Week 7: Bijections

MSJ Math Club

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Bijections! Now that may sound like a scary word, but the concept is actually pretty simple: you want to count something, but that something is rather complicated, but it is in one-to-one correspondence with a set which is simpler to count. By far the best way to illustrate this principle is with a few examples:

1 Examples

- How many paths are there from $(0, 0)$ to $(4, 5)$ if each step in the path can only go 1 unit up or right?
- Suppose we have 10 urns and 20 balls which will be placed in these urns. How many ways are there to place the 20 balls if the balls are indistinguishable, the urns are distinguishable, and there must be at least one ball per urn?
- If n is a positive integer, then prove that the number of partitions of n is equal to the number of partitions of $2n$ with n parts. (A Ferrers diagram is helpful in this problem; we will explain what that is.)

2 Tips and Tricks

We have none; the best way to learn combinatorics techniques like bijections is to do practice problems.

3 Practice Problems

1. *(Intermediate C/P)* Find the number of positive integer solutions to $w + x + y + z < 25$.
2. *(Intermediate C/P)* Andrew has 10 candy bars, 10 packages of jelly beans, 10 lollipops, and 10 packs of chewing gum, and Andrew has two sisters. In how many ways can Andrew distribute the candies between his sisters, so that each sister gets 20 items total?
3. Construct an equilateral triangle with sides n by building successively larger rows of equilateral triangles of side length 1. The first row would then have one triangle, the second row would have three, and so on, with a total of n^2 small triangles. How many parallelograms can be formed from the boundaries of the triangles?
4. *(Mandelbrot)* In a certain lottery, 7 balls are drawn at random from n balls numbered 1 through n. If the probability that no pair of consecutive numbers is drawn equals the probability of drawing exactly one pair of consecutive numbers, find n.
5. *(HMMT)* How many sequences of 5 positive integers (a, b, c, d, e) satisfy $abcde \leq a + b + c + d + e \leq 10$?
6. *(Canada)* Consider an equilateral triangle of side length n, which is divided into unit triangles similarly to the third practice problem on this list. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. Determine the value of $f(2005)$.
7. *(Putnam)* How many possible bijections f on $1, 2, \ldots, n$ are there such that for each $i = 2, 3, \ldots, n$ we can find $j < i$ with $f(i) - f(j) = \pm 1$? (Heh this isn't really a bijection problem)