MSJ Math Club

Vieta's Formulas

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1 Introduction

(For convenience, we just took the introduction for last year and added/removed some stuff as we thought neccessary)

Vieta's Formulae (also called Viete's Formulae) are a quick way to determine the sum, product, etc. of the roots of a polynomial. The derivation comes from the Fundamental Theorem of Algebra.

Suppose we have an nth-degree polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

which we factor as

$$a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

If we expand the latter, we will find:

$$p(x) = a_n(x^n - (r_1 + r_2 + \cdots + r_n)x + (r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n)x^2 - \cdots \pm r_1r_2\cdots r_n)$$

where the $\pm$ is $+$ for even n and $-$ for odd n.

Taking into account the alternating signs, we find that the sum of the roots taken k at a time is

$$(-1)^k \frac{a_{n-k}}{a_n}$$

That is, it's the coefficient of x^k divided by the leading coefficient, positive if k is even and negative if k is odd. And by taking the roots k at a time, I mean we multiply k of the roots together, and add up all of these products in a symmetric sum. (This is also called the kth elementary symmetric polynomial)

2 Tips and Tricks

- The applications of Vieta's Formulas are usually fairly obvious especially at the AMC/AIME level.
- If you have some symmetric polynomials, (i.e. if you switch two variables in the polynomial, the expression doesn't change) then its probably solved with Vieta. Notice that all symmetric polynomials are also polynomials of the elementary symmetric polynomials.
- You can also save lots of time when otherwise you may have to use quadratic, cubic, or quartic equations.
- Vieta's formulas can also be used along with the Taylor Series of the sine function to solve the Basel problem. It can also be used to prove a lot of other identites about π.
- Vieta's formula includes complex roots.
- Elementary symmetric polynomials is one of the entry points to abstract algebra and Galois theory.

3 Examples

1. Find the sum of the cubes of the roots of the polynomial $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6$.
2. (1996 AIME) Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$. Find t.
3. (2003 AIME 2) Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.
4. Express $sin(x)$ as a infinite product of linear polynomials.

4 Practice Problems

1. (2003 AMC 10A) What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?
2. (2005 AIME 1) The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.
3. (2005 AMC 12B) The quadratic equation $x^2 + mx + n$ has roots twice those of $x^2 + px + m$, and none of m, n, and p is zero. What is the value of n/p?
4. (1995 AIME) For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.
5. (Overused SMT problem) Solve the system of equations below:

$$\begin{cases} a^3 + b^3 + c^3 = 34 \\ a^2 + b^2 + c^2 = 14 \\ a + b + c = 4 \end{cases}$$

6. (Classic) Given a quadratic $Q(x) = x^2 + ax + b$, find $\sqrt{x_1} + \sqrt{x_2}$ in terms of a and b in a simple expression.
7. (Classic) Use the above result and the quadriatic formula to show that

$$\sqrt{x + \sqrt{y}} + \sqrt{x - \sqrt{y}} = \sqrt{2x + 2\sqrt{x^2 - y}}$$

   This result may be useful for simplifying square roots.

8. (John Wallis) Show that $\frac{\pi}{2} = \frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \cdots \times \frac{4n^2}{4n^2 - 1}$. (Hint: It was discovered before the 1700's)