Update Rules

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1 Cost Function

$$J = \sum_{j:G(i,j)=1} \left(\mathbf{W}_{i,j} \odot (\mathbf{D}^{+}_{\mathbf{i,j}} - \mathbf{F_i} \mathbf{P}^{+}_{\mathbf{i,j}} \mathbf{F}'_{\mathbf{j}}) \right)^2_F + \sum_{j:G(i,j)=1} \left(\mathbf{W}_{i,j} \odot (\mathbf{D}^{-}_{\mathbf{i,j}} - \mathbf{F_i} \mathbf{P}^{-}_{\mathbf{i,j}} \mathbf{F}'_{\mathbf{j}}) \right)^2_F + \tag{1}$$

$$+\alpha \sum_{i=1}^{g} tr(\mathbf{F}'_{\mathbf{i}}(\mathbf{T_i} - \mathbf{A_i})\mathbf{F_i}) + \beta \sum \mathbf{F_i}^2_F \tag{2}$$

Definitions of the symbols

$$\mathbf{D}_{i,j} = \begin{cases} 1 & \text{if relation observed} \\ 0 & \text{if relation unknown or not observed} \end{cases}$$

$$\mathbf{W}_{i,j} = \begin{cases} 1 & \text{if} \quad \mathbf{D}_{i,j} = 1 \\ \sqrt{w} & \text{if} \quad \mathbf{D}_{i,j} = 0 \end{cases}$$

$\mathbf{F_i}$ low-level representation of layer i (3)

$\mathbf{P}^{+}_{\mathbf{i,j}}$ low-level matrix of "+" interactions between layers i and j (4)

$\mathbf{P}^{-}_{\mathbf{i,j}}$ low-level matrix of "-" interactions between layers i and j (5)

$\mathbf{A_i}$ connectivity within layer i (6)

$\mathbf{T_i}$ diagonal degree matrix of $\mathbf{A_i}$ (7)

α regularization parameter (8)

β regularization parameter (9)

2 Update Rules for $\mathbf{F_i}$, $\mathbf{P^+_{i,j}}$ and $\mathbf{P^-_{i,j}}$

Update rule for $\mathbf{F_i}$

$$\mathbf{F_i}(u,v) \leftarrow \mathbf{F_i}(u,v) \sqrt{\frac{[\sum_{G(i,j)=1}(((\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{D_{i,j}}^+)\mathbf{F_j}\mathbf{P'_{i,j}}^+) + (\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{D_{i,j}}^-)\mathbf{F_j}\mathbf{P'_{i,j}}^-) + \alpha \mathbf{A_i}\mathbf{F_i}](u,v)}{[\sum(((\mathbf{W_{i,j}} \odot \mathbf{W}_{i,j} \odot \mathbf{F_i}\mathbf{P_{i,j}}^+\mathbf{F'_j})\mathbf{F_j}\mathbf{P'_{i,j}}^+) + (\mathbf{W_{i,j}} \odot \mathbf{W}_{i,j} \odot \mathbf{F_i}\mathbf{P_{i,j}}^-\mathbf{F'_j})\mathbf{F_j}\mathbf{P'_{i,j}}^-) + \alpha \mathbf{T_i}\mathbf{F_i} + \beta \mathbf{F_i}](u,v)}} \tag{10}$$

Update rule for $\mathbf{P^+_{i,j}}$

$$\mathbf{P^+_{i,j}}(u,v) \leftarrow \mathbf{P^+_{i,j}}(u,v) \sqrt{\frac{(\mathbf{F'_i}(\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{D_{i,j}}^+)\mathbf{F_j})(u,v)}{[(\mathbf{F'_i}(\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{F_i}\mathbf{P^+_{i,j}}\mathbf{F'_j})\mathbf{F_j})](u,v)}} \tag{11}$$

Update rule for $\mathbf{P^-_{i,j}}$

$$\mathbf{P^-_{i,j}}(u,v) \leftarrow \mathbf{P^-_{i,j}}(u,v) \sqrt{\frac{(\mathbf{F'_i}(\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{D_{i,j}}^-)\mathbf{F_j})(u,v)}{[(\mathbf{F'_i}(\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{F_i}\mathbf{P^-_{i,j}}\mathbf{F'_j})\mathbf{F_j})](u,v)}} \tag{12}$$

3 Additional definitions for the New Update Rules

Identity matrices for observed, unobserved and all relations: $I^A = I^O + I^{O^C}$

Let $\widetilde{\mathbf{D}}^+_{i,j}$ be $\mathbf{D_{i,j}}^+$ estimated by $\mathbf{F_i}\mathbf{P_{i,j}}^+\mathbf{F'_j}$

Then $\widetilde{\mathbf{D}}^+_{i,j} = \widetilde{\mathbf{D}}^+_{i,j_{observed}} + \widetilde{\mathbf{D}}^+_{i,j_{unobserved}} = \widetilde{\mathbf{D}}^+_{i,j_O} + \widetilde{\mathbf{D}}^+_{i,j_{O^C}}$

Additional definitions:

$$\widetilde{\mathbf{D}}^-_{i,j_O}(u,v) = \begin{cases} \mathbf{F_i}\mathbf{P_{i,j}}^-\mathbf{F'_j} & \text{if "-" relation (u,v) is observed} \\ 0 & \text{otherwise} \end{cases}$$

4 Derivation of New Update Rules

$$(\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{D_{i,j}}^{+})\mathbf{F_j}\mathbf{P'_{i,j}}^{+} = \tag{13}$$

$$= \mathbf{D_{i,j}}^{+}\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{14}$$

$$(\mathbf{W_{i,j}} \odot \mathbf{W}_{i,j} \odot \mathbf{F_i}\mathbf{P_{i,j}}^{+}\mathbf{F'_j})\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{15}$$

$$= (\mathbf{W_{i,j}} \odot \mathbf{W}_{i,j} \odot (\widetilde{\mathbf{D}}^{+}_{i,j_O} + \widetilde{\mathbf{D}}^{+}_{i,j_{O^C}}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{16}$$

$$= (\mathbf{W_{i,j}} \odot \mathbf{W}_{i,j} \odot (\widetilde{\mathbf{D}}^{+}_{i,j_O} + \widetilde{\mathbf{D}}^{+}_{i,j_{O^C}}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{17}$$

$$= ((\mathbf{W_{i,j}} \odot \mathbf{W}_{i,j} \odot \widetilde{\mathbf{D}}^{+}_{i,j_O}) + (\mathbf{W_{i,j}} \odot \mathbf{W}_{i,j} \odot \widetilde{\mathbf{D}}^{+}_{i,j_{O^C}}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{18}$$

$$= (\widetilde{\mathbf{D}}^{+}_{i,j_O} + (\mathbf{W_{i,j}} \odot \mathbf{W}_{i,j} \odot \widetilde{\mathbf{D}}^{+}_{i,j_{O^C}}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{19}$$

$$= (\widetilde{\mathbf{D}}^{+}_{i,j_O} + (wI^{O^C} \odot \widetilde{\mathbf{D}}^{+}_{i,j_{O^C}}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{20}$$

$$= (\widetilde{\mathbf{D}}^{+}_{i,j_O} + (wI^{A} \odot \widetilde{\mathbf{D}}_{i,j} - wI^{O} \odot \widetilde{\mathbf{D}}^{+}_{i,j_O}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{21}$$

$$= (\widetilde{\mathbf{D}}^{+}_{i,j_O} + (wI^{A} \odot \widetilde{\mathbf{D}}_{i,j} - wI^{O} \odot \widetilde{\mathbf{D}}^{+}_{i,j_O}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{22}$$

$$= (\widetilde{\mathbf{D}}^{+}_{i,j_O} + (wI^{A} \odot (\mathbf{F_i}\mathbf{P_{i,j}}^{+}\mathbf{F'_j}) - wI^{O} \odot \widetilde{\mathbf{D}}^{+}_{i,j_O}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{23}$$

$$= (\widetilde{\mathbf{D}}^{+}_{i,j_O} + w * (\mathbf{F_i}\mathbf{P_{i,j}}^{+}\mathbf{F'_j}) - w * \widetilde{\mathbf{D}}^{+}_{i,j_O})\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{24}$$

$$= ((1-w) * \widetilde{\mathbf{D}}^{+}_{i,j_O} + w * (\mathbf{F_i}\mathbf{P_{i,j}}^{+}\mathbf{F'_j}))\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{25}$$

$$= (1-w) * \widetilde{\mathbf{D}}^{+}_{i,j_O}\mathbf{F_j}\mathbf{P'_{i,j}}^{+} + w * (\mathbf{F_i}\mathbf{P_{i,j}}^{+}\mathbf{F'_j})\mathbf{F_j}\mathbf{P'_{i,j}}^{+} \tag{26}$$

$$\mathbf{F'_i}(\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{F_i}\mathbf{P}^{+}_{\mathbf{i,j}}\mathbf{F'_j})\mathbf{F_j} = \tag{27}$$

$$= (1-w) * \mathbf{F'_i}\widetilde{\mathbf{D}}^{+}_{i,j_O}\mathbf{F_j} + w * \mathbf{F'_i}(\mathbf{F_i}\mathbf{P_{i,j}}^{+}\mathbf{F'_j})\mathbf{F_j} \tag{28}$$

$$(\mathbf{F'_i}(\mathbf{W}_{i,j} \odot \mathbf{W}_{i,j} \odot \mathbf{D_{i,j}}^{-})\mathbf{F_j}) = \tag{29}$$

$$= \mathbf{F'_i}\mathbf{D_{i,j}}^{-}\mathbf{F_j} \tag{30}$$

5 New Update Rules

Update Rule: $\mathbf{F_i}(u,v) \leftarrow \mathbf{F_i}(u,v)$

$$\mathbf{F_i}(u,v) \leftarrow \mathbf{F_i}(u,v)\sqrt{\frac{A}{B}} \tag{31}$$

$$A = [\sum_{G(i,j)=1} ((\mathbf{D_{i,j}}^{+}\mathbf{F_j}\mathbf{P'_{i,j}}^{+}) + (\mathbf{D_{i,j}}^{-}\mathbf{F_j}\mathbf{P'_{i,j}}^{-})) + \alpha\mathbf{A_i}\mathbf{F_i}](u,v) \tag{32}$$

$$B = [\sum_{G(i,j)=1} (((1-w) * \widetilde{\mathbf{D}}^{+}_{i,j_O}\mathbf{F_j}\mathbf{P'_{i,j}}^{+} + w * (\mathbf{F_i}\mathbf{P_{i,j}}^{+}\mathbf{F'_j})\mathbf{F_j}\mathbf{P'_{i,j}}^{+}) \tag{33}$$

$$+ ((1-w) * \widetilde{\mathbf{D}}^{-}_{i,j_O}\mathbf{F_j}\mathbf{P'_{i,j}}^{-} + w * (\mathbf{F_i}\mathbf{P_{i,j}}^{-}\mathbf{F'_j})\mathbf{F_j}\mathbf{P'_{i,j}}^{-}))) \tag{34}$$

$$+ \alpha\mathbf{T_i}\mathbf{F_i} + \beta\mathbf{F_i}](u,v) \tag{35}$$

Update Rule: $\mathbf{P^{+}_{i,j}} \leftarrow \mathbf{P^{+}_{i,j}}$

$$\mathbf{P^{+}_{i,j}}(u,v) \leftarrow \mathbf{P^{+}_{i,j}}(u,v)\sqrt{\frac{(\mathbf{F'_i}\mathbf{D_{i,j}}^{+}\mathbf{F_j})(u,v)}{\mathbf{F'_i}((1-w) * \widetilde{\mathbf{D}}^{+}_{i,j_O} + w * (\mathbf{F_i}\mathbf{P_{i,j}}^{+}\mathbf{F'_j}))\mathbf{F_j}(u,v)}} \tag{36}$$

Update Rule: $\mathbf{P^{-}_{i,j}} \leftarrow \mathbf{P^{-}_{i,j}}$

$$\mathbf{P^{-}_{i,j}}(u,v) \leftarrow \mathbf{P^{-}_{i,j}}(u,v)\sqrt{\frac{(\mathbf{F'_i}\mathbf{D_{i,j}}^{-}\mathbf{F_j})(u,v)}{\mathbf{F'_i}((1-w) * \widetilde{\mathbf{D}}^{-}_{i,j_O} + w * (\mathbf{F_i}\mathbf{P_{i,j}}^{-}\mathbf{F'_j}))\mathbf{F_j}(u,v)}} \tag{37}$$

Additional definitions:

$$\widetilde{\mathbf{D}}^{-}_{i,j_O}(u,v) = \begin{cases} \mathbf{F_i}\mathbf{P_{i,j}}^{-}\mathbf{F'_j} & \text{if "-" relation (u,v) is observed} \\ 0 & \text{otherwise} \end{cases}$$

6 Additional Hyper-parameter

Additional hyper-parameter to balance the amount of positive and negative relations in signed layer-layer relation: $\mathbf{b}$:

$$J = \sum_{j:G(i,j)=1} \mathbf{b}\Big(\mathbf{W}_{i,j} \odot (\mathbf{D^+_{i,j}} - \mathbf{F_i P^+_{i,j} F'_j})\Big)^2_F + \sum_{j:G(i,j)=1} (1-\mathbf{b})\Big(\mathbf{W}_{i,j} \odot (\mathbf{D^-_{i,j}} - \mathbf{F_i P^-_{i,j} F'_j})\Big)^2_F + \tag{38}$$

$$+\alpha \sum_{i=1}^{g} tr(\mathbf{F'_i}(\mathbf{T_i} - \mathbf{A_i})\mathbf{F_i}) + \beta \sum \mathbf{F_i}^2_F \tag{39}$$

Update Rules with b hyper-parameter (only rule for F changes)

$$\mathbf{F_i}(u,v) \leftarrow \mathbf{F_i}(u,v)\sqrt{\frac{A}{B}} \tag{40}$$

$$A = [\sum_{G(i,j)=1} (\mathbf{b}(\mathbf{D_{i,j}}^+ \mathbf{F_j P'_{i,j}}^+) + (1-\mathbf{b})(\mathbf{D_{i,j}}^- \mathbf{F_j P'_{i,j}}^-)) + \alpha \mathbf{A_i F_i}](u,v) \tag{41}$$

$$B = [\sum_{G(i,j)=1} (\mathbf{b}((1-w) * \widetilde{\mathbf{D}}^+_{i,j_O} \mathbf{F_j P'_{i,j}}^+ + w * (\mathbf{F_i P_{i,j}}^+ \mathbf{F'_j}) \mathbf{F_j P'_{i,j}}^+) \tag{42}$$

$$+ (1-\mathbf{b})(1-w) * (\widetilde{\mathbf{D}}^-_{i,j_O} \mathbf{F_j P'_{i,j}}^+ + w * (\mathbf{F_i P_{i,j}}^- \mathbf{F'_j}) \mathbf{F_j P'_{i,j}}^-)) \tag{43}$$

$$+ \alpha \mathbf{T_i F_i} + \beta \mathbf{F_i}](u,v) \tag{44}$$

Update Rule: $\mathbf{P^+_{i,j}} \leftarrow \mathbf{P^+_{i,j}}$

$$\mathbf{P^+_{i,j}}(u,v) \leftarrow \mathbf{P^+_{i,j}}(u,v)\sqrt{\frac{(\mathbf{F'_i D_{i,j}}^+ \mathbf{F_j})(u,v)}{\mathbf{F'_i}((1-w) * \widetilde{\mathbf{D}}^+_{i,j_O} + w * (\mathbf{F_i P_{i,j}}^+ \mathbf{F'_j}))\mathbf{F_j}(u,v)}} \tag{45}$$

Update Rule: $\mathbf{P^-_{i,j}} \leftarrow \mathbf{P^-_{i,j}}$

$$\mathbf{P^-_{i,j}}(u,v) \leftarrow \mathbf{P^-_{i,j}}(u,v)\sqrt{\frac{(\mathbf{F'_i D_{i,j}}^- \mathbf{F_j})(u,v)}{\mathbf{F'_i}((1-w) * \widetilde{\mathbf{D}}^-_{i,j_O} + w * (\mathbf{F_i P_{i,j}}^- \mathbf{F'_j}))\mathbf{F_j}(u,v)}} \tag{46}$$