MSJ Math Club

The Stable Marriage Problem

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1 Introduction

Problem: N single men and N single women are desperate to get married, and they need your advice as to how to pair up. Each person has ranked the members of the opposite gender in order of who they want their partner to be (there are no ties). How do you make everyone as happy as possible, or at least create a stable arrangement? (A pairing is a bijection between the men and the women, meaning every man is paired with exactly one woman, and every woman is paired with exactly one man.)

In order to solve this problem, let's first define what a stable pairing is.

Definition. A pairing is *unstable* if there exist a man and a woman who prefer each other to their current partners. We will call this man and woman a *run-away couple*. (Consequently, a stable pairing is one without any run-away couples.)

Here is an example to illustrate the problem (any resemblance to real persons is purely coincidental). The men are William, Xavier, Yancy, and Zach, and the women are Ava, Beth, Chloe, and Dawn.

- William: Chloe>Dawn>Beth>Ava
- Xavier: Ava>Chloe>Dawn>Beth
- Yancy: Ava>Dawn>Chloe>Beth
- Zach: Beth>Ava>Chloe>Dawn

- Ava: William>Zach>Yancy>Xavier
- Beth: Yancy>Xavier>William>Zach
- Chloe: Yancy>Zach>Xavier>William
- Dawn: Yancy>Zach>William>Xavier

2 The Jane Austen Algorithm

Given these conditions, it seems difficult to find a stable pairing, much less prove that one always exists. However, the following algorithm, discovered by Gale and Shapley in 1962, not only proves that a stable pairing always exists, it describes a way to find one. This algorithm is known by several names, including the Gale-Shapley algorithm, traditional algorithm, and Jane Austen algorithm, and it works as follows:

1. Each day, every man not currently engaged proposes to the first woman on his list who has not yet rejected him, starting from his top choice and working his way down.
2. Each woman will immediately accept a proposal if she is not engaged. If a woman is already engaged, she will compare the man who is proposing with her current fiancé, taking the one she prefers as her new fiancé and rejecting the other man.
3. When every man is engaged to a woman, each woman will marry the man she is engaged to.

3 Problems to Consider

1. Find a stable matching for the example mentioned above using the Gale-Shapley algorithm.

2. Prove that the Gale-Shapley algorithm always terminates.

3. Prove that the Gale-Shapley algorithm always produces a stable pairing.

4. Prove that if a stable pairing is male-optimal (meaning that every man gets the best possible woman that he could have gotten in any stable pairing), then it is also female-pessimal (every woman gets the worst possible man she could have gotten in any stable pairing).

5. Prove that the Gale-Shapley algorithm produces a male-optimal pairing (which is also female-pessimal, as proved in Question 4).

6. After doing the first five problems, you might feel really bad for the women in this algorithm, since they're always getting the worst possible outcome. However, it turns out that the women can force the algorithm to favor them by lying about their preferences. Find a way the women can falsify their preferences to make the algorithm female-optimal instead.

7. Find an algorithm that produces a female-optimal pairing (every woman gets the best possible man she could have gotten in the set of all stable pairings).

8. The *stable roommates problem* (also known as the stable same-sex marriage problem) is the same as the stable marriage problem, except that each person ranks the other $2N - 1$ people, not just people of the same gender. Given a set of roommate preferences, prove that it is not possible to always find a stable pairing. Note: The stable roommate problem is much more complicated than the stable marriage problem; a general solution in polynomial time was only discovered in 1985.

9. The stable marriage problem has also been studied in polygamous cases, in which the men want multiple wives, the women want multiple husbands, or both. This is also known as the hospitals-residents problem because it was actually used in the NRMP, or National Residency Matching Program, which pairs medical school graduates to internship spots at hospitals. The hospitals each had a certain quota of spots they wanted to fill, so both the hospitals and graduates submitted their preference lists, and a computer algorithm assigned students to hospitals.

   At first, the computer used the Gale-Shapley algorithm and produced a hospital-optimal pairing, and it was not until the 1997 that they switched to a student-optimal pairing and made accommodations for married couples to work at nearby hospitals. This algorithm has since been adopted in many other places, including assigning students to New York public schools starting in 2003. Demonstrate how the Gale-Shapley algorithm can be applied to the hospitals-residents problem.

10. How could the Gale-Shapley algorithm apply to college admissions?