Symmedians

Power Round

Mission Math Tournament Fall 2012

1 Instructions

These rules supersede any rules appearing elsewhere about the Power Round.

1. The Power Round is a 50-minute test that requires written solutions to each problem. Solutions will be graded based on accuracy.
2. It is recommended that diagrams are included on all solutions which involve them.
3. Teams may work together to solve the problems.
4. Solutions to different problems may be written on the same page; however, there should be a clear divider between solutions.
5. On any problem, you may use without proof any result or remark from *earlier* in the test, even if it's a problem your team has not solved. You may not cite results from later problems.

This test contains a total of 80 points. Each problem has a certain number of points assigned to it, indicated at the beginning of the problem in brackets. For a problem with multiple parts, the value of each part is indicated, and the total number of points for that problem is indicated next to the problem number.

2 Background

In this Power Round, we will be exploring some cool properties of symmedians of triangles. In order to prove some parts, you will need the definition of the **sine** of an angle.

Definition 2.1. Denoted by $\sin\theta$, sine is a function of an angle θ defined in terms of right triangles: the ratio of the side opposite of the angle to the hypotenuse. Note that values of θ are not restricted to 0 to $\pi/2$, but all angle measures.

We now introduce the notion of a symmedian.

Definition 2.2. Let ABC be a triangle, and point M be the midpoint of side BC. Define point D to be the unique point on segment BC such that $\angle CAM = \angle BAD$. The segment AD is called the A-**symmedian** of the triangle. Likewise, there is a B-symmedian and C-symmedian.

You may cite the following theorem without proof at any point in the test.

Theorem 2.1 (Ceva's Theorem)**.** *In a triangle ABC with points D, E, and F on sides BC, AC, and AB respectively, the cevians AD, BE, and CF are concurrent at a single point P if and only if*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

3 Law of Sines

1. [**4**] In triangle ABC, let D be the foot of the altitude of point A onto side BC.
   (a) [**3**] Express the length of AD in two ways to show that $\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C}$.
   (b) [**1**] Prove the **Law of Sines**: $\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C} = \dfrac{BC}{\sin A}$.

2. [**5**] In triangle ABC, let O be the circumcenter of the triangle with circumradius R. Let point A' be the reflection of point A about O. Show that $\dfrac{AC}{\sin B} = 2R$. This is called the **Extended Law of Sines**.

3. [**6**] Prove the trigonometric form of Ceva's Theorem: the cevians AD, BE, and CF are concurrent at a single point P if and only if

$$\frac{\sin \angle ABE}{\sin \angle CBE} \cdot \frac{\sin \angle BCF}{\sin \angle ACF} \cdot \frac{\sin \angle CAD}{\sin \angle BAD} = 1.$$

4 Symmedians

4. [**6**] Points E and F are on sides AC and AB respectively of triangle ABC. Show that if $BCEF$ is cyclic, then the A-symmedian of triangle ABC passes through the midpoint of EF.

5. [**6**] The A-symmedian of the triangle ABC intersects side BC at point D. Show that $BD : DC = c^2/b^2$, where b and c are the side lengths of AC and AB respectively.

6. [**5**] Show that the symmedians of a triangle concur at a point in the triangle.

7. [**6**] Tangents to the circumcircle ω of triangle ABC at points B and C intersect at point P. Show that AP coincides with the A-symmedian of triangle ABC.

8. [**7**] Points B and D are on circle ω, and point P is a point outside of ω such that PB and PD are tangent to the circle. A line through P intersects the circle again at two points A and C. Show that $AB/BC = AD/DC$.

9. [**9**] Let P be a point in triangle ABC such that $\Delta PBA \sim \Delta PAC$ and O be the circumcenter of the triangle.
   (a) [**4**] Show that $BPOC$ is cyclic.
   (b) [**5**] Show that P lies on the A-symmedian of triangle ABC.

10. [**11**] On the circle ω with center O and radius R, consider two fixed points A ad B, and a variable point C. Let ω_1 be the circle through A tangent to BC at C. Similarly, let ω_2 be the circle passing through B, which is tangent to AC at C. Let D be the second point of intersection (other than C) of ω_1 and ω_2.
    (a) [**5**] Show that the line CD passes through a fixed point.
    (b) [**6**] Show that $CD \leq R$.

11. [**15**] Given triangle ABC, define points M and N on sides AB and AC respectively such that $MN||BC$. Segments BN and CM intersect at point P. The circumcircles of triangles BMP and CNP intersect again at point Q distinct from P.
    (a) [**5**] Prove that quadrilaterals $AMQC$ and $ANQB$ are cyclic.
    (b) [**5**] Show that $\Delta ABQ \sim \Delta CPQ$ and $\Delta QNB \sim \Delta QCM$.
    (c) [**5**] Prove that AQ coincides with the A-symmedian of triangle ABC.