MSJ Math Club

Probability and Expected Value

1 October 2015

1 A Warm-Up to Ponder

What is the probability that a randomly chosen chord in a circle has a distance from the center less than half the radius?

2 The Basics

Probability is the chance that something will be true, and for anything probability space, with w_i being an individual possible state, we have

$$\sum_{i=0} p(w_i) = 1$$

This is just a fancy way of saying that the sum of all probabilities for any situation must be 1.

Consider rolling two n-sided dice. What is the probability that the first roll is greater than the second roll?

We can do this by casework on what the first roll is, and then summing up the total number of possibilities where this is true. Alternatively, we can realize that $p(\text{first roll} < \text{second roll}) + p(\text{first roll} = \text{second roll}) + p(\text{first roll} > \text{second roll}) = 1$. But notice that since the two dice are identical, the probability that the first roll is greater than the second roll is **equal** to the probability that the first roll is less than the second roll. Therefore we can solve the equation to get that

$$p(\text{first roll} > \text{second roll}) = \frac{1 - p(\text{first roll} = \text{second roll})}{2}$$

Note that the probability that the two rolls are the same is simply $\frac{1}{n}$, so our desired probability is $\boxed{\frac{n-1}{2n}}$.

Now, consider an example of simple geometric probability. A man goes to an airport at a random time between 11 and 11:50, and waits for 20 minutes for the woman. The woman arrives at a random time between 11 and 12, and waits for 10 minutes for the man. What is the probability that they meet?

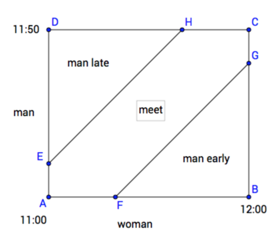

Figure 1: Note that the area of the ”meet” over the area of the rectangle is our desired answer.

3 Tips and Tricks

- Don't be afraid to assign variables! When you are given a problem that involves a bunch of states, go ahead and assign variables - it may make your life easier.
- Watch out for symmetries! If you can exploit similarities between two possible states, you can give yourself a huge leg up on your opponents.
- Venn diagrams can help with things like conditional probability.
- There is an extremely powerful tool known as the **linearity of expectation**. This basically means that the expected value of a series of events is equal to the sum of the expected values of the individual events.

4 Examples

1. (2006 AMC 10B) For a particular peculiar die, the probabilities of rolling 1, 2, 3, 4, 5, and 6 are in the ratio of $1:2:3:4:5:6$. What is the probability that after 2 rolls, you get a sum of 7?
2. (1998 AIME) Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is still in the cafeteria is 40%. Given that $m = a - b\sqrt{c}$ where a, b, c are positive integers, and c is not divisible by the square of any prime, find $a + b + c$.
3. (2015 BMT) On a 2 x 40 chessboard colored black and white in the standard alternating pattern, 20 rooks are randomly placed on the black squares. The expected number of white squares with only rooks as neighbors can be expressed as $\frac{p}{q}$, where p and q are coprime positive integers. What is $p + q$? (Two squares are said to be neighbors if they share an edge.)

5 Practice Problems

1. (2013 SMT) Nick has a terrible sleep schedule. He randomly picks a time between 4 AM and 6 AM to fall asleep, and wakes up at a random time between 11 AM and 1 PM of the same day. What is the probability that Nick gets between 6 and 7 hours of sleep?
2. (2011 HMMT) Let p be the answer to this question. If a point is chosen randomly from the square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, what is the probability that at least one of its coordinates is greater than p?
3. (2013 SMT) An unfair coin lands heads with probability $\frac{1}{17}$. Matt flips the coin repeatedly until he flips at least one head and one tail. What is the expected number of times that Matt flips the coin?
4. (2004 AIME) A circle of radius 1 is randomly placed inside a 15-by-36 rectangle ABCD so that the circle is contained entirely within the rectangle. Given that the probability that the circle will not touch diagonal AC is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.
5. (2014 AMC 10B) In a small pond there are eleven lily pads in a row labeled 0 through 10. A frog is sitting on pad 1. When the frog is on pad N, $0 < N < 10$, it will jump to pad $N - 1$ with probability $\frac{N}{10}$ and to pad $N + 1$ with probability $1 - \frac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape being eaten by the snake?
6. **Puzzle of the Week:** An odd number of soldiers are on a field, with all pairwise distance distinct. Each soldier is told to keep an eye on the nearest soldier. Prove that at least one soldier is not being watched.