MSJ Math Club

Markov Chains

4 December 2015

1 Introduction

A *markov chain* is a set of states that, after an event, can transform into each other with certain weights that sum to one. Often, these weights represent probabilities; each state will transform into a different state with a certain probability.

The states in a markov chain can be anything. For example, if your event is flipping a coin and your "state" is the value of the last two coins you flipped, one can see that the state "HH" has a 50% chance of transforming into "HT" and a 0% chance of transforming into "TT". These probabilities are your weights in the markov chain.

Suppose we had a system with two states such that state A has a 70% chance of transforming into itself, and state B has a 40% chance of transforming into itself. Suppose the system begins in state A. How does one find the probability that the system will be in a certain state after a certain number of transformations? After zero iterations, the system will be in state A with probability 1. After one iteration, the system has a 0.7 probability of being in state A and a 0.3 probability of being in state B. After two iterations, the system will have a $0.7 \times 0.7 + 0.3 \times 0.6 = 0.67$ chance of being in state A and a $0.7 \times 0.3 + 0.3 * 0.4 = 0.33$ chance of being in state B. We can continue this. for any number of iterations.

What if we wanted to find the result after a very large number of iterations? We can "solve" for the equilibrium of this system. After a very large number of iterations, the probabilities don't change between successive iterations. Then we have $p(A) = p(A) \times 0.7 + p(B) \times 0.6$ and $p(B) = p(A) \times 0.3 + p(B) \times 0.4$. With a system of 2 linear equations and 2 variables, we get that $p(A) = 2 \times p(B)$, and since $p(A) + p(B) = 1$, we get that $p(A) = \frac{2}{3}$ and $p(B) = \frac{1}{3}$.

Consider a tennis match between Athena and Grendel that is currently tied at deuce. Athena has a 60% chance of winning any given point. What is the probability that Athena will win the match?

You can imagine that there are five different states in this scenario.

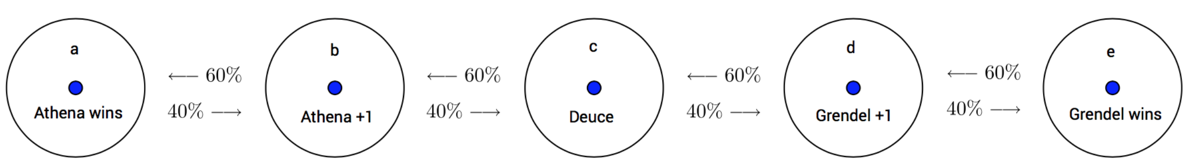

Figure 1: Note that there are 5 distinct states.

Let a be the probability that Athena wins in the first circle, b be the probability that Athena wins in the second circle, and so on. Then clearly we have $a = 1$, $b = \frac{6}{10}a + \frac{4}{10}c$, $c = \frac{6}{10}b + \frac{4}{10}d$, $d = \frac{6}{10}c + \frac{4}{10}e$, $e = 0$.

Now we just have a system of 5 linear equations with 5 unknowns, so we just solve. We have $b = \frac{3}{5} + \frac{2}{5}c$ and $d = \frac{3}{5}c$. Combining this with the middle equation, we get that $c = \frac{3}{5}(\frac{3}{5} + \frac{2}{5}c) + \frac{2}{5}(\frac{3}{5}c)$. Therefore we have $c = \frac{9}{25} + \frac{6}{25}c + \frac{6}{25}c$.

Multiplying by 25 on both sides, we get that $25c = 9 + 6c + 6c = 12c + 9$. Thus we have that $c = \boxed{\frac{9}{13}}$.

2 Tips and Tricks

- Some Markov Chains can be infinite. Find creative ways to solve this!
- Many times, as in coin flipping, one state is simply the average of its surrounding states.

3 Examples

1. (SMT 2014) Robin is playing notes on an 88-key piano. He starts by playing middle C, which is actually the 40th lowest note on the piano (i.e. there are 39 notes lower than middle C). After playing a note, Robin plays with probability 1/2 the lowest note that is higher than the note he just played, and with probability 1/2 the highest note that is lower than the note he just played. What is the probability that he plays the highest note on the piano before playing the lowest note?

4 Practice Problems

1. On a weird planet, the sunny and rainy days depend solely on whether the previous day was sunny or rainy. Every sunny day is followed by a sunny day 80% of the time. Every rainy day is followed by a another rainy day 60% of the time. It is known that on Day 1, it is sunny. After an infinite number of days, what is the probability that it is sunny on a random day? Does our information of the weather on Day 1 even matter?
2. (Classic) A certain bug has a probability p of making an identical copy of itself, a probability of q of not doing anything, and a probability $1 - p - q$ of dying in any iteration. What is the probability that a population of bugs, starting from a single bug, will survive indefinitely?
3. (ASMT 2015, generalized) A certain bug has a probability n of making two extra copies of itself, p of making an identical copy of itself, a probability of q of not doing anything, and a probability $1-n-p-q$ of dying in any iteration. What is the probability that a population of bugs, starting from a single bug, will survive indefinitely? (You don't need to solve the cubic if you don't want to.)
4. (Po-Shen Loh at ASMT 2015 lecture) Two players play a game in which they continue to flip coins until the values of the last three coins matches their guess of three coin flips. If player A chooses HTH, and player B chooses THH, what is teh probability player A will win?
5. (Similar to above) What is the expected number of coin flips until one of the players win?
6. (SMT 2012) Two ants are on opposite vertices of a regular octahedron (an 8-sized polyhedron with 6 vertices, each of which is adjacent to 4 others), and make moves simultaneously and continuously until they meet. At every move, each ant randomly chooses one of the four adjacent vertices to move to. Eventually, they will meet either at a vertex (that is, at the completion of a move) or on an edge (that is, in the middle of a move). Find the probability that they meet on an edge.
7. **Puzzle of the Week:** Three spiders are trying to catch an ant. All are constrained to the edges of a cube. Each spider can move at least one third as fast as the ant can. Prove that the spiders can catch the ant after a finite amount of time.