Solutions to the 2015 MMT: Individual Round Day 2.

Problem 1: Let O be the midpoint of $\overline{AC}$ in rectangle $ABCD$. Find the probability that a randomly chosen point inside rectangle $ABCD$ is closer to point O than any vertex of the rectangle.

Solution: Draw the rectangle with dimensions one-half of the larger rectangle. (Split up the larger one into 4 smaller ones) Then clearly the region closer to the center of the perpendicular bisector of the diagonal of these smaller rectangles is symmetric with the region closer to the outside vertices. Therefore the answer is $\boxed{\frac{1}{2}}$.

Problem 2: Find the value of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \cdots$.

Solution: We have $\dfrac{1}{n^2-1} = \dfrac{1}{2} * (\dfrac{1}{n-1} - \dfrac{1}{n+1})$. Therefore our sum telescopes into

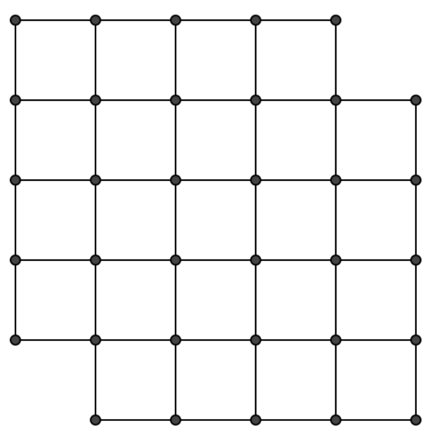
$$\frac{1}{2} * (\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots) = \frac{1}{2} * (\frac{1}{1} + \frac{1}{2}) = \boxed{\frac{3}{4}}$$

.

Problem 3: The function $f(x)$ outputs the truncation of the result when x is divided by 5. For example, $f(4) = 0$ and $f(12) = 2$ and $f(20) = 4$. The function $g(x)$ is defined to be f applied enough times until the result is less than 5. For example, $g(2) = 0$ and $g(33) = g(6) = g(1) = 1$ and $g(64) = g(12) = g(2) = 2$. For how many positive integers x such that $1 \le x \le 300$ is it true that $g(x) = 2$?

Solution: Note that $g(x)$ gives the first digit of x when written in base 5. Since there are an equal number of 1's, 2's, 3's, and 4's, the number of values of x such that $1 \le x \le 125 - 1$ is $\dfrac{124}{4} = 31$. Then, we have to count 250 to 300, inclusive, for a total of $\boxed{82}$.

Problem 4: How many rectangles are in the figure below? (Remember that squares are rectangles.)



Solution: There are a total of $\binom{6}{2}^2$ rectangles in the full figure, since we choose 2 out of the 6 vertices at the top and 2 out of the 6 vertices at the sides to intersect, forming any rectangle. Then we have to subtract 25 for each corner, and add back 1 for a total of $225 - 25 - 25 + 1 = \boxed{176}$.

Problem 5: Given that x is a real number that satisfies $x^3 + 4x = 8$, find the value of $x^7 + 64x^2$.

Solution: We have

$$x^7 + 4x^5 - 8x^4 = 0$$
$$-4x^5 - 16x^3 + 32x^2 = 0$$
$$8x^4 + 32x^2 - 64x = 0$$
$$16x^3 + 64x - 128 = 0$$

Adding all these equations gives $x^7 + 64x^2 = \boxed{128}$.

Problem 6: Circles ω_1, ω_2, and ω_3 are mutually externally tangent, and their radii are 3, 4, and 5, respectively. Let their points of tangency be A, B, and C. Find the area of $\triangle ABC$.

Solution: By Heron's the area of the outer 7-8-9 triangle is $12\sqrt{5}$. Now, we can find the ratios of the 3 outer triangles to the large triangle, and subtract those out:

$$12\sqrt{5} \times \left(1 - \frac{3*3}{7*8} - \frac{4*4}{7*9} - \frac{5*5}{8*9}\right) = \boxed{\frac{20\sqrt{5}}{7}}$$

.