Calculus has far-reaching consequences in algebra, combinatorics, geometry, physics, and beyond. Integral calculus rules over the vast majority of these cases, and proper integration technique will open a whole range of new problems. Almost all integration problems may be solved with some combination of five techniques:

- (1) u-substitutions (e.g., reflections or trigonometric substitutions)
- (2) integration by parts (n.b., self-referential applications)
- (3) series decompositions (e.g., Taylor Series or Fourier Series)
- (4) differentiation under the integral sign (i.e., double integrals)
- (5) complex analysis (e.g., residue theory or harmonic conjugates)

Geometric Interpretations.

1. Suppose f is continuous and $f(-x) + f(x) = x^2$. Find $\displaystyle\int_{-1}^{1} f(x)\, dx$.
2. Define a one-to-one function f such that $f(2) = 3$ and $f(5) = 7$. If $\displaystyle\int_{2}^{5} f(x)\, dx = 17$, find $\displaystyle\int_{3}^{7} f^{-1}(x)\, dx$.
3. Define a one-to-one function f such that $f(1) = 4$ and $f(6) = 2$. If $\displaystyle\int_{1}^{6} f(x)\, dx = 15$, find $\displaystyle\int_{2}^{4} f^{-1}(x)\, dx$.
4. Evaluate $\displaystyle\int_{0}^{1} e^{x^2} + (e-1) \cdot \sqrt{\ln((e-1)x+1)}\, dx$.
5. Given real $a > 1$, evaluate $\displaystyle\int_{0}^{\pi} \ln(1 - 2a\cos x + a^2)\, dx$.
6. Evaluate $\displaystyle\int_{0}^{\pi/2} \frac{\cos\sqrt{1-x}\cosh\sqrt{x}}{\sqrt{x-x^2}}\, dx$.

Combinatorial Applications.

1. If two real numbers between 0 and 4 are selected uniformly and at random, what is the probability that their sum is greater than their product?
2. If four points are placed randomly on the circumference of a unit circle, what is the probability that all four can be contained within one of its semicircles?
3. Divide a given line segment into two other line segments. Then, cut each of these new line segments into two more line segments. Given each cut is made with all possible outcomes equally likely, what is the probability that the resulting four line segments are the sides of a quadrilateral?
4. If three real numbers between 0 and 1 are selected uniformly and at random, what is the probability that their product is greater than $\dfrac{1}{2}$?
5. Brian generates a real number p uniformly at random from the interval $[0, 1]$, then creates a coin that lands heads with probability p. He flips the coin until he has seen either two more heads than tails or two more tails than heads. He observes that this process ends with him seeing two more heads than tails. What is the probability that $p \geq \dfrac{1}{2}$?
6. Ronald throws a dart at a circular dartboard such that it hits the board at point P, where P is chosen with uniform probability. His score is the minimum distance from P to the edge of the dartboard divided b the minimum distance from P to the center of the dartboard. What is the expected value of his shot?

For each of the following problems, assume that f is a continuous doubly-differentiable function.

Algebraic Manipulations.

1. If $\int_0^1 f(x)f'(x)\,dx = 0$ and $\int_0^1 (f(x))^2 f'(x)\,dx = 18$, what is $\int_0^1 (f(x))^4 f'(x)\,dx$?
2. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{n}\sqrt{1+\frac{3k}{n}}$.
3. If $f(3) = 1$, $f'(3) = 2$, and $\int_0^3 f(x)\,dx = 7$, what is $\int_0^3 x^2 f''(x)\,dx$?
4. For an integer $n \geq 0$, find a closed form expression for $\int_0^\infty x^n e^{-x}\,dx$.
5. If $f(2x) = 3f(x)$ for all x and $\int_0^1 f(x)\,dx = 1$, what is $\int_1^2 f(x)\,dx$?
6. Find $\lim_{n\to\infty} \int_0^1 \frac{nx^{n-1}}{1+x}\,dx$.

Assorted Integrals.

1. $\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$
2. $\int_{32}^{243} \frac{1}{x - x^{3/5}}\,dx$
3. $\int_0^{10} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx$
4. $\int_0^{\pi/2} \sin x \ln(\sin x)\,dx$
5. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{e^x+1}\,dx$
6. $\int_0^1 (\sin(\ln x))^2\,dx$
7. $\int_0^\infty \frac{x\tan^{-1} x}{(x^2+1)^2}\,dx$
8. $\int_0^1 \ln(x)\ln(1-x)\,dx$
9. $\int_0^\infty \frac{\tan^{-1}(\pi x) - \tan^{-1} x}{x}\,dx$
10. $\int_0^\infty \frac{e^{e\cdot x} - e^x}{x(e^{e\cdot x}+1)(e^x+1)}$
11. $\int_0^1 \frac{\ln(1+x)}{x\sqrt{1-x^2}}\,dx$
12. $\int_0^{2\pi} \tan^{-1}\left(\frac{1+\cos x}{1+\sin x}\right)\,dx$
13. $\int_0^{\frac{\pi}{2}} \cos(\tan x)\,dx$