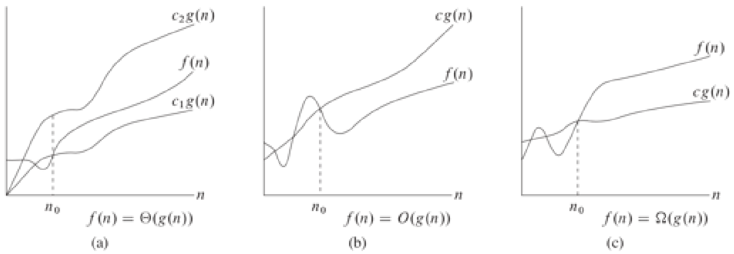
점근적 표기(Asymptotic Notation)

- To describe growth of functions and to compare functions
- 함수들의 독립변수가 아주 커졌을 때, 함수값의 크기를 비교하는데 사용

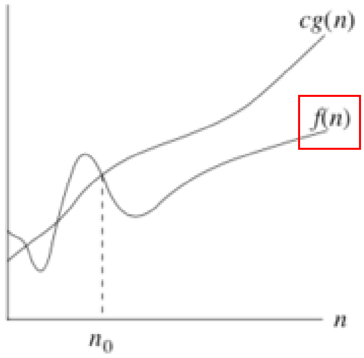


$f(n) = \boxed{O}(g(n))$	is like	$a \le b$,
$f(n) = \boxed{\Omega}(g(n))$	is like	$a \ge b$,
$f(n) = \boxed{\Theta}(g(n))$	is like	$a = b$,
$f(n) = \boxed{o}(g(n))$	is like	$a < b$,
$f(n) = \boxed{\omega}(g(n))$	is like	$a > b$.

O-Notation

***O*-notation**

$O(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that
$0 \le \boxed{f(n)} \le cg(n)$ for all $n \ge n_0\}$.

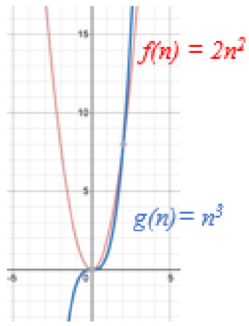


$g(n)$ is an ***asymptotic upper bound*** for $\boxed{f(n)}$.

If $f(n) \in O(g(n))$, we write $f(n) = O(g(n))$

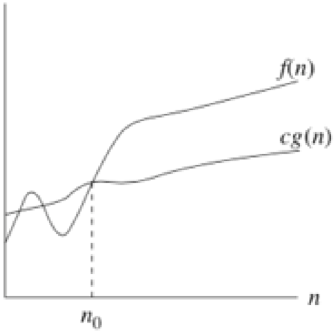
Example

$\boxed{2n^2} = O(n^3)$, with $c = 1$ and $n_0 = 2$.



Ω-Notation

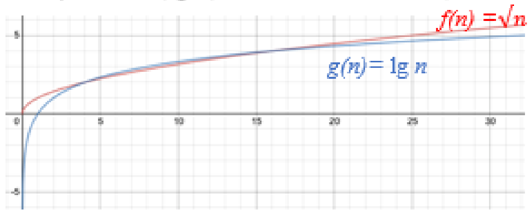
$\Omega(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$.



$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Example

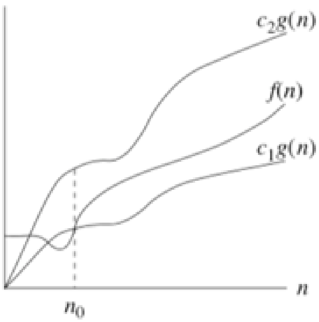
$\sqrt{n} = \Omega(\lg n)$, with $c = 1$ and $n_0 = 16$

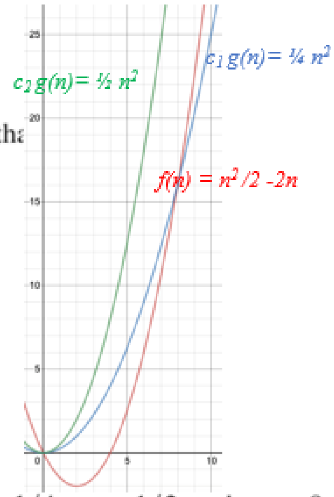


Θ-Notation

Θ-notation

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such tha $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.





Example

$n^2/2 - 2n = \Theta(n^2)$, with $c_1 = 1/4$, $c_2 = 1/2$, and $n_0 = 8$

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Theorem

$f(n) = \Theta(g(n))$ if and only if $f = O(g(n))$ and $f = \Omega(g(n))$

o-Notation(Little O)

$o(g(n)) = \{f(n) :$ for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $0 \le f(n) < cg(n)$ for all $n \ge n_0\}$.

Another view, probably easier to use: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

$n^{1.9999} = o(n^2)$
$n^2/\lg n = o(n^2)$
$n^2 \ne o(n^2)$ (just like $2 \not< 2$)
$n^2/1000 \ne o(n^2)$

ω-Notation(Little Omega)

$\omega(g(n)) = \{f(n) :$ for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $0 \le cg(n) < f(n)$ for all $n \ge n_0\}$.

Another view, again, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

$n^{2.0001} = \omega(n^2)$
$n^2 \lg n = \omega(n^2)$
$n^2 \ne \omega(n^2)$

Comparing Functions

Transitivity:

$$
\begin{aligned}
f(n) &= \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) \text{ imply } f(n) = \Theta(h(n)) , \\
f(n) &= O(g(n)) \text{ and } g(n) = O(h(n)) \text{ imply } f(n) = O(h(n)) , \\
f(n) &= \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) \text{ imply } f(n) = \Omega(h(n)) , \\
f(n) &= o(g(n)) \text{ and } g(n) = o(h(n)) \text{ imply } f(n) = o(h(n)) , \\
f(n) &= \omega(g(n)) \text{ and } g(n) = \omega(h(n)) \text{ imply } f(n) = \omega(h(n)) .
\end{aligned}
$$

Reflexivity:

$$
\begin{aligned}
f(n) &= \Theta(f(n)) , \\
f(n) &= O(f(n)) , \\
f(n) &= \Omega(f(n)) .
\end{aligned}
$$

Symmetry:

$$f(n) = \Theta(g(n)) \text{ if and only if } g(n) = \Theta(f(n)) .$$

Transpose symmetry:

$$
\begin{aligned}
f(n) &= O(g(n)) \text{ if and only if } g(n) = \Omega(f(n)) , \\
f(n) &= o(g(n)) \text{ if and only if } g(n) = \omega(f(n)) .
\end{aligned}
$$

$$
\begin{aligned}
f(n) &= O(g(n)) &&\text{is like} && a \le b , \\
f(n) &= \Omega(g(n)) &&\text{is like} && a \ge b , \\
f(n) &= \Theta(g(n)) &&\text{is like} && a = b , \\
f(n) &= o(g(n)) &&\text{is like} && a < b , \\
f(n) &= \omega(g(n)) &&\text{is like} && a > b .
\end{aligned}
$$

- 실수와는 다르게, 모든 Function이 점근적으로 비교가능한건 아니다