MSJ Math Club

Fractals

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1 Introduction

To understand what fractals are, we first have to define a term known as the *fractal dimension*.

Consider the set of points forming the Cantor Dust fractal, as we see below:

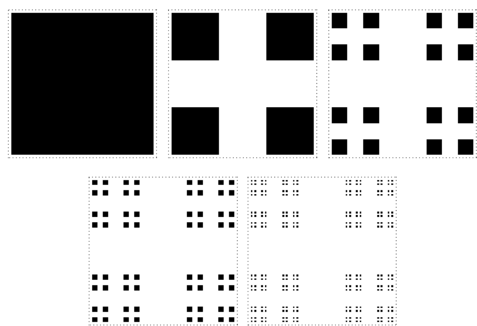

Figure 1: http://www.robertdickau.com/

Every step in the generation of this fractal, we have an object with a defined 2D area. However, this area becomes arbitrarily small as the number of iterations goes to infinity. At the very end, the object just becomes a sparse set of discrete points!

How does one assign a "dimension" to such an object? We know a finite set of points has dimension zero. But this set is infinite (and uncountable)! We know that a line is "one dimensional". We know that a square is 2D. But how exactly do we know this; what exactly is a "dimension"?

2 Box-Counting Dimension

Suppose we were to cover a set of points with a number of circles (or spheres, or hyperspheres, etc) with radius $1/n$. What is the minimum number of circles we need to do this? Call this number $f(n)$.

The *box-counting dimension* of a fractal is the limit of

$$d = \frac{\ln f(n)}{\ln n}$$

as n approaches infinity (i.e. "gets very very large").

Let's compute this for a few objects we know about already. A line with length k would need at least $\frac{nk}{2}$ circles to cover (each circle can cover a length of $2/n$). When n is very big,

$$d = \frac{\ln n + \ln k/2}{\ln n} = 1$$

because the $\ln k/2$ becomes negligible with respect to $\ln n$.

What about a square? Well, we know that arranging the circles in some sort of grid means that the the number of circles we need to cover a square with side length k is at most $f(n) < (\frac{nk}{\sqrt{2}})^2$. We also know that since the area of each circle is π/n^2, the number of circles we need is at least $f(n) > \frac{n^2k^2}{\pi}$. Notice how both of these have are degree two polynomials in terms of n. It's not difficult to show that, based on the definition, a square has dimension 2.

The interesting thing about a fractal is that it has *fractional* dimension, as we will see in the example problems.

3 Tips and Tricks

- It's enough to find an upper and a lower bound for the number of circles that are needed to cover an object, and then show that both of these have logarithms that converge to the same number.
- Notice that if $m \geq n$, then $f(m) \geq f(n)$.
- The box-counting dimension is not generally considered the most "accurate" measure of a fractal's dimension. To do that, you would need the concept of *Hausdorff dimension*, which requires a bit of set theory to work with.
- Fractal dimension has nothing to do with the "size" of a set of points. A square has essentially the same number of points as a line segment; we can prove this using transfinites.
- Box-counting is not a good mesure for fractals that are unbounded (i.e. a full infinite line would have an undefined box-counting dimension).

4 Examples

1. Find the box-counting dimension of the Cantor Dust fractal.
2. Find the box counting dimension of the set $\{1/n \mid n \in \mathbb{Z} \textit{ and } n > 0\}$

5 Practice Problems

1. Find the box-counting dimension of the Koch Snowflake.
2. Find the box-counting dimension of the set of rational numbers between 0 and 1. This shows some of the limitations of the box-counting method.
3. Length is defined as $2\frac{f(n)}{n}$ as n approaches infinity. Prove or find a counterexample to the claim that any set of points with an undefined length has a box-counting dimension not equal to 1.

6 Problem of the Week

The numbers 1,2,3,4,5,6,7, and 8 are randomly placed on the vertices of a cube so that each vertex has a different number. Find the probability that no two consecutive numbers are written on vertices with a common edge, given that 1 and 8 are considered consecutive.