Team Round Solutions

MagMaR 2014

January 26, 2014

1. Aaron says, "I have been alive for all or part of three separate decades". To the nearest year, what is the minimum possible value of his age?

1. **10**

Solution: To minimize Aaron's age, suppose that he was born at the end of one decade, lived through the next decade, and that now is the very beginning of the third decade. Aaron is slightly older than 10 years old, so the answer is 10.

2. In the table below, the average wealth for the richest 1%, middle 19%, and bottom 80% of Americans are each shown in the table below. What is the average wealth of all Americans, in dollars?

2. **250000**

Class of population	Average wealth
Top 1%	$10,500,000
Middle 19%	$500,000
Bottom 80%	$62,500

Solution: Let P be the population of the United States. The total wealth of the top 1% is $0.01P \cdot 10,500,000 = \$105,000P$. The total wealth of the middle 19% is $0.19P \cdot 500,000 = \$95,000P$. The total wealth of the bottom 80% is $0.80P \cdot 62,500 = \$50,000P$. Thus, the total wealth is $105,000P + 95,000P + 50,000P = 250,000P$, so the average wealth per person is $\$250,000$.

Note: These statistics are approximate values based off of the actual 2010 statistics.

3. A rectangle's length and width are both prime numbers and has a perimeter of 120 units. How many different possible areas are there?

3. **6**

Solution: Let p be the rectangle's length and q be the rectangle's width. We can assume that $p > q$. We have that $2(p+q) = 120$, so $p + q = 60$. Testing the values of p for primes greater than 30 and less than 60, we find that only $p = 31, 37, 41, 43, 47, 53$ satisfy that $60 - p$ is prime as well, so there are 6 answers. (Note that $p = 59$ does not work because $q = 1$ is not prime.)

4. Three vertices of a parallelogram are $(1, 2)$, $(5, 3)$, and $(2, 7)$. If the fourth vertex lies in the first quadrant, what is the sum of its x and y coordinates?

4. **14**

Solution: A parallelogram $ABCD$ satisfies that $B - A = C - D$ and $B - C = A - D$. (Here, the difference between two points is the difference between the x and y coordinates. For example, $(5, 4) - (1, 2) = (5 - 1, 4 - 2) = (4, 2)$.) Thus, the three possible coordinates for the fourth vertex of the parallelogram are $(1+5-2, 2+3-7) = (4, -2)$, $(5+2-1, 3+7-2) = (6, 8)$, and $(2 + 1 - 5, 7 + 2 - 3) = (-2, 6)$. Of this, only $(6, 8)$ is in the first quadrant, so the answer is $6 + 8 = 14$.

5. Maggie and Carl and playing chess. For every game that Maggie wins, she takes \$12 from Carl. For every game that Carl wins or draws, Carl takes \$15 from Maggie. After a positive number of games, each of them has the same amount that they each started with. What is the minimum number of games they could have played?

5. **9**

Solution: The least common multiple of 15 and 12 is 60, so there was a minimum of \$60 transferred among Maggie and Carl. Thus, Maggie won $60/12 = 5$ times, and Carl won $60/15 = 4$ times, so there were a total of 9 games.

6. Compute $-1^2 + 2^2 - 3^2 + 4^2 - \cdots - 99^2 + 100^2$.

6. **5050**

Solution: We claim the the difference between the squares of two consecutive numbers is the sum of the two consecutive numbers. Using algebra, we can see that $(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 = n + (n + 1)$. Thus, the sum is equal to $S = (2^2 - 1^2) + (4^2 - 3^2) + \dots (100^2 - 99^2) = (1 + 2) + (3 + 4) + \cdots + (99 + 100)$. It is easy to show that this sum is equal to $100(101)/2 = 5050$.

Note: The first claim can be proven by considering an $n \times n$ square inside of an $(n+1)\times(n+1)$ square. Additionally, the sum of the arithmetic sequence can be proven by reversing the sum. For example, if $S = 1+2+\cdots+99+100$, we also have that $S = 100+99+\cdots+2+1$. Adding these two equations term-for-term gives that $2S = (100+1)+(99+2)+\cdots+(2+99)+(1+100) = 101 \cdot 100/2$.

7. A very small fish tank requires 60 square inches of wrapping paper to fully wrap and can hold 1/2 liter of water. If another fish tank with dimensions in a similar ratio requires 240 square inches of wrapping paper, how many liters of water can it hold inside?

7. **4**

Solution: Surface area has a dimension of length^2 and while volume has a dimension of length^3. Thus, the larger fish tank has side lengths that are $\sqrt{(240/60)} = 2$ times that of the original, so it has a volume of $\frac{1}{2} \cdot (2)^2 = 4$ liters of water.

8. The side of a new roll of toilet paper is a circle with a radius of 3 inches, with a hollow circular center with a radius of 1 inch. Calvin completely unrolls a new roll of toilet paper, uses half of the toilet paper, and flushes it down the toilet. If he rolls the remaining half of the toilet paper back into a cylinder, what is the radius of the used roll?

8. $\underline{\quad\sqrt{5}\quad}$

Solution: Since Calvin uses up half of the length of toilet paper, from the side view, the area of the used portion is equal to the area of the remaining portion of toilet paper. The area of a new roll of toilet paper is $\pi(3^2 - 1^1) = 8\pi$, so the area of each of the remaining portion of toilet paper must be 4π.

Hence the area of the side view of the used roll of toilet paper including the hollow cardboard center is $4\pi + \pi = 5\pi$, so the radius is $\sqrt{5}$.

9. In a five-pointed star, four of the angles are marked as shown. What is the angle of the fifth point?

9. $\underline{\quad\mathbf{57}\quad}$

Solution: We claim that the sum of the angles at the five points of the star add up to 180°.

The points have been labeled in the diagram above. Notice that $\angle JHA = \angle BHD = \angle HBC + \angle BCH = \angle B + \angle C$ and $\angle HJA = \angle CJE = \angle JED + \angle EDJ = \angle E + \angle D$. Since $\angle HJA + \angle JHA + \angle A = 180°$, we know that $\angle A + \angle B + \angle C + \angle D + \angle E = 180°$, so the missing angle has a measure of $180 - 41 - 18 - 30 - 34 = 57°$.

10. A factorial, or $n!$, is equal to $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. Define a "reverse factorial", which is denoted as $n¡$, to be $n \div ((n-1) \div ((n-2) \div \cdots 3 \div (2 \div 1)))$. For example, $5¡= 5/(4/(3/(2/1))) = \dfrac{5 \cdot 3 \cdot 1}{4 \cdot 2} = 15/8$.
What is the largest positive integer m such that $m!$ is a factor of $100! \cdot 100¡$?

10. $\underline{\quad\mathbf{52}\quad}$

Solution: We have that $100! \cdot 100$¡$= (100 \times 99 \times \cdots \times 2 \times 1) \cdot \left(\dfrac{100 \times 98 \times \cdots \times 2}{99 \times 97 \times \cdots \times 1} \right) = (100 \times 98 \times 96 \times \cdots \times 4 \times 2)^2$.

Notice that $(100 \times 98 \times 96 \times \cdots \times 4 \times 2)^2 = \left(2^{50} \cdot 50!\right)^2$
$= 50! \cdot 2^{100} \cdot (50 \times 49 \times \cdots \times 17 \times \cdots \times 13 \times 12 \times \cdots \times 2 \times 1)$. Since $17 \times 13 \times 12 = 51 \times 52$, we know that $52!$ divides the entire expression. However, because 53 is prime, there are no factors of 53 in the expression, so the answer is 52.

11. Carolyn and her younger brother are splitting a 4×4 square cake, as shown below. She tricks him into thinking that it is fair if and only if the perimeters of their pieces are equal. If she must cut along the gridlines, what is the maximum area of cake that she can take?

11. **10**

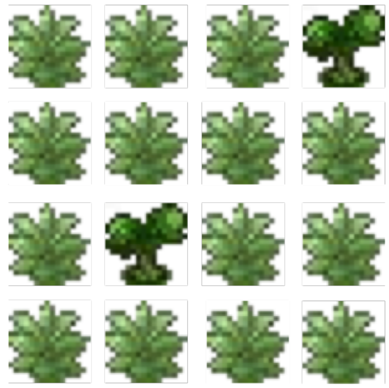
Solution: Carolyn can take up to 10 square units of cake by cutting along the bolded line:

To show that Carolyn cannot take more than 10 square units of cake, we first claim that if the cake had a "crust" along its edge, then two contiguous (connected) pieces of cake have equal perimeter if and only if they contain the same perimeter of crust. Each unit of a cut through the cake leaves adds 1 unit of perimeter to the pieces on either side of the cut. Because these two sides must have different pieces (or else the cut would not add perimeter to either piece), the intricacy of the cut does not change the difference between the perimeters of the two slices. Thus, both slices must contain half of the crust. Since the cakes must be contiguous, the crust on each piece must be connected, and with this lemma, it is easy to show that Carolyn cannot take 11 units of cake.

Note: If the pieces of cake did not have to be contiguous, then Carolyn can take all of the cake excluding the four corners.

12. A Youngster, Lass, Bug Catcher, and School Kid each want to stand in the 4×4 grid of grass patches shown below, but no two want to stand in the same row or same column as another person. If no one may stand on the two squares occupied by trees, how many different arrangements are possible?

12. **336**



Solution: We first consider the number of ways to choose four squares that can be occupied by the four individuals by doing casework on the square that is occupied in the first row.

If the patch of grass in the first column, then after removing all squares in the occupied column and row, we are left with a 3×3 square with a tree-square. To fill this 3×3 square, we note that there are 2 ways to place an individual in the same row as the tree, and 2 ways to choose the remaining two squares, so there are 4 ways for this first case.

In the second case, if the square in the second column and first row is occupied, then removing all of the squares in the first row and second column results in a 3×3 square with no restrictions. There are 3 ways to choose a square in the remaining first row and 2 ways to choose the remaining two squares, so there are 6 outcomes in this case.

If the square in the third column and first row is occupied, we obtain a result similar to that of the first case.

Thus, there are $4+6+4=14$ ways to choose the four occupied squares. After this, there are $4! = 4 \times 3 \times 2 \times 1$ to rearrange the four individuals among these chosen squares, so there are a total of $14 \cdot 24 = 336$ outcomes.

13. Peter is buying a car that costs \$100,000, and uses two 20% off coupons and two 25% off coupons (because they expire today). If Peter uses all four coupons in an order that minimizes the final cost, how much does he pay?

13. 36000

Solution: When Peter uses a 20% off coupon, he is multiplying his final price by 0.8. When Peter uses a 25% off coupon, he is multiplying the final price by 0.75. Since multiplication is commutative (order doesn't matter), the order in which Peter uses the coupons doesn't affect the final price, so he ends up paying $100,000 \cdot 0.8^2 \cdot 0.75^2 = \$36,000$.

14. If $a+b+c=7$, $ab+bc+ca=14$, $abc=8$, what is $(1+a)(1+b)(1+c)$?

14. **30**

Solution: By expanding, we find that $(1+a)(1+b)(1+c) = (1+a+b+ab)(1+c) = 1+a+b+ab+c+ac+bc+abc = 1+(a+b+c)+(ab+bc+ca)+abc = 1+7+14+8 = 30$.

15. Every second, you will have a fixed probability p of falling asleep. In six minutes, you have a 37/64 chance of being asleep. What is the probability that you will be asleep in eight minutes?

15. 175/256

Solution: The probability that you fall asleep at any given time is constant, so we let q be the probability that you fall asleep in any minute. The probability that you are awake after 6 minutes is $(1-q)^6$, so $1-(1-q)^6 = 37/64$. Solving yields that $(1-q)^6 = 27/64$, so $(1-q)^2 = 3/4$. The probability that you are asleep after 8 minutes is $1-(1-q)^8 = 1-\left((1-q)^2\right)^4 = 1-(3/4)^4 = 175/256$.