MSJ Math Club

Week 10 (?): Cubics and an Identity

February 6, 2013

1 Tips and Tricks

- Here is a very useful identity:

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=\frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)$$

  (To prove, you may consider the monic cubic $P(x)$ with roots a, b, and c, and analyze $P(a)+P(b)+P(c)$.) One of the immediate implications is that $a^3+b^3+c^3-3abc=0$ if $a+b+c=0$. From the last expression in the factorization above, the converse holds when $a+b+c=0$ or $a=b=c$. This is powerful conclusion, and can be used in very creative ways.

- Cubics are not quite as nice as quartic (from an "objective view", of course :P), since they can't be decomposed into quadratics. On the other hand, the power is smaller, so if you choose to guess and check roots, once you find one, you are done. Also, cubics always have a real root, so it might make guessing and checking roots easier, if you *really* need to.

- The cubic formula is almost as fun to memorize as the quartic formula! The margins of this page are too small to include the song. Here is a real root of the equation $ax^3+bx^2+cx+d=0$:

$$\begin{aligned} x_1 = &-\frac{b}{3a} \\ &-\frac{1}{3a}\sqrt[3]{\frac{2b^3-9abc+27a^2d+\sqrt{\left(2b^3-9abc+27a^2d\right)^2-4\left(b^2-3ac\right)^3}}{2}} \\ &-\frac{1}{3a}\sqrt[3]{\frac{2b^3-9abc+27a^2d-\sqrt{\left(2b^3-9abc+27a^2d\right)^2-4\left(b^2-3ac\right)^3}}{2}} \end{aligned}$$

  The other two roots look very similar to the expression above.

- A well-known technique for deriving the above equation is called Cardano's Method. However the steps are very unmotivated, so a better derivation will be presented here. First, we divide by the leading coefficient, and work with the polynomial $f(x)=x^3+bx^2+cx+d=0$. We then depress the cubic by considering $f\left(x-\frac{b}{3}\right)=x^3+ex+f$ for some values of e and f. Notice that there is no x^2 term.

  Now suppose that a root of the new polynomial is in the form $r=\sqrt[3]{p}+\sqrt[3]{q}$. We move r to the right hand side of the equation and obtain $\sqrt[3]{p}+\sqrt[3]{q}-r=0$. By the identity in the first bullet point, we have $p+q-r^3+(3\sqrt[3]{pq})r=0$ or $r^3-3(\sqrt[3]{pq})r-(p+q)=0$. Notice that this is a cubic in terms of r, so we can set $3\sqrt[3]{pq}=-e$ and $p+q=-f$. The variables e and f are known, and this two variable system has a fairly standard solving procedure.

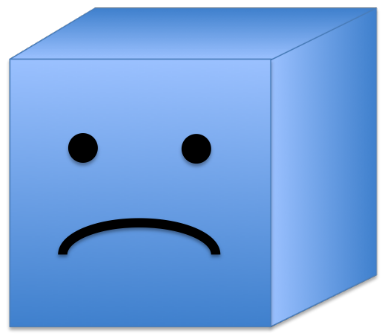

Figure 1: A Depressed Cube

2 Examples

1. Show that $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=1$.

3 Practice Problems

1. (Rice/Harvard/MIT/Stanford/John Hopkins Math Tournament 2000) Evaluate $2000^3-1999\cdot 2000^2-1999^2\cdot 2000+1999^3$.

2. Let $r=5+\sqrt[3]{7}$ be the root of a polynomial equation of the smallest possible degree in which all coefficients are integers. If the coefficient of the term of the highest degree is 1, find the sum of the coefficients.

3. (RMT 2011) Find all rational roots to $|x-1|\times|x^2-2|-2=0$.

4. (AIME 1993) Let $P_0(x)=x^3+313x^2-77x-8$. For integers $n\geq 1$, define $P_n(x)=P_{n-1}(x-n)$. What is the coefficient of x in $P_{20}(x)$?

5. (Turkey MC-2005) What is the largest real number x satisfying $x^3-x^2-x-\dfrac{1}{3}=0$?

6. Find all pairs of integers (x,y) such that $xy+\dfrac{x^3+y^3}{3}=2007$.

7. (All Russian MO, 1998) Two lines parallel to the x-axis cut the graph of $y=ax^3+bx^2+cx+d$ in points A,C,E and B,D,F respectively, in that order from left to right. Prove that the length of the projection of the segment CD onto the x-axis equals the sum of the lengths of the projections of AB and EF.

4 A Bad Problem

This is from the infamous 2011 AIME A. It's not a very good problem, but I'll give it to you anyway.

8. For some integer m, the polynomial $x^3-2011x+m$ has the three integer roots a, b, and c. Find $|a|+|b|+|c|$.