MSJ Math Club

The Fermat Point

21 May 2015

1 Warm Up

What is the shortest distance between 2 points in a plane?

2 Exercise

What if I have two points $(1, 4)$ and $(10, 8)$ and I want to find the shortest path between the two such that the path touches the x-axis?

Many people would know that we can reflect the point $(10, 8)$ across the x-axis to the point $(10, -8)$ and then connect the two resulting points. This strategy works because after we draw the line connecting $(1, 4)$ and $(10, -8)$, we can simply reflect the line under the x-axis back up so that the ending point is $(10, 8)$ again.

3 The Fermat Point

Now, what if, in $\triangle ABC$, I wanted to find the point P such that the sum $AP + BP + CP$ is minimized? The answer to this problem is significantly more complicated that the solution to the first two. The fact is, such a point does indeed exist, and is known as the Fermat Point of $\triangle ABC$. (Pronounced: fair'-ma)

The idea here is the do a similar argument as the reflection argument shown in the second example. But here, consider the point P. Now consider the rotation of $\triangle ABC$ $60°$ about point B, so that A gets mapped to A', C gets mapped to C', and P gets mapped to P'.

Since $\triangle BPP'$ is an equilateral triangle, we have $BP = PP'$. Also, since we are just rotating the triangle, all the lengths are preserved. In particular, we have $PC = P'C'$. Therefore, we have $AP + BP + CP = AP + PP' + P'C'$.

Now, the problem becomes identical to the first example! What is the shortest possible value of $AP + PP' + P'C'$? The answer is just the distance between point A and point C'! Therefore, we must have A, P, P', C' lie on a line. Since $\triangle BPP'$ is equilateral, we must have $\angle BPP' = 60°$, so $\angle APB = 120°$. Similarly, we must have $\angle BPC = \angle CPA = 120°$. Therefore, the Fermat Point P is the point within $\triangle ABC$ such that $\angle APB = \angle BPC = \angle CPA = 120°$. This point P is the point that minimizes the value of $AP + BP + CP$.

Notice how we reduced such a complicated seeming problem into a simpler problem that we all know how to solve just by a simple transformation. This is the beauty of math - that many complex ideas are actually reformulations of simple problems that we all know how to approach.