Introduction to Error Analysis

Dustin Wheeler

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Homework Problems: Error Propagation

1. In each of the following cases, determine the answer and its error, assuming that the errors to the relevant quantities involved in the calculation are uncorrelated (*i.e.,* the errors add in quadrature).

This problem is from [1].

   (a) Determine the distance between the points $(0.0 \pm 0.2, 0.0 \pm 0.3)$ and $(3.0 \pm 0.3, 4.0 \pm 0.2)$, and the angle that the line joining them makes with the x axis.

   (b) The number, N, of particles surviving a distance, x, in a medium is given by

$$N_0 \exp(-x/\lambda) \ ,$$

   where N_0 is the number of particles at $x = 0$ and λ is the mean free path of a particle. What is N if $N_0 = (1000 \pm 5) \times 10^6$, $x = 1.00 \pm 0.01$ m, and $\lambda = 0.25 \pm 0.06$ m?

   (c) A particle travels along a straight-line trajectory given by

$$y = a + bx \, .$$

   If $a = 2.5 \pm 0.3$ m and $b = (5.0 \pm 0.1) \times 10^{-2}$, what is the value of y at:

   i. $x = 4$ m and
   ii. $x = 4.0 \pm 0.1$ m?

The following problems come from [2].

2. Evaluate each of the following:

   (a) $(5.6 \pm 0.7 + 3.70 \pm 0.03)$
   (b) $(5.6 \pm 0.7 + 2.3 \pm 0.1)$
   (c) $(5.6 \pm 0.7 + 4.1 \pm 0.2)$
   (d) $(5.6 \pm 0.7 + 1.9 \pm 0.3)$

   For each sum, consider both the case that the original uncertainties are independent and random (“errors add in quadrature”) and that they are not (“errors add directly”). Assuming the uncertainties are needed with only one significant figure, identify those cases in which the second of the original uncertainties can be ignored entirely. If you decide to do the additions in quadrature on a calculator, note that the conversion from rectangular to polar coordinates automatically calculates $\sqrt{x^2 + y^2}$ for a given x and y.

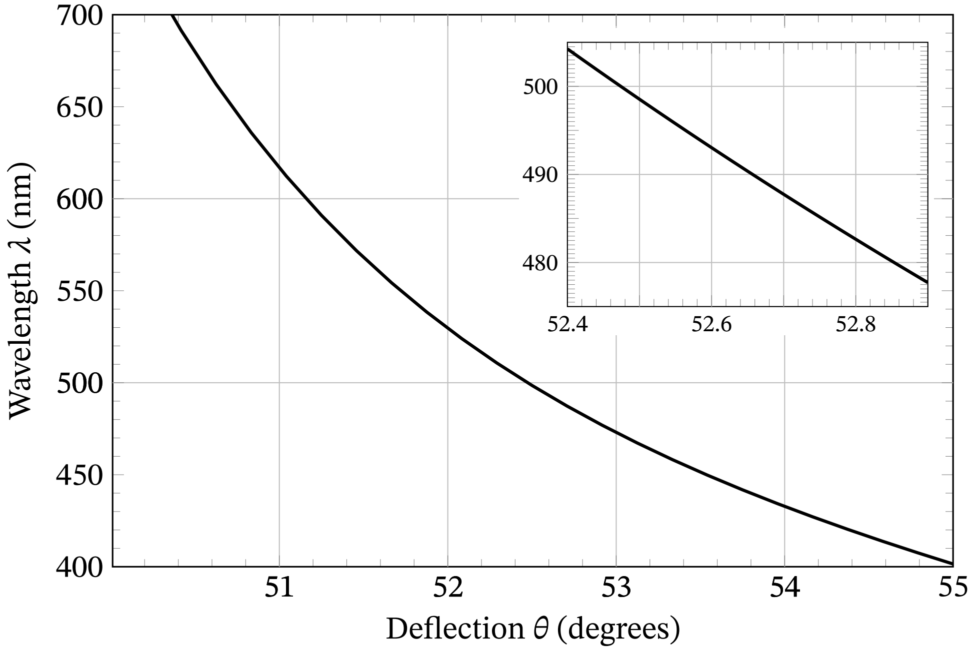

Figure 1: Calibration curve of wavelength λ against deflection θ for prob. (4).

3. A student makes the following measurements:

$$a = 5 \pm 1\,\text{cm}$$
$$b = 18 \pm 2\,\text{cm}$$
$$c = 12 \pm 1\,\text{cm}$$
$$t = 2.0 \pm 0.5\,\text{s}$$
$$m = 18 \pm 1\,\text{g}$$

   (a) Use the provisional rules (3.4 and 3.8 in the text) to compute the following quantities with their absolute and relative uncertainties (assuming the errors are correlated, *i.e., not* independent and random):

      i. $a + b + c$
      ii. $a + b - c$
      iii. ct
      iv. mb/t

   (b) Repeat the problem assuming that the original uncertainties *are* independent and random. Arrange your answers in a table to compare the two different methods of propagating errors.

4. A spectrometer is a device for separating the different wavelengths in a beam of light and measuring the wavelengths. It deflects the different wavelengths through different angles θ, and, if the relation between the angle θ and the wavelength λ is known, the experimenter can find λ by

measuring θ. Careful measurements with a certain spectrometer have established the calibration curve shown in fig. 1; this figure is simply a graph of λ (in nanometers, nm) agains θ, obtained by measuring θ for several accurately known wavelengths λ. A student directs a narrow beam of light from a hydrogen lamp through this spectrometer and finds that the light consists of just three well-defined wavelengths; that is, she sees three narrow beams (one red, one turquoise, and one violet) emerging at three different angles. She measures those angles as

$$\theta_1 = 51.0 \pm 0.2°$$
$$\theta_2 = 52.6 \pm 0.2°$$
$$\theta_3 = 54.0 \pm 0.2°$$

(a) Use the calibration curve in fig. 1 to find the corresponding wavelengths λ_1, λ_2, and λ_3 with their uncertainties.

(b) According to theory, these wavelengths should be 656 nm, 486 nm and 434 nm. Are the student's measurements in satisfactory agreement with these theoretical values?

(c) If the spectrometer has a vernier scale to read the angles, the angles can be measured with an uncertainty of 0.05° or even less. Let us suppose the three measurements above have uncertainties of ±0.05°. Given this new, smaller uncertainty in the angles and *without drawing any more lines on the graph,* use your answers from prob. (4a) to find the new uncertainties in the three wavelengths, explaining clearly how you do it.[1]

(d) To take advantage of more accurate measurements, an experimenter may need to enlarge the calibration curve. The inset in fig. 1 is an enlargement in the vicinity of the angle θ_2. Use this graph to find the wavelength λ_2 if θ_2 has been measured as $52.72 \pm 0.05°$; check that your prediction for the uncertainty of λ_2 in prob. (4c) was correct.

[1] Hint: the calibration curve is nearly straight in the vicinity of any one measurement.

5. If you measure two independent variables as

$$x = 6.0 \pm 0.1 \quad \text{and} \quad y = 3.0 \pm 0.1\,,$$

and use these values to calculate $q = xy + x^2/y$, what will be your answer and its uncertainty?[2]

[2] You must use the general rule (3.47 in [2] to find δq. To simplify your calculation, do it first by finding the two separate contributions δq_x and δq_y as defined in rules 3.50 and 3.51, then combine them inquadrature.

6. If an object is placed at a distance p from a lens and an image is formed at a distance q from the lens, the lens's focal length can be found as[3]

$$f = \frac{pq}{p+q}\,. \tag{1}$$

[3] This equation follows from the "lens equation", $1/f = (1/p) + (1/q)$.

(a) Use the general rule (3.47) to derive a formula for the uncertainty δf in terms of p, q, and their uncertainties.

(b) Starting from eq. (1) directly, you cannot find δf in steps because p and q both appear in numerator and denominator. Show, however, that f can be rewritten as

$$f = \frac{1}{1/p + 1/q}\,.$$

Starting from this form, you *can* evaluate δf in steps. Do so, and verify that you get the same answer as in prob. (6a).

References

[1] Louis Lyons. *Data Analysis for Physical Science Students (Cambridge*. Cambridge University Press, 1991.

[2] John R Taylor. *An Introduction to Error Analysis*. 2nd ed. University Science Books, Sausalito, CA, 1997.