Symmedians

Power Round Solutions

Mission Math Tournament Fall 2012

1 Law of Sines

1. In triangle ABC, let D be the foot of the altitude of point A onto side BC.

   (a) Express the length of AD in two ways to show that $\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C}$.

   (b) Prove the **Law of Sines**: $\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C} = \dfrac{BC}{\sin A}$.

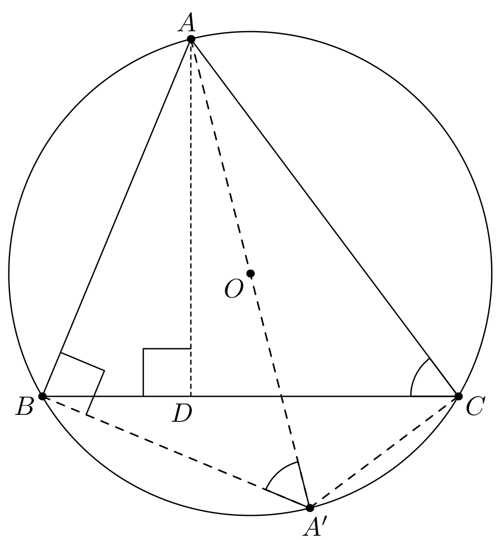

Figure 1: Diagram for problems 1 and 2

Solution. By the definition of sines, we have that $AD = AB\sin B$ and $AD = AC\sin C$. Equating these two expressions yields $\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C}$. Doing a similar computation with the altitude from point B yields the Law of Sines. □

2. In triangle ABC, let O be the circumcenter of the triangle with circumradius R. Let point A' be the reflection of point A about O. Show that $\dfrac{AC}{\sin B} = 2R$. This is called the **Extended Law of Sines**.

Solution. Notice that $AB = AA'\sin AA'B$. Because AA' is the diameter of the circle and $\angle AA'B = \angle ACB$ by inscribed arcs, $AB = 2R\sin C$, or $\dfrac{AB}{\sin C} = 2R$. By the Law of Sines, we can conclude that $\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C} = \dfrac{BC}{\sin A} = 2R$. □

3. Prove the trigonometric form of Ceva's Theorem: the cevians AD, BE, and CF are concurrent at a single point P if and only if

$$\frac{\sin\angle ABE}{\sin\angle CBE}\cdot\frac{\sin\angle BCF}{\sin\angle ACF}\cdot\frac{\sin\angle CAD}{\sin\angle BAD}=1.$$

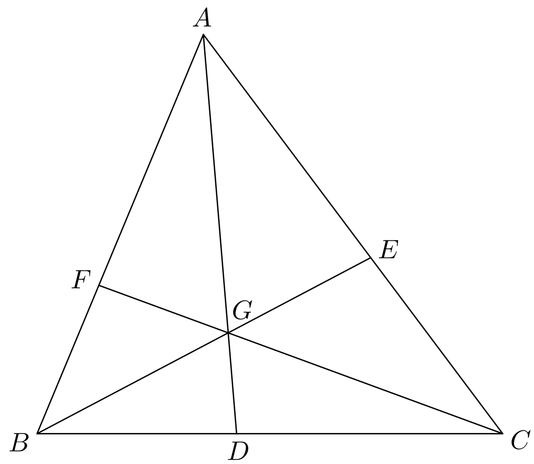

Figure 2: Diagram for Problem 3

Solution. Suppose that cevians BE and CF intersect at point G. We will show that if Ceva's Trigonometric Criterion holds, then the third cevian must pass through point G. From the Law of Sines, we have:

$$\begin{aligned}1&=\frac{\sin\angle ABE}{\sin\angle CBE}\cdot\frac{\sin\angle BCF}{\sin\angle ACF}\cdot\frac{\sin\angle CAD}{\sin\angle BAD}\\&=\frac{GB}{GC}\cdot\frac{GC\cdot\sin\angle BAG}{GB\cdot\sin\angle CAG}\cdot\frac{\sin\angle CAD}{\sin\angle BAD}\end{aligned}$$

which implies

$$\sin\angle BAG\sin\angle CAD=\sin\angle CAG\sin\angle BAD$$

By applying the product-to-sum-identity $\sin x\sin y=\frac{1}{2}(\cos(x-y)-\cos(x+y))$ on both sides of the equation, we get:

$$\cos(\angle BAG-\angle CAD)-\cos(\angle BAG+\angle CAD)=\cos(\angle CAG-\angle BAD)-\cos(\angle CAG+\angle BAD)$$

Notice that since $\angle BAG-\angle CAD=\angle CAG-\angle BAD$, we obtain that $\angle BAG+\angle CAD=\angle CAG+\angle BAD$, implying that $\angle DAG=0$. Thus, for the criterion to hold true, the three cevians must be concurrent.

The converse of the statement is clear by this proof. □

2 Symmedians

4. Points E and F are on sides AC and AB respectively of triangle ABC. Show that if $BCEF$ is cyclic, then the A-symmedian of triangle ABC passes through the midpoint of EF.

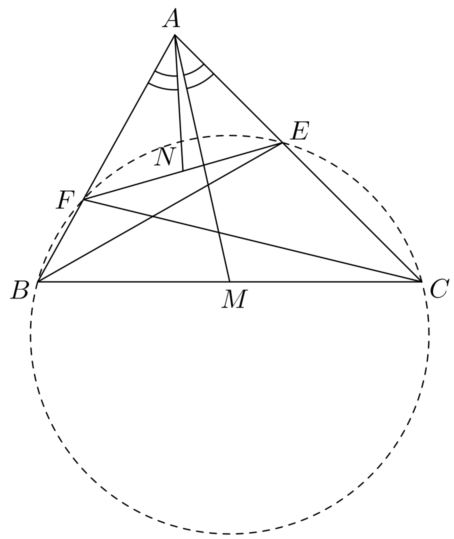

Figure 3: Diagram for Problem 4

Solution. Let M and N be the midpoints of segments BC and EF respectively. We will show that the A-symmedian of triangle ABC passes through point N.

Because quadrilateral $BCEF$ is cyclic, we have that $\angle AFE = \angle ACB$, and consequently $\Delta AFE \sim \Delta ACB$ by AA. By similar triangles and the fact that M and N are midpoints, we can write:

$$\frac{AF}{FE} = \frac{AC}{CB} \Longrightarrow \frac{AF}{FN} = \frac{AC}{CM}$$

Thus, we have that $\Delta AFN \sim \Delta ACM$ by SAS. It follows that $\angle BAN = \angle CAM$, so line AN is the A-symmedian of triangle ABC, as desired. □

5. The A-symmedian of the triangle ABC intersects side BC at point D. Show that $BD : DC = c^2/b^2$, where b and c are the side lengths of AC and AB respectively.

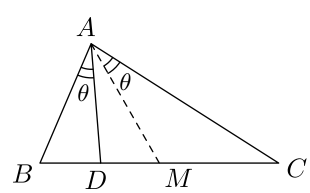

Figure 4: Diagram for Problem 5

Solution. Let M be the midpoint of side BC. With a few applications of the Law of Sines, we get:

$$\begin{aligned}
\frac{BD}{CD} &= \frac{\sin\angle BAD \cdot AD/\sin\angle ABC}{\sin\angle CAD \cdot AD/\sin\angle ACB} \\
&= \left(\frac{\sin\angle ACB}{\sin\angle ABC}\right)\left(\frac{\sin\angle MAC}{\sin\angle MAB}\right) \\
&= \left(\frac{AB}{AC}\right)\left(\frac{MC \cdot \sin\angle AMC/AC}{MB \cdot \sin\angle AMB/AB}\right)
\end{aligned}$$

Since $MB = MC$ and $\sin\angle AMC = \sin\angle AMB$, we get the desired ratio. □

6. Show that the symmedians of a triangle concur at a point in the triangle.

   Solution. (There is no diagram provided for this solution.) By the result from the previous problem and Ceva's Theorem, it follows that the symmedians of a triangle are concurrent. □

7. Tangents to the circumcircle ω of triangle ABC at points B and C intersect at point P. Show that AP coincides with the A-symmedian of triangle ABC.

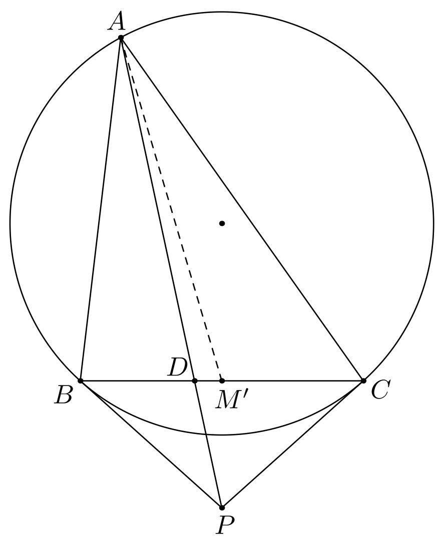

Figure 5: Diagram for problems 7 and 8

   Solution. Let M' be the point on side BC such that $\angle M'AC = \angle PAB$. By the Law of Sines, we have:

$$\begin{aligned}\frac{BM'}{CM'} &= \frac{AM' \sin\angle BAM' / \sin\angle ABC}{AM' \sin\angle CAM' / \sin\angle ACB} \\ &= \frac{\sin\angle BAM' \sin\angle ABP}{\sin\angle ACP \sin\angle CAM'} \\ &= \frac{\sin\angle CAP \sin\angle ABP}{\sin\angle ACP \sin\angle BAP} \\ &= \frac{AP \cdot BP}{CP \cdot AP} \\ &= 1\end{aligned}$$

   Thus, AM' is the median of the triangle, so AP coincides with the symmedian. □

8. Points B and D are on circle ω, and point P is a point outside of ω such that PB and PD are tangent to the circle. A line through P intersects the circle again at two points A and C. Show that $AB/BC = AD/DC$.

   Solution. Notice that $\angle BAC = \angle BDC = \angle PBC$ and $\angle CAD = \angle CBD = \angle CDP$, so by AA similarity, $\Delta PCB \sim \Delta PBA$ and $\Delta PCD \sim \Delta PDA$. Thus, $\dfrac{CB}{BA} = \dfrac{PC}{PB}$ and $\dfrac{CD}{DA} = \dfrac{PC}{PD}$. Since $PB = PD$, the result follows. □

9. Let P be a point in triangle ABC such that $\Delta PBA \sim \Delta PAC$ and O be the circumcenter of the triangle.
   (a) Show that $BPOC$ is cyclic.

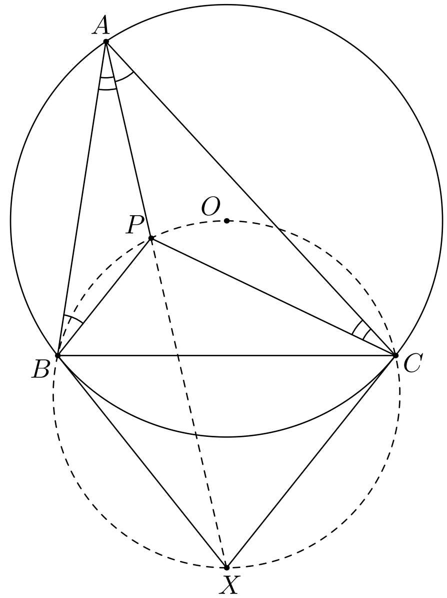

Diagram for Problem 9

(b) Show that P lies on the A-symmedian of triangle ABC.

Solution. Notice that $\angle BPA = 180 - \angle PAB - \angle ABP = 180 - \angle PAB - \angle CAP = 180 - \angle CAB$. Since $\Delta PBA \sim \Delta PCA$, we have that $\angle BPC = 360 - \angle BPA - \angle APC = 360 - 2(180 - \angle BAC) = 2\angle BAC$. Since $\angle BOC$ faces the same arc that $\angle BAC$ subtends to, $\angle BOC = 2\angle BAC$, so $BPOC$ is cyclic.

Now consider point X such that XB and XC are tangents to the circumcircle of triangle ABC. Because $BPOC$ is cyclic and $\angle OCX + \angle XBO = 90 + 90 = 180°$, points B, C, X, P, and O all lie on the same circle.

Since $\angle BPA + \angle BPX = (180 - \angle BAC) + (\angle BCX) = 180 - \angle BAC + \angle BAC = 180°$, points A, P, and X are collinear, so AP coincides with the A-symmedian of triangle ABC. $\square$

10. On the circle ω with center O and radius R, consider two fixed points A ad B, and a variable point C. Let ω_1 be the circle through A tangent to BC at C. Similarly, let ω_2 be the circle passing through B, which is tangent to AC at C. Let D be the second point of intersection (other than C) of ω_1 and ω_2.

    (a) Show that the line CD passes through a fixed point.

    (b) Show that $CD \leq R$.

    Solution. Notice that because ω_1 and ω_2 are tangent to sides BC and AC respectively, we have that $\angle DBC = \angle DCA$ and $\angle DAC = \angle DCB$, so $\Delta CAD \sim \Delta BCD$ by AA.

    We let X be the intersection point of the tangents to ω at points A and B. From the previous problem, it is clear that CD passes through point X, which is fixed.

    Since $AODBX$ is cyclic with OX as the diameter, we have that $Area(ADB) \leq Area(AOB)$, or $\frac{1}{2}AD \cdot BD \sin\angle ADB \leq \frac{1}{2}OA \cdot OB \sin\angle AOB$. Because $\angle ADB = \angle AOB$, we are left with the inequality $AD \cdot BD \leq R^2$. By similar triangles ADC and CDB, we have that $CD^2 = AD \cdot BD$, so $CD \leq R$. $\square$

11. Given triangle ABC, define points M and N on sides AB and AC respectively such that $MN||BC$. Segments BN and CM intersect at point P. The circumcircles of triangles BMP and CNP intersect again at point Q distinct from P.

    (a) Prove that quadrilaterals $AMQC$ and $ANQB$ are cyclic.

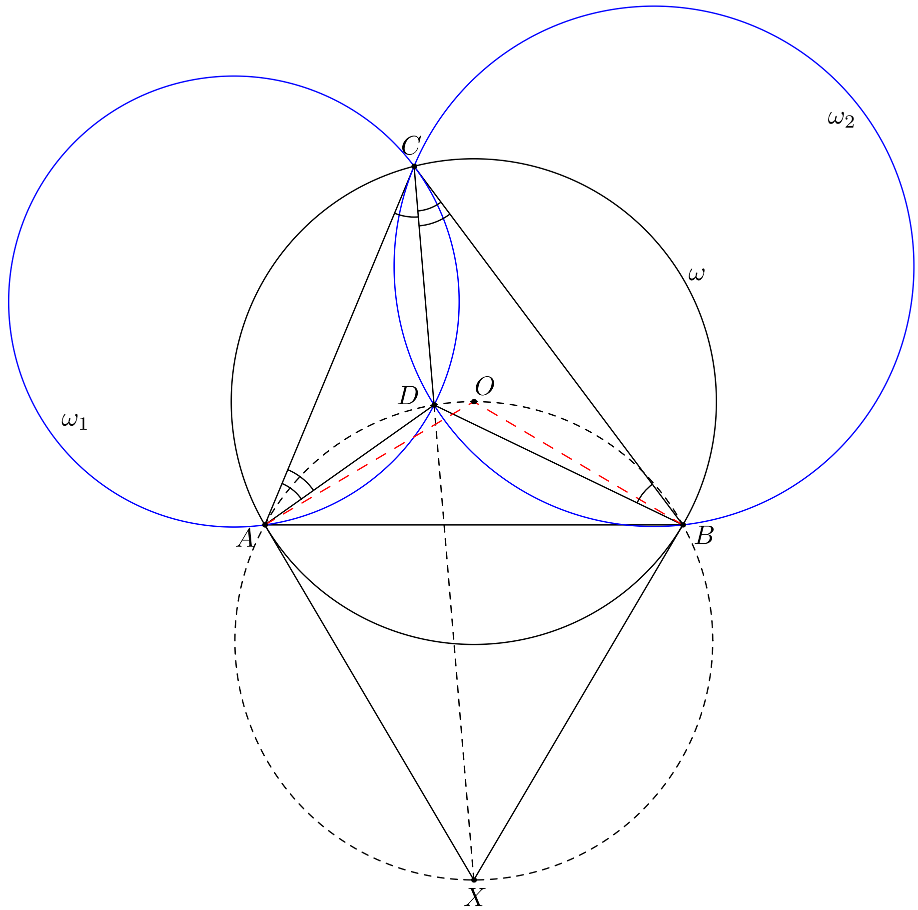

Figure 6: Diagram for Problem 10

(b) Show that $\Delta ABQ \sim \Delta CPQ$ and $\Delta QNB \sim \Delta QCM$.

(c) Prove that AQ coincides with the A-symmedian of triangle ABC.

Solution. From angle chasing, we have that $\angle QBA = \angle QPC = \angle QNC$, and $\angle NCQ = \angle BPQ = \angle BMQ$, so quadrilaterals $AMQC$ and $ANQB$ are cyclic. Thus, $\angle BAQ = \angle MCQ$ and $\angle ABQ = \angle CPQ$, so $\Delta ABQ \sim \Delta CPQ$. Similarly, $\Delta QNB \sim \Delta QCM$.

Notice that because $MN \parallel BC$, we have two sets of similar triangles: $\Delta PNM \sim \Delta PBC$ and $\Delta AMN \sim \Delta ABC$. With this and that $\Delta BAQ \sim \Delta PCQ$, we can write the following ratios:

$$\frac{PC}{MP} = \frac{BC}{MN} = \frac{AB}{AM}$$
$$\frac{AM}{MP} = \frac{AB}{PC} = \frac{AQ}{QC}$$

Since $\angle AMP = \angle AQC$ by cyclic quadrilaterals, we have that $\Delta AMP \sim \Delta AQC$, so $\angle MAP = \angle QAC$. By Ceva's Theorem on point P with respect to triangle ABC, we have that AP is the median of triangle ABC, so AQ must coincide with the symmedian of the triangle. $\square$

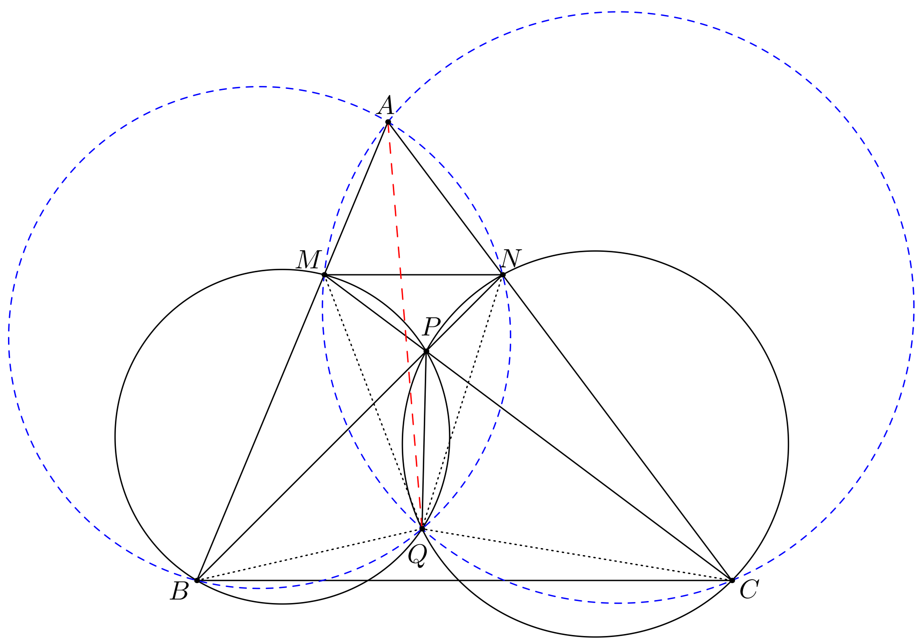

Figure 7: Diagram for Problem 11