MSJ Math Club

How to Write Proofs

March 21, 2013

Learning proper style for writing proofs is incredibly important for not only those participating in olympiads, but also for general presentation skills. This week's handout will discuss some general guidelines. Most of these rules follow the motto that you should try to make grading as easy as possible for whoever reads your paper.

1 Logical Presentation

- The most important part of your proof is to be rigorous. There are some problems that may seem "intuitive," but are hard to prove. You should avoid citing some things as obvious, but there are some steps that you can (and should) skip. For example, you can simply state that "since AC and BD mutually bisect each other, then $ABCD$ is a parallelogram," without having to cite a specific theorem or proving the claim. If you know that you can easily prove a certain step if you weren't so lazy, then go ahead and skip it, but when in doubt, include a step in your proof.

- Avoid using "you" in your proof. If you must, use "we," but it is still best to avoid both if possible. (Passive voice is better.)

- Make sure you write using both words and equations (when appropriate). Equations only without any word explanations is unclear, and using paragraphs only can sometimes be ambiguous.

- When using proof by contradiction, it is important to mention first: "We assume that aoeu is false for the sake on contradiction." You should also conclude your proof by contradiction by stating what you have just proved.

- When using induction, it is best to mark each part of the induction clearly, and have the following elements:

  - Start your induction saying something like "We proceed by induction."
  - Discuss the base case clearly. Usually this should only needs 1 or 2 sentences.
  - When you are moving to your inductive step, write something to the effect of "for the inductive step, we assume that the result holds for $n = k$, for some k".
  - At some point in the inductive step, you should cite your "inductive hypothesis," or your assumption that the result holds for $n = k$.
  - Conclude with something like: "Our induction is complete, proving that aoeu."

- Remember to write your proofs forward! Let's say you are given A and are asked to prove C. If B is a step in between, then you should write your proof in the order $A \Rightarrow B \Rightarrow C$. Sometimes when teaching, we present proofs in the reverse order to show you our thinking behind the problem, but you should avoid writing the way solutions are presented. Also it is best to avoid something along the structure of "B is true because of A, so C is true." Notice the difference: in the bad example above, the logical structure jumps around, which makes it harder to follow.

  Overall, this part of proof-writing is usually one of the more difficult parts to master. It might help to read official solutions (but not excessively!) and ask for some feedback overall.

- In addition to the point above, it is often useful to define a lemma (plural: lemmata, but no one uses this) for well-known results that don't have a name or a mini-theorem of the problem that you may need to use a few times. For example, if you need to use the symmedian lemma, you should write your proofs along the lines of this:

  "To proceed, we first prove a lemma:

  Lemma 1: Statement.
  Proof: Proof of lemma.

  In the original problem, we have that blah. By lemma 1 above, ..."

- Define your terms! If you talk about $x^2 + y^2$ as a factor, you should either rewrite it the next time you mention it, or mention something like "Let $a = x^2 + y^2$." Avoid using ambiguous pronouns.

- If you have an important equation or a block of equations, isolate it from the rest of your paragraphs by putting them on their own lines, indented.

2 Formatting Style

- Use 1-inch margins around the page. Some people like drawing the actual margin lines. Also, try to space your lines a bit. If your handwritten proofs were typed, aim for 12-pt font and 1.5× or 2× spacing. Cluttered work can be unbelievably annoying to read.

- Write appropriate information in the margin-corners of your proofs. On the USA(J)MO, they provide blanks for you. It is generally a good idea to include your ID/name, and "Page m of n" on each page.

- If you screw up, cross your stuff out. Don't scribble. Don't draw arrows all over the place.

- Draw pretty diagrams! On USA(J)MO, you will be required to draw a good diagram on the first page for 1 point, but you should do this anyway as a good habit. Let's pretend for a solution, you choose to extend two lines and create a new point P. If you do not mark it on your diagram, when you mention your point P, make sure to put "not marked on diagram". Even if you draw it on the diagram, it is still good to define how the point is constructed right underneath the diagram in a paragraph.

- Two column proofs are usually bad, unless you want to bother your grader.

- If you type up solutions, please use LaTeX.

3 Practice Problems

Feel free to handwrite up these problems and send them to me (Aaron) to check. Notice that the second example problem has been changed.

1. There are 102 points in the plane with no three in a line. Show that there exists a line connecting two of the points that divides 50 points from the other 50 points.

2. (Week 8) In triangle ABC, let D be an arbitrary point on side BC. Let ω_1 and ω_2 denote the incircles of triangles ABD and ACD respectively. The common external tangent to ω_1 and ω_2 aside from BC intersects segment AD again at point K. Show that the length of AK is independent of your choice of point D.