MSJ Math Club

Mass Points

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1 Introduction

Mass points is a technique that simplifies the process of finding ratios between lengths of segments in a geometric problem, and it is actually based on the concept of torque (from physics). Problems that use mass points can usually be solved using similar triangles, area ratios, or vector bashing, but mass points gives us a cleaner, faster solution.

If you have some physics knowledge, here is some background as to where the idea of mass points came from (using levers):

$$\tau = rF$$

where τ is the magnitude of torque, r is the distance between the fulcrum and the force, and F is the magnitude of the force. If two objects are placed at opposite ends of a lever, they are in equilibrium if the two torque forces are equal, or

$$r_1F_1 = r_2F_2$$

If we divide by g, we get that

$$m_1r_1 = m_2r_2$$

Mass point geometry works by assigning masses to points using ratios between lengths of segments, using the above fact that the lengths are inversely proportional to the masses. In addition, we give the point dividing the two segments a mass equal to the sum of the masses at the two endpoints, since this point acts as a fulcrum.

Here is an example to make this clear:

Example 1: In $\triangle ABC, D$ is on $\overline{BC}$ such that $BD = DC$ and E is on $\overline{AC}$ such that $EC = 5AE$. $\overline{AD}$ and $\overline{BE}$ intersect at a point G. If F is a point on $\overline{AB}$ such that points C, G, and F are collinear, then calculate $\frac{AG}{GD}$ and $\frac{AF}{FB}$.

Solution:

Without loss of generality, let's assign a mass of 1 to C. Then, A will have a mass of 5 (since $EC = 5AE$), and B will have a mass of 1 (since $BD = DC$). Points D, E, and F are all fulcrums of line segments, so we assign each of their masses as the sum of the endpoints of the line segment they're on. Therefore, they will have masses of 2, 6, and 6 respectively. Finally, G is also a fulcrum, and we find that its mass is 7. Therefore, $\frac{AG}{GD} = \frac{2}{5}$ (since there's a mass of 5 on A and a mass of 2 on D) and $\frac{AF}{FB} = \frac{1}{5}$ (since there's a mass of 5 on A and a mass of 1 on B).

One of the hardest parts about using mass points is knowing when to use them; often, if a problem involves only ratios of lengths (as the one above did), mass points is a good way to reduce the amount of work you have to do. Recently, however, since mass points are becoming more well-known, contest problems involving finding ratios generally use other techniques in conjunction with mass points. (You will see some of these in the practice problems below).

2 Practice Problems

1. (2004 AMC 10B) In $\triangle ABC$ points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?

2. (AIME 1992) In $\triangle ABC$, A', B', and C' are on $\overline{BC}, \overline{AC}$, and $\overline{AB}$ respectively. Given that $\overline{AA'}, \overline{BB'}$, and $\overline{CC'}$ are concurrent at P, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

3. (Varignon's Theorem) If the midpoints of consecutive sides of a quadrilateral are connected, the resulting quadrilateral is a parallelogram.

4. In quadrilateral $ABCD, E, F, G$, and H are the trisection points of $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$ nearer A, C, C, A respectively. Show that $EFGH$ is a parallelogram.

5. In $\triangle ABC$, D is on $\overline{BC}$ such that $BD = DC$, E is on $\overline{AC}$ such that $AE = 3EC$, and F and G are on $\overline{AC}$ such that $AF : FG : GB = 2 : 3 : 3$. $\overline{CF}$ and $\overline{ED}$ intersect at a point H and $\overline{CG}$ and $\overline{ED}$ intersect at a point I. Compute $EH : HI : ID$.

6. (East Bay Mathletes 1999) In $\triangle ABC, D$ is on $\overline{AB}$ and E is on $\overline{BC}$. Let $F = \overline{AE} \cap \overline{CD}, AD = 3, DB = 2, BE = 3$, and $EC = 4$. Find $EF : FA$.

7. (AHSME 1964) The sides of a triangle are lengths 13, 14, and 15. The altitudes of the triangle meet at point H. If $\overline{AD}$ is the altitude to the side of length 14, what is the ratio $HD : HA$?

8. (AIME 1985) In triangle ABC, cevians $\overline{AD}, \overline{BE}$, and $\overline{CF}$ intersect at point P. The areas of triangles PAF, PFB, PBD, and PCE are 40, 30, 35, and 84, respectively. Find the area of triangle ABC.

9. (AIME 1989) P is inside $\triangle ABC$. $\overline{APD}, \overline{BPE}$, and $\overline{CPF}$ are drawn with D, E, and F on $\overline{BC}, \overline{AC}$, and $\overline{AB}$. Given that $AP = 6, BP = 9, PD = 6, PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.