Single-Source Shortest Paths

Problem

The problem of finding shortest paths from a source vertex s to other vertices in the graph

In a weighted graph $G = (E,V)$, find all $\delta(s, v)$
from a source vertex $s \in V$ to all vertices $v \in V$ where

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i) .$$ The ***weight*** $w(p)$ of path $p = \langle v_0, v_1, \ldots, v_k \rangle$

We define the ***shortest-path weight*** $\delta(u, v)$ from u to v by

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\rightsquigarrow} v\} & \text{if there is a path from } u \text{ to } v , \\ \infty & \text{otherwise .} \end{cases}$$

- Bellman-Ford algorithm : works in a graph with negative weights
- Dijkstra's algorithm : works in a graph with non-negative weights

Variants

- Single destination shortest-paths problem
  - edge 방향을 반대로 하고 single-source shortest-paths problem을 푼다
- Single-pair shortest-path problem
  - single-source shortest paths problem을 풀면 그 안에 해가 포함되어 있다.
  - single-pair shortest path problem만 푸는 알고리즘의 worst-case running time은 가장 좋은 single-source shortest-paths problem의 worst-case running time과 점근적으로 같다
- All-pairs shortest-paths problem
  - 모든 vertices에 대해 single-source shortest-paths problem을 푼다.
    그러나 Floyd-Warshall algorithm과 같이 더 효율적인 방법도 있다

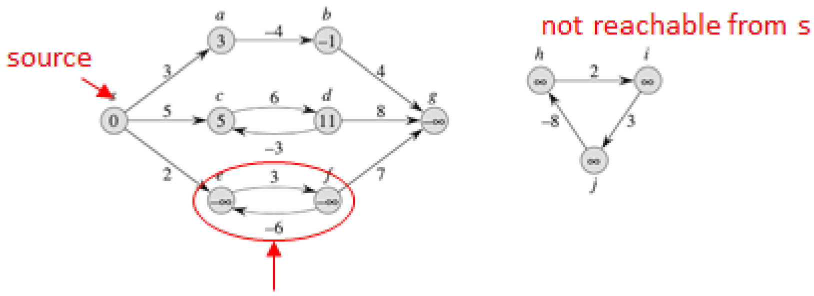
Optimal substructure of a shortest path

Lemma 24.1 (Subpaths of shortest paths are shortest paths)
Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from vertex v_0 to vertex v_k and, for any i and j such that $0 \le i \le j \le k$, let $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ be the subpath of p from vertex v_i to vertex v_j. Then, p_{ij} is a shortest path from v_i to v_j.

Proof If we decompose path p into $v_0 \overset{p_{0i}}{\rightsquigarrow} v_i \overset{p_{ij}}{\rightsquigarrow} v_j \overset{p_{jk}}{\rightsquigarrow} v_k$, then we have that $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$. Now, assume that there is a path p'_{ij} from v_i to v_j with weight $w(p'_{ij}) < w(p_{ij})$. Then, $v_0 \overset{p_{0i}}{\rightsquigarrow} v_i \overset{p'_{ij}}{\rightsquigarrow} v_j \overset{p_{jk}}{\rightsquigarrow} v_k$ is a path from v_0 to v_k whose weight $w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$ is less than $w(p)$, which contradicts the assumption that p is a shortest path from v_0 to v_k. ■

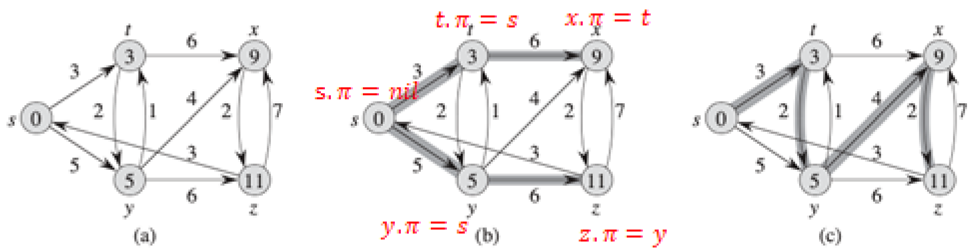
Thus, Dijkstra's algorithm : greedy algorithm, Floyd-Warshall algorithm : dynamic programming

Negative-weight edges



- Graph G에 negative-weight **cycle**이 있으면 shortest-path problem은 well-defined가 아니다
  - s -> e, s -> f, s -> g 때문
- Bellman-Ford algorithm
  - works in a graph with **negative weights**, unless there is a **negative cycle**
- Dijkstra's algorithm
  - works in a graph with **non-negative weights**

Shortest-paths tree



A ***shortest-paths tree*** rooted at s is a directed subgraph $G' = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$, such that

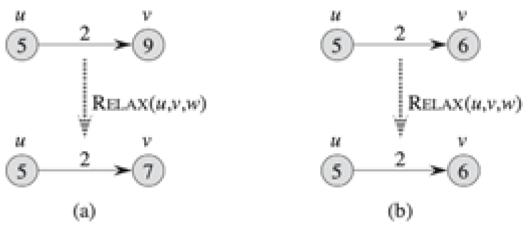
1. V' is the set of vertices reachable from s in G,
2. G' forms a rooted tree with root s, and
3. for all $v \in V'$, the unique simple path from s to v in G' is a shortest path from s to v in G.

- **predecessor subgraph** defined by "$v.\pi$: v의 predecessor in shortest-paths tree"와 같다

Relaxation

Update operation of $v.d(shortest - path\ estimate)$

RELAX(u, v, w)
1 **if** $v.d > u.d + w(u, v)$
2 $\quad v.d = u.d + w(u, v)$
3 $\quad v.\pi = u$



Bellman-Ford Algorithm

- Single source shortest path algorithm
- Unlike Dijkstra's algorithm, edges can have negative weight
- The algorithm returns false when there is a negative cycle

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| - 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

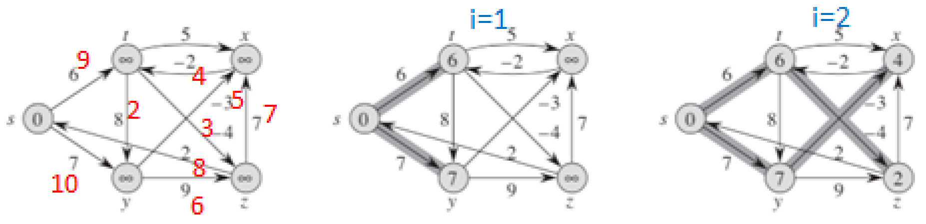
main

check for negative cycle

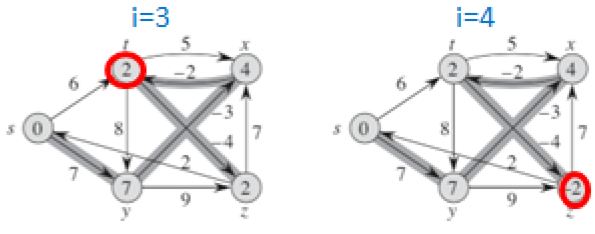
```
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex v ∈ G.V
    v.d = ∞
    v.π = NIL
s.d = 0
```

source : s

edge order : $(t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)$



(a) (b) (c)



(d) (e)

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| - 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

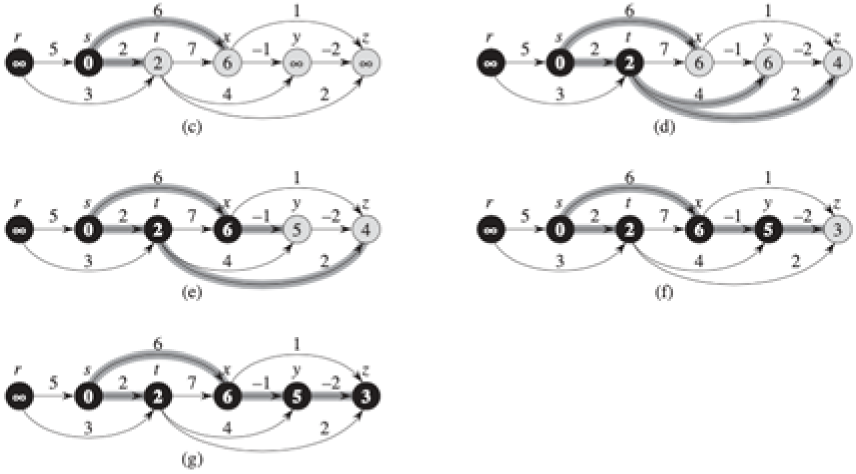
$$= O(VE)$$

Topological sort를 이용한 single-source shortest path

- G가 directed acyclic graph일 때(cycle이 없으므로 negative-weight cycle도 없음) 사용 가능

```
DAG-SHORTEST-PATHS(G, w, s)
topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex u, taken in topologically sorted order
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

(a) (b)



Dijkstra's Algorithm

- used when G has no negative edges
- Greedy Algorithm

```
MST-PRIM(G, w, r)
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = G.V                          : BUILD_MIN_HEAP( ) : O(V)
while Q ≠ ∅
    u = EXTRACT-MIN(Q)           ← greedy choice : V x O(lg V)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)           : DECREASE_KEY( ) : E x O(lg V)
```

priority queue 가 binary min heap 으로 구현된 경우 running time (refer to chapter 5) = O((V+E)lg V)