MSJ Math Club

Floors and Ceilings

13 November 2014

1 Introduction

The floor function, also known as the greatest integer function, gives the greatest integer less than or equal to its argument and is denoted by $\lfloor x \rfloor$. For example, $\lfloor 3.1 \rfloor = 3$, and $\lfloor -2 \rfloor = -2$. Similarly, the ceiling function (least integer function), gives the least integer greater than or equal to its argument, and is denoted by $\lceil x \rceil$.

There's also something called the fractional part, or sawtooth, function that comes up fairly often when dealing with floors or ceilings. It's just what it sounds like, the leftover stuff after the decimal part. More formally:

$$\{x\} = x - \lfloor x \rfloor ,$$

where $\{x\}$ is the fractional part of x.

2 Tips and Tricks

- A common thing to do with floor functions is split up x into integer and fractional parts, which can simplify a lot of problems. (See the second example problem.)
- Since floors take discrete values, it often helps to consider the points where the function jumps to another integer.
- Hermite's identity: for every real number x and positive integer n,

$$\sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor nx \rfloor$$

3 Examples

1. Find all positive integers n such that $\left\lfloor \sqrt[n]{111} \right\rfloor$ divides 111.
2. (2014 AMC 12A) For every real number x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, and let

$$f(x) = \lfloor x \rfloor (2014^{x - \lfloor x \rfloor} - 1).$$

   The set of all numbers x such that $1 \le x \le 2014$ and $f(x) \le 1$ is a union of disjoint intervals. What is the sum of the lengths of those intervals?
3. (1968 IMO) Evaluate the sum

$$\sum_{k=0}^{\infty} \left\lfloor \frac{n + 2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \cdots + \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor + \cdots$$

4 Practice

1. (Classic) Find the largest integer value of p such that 3^p divides 2014!.
2. Find all values of x for which
$$\lfloor x \lfloor x \rfloor \rfloor = 1$$
.
3. (HMMT 2007) Compute the value of $\left\lfloor \frac{2007!+2004!}{2006!+2005!} \right\rfloor$.
4. (2014 OMO) We select a real number α uniformly and at random from the interval (0, 500). Define
$$S = \frac{1}{a} \sum_{m=1}^{1000} \sum_{n=m}^{1000} \left\lfloor \frac{m+\alpha}{n} \right\rfloor .$$
Let p denote the probability that $S \geq 1200$. Compute $1000p$.
5. (AIME 1985) How many of the first 1000 positive integers can be expressed in the form $\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor$, where x is a real number?
6. (AIME 2002)Find the least positive integer k for which the equation $\left\lfloor \frac{2002}{n} \right\rfloor = k$ has no integer solutions for n.
7. (Classic)Evaluate the sum $\{\frac{p}{q}\} + \{\frac{2p}{q}\} + \cdots + \{\frac{(q-1)p}{q}\}$.
8. (2011 AMC 12B)For every m and k integers with k odd, denote by $\left[\frac{m}{k}\right]$ the integer closest to $\frac{m}{k}$. For every odd integer k, let $P(k)$ be the probability that
$$\left[\frac{n}{k}\right] + \left[\frac{100-n}{k}\right] = \left[\frac{100}{k}\right]$$
for an integer n randomly chosen from the interval $1 \leq n \leq 99!$. What is the minimum possible value of $P(k)$ over the odd integers k in the interval $1 \leq k \leq 99$?
9. (2012 AMC 12A)Let $f(x) = |2\{x\} - 1|$ where $\{x\}$ denotes the fractional part of x. The number n is the smallest positive integer such that the equation $nf(xf(x)) = x$ has at least 2012 real solutions. What is n?