# Team Round Solutions

Mission Math Regional

January 19, 2013

1. Carl Ray Jensen is naming his newborn daughter either "Yes", "No", or "Maybe." The probability that Carl chooses a certain name is directly proportional to the number of letters in the name. What is the probability that he calls her "Maybe"?

   **Solution:** There are 3, 2, and 5 letters in the names "Yes," "No," and "Maybe," respectively. Thus, the probability that Carl calls her "Maybe" is $5/(3+2+5) = \boxed{1/2}$.

2. What are all possible months that the $30^{\text{th}}$ Sunday can be in? Assume that it is not a leap year.

   **Solution:** We list out the number of days elapsed in the year at the end of each month:

   | | |
   |---|---|
   | January: | 31 |
   | February: | 59 |
   | March: | 90 |
   | April: | 120 |
   | May: | 151 |
   | June: | 181 |
   | July: | 212 |
   | ... | |

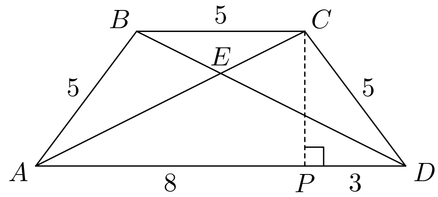
   The $30^{\text{th}}$ Sunday in the year must be between the $204^{\text{th}}$ and $210^{\text{th}}$ days of the year. Because days $182 - 212$ are all in July, the answer is $\boxed{\text{July only}}$.

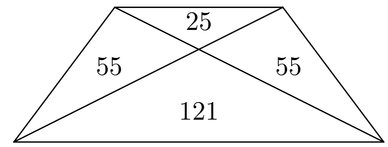
3. Kurt can type at an average of 50 words per minute on his QWERTY keyboard. Eleanor can type at an average speed of 80 words per minute on her Dvorak keyboard. How much more time, in seconds, does Kurt spend typing a 1000 word essay than Eleanor?

   **Solution:** Kurt spends $1000/50 = 20$ minutes typing, while Eleanor spends $1000/80 = 12.5$ minutes typing. Thus, Kurt spends 7.5 more minutes typing than Eleanor, which is equivalent to $\boxed{\text{450 seconds}}$.

4. An isosceles trapezoid has side lengths of $5, 5, 5,$ and $11$. Marty draws in the two diagonals of the trapezoid and receives four small triangles. What is the area of the largest of these four triangles?

   **Solution:**




We label the points of the trapezoid as shown above. First, we find the area of the entire trapezoid by dropping an altitude from point $C$. Because the trapezoid is isosceles, we know that $DP = (11-5)/2 = 3$. By the Pythagorean Theorem, $CP = \sqrt{CD^2 - DP^2} = \sqrt{25-9} = 4$. Thus, the area of the trapezoid is $\frac{1}{2}(5+11)(4) = 32$.

Next, we use area ratios to find the fraction of the total trapezoid area that each triangle takes up. By similar triangles $ADE$ and $CBE$, $CE/EA = CB/AD = 5/11$. Because triangles $ABE$ and $BCE$ share a common vertex (point $B$) and their bases are along the same line, they have the same height, so the ratio of their areas is the ratio of their bases, which is $CE/EA = 5/11$. Similarly, the ratios of the areas of $CED$ to $EAD$ is $CE/EA = 5/11$. By symmetry, the area of triangles $ABE$ and $CDE$ are the same. Thus, $[BCE] : [ABE] : [CDE] : [ADE] = 25 : 55 : 55 : 121$, where $[SHAPE]$ denotes the area of a shape. Thus, the area of the large triangle, $ADE$, is $(121/(121+55+55+25))(32) = \boxed{121/8}$.

5. Find the sum of the positive odd factors of 2,700.

**Solution:** The prime factorization of 2700 is $2^2 \cdot 3^3 \cdot 5^2$. The sum of the odd factors is $3^0 5^0 + 3^0 5^1 + 3^0 5^2 + 3^1 5^0 + \cdots + 3^2 5^2 + 3^3 5^0 + 3^3 5^1 + 3^3 5^2 = (3^0 + 3^1 + 3^2 + 3^3)(5^0 + 5^1 + 5^2) = 40 \times 31 = \boxed{1240}$.

6. It costs \$4.15 to buy 3 erasers, 4 notebooks, and 5 pencils, \$4.40 to buy 4 erasers, 5 notebooks, and 3 pencils, and \$3.45 to buy 5 erasers, 3 notebooks, and 4 pencils. How much does it cost to buy 1 eraser, 2 notebooks, and 3 pencils?

**Solution:** Let $e$, $n$, and $p$ be the number of erasers, notebooks, and pencils, respectively, that are valued in each case. We can write the following equations:

$$\begin{aligned} 3e + 4n + 5p &= 4.15 \\ 4e + 5n + 3p &= 4.40 \\ 5e + 3n + 4p &= 3.45 \end{aligned}$$

Adding these three equations yields:

$$12e + 12n + 12p = 12.00$$

$$e + n + p = 1.00$$

Subtracting twice the last equation from the first given equation yields:

$$(3e + 4n + 5p) - 2(e + n + p) = 4.15 - 2(1.00)$$
$$e + 2n + 3p = \boxed{\$2.15}$$

7. A duck walked up to a lemonade stand, and it said to the man running the stand, "Hey, if you pour half of the lemonade in your pitcher into my jug of grape juice, the mixture will be 40% grape juice." The lemonade man did so. The duck then poured 1/3 of his new mixture back into the lemonade pitcher. What fraction of the mixture in the pitcher is lemonade?

   **Solution:** Suppose that the duck started with $x$ liters of grape juice. Since 60% of the first mixture is half of the original lemonade, the man started with $3x$ liters of lemonade. After the first mixture, the duck has $x$ liters of grape juice and $3x/2$ liters of lemonade. When the duck returns 1/3 of his new mixture into the lemonade, a total of $5x/6$ liters is poured back, of which is $x/2$ liters of lemonade. Because the mixture is retransferred into a pitcher with $3x/2$ liters of lemonade, the fraction of the final mixture that is lemonade is $(3x/2 + x/2) \div (3x/2 + 5x/6) = \boxed{6/7}$.

8. Three parallel cuts are made through a sphere of radius 2 such that each of the four pieces has a width of 1 unit. What is the total surface area of the four pieces?

   **Solution:** After the four cuts are made, we are left with two end-pieces and two middle-pieces. The total surface area of the curved surfaces is the surface area of the entire sphere, which is $4\pi(2)^2 = 16\pi$. Furthermore, we must account for two larger diametric cross-sections and four small cross-sectional areas. The area of the two large cross-sections is the twice the area of a circle with radius two, which is $2 \times \pi(2)^2 = 8\pi$. Next, we find the area of each of the smaller cross sections, which are each circles. Because the cuts have equal width, the radius of each of these four surfaces is $\sqrt{r^2 - (r/2)^2} = \sqrt{3}$, by the Pythagorean Theorem. Thus, the total area of these four sections is $4 \times \pi(\sqrt{3})^2 = 12\pi$. Thus, the total surface area is $16\pi + 8\pi + 12\pi = \boxed{36\pi}$.

9. The fourth house on a street has address number of 101 and the ninth house has an address number of 201. Given that the house numbers on the street form an arithmetic progression, what is the address of the first house on the street?

   **Solution:** The common difference between the house numbers is $(201 - 101)/(9 - 4) = 20$. Thus, the first house's number is $101 - (4 - 1)(20) = \boxed{41}$.

10. Lewis was asked to cut a deck of cards for a magician's trick, so he took a pair of scissors and cut each of his 20 cards in half, leaving 20 red cards and 20 black cards. What is the probability that two cards drawn (without replacement) from Lewis's newly made deck are both red?

**Solution:** There are $20 \times 19/2 = 190$ ways for Lewis to choose two cards that are both red. There are $40 \times 39/2 = 780$ ways for Lewis to choose any two cards. Thus, the probability is $190/780 = \boxed{19/78}$.

11. Let $n$ be a positive integer. The positive integers $a_0, a_1, \cdots, a_n$ satisfy

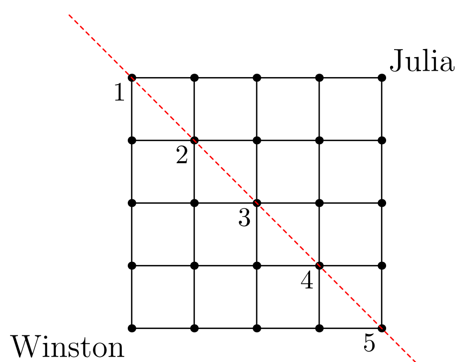
$$11a_0 - 1 = a_1^{10} + a_2^{10} + \cdots + a_n^{10}.$$

What is the smallest possible value of $n$?

**Solution:** We consider the remainder when each side of the equation is divided by 11. Because $11a_0$ is always divisible by 11, the left hand side leaves a remainder of 10. On the other hand, it can be checked that $a^{10}$ always leaves a remainder of 0 or 1 when divided by 11. (One can test each value, or use a more advanced theorem called Fermat's Little Theorem.) Thus, we need at least 10 terms on the right hand side in order for the expression to leave a remainder of 10 when divided by 11. Letting $a_0 = a_1 = \cdots = a_{10} = 1$ indeed satisfies the equation, so the answer is $\boxed{10}$.

12. Winston is in the bottom left hand corner of a $4 \times 4$ grid, and Julia is in the top right hand corner of the same grid. Each minute, Winston randomly moves 1 step up or to the right along the edges, and Julia simultaneously randomly moves 1 step down or to the left along the edges. After four minutes, find the probability that they end up on the same point.

**Solution:**




After four steps, both Winston and Julia must be on a point on the main diagonal. We break this problem into cases, count the number of ways that they can arrive at each point. The points have been labeled above.

**Case 1:** They end up on point 1.
There is 1 way for each of them to arrive at that point (that is, to go straight up for Winston, or directly to the left for Julia), so the answer for this case is 1.

**Case 2:** They end up on point 2.
Winston must take 3 steps up and 1 step to the right. Thus, there are $\binom{4}{1} = 4$ ways for

Winston to choose when to make the right turn. A similar argument holds for Julia. Thus, there are 4 ways for each of them to arrive at that point, so the answer for this case is $4^2 = 16$.

**Case 3:** They end up on point 3.
Winston must take 2 steps up and 2 step to the right. Thus, there are $\binom{4}{2} = 6$ ways for Winston to choose when to make his two right turns. A similar argument holds for Julia. There are 6 ways for each of them to arrive at that point, so the answer for this case is $6^2 = 36$.

**Case 4:** They end up on point 4.
This is identical to case 2.

**Case 5:** They end up on point 5.
This is identical to case 1.

Thus, there are a total of $1+16+36+16+1 = 70$ different ways for them to end at the same point. Since on each turn, both Winston and Julia can either travel up or down, so each of them have $2^4 = 16$ different ways to travel. Thus, the probability is $70/(16^2) = \boxed{\frac{35}{128}}$.

13. All possible rearrangements of the letters 'AAGMMR' are placed in alphabetical order. On this list of 'words,' in what place is the word 'MAGMAR'?

**Solution:** There are $6!/(2 \cdot 2) = 180$ total permutations. We consider the words that start with $A$ and $G$ first. There are $5!/2 = 60$ words starting with $A$ (the number of ways to permute the remaining letters), and $5!/(2 \cdot 2) = 30$ words starting with $G$. Thus, words starting with $M$ start with word number 91. The first few words starting from number 91 also start with the substring $MA$.

Next, we find the index of $GMAR$ in the list of all possible permutations of the four letters. We can simply list them out:

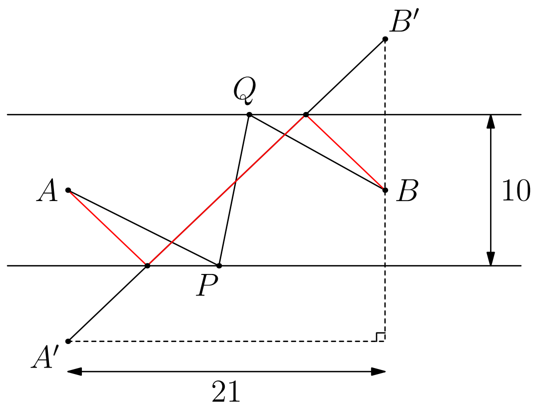
AGMR
AGRM
AMGR
AMRG
ARGM
ARMG
GAMR
GARM
GMAR
...

Because $GMAR$ is the $9^{\text{th}}$ term on this list, the index of $MAGMAR$ is $91 + 9 - 1 = \boxed{99}$.

14. You are doing a quest in the game of Run Escape. You are standing at point $A$, in between two parallel rivers (River Lum and River Salve) that are 10 kilometers apart. You are 5 kilometers away from both rivers. To complete the quest, you must get a bucket of water from River Lum, get a bucket of water from River Salve, and talk to a person who is standing at point $B$, in

that order. If the line $AB$ is parallel to both rivers, and the distance between points $A$ and $B$ is 21 kilometers, what is length of the shortest path that you can take from point $A$ to one river, then the other, and finally to point $B$, in kilometers?

**Solution:**




Suppose that you go to point $P$ on River Lum (the first river) and point $Q$ on the River Salve (the second one) before talking to the person at point $B$. Let $A'$ and $B'$ the reflections of points $A$ and $B$ about Rivers Lum and Salve respectively. Because $AP = A'P$ and $BQ = B'Q$, the total distance that you travel is $AP + PQ + QB = A'P + PQ + QB'$. This distance is minimized when the four points $A'$, $P$, $Q$, and $B'$ are collinear, as shown above.

To find the distance from points $A'$ and $B'$, we construct a right triangle with hypotenuse as $A'B'$ and legs parallel and perpendicular to the rivers. The distance between points $A$ and $B$ is 21 and the distance between the rivers is 10. By the Pythagorean Theorem, the minimum possible distance that you must traverse is $A'B' = \sqrt{21^2 + (2 \times 10)^2} = \boxed{29 \text{ kilometers}}$.

15. A polynomial function $P(x)$ with integer coefficients satisfies that $P(20) = P(13) = 2013$. What is the maximum number of integer roots that $P(x)$ can have?

**Solution:** We claim that if $P(a) = m$ and $P(b) = n$ for some integer polynomial $P(x)$ and integers $a$ and $b$, then $m - n$ is divisible by $a - b$. We can write out $P(x)$ in the general form of a polynomial:

$$P(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x^1 + c_0$$

Notice that:

$$\begin{aligned} m - n &= P(a) - P(b) \\ &= (c_k a^k + c_{k-1} a^{k-1} + \cdots + c_1 a^1 + c_0) - (c_k b^k + c_{k-1} b^{k-1} + \cdots + c_1 b^1 + c_0) \\ &= c_k(a^k - b^k) + c_{k-1}(a^{k-1} - b^{k-1}) + \cdots + c_1(a - b) \end{aligned}$$

Since $a - b$ is a factor of $a^\ell - b^\ell$ for all $\ell$, then $a - b$ divides the left-hand side of the equation, as desired.

We now claim that there are no integer roots to the polynomial. For the sake of contradiction, assume that there exists an integer $r$ such that $P(r) = 0$. Then we must have that $r - 20$ and $r - 13$ both divide 2013. Notice that the difference between $r - 20$ and $r - 13$ is 7. Because 2013 is an odd number, it can not have two factors that differ by an odd number, so there are $\boxed{\text{0 integer roots}}$.