, AIME, and perhaps even the USA(J)MOs.
- One common way of dealing with combinatorial problems is to exploit the symmetry.
- When dealing with problems that involve multiple parts, split up the problem into separate, easier-to-manage parts, and combine the parts at the end.
- Learning to manipulate and transform identities and formulas you already know can be very useful.
- As with all other types of math, the way to build intuition on combinatorics is simply to do lots and lots of problems.

3 Examples

1. Find $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots$
2. How many ways are there to arrange n distinct keys on a keychain?
3. (2007 AIME I) In a 6×4 grid (6 rows, 4 columns), 12 of the 24 squares are to be shaded so that there are two shaded squares in each row and three shaded squares in each column. Let N be the number of shadings with this property. Find the remainder when N is divided by 1000.
4. (SMT) An unfair coin has a 2/3 probability of landing on heads. If the coin is flipped 50 times, what is the probability that the total number of heads is even?

4 Practice Problems

1. (2005 AIME I) Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of one face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.
2. (Classic) How many rectangular prisms can be found on a $m \times n \times p$ 3-dimensional grid?
3. (Common Question) How many ways are there to put 10 balls in 5 urns, assuming that the balls are indistinguishable, while the urns are? What if the balls are distinguishable?
4. (AMC 10) Amy takes three digits from the the set $(0, 1, 2, 3, 4, 5, 6, 7, 8)$ and arranges them in ascending order to get a three digit number. Bob does the same except with $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$. What is the probability that Bob's number is greater than that of Amy's?
5. (2001 AIME I) A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.
6. (2006 AIME II) Seven teams play a soccer tournament in which each team plays every other team exactly once. No ties occur, each team has a 50% chance of winning each game it plays, and the outcomes of the games are independent. In each game, the winner is awarded a point and the loser gets 0 points. In the first game of the tournament, team A beats team B. The probability that team A finishes with more points than team B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
7. (SMT parody Q) All the permutations of the letters in STANFORD are put in alphabetical order. What is the index of DANFROST in this list?
8. (2006 AIME II) Let $(a_1, a_2, a_3, \ldots, a_{12})$ be a permutation of $(1, 2, 3, \ldots, 12)$ for which $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and $a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}$. An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.
9. (ASDAN) Compute how many permutations of the numbers $1, 2, \cdots, 8$ have no adjacent numbers that sum to 9.