Recurrence(점화식)

- 더 작은 입력에 대한 자신의 값으로 함수를 나타내는 방정식 혹은 부등식

치환법(Substitution Method)

- 해의 모양을 추측한다
- 상수들의 값을 찾아내기 위해 수학적 귀납법을 사용하고 그 해가 제대로 동작함을 보인다

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n > 1 \end{cases}$$

1. *Guess:* $T(n) = n \lg n + n$
2. *Induction:*

   Basis: $n = 1 \Rightarrow n \lg n + n = 1 = T(n)$

   Inductive step: Inductive hypothesis is that $T(k) = k \lg k + k$ for all $k < n$. We'll use this inductive hypothesis for $T(n/2)$.

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + n \\ &= 2\left(\frac{n}{2}\lg\frac{n}{2} + \frac{n}{2}\right) + n \qquad \text{(by inductive hypothesis)} \\ &= n\lg\frac{n}{2} + n + n \\ &= n(\lg n - \lg 2) + n + n \\ &= n\lg n - n + n + n \\ &= n\lg n + n . \qquad \blacksquare \end{aligned}$$

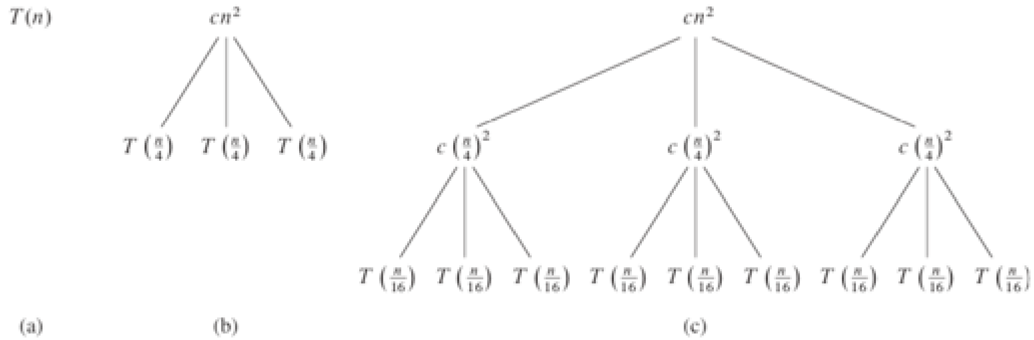
재귀 트리 방법(Recursion tree Method)

- 치환법의 1번 단계의 추측을 하기 위해 재귀 트리를 그린다
  - 재귀 트리의 각 노드는 재귀 호출되는 하위 문제 하나의 비용을 나타냄
- 상수들의 값을 찾아내기 위해 수학적 귀납법을 사용하고 그 해가 제대로 동작함을 보인다

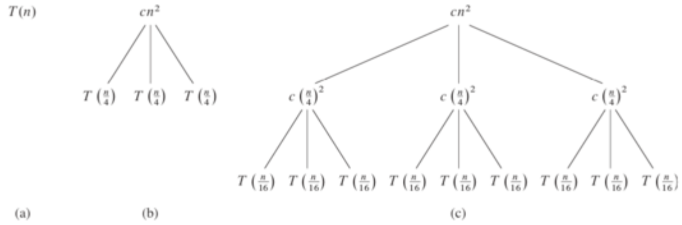
$$T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$$

rewrite as

$$T(n) = 3T(n/4) + cn^2$$



$$T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$$



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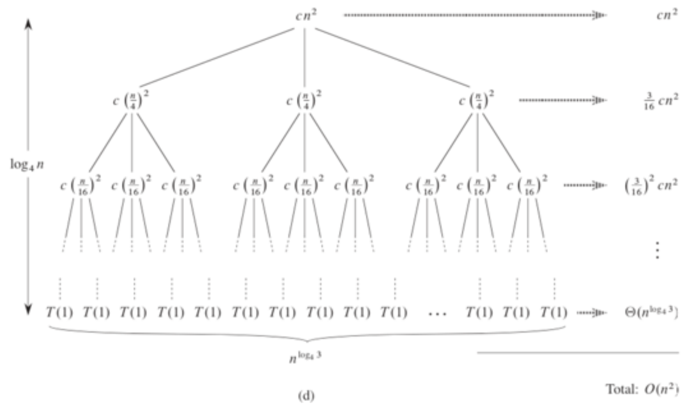
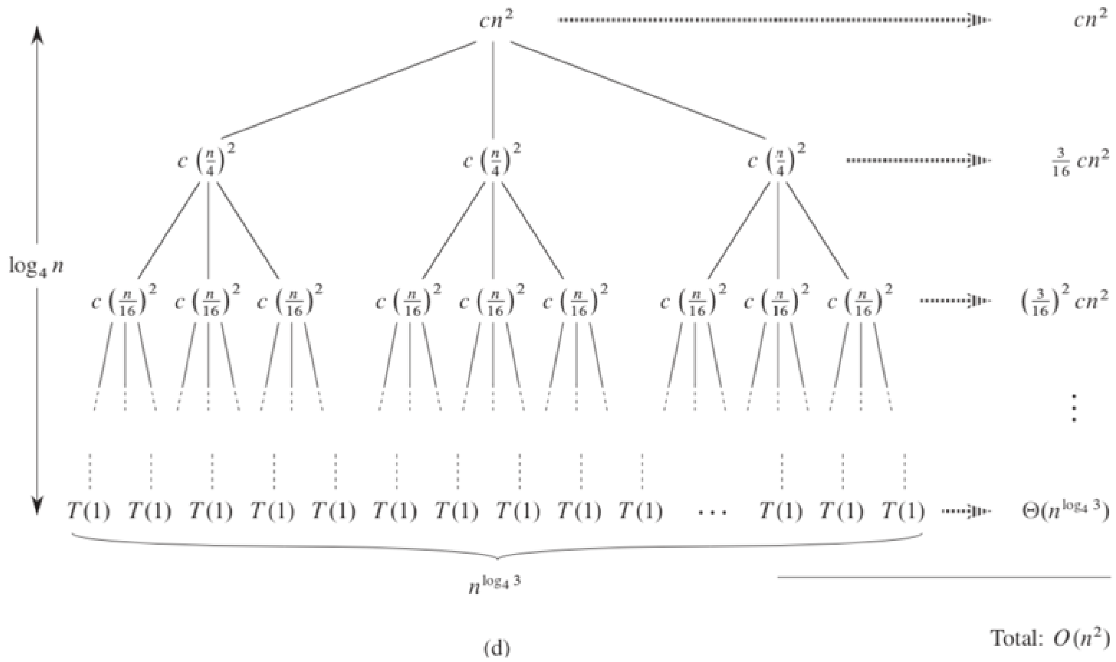
$$T(n) = 3T(n/4) + cn^2$$



$$T(n) = 3T(n/4) + cn^2$$



$$
\begin{aligned}
T(n) &= cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n - 1} cn^2 + \Theta(n^{\log_4 3}) \\
&= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{(3/16)^{\log_4 n} - 1}{(3/16) - 1} cn^2 + \Theta(n^{\log_4 3}) \qquad \text{(by equation (A.5))} .
\end{aligned}
$$

or,

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{1}{1 - (3/16)} cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{16}{13} cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2) .
\end{aligned}
$$

- 수학적 귀납법을 통한 확인

Show that $T(n) <= dn^2$ for some constant d

$$
\begin{aligned}
T(n) &\le 3T(\lfloor n/4 \rfloor) + cn^2 \\
&\le 3d \lfloor n/4 \rfloor^2 + cn^2 \\
&\le 3d(n/4)^2 + cn^2 \\
&= \frac{3}{16} dn^2 + cn^2 \\
&\le dn^2 ,
\end{aligned}
$$

where the last step holds as long as $d \ge (16/13)c$.

마스터 방법(Master Method)

$T(n) = aT(n/b) + f(n)$,

where $a \ge 1$, $b > 1$, and $f(n) > 0$.

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

- Examples

- $T(n) = 9T(n/3) + n$ case 1. Since $f(n) = O(n^{\log_3 9-\epsilon})$, where $\epsilon = 1$

→ $T(n) = \Theta(n^2)$.

- $T(n) = T(2n/3) + 1$ case 2. since $f(n) = \Theta(n^{\log_b a}) = \Theta(1)$

→ $T(n) = \Theta(\lg n)$.

- $T(n) = 5T(n/2) + \Theta(n^3)$ case 3. $\lg 5 + \epsilon = 3$ for some constant $\epsilon > 0$

$af(n/b) = 5(n/2)^3 = 5n^3/8 \le cn^3$ for $c = 5/8 < 1$

→ $T(n) = \Theta(n^3)$