Optimization and Algorithms
Part 3 of the Project

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1 Projecting onto a disk

In Task 5, Part 1 of the project, you were asked to find a closed-form expression for the distance from a point p to a disk $D(c,r) = \{x\colon\ \|x-c\|_2 \leq r\}$.

You can solve this problem by first computing the projection of the point onto the disk, that is, by first finding a closed-form solution to the problem

$$\begin{array}{ll} \underset{y \in \mathbf{R}^n}{\text{minimize}} & \|p-y\|_2 \\ \text{subject to} & y \in D(c,r). \end{array} \tag{1}$$

With the solution for problem (1) in hand—call it $y^\star$—the distance from p to $D(c,r)$ is $\|p-y^\star\|_2$.

Task 1. Use the KKT conditions to solve problem (1), that is, to find a closed-form solution for $y^\star$.

2 Solving quickly a trajectory problem

Consider the following three problems:

- Problem A:

$$
\begin{array}{ll}
\underset{x,u}{\text{minimize}} & \sum_{t=1}^{T-1} \|x(t) - x_{\text{des}}(t)\|_2^2 + \lambda \sum_{t=0}^{T-1} \|u(t)\|_2^2 \\
\text{subject to} & x(0) = x_{\text{initial}} \\
& x(T) = x_{\text{final}} \\
& x(t+1) = Ax(t) + Bu(t), \quad \text{for } 0 \le t \le T-1;
\end{array}
\tag{2}
$$

- Problem B:

$$
\begin{array}{ll}
\underset{x,u}{\text{minimize}} & \sum_{t=1}^{T-1} \|x(t) - x_{\text{des}}(t)\|_2^2 + \lambda \sum_{t=0}^{T-1} \|u(t)\|_2 \\
\text{subject to} & x(0) = x_{\text{initial}} \\
& x(T) = x_{\text{final}} \\
& x(t+1) = Ax(t) + Bu(t), \quad \text{for } 0 \le t \le T-1;
\end{array}
\tag{3}
$$

- Problem C:

$$
\begin{array}{ll}
\underset{x,u}{\text{minimize}} & \sum_{t=1}^{T-1} \|x(t) - x_{\text{des}}(t)\|_2^2 + \lambda \sum_{t=0}^{T-1} \|u(t)\|_1 \\
\text{subject to} & x(0) = x_{\text{initial}} \\
& x(T) = x_{\text{final}} \\
& x(t+1) = Ax(t) + Bu(t), \quad \text{for } 0 \le t \le T-1.
\end{array}
\tag{4}
$$

In these three problems (which can be interpreted as variations of the problems (2), (6), and (7) from Part 1 of the project), the vector x_{initial}, the vector x_{final}, the vectors $x_{\text{des}}(t)$ for $1 \le t \le T-1$, and the positive number λ are given constants.

Task 2. Find a closed-form solution to one of the problems A, B, or C. You have to solve only one of these three problems; you choose which one.