MSJ Math Club

Hat Problems

8 January 2015

1 Introduction

Hat problems are a combination of logic, mathematics, and reasoning about the actions of other people. Here, I'll present several puzzles that have surprising applications to game theory and coding theory. They're also pretty fun to think about!

Here's a term that will be used in the problems:

- A strategy is *guaranteed to save* n people if a position playing by the strategy always saves at least n prisoners, no matter what the configuration.

2 Problems

1. TC and Austin are playing a game in which hats, either black or white, are placed on their heads. Each person can see the color of the other person's hat, but not the color of his own. TC and Austin will simultaneously make a guess as to the color of his own hat, and they win if at least one of them guesses correctly. What strategy can TC and Austin agree upon to guarantee that they win?

2. N prisoners are in a circle, each wearing either a black or a white hat. The prisoners can see everyone else's hat colors, but not their own. Starting from some initial prisoner, each prisoner will be asked to guess the color of their own hat. If they're correct, they're freed; otherwise, they're executed. Each prisoner can hear the guesses of those before him. The prisoners can meet beforehand to discuss a strategy. Find a strategy that maximizes the number of people guaranteed to live.

   (a) Solve the problem for $N = 2$.

   (b) Solve the problem for $N = 3$.

   (c) Solve the problem for an arbitrary value of N. What is the maximum number of people that can be saved in terms of N?

   (d) Solve the problem for n prisoners and k colors of hats.

3. Consider the scenario in problem #2, except that each prisoner can't hear the guesses before him.

   (a) Show that if there are ten people, there exists a strategy that is guaranteed to save at least 5 of the prisoners.

   (b) Show that if there are n people, there exists a strategy that is guaranteed to save at least $\left\lfloor \frac{n}{2} \right\rfloor$ prisoners.

   (c) Solve the problem for 100 people and 100 colors.

   (d) (Challenge!) Solve the problem for n people and k colors.

4. Consider the scenario in problem #2, except that each prisoner can't hear the guesses before him, and they must all follow the same strategy.

(a) Show that for $N = 2$, no strategy is guaranteed to save any prisoners.

(b) Show that for $N = 3$, there exists a strategy that is guaranteed to save 1 prisoner.

(c) Show that for $N > 2$, there exists a strategy that saves one prisoner.

(d) Show that for $N = 6$, there exists a strategy that is guaranteed to save 2 prisoners.

(e) Show that if a strategy exists to save k out of n prisoners, then a strategy exists to save mk out of mn prisoners for a positive integer m.

(f) (Challenge!) Show that no strategy is guaranteed to save at least half the prisoners. (Look carefully: this does not contradict #3)

5. Consider the scenario in problem #2, except that each prisoner can't hear the guesses before him, and if one of them guesses incorrectly, they all die. However, you are now allowed to not make a guess, and the prisoners now only need at least one correct guess to win. (Note: If you're interested in this problem, look up Hamming Codes, which are used in data transmission in error detection.)

   (a) Show that for $N = 1$ or $N = 2$, the optimal strategy has a $\frac{1}{2}$ chance of winning.

   (b) Show that for $N = 3$, the optimal strategy has a $\frac{3}{4}$ chance of winning.

   (c) (Challenge!) Show that for $N = 2^k - 1$, the optimal strategy has a $\frac{n}{n+1}$ chance of winning.