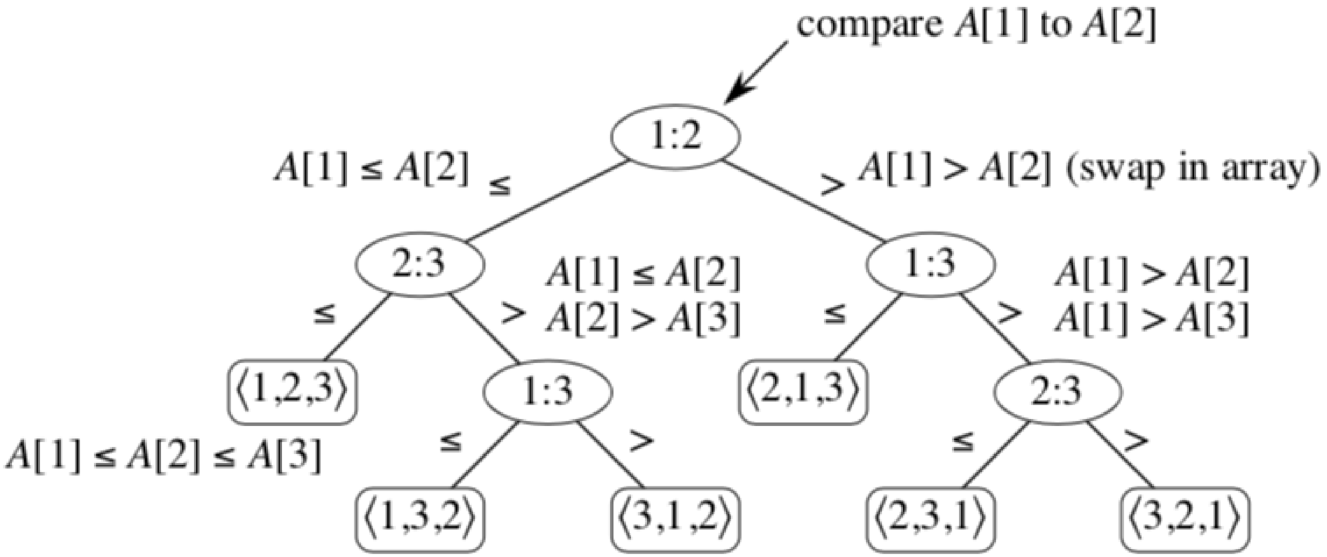
Sorting in Linear Time

counting sort, radix sort, bucket sort

모든 비교 정렬 알고리즘은 최악의 경우 $\Omega(n \lg n)$ 번의 비교가 필요하다.

the decision-tree model for comparison sort



l : # of leaves

h : height of decision-tree = 최악의 경우 비교 횟수

정렬 알고리즘의 실행은 결정 트리의 루트에서 하나의 리프까지 경로를 따라가는 것

$$n! \le l \le 2^h$$

$$\begin{aligned} h &\ge \lg(n!) \\ &= \Omega(n \lg n) \end{aligned}$$

by eq.3.19 $\lg(n!) = \Theta(n \lg n)$

Counting Sort

- 각 원소는 [0, k] 인 정수인 경우
- 각 원소에 대해 그보다 작은 원소의 갯수를 세면 정렬 후 원소의 위치를 알 수 있다.

```
let C[0..k] be a new array
for i = 0 to k
    C[i] = 0
for j = 1 to A.length
    C[A[j]] = C[A[j]] + 1
```

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C	2	0	2	3	0	1

Counting Sort - 2

```
for i = 1 to k
        C[i] = C[i] + C[i - 1]
```

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C	2	0	2	3	0	1

(a)

	0	1	2	3	4	5
C	2	2	4	7	7	8

(b)

Counting Sort - 3

```
for j = A.length downto 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1
```

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C	2	2	4	7	7	8

	1	2	3	4	5	6	7	8
B							3	

	0	1	2	3	4	5
C	2	2	4	6	7	8

	1	2	3	4	5	6	7	8
B		0					3	

	0	1	2	3	4	5
C	1	2	4	6	7	8

	1	2	3	4	5	6	7	8
B		0				3	3	

	0	1	2	3	4	5
C	1	2	4	5	7	8

	1	2	3	4	5	6	7	8
B	0	0	2	2	3	3	3	5

Counting Sort

- stable sort : 출력 배열에서 값이 같은 숫자가 입력 배열에 있던 것과 같은 순서를 유지하는 정렬
- running time : $\Theta(k+n)$

if $k = O(n)$, $\Theta(n)$

COUNTING-SORT(A, B, k)

```
let C[0..k] be a new array
for i = 0 to k
    C[i] = 0
for j = 1 to A.length
    C[A[j]] = C[A[j]] + 1
// C[i] now contains the number of elements equal to i.
for i = 1 to k
    C[i] = C[i] + C[i - 1]
// C[i] now contains the number of elements less than or equal to i.
for j = A.length downto 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1
```

Radix Sort

- 가장 낮은 자리 숫자부터 정렬하는 것을 자리수만큼 반복

```
RADIX-SORT(A, d)
for i = 1 to d
    use a stable sort to sort array A on digit i
```

329		720		720		329
457		355		329		355
657		436		436		436
839	→	457	→	839	→	457
436		657		355		657
720		329		457		720
355		839		657		839

- 각각의 정렬은 stable 해야 함. counting sort 사용
- running time : $\Theta(d(n+k))$ total

  If $k = O(n)$, time $= \Theta(dn)$

높은 자리부터 정렬하면...

34	17	31
17	25	34
25	34	25
31	31	17

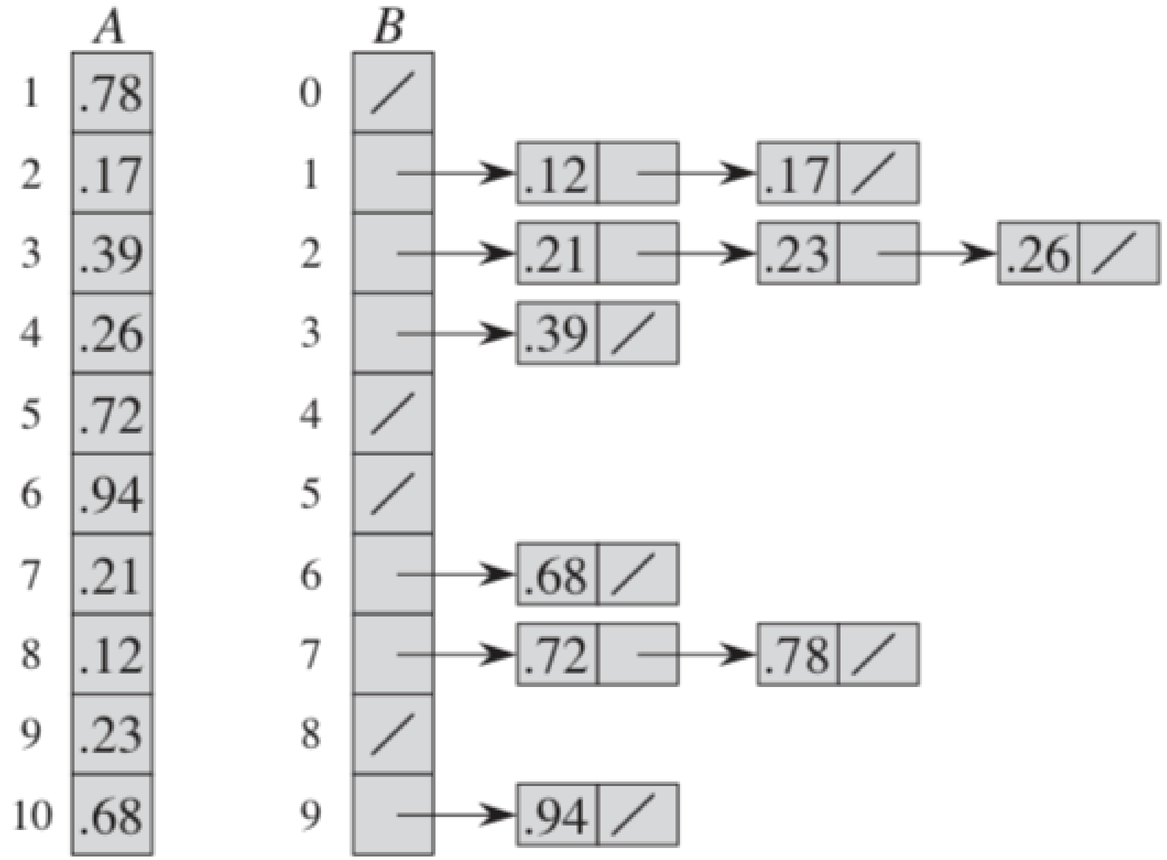
Bucket Sort

A.length = B.length 라는 것이 복잡도 계산의 핵심

- 입력이 uniformly distributed in [0,1) 인 경우

BUCKET-SORT(A)

let $B[0\,..\,\boxed{n} - 1]$ be a new array
$n = \boxed{A.length}$
for $i = 0$ **to** $n - 1$
    make $B[i]$ an empty list
for $i = 1$ **to** n
    insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
for $i = 0$ **to** $n - 1$
    sort list $B[i]$ with insertion sort
concatenate the lists $B[0], B[1], \ldots, B[n-1]$ together in order



Time complexity of bucket sort

- n_i : # of elements in B[i]
- insertion sort on runs in $O(n^2)$

→ $$T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2) .$$

- average case running time

- $\mathrm{E}\left[n_i^2\right] = 2 - 1/n$ 이므로

```
BUCKET-SORT(A)
let B[0..n − 1] be a new array
n = A.length
for i = 0 to n − 1
    make B[i] an empty list
for i = 1 to n
    insert A[i] into list B[⌊nA[i]⌋]
for i = 0 to n − 1
    sort list B[i] with insertion sort
concatenate the lists B[0], B[1], ..., B[n − 1] together in order
```

$$\begin{aligned}
\mathrm{E}[T(n)] &= \mathrm{E}\left[\Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)\right] \\
&= \Theta(n) + \sum_{i=0}^{n-1} \mathrm{E}\left[O(n_i^2)\right] \\
&= \Theta(n) + \sum_{i=0}^{n-1} O\left(\mathrm{E}\left[n_i^2\right]\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[T(n)] &= \Theta(n) + \sum_{i=0}^{n-1} O(2 - 1/n) \\
&= \Theta(n) + O(n) \\
&= \Theta(n)
\end{aligned}$$

$\mathrm{E}[n_i^2] = 2 - 1/n$ in bucket sort

$$X_{ij} = \mathrm{I}\{A[j] \text{ falls in bucket } i\}$$

for $i = 0, 1, \ldots, n-1$ and $j = 1, 2, \ldots, n$. Thus,

$$n_i = \sum_{j=1}^{n} X_{ij} .$$

$$\begin{aligned}
\mathrm{E}[n_i^2] &= \mathrm{E}\left[\left(\sum_{j=1}^{n} X_{ij}\right)^2\right] \\
&= \mathrm{E}\left[\sum_{j=1}^{n}\sum_{k=1}^{n} X_{ij} X_{ik}\right] \\
&= \mathrm{E}\left[\sum_{j=1}^{n} X_{ij}^2 + \sum_{1 \le j \le n} \sum_{\substack{1 \le k \le n \\ k \ne j}} X_{ij} X_{ik}\right] \\
&= \sum_{j=1}^{n} \mathrm{E}[X_{ij}^2] + \sum_{1 \le j \le n} \sum_{\substack{1 \le k \le n \\ k \ne j}} \mathrm{E}[X_{ij} X_{ik}]
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[X_{ij}^2] &= 1^2 \cdot \frac{1}{n} + 0^2 \cdot \left(1 - \frac{1}{n}\right) \\
&= \frac{1}{n} .
\end{aligned}$$

When $k \ne j$, the variables X_{ij} and X_{ik} are independent

$$\begin{aligned}
\mathrm{E}[X_{ij} X_{ik}] &= \mathrm{E}[X_{ij}]\,\mathrm{E}[X_{ik}] \\
&= \frac{1}{n} \cdot \frac{1}{n} \\
&= \frac{1}{n^2} .
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[n_i^2] &= \sum_{j=1}^{n} \frac{1}{n} + \sum_{1 \le j \le n} \sum_{\substack{1 \le k \le n \\ k \ne j}} \frac{1}{n^2} \\
&= n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n^2} \\
&= 1 + \frac{n-1}{n} \\
&= 2 - \frac{1}{n} ,
\end{aligned}$$

worst-case running time of bucket sort

- n_i : # of elements in B[i]
- insertion sort on runs in $O(n^2)$

→ $$T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2) .$$

- worst case running time $\Theta(n^2)$ when ?

```
BUCKET-SORT(A)
let B[0..n-1] be a new array
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    make B[i] an empty list
for i = 1 to n
    insert A[i] into list B[⌊nA[i]⌋]
for i = 0 to n-1
    sort list B[i] with insertion sort
concatenate the lists B[0], B[1], ..., B[n-1] together in order
```

Running times of sorting algorithms

Algorithm	Worst-case running time	Average-case/expected running time
Insertion sort	$\Theta(n^2)$	$\Theta(n^2)$
Merge sort	$\Theta(n \lg n)$	$\Theta(n \lg n)$
Heapsort	$O(n \lg n)$	—
Quicksort	$\Theta(n^2)$	$\Theta(n \lg n)$ (expected)
Counting sort	$\Theta(k + n)$	$\Theta(k + n)$
Radix sort	$\Theta(d(n + k))$	$\Theta(d(n + k))$
Bucket sort	$\Theta(n^2)$	$\Theta(n)$ (average-case)