MSJ Math Club

MMT Power Round: Generating Functions

15 December 2014

Some of the following information may help you:

- Unless otherwise stated, all answers must be in closed form. That is, your answer cannot contain any summations or products, and must be in one line.
- Unless otherwise stated, all answers must be proven.
- You may use the result of a previous problem to write your proof, even if you did not prove it.

1 Introduction

Some introductory problems:

1.1 [2] Let $f(x) = 1 + x + x^2 + x^3 + x^4 + \ldots$. Show that for $|x| < 1$, $f(x) = \frac{1}{1-x}$

1.2 [1] Evaluate 10101010101010101^2. (No proof required)

1.3 [3] Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$. Show that for $|x| < 1$, $f(x) = \frac{1}{(1-x)^2}$.

1.4 [1] Find the coefficient of x^2 in the expansion of $(1 + x)^{101}$.

1.5 [2] Find the coefficient of x^{19} in the expansion of $(1 + x^5 + x^7)^{10}$.

2 Definitions

A *generating function* for a sequence (usually of integers) $a_0, a_1, a_2, a_3, \ldots$ is equal to

$$p(x) = \sum_{n=0}^{\infty} a_n x^n$$

. That is, it is the polynomial, with the sequence a_n as its set of coefficients.

2.1 [1] Find the sequence $a_0, a_1, a_2, a_3, \ldots$ for the generating function $1/(1 - x)$.

2.2 [1] Let $a(x), b(x)$ be the generating functions for $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$. Prove that $a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots = a(x) + b(x)$.

2.3 [1] Show that, for a constant n, $na_0, na_1, na_2, na_3, \ldots = na(x)$.

2.4 [2] Show that shifting a sequence n terms to the right ($a'_{k+n} = a_k$) equals $x^n a(x)$.

2.5 [1] Find the closed form of the generating function of the sequence a_n, where $a_0 = 1$ and $a_k = 3a_{k-1}$. (no proof required)

2.6 [2] Find the closed form of the generating function of the sequence a_n, where $a_0 = 3$ and $a_k = a_{k-1} + 5$. (no proof required)

2.7 [2] Find the closed form of the generating function for a sequence x_n, where $x_0 = a$, and $x_k = bx_{k-1}$ in terms of a and b.

2.8 [2] Find the closed form of the generating function for a sequence x_n, where $x_0 = a$, and $x_k = x_{k-1} + b$ in terms of a, b.

2.9 [4] Find the closed form of the generating function for a sequence x_n, where $x_0 = a$, and $x_k = cx_{k-1} + b$ in terms of a, b, c.

3 Solving

Sometimes you can also solve for generating functions. For example if the sequence b_n is defined such that $b_0 = 1$ and $b_n = 2b_{n-1}$, one may solve for the generating function $f(x)$ as follows:

$$f(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} 2b_{n-1}x^n + 1 = \sum_{n=0}^{\infty} 2b_n x^{n+1} + 1 = 2xf(x) + 1$$

$$f(x) = 2xf(x) + 1, \ (1 - 2x)f(x) = 1, \ f(x) = \frac{1}{1 - 2x}$$

3.1 [1] Find the closed form of the generating function of the sequence defined by $a_0 = 1$, $a_1 = 1$, and $a_n = 2a_{n-1} + a_{n-2}$.

3.2 [2] Find the generating function $a(x) = \sum a_n x^n$ of the sequence defined by $a_0 = 1$, $a_1 = 1$, and $a_n = k_1 a_{n-1} + k_2 a_{n-2} + k_3 a_{n-3} + \ldots$ in terms of the generating function $k(x)$ of k_n.

3.3 [4] Find the generating function $a(x) = \sum a_n x^n$ of the sequence defined by $a_0 = 1$, $a_1 = 1$, and $a_n = q + k_1 a_{n-1} + k_2 a_{n-2} + k_3 a_{n-3} + \ldots$ in terms of q and the generating function $k(x)$ of k_n.

4 Catalan Numbers

One especially nice way to use solving for generating functions is solving for the generating function of the Catalan numbers, defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$.

4.1 [7] Show that $C_{n+1} = C_n C_0 + C_{n-1} C_1 + C_{n-2} C_2 + ... + C_1 C_{n-1} + C_0 C_n$

4.2 [2] If $f(x)$ is the generating function for a sequence a_n, find an explicit formula for the sequence b_n which has a generating function: $(f(x))^2$. (doesn't have to be in closed form)

4.3 [5] Find the generating function of the Catalan numbers.

5 Properties of Exponential Generating Functions

An *exponential generating function* for a sequence $a_0, a_1, a_2, a_3, \ldots$ is equal to

$$p(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

The natural logarithm of base e, $\ln n$, is defined as the area under the curve $f(x) = 1/x$ from $x = 1$ to $x = n$. The number e has the following property:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

5.1 [1] Find the sequence $a_0, a_1, a_2, a_3, \ldots$ for the exponential generating function e^x.

5.2 [2] For a constant n, find the sequence $a_0, a_1, a_2, a_3, \ldots$ for e^{nx}.

5.3 [1] Let $a(x), b(x)$ be the exponential generating functions for $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$. Prove that $a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots = a(x) + b(x)$.

5.4 [1] Show that, for a constant n, $na_0, na_1, na_2, na_3, \ldots = na(x)$.

5.5 [2] Show that shifting a sequence n terms to the right does not equal $x^n a(x)$.

5.6 [4] Show that for exponential generating functions $a(x), b(x), c(x)$ such that $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$, we have $c(x) = a(x)b(x)$.

Note: This is extremely useful for counting involving ordered objects. Given arrangements of type A (a_n arrangements of type A for n objects) and type B (b_n arrangements of type B for n objects), define arrangements of type C for n labeled objects as follows: Divide the n objects into two groups, and arrange the first group by an arrangement of type A and arrange the second group by an arrangement of type B. Then

$$c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$$

Thus

$$c(x) = a(x)b(x)$$

We can generalize this to

$$p(x) = a_1(x)a_2(x)a_3(x)\ldots a_k(x)$$

for counting the number of ways to order n distinguishable objects subject to the arrangements of $a_1(x)$, $a_2(x), \ldots a_k(x)$.

5.7 [1] Show that the exponential generating function for the number of n-letter words composed entirely of a's is e^x.

5.8 [2] Show that the exponential generating function for the number of n-letter words composed of an even number of a's is $\frac{e^x + e^{-x}}{2}$.

5.9 [2] Show that the exponential generating function for the number of n-letter words composed of an odd number of a's is $\frac{e^x - e^{-x}}{2}$.

5.10 [3] Using exponential generating functions, find a closed form for the number of n-letter words composed of any nonnegative integer number of m letters.

5.11 [2] Using exponential generating functions, find a closed form for the number of n-letter words composed of a′s, an even number of b′s, and an odd number of c′s.

5.12 [5] Find the exponential generating function for the number of n-letter words composed of a multiple of 4 number of a′s.

5.13 [3] Using exponential generating functions, find a closed form for the number of n-letter words composed of a′s, an odd number of b′s, and a multiple of 4 number of c′s.

6 Bonus Section!

6.1 [5] We can also make multivariate generating functions! Find the closed form of

$$p(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} x^i y^j$$

(the $i + j$ is put to ensure nothing is left undefined)

6.2 [5] Find the generating function for the number of ways to split n indistinguishable objects into a number of groups. Order does not matter.

6.3 [5] Find the generating function for the number of ways to split n indistinguishable objects into a number of groups containing odd numbers of items. Order does not matter.

6.4 [5] Find the generating function for the number of ways to split n indistinguishable objects into a number of distinct groups. Order does not matter.

6.5 [3] Show that the two above are equal.