MSJ Math Club

Transfinite Cardinals

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1 Introduction

Definition
Two sets A, B are of *equal cardinality* (cardinality means size) if there exists a bijective (both surjective and injective) function $f : A \longrightarrow B$. (We will refer to this as $A \equiv B$.)

This is basically the same as drawing lines connecting every element in A to every element in B.

However, what is interesting about this is that it can be generalized to infinite sets!

Example 1
Show that the set of non-negative integers (0,1,2,3...) and the set of perfect squares (0,1,4,9...) have equal cardinality.

Proof
We can use the function $f(x) = x^2 (x \in \mathbb{Z})$ as the bijection. Clearly, as the domain is restricted to non-negative numbers, the function is one-to-one and invertible.

Example 2
Show that the set of all rational numbers between 0 and 1 is equal (in cardinality) to the set of all positive integers.

Proof
We can write any rational number as $\frac{p}{q}$, where p and q are relatively prime integers. We can arrange the rational numbers so that 1 corresponds to $\frac{1}{1}$, 2 corresponds to $\frac{1}{2}$, and so on as follows:

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$$

(basically counting in diagonal rows, and skipping fractions that can be reduced)

Here is a useful and simple lemma that needs to be rigorously proven.

Transitive Property
If $A \equiv B$ and $B \equiv C$, then $A \equiv C$.

Proof
Let f be the function that bijects A to B and g be the function that bijects B to C. Then we take the function composition of f and g to biject A to C.

From here on, we will can the cardinality of the integers (and rationals) $\aleph_0$

However, not all sets have equal cardinality:

Uncountability of Reals
The set of all reals between 0 and 1 has greater cardinality than the set of integers.

To do this we must show that there does not exist a function f bijecting $\mathbb{R}$ and $\mathbb{Z}^+$. Cantor's very elegant proof uses diagonalizations.

Proof
Suppose there were a function f bijecting the reals and integers. Then we can list the reals in order a_1, a_2, a_3, a_4.

We want to show that no matter what numbers we put into the table, we can always pick a new real that is not in the list. We do this by picking a number a such that the first digit of a is different from the first digit of a_1, the second digit of a is different from the second digit of a_2, the third digit of a is different from the third digit of a_3 and so on...

By continuing this process, we can construct a number that is different in at least one digit from all the other numbers on the list. Therefore, this number is not on the list, which is a contradiction. Therefore the set of reals is greater that the set of integers.

This theorem is a classical theorem in transfinite set theory. Cantor did have a more powerful theorem, however:

Cardinality of Subsets
The set of all the subsets of a set A has greater cardinality than the set A

Proof
We do it the same way as the previous proof (kind of)! Except now its binary, and instead of digits, these now represent if q is inside a set.

Assume for sake of contradiction that we had a function $f : A \longrightarrow \{q|q \subset A\}$

We construct a set Q as follows: for every $x \in A$, we find $f(x)$. If $x \in f(x)$, then don't put x in Q. If $x \notin f(x)$, then $x \in Q$. We now have a Q such that, $\forall x(Q \neq f(x))$, which is a contradiction!

2 Tips and Tricks

- Transfinites are sometimes necessary and useful to prove problems involving infinite sets.
- You may use that if there exist a surjective $f : A \longrightarrow B$ and there is an injective $f : A \longrightarrow B$, then there is a bijection from A to B
- There is a big difference between transfinite cardinals and transfinite ordinals.
- There are an infinite amount of transfinite cardinals. For every transfinite cardinal you can find, you can get a new one just by finding the set of all subsets of it.
- Infinity plus one is the same thing as infinity in terms of cardinals. Ordinals... are trickier.
- The continuum hypothesis states that there are no cardinals between that of the real numbers and that of the integers. The generalized continuum hypothesis states that there are no cardinals between a cardinal A and the set of its subsets. Both of these have been shown to be independent of the ZFC axioms.

3 Practice Problems

1. Show that the set of all rational numbers between 0 and 1 is equal (in cardinality) to the set of all positive rational numbers.
2. Show that the cardinality of the set of all possible math problems is at most equal to $\aleph_0$.
3. Show that the cardinality of the set of points in a 1 by 1 square is the same as that of the points in a line segment.
4. Show that the number of solutions to polynomials with rational coefficients is $\aleph_0$.
5. *Cantor dust* is created by starting with a line segment, and taking the middle 1/3 of the segment. This results in two segments. We then take the middle 1/3 of those two segments, and we keep on taking the middle 1/3 of the resulting segments ad infinitum. Show that the set of points you end up with has cardinality equal to that of the reals.
6. Use Cantor set theory to show that a universal set (set of all sets) cannot exist.
7. (MOSP) Construct a function f such that for every open interval (a, b) for every $y \in \mathbb{R}$, there exists a $c \in (a, b)$ such that $f(c) = y$.