MSJ Math Club

Complex Numbers

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1 Introduction

A complex number $a + bi$ can be intuitively thought of as a real number a added to an imaginary part bi. A complex number can also be located and graphed in the complex plane with axes $Re(z)$ and $Im(z)$.

The magnitude of a complex number $a + bi$ is simply the distance from the origin, or $\sqrt{a^2 + b^2}$. We can determine the location of a complex number in polar coordinates using the magnitude of the number and the angle it makes with the positive real axis going counter clockwise.

$$a + bi = \sqrt{a^2 + b^2}(\cos(\theta) + i\sin(\theta))$$

Make sure you see that these two definitions uniquely define the same complex number.

Euler's formula: A complex number $a + bi$ can also be written as $Ae^{i\theta}$, where A is the magnitude and θ is the angle the complex number makes with the positive real axis.

When adding or subtracting complex numbers, simply add and/or subtract the corresponding real and imaginary parts of the number.

Multiplying and dividing complex numbers can also be done by using the distributive property. However, there is a simple way to multiply two numbers written in Euler's form: *multiply* the magnitudes and *add* the angles.

Proof: Consider the following two complex numbers

$$A_1e^{i\theta_1} = A_1(\cos(\theta_1) + i\sin(\theta_1))$$

$$A_2e^{i\theta_2} = A_2(\cos(\theta_2) + i\sin(\theta_2))$$

Then multiplying them together yields:

$$(A_1e^{i\theta_1})(A_2e^{i\theta_2}) = A_1(\cos(\theta_1) + i\sin(\theta_1))A_2(\cos(\theta_2) + i\sin(\theta_2))$$

$$= A_1A_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2)$$

$$= A_1A_2(\cos\theta_1 + \theta_2 + i\sin\theta_1 + \theta_2)$$

$$= A_1A_2e^{i(\theta_1+\theta_2)}$$

The nth roots of unity are the n solutions to the equation $x^n = 1$. It can be shown that these n roots are of the form

$$e^{i(\frac{2\pi k}{n})} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$$

for $k = 0, 1, 2, \cdots n - 1$

2 Tips and Tricks

- The idea of complex numbers can seem really theoretical at first, but once you get used to it, they can begin to feel really natural to use.
- Complex numbers don't show up on the AMC 10, but they definitely find their way into the AMC 12, AIME, and have applications in the olympiads.

3 Examples

1. (Classic) Express $\sin 5\theta$ and $\cos 5\theta$ in terms of single trigonometric functions.
2. (2014 AIME I) Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = arg(\frac{w-z}{z})$. Find the maximum possible value of $\tan^2\theta$.
3. (Classic) Evaluate
$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n}$$

4 Practice Problems

1. (1990 AIME) The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C?
2. (2008 AIME II) A particle is located on the coordinate plane at $(5, 0)$. Define a move for the particle as a counterclockwise rotation of $\pi/4$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p, q), find the greatest integer less than or equal to $|p| + |q|$.
3. (2005 AIME II) For how many positive integers n less than or equal to 1000 is $(\sin t + i\cos t)^n = \sin nt + i\cos nt$ true for all real t?
4. (2008 AMC 12A) A sequence $(a_1, b_1), (a_2, b_2), (a_3, b_3), \cdots$ of points in the coordinate plane satisfies
$$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n).$$
Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?
5. (2005 AMC 12B) A sequence of complex numbers $z_0, z_1, z_2, \cdots$ is defined by the rule
$$z_{n+1} = \frac{iz_n}{\overline{z_n}},$$
where $\overline{z_n}$ is the complex conjugate of z_n and $i^2 = -1$. Suppose that $|z_0| = 1$ and $z_{2005} = 1$. How many possible values are there for z_0?
6. (2000 AIME II) Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.