MSJ Math Club

Circle Tricks

22 September 2015

1 Topic Guide

1.1 Angle Chasing on Circles

Def: The *measure* of an arc AB is equal to the angle AOB, where O is the center of the circle.

A: If A, B, C, D are all on the circle O, then $\sin \angle ABC = \sin \angle DBC = \sin(\frac{1}{2}\angle BOC)$. The converse of this is also true, which is very helpful in Olympiads.

B: If AB is a chord in a circle, and l is a line tangent to that circle at A, then the angle between AB and l is equal to half the measure of the arc AB.

Most other angle-chasing tools can be easily derived from these two.

1.2 Circle Theorems

(Common) If two circles are tangent to one another, then the centers of those circles and the point of tangency are collinear.

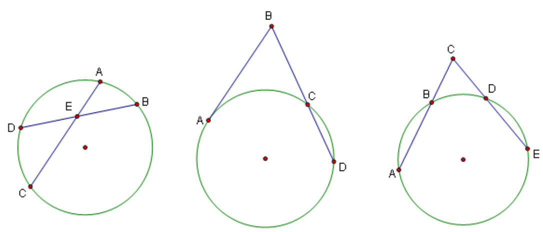
(Ptolemy) Given a cyclic quadrilateral $ABCD$ with side lengths a, b, c, d and diagonals e, f: $ac + bd = ef$

(Pascal) If a hexagon is inscribed in a conic section, then the points of intersection of the pairs of its opposite sides are collinear.

(Brianchon) The diagonals of a hexagon circumscribed about a conic concur at a single point.

(Power of a Point) There are three possibilities as displayed in the figures below.

The two lines are secants of the circle and intersect inside the circle (figure on the left). In this case, we have $AE \cdot CE = BE \cdot DE$.
One of the lines is tangent to the circle while the other is a secant (middle figure). In this case, we have $AB^2 = BC \cdot BD$.
Both lines are secants of the circle and intersect outside of it (figure on the right). In this case, we have $CB \cdot CA = CD \cdot CE$.



2 Tips and Tricks

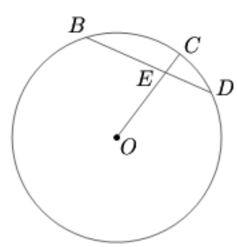
- Note that Pascal's and Brianchon's theorems have interesting degenerate cases which can be very useful.

3 Examples

1. Prove Ptolemy's theorem.
2. Circles A, B, have radii r_1 and r_2. What is the length of their common external tangent (between the points of tangency)?
3. (SMT) O is a circle with radius 1. A and B are fixed points on the circle such that $AB = \sqrt{2}$. Let C be any point on the circle, and let M and N be the midpoints of AC and BC, respectively. As C travels around circle O, find the area of the locus of points on MN.

4 Practice Problems

1. (AMC 12) Quadrilateral $ABCD$ is inscribed in a circle with $\angle BAC = 70°, \angle ADB = 40°, AD = 4$, and $BC = 6$. What is AC?
2. (AMC 12) Four circles, no two of which are congruent, have centers at A, B, C, and D, and points P and Q lie on all four circles. The radius of circle A is $\frac{5}{8}$ times the radius of circle B, and the radius of circle C is $\frac{5}{8}$ times the radius of circle D. Furthermore, $AB = CD = 39$ and $PQ = 48$. Let R be the midpoint of $\overline{PQ}$. What is $AR + BR + CR + DR$?
3. (SMT) Let ABC be a triangle where $\angle BAC = 30°$. Construct D in $\triangle ABC$ such that $\angle ABD = \angle ACD = 30°$. Let the circumcircle of $\triangle ABD$ intersect AC at X. Let the circumcircle of $\triangle ACD$ intersect AB at Y. Given that $DB - DC = 10$ and $BC = 20$, find $AX \cdot AY$.
4. (Canadian MO 1971) DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the center of the circle. Join OE and extend OE to cut the circle at C. Given $EC = 1$, find the radius of the circle.



5. (AMC 12) A collection of circles in the upper half-plane, all tangent to the x-axis, is constructed in layers as follows. Layer L_0 consists of two circles of radii 70^2 and 73^2 that are externally tangent. For $k \geq 1$, the circles in $\bigcup_{j=0}^{k-1} L_j$ are ordered according to their points of tangency with the x-axis. For every pair of consecutive circles in this order, a new circle is constructed externally tangent to each of the two circles in the pair. Layer L_k consists of the 2^{k-1} circles constructed in this way. Let $S = \bigcup_{j=0}^{6} L_j$, and for every circle C denote by $r(C)$ its radius. What is $\sum_{C \in S} \frac{1}{\sqrt{r(C)}}$?
6. (1991 AIME) A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.