MSJ Math Club

AMC Review

11 February 2016

1 Warm Up

Given that x and y are real numbers randomly and independently chosen in the interval [0, 1]. What is the probability that one of them is at least twice as large as the other?

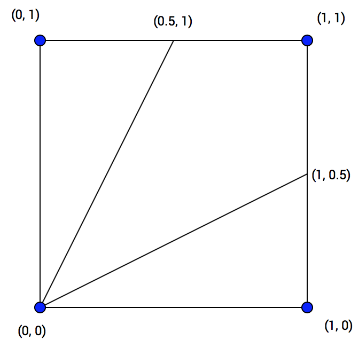

Figure 1: Let the x-axis denote x and let the y-axis denote y.

Then we see that the probability that one is at least twice the other is exactly the area to the left of the $y = 2x$ line plus the area under the $y = \frac{1}{2}x$ line. Therefore our answer must be $2 \cdot \frac{1 \cdot \frac{1}{2}}{2}$ all over the entire probability space, which is just 1. Thus our answer is $\boxed{\frac{1}{2}}$.

Another way you could do this problem would be to assume, without loss of generality, that x were the larger one. Then the probability that y is at most half of x is exactly one-half, so we get the same answer either way.

2 #23 on the AMC 12A

Three numbers in the interval [0, 1] are chosen independently and at random. What is the probability that the chosen numbers are the side lengths of a triangle with positive area?

We can put this problem onto the 3-dimensional coordinate system, and we note that our entire probability space is a cube with side length one. We then notice that the probabilities that each variable is greater than the sum of the other two are each $\frac{1}{6}$, which means our final answer must be $1 - 3 \cdot \frac{1}{6} = \boxed{\frac{1}{2}}$.

Alternatively, we could assume, without loss of generality, that x were the largest number. Then the probability that $y + z$ is at least x is precisely $\frac{1}{2}$, which gives us our answer in another way.

3 Tips and Tricks

- When you are faced with a problem that you have never seen before, try to reduce the problem into a problem that you **have** seen before.
- Don't get flustered on the AMC! Just pretend you have no time limit, and focus on the problem.
- Don't get too absorbed by what your "current score" is. Just try to solve as many problem as you can.
- Keep in mind that the later problems are not necessarily harder for you!

4 Practice Problems

1. (AMC 10A #18 and AMC 12A #14) Each vertex of a cube is to be labeled with an integer 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?
2. (AMC 10A #20) For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c, and d, each to some positive power. What is N?
3. (AMC 10A #23 and AMC 12A #20) A binary operation $\diamondsuit$ has the properties that $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ and that $a \diamondsuit a = 1$ for all nonzero real numbers a, b and c. (Here the dot $\cdot$ represents the usual multiplication operation.) The solution to the equation $2016 \diamondsuit (6 \diamondsuit x) = 100$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p+q$?
4. (AMC 10A #25 and AMC 12A #22) How many ordered triples (x, y, z) of positive integers satisfy $\operatorname{lcm}(x, y) = 72, \operatorname{lcm}(x, z) = 600$ and $\operatorname{lcm}(y, z) = 900$?
5. (HMMT) Four unit circles are each centered at different corners of a unit square. What is the area enclosed by every circle?
6. (AMC 12A #24) There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial $a^3 - ax^2 + bx - a$ are real. In fact, for this value of a, the value of b is unique. What is this value of b?
7. (2013 AMC 10A #25 and AMC 12A #23) Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S. For example, if $N = 749$, Bernardo writes the numbers $10,444$ and $3,245$, and LeRoy obtains the sum $S = 13,689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?
8. (AMC 12A #25) Let k be a positive integer. Bernardo and Silvia take turns writing and erasing numbers on a blackboard as follows: Bernardo starts by writing the smallest perfect square with $k+1$ digits. Every time Bernardo writes a number, Silvia erases the last k digits of it. Bernardo then writes the next perfect square, Silvia erases the last k digits of it, and this process continues until the last two numbers that remain on the board differ by at least 2. Let $f(k)$ be the smallest positive integer not written on the board. For example, if $k = 1$, then the numbers that Bernardo writes are $16, 25, 36, 49, 64$, and the numbers showing on the board after Silvia erases are $1, 2, 3, 4$, and 6, and thus $f(1) = 5$. What is the sum of the digits of $f(2) + f(4) + f(6) + ... + f(2016)$?
9. **Puzzle of the Week:** "Boss, I have a complaint," said Max. "Jones and I were hired at the same time, handled the same number of assignments, and each required a yes-or-no decision. I've been correct 70% of the time, but Jones has only been correct 10% of the time. How come you have given Jones a promotion and a raise, while turning me down?" (Puzzles in Math and Logic)