MSJ Math Club

Multidimensional Geometry

10 March 2016

1 Introduction

Multidimensional geometry is often much harder to visualize and much harder to understand than simple 2D geometry. However, a number of methods can help you solve 3D geometry problems without trying to bash functions of two variables!

Question: A square-based pyramid is formed by connecting the square $ABCD$ to the apex X. A plane intersects the lines AX, BX, CX, DX at the points A, E, F, G, respectively. If $XE = EB$ and $XG = 2GD$, what is $\frac{XF}{FC}$?

Solution: Let the line EF intersect BC at Q, and let FG intersect CD at R. Consider the plane of the triangle XBC.

By Menelaus's theorem (or just use mass points), $\frac{XF}{CF}\frac{CQ}{BQ}\frac{BE}{XE} = 1$. Therefore,

$$\frac{XF}{CF} = \frac{BQ}{CQ}$$

Similarly, taking the plane of XCD,

$$\frac{XF}{CF} = 2\frac{DR}{CR}$$

Equating the two equations and inverting both sides, we get:

$$2\frac{CQ}{BQ} = \frac{CR}{DR} \longrightarrow 2\frac{DR}{BQ} = 2\frac{DR}{DA}\frac{BA}{BQ} = \frac{CR}{CQ}$$

Since Q, A, R are collinear (they're all on the intersection of the plane and the extended base of the pyramid), we know that QBA, ADR, and QCR are similar, we know that $DR/DA = BA/BQ = CR/CQ$, and using the previous equation, we know that they're all equal to $1/2$.

Therefore,

$$\frac{XF}{CF} = \frac{BQ}{CQ} = \frac{2AB}{2AB + BC} = \frac{2}{3}$$

While this may seem somewhat long at first, it's definitely faster than trying to bash out the equations of a shape whose dimensions are not all given!

2 Tips and Tricks

- Multidimensional geometry can often be done by extracting particular planes, or by projecting the configuration onto a few planes and working with the lower amount of dimensions.
- Most of the time, one can generalize theorems in 2D geometry and apply them in three dimensions.
- Usually, it is a bad idea to coordinate bash 3D geometry. There are certain exceptions.
- Projective geometry is also sometimes helpful.

3 Examples

1. (AMC 2007) A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?
2. (Classic) A sphere is inscribed in an octahedron with side length 1. What is its radius?
3. (MMT 2014) An equilateral-triangle-based pyramid $ABCD$ (apex is A and is directly above the center of the equilateral triangle) is cut with a plane m. m intersects edge AB at E such that $AE = 2BE$, and intersects AD at F such that $AF = DF$ and intersects edge BC at G such that C is between B and G. $BG = 3CG$. If m intersects DC at H, find DH/CH.

4 Practice Problems

1. (AHSME 1996) Triangle PAB and square $ABCD$ are in perpendicular planes. Given that $PA = 3$, $PB = 4$ and $AB = 5$, what is PD?
2. (AMC 2004) Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?
3. (Classic) Two distinct regular tetrahedrons with the maximum possible side length are inscribed inside a unit cube. What is the volume of their union?
4. (AIME 2012) Cube $ABCDEFGH$, (CG is normal (perpendicular) to square $ABCD$) has edge length 1 and is cut by a plane passing through vertex D and the midpoints M and N of $\overline{AB}$ and $\overline{CG}$ respectively. The plane divides the cube into two solids. The volume of the larger of the two solids can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
5. (AIME 2007) A square pyramid with base $ABCD$ and vertex E has eight edges of length 4. A plane passes through the midpoints of AE, BC, and CD. The plane's intersection with the pyramid has an area that can be expressed as $\sqrt{p}$. Find p.
6. (MMT 2014) A hypersphere is inscribed in a four-dimensional simplex of edge-length 12 (5 vertices, all edges, faces and volumes are equal and regular). Find its radius.

5 Problem of the Week

The figure consists of three concentric triangles. The lengths of the sides of the middle triangle are 9, 10, and 11. Also, the distance between the sides of the middle triangle and each of the other two triangles is 1, as shown in the diagram. Find the shaded area.