Week 5: Sequences and Series

MSJ Math Club

October 25, 2012

1 Important Stuff

- An **arithmetic sequence** is a sequence of numbers $\{a_n\}$ such that for all i, the difference between a_{i+1} and a_i is constant. The sum of such a sequence is

$$an + \left[\frac{n(n-1)}{2}\right] d$$

where a is the initial term, n is the number of terms, and d is the common difference.

- A **geometric sequence** is a sequence of numbers $\{a_n\}$ such that for all i, the ratio a_{i+1}/a_i is constant. The sum of such a sequence is

$$a\left(\frac{1-r^n}{1-r}\right)$$

where a is the initial term, n is the number of terms, and r is the common ratio. If we let n go to infinity, we find that the sum of the series is

$$\frac{a}{1-r}$$

which is only valid for $|r| < 1$ (why?).

- A **telescoping sequence** is a sequence $\{a_n\}$ such that for another sequence $\{b_n\}$, a_n can be written in the form $b_{n+1} - b_n$. So the series would be:

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_{k+1} - b_k = b_{n+1} - b_1.$$

The hard part is usually finding $\{b_n\}$; try partial fraction decomposition if the expression to be summed is a rational function.

- Remember that most series problems at the AMC/AIME level involve arithmetic or geometric series. But if the series in question doesn't seem to fit one of these molds, try starting from scratch when approaching the problem.

2 Examples

- *(AIME)* Let S be the sum of all numbers of the form a/b, where a and b are relatively prime positive divisors of 1000. What is the greatest integer that does not exceed $S/10$?
- *(AIME)* For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \leq 300$.

3 Practice Problems

1. *(SMT)* Evaluate
$$\sum_{k=1}^{\infty} \left\lfloor \frac{k}{60} \right\rfloor.$$

2. Evaluate
$$\sum_{k=1}^{\infty} \frac{k}{5^k}.$$

3. *(HMMT)* Let $f(r) = \sum_{j=2}^{2008} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

4. *(HMMT)* Compute
$$\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}.$$

5. *(Mock ARML 2)* Given that $\sum_{i=0}^{n} a_i a_{n-i} = 1$ and $a_n > 0$ for all non-negative integers n, evaluate
$$\sum_{j=0}^{\infty} \frac{a_j}{2^j}.$$

6. *(SMT)* Let $\delta(n)$ be the number of 1s in the binary expansion of n (e.g. $\delta(1) = 1$, $\delta(2) = 1$, $\delta(3) = 2$, $\delta(4) = 1$). Evaluate:
$$10 \left(\frac{\sum_{n=1}^{\infty} \frac{\delta(n)}{n^2}}{\sum_{n=0}^{\infty} \frac{(-1)^{n-1}\delta(n)}{n^2}} \right).$$