MSJ Math Club

Polynomial Tricks II (Other Solution Methods)

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1 Introduction

1.1 Mirrored Polynomials

Consider the polynomial $P(x) = x^4 - 2x^3 + x^2 - 2x + 1 = 0$. Notice how the coefficients are symmetrical. Since $x = 0$ is not a solution, we can divide by x^2 to get $\frac{P(x)}{x^2} = x^2 - 2x + 1 - \frac{2}{x} + \frac{1}{x^2} = 0$.

Now let $y = x + \frac{1}{x}$. Then $y^2 = x^2 + 2 + \frac{1}{x^2}$. And so we can write our polynomial

$$\frac{P(x)}{x^2} = x^2 - 2x + 1 - \frac{2}{x} + \frac{1}{x^2} = (x^2 + 2 + \frac{1}{x^2}) - 2(x + \frac{1}{x}) - 1 = y^2 - 2y - 1$$

Then we simply have to solve $y^2 - 2y - 1 = 0$ to get the two roots

$$y_1 = \frac{2 + \sqrt{8}}{2} = 1 + \sqrt{2} \qquad y_2 = \frac{2 - \sqrt{8}}{2} = 1 - \sqrt{2}$$

Now, we simply use the fact that we set $y = x + \frac{1}{x}$, so we now have two quadratics

$$x + \frac{1}{x} = 1 + \sqrt{2} \qquad x + \frac{1}{x} = 1 - \sqrt{2}$$

which we all know how to solve, to get at most 4 real solutions. This easily generalizes to all polynomials that have symmetric coefficients. The important thing to note is that setting $y = x + \frac{1}{x}$ gives us nice results.

1.2 Something More

Another nice thing above symmetric polynomials is that if x is a root, then $\frac{1}{x}$ must also be a root. For example, in our previous case, if a were a root of $P(x) = x^4 - 2x^3 + x^2 - 2x + 1$, then $a^4 - 2a^3 + a^2 - 2a + 1 = 0$. However, if we plug in $x = \frac{1}{a}$, we get

$$P\Big(\frac{1}{a}\Big) = \frac{1}{a^4} - \frac{2}{a^3} + \frac{1}{a^2} - \frac{2}{a} + 1 = \frac{1}{a^4}\Big(1 - 2a + a^2 - 2a^3 + a^4\Big) = \frac{1}{a^4}\Big(a^4 - 2a^3 + a^2 - 2a + 1\Big) = 0$$

1.3 Factoring

Never dismiss factoring as a way of solving polynomials! Sometimes an impossible-looking polynomial has a very simple solution if you realize an elegant way to factorize it. More about this in the examples.

2 Tips and Tricks

- As mentioned before, factoring a polynomial can be an extremely powerful way to simply a problem.
- Sometimes, graphing can also help with solving polynomials. Consider Example #2.
- If you ever see that two polynomials are equal (i.e. $ax^2 + bx + c = dx^2 + ex + f$), then it *must* be the case that $a = d$, $b = e$, and $c = f$.
- Never go into a problem set on using a specific method. Keep yourself flexible: write down all possible methods, and spend a little bit of time trying each one to see if it works.

3 Examples

1. (SMT 2013) Compute the largest root of $x^4 - x^3 - 5x^2 + 2x + 6$.
2. (AIME 2010) Let $P(x)$ be a quadratic polynomial with real coefficients satisfying $x^2 - 2x + 2 \leq P(x) \leq 2x^2 - 4x + 3$ for all real numbers x, and it is given that $P(11) = 181$. Find $P(16)$.
3. The function $f(x)$ is a monic quartic polynomial satisfying $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, $f(4) = 16$. What is $f(5)$?
4. There exists a monic cubic function $f(x)$ such that $f(1) = 1$, $f(2) = \frac{1}{2}$, and $f(3) = \frac{1}{4}$. Find $f(x)$.

4 Practice Problems

1. (HMMT 2010) Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(1) = 1$. Compute the value of $P(3)$.
2. (ASMT 2015) Find all pairs (x, y) that satisfy the following equations:

$$x^2 + y^2 = 1 \qquad x + 2y = 2$$

3. Let $f(x)$ be a monic cubic polynomial such that $f(0) = 1$, $f(1) = 2$, $f(2) = 3$. Find the value of $f(3)$.
4. (RMT 2012) The quartic (4th degree) polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.
5. (ASMT 2015) Find the sum of all real roots of $x^5 + 4x^4 + x^3 - x^2 - 4x - 1$.
6. (SMT 2013) Find the sum of all real x such that

$$\frac{4x^2 + 15x + 17}{x^2 + 4x + 12} = \frac{5x^2 + 16x + 18}{2x^2 + 5x + 13}$$

7. **Puzzle of the Week:** Alice and Bob are playing the following game: Alice and Bob each take turns selecting a whole number from 1 through 9, inclusive, without replacement. The first person to have a set of exactly three numbers that sum to 15 wins the game. For example, in the following scenario, Alice wins since she is the first to have three numbers that sum to 5 (4,5, an 6): Alice picks 8, Bob picks 3, Alice picks 5, Bob picks 2, Alice picks 6, Bob picks 1, and Alice picks 4.

   Does Alice have a strategy to guarantee her winning the game? Does Bob have a strategy to guarantee him winning the game?