Individual Round Solutions

MagMaR 2014

January 26, 2014

1. A brand of hand sanitizer claims to kill 99.99% of germs. If a drop of sneeze contains 200 million germs, how many germs in a single drop of sneeze are not killed from using the hand sanitizer?

1. **20000**

Solution: Since 99.99% of germs are killed in one drop of sneeze, 0.01% are not killed, so the answer is $200,000,000 \times 0.0001 = 20,000$ germs.

2. If x is the answer to this question, what is $11x - 70$?

2. **7**

Solution: We let $x = 11x - 70$, so $10x = 70$ or $x = 7$.

3. Define $a \star b = b^2 - (a^2 + a)$. Compute $x = ((((1 \star 2) \star 3) \star 4) \star 5) \star 6$.

3. **6**

Solution: We have that:

$$
\begin{aligned}
x &= ((((1 \star 2) \star 3) \star 4) \star 5) \star 6 \\
&= (((2 \star 3) \star 4) \star 5) \star 6 \\
&= ((3 \star 4) \star 5) \star 6 \\
&= (4 \star 5) \star 6 \\
&= 5 \star 6 \\
&= 6
\end{aligned}
$$

4. The height and circumference of the base of a cylindrical soda can are both equal to 2π units. What is the volume of the can in units^3?

4. $2\pi^2$

Solution: If r is the radius of the base, then the circumference is $2\pi = 2\pi r$, so $r = 1$. Thus, the volume is $(\pi r^2)h = (\pi 1^2)2\pi = 2\pi^2$.

5. The mass of five U.S. dimes is equal to the mass of two U.S. quarters. If Danny has 42 kilograms of dimes and 21 kilograms of quarters, what is the ratio of the number of dimes to the number of quarters that Danny has?

5. **5**

Solution: If a pile of dimes and a pile of quarters have the same mass, then the values of both piles are also equal. Since Danny's pile of dimes is twice as heavy as his pile of quarters, the amount of money he has in dimes is twice the amount of money he has in quarters.

Let \$$x$ be the amount money that Danny has in quarters. Then Danny has $x/0.25 = 4x$ quarters and $2x/.1 = 20x$ dimes, so the final answer is $20/4 = 5$.

6. Solve for x: $\sqrt{1x} + \sqrt{4x} + \sqrt{9x} + \sqrt{16x} = 1 + 4 + 9 + 16$.

6. **9**

Solution: We have that $\sqrt{1x}+\sqrt{4x}+\sqrt{9x}+\sqrt{16x} = \sqrt{x}+2\sqrt{x}+3\sqrt{x}+4\sqrt{x} = 10\sqrt{x} = 30$, so $\sqrt{x} = 3$ or $x = 9$.

7. A perfect power is a positive integer that can be written in the form n^m for a positive integer n and $m \geq 2$. Catalan's Conjecture states that $2^3 = 8$ and $3^2 = 9$ are the only two consecutive positive perfect powers among all positive integers. Find the average of the smallest pair of perfect powers that differ by exactly 2.

7. **26**

Solution: The first six perfect squares are $1, 4, 9, 16, 25, 36$ and the first four perfect cubes are $1, 8, 27, 64$. Since 25 and 27 differ by exactly two, and there are no other perfect powers under 25 not listed already, the answer is 26.

8. Quadrilateral $GRAM$ is inscribed in a circle, and $\angle GAM \cong \angle RMA$. If $RM = 8$, what is the ratio of the area of triangle MAG to that of triangle MAR?

8. **1**

Solution: Let r be the distance from point R to line AM and g the distance from point G to line AM. Notice that $\angle GAM$ faces chord GM and $\angle RMA$ faces chord RA of the same circle. Since the measures of the two angles are equal, the chords also have the same length, so $GRAM$ is an isosceles trapezoid with $GR \parallel AM$. Hence, $r = g$.

The desired answer is $\dfrac{[MAG]}{[MAR]} = \dfrac{AM \cdot g/2}{AM \cdot r/2} = 1$. Note that the given length $RM = 8$ was extraneous.

9. An arithmetic sequence is a list of terms such that the difference any between two consecutive terms is constant. For example, $\cdots, 1, 5, 9, 13, \cdots$ is an arithmetic sequence. How many decreasing arithmetic sequences with only integer terms have 20 and 14 as two of the terms?

9. **4**

Solution: We let r be the *common difference* of the series, or the difference between two consecutive terms of the sequence. Any pair of two numbers in a sequence are separated by an integer multiple of the common difference, so we know that $nr = 20 - 14 = 6$ for some positive integer n. Thus, r must be a factor of 6, and since the sequence is decreasing, the possible values for r are $-1, -2, -3, -6$, leaving an answer of 4.

The four sequences are: (a) $\cdots, 20, 19, 18, 17, 16, 15, 14, \cdots$, (b) $\cdots, 20, 18, 16, 14, \cdots$, (c) $\cdots, 23, 20, 17, 14, 11, \cdots$, and (d) $\cdots, 26, 20, 14, 8, \cdots$.

10. Alex and Bob are counting numbers. Alex starts at 2014 and counts down by 2's while Bob starts at 9 and counts up by 3's. If they count at the same rate, at what number do they meet?

10. **1212**

Solution: On each turn, the positive difference between their numbers decreases by $2+3=5$. They meet when the positive difference between their numbers equals to 0. Thus, they will meet after $(2014-9)/5 = 401$ turns, so they will count the number $9 + 3(401) = 1212$ at the same time.

11. Al and Cy are standing together, 60 meters away from Bo. When the clock strikes midnight, Al and Cy will each begin moving towards Bo, at 2 m/s and 4 m/s respectively. Simultaneously, Bo will begin moving towards Al at 1 m/s. When Cy meets Bo, he will turn around and begin moving towards Al at 4 m/s. How many seconds after midnight will Cy meet Al?

11. **16**

Solution: Suppose that Al, Bo, and Cy are positioned on a number line, with Al and Cy starting at $x = 0$ and Bo starting at $x = 60$. We first try to determine their positions when Bo and Cy meet. Since Cy moves at 4 m/s and Bo at 2 m/s, they will meet $60/(4+2) = 10$ seconds in, so they will be at $x = 40$ when they meet. Al will be at $x = 1 \cdot 10 = 10$ when Bo and Cy meet.

Since Cy and Al are separated by $40-10 = 30$ meters, they will take an additional $30/(4+1) = 6$ seconds to meet up in the middle. Thus, the answer is $10 + 6 = 16$ seconds.

12. Penny flips a fair coin five times. Let p be the probability that she flips 5 heads in a row, and let q be the probability that she flips $HTTHH$ (here, H denotes a head and T denotes a tail flip). Compute q/p.

12. **1**

Solution: Each flip is independent of each prior flip. Since the probability of flipping a head is 1/2, the probability of flipping 5 heads in a row is $p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$. Similarly, $q = \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{2}\right) \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$. Thus, $q/p = 1$.

13. You have an unopened bag of chips and can of salsa. You realize that each bag of chips uses up 5/7 of a can of salsa. If you finish a bag of chips but have leftover salsa, you buy another bag of chips. If you finish a can of salsa but have leftover chips, you buy another can of salsa. By the time that you run out of chips and salsa at the same time, how many bags of chips will you have consumed?

13. **7**

Solution: We split up one bag of chips into five mini-bags of chips, and one can of salsa into seven mini-cans of salsa. Since one bag of chips uses of 5/7 of a can of nachos, then then one mini-bag of chips uses up one mini-can of salsa. In the end, since we run out of chips and salsa at the same time, the total number of mini-units must be a multiple of both five and seven. Since the smallest such number is 35, you will have consumed $35/5 = 7$ bags of chips at that point.

14. Find the smallest positive integer n that satisfies the following:

$$\begin{aligned} n &\equiv 14 \pmod{20} \\ n &\equiv 20 \pmod{14} \end{aligned}$$

(Here, $a \equiv b \pmod{m}$ means that a and b leave the same remainder when each is divided by m. For example, $20 \equiv 14 \pmod{6}$ because both 20 and 14 leave a remainder of 2 upon division by 6.)

14. **34**

Solution: From the first congruence relation above, we know that n leaves a remainder of 14 upon division by 20, so the last two digits of n must be $14, 34, 54, 74$, or 94. From the second congruence relation, we know that n leaves a remainder of $20 - 14 = 6$ upon division by 14. The first few positive integers that fit this criteria are $6, 20, 34, 48, \cdots$. Thus, 34 is the answer.

15. Six points on the plane form a regular hexagon. Let a be the number of distinct triangles with three vertices as three distinct points. Let b be the number of distinct triangles with three vertices on these points such that the center of the hexagon lies on or in the triangle. Compute $100a + b$.

15. **2014**

Solution: As shown in the figure below, there are only three types of triangles that can be drawn using the dots. In total, there are $a = (6 \times 5 \times 4)/(3 \times 2 \times 1) = 20$ ways to choose three dots. Of these 20 triangles, only those congruent to Triangle 1 do not contain the center. Since there are 6 such triangles, $b = 20 - 6 = 14$, so the answer is 2014.

16. In the card game War, two players simultaneously play a card from his or her deck. The person who plays the higher card takes both played cards and adds them to his or her deck. This process is repeated until one person, the winner, holds all of the cards. Maxie starts with the cards 3, 4, 5, and 10 and Minnie starts with 6, 7, 8, 9. If one of them wins after two days of playing War, what is the probability that Maxie was the winner?

16. **1**

Solution: Maxie's 10 cannot be taken. Since one of them wins, Maxie must be the winner, so the probability is 1.

17. Let $f(x)$ be a function such that $f(f(x)) = x$ for all x. If $f(a^2) = f(a + 20)$ for some $a > 0$, what is a?

17. **5**

Solution: Let $y = f(a^2) = f(a+20)$. Then $f(y) = f(f(a^2)) = f(f(a+20))$, so $a^2 = a + 20$. This quadratic is equivalent to $a^2 - a - 20 = 0$ or $(a-5)(a+4) = 0$. Since $a > 0$, the value of a is 5.

18. Let r be the sum of the positive factors of 108 and let s be the sum of the squares of the positive factors of 108. Compute s/r.

18. **123/2**

Solution: Since $108 = 2^2 \cdot 3^3$, each of its factors is in the form of $2^a \cdot 3^b$ for some $0 \le a \le 2$ and $0 \le b \le 3$. Thus, $r = (2^0 + 2^1 + 2^2)(3^0 + 3^1 + 3^2 + 3^3) = \frac{2^3-1}{2-1} \cdot \frac{3^4-1}{3-1}$.

Additionally, the squares of the factors of 108 must be in the form of $2^c \cdot 3^d$ for $c = 0, 2, 4$ and $d = 0, 2, 4, 6$. Thus, $s = (2^0 + 2^2 + 2^4)(3^0 + 3^2 + 3^4 + 3^6) = \frac{2^6-1}{2^2-1} \cdot \frac{3^8-1}{3^2-1}$. Using the difference of squares identity to simplify the computation, we find that the answer is:

$$\begin{aligned}
\frac{s}{r} &= \frac{2^6-1}{2^2-1}\cdot\frac{2-1}{2^3-1}\cdot\frac{3^8-1}{3^2-1}\cdot\frac{3-1}{3^4-1}\\
&= \frac{2^6-1}{2^3-1}\cdot\frac{2-1}{2^2-1}\cdot\frac{3^8-1}{3^4-1}\cdot\frac{3-1}{3^2-1}\\
&= (2^3+1)\cdot\frac{1}{2+1}\cdot(3^4+1)\cdot\frac{1}{3+1}\\
&= 123/2
\end{aligned}$$

19. Let p be the probability that an eight-digit number in the form of $AB2014CD$ is divisible by 9 (the digits are not necessarily distinct). Let q be the probability that an eight-digit number in the form of $2014ABCD$ is divisible by 9. Compute $|p-q|$.

19. **1/90000**

Solution: The divisibility rule for 9 is that if the sum of the digits is divisible by 9, then the entire number is divisible by 9. To compute p, choose any values for B, C, and D. Since A can take on any of the values $1, 2, 3, \cdots, 8, 9$ (but not 0) and each of these values leaves a different remainder upon division by 9, there will always be exactly one value of A that makes the number divisible by 9. Thus, $p = 1/9$.

To compute q, we note that the smallest and largest multiples of 9 between 20140000 and 20149999, inclusive, are 20140002 and 20149992 respectively. There are $(20149992-20140002)/9+1 = 1111$ multiples of 9 in the range. Since there are 10000 numbers in the range, $q = 1111/10000$. Thus,

$$|p-q| = \frac{1}{9} - \frac{1111}{10000} = \frac{10000-9999}{90000} = \frac{1}{90000}$$

20. In triangle ABC, points D and E are on sides AB and AC respectively so that DE is parallel to side BC. Let F be the intersection of segments BE and CD. If the ratio of the areas of triangles FBC and ABC is $13:37$, compute the ratio $DE:BC$.

20. 12 : 25

Solution: Since triangles FBC and ABC share the same base BC, then the ratio of the distance from F to BC the distance from A to BC is $13:37$. Let $13x$ be the distance between F and BC and let $37x$ be the distance from A to BC. Additionally, let the distance from F to line DE be ax; we will attempt to solve for a.

The answer we seek is DE/BC. Since $\triangle FDE \sim \triangle FCB$, we have that $DE/CB = ax/13x = a/13$. Additionally, $\triangle ADE \sim \triangle ABC$, so $DE/CB = (37x - ax - 13x)/37x = (24-a)/37$. Thus, $\frac{a}{13} = \frac{24-a}{37}$. Solving for this equation yields that $37a = 24\cdot 13 - 13a$, or $a = 24\cdot 13/50$. Substituting this value back in gives an answer of $DE/BC = a/13 = 24/50 = 12/25$.