MSJ Math Club

Polynomial Tricks I (Standard Methods)

10 September 2015

1 Topic Guide

This handout will mostly concern itself with solving polynomials. Let's start with something simple.

1.1 The Linear Equation

Consider the scenario where you are met with a linear equation $ax + b = 0$. How do you solve it?

1.2 The Quadratic

Now consider the quadratic $ax^2 + bx + c = 0$. You probably know of something called "completing the square" as follows: $ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c = 0$. We can then rearrange the terms to get $a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c = \frac{b^2-4ac}{4a}$. Dividing by a and taking the square root gives us $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a}$, and so our two solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.3 Monic Depressed Polynomial

What about a cubic? Let's define some terms first. A *monic* polynomial is a polynomial $P(x)$ of degree n such that the leading coefficient is 1. You can make a polynomial monic by dividing the entire polynomial by the coefficient of x^n.

A *depressed* polynomial is a polynomial $P(x)$ of degree n such that the coefficient of x^{n-1} is 0. For a monic cubic $P(x) = x^3 + bx^2 + cx + d = 0$, you can make it depressed by considering $P(x - \frac{b}{3}) = x^3 + ex + f$ for some values of e and f. Notice the x^2 term is gone.

For example, the polynomial $P(x) = x^3 + 3x^2 + 4x + 5$ becomes $(x + 1)^3 + (x + 1) + 3$, which is a depressed monic cubic in terms of $x + 1$ because there is no $(x + 1)^2$ term (this is also called "completing the cube"). So if we set $y = x + 1$, then if we find a solution of the equation $y^3 + y + 3 = 0$, then we know that $y - 1$ is a solution to the original equation.

1.4 Cardano's Method

An extremely useful identity to remember is the following (which you can verify by expanding):

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Now consider a monic depressed cubic polynomial $P(x) = x^3 - px + q$.

To solve this, we let $q = y^3 + z^3$ and $p = 3yz$ for some real numbers y and z. Then $y^3z^3 = \dfrac{p^3}{27}$ by the second equation.

Then $P(x) = x^3 - px + q = x^3 - (3yz)x + (y^3 + z^3) = x^3 + y^3 + z^3 - 3xyz$. By the identity, this equals 0 when $x + y + z = 0$! If we solve the previous equations for $a = y^3$ and $b = z^3$, then we get $a + b = q$ and $\frac{p}{27} = ab$. Now we have two variables (a and b) and 2 equations, we can solve this very easily (they are actually roots of $x^2 - qx + \frac{p}{27}$). Therefore we can find a and b. Then $y = \sqrt[3]{a}$ and $z = \sqrt[3]{b}$.

Finally, since $x + y + z = 0$, we get that one of the roots is $x = -\sqrt[3]{a} - \sqrt[3]{b}$.

Putting *everything* together, for the cubic polynomial $ax^3 + bx^2 + cx + d = 0$, we have our cubic formula:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

2 Tips and Tricks

- We *highly* recommend not memorizing the above formula. Read through how we derived the formula, and try to understand *why* it works.

3 Examples

1. (2013 AIME I) The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\dfrac{\sqrt[3]{a} + \sqrt[3]{b} + 1}{c}$, where a, b, and c are positive integers. Find $a + b + c$.

4 Practice Problems

1. Find all real roots of the polynomial $x^3 + 3x^2 + 3x + 9$.
2. (Rice/Harvard/MIT/Stanford/John Hopkins Math Tournament 2000) Evaluate $2000^3 - 1999 \cdot 2000^2 - 1999^2 \cdot 2000 + 1999^3$.
3. Show that $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}} = 1$.
4. (Turkey MC-2005) What is the largest real number x satisfying $x^3 - x^2 - x - \dfrac{1}{3} = 0$?
5. **Puzzle of the Week:** You have an m x n piece of chocolate. On one break you may choose one piece of chocolate greater than 1 x 1 and break that piece along a grid line. What is the smallest number of breaks you must make in order to end up with all 1 x 1 blocks?
6. **Extra:** If you are interested in coding, check out www.easyctf.com (a high school hacking competition)!