Individual Round Solutions

Mission Math Regional

January 19, 2013

1. SpongeRobert is a rectangular prism 4 inches thick, 18 inches tall, and 12 inches wide. What is the volume of SpongeRobert in cubic feet?

   Solution: The dimensions of SpongeRobert are 1/3 foot by 3/2 foot by 1 foot, so the volume is $1/3 \times 3/2 \times 1 = \boxed{1/2 \text{ ft}^3}$.

2. Ash chooses one of his six Pokemon at random for a Pokemon battle. What is the probability that in his next battle, he will choose the same Pokemon?

   Solution: The Pokemon that Ash chooses in his first battle is independent of the choice for the second battle, so the answer is $\boxed{1/6}$.

3. This test was printed out 2 days ago on January 17, at 3:00 P.M. Right now, it is January 19, and the time is 10:40 A.M. (No really, it is.) How many minutes ago was this test printed out?

   Solution: We can break up this problem into counting the number of minutes passed in each of the three days (Jan. 17, 18, 19), and adding them up. On January 17, $9 \times 60 = 540$ minutes passed. On January 18, $24 \times 60 = 1440$ minutes passed. On January 19, $10 \times 60 + 40 = 640$ minutes passed. Thus, the final answer is $540 + 1440 + 640 = \boxed{2620 \text{ minutes}}$.

4. The number $3 \cdot \sqrt{3} \cdot \sqrt[3]{3} \cdot \sqrt[4]{3}$ can be expressed in the form 3^a for some value of value of a. Find a.

   Solution: The expression can be written as $3^1 \cdot 3^{1/2} \cdot 3^{1/3} \cdot 3^{1/4} = 3^{1+1/2+1/3+1/4} = 3^{25/12}$, so $\boxed{a = 25/12}$.

5. Evaluate the sum $1 + 2 + 4 + 5 + 7 + 8 + \cdots + 1000 + 1001$, in which multiples of 3 are excluded from the first 1001 positive integers.

   Solution: We can write this sum as:

$$
\begin{aligned}
& (1 + 2 + 3 + 4 \cdots + 999 + 1000 + 1001) - (3 + 6 + 9 + \cdots + 996 + 999) \\
= \quad & \frac{1001 \cdot 1002}{2} - 3\left(\frac{333 \cdot 334}{2}\right) \\
= \quad & 501(1001 - 333) \\
= \quad & 501(668) \\
= \quad & \boxed{334668}
\end{aligned}
$$

6. Professor Oak's 3rd grade class is trying to split into groups of equal size for a class project. However, whether they have groups of 2, 3, or 4 people, there is always exactly one person who does not have a group. If there are between 15 and 30 students in the class, how many students are in Professor Oak's class?

   Solution: If the number of people in the class was a multiple of 12, then there would not be any "leftover" people. If one person joined the class from there, then there would always be one person without a group in all cases. Thus, the number of people in the class leaves a remainder of 1 when divided by 12. The only number of people between 15 and 30 satisfying this condition is $\boxed{\text{25 people}}$.

7. How many positive integers between 1 and 2013, inclusive, share at least one prime factor with 2013?

   Solution: The prime factorization of 2013 is $3 \times 11 \times 61$. Let $M(n)$ denote the number of multiples of n from 1 to 2013. By the Principle of Inclusion-Exclusion, the desired answer is:

$$\begin{aligned} & M(3) + M(11) + M(61) - M(3 \cdot 11) - M(11 \cdot 61) - M(61 \cdot 3) + M(3 \cdot 11 \cdot 61) \\ = \; & 33 + 671 + 183 - 3 - 11 - 61 + 1 \\ = \; & \boxed{813} \end{aligned}$$

8. A row of houses is labeled from 1 to 100. Woody rings the doorbell of every third house, starting with houses $1, 4, 7, \cdots$. Meanwhile, Buzz Lightyear rings the doorbell of every fourth house starting from the back, starting with houses $100, 96, 92, \cdots$. How many houses are left undisturbed?

   Solution: Among the houses from 1 to 12, Woody bothers houses number $1, 4, 7, 10$ and Buzz bothers $4, 8, 12$. Thus, the six houses $2, 3, 5, 6, 9, 11$ are all left undisturbed.

   Notice that if house number n is undisturbed, then house number $n + 12$ is also undisturbed, because Woody only rings every third house and Buzz only rings every fourth house. Thus, among houses 1 to 96, we have a total of $6 \times 8 = 48$ undisturbed houses. In addition, by the same argument, houses number 98 and 99 are not disturbed, leaving us a total of $\boxed{50}$ angry neighbors.

9. For positive numbers a, b, c, and d, we have that $|a - b| = 1$, $|b - c| = 3$, and $|c - d| = 5$. If $a = 10$, what is the sum of all possible distinct values of d?

   Solution: Suppose that for some chosen values of b, c, and d that satisfy the equations given, we have that:

$$a - b = x = 1 \text{ or } -1$$

$$b - c = y = 3 \text{ or } -3$$
$$c - d = z = 5 \text{ or } -5$$

Then we have that $x + y + z = a - b + b - c + c - d = a - d$, so $d = 10 - (x + y + z)$. Since each of the numbers x, y, and z have two possible values, there are a total of $2 \times 2 \times 2 = 8$ different ways to get values of d. We can list them out as follows:

$$\begin{aligned}
d &= 10 - (1 + 3 + 5) \\
d &= 10 - (1 + 3 - 5) \\
d &= 10 - (1 - 3 + 5) \\
d &= 10 - (1 - 3 - 5) \\
d &= 10 - (-1 + 3 + 5) \\
d &= 10 - (-1 + 3 - 5) \\
d &= 10 - (-1 - 3 + 5) \\
d &= 10 - (-1 - 3 - 5)
\end{aligned}$$

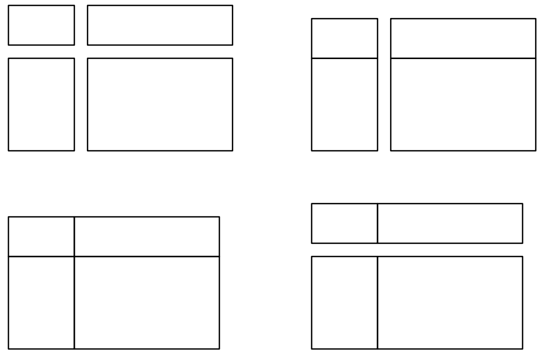
Notice that the values in the parentheses all add up to 0, so the total sum is $10 \times 8 = \boxed{80}$.

10. Evan gets a large paycheck at the beginning of each month. Of his monthly salary, half of it is spent on honey, one-sixth of it on toothpaste, and one-fourth of it on berries. He also spends \$9001 each month on books. If Evan breaks even each month (i.e. he does not gain or lose money in the long run), how large is his paycheck in dollars?

    Solution: Evan spends $\frac{1}{2} + \frac{1}{6} + \frac{1}{4} = \frac{11}{12}$ of his monthly paycheck on honey, toothpaste, and berries. He spends the remaining $\frac{1}{12}$ of his paycheck on books, which equals to \$9001. Thus, his total paycheck is $\$9001 \times 12 = \boxed{\$108,012}$.

11. One vertical line and one horizontal line is drawn in a rectangle with perimeter 1, forming four small rectangles. Find sum of the perimeters of all of the rectangles of all sizes, including the original rectangle.

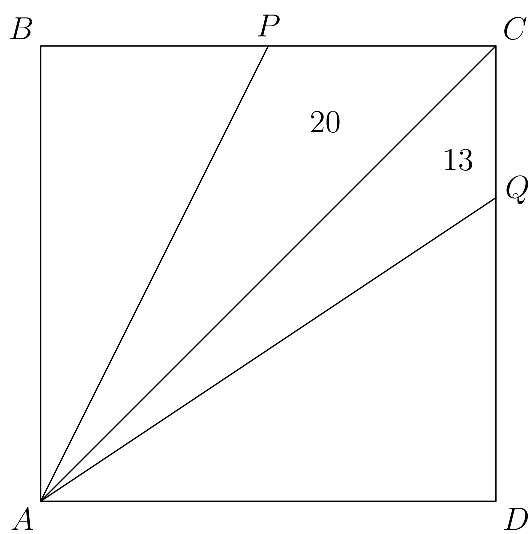
    Solution: There are a total of 9 rectangles: four small ones, four that are composed of two small rectangles, and the large rectangle formed by all four small rectangles. We consider the perimeters in each of the four scenarios shown below:



In the top left figure, the sum of the perimeters is twice of the perimeter of the big rectangle. In the top right and bottom right figures, the perimeters together yield three times of the total perimeter. Thus, the sum of the perimeters is $2 + 3 + 1 = \boxed{6}$.

12. Square $ABCD$, points P and Q are on segments BC and CD respectively. If $[APC] = 20$, $[AQC] = 13$, and $[AQD] = 2[APB]$, find area of the square. (Here, $[SHAPE]$ denotes the area of the shape.)

Solution:



Let $[ABP] = x$ and $[ADQ] = y$. From the givens, we have that $y = 2x$. Furthermore, we know that $[ABP] + [APC] = [ABC] = [ADC] = [ADQ] + [AQC]$, which gives us $y + 13 = x + 20$, or $y = x + 7$. Subtracting the two equations gives us that $x = 7$ and $y = 14$. Thus, the total area of the square is $\boxed{54}$.

13. What is the tens digit of the number 2^{2013}?

Solution: We will examine the last two digits of the number. Notice that $2^{20} = (2^{10})^2 = 1024^2 = 1048576$ leaves a remainder of 1 when divided by 25. For $a \geq 2$, 2^a is always divisible by 4, so that means that 2^{2013} leaves the same remainder as 2^{1993} upon division by 100. We can continue this thought process, until we reduce 2^{1993} to 2^{1973} and so on, until it is simplified to finding the last two digits of 2^{13}. It is easy to find that $2^{13} = 8192$, so the final answer is $\boxed{9}$.

14. A book with $4n$ pages is made by stacking multiple sheets of paper on top of each other, and folding the stack in half. (For example, a book with 12 pages requires three sheets of paper: one with the pages 1, 2, 11, 12, another with the pages 3, 4, 9, 10, and the last one with pages 5, 6, 7, 8.) Victoria notices that in her book, the sheet with pages 36 and 93 are missing. What is the page number of the last page of the book?

    Solution: On any given sheet of paper, the four page numbers in increasing order are odd, even, odd, even. Because $93 > 36$, we know that the page numbers are $35, 36, 93,$ and 94. Also the sum of the page numbers on a sheet of the book must be two more than twice the total number of pages. Because the sum of the page numbers is $35 + 36 + 93 + 94 = 258$, there are $\boxed{128 \text{ pages}}$ in the book.

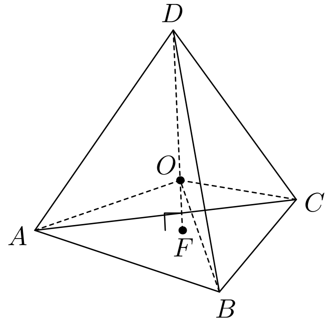
15. One morning, Romeo leaves the city of Verona at 9:00 in the morning, driving at a constant speed of 40 miles per hour. An hour later at 10:00, Juliet notices that Romeo is missing, and chases after Romeo at a constant speed of 50 miles per hour. If their parents realize that both of them are missing at 11:00 in the morning, how fast must they drive, in miles per hour, if they want to catch up to Romeo and Juliet at the same time?

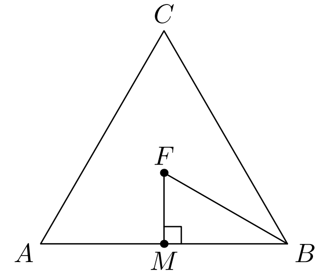
    Solution: We first calculate when Romeo and Juliet meet. The relative speed between the two of them is $50 - 40 = 10$ mph. Because Juliet is 1 hour behind, the relative distance she must drive $40 \times 1 = 40$ miles, so the two of them will meet $40/10 = 4$ hours after Juliet's departure. Their meeting point is $50 \times 4 = 200$ miles away from Verona.

    Because the parents have 3 hours to drive to the meeting spot, and 200 miles to cover, they must drive at $\boxed{200/3 \text{ mph}}$.

16. What is the volume of a regular tetrahedron inscribed in a sphere of radius 1?

    Solution:





Let A, B, C, D be the vertices of the tetrahedron, O the center of the tetrahedron, and F the foot of D on side ABC. (F is also the center of face ABC.). First, we claim that $DO : OF = 3 : 1$. To prove this, consider the volumes of the tetrahedra $OABC$, $OABD$, $OACD$, and $OBCD$. Since all of these constitute $1/4$ of the volume of the large tetrahedron, the height of $OABC$ must be $1/4$ of that of $DABC$. Since D, O, and F are collinear, $DO : OF = 3 : 1$.

Since the radius of the sphere DO has length of 1, the height h of the tetrahedron, DF, is $4/3$. To compute the area of the base, we use the Pythagorean Theorem on triangle OFB. We have $OB^2 = OF^2 + FB^2$, or $1^2 = (1/3)^2 + FB^2$, so $FB = 2\sqrt{2}/3$.

Let M be the midpoint of side AB. We consider the $30 - 60 - 90$ triangle FMB. Notice Since $FB = 2\sqrt{2}/3$, $MB = \sqrt{6}/3$, so the side length of the tetrahedron is $2\sqrt{6}/3$. Using the formula $A = \sqrt{3}s^2/4$ for an equilateral triangle with side length s, we obtain that the area of the base of the tetrahedron is $2\sqrt{3}/3$. Because the height is $4/3$, the final volume is $(1/3) \cdot (2\sqrt{3}/3) \cdot (4/3) = \boxed{8\sqrt{3}/27}$.

17. The points on the spiral are numbered in increasing order as shown below. What is the number of the point that is 10 units to the left of point 0?

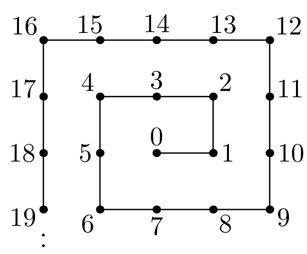


Figure 1: Problem 17

Solution: Notice that the numbers in the top left corners of the spiral are squares of even numbers. Thus, the number 10 units northwest of the point labeled "0" is $(2 \cdot 10)^2 = 400$. To obtain the number 10 units to the left of point 0, we move ten units below point 400, yielding a number of $\boxed{410}$.

18. The diagram shown below consists of four isosceles triangles with congruent side lengths of 20 around a square of side length 24. When the four triangles are folded up, the resulting object is a triangular pyramid. Find the volume of the pyramid.

Solution:

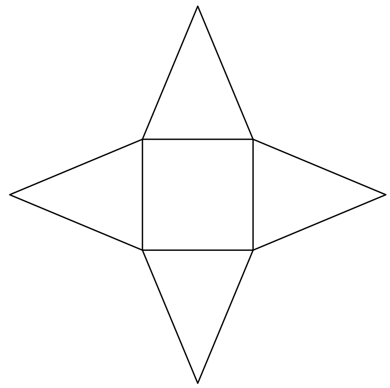
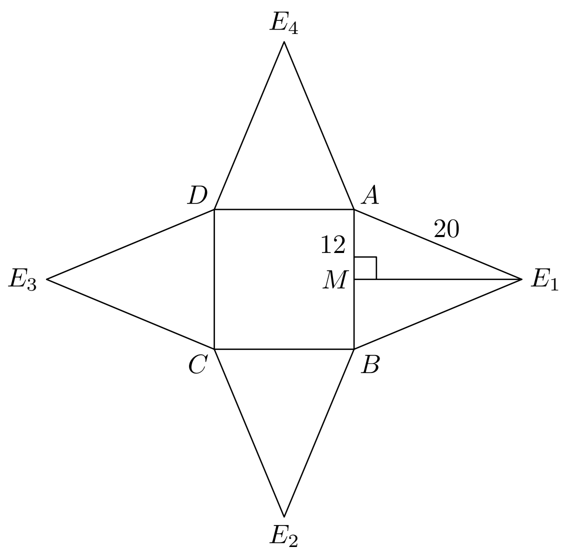
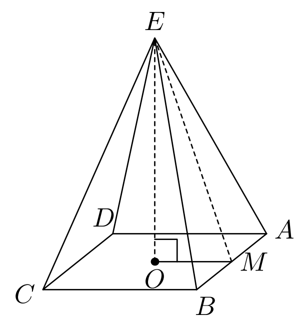


Figure 2: Problem 18





Let M be the midpoint of side AB, O be the center of the face $ABCD$, and E be the vertex of the pyramid. By the Pythagorean Theorem, $EA^2 = AM^2 + ME^2$, so $ME^2 = 20^2 - 12^2 = 16^2$. We use the Pythagorean Theorem again on triangle EOM to get $ME^2 = EO^2 + OM^2$. Because OM is half of the side length of the square, we obtain that $EO = \sqrt{ME^2 - OM^2} =$ $16^2 - 12^2 = \sqrt{112} = 4\sqrt{7}$. The volume of the pyramid is $(1/3)bh = (1/3)(AB^2)(EO) = (1/3) \times 576 \times 4\sqrt{7} = \boxed{768\sqrt{7}}$.

19. Compute the sum $0.\overline{01} + 0.\overline{03} + 0.\overline{05} + \cdots + 0.\overline{99}$.

Solution: We wish to express each of the terms as a common fraction. Suppose we have some number $X = 0.\overline{ab}$. Then $100X = ab.\overline{ab}$. Subtracting the two equations yields that $99X = \underline{a}\,\underline{b}$, so X is equal to the two digit number $\underline{a}\,\underline{b}/99$. Thus, the sum that we wish to compute is $(1+3+5+\cdots+97+99)/99$. It is well-known that the sum of the first n positive odd numbers is n^2, so our answer is $50^2/99 = \boxed{2500/99}$.

20. *Problem Removed.*