MSJ Math Club

Week 18: "Fact 5"

April 25, 2013

1 "Fact 5"

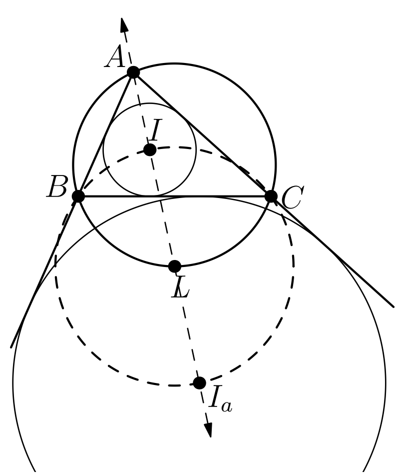
The lemma in geometry which is commonly referred to as "Fact 5" comes from a 2011 MOP Handout by Carlos Shine. The 5^{th} fact listed is the following lemma, which comes up *very* frequently in olympiad geometry.

Lemma: In a triangle ABC, let I be the incenter of the triangle, and L be the midpoint of the arc BC of the circumcircle of ABC. Furthermore, let I_a be the A-excenter of triangle ABC. (The excircle of a triangle is a circle that is tangent to the extensions of the sides of the triangle.) Then:

(i) Point A, I, L, and I_a are collinear

(ii) Quadrilateral $BICI_a$ is cyclic

(iii) Point L is the center of the circumcircle of $BICI_a$



Proof:

(i) Because L is the midpoint of the arc BC, line AL is the angle bisector of angle BAC. Furthermore, since both the incircle and excircle are tangent to the sides AB and AC, by symmetry, their centers also lie on the angle bisector.

(ii) We will prove that the point L is equidistant from all four points. See the next proof.

(iii) Because L is the midpoint of the arc BC, $LB = LC$. Also, we have that $\angle IBL = \angle IBC + \angle CBL = \angle ABI + \angle CAL = \angle ABI + \angle IAB = \angle LIB$, so triangle BLI is isosceles, showing that $LI = LB = LC$. A similar angle chasing procedure can be used to show that $LI_a = LI = LB = LC$.

2 Tips and Tricks

- There isn't much to say besides to recognize this configuration. When you suspect that such a lemma can be useful (for example if you see a midpoint of an arc), complete the configuration, and perhaps it will come in handy.

3 Practice Problems

1. (CGMO 2012) The incircle of ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of BCI. Prove that $\angle ODB = \angle OEC$.
2. (HMMT 2011) Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.
3. In triangle ABC, the angle bisector AD (with D on side BC) hits the circumcircle of ABC at point L. Show that $\triangle LAB \sim \triangle LBD$.
4. In cyclic quadrilateral $ABCD$, AC bisects angle BAD. Point F is on AB such that CF and AB are perpendicular. If $AF = 2005$ and $AB = 2006$, find AD.
5. (CHMMC Spring 2012) In triangle ABC, the angle bisector from A and the perpendicular bisector of BC meet at point D, the angle bisector from B and the perpendicular bisector of AC meet at point E, and the perpendicular bisectors of BC and AC meet at point F. Given that $\angle ADF = 5°$, $\angle BEF = 10°$, and $AC = 3$, find the length of DF.
6. (AIME 1983) Chords AD and BC of the same circle intersect. Suppose that the radius of the circle is 5, that $BC = 6$, and that AD is bisected by BC. Suppose further that AD is the only chord starting at A which is bisected by BC. Find the sine of the minor arc AB.
7. (NIMO 2012) In cyclic quadrilateral $ABXC$, $\measuredangle XAB = \measuredangle XAC$. Denote by I the incenter of $\triangle ABC$ and by D the projection of I on $\overline{BC}$. If $AI = 25$, $ID = 7$, and $BC = 14$, then find the length XI.
8. (OMO 2012) Let ABC be a triangle with circumcircle ω. Let the bisector of $\angle ABC$ meet segment AC at D and circle ω at $M \neq B$. The circumcircle of $\triangle BDC$ meets line AB at $E \neq B$, and CE meets ω at $P \neq C$. The bisector of $\angle PMC$ meets segment AC at $Q \neq C$. Given that $PQ = MC$, determine the degree measure of $\angle ABC$.
9. (HMMT 2013) Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD.
10. (ISL 2006) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

    $$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

    Show that $AP \geq AI$, and that equality holds if and only if $P = I$.