Wild Round Solutions

MagMaR 2014

January 26, 2014

1. Compute $(20 + 15)14 - (14 + 15)20$.

1. -90

Solution: This is $20 \cdot 14 + 15 \cdot 14 - 14 \cdot 20 - 15 \cdot 20 = 15(14 - 20) = -90$.

2. In a future world, regular years (365 days) are to be split up into 5 regular-months and leap years (366 days) are to be split into 6 leap-months, so that each regular-month and leap-month has a fixed number of days in it. How many more days will there be in a regular-month than a leap-month?

2. **73**

Solution: In a regular month, there are $365/5 = 73$ days. In a leap month, there are $366/6 = 61$ days, so the answer is 12.

3. Triangle ABC has side lengths of $3, 4, 5$. What is its area?

3. **6**

Solution: Since $3^2 + 4^2 = 5^2$, the triangle is right, so the area is $3 \cdot 4/2 = 6$.

4. Steven spends $2/5$ of his monthly salary on rent while Amy spends $1/3$ of her monthly salary on rent. If both spend the same amount of money on rent every month, what is the ratio of Steven's salary to Amy's salary?

4. $5/6$

Solution: Let s be Steven's salary and a be Amy's salary. Since $2s/5 = a/3$, $s/a = 5/6$.

5. How many of the first 2014 positive integers are divisible by 8?

5. **251**

Solution: The greatest multiple of 8 under 2014 is 2008, so the answer is $2008/8 = 126$. (Note: this trick only works because we start counting from the first positive multiple of 8, which is 8.)

6. What is the last digit of the sum $1 + 22 + 333 + 4444 + \cdots + 999999999$?

6. **5**

Solution: The last digit of the desired sum is the last digit of each term's units digit, or $1 + 2 + 3 + \cdots + 8 + 9 = 45$. Thus, the answer is 45.

7. While computing 2^{25} for a certain math contest, Aaron received an answer of 33554433. If his answer is within 10 of the real value, compute actual value of 2^{25}.

7. **33554432**

Solution: We know that the number must be divisible by 16 because it is a large power of 2, so we test for the remainder of the last four digits upon division by 16:

$$4433 \equiv 4000 + 400 + 33 \equiv 8(500) + 8(50) + 32 + 1 \equiv 1 \pmod{16}$$

The three closest multiples of 16 to the original number are 33554416, 33554432, and 33554448. Only the second one falls in the desired bounds, so 33554432 is the answer.

Note: To test if a number is divisible by 2^n for some $n \geq 1$, you can consider the last n digits only. This is because the digits beyond the $n+1$-st place can be written as $10^n \cdot A = 2^n \cdot (5^n A)$ for some cluster of digits A.

8. For every day that you are alive, starting today, you have a 1/2 chance of dying. What is the probability that you are still alive after the fourth day?

8. 1/16

Solution: Every day, you have a $1 - \frac{1}{2}$ chance of staying alive, so the probability is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$.

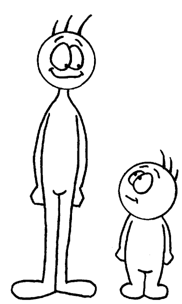
9. Five sheep are standing at the points $(0, 0)$, $(1, 3)$, $(3, 0)$, $(3, 4)$, and $(4, 0)$ on the plane. When the scientist presses "Start," each sheep immediately start walking towards its closest neighbor. How many loops (instances of sheep A following sheep B following other sheep following sheep A) are there?

9. **2**

Solution: The sheep at $(0, 0)$ follows the sheep at $(1, 3)$, which follows the sheep at $(0, 0)$. This is one loop. The sheep at $(3, 0)$ follows the sheep at $(4, 0)$, which follows the sheep at $(3, 0)$, which is another loop. The last sheep at $(3, 4)$ follows the one at $(1, 3)$, so he or she is not part of a loop.

10. At Asdfghjkl Middle School, 20% of students are short, 50% are happy, and 40% are neither short nor happy. If you select a short person from the school at random, what is the probability that he or she is happy?

10. <u>1/2 **or** 50%</u>



Solution: Since 40% of students are neither short nor happy, 60% are short or happy or both. Also, 50% are happy, so $60 - 50 = 10\%$ are short but not happy, making $20 - 10 = 10\%$ of short students happy. Thus, a randomly selected short student has a $10/20 = 50\%$ chance of being happy.

Note: Since the probability that you are happy regardless of height is equal to the probability that you are happy given that you are short, we might conclude that being short and being happy are independent. Probability in statistics deals with testing for whether two variables are independent or not.

11. Your teacher is thinking of an integer between 1 and 100 inclusive, and whoever selects the closer number among you and your best friend wins. Selected integers must be distinct. Your

best friend chooses the number 14. Assuming that you choose the number that maximizes your winning chances, what is the probability that you will win?

11. 43/50

Solution: You should choose the number 15, so that you get all numbers from 15 to 100 inclusive. Thus, the probability is $(100 - 15 + 1)/100 = 43/50$.

12. You have four pencils in your backpack. Two are yellow and two are orange. You stick your hand into your backpack and grab two at once. What is the probability that one is orange and one is yellow?

12. 2/3

Solution: There are 6 ways to choose any two pencils. Since there are 2 ways for you to choose two pencils of the same color, then the probability that you get one orange and one yellow pencil is $(6 - 2)/6 = 2/3$.

13. Nine coins are flipped. Find the probability that there are at least five heads.

13. 1/2

Solution: Since the coin is fair, flipping 5 heads has the same probability as flipping 5 tails, flipping 6 heads has the same probability as flipping 6 tails, and so on. Thus, the answer is $1/2$.

Note: The problem would be different if the coin were not fair, or if there were an even number of coins. In this case, since flipping five or more heads and flipping five or more tails are mutually disjoint (cannot occur simultaneously) and cover the space of outcomes (i.e. there are no third options), we can use this symmetry argument.

14. Compute $201 \cdot 4 + 20 \cdot 14 + 2 \cdot 014$.

14. **1112**

Solution: This is $804 + 280 + 28 = 1112$.

15. Brenda tries to compute the value of $2014 - 14 \times 20$, but does order of operations in the wrong order. What answer did she obtain?

15. **40000**

Solution: Brenda computed $(2014 - 14) \times 20 = 2000 \times 20 = 40000$.

16. Jacob, Paul, and Andrew each start with \$80 in their pockets. Andrew steals half of Paul's money, then Jacob steals half of Andrew's money, and finally Paul steals half of Jacob's money. How much money does Paul have in the end?

16. **110**

Solution: After Andrew makes his steal, Paul has \$40 and Andrew has \$120. After Jacob makes his steal, Jacob has \$140. Thus, Paul ends up with $\$40 + \$70 = \$110$.

17. At the local supermarket, Brandon purchases an Apple Macbook on sale for \$10. If the original price is \$800, what percent *discount* did Brandon receive?

17. 98.75

Solution: Brandon received a $100\% \times 79/80 = 98.75\%$ discount.

18. Suppose that $2014L + 1000 = 2014 + 1000L$. What is L?

18. **1**

Solution: Moving the L's to the left-hand side and the constants to the other yields $1014L = 1014$, so $L = 1$.

19. If the largest element of the set $\left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots, \frac{99}{100}\right\}$ is $\frac{m}{n}$ in lowest terms, find $m + n$.

19. **199**

Solution: The largest number in the set is the number that is closest to 1. Since the smallest of $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots, \frac{1}{100}\right\}$ is the last term, the largest fraction is $99/100$, so the answer is 199.

20. Alex dyes his hair a different color every day. If he can choose from cyan, magenta, yellow, or black every morning, and cannot have the same color hair on two consecutive days, how many different hair color schemes can he have in a 4-day block?

20. **108**

Solution: On the first day he has 4 choices. On each subsequent day, he can choose any color except for the one he used on the previous day, so the answer is $4 \times 3 \times 3 \times 3 = 108$.

21. A square $ABCD$ has integer side lengths and a diagonal of length at most 30. What is the maximal possible area of the square?

21. **441**

Solution: If the square has a side length of x, then the diagonal has a length of $d = x\sqrt{2}$ by Pythagorean Theorem, so the area of the square is $d^2/2$. Thus, we know that the area of the square is less than or equal to $30^2/2 = 450$. The largest perfect square less than this is 441.

22. Alvin the Cow eats 10 kilograms of grass every day. If Alvin burns off 30 grams of excess weight in one minute of running, how long must Alvin run each day (in seconds) in order to not gain weight?

22. **20000**

Solution: Alvin burns off 1 gram of excess weight in 2 seconds of running, so he must run for $2 \times 10000 = 20000$ seconds every day. (Approximately 5.5 hours.)

23. The mean of four distinct positive integers is 5. What is the largest possible value for the second-largest number?

23. **8**

Solution: The sum is $5 \cdot 4 = 20$. We wish to maximize the two largest numbers, so we let the two smallest numbers be 1 and 2. The sum of the other two numbers is 17, so we let the largest number be 9 and the second largest be 8.

24. A random permutation (or rearrangement) of the letters 'AAGMMRR' is chosen. What is the probability that either 'GRAMMAR' or 'MAGMARR' is selected?

24. 1/315

Solution: There are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways to arrange any 7 letters (or objects) in a row. However, since there are two M's, two A's and two R's, there are $\frac{5040}{2! \cdot 2! \cdot 2!} = 630$ orderings of the letters. Thus, the answer is $2/630 = 1/315$.

25. The greatest common divisor (GCD) of two positive integers is 2, and the least common multiple (LCM) of the same two numbers is 140. How many different unordered pairs of integers satisfy this?

25. **4**

Solution: If each number is divided by 2, then we have two relatively prime numbers that multiply to $140/2 = 70 = 2 \cdot 5 \cdot 7$. Let a be the number with the extra factor of 2 and b be the number without that extra factor. The factors of 5 and 7 can go to either a or b, so there are $2 \times 2 = 4$ such unordered pairs of numbers.

26. A triangle has side lengths of $20, 14$, and m, where m is an integer. How many different values for m are there?

26. **27**

Solution: By the Triangle Inequality, the minimum possible value of m is $20 - 14 + 1 = 7$ and the maximum is $20 + 14 - 1 = 33$, so there are $33 - 7 + 1 = 27$ values for m.

27. Given a triangle with two side lengths of 4 and 6 and an area of 12, find the length of the third side of the triangle.

27. $2\sqrt{13}$

Solution: The triangle's maximal area is $4 \cdot 6/2 = 12$, so the angle between the two known side lengths is $90°$. Thus, the side length of the third side is $\sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$.

28. The slope of the line connecting the points $(1, 3)$ and $(4, 7)$ is equal to the slope of the line connecting the points $(-2, 5)$ and $(x, 13)$. Compute x.

28. **4**

Solution: The slope of the line is $(7-3)/(4-1) = 4/3$. Thus, $(13-5)/(x-(-2)) = 4/3$, so $x - (-2) = 6$ or $x = 4$.

29. An isosceles trapezoid has bases of length 20 and 14. If the area of the trapezoid is 102, what is the length of each of the legs (the sides of the trapezoid that are not parallel)?

29. $3\sqrt{5}$

Solution: The area of a trapezoid is $(b_1 + b_2) \cdot h/2 = (20 + 14)/2 \cdot h = 102$, so $h = 6$. If we draw a height from one of the vertices of the shorter base to the longer base, we can form a right triangle with the trapezoid's leg ℓ as the hypotenuse. By the Pythagorean Theorem, we have that $\ell = \sqrt{((b_1 - b_2)/2)^2 + h^2} = \sqrt{(6/2)^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$.

30. How many ways can \$15 be paid off with some combination of quarters and dimes if at least one of each coin must be used?

30. **29**

Solution: We must use an even number of quarters in order to have a total of \$15 in dimes and quarters only. Thus, we can have $2, 4, 6, \cdots, 58$ quarters (since we must have at least one of each coin, so there are 29 ways.

31. Two mangoes and three pencils cost \$1714. Two mangoes and five pencils cost \$2014. How much would two mangoes and sevens pencils cost?

31. \$2314

Solution: Let X be the cost of two mangoes and seven pencils. Notice that $X + 1714$ is the cost of four mangoes and ten pencils, or $2 \cdot 2014$, so $X = 4028 - 1714 = \$2314$.

32. During detention, Anna lists all ten-digit positive integers that use each of the ten digits exactly once, such as 1234567890 and 9876543210. What is the greatest common divisor (GCD) of all of the numbers that Anna writes?

32. **9**

Solution: Since the sum of the digits is always $0 + 1 + 2 + \cdots + 8 + 9 = 45$, the number is always divisible by 9. Furthermore, 1023456789 and 1023456798 are two consecutive multiples of 9, so they cannot have a GCD greater than 9. Thus, the answer is 9.

33. On the n^{th} day of January 2014, your mean friend steals $\$n$ dollars from you and your average friend donates $\$(16 - n)$ to you. If you had \$100 at the start of the year, how much money will you have at the end of the fifteenth day of January?

33. \$100

Solution: On the first fifteen days of January, from your mean friend you lose $1, 2, 3, \cdots, 14, 15$ dollars. At the same time, from your average friend you gain $15, 14, 13, \cdots, 2, 1$ dollars. Thus, you don't make or lose any money, so you have \$100 in the end.

34. Ryan sets his clock to emit a *ring* every 25 minutes and a *ding* every 45 minutes, starting at midnight. Later that day, he looks over to the clock just as it goes *ring* and *ding* at the same time, for the first time on that day. What does the clock say? (Express your answer in $HH : MM$ notation, where HH represents the hour and MM represents the minutes.)

34. $3:45$

Solution: The least common multiple of 25 and 45 is 225, so after 225 minutes, both alarms will ring at the same time. Thus, at $03:45$, your clock says *ring-ding-ding-ding.*

35. A 1-watt lightbulb is connected to several transformers, each of which doubles the power output. What is the fewest number of transformers needed in order for the final power level to be over 9000 watts?

35. **14**

Solution: We want to find n for which $2^n > 9000$. Using $2^{10} = 1024$ as a checkpoint, we find that $2^{13} = 8192$ and $2^{014} = 16384$, so the answer is 14.

36. An Online Problemsolving Site (AOPS) and Acronym of Problemsolving Society (AoPS) both started in 2003, each with 1000 community members. The AOPS community grows by 10% every year, while AoPS attracts 100 new members every year. At the end of the third year, how many more members will AOPS have than AoPS?

36. **31**

Solution: At the end of the third year, AOPS will have $1000 \cdot 1.10^3 = 1331$ members while AoPS will have $1000 + 3(100) = 1300$ members. The answer is $1331 - 1300 = 31$ members.

37. How many of the fractions in the set $\left\{\frac{1}{7}, \frac{2}{8}, \frac{3}{9}, \cdots, \frac{93}{99}, \frac{94}{100}\right\}$ are fully reduced?

37. **31**

Solution: Since the numerator and denominator differ by 6, the fractions are fully reduced when the numerator and denominator are not both multiples of 2 or 3. Among the first 6 fractions, $\frac{1}{7}, \frac{2}{8}, \frac{3}{9}, \frac{4}{10}, \frac{5}{11}$, and $\frac{6}{12}$, 2 are fully reduced. This pattern repeats every 6 fractions (because we only need to consider the remainders that the numerators/denominators leave when divided by 6), and applies up to $\frac{90}{96}$. Of the last four fractions, only one is fully reduced. Since there were 15 sets of 6 fractions, the final answer is $15 \cdot 2 + 1 = 31$.

38. Compute $\sqrt{2^{16} + 2^9 + 1}$.

38. **257**

Solution: Note that $2^{16} + 2 \cdot 2^8 + 1 = (2^8 + 1)^2$.

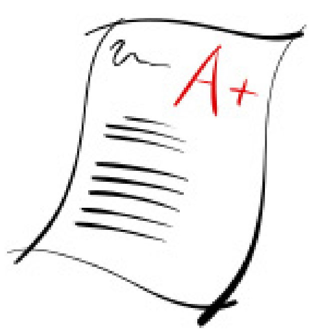
39. The problem writers for a self-centered math contest count the total number of problems that mention the contest year, and divide that number by 135, the total number of problems. What is the probability that the computed decimal has an infinite number of decimal digits? (For example, $1/135 = 0.00740740740\ldots$, which has an infinite number of decimal digits.)

39. 26/27

Solution: A fraction does not have repeating digits if and only if the denominator only contains factors of 2 and 5. Let x be the number of contests that mention the contest year (the numerator of the fraction $x/135$). Since $135 = 3^3 \cdot 5$, x must have a factor of 3^3 in order to cancel out with the 3^3 in the denominator. There are $135/27 = 5$ such numbers that satisfy this, so the answer is $(135 - 5)/135 = 26/27$.

40. Your English teacher is very slow at grading papers. On her own, she can finish grading the essays in 30 days, but if both of you grade the papers together, you two can finish in 20 days. How many days would you take you to grade the essays on your own, assuming that each of you work at a constant rate?

40. **60**



Solution: In 20 days, your teacher can finish 2/3 of the papers on her own, so you can finish 1/3 of the papers in the same amount of time. Thus, you work twice as slowly as your teacher, so you would take 60 days.