MSJ Math Club

Game Theory

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1 Introduction

Let's say a group of children are playing in the rain, and all we know is that some of the children fell down and got their foreheads muddy. Since the mud is on their foreheads, nobody knows whether his or her own head is muddy. Now when they re-enter the classroom, the teacher says: "In the next 5 seconds, tell me how many of you have muddy foreheads." Assuming the children are perfect mathematicians, what would happen in this case? What would the teacher have to say in order to get a correct response?

2 The Basics

Theorem: Problems that involve strategies can usually be solved through a combination of simple logic and casework.

Definition: In strategy games, a winning position is a position that can lead to a losing position for your opponent. A losing position is a position that always leads to a winning position for your opponent.

Consider a reformulation of a classic problem:

Two people are playing a game with n stones, where $1 \leq n \leq 1000$. On a turn, a person may take 1, 4, or 6 stones from the pile of stones. The person who takes the last stone wins. For how many n does the first person have the winning strategy?

Solution We can make a table to keep track of winning and losing position, with W marking a win position and L marking a losing position.

1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
W	L	W	W	L	W	L	W	W	L	W	L	$\cdots$

We see that the same pattern, $WLWWL$ repeats over and over again. Thus in every 5 possible starting positions, there are 3 winning positions. Therefore our answer should be $\frac{3}{5} \cdot 1000 = \boxed{600}$.

3 Tips and Tricks

- We *highly* recommend not memorizing the above formula. Read through how we derived the formula, and try to understand *why* it works.
- Almost every combinatorial game can be analyzed with the same strategy as the game above. Just start with small, special cases and work your way up. Finding symmetry is also useful.

4 Examples

1. (Classic) Two players take turns placing coins of radius r on a circular table with radius R. Coins may not overlap. The last player to be able to carry out a legal move wins. Which player has a winning strategy?

2. Suppose there are 5 pirates and 1000 gold coins in a chest. The pirates line up in order, and starting from the first pirate, he proposes a way to split up the gold coins. If and only if a majority of the pirates agree with his scheme, the pirates split up according to his plan. Otherwise, the pirate is forced to walk the plank. Then, control goes over to the next pirate in line. We keep doing this until an agreement is reached by a majority of the pirates remaining. What should the fifth pirate propose? In order, the pirates have these priorities: living, getting the most gold coins, and finally killing off as many other pirates as possible.

5 Practice Problems

1. In a two-player game, each person takes away either 1, 3, or 8 stones away from a pile. The person who takes the last stone wins. For how many starting values of n such that $1 \leq n \leq 1000$ does the first person have the winning strategy?

2. What happens when we extend the game above into a three player game? How can we set up the game such that there still is a winning strategy? What is the winning strategy?

3. (Nim) We have n piles of pebbles (not necessarily of the same size). Every turn, each player can remove any number of pebbles from any one pile (but only one pile). The player to remove the last pebble wins. Find which pile combinations lead to winning games for the first player.

4. (AOPS forums) There are numbers 1 to 100 on a black board. On each turn, players must take turns removing one number from the black board until there is two left. Assuming both players are perfectly rational, and that the first player wants to maximize the difference of the two numbers while the second player wants to minimize that difference, what will be the final difference of the two numbers?

5. (BMT 2015) Two players play a game with a pile with N coins is on a table. On a players turn, if there are n coins, the player can take at most $n/2+1$ coins, and must take at least one coin. The player who grabs the last coin wins. For how many values of N between 1 and 100 (inclusive) does the first player have a winning strategy?

6. (USAMO 2014) Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

7. (RMM 2015) For an integer $n \geq 5$, two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is legal if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

8. **Puzzle of the Week:** Find two unequal numbers, A and B, such that $A+n$ is a factor of $B+n$ for all values of n from 0 through 1000, inclusive.