Week 13: Recursion

MSJ Math Club

February 28, 2013

We're going to drop down to AIME-level material for at least a week. This week, we will introduce recursion, a combinatorial technique that comes up not infrequently. There is a rich body of problems and theory that concerns itself with the solutions to recurrences, but we'll focus on setting recursions up in the first place and extracting values out of them. We'll throw in a tiny bit of linear recurrence theory for fun.

1 The Basic Method

Many problems can be broken down into problems of a lower "order" so to speak. When you try to enumerate the possibilities in a given problem, you may find it easier to express your answer in terms of these lower order answers, instead of finding the total directly. In other words, you may find it easier to find an equation of the form

$$f(n) = f(n-1) + f(n-2) + f(n-3) + f(n-4)$$

than an equation like this:

$$f(n) = n \cdot n! \cdot 2^{2^n} + \lfloor 3\sqrt{\pi} + n/e \rfloor.$$

The challenge then lies in actually getting a numerical value. To start off, since $f(n)$ depends on at least one $f(k)$ for $k < n$, it follows that we need to define base cases. For example, if we take the domain of f to be all nonnegative integers, we would need the base cases $f(0)$, $f(1)$, $f(2)$, and $f(3)$, because otherwise if we plug $0, 1, 2$, or 3 for n, we'd feed negative numbers into f and that is not good.

Protip: if a problem asks you to find $f(n)$ for low values of n, you might as well calculate all the previous values of f.

Another protip: don't bash that all out if n is large! Contest problems are almost never cruel enough to make you do that. Either find a pattern or look for an alternate solution without recursion.

Further protip: don't get locked into the mentality of forcing a connection between $f(n)$ and $f(n-1)$, say. If the problem can be decomposed into a much more natural way, do it that way instead.

2 Linear Recurrences

These don't show up a whole lot on the AIME, but the theory gives you an idea of the different methods that have cropped up to solve recurrences and it shows that recurrences are not totally an ad-hoc domain. I don't want to use up more than two pages, so I will say this: suppose we have a linear recurrence relation

$$f(n) = c_1 f(n-1) + c_2 f(n-2) + \cdots + c_k f(n-k).$$

This grows exponentially, which you can see if you try different values, so it's reasonable to plug $f(n) = kr^n$ to see what happens. You get a **characteristic equation** $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$, so you can find the roots and eventually get tentative answers. It turns out that none of these work in general, but for some constants a_i which you can find by solving the system formed by $f(0), f(1), \cdots f(n-1)$, $f(n) = a_1 r_1^n + a_2 r_2^n + \cdots + a_k r_k^n$ where the r_i are the roots of the characteristic equation. Multiple roots have further (consistent) complications.

3 Problems

As with all math problems (and especially combinatorics problems), practice is crucial. Warning: the problems vary widely in difficulty.

1. *(SMT)* The triangular numbers $T_n = 1, 3, 6, 10, \cdots$ are defined by $T_1 = 1$ and $T_{n+1} = T_n + (n+1)$. The square numbers $S_n = 1, 4, 9, 16, \ldots$ are defined by $S_1 = 1$ and $S_{n+1} = T_{n+1} + T_n$." The pentagonal numbers $P_n = 1, 5, 12, 22, \cdots$ are defined by $P_1 = 1$ and $P_{n+1} = S_{n+1} + T_n$. What is the 20th pentagonal number P_{20}?

2. *(Mock AIME)* Let S_n be the set of strings with only 0s or 1s with length n such that any 3 adjacent place numbers sum to at least 1. For example, 00100 works, but 10001 does not. Find the number of elements in S_{11}.

3. *(SMT)* A function f maps every sequence of integers to another sequence of integers as follows:

$$(f(a))_n = \begin{cases} 1 & \text{if } n = 0 \\ na_{n-1} & \text{if } n \neq 0. \end{cases}$$

   If the sequence b_n is a fixed point of f, what is $b(2008)$?

4. *(SMT)* Lord Voldemort only does two things all day: curse Muggles, and kick puppies. Each Muggle he curses has a 50% chance of dying while a puppy kick is always successful. Each dead Muggle gives him 3 units of satisfaction and each kicked puppy gives him 2 units. If an even number of Muggles die, he doubles his satisfaction from each of them. If he can curse one Muggle or kick one puppy per hour, how many Muggles should he curse in a day to maximize his expected satisfaction?

5. *(HMMT)* How many ways can one color the squares of a 6x6 grid red and blue such that the number of red squares in each row and column is exactly 2?

6. Look up Catalan numbers, especially the derivations of closed-form formulas for them.

7. Derive a closed-form formula for the Fibonacci numbers. This is known as **Binet's formula**.

8. *(Lewis)* A bug either splits into two perfect copies of itself or dies. If the probability of splitting is $p > \frac{1}{2}$ (and is independent of the bug's ancestry), what is the probability that a bug's descendants die out? Express your answer as a function in terms of p. (Note: you will probably get multiple answers in the beginning without new ideas, but how would you prove, using recursive relations, that one of these answers is the correct one?)

9. *(HMMT)* A parking lot consists of 2012 parking spots equally spaced in a line, numbered 1 through 2012. One by one, 2012 cars park in these spots under the following procedure: the first car picks from the 2012 spots uniformly randomly, and each following car picks uniformly randomly among all possible choices which maximize the minimal distance from an already parked car. What is the probability that the last car to park must choose spot 1?