Primes, Prime Factorization

1. Find the prime factorization of 999999.
2. Find the smallest positive common multiple of 18 and 30 that is both a perfect square and perfect cube.
3. The product of a set of positive integers is 140. What is their least possible sum?
4. [Mandelbrot] Which of 1999, 2000, or 2001 has the largest proper divisor?
5. What is the greatest prime divisor of the following arithmetic series: $1 + 2 + 3 + \cdots + 70$?
6. What is the greatest prime divisor of $59! + 60!$?
7. [SMT] What is the greatest prime divisor of 3599?
8. [Mandelbrot] Find the smallest positive integer greater than 1 which leaves a remainder of 1 when divided by $2, 3, 4, \ldots, 9$.
9. [Mandelbrot] How many possible values are there for the sum $a+b+c$ if a, b, and c are positive integers such that $abc = 72$?
10. Find the number of distinct pairs of integers (x, y) such that $0 < x < y$ and $\sqrt{1984} = \sqrt{x} + \sqrt{y}$.
11. Compute the largest prime factor of $3(3(3(3(3(3(3(3(3(3+1)+1)+1)+1)+1)+1)+1)+1)+1)+1$.
12. Prove that there is a unique positive integer n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

GCD and LCM

13. Find n if $\gcd(n, 40) = 10$ and $\mathrm{lcm}[n, 40] = 280$.
14. [AHSME] m and n are positive integers such that the GCD of m and n has 5 divisors, and the LCM of m and n has 30 divisors. What is the greatest possible number of *prime* divisors that n could have?
15. How many distinct pairs of positive integers are there which have a GCD of 6 and an LCM of 600?

Divisibility

16. What is the smallest positive integer N such that the value $7 + 30N$ is not a prime number?
17. [Mandelbrot] Jayne writes the integers from 1 to 2000 on a piece of paper. She erases all the multiples of 3, then all the multiples of 5, and so on, erasing all the multiples of each odd prime. How many numbers are left when she finishes?
18. What is the smallest positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers?
19. If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number will be divisible by 11?
20. [Mandelbrot] The product of any two of the positive integers 30, 72, and N is divisible by the third. What is the smallest possible value of N?
21. [AHSME] There are unique integers $a_2, a_3, a_4, a_5, a_6, a_7$ such that $\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!}$, where $0 \le a_i < i$ for $i = 2, 3, \ldots, 7$. Find $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$.
22. [AIME] What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?
23. Prove that the number $2^{2^n} + 2^{2^{n-1}} + 1$ can be expressed as the product of at least n prime factors, not necessarily distinct.

24. Prove that any two numbers of the following sequence are relatively prime: $2+1, 2^2+1, 2^4+1, 2^8+1, \ldots, 2^{2^n}+1, \ldots$

Digit Problems

25. [IMO] When 4444^{4444} is written in decimal notation, the sum of its digit is A. Let B be the sum of the digits of A. Find the sum of the digits of B.

26. [NIMO] The number $\left(2+2^{96}\right)!$ has 2^{93} trailing zeroes when expressed in base B.
    (a) Find the minimum possible B.
    (b) Find the maximum possible B.
    (c) Find the total number of possible B.

Diophantine Equations

27. Find a primitive Pythagorean triangle one of whose legs is 270, or prove that no such triangle exists.

28. Let n be a positive integer. Prove that if $2n$ can be written as the sum of two squares, then n can also be written as the sum of two squares.

29. How many ordered pairs of integers (m, n) satisfy $5m^2 + 9n^2 = 1234567$?

30. Show that $n^4 + 3n^2 + 1$ is never a perfect square for positive integers n.

31. Prove that if $pn+1$ is the square of an integer, where n is a positive integer and p is a prime, then $n+1$ can be written as the sum of p squares.

32. Let n be an integer such that $N = 2 + 2\sqrt{28n^2+1}$ is also an integer. Prove that N is also a perfect square.

33. [IMO] Are there integers m and n such that $5m^2 - 6mn + 7n^2 = 1985$?