MSJ Math Club

MMT 2015: Power Round

Permutations

1 The Power Round

The **Power Round** is a 60 minute proof-based round with a maximum score of 100 points. All problems requires justification unless otherwise specified. You may freely use results of previous problems in proving later problems, even if you have not proved the previous result (but not the other way around). Please write your team name on every sheet of paper you turn in, and *clearly label* each problem number on every sheet. You may use both sides of the paper. Partial credit may be given for partial progress on a problem, but the decisions of the judges will be final. Good luck!

2 Basics

The *cardinality* of a set is the size of the set. For example, the cardinality of $\{1, 2, 3, 4\}$ is 4.

A function is *bijective* if you can essentially draw a line between every element x in the domain of the function and every element $f(x)$ in the range of the function. A more rigorous statement is that a function f is bijective if for every element y in its range, there is exactly one x such that $f(x) = y$, and for every element x in its domain, there is exactly one y such that $f(x) = y$. A function is bijective if and only if it has an inverse function.

A *permutation* is defined a bijective (one to one) function f from a set S of n elements to itself. Essentially, f switches around the elements of S. For example, the function f such that $f(1) = 2, f(2) = 4, f(3) = 1$,and $f(4) = 3$ is a permutation on the set $\{1, 2, 3, 4\}$.

Two permutations f and g are considered distinct if for at least one $x \in S$, $f(x) \neq g(x)$. If the permutations have the same values for the same inputs, then they are the same.

Problem 2.1: How many permutations are there over a set of four elements? [8pt]

Problem 2.2: Let $n!$ be defined such that $0! = 1$ and $n! = n(n-1)!$ for all positive integers n. Prove that the number of permutations over a set of n elements is equal to $n!$. [4pt]

We define the composition $h = f.g$ of two permutations f and g to be the function $h(x) = f(g(x))$. By combining two permutations, one is basically permuting a permutation.

Problem 2.3: Show that the composition of two permutations is also a permutation. (i.e. show it's a bijective function from S to S.) [2pt]

Problem 2.4: Prove, or disprove, the claim that $f.g$ is always equal to $g.f$. [2pt]

The identity permutation is the permutation such that $I(x) = x$ for $x \in S$. It doesn't switch around any of the elements in S. We define the *inverse* of a permutation to be the permutation g such that $g.f = I$.

Problem 2.5: Show that the inverse of a permutation always exists and is always a permutation. [2pt]

Problem 2.6: Show that no two distinct permutations have the same inverse; show that the inverse of a permutation is unique. [4pt]

The kth ($k \in \mathbb{Z}$) power of a permutation f, f^k is defined by $f^0 = I$, and $f^k = f.f^{k-1}$ for integer $k > 0$.

Problem 2.7: Show that for every permutation, there exists a $k > 0$ such that $f^k = I$. [8pt]

3 Swapping

A $swap(i, j)$, $i, j \in S$, $i \neq j$ is defined as the permutation f such that $f(i) = j$, $f(j) = i$, and $f(x) = x$ for all other $x \in S$.

Problem 3.1: Show that any permutation can be expressed as the composition of a number of swaps. (i.e. For every permutation g there exist a sequence of swaps $f_1, f_2, f_3, \ldots$ such that $f_1.f_2.f_3 \ldots = g$.) [4pt]

Problem 3.2: Suppose you had a set A of swaps (over S). Prove that for every permutation g there exist $f_1, f_2, f_3, \ldots \in A$ such that $f_1.f_2.f_3 \ldots = g$ if and only if for every x and y in S, there exist $z_1, z_2, z_3, ..., z_n \in S$ such that $swap(x, z_1), swap(z_1, z_2), swap(z_2, z_3), ..., swap(z_{n-1}, z_n), swap(z_n, y) \in A$. [8pt]

Problem 3.3: Suppose the condition (for every x and y in S, there exist $z_1, z_2, z_3, ..., z_n \in S$ such that $swap(x, z_1), swap(z_1, z_2), swap(z_2, z_3), ..., swap(z_{n-1}, z_n), swap(z_n, y) \in A$.) in the previous problem were not true; what is the maximum number of permutations that can possibly be expressible as the composition of terms in A? Express your answer in terms of n, the cardinality (size) of the set S. (no proof needed) [2pt]

4 Parity

We now introduce a concept known as the *parity* of a permutation. An *odd* permutation is one that is expressible as the composition of an odd number of swaps. An *even* permutation is one that is expressible as the composition of an even number of swaps.

Problem 4.1: Show that *even* permutations are never *odd* permutations, and vice versa. [4pt]

Problem 4.2: Show that the composition of two permutations with the same parity (both odd or both even) is even. Show that the composition of two permutations with different parities (both odd or both even) is odd. [2pt]

Problem 4.3: How many even permutations are there over a certain set S with cardinality (size) n? Prove your answer. [8pt]

5 Composition Sets

Let us have a set A of permutations. Let $Q(A)$ be the set of all permutations that can be expressed as compositions of elements of A.

Problem 5.1: Show that $Q(Q(A))$ is equal to $Q(A)$. [4pt]

Problem 5.2: Suppose $swap(x, y) \notin Q(A)$ for a pair of distinct $x, y \in S$. Show that the cardinality of $Q(A)$ is at most $n!/2$. [8pt]

Problem 5.3: Suppose the set A contains n distinct swaps. What is the minimum possible cardinality of $Q(A)$? (no proof needed) [2pt]

Problem 5.4: Suppose the set A contains n distinct swaps. What is the maximum possible cardinality of $Q(A)$? (no proof needed) [2pt]

6 Cycles and Permutations

A *cycle* of size n is a permutation f such that for distinct $x_1, x_2, ..., x_n \in S$, $f(x_1) = x_2, f(x_2) = x_3, ...,$ $f(x_{n-1}) = x_n, f(x_n) = x_1$.

The *order* of a permutation f is the minimum k such that $f^k = I$.

Problem 6.1: Show that a cycle of size n has order n. [2pt]

A permutation f acts on an element x if $f(x) \neq x$. The *operating set* of a permutation f is the set of all the elements x such that f acts on x.

Problem 6.2: Show that a permutation cannot act on only one element. [4pt]

Problem 6.3: Show that if f and g have disjoint operating sets (i.e. the intersection of the sets is the empty set), then $f.g = g.f$. [4pt]

Problem 6.4: Show that any permutation on a finite set S can be expressed as the composition of a number of cycles with disjoint operating sets. [16pt]