MATH 135: Final Review – Polynomial problems

1. Let $f(x) = 3x^3 - 8x + 5 \in \mathbb{C}[x]$.

    a. Find all rational roots of *f(x)*

    b. Factor *f(x)* completely in $\mathbb{C}[x]$.

2. Factor $f(x) = 10x^3 - 39x^2 + 29x - 6$ as product of linear terms in $\mathbb{Q}[x]$.

3. Let $f(x) = x^6 - 27 \in \mathbb{R}[x]$. Find all complex roots of *f(x)*.

4. Factor $x^4 + 2x^3 + 4x - 4$ in $\mathbb{Q}[x]$.

5. Factor $x^2 + 5x - 3$ in $\mathbb{Z}_{11}[x]$.

6. Show that $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.

7. Let *n* be any positive integer which is not a multiple of 5. Prove that $x^4 + x^3 + x^2 + x + 1$

    divides $x^{4n} + x^{3n} + x^{2n} + x^n + 1$ in $\mathbb{Q}[x]$.

Bonus: Find the sixth root of $-8i$. Express your answer in polar form (i.e. $\pi e^{i\theta}$ where $0 \leq \theta < 2\pi$).

Math 135: Final Review – GCD

Part 1: Basics GCD

1. Suppose that $a, b, c, d \in \mathbb{Z}$ satisfy $\gcd(a, b) = 1$, $c|a$ and $d|b$. Prove that $\gcd(c, d) = 1$.

2. Suppose that $a, b \in \mathbb{Z}$ are coprime. Prove that $\gcd(a + b, a - b) \in \{1, 2\}$.

3. Let $a, b, c, d \in \mathbb{Z}$. Suppose that $d|ab, d|ac$ and $gcd(b, c) = 1$. Prove that $d|a$.

4. Prove that for any $h \in \mathbb{Z}, \gcd(21h - 5, 6h - 2) \in \{1, 2, 4\}$, more precisely $\gcd(21h - 5, 6h - 2) = \gcd(h - 1, 4)$.

5. Let $a, b, c \in \mathbb{Z}$. Prove that if $gcd(a, b) = 1$, then $gcd(a, bc) = \gcd(a, c)$. Deduce that if $\gcd(a, b) = \gcd(a, c) = 1$, then $gcd(a, bc) = 1$.

6. Let $u, v \in \mathbb{Z}$. Prove that if *u* and *v* are coprime, then $\gcd(u + v, uv) = 1$.

Part 2: Linear Diophantine Equations

7. Find a general solution to the following LDEs:

   a. $6x + 4y = 60$

   b. $27x + 72y = 12$

   c. $81x - 24y = 6$

8. When Laurie cashed a cheque for *x* dollars and *y* cents, she received instead *y* dollars and *x* cents. They found that they had two cents more than twice the proper amount. How much was the cheque written?

9. Roosters cost \$5 each, hens cost \$3 each, and chicks cost \$1 for three. If \$100 fowls are brought for \$100. How many roosters, hens, and chickens are there? Find all positive solutions.

10. Express 100 as a sum of two positive integers such that one is divisible by 11 and the other is divisible by 7.

11. Find the smallest positive integer which leaves a remainder of x when divided by 13 and a remainder of 2 when divided by 8.

Math 135: Final Review – Modular

1. Show that $|a^2 - 10b^2| = 2$ has no integer solutions for *a, b*.

2. Solve the following system of linear congruences:

$$4x \equiv 7 \ (mod\ 9)$$

$$3x \equiv 2 \ (mod\ 11)$$

3. Solve the following system of congruences:

$$x^3 \equiv 5 \ (mod\ 8)$$

$$3x^2 \equiv 3 \ (mod\ 9)$$

$$2x \equiv 0 \ (mod\ 10)$$

4. Define a sequence as follows: $a_0 = 3, a_1 = 7, a_n = 5(a_{n-1} + a_{n-2}) + 4a_{n-1}^2 + 1$ for $n \geq 2$.

   Prove that $a_n \equiv 3\ (mod\ 4)\ \forall\ n \geq 0$.

5. Prove that for any integer $n \geq 25$, there exists non-negative integers a, b such that

$$5a + 7b = n.$$

6. Prove that $n^3 - n$ is divisible by $3, \forall\ n \in \mathbb{Z}$.

7. Prove that $n^7 - n$ is is divisible by $42, \forall\ n \in \mathbb{Z}$.

8. Prove that if $a \equiv b\ (mod\ n)$ then for all positive integers c that divide a and b,

$$\frac{a}{c} \equiv \frac{b}{c} \left(mod \frac{n}{\gcd(a, c)} \right)$$

9. Let $n \in \mathbb{Z}$. Prove that $(a + b)^n \equiv a^n + b^n\ (mod\ n), \forall\ a, b \in \mathbb{Z}$.

Math 135: Final Review – Primes

1. Prove that there are infinitely many primes of the form $6n + 5$.

2. Let p be a prime. Prove that $\sqrt{p}$ is irrational.

3. Prove that if $2^n - 1$ is prime, then n is prime.

4. Let p be a prime, such that $p \equiv 3\ (mod\ 4)$. Prove that $b \equiv a^{\frac{p+1}{4}}\ (mod\ p)$ satisfies

$$b^2 \equiv a\ (mod\ p)$$

5. Let p be a prime number and let $q \in \mathbb{N}$. Prove that if $q \neq p$, then $(\gcd(q^2, p) = 1$.

   Prove that for distinct primes p, q that $p^{q-1} + q^{p-1} \equiv 1\ (mod\ pq)$.

6. Let $a, b \in \mathbb{Z}$. Prove that $\gcd(a, b)^n \equiv \gcd(a^n, b^n), \forall\, n \in \mathbb{N}$.

7. If $k \in \mathbb{N}$ and 2^{k-1} is prime, then $2^{k-1}(2^k - 1)$ is perfect. i.e. the sum of its positive divisors is

   $2\left(2^{k-1}(2^k - 1)\right)$.

8. Prove that $\forall\, k \in \mathbb{N}, \exists\, n \in \mathbb{N}$ such that $2^k | (3^n + 5)$.

9. Let $b \in \mathbb{Z}$. Prove that $\forall\, n \in \mathbb{N}$ that if $\{a_1, \dots, a_n\}$ is a set of n integers such that

   $\gcd(b, a_i) = 1,\ 1 \leq i \leq n$, then $\gcd(b, \prod_{i=1}^{n} a_i) = 1$.