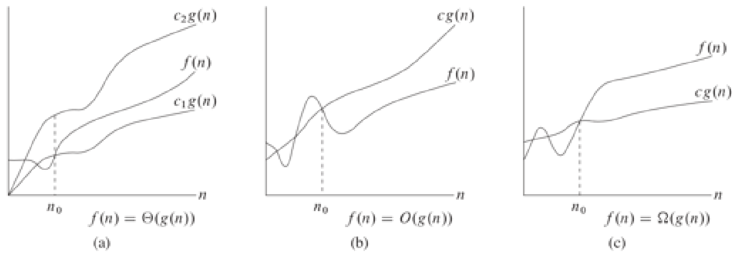
점근적 표기(Asymptotic Notation)

- To describe growth of functions and to compare functions
- 함수들의 독립변수가 아주 커졌을 때, 함수값의 크기를 비교하는데 사용

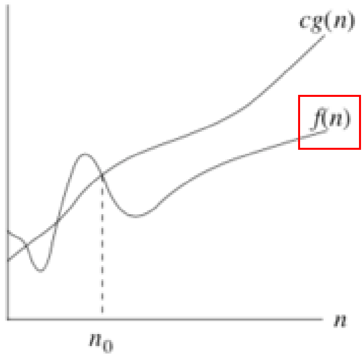


$$
\begin{aligned}
f(n) &= O(g(n)) && \text{is like} && a \le b\,, \\
f(n) &= \Omega(g(n)) && \text{is like} && a \ge b\,, \\
f(n) &= \Theta(g(n)) && \text{is like} && a = b\,, \\
f(n) &= o(g(n)) && \text{is like} && a < b\,, \\
f(n) &= \omega(g(n)) && \text{is like} && a > b\,.
\end{aligned}
$$

O-Notation

O-notation

$$O(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le \boxed{f(n)} \le cg(n) \text{ for all } n \ge n_0\}\,.$$

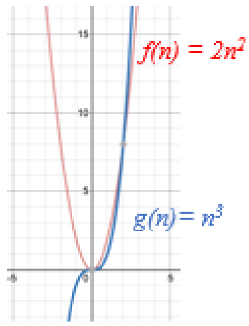


$g(n)$ is an ***asymptotic upper bound*** for $\boxed{f(n)}$.

If $f(n) \in O(g(n))$, we write $f(n) = O(g(n))$

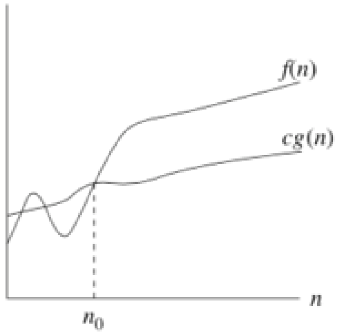
Example

$\boxed{2n^2} = O(n^3)$, with $c = 1$ and $n_0 = 2$.



Ω-Notation

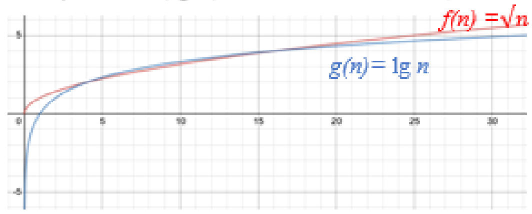
$\Omega(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that
$0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$.



$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Example

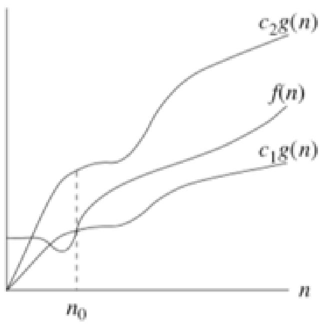
$\sqrt{n} = \Omega(\lg n)$, with $c = 1$ and $n_0 = 16$

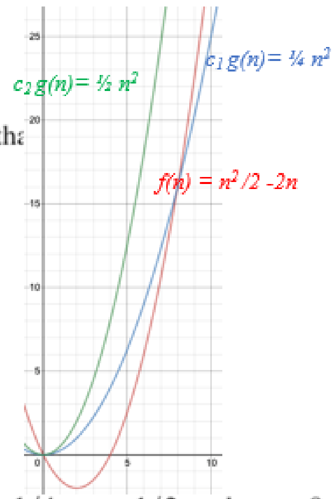


Θ-Notation

Θ-notation

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such tha
$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.





Example

$n^2/2 - 2n = \Theta(n^2)$, with $c_1 = 1/4$, $c_2 = 1/2$, and $n_0 = 8$

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Theorem

$f(n) = \Theta(g(n))$ if and only if $f = O(g(n))$ and $f = \Omega(g(n))$

o-Notation(Little O)

$o(g(n)) = \{f(n) :$ for all constants $c > 0$, there exists a constant
$n_0 > 0$ such that $0 \le f(n) < cg(n)$ for all $n \ge n_0\}$.

Another view, probably easier to use: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

$n^{1.9999} = o(n^2)$
$n^2/\lg n = o(n^2)$
$n^2 \ne o(n^2)$ (just like $2 \not< 2$)
$n^2/1000 \ne o(n^2)$

ω-Notation(Little Omega)

$$\omega(g(n)) = \{f(n) : \text{for all constants } c > 0, \text{there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\} .$$

Another view, again, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

$n^{2.0001} = \omega(n^2)$
$n^2 \lg n = \omega(n^2)$
$n^2 \ne \omega(n^2)$

Comparing Functions

Transitivity:

$$\begin{array}{lllllll}
f(n) = \Theta(g(n)) & \text{and} & g(n) = \Theta(h(n)) & \text{imply} & f(n) = \Theta(h(n)) , \\
f(n) = O(g(n)) & \text{and} & g(n) = O(h(n)) & \text{imply} & f(n) = O(h(n)) , \\
f(n) = \Omega(g(n)) & \text{and} & g(n) = \Omega(h(n)) & \text{imply} & f(n) = \Omega(h(n)) , \\
f(n) = o(g(n)) & \text{and} & g(n) = o(h(n)) & \text{imply} & f(n) = o(h(n)) , \\
f(n) = \omega(g(n)) & \text{and} & g(n) = \omega(h(n)) & \text{imply} & f(n) = \omega(h(n)) .
\end{array}$$

Reflexivity:

$$\begin{aligned}
f(n) &= \Theta(f(n)) , \\
f(n) &= O(f(n)) , \\
f(n) &= \Omega(f(n)) .
\end{aligned}$$

Symmetry:

$$f(n) = \Theta(g(n)) \text{ if and only if } g(n) = \Theta(f(n)) .$$

Transpose symmetry:

$$\begin{array}{lll}
f(n) = O(g(n)) & \text{if and only if} & g(n) = \Omega(f(n)) , \\
f(n) = o(g(n)) & \text{if and only if} & g(n) = \omega(f(n)) .
\end{array}$$

$$\begin{array}{lll}
f(n) = O(g(n)) & \text{is like} & a \le b , \\
f(n) = \Omega(g(n)) & \text{is like} & a \ge b , \\
f(n) = \Theta(g(n)) & \text{is like} & a = b , \\
f(n) = o(g(n)) & \text{is like} & a < b , \\
f(n) = \omega(g(n)) & \text{is like} & a > b .
\end{array}$$

- 실수와는 다르게, 모든 Function이 점근적으로 비교가능한건 아니다

Standard Notations and Common Functions

- Floor Function(내림), Ceiling Function(올림)

$$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$$

- modulo 연산

$$a \bmod n = a - n\lfloor a/n \rfloor$$

- d차 다항식

$$p(n) = \sum_{i=0}^{d} a_i n^i \quad = \Theta(n^d)$$

  - 최고차항만 보면 된다
- a>1인 모든 실수 a와 b에 대하여

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0 \quad \rightarrow \quad n^b = o(a^n)$$

  - 다항식으로 나오는게 더 빨리 수행됨
- 모든 실수 x에 대하여

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!} \qquad e^x \ge 1 + x$$

- $$|x| \le 1, \qquad 1 + x \le e^x \le 1 + x + x^2$$

  - e^x를 approximate 할 때 활용
- 모든 x에 대하여

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$$

Logarithms

$$\begin{aligned}
\lg n &= \log_2 n && \text{(binary logarithm)} ,\\
\ln n &= \log_e n && \text{(natural logarithm)} ,\\
\lg^k n &= (\lg n)^k && \text{(exponentiation)} ,\\
\lg\lg n &= \lg(\lg n) && \text{(composition)} .
\end{aligned}$$

$$\begin{aligned}
a &= b^{\log_b a} ,\\
\log_c(ab) &= \log_c a + \log_c b ,\\
\log_b a^n &= n \log_b a ,\\
\log_b a &= \frac{\log_c a}{\log_c b} ,\\
\log_b(1/a) &= -\log_b a ,\\
\log_b a &= \frac{1}{\log_a b} ,\\
a^{\log_b c} &= c^{\log_b a} ,
\end{aligned}$$

- when $|x| < 1$ $\quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots .$
- for $x > -1$ $\quad \frac{x}{1+x} \le \ln(1+x) \le x$
- $\lim_{n\to\infty} \frac{n^b}{a^n} = 0 \rightarrow \lim_{n\to\infty} \frac{\lg^b n}{(2^a)^{\lg n}} = \lim_{n\to\infty} \frac{\lg^b n}{n^a} = 0 \rightarrow \lg^b n = o(n^a)$

Factorials

- Stirling's approximation : $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$

→
$$\begin{aligned} n! &= o(n^n)\,, \\ n! &= \omega(2^n)\,, \\ \lg(n!) &= \Theta(n \lg n) \end{aligned}$$

The iterated logarithm function

- $\lg^{(i)} n$ = lg (lg (lg … n …))) vs. $\lg^{i} n$ = lg n x lg n x … x lg n
- $\lg^* n = \min\{i \geq 0 : \lg^{(i)} n \leq 1\}$

VERY sloooooooowly growing function

$$\begin{aligned} \lg^* 2 &= 1 \\ \lg^* 4 &= 2 \\ \lg^* 16 &= 3 \\ \lg^* 65536 &= 4 \\ \lg^*(2^{65536}) &= 5 \end{aligned}$$

Fibonacci numbers

- 지수적으로 증가함

$$\begin{aligned} F_0 &= 0\,, \\ F_1 &= 1\,, \\ F_i &= F_{i-1} + F_{i-2} \qquad \text{for } i \geq 2 \end{aligned}$$

- $x^2 = x + 1$ 의 해 $\phi = \frac{1+\sqrt{5}}{2} = 1.61803\ldots\,,$ $\hat{\phi} = \frac{1-\sqrt{5}}{2} = -.61803\ldots\,.$

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

→ $F_i = \left\lfloor \frac{\phi^i}{\sqrt{5}} + \frac{1}{2} \right\rfloor$