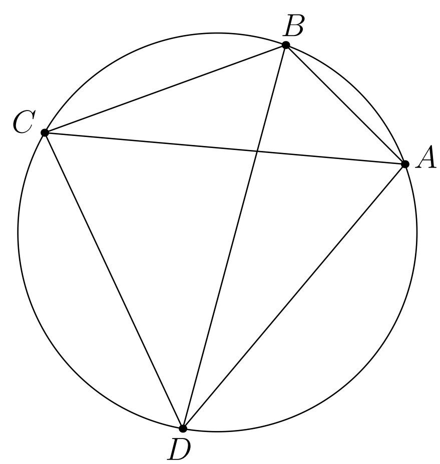
MSJ Math Club

Olympiad Angle Chasing

April 11, 2013

1 Tips and Tricks

- Angle chasing is undoubtedly one of the most useful skills you can learn when preparing for USA(J)MO. This isn't really a tip or trick, but just something you may want to know.
- Know which configurations are "angle-chase-able." Often, if you can recognize whether you can express certain angles in terms of other angles, you can get a pretty good idea of what kind of techniques you may want to use. This takes practice.
- Cyclic quadrilaterals are very useful for angle chasing, as they have two important angle relationships: (1) Opposite angles add up to $180°$. (2) Angles that subtend the same arc are congruent. The converse holds: if one of these statements is true, then you can conclude that the quadrilateral is cyclic.



- Midpoints are generally kind of ugly for angle chasing. If angles are nice for a diagram, sometimes reflecting a point about a midpoint (to create a parallelogram is useful).
- If you have length relations given, but you wish to use angles, sometimes you will need similar triangles to establish this connection.

2 Useful Results

1. Prove that the altitudes of a triangle are concurrent.
2. Let O be the circumcenter of a triangle ABC. Prove that $\angle BCO = 90 - \angle A$.

3. Prove that when the orthocenter H is reflected about side BC of triangle ABC and midpoint M of side BC, the resulting points lie on the circumcircle of ABC.

4. (Simson Line) Given triangle ABC, and a point P, denote X, Y, and Z to be the feet of P onto lines BC, CA, and AB respectively. Show that X, Y, Z are collinear if and only if P is on the circumcircle of ABC.

5. (9-point circle) For a triangle ABC, let H be the orthocenter. Show that the midpoints of the sides, the feet of the altitudes on the sides, and the midpoints of HA, HB, and HC all lie on a circle.

3 Practice Problems

1. In angle XOY, points P and Q are chosen such that $\angle POX = \angle QOY$. The feet of P on OX and OY are denoted A_1 and A_2, while the feet of Q on OX and OY are denoted B_1 and B_2 respectively. Show that the points A_1, A_2, B_1, and B_2 all lie on a circle.

2. (Russia 1996) Points E and F are on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.

3. (Iran 2010) Let M be an arbitrary point on side BC of triangle ABC. W is a circle which is tangent to AB and BM at T and K and is tangent to circumcircle of AMC at P. Prove that if $TK||AM$, circumcircles of APT and KPC are tangent together.

4. (USAJMO 2010) Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

5. (USAJMO 2011) Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $DE \parallel AC$. Prove that BE bisects AC.

6. Given triangle ABC, define points M and N on sides AB and AC respectively such that $MN||BC$. Segments BN and CM intersect at point P. The circumcircles of triangles BMP and CNP intersect again at point Q distinct from P. Prove that AQ coincides with the A-symmedian of triangle ABC. (Solution in MMT Power.)