MSJ Math Club

Week 6: Qua(d)r(a)tics?

November 1, 2012

1 Tricks and/or Treats

- First method for solving: WolframAlpha. Not recommended.
- Second method: Memorize the quartic formula. Recommended, but probably very painful process.
- Many times on contests, if the problem explicitly gives you a quartic equation with the coefficients given, try splitting this into two quadratics. There are a sundry (yay SAT vocab) of ways that you can approach this, but a very common method is to take the quartic and cubic terms and find the appropriate quadratic term to complete the square. You can see this in action in example 1.
- The Sophie-Germain identity is a particularly well known factorization trick for polynomials of the form $x^4 + 4k^2 = (x^2 + 2kx + 2k)(x^2 - 2kx + 2k)$ for some given k. Whenever you see that lone x^4 term and a constant, it might help to think about how you could use this.
- It is also worth noting why they Sophie-Germain Identity works: difference of squares. The neat thing about this case is that the middle term cancels out. Using this technique is harder to think of, but it is a handy trick to keep in mind.
- When approaching polynomials in general, be sure to look at the coefficients closely, and ask yourself if you see anything recurring or any patterns. For example, the equal x^3 and x coefficients can often hint something... (see example 3).

2 Examples

1. Once upon a time, there was a pumpkin who was sad because he couldn't attend MSJ Math Club because he was too slow of a walker. To distract himself from his depression, he decided to expand a few polynomials. One day, he finished expanding a quartic, and fell asleep. When he woke up, he found that Hurricane Sandy blew his answer key away with all of the roots, but remembered the quartic was $4x^4 + 8n^3 + -5n^2 + 6n - 1 = 0$. Help the pumpkin factor his beloved quartic!
2. Solve for x:
$$5 - x^2 = \sqrt{5 - x}$$
3. Find the roots of $x^4 + 5x^3 + 4x^2 - 5x + 1 = 0$.

3 Practice Problems

1. Find the prime factors of $25^2 + 72^2$.
2. Factor $x^4 + 2x^3 + 3x^2 + 2x + 1$.
3. (SMT 2010) Bob sends a secret message to Alice using her RSA public key $n = 400000001$. Eve wants to listen in on their conversation. But to do this, she needs Alice's private key, which is the factorization of n. Eve knows that $n = pq$, a product of two prime factors. Find p and q.
4. (AIME 2005) Let P be the product of the non-real roots of $x^4 + 4x^3 + 6x^2 + 4x = 2005$. Find the largest integer less than or equal to P.
5. (SMT 2001) Find all solutions to $(x-3)(x-1)(x+3)(x+5) = 13$.

4 More Practice Problems!

This section is a demonstration of how powerful these techniques could be. All of these problems are from HMMT 2005 Algebra. That's right - 5 out of 10 problems on a single test involve quartics. In fact, these are problems 6-10 on the test - the hardest of the set.

6. Find the sum of the x-coordinates of the distinct points of intersection of the plane curves given by $x^2 = x + y + 4$ and $y^2 = y - 15x + 36$.
7. Let x be a positive real number. Find the maximum possible value of
$$\frac{x^2 + 2 - \sqrt{x^4 + 4}}{x}$$
8. Compute
$$\frac{0}{0^4 + 0^2 + 1} + \frac{1}{1^4 + 1^2 + 1} + \frac{2}{2^4 + 2^2 + 1} + \cdots + \frac{n}{n^4 + n^2 + 1} + \cdots .$$
9. The number $27,000,001$ has exactly four prime divisors. Find their sum.
10. Find the sum of the absolute values of the roots of $x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$.

5 :P

11. Show that all quartics with coefficients of $1, -2, 3, 4,$ and -6 in any order has a rational root.