Second report

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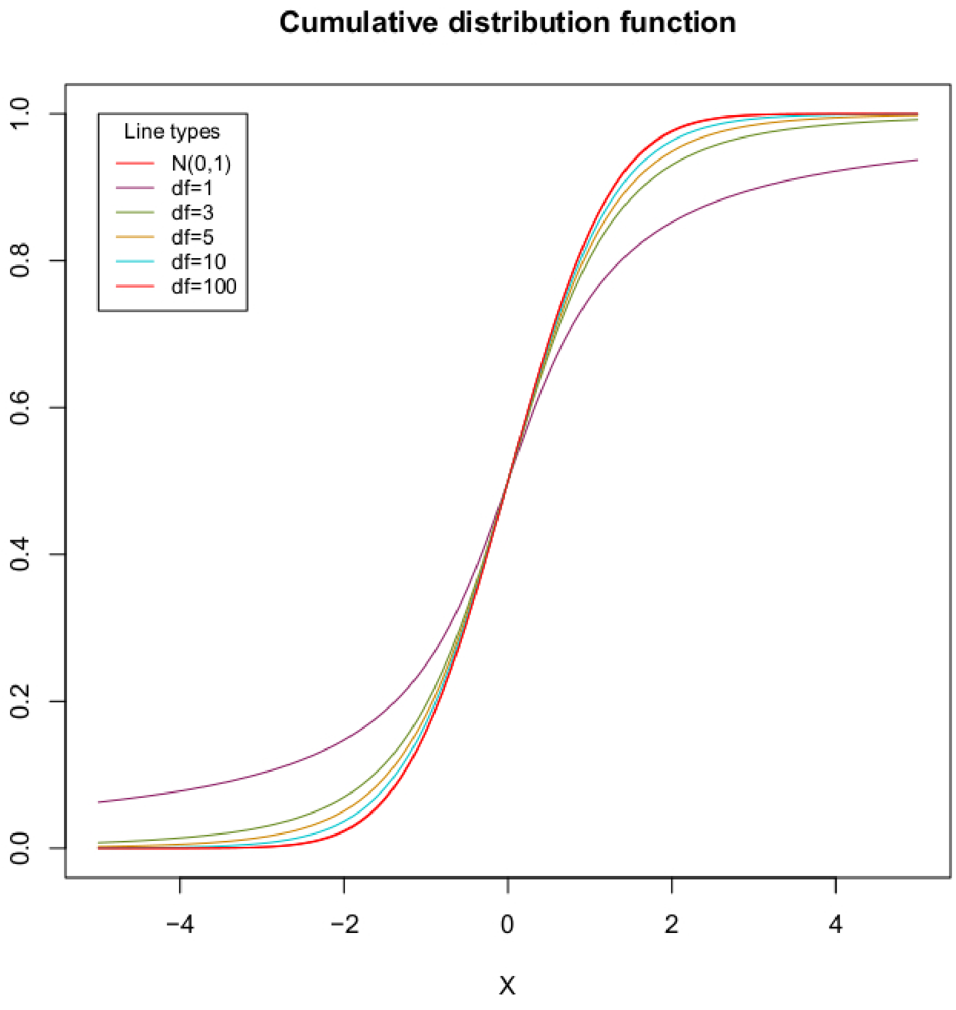
Exercise 1.

We draw one graph with cumulative distribution function (cdf) of the standard normal distribution and the Student's distribution with different degrees of freedom.

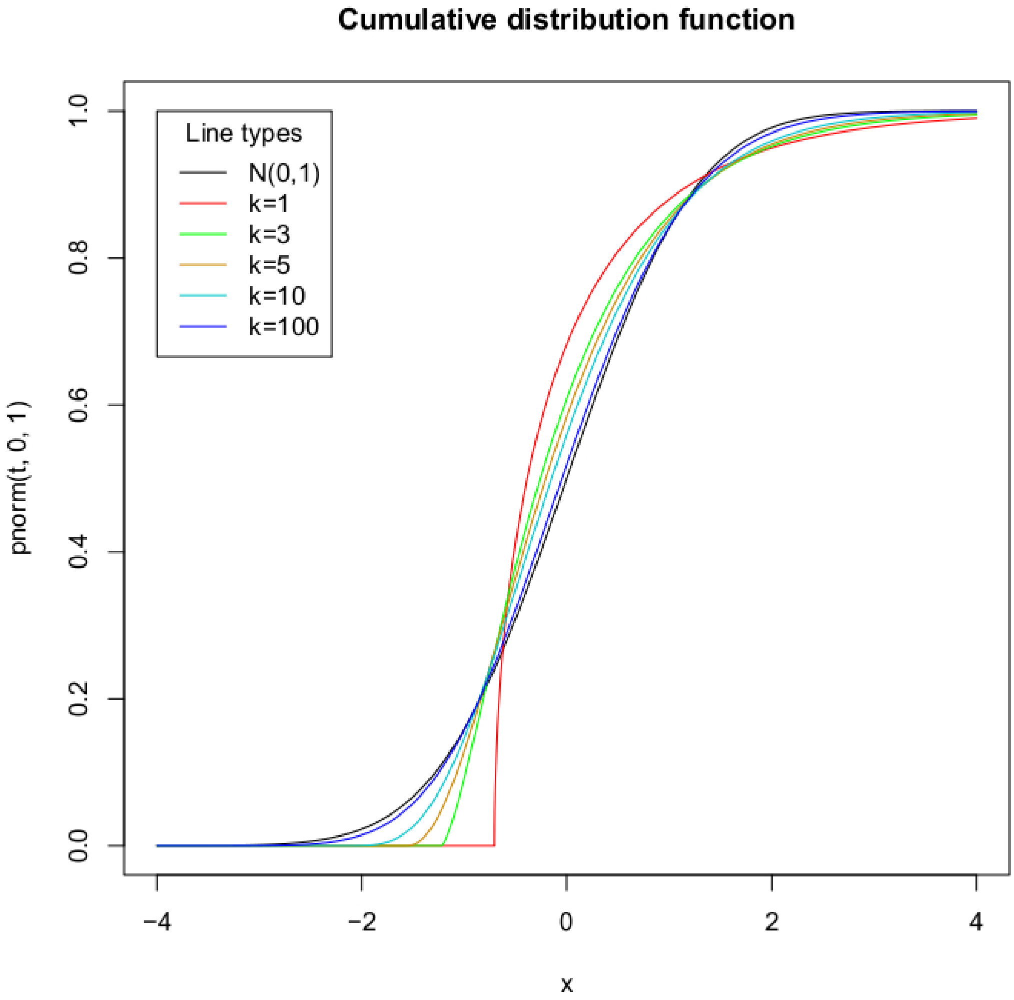


As you can see, the student's t-distribution converges to the normal distribution with increasing degrees of freedom. Therefore, in the graph, we have marked with the same color cdf for the standard normal distribution and for a student's t-distribution with 100 degrees of freedom.

Exercise 2.

We draw one graph with the cumulative distribution function (cdf) of the standard normal distribution and the chi-square distribution with different degrees of freedom. We use standardization for this:

$$T = \frac{\chi^2_{df} - df}{\sqrt{2df}}.$$



As you can see, the chi-square distribution coincides with the normal distribution with increasing degrees of freedom.

Exercise 3.

Let $X_1, \ldots, X_{100}$ be the sample from the Poisson distribution. Let's consider the test for the hypothesis:

$$H_{0i} : \mathrm{E}(X_i) = 5 \text{ against } H_{Ai} : \mathrm{E}(X_i) > 5$$

which rejects the null hypothesis for large values of $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Let's find the test statistic:

$$\frac{\prod_{i=1}^{n} \lambda_0^{x_i} e^{-\lambda_0}}{\prod_{i=1}^{n} \lambda_A^{x_i} e^{-\lambda_A}} = \frac{\lambda_0^{\sum_{i=1}^{n} x_i} e^{-\lambda_0}}{\lambda_A^{\sum_{i=1}^{n} x_i} e^{-\lambda_A}} = \left(\frac{\lambda_0}{\lambda_A}\right)^{\sum_{i=1}^{n} x_i} e^{\lambda_A - \lambda_0}$$

As you can see, our test statistic is :

$$T = \sum_{i=1}^{100} X_i$$

and

$$\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{100} \mathrm{E}(X_i) = 100 \cdot 5.$$

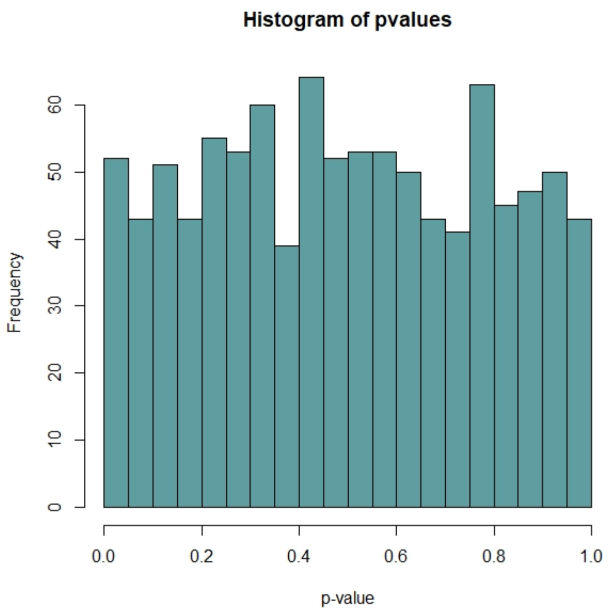
Now we're writing a function in R to compute the p-value for this test:

```
pv = ppois(sum(X), 100*5, lower.tail=FALSE)
```

For a random sample with $\lambda = 5$, we got a p-value of 0.9627065.
Now, let's consider 1000 of similar hypothesis and the meta problem of testing the global hypothesis $H_0 = \bigcup_{j=1}^{1000} H_{0j}$ and let's use simulations to estimate the probability of the type I error for the Bonferroni and Fisher tests at the significance level $\alpha = 0,05$.

	Bonferroni test	Fisher test
probability of the type I error	0.054	0.147

The probability of type 1 error for the Bonferroni test is close to α, but probability of type 1 error for the Fisher test is different from α. Let's see why? Let's draw histogram of p-values and discuss their distribution.



The Poisson distribution is not continuous, therefore p-values are not from uniform distribution. Hence the measurement error in the Fisher test.

Fisher's Test aggregates all of the p-values, rather than just looking at the minimum p-value. Hence, we expect this test to be powerful when there are many small effects, and less powerful when there are a few strong effects. In this sense, Fisher's Combination Test is the opposite of Bonferroni's test.

Let's use simulations to compare the power of the Bonferroni and Fisher tests for two alternatives:

- Needle in the haystack

$$\mathrm{E}(X_1) = 7; \text{ for } j \in \{2, \ldots, 1000\} \ \mathrm{E}(X_j) = 5$$

power of Bonferroni test	power of Fisher test
1	0.72

- Many small effects

$$\text{for } j \in \{2, \ldots, 100\} \ \mathrm{E}(X_j) = 5.5 \ \text{ and for } j \in \{101, \ldots, 1000\}\, \mathrm{E}(X_j) = 5$$

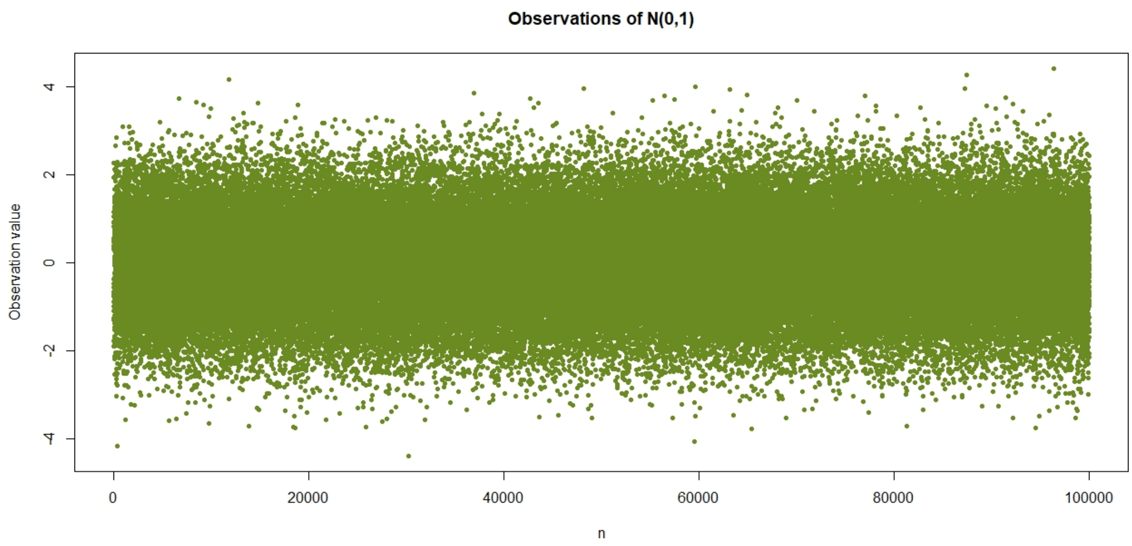
power of Bonferroni test	power of Fisher test
0.91	1

Due to the use of the minimum in the construction of the critical area, the Bonferonni's test is sensitive to a single small deviation from the null hypothesis and thus is not susceptible to many small deviations from the null hypothesis.

The Fisher test works quite the opposite, it is not susceptible to a single, strong premise for rejecting the null hypothesis, but it is perfect for testing a situation where you have many groups that are close to the null hypothesis.

Exercise 4.

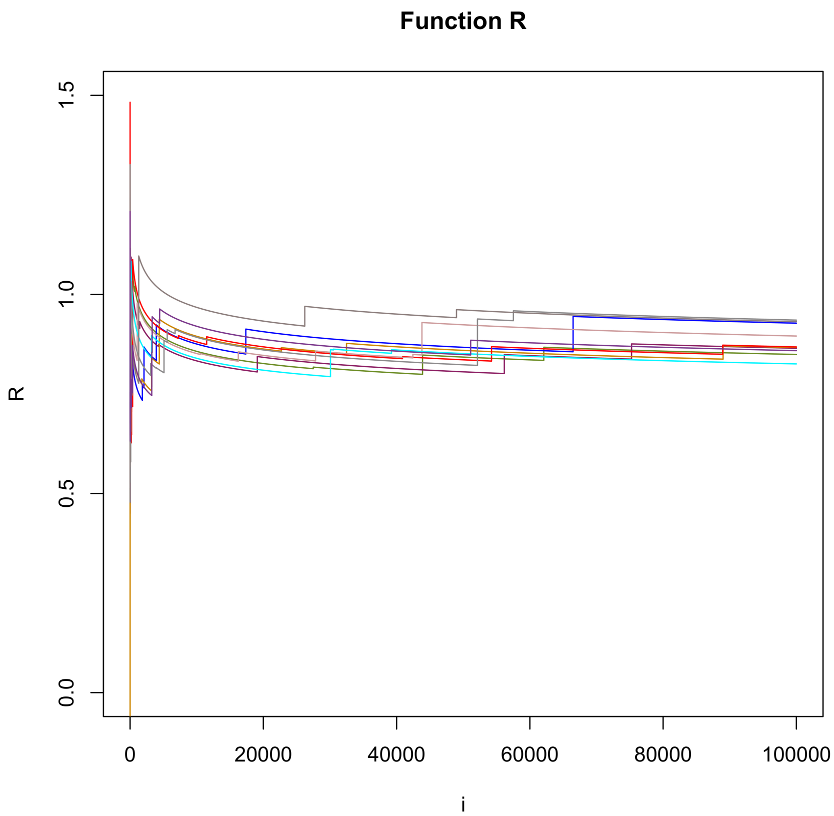
Let's draw 100000 iid variables $X_1, \ldots, X_{100000}$ from $N(0,1)$.



Now, for $i \in \{2, \ldots, 100000\}$ let's plot the graph of the function

$$R_i = \frac{\max\{X_j, j = 1, \ldots, i\}}{\sqrt{2 \log i}}.$$

Let's repeat the above experiment 10 times.



There is some convergence in the above graph, but it cannot be clearly identified. Perhaps we have too little observation. However, we know from the lecture theories that: $\frac{\max X_j}{\sqrt{2 \log[i]}} \to 1$.

Exercise 5.

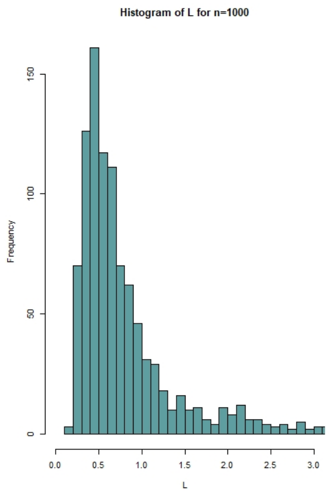
Let $Y = (Y_1, \dots, Y_n)$ be the random vector from $N(\mu, I)$ distribution. Let

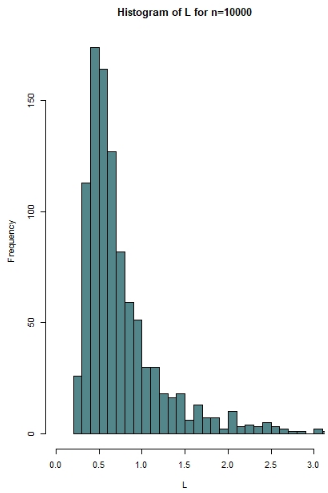
$$L(Y) = \frac{1}{n} \sum_{i=1}^{n} e^{\gamma Y_i - \gamma^2/2}$$

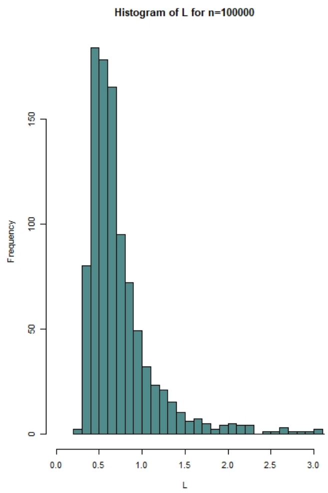
is the statistics of the optimal Neyman-Pearson test for the hypothesis $H_0 : \mu = 0$ for the classical needle in haystack problem H_A - one of the elements of mu is equal to γ. And let

$$\hat{L}(Y) = \frac{1}{n} \sum_{i=1}^{n} \left[e^{\gamma Y_i - \gamma^2/2} \mathbf{1}_{Y_i < \sqrt{2 \log n}} \right]$$

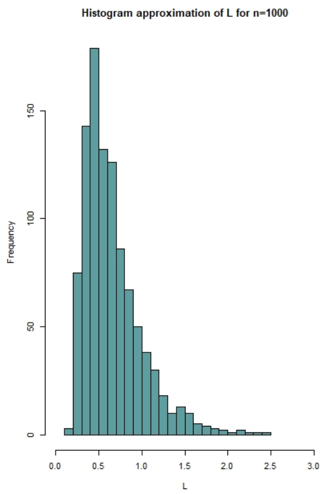
it's approximation of L. Let's use 1000 replicates to draw the histogram and calculate variances of L and $\hat{L}$ under the null hypothesis for $\gamma = (1 - \epsilon)\sqrt{2 \log n}$ with $\epsilon = 0.1$ and $n = \{1000, 10000, 100000\}$.

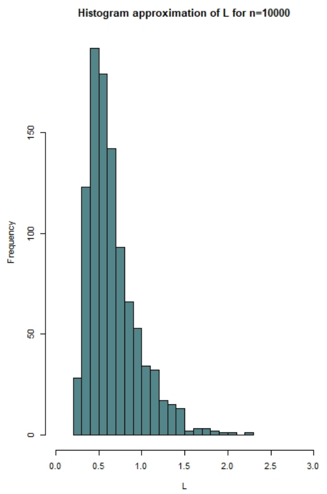


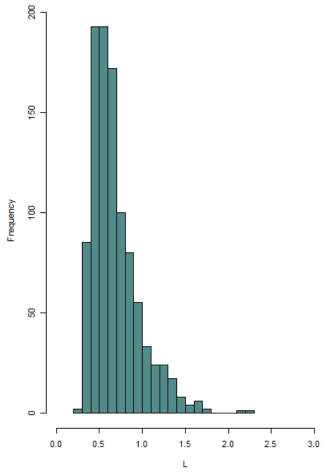




$Var(L_{1000}) = 2.214$ $\quad$ $Var(L_{10000}) = 1.5$ $\quad$ $Var(L_{100000}) = 1.05$







$Var(\hat{L}_{1000}) = 0.09787$ $\quad$ $Var(\hat{L}_{10000}) = 0.08135$ $\quad$ $Var(\hat{L}_{100000}) = 0.0715$

As you can see, the histograms for the respective ns hardly differ.

	n=1000	n=10000	n=100000
$P_{H0}(L = \hat{L})$	0.907	0.9173	0.9238

And this agrees with the theory that $P_{H0}(L = \hat{L}) \to 1$.

Exercise 6.

Let's use simulations to find the critical value of the optimal Neyman Pearson test and let's compare the power of this test and the Bonferroni test for the needle in haystack problem with $n \in \{500, 5000, 50000\}$ and the needles:

- $\mu_1 = 1.05\sqrt{2\log n}$ and $\mu_2 = \ldots = \mu_n = 0$

	$n = 500$	$n = 5000$	$n = 50000$
Neyman Pearson power	0.5186	0.5562	0.5791
Bonferroni power	0.4532	0.4919	0.5183

- $\mu_1 = 1.2\sqrt{2\log n}$ and $\mu_2 = \ldots = \mu_n = 0$

	$n = 500$	$n = 5000$	$n = 50000$
Neyman Pearson power	0.7096	0.7622	0.8089
Bonferroni power	0.654	0.7229	0.7678

As you can see, regardless of μ_1, the power of the Neyman Pearson test is greater than that of the Bonferroni test. So the Neyman Pearson test is better than the Bonferroni test because it is better suited for testing single deviations from the null hypothesis (higher probability of rejecting the null hypothesis when it is false).

It also follows that the larger the single outlier, the greater the power of both tests.