MSJ Math Club

Recursion

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1 Introduction

Recursion is the repeated application of a function, and recursion is the repeated application of a function, and recursion is the repeated application of a function, and recursion is the repeated application of a function, and...

Recursion is a powerful tool that can be used to solve a wide variety of computational and olympiad problems alike.

As a simple example, consider the sequence $\{a_n\}$ where $a_0 = 3$ and $a_n = a_{n-1} + n^2$. If we wanted to compute a closed formula for a_n, it would be a hassle to try to evaluate a_{n-1}, and a_{n-2}, and so on. An easy way to get rid of this is to plug the equation back into itself, hence, recursion. If we do so, we get

$$\begin{aligned} a_n &= a_{n-1} + n^2 = a_{n-2} + (n-1)^2 + n^2 \\ &= \cdots = a_0 + 1^2 + 2^2 + 3^2 \cdots + n^2 \\ &= 3 + \frac{(n)(n+1)(2n+1)}{6} \end{aligned}$$

Consider another applied example. Assume that A (Aurich) and B (Brian) are playing a tennis match. It is deuce. Aurich has a $\frac{3}{7}$ chance of winning any point, and Brian has a $\frac{4}{7}$ chance of winning any point. What is the probability that Aurich wins the game?

Consider the following scenarios.

Aurich wins 2 games in a row (Probability: $(\frac{3}{7})^2$)

Aurich wins one game, then loses (Probability: $(\frac{3}{7}) * (\frac{4}{7})$)

Aurich loses one game, then wins (Probability: $(\frac{4}{7}) * (\frac{3}{7})$)

Aurich gets nervous and loses 2 games in a row (Probability: $(\frac{4}{7})^2$)

If we look at the second and third scenarios, we notice that afterwards, the game is at deuce again, and so the algorithm runs exactly as it did before. Thus, letting x equal the probability that Aurich wins the game, we have

$$x = (\frac{3}{7})^2 + (\frac{3}{7})(\frac{4}{7})x + (\frac{4}{7})(\frac{3}{7})x$$

We have reduced the problem down to solving for x in a linear equation, in which we find that $x = \frac{9}{25}$, indicating that Aurich has less than a 37% probability of winning the game. Notice how we have used recursion to simplify the problem from a seemingly complicated morass of casework to a simple linear equation.

2 Tips and Tricks

- Realizing that a problem can be solved using recursion is usually half the battle. Afterwards, all you need is to clearly write out the recursion equation in order to greatly simplify the problem into something easily approachable.
- If you find out that after a few steps, the resulting problem is comparable to the original problem, then you can probably use recursion to write out a recurrence relation.
- In problems involving sequences, it may be useful to write a_n in terms of a_0, or preferably in a closed form.
- The Fibonacci numbers (usually defined as $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$) involve a recurrence relation as well, and F_n can actually be written in a closed form.

3 Examples

1. The Fibonacci numbers are defined as $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Find a closed form for F_n.
2. (2014 AMC 10B) In a small pond there are eleven lily pads in a row labeled 0 to 10. A frog is sitting on pad 1. When the frog is on pad $0 < N < 10$, it will jump to pad $N - 1$ with probability $\frac{N}{10}$ and to pad $N + 1$ with probability $1 - \frac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape being eaten by the snake?

4 Practice Problems

1. (2001 AIME II) Given that $x_1 = 211$, $x_2 = 375$, $x_3 = 420$, $x_4 = 523$, and $x_n = x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4}$ when $n \geq 5$, find the value of $x_{531} + x_{752} + x_{975}$.
2. (2001 AIME II) A certain function f has the properties that $f(3x) = 3f(x)$ for all positive real values of x, and that $f(x) = 1 - |x - 2|$ for $1 \leq x \leq 3$. Find the smallest x for which $f(x) = 2001$.
3. (2003 AIME II) Define a *good word* as a sequence of letters that consisits only of the letters A, B, and C - some of these letters may not appear in the sequence - and in which A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many seven-letter *good words* are there?
4. (2014 SMT) $a_0, a_1, a_2, \cdots$ is a sequence of positive integers where $a_n = n!$ for all $n \leq 3$. Moreover, for all $n \geq 4$, a_n is the smallest positive integer such that $\frac{a_n}{a_i a_{n-i}}$ is an integer for all integers i, $0 \leq i \leq n$. Find a_{2014}.
5. (2004 AIME I) An integer is called *snakelike* if its decimal representation $a_1 a_2 a_3 \cdots a_k$ satisfies $a_i < a_{i+1}$ if i is odd and $a_i > a_{i+1}$ if i is even. How many *snakelike* integers between 1000 and 9999 have four distinct digits?
6. (2014 BMT) Albert and Kevin are playing a game. Kevin has a 10% chance of winning any given round in the match. If Kevin wins the first game, he wins the match. If not, he requests that the match be extended to a best of 3. If he wins the best of 3, he wins the match. If not, then he requests the match be extended to a best of 5, and so forth. What is the probability that Kevin eventually wins the match? (A best of $2n + 1$ match consists of a series of rounds. The first person to reach $n + 1$ winning games wins the match.)