Solutions to the 2015 MMT: Individual Round Day 1.

Problem 1: What is the greatest 9-digit number that is divisible by 9, all of whose digits are distinct?

Solution: The greatest 9-digit number with distinct digits is 987654321, and this is also divisible by 9, so our answer is $\boxed{987654321}$.

Problem 2: What is the probability that the sum of 2 independent dice rolls is greater than 7?

Solution: The probability that the sum is greater than 7 is the same as the probability that the sum is less than 7. The probability that the sum is exactly 7 is $\frac{1}{6}$, so our desired answer is $\frac{1-\frac{1}{6}}{2} = \boxed{\frac{5}{12}}$.

Problem 3: TC can buy packs of playing cards in the following deals:

- 2 packs for $10
- 3 packs for $14
- 7 packs for $29

What is the least amount of money TC needs to buy at least 22 packs of cards?

Solution: TC can buy the 7-pack twice, the 3-pack twice, and the 2-pack once for a total of $58+28+10 = \boxed{96}$.

Problem 4: In right triangle $\triangle ABC$ with $\angle C = 90°$, we have D, E, F on AB such that, in order, we have A, D, E, F, B and $AD : DE : EF : FB = 1 : 2 : 3 : 4$. Given that $BC = 5$ and $AC = 12$, find [DEC] + [FBC], where brackets indicate area.

Solution: The ratio of the areas of [ADC]:[DEC]:[EFC]:[FBC] is also 1 : 2 : 3 : 4, so the ratio of our desired areas to the total area is $\frac{6}{10}$, and since the total area is 30, our answer is $\boxed{18}$.

Problem 5: Points A, B, and C lie on the parabola $y = x^2$. If the slope of AB is 4, the slope of BC is 8, and the slope of AC is 16, what are the possible **x-coordinates** of A?

Solution: The slope of a line is $\frac{y_1 - y_2}{x_1 - x_2} = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2$. Therefore, if x_1 is the x-coordinate of A, x_2 the x-coordinate of B, and x_3 the x-coordinate of C, then we have $x_1 + x_2 = 4$, $x_2 + x_3 = 8$, and $x_3 + x_1 = 16$. Solving this system of equations gives $x_1 = \boxed{6}$.

Problem 6: A rectangle $ABCD$ has $AB = 2$ and $BC = 3$. A circle with radius 1 is inscribed inside, externally tangent to DA, AB, and BC. Another circle ω is also inscribed inside, externally tangent to the first circle, BC, and CD. Find the radius r of ω.

Solution: Connect the centers of the circles as the hypotenuse of a right triangle. The hypotenuse is equal to $1 + r$. The lengths of the legs are $2 - r$ and $1 - r$. We then use Pythagorean Theorem to get that $4 - 4r + r^2 + 1 - 2r + r^2 = 1 + 2r + r^2$. Solving this gives $r = \boxed{4 - 2\sqrt{3}}$.

Problem 7: A 5×5 grid of squares is randomly filled in with 0's and 1's. A square is *happy* if all of its edge-neighbors have the same value as itself. What is the expected number of *happy* squares in the 5×5 grid?

Solution: The probability that a corner square is happy is $\frac{1}{4}$. The probability that an edge square is happy is $\frac{1}{8}$. The probability that a center square is happy is $\frac{1}{16}$. Therefore the expected number of happy squares is $4 * \frac{1}{4} + 12 * \frac{1}{8} + 9 * \frac{1}{16} = \boxed{\frac{49}{16}}$.

Problem 8: Triangle $\triangle ABC$ has D, E, F the midpoints of BC, CA, and AB, respectively. Also, let G be the intersection of AD and FE, let H be the intersection of BE and DF, and let I be the intersection of CF and DE. Given that [ABC] = 112, find [GHI], where brackets indicate area.

Solution: Note that $\triangle GHI$ is the medial triangle of $\triangle DEF$, and that $\triangle DEF$ is the medial triangle of $\triangle ABC$. Since medial triangles have $\frac{1}{4}$ of its outer triangle, the area of the inner triangle is $112 * \frac{1}{16} = \boxed{7}$.

Problem 9: A sequence is given by $a_1 = 5$ and $a_n = 3a_{n-1} + 8$ for $n > 1$. Find a closed form for a_n (one without summations or references to previous terms in the sequence).

Solution: Since there is a 3 in the formula, it makes sense that the closed form is something of the form $a_n = 3^n + C$ for some constant C. It is not hard to check that $a_n = \boxed{3^{n+1} - 4}$.

Problem 10: Find all real solutions x that satisfy the equation $x^4 - 8 = 2x(2x^2 - 3x + 2)$.

Solution: This simplifies to $x^4 - 4x^3 + 6x^2 - 4x + 1 = 9$. Therefore we have $(x-1)^4 = 9$, so $x - 1 = \pm\sqrt{3}$. Therefore our answers are $\boxed{1 \pm \sqrt{3}}$.

Problem 11: A rectangular piece of paper has length 2 and width 1. A dotted line is drawn from two opposite vertices. The paper is then folded flat along the dotted line to create a new shape. What is the area of this new shape?

Solution: The folded shape is clearly symmetric, so that means that the triangle on one of its ends is a right triangle with hypotenuse x and legs 1 and $2 - x$. Solving this quadratic gives $x = \frac{5}{4}$. That means that the area of overlap is $\frac{1}{2} * \frac{5}{4} * 1 = \frac{5}{8}$. Therefore the total area is $\boxed{\frac{11}{8}}$.

Problem 12: A permutation σ is a function that maps a finite set to itself. How many permutations σ acting on the set $\{1, 2, 3, 4, 5, 6, 7\}$ are there such that $\sigma(\sigma(\{1, 2, 3, 4, 5, 6, 7\})) = \{1, 2, 3, 4, 5, 6, 7\}$? In other words, how many self-inverse permutations that act on a set of 7 distinct elements are there? For example, if our permutation were $\delta = (3, 2, 1)$, then $\delta(\delta(\{1, 2, 3\})) = \delta(\{3, 2, 1\}) = \{1, 2, 3\}$. Therefore $\delta = (3, 2, 1)$ is a self-inverse permutation that acts on a set of 3 distinct elements.

Solution: Let $f(n)$ be the number of self-inverse permutations on n elements. If the first element goes to itself, there are $f(n-1)$ ways to choose the other $n-1$ elements. If the first element goes to any of the $n-1$ other elements, then that element must got back to 1, so that means we have the recurrence $f(n) = f(n-1) + (n-1) * f(n-2)$. Since $f(1) = 1$ and $f(2) = 2$, we can use the recurrence to build $f(7) = \boxed{232}$.