Gauss's Law

MSJ Physics Club

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It should be noted that these handouts are concise and do not take the place of a full physics class. A bit of familiarity with calculus is assumed.

2 Gauss's Law

2.1 Electric Flux

The electric flux through a surface is equivalent to the area of the surface multiplied by the component of the electric field *normal to the surface.*

$$\Phi = A \cdot E$$

If the surface is curved or the electric field is not constant, we can just add up the fluxes of a lot of little surfaces

$$\Phi = \sum E \cdot \Delta S = \int E \cdot dS$$

Flux is a very useful tool as we will see in a later section (next week).

2.2 Gauss's Law

Let us try to calculate the total electric flux through a sphere enclosing a point charge $+q$. We have

$$\Phi = E \cdot A = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

Notice that this value does not depend on the radius of the sphere we take. In fact for any closed surface enclosing a charge q, we have

$$\Phi = \frac{q}{\epsilon_0}$$

2.2.1 Spherical Shell of Charge

Suppose we had a spherical shell of charge with radius R and charge Q, and we wanted to find the electric field in the environment of the sphere. Outside the sphere, the electric field is the same as that of a point charge

$$E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \longrightarrow E = \frac{Q}{\epsilon_0 4\pi r^2}$$

However, inside the sphere of charge, Q is zero (there is no enclosed charge), so

$$E = \frac{Q}{\epsilon_0 4\pi r^2} = 0$$

2.2.2 Thin, Infinite, and Uniform Rod of Charge

Suppose we had a infinite, thin, and uniform rod of charge, with charge density k (the amount of charge per unit length of rod) and we wanted to find the electric field in the environment of the rod. The closed surface we choose is that of a cylinder whose axis is the rod. Suppose the cylinder has height h and radius r. (Draw a picture)

The charge enclosed within the cylinder is equal to $k \cdot h$. We also know that, by symmetry, the electric flux through the caps of the cylinder is zero (as the electric field is parallel to the caps), and that the electric field at any two points on the curved surface are equal. Therefore, the electric field can be solved as follows:

$$E \cdot 2\pi rh = \frac{kh}{\epsilon_0} \longrightarrow E = \frac{k}{\epsilon_0 2\pi r}$$

2.2.3 Conductors and Gauss's Law

We claim that the electric field inside of a conductor at equilibrium is zero. Suppose there were an electric field within the conductor. Then since charge flow freely within a conductor and electrons are present everywhere inside a conductor, the charges must flow, and so the conductor is not in equilibrium (R.A.A.). Suppose we took a Gaussian surface that is almost equal to the surface of a charged conductor, but below the surface by an infinitesimal length dx. Because there is no electric field inside a conductor, the electric field at all points on the surface is zero, so the electric flux is zero, and therefore the enclosed charge is zero.

$$\Phi = 0 \longrightarrow Q = \Phi \cdot \epsilon_0 = 0$$

But the conductor has charge, so we conclude that all the charges in a conductor must be at the surface of the conductor.

3 Problems

1. An electron is fired from the origin from a plate (in the x-z plane) with initial velocity $\vec{v_0} = 2.83 \times 10^6(\hat{\imath} + \hat{\jmath})$ m/s. The electric field is in the positive y-direction and has a magnitude of 2.5×10^3 N/C. Where will the electron strike the plate?

2. Two small spheres of mass m are suspended from a common point by threads of length L. When each sphere carries a charge q, each thread makes an angle of θ with the vertical. Determine the charge q in terms of m, L, θ, and other fundamental constants.

3. A small bead of mass m carrying a negative charge q is constrained to move along a thin, frictionless rod. A distance L from the rod is a positive charge Q. Show that if the bead is displaced a distance x, where $x \ll L$, and released, it will exhibit simple harmonic motion. Obtain an expression for the period of this motion in terms of the parameters L, Q, q, and m.

4. Consider two infinitely long, concentric cylindrical shells. The inner shell has a radius R_1 and carries a uniform surface charge density of σ_1, and the outer shell has a radius R_2 and carries a uniform surface charge density of σ_2.

   (a) Find the electric field everywhere.

   (b) What is the ratio of the surface charge densities $\frac{\sigma_2}{\sigma_1}$ and their relative signs if the electric field is zero at $r > R_2$?

   (c) Sketch the electric field lines for the situation in part (b) if σ_1 is positive.

5. A thin nonconducting uniformly charged spherical shell of radius r has a total charge of Q. A small circular plug is removed from the surface.

   (a) What is the magnitude and direction of the electric field at the center of the hole?

   (b) Calculate the force acting on the plug.

   (c) Calculate the electrostatic pressure (force / unit area) tending to expand the sphere.

6. A charge Q is placed at the origin. Find the electric flux through the plane defined by $x = 2$, $|y| < 2$, $|z| < 2$.

7. A uniformly charged nonconducting sphere of radius a with center at the origin has volume charge density ρ.

   (a) Show that at a point within the sphere a distance r from the center $\vec{E} = \frac{\rho}{3\epsilon_0} r$.

   (b) Material is removed from the sphere, leaving a spherical cavity of radius $b = a/2$ with its center at $x = b$ on the x axis. Calculate the electric field at $(0, 0)$ and $(a, 0)$.

   (c) Show that the electric field throughout the cavity is uniform and is given by $\vec{E} = \frac{\rho}{3\epsilon_0} b\hat{\imath}$. (Hint: Replace the sphere-with-cavity with two uniform spheres of equal positive and negative charge densities)