MSJ Math Club Week 15

Trig Problems Compilation

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Disclaimer: Solution not available for all problems yet! Also, problem difficulties may be off.

1 Easy

1. *(HMMT 2003)* Compute $\dfrac{\tan^2(20^\circ) - \sin^2(20^\circ)}{\tan^2(20^\circ)\sin^2(20^\circ)}$.

2. Compute:

$$\frac{1}{1+\cot 1^\circ} + \frac{1}{1+\cot 5^\circ} + \frac{1}{1+\cot 9^\circ} + ... + \frac{1}{1+\cot 85^\circ} + \frac{1}{1+\cot 89^\circ}$$

3. Given that $\sin(x) + \sin(4x) = \sin(2x) + \sin(3x)$, solve for x in the range $[-2\pi, 2\pi]$.

4. For any real number x such that $|x| \geq 1$, simplify the expression $\sec^2(\tan^{-1} x) - \tan^2(\sec^{-1} x)$.

5. *(CAML 2012-2013)* For some real number t, the infinite series $\cos^2 t + \cos^4 t + \cos^6 t + \cdots + \cos^{2n} t + \cdots =$ 2013. Compute the infinite sum $\sin^2 t + \sin^4 t + \sin^6 t + \cdots + \sin^{2n} t + \cdots$.

6. *(HMMT 2001)* Find all x between $-\pi/2$ and $\pi/2$ such that $1 - \sin^4 x - \cos^2 x = 1/16$.

7. *(HMMT 2000)* Given that $\cos(\alpha + \beta) + \sin(\alpha - \beta) = 0$ and $\tan\beta = 1/2000$, find $\tan\alpha$.

8. *(HMMT 2002)* Find all values of x such that $\sin x + \cos x = \sqrt{2}$.

9. *(Lazar)* Solve:

$$\sin 5t = \sin t$$
$$\cos 5t = \cos t$$

(Find a clever solution.)

10. *(SCU 2013)* If $\sin\theta + 2\cos\theta = 2.2$, then find all possible values of $2\sin\theta + \cos\theta$.

11. *(AIME 2003)* Given that $\log_{10}\sin x + \log_{10}\cos x = -1$ and that $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, find n.

12. *(AIME 1995)* Given that $(1 + \sin t)(1 + \cos t) = 5/4$ for some t, find $(1 - \sin t)(1 - \cos t)$.

2 Medium

1. *(Math Prize for Girls, 2010)* Compute the value of the sum:

$$\sum_{i=0}^{8} \frac{1}{1 + \tan^3 10i}$$

2. *(Math Prize for Girls, 2010)* If a and b are positive integers such that $\sqrt{8+\sqrt{32+\sqrt{768}}} = a\cos\frac{\pi}{b}$, find the ordered pair (a, b).

3. Compute the value of $\cot 10^\circ \cdot \cot 30^\circ \cdot \cot 50^\circ \cdot \cot 70^\circ$.

4. *(HMMT 2008)* Compute $\arctan(\tan 65^\circ - 2\tan 40^\circ)$ in degrees.

5. *(HMMT 2009)* Let x and y be positive real numbers and θ an angle such that $\theta \neq \frac{\pi}{2}n$ for any integer n. Suppose
$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y}$$
and
$$\frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{97\sin 2\theta}{x^3y+y^3x}.$$
Compute $\frac{x}{y}+\frac{y}{x}$.

6. *(HMMT 2001)* Compute the sum:
$$\frac{\sin 10+\sin 20+\sin 30+\sin 40+\sin 50+\sin 60+\sin 70+\sin 80}{\cos 5\cos 10\cos 20}$$
All angles are in degrees.

7. Prove that $\dfrac{\cos(7x)}{2\cos x} - \cos(2x) + \cos(4x) - \cos(6x) = -\dfrac{1}{2}$ for all x.

8. Compute $\tan 1^\circ + \tan 5^\circ + \tan 9^\circ + \cdots + \tan 177^\circ$.

9. Evaluate $\arcsin(\tan 12^\circ \tan 48^\circ \tan 54^\circ \tan 72^\circ)$.

10. Evaluate the sum $\tan 1\tan 2 + \tan 2\tan 3 + \cdots + \tan 2004\tan 2005$. All angles are in degrees.

11. Prove that $2\left(\cos\dfrac{4\pi}{19} + \cos\dfrac{6\pi}{19} + \cos\dfrac{10\pi}{19}\right)$ is a root of the equation:
$$\sqrt{4+\sqrt{4+\sqrt{4-x}}} = x$$

12. Solve the following equation over the reals:
$$\sqrt{7+2\sqrt{7-2\sqrt{7-2x}}} = x$$

13. *(AIME 1984)* Find the value of $10\cot(\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\cot^{-1}21)$.

14. *(AIME 1989)* Let a, b, c be the three sides of a triangle, and let α, β, γ, be the angles opposite them. If $a^2+b^2 = 1989c^2$, find
$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}$$

15. *(AIME 1991)* Suppose that $\sec x + \tan x = \frac{22}{7}$ for some x. Find $\csc x + \cot x$.

16. *(AIME 1996)* Find the smallest positive integer solution to
$$\tan 19x^\circ = \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}.$$

17. *(AIME 1997)* Let $x = \dfrac{\sum_{n=1}^{44}\cos n^\circ}{\sum_{n=1}^{44}\sin n^\circ}$. What is the greatest integer that does not exceed $100x$?

18. *(AIME 1998)* Given that $A_k = \frac{k(k-1)}{2}\cos\frac{k(k-1)\pi}{2}$, find $|A_{19} + A_{20} + \cdots + A_{98}|$.

19. *(AIME 1999)* Compute $\tan^{-1}\left(\sum_{k=1}^{35}\sin 5k\right)$.

20. *(AIME 2000)* Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \dfrac{1}{z^{2000}}$.

21. *(AIME 2002)* While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of x for which the sine of x degrees is the same as the sine of x radians are $\frac{m\pi}{n-\pi}$ and $\frac{p\pi}{q+\pi}$, where m, n, p and q are positive integers. Find $m+n+p+q$.

22. *(AIME 2008)* Find the positive integer n such that

$$\arctan\frac{1}{3} + \arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{n} = \frac{\pi}{4}$$

23. *(AIME 2011)* Suppose x is in the interval $[0, \pi/2]$ and $\log_{24\sin x}(24\cos x) = \frac{3}{2}$. Find $24\cot^2 x$.

24. *(AIME 2012)* Let x and y be real numbers such that $\dfrac{\sin x}{\sin y} = 3$ and $\dfrac{\cos x}{\cos y} = \frac{1}{2}$. Find the value of $\dfrac{\sin 2x}{\sin 2y} + \dfrac{\cos 2x}{\cos 2y}$.

3 Hard

1. Let $X = \sin 1^\circ \sin 3^\circ \sin 5^\circ \cdots \sin 87^\circ \sin 89^\circ$. Compute $\log_2 X$.

2. Compute:
$$\frac{\sin(x) + \sin(3x) + \cdots + \sin((2n+1)x)}{\cos(x) + \cos(3x) + \cdots + \cos((2n+1)x)}$$

3. Compute $\cos\dfrac{\pi}{13}\cos\dfrac{3\pi}{13}\cos\dfrac{4\pi}{13}$.

4. Find the value of $\tan x \tan 2x + \tan 2x \tan 4x + \tan 4x \tan x$ when $x = \frac{\pi}{7}$.

5. Prove that $\tan^2 1^\circ + \tan^2 3^\circ + \tan^2 5^\circ + .. + \tan^2 89^\circ = 5004$.

6. *(AIME 2000)* Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

7. *(AIME 2006)* Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$.

8. *(AIME 2013)* For $\pi \le \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \ldots$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \ldots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Find $\sin\theta$.