GCDs'

6. Suppose $a, b \in \mathbb{Z}$ are coprime. Prove that $\gcd(a+b, a-b) \in \{1, 2\}$

Solution: Since a and b are coprime then $\gcd(a,b) = 1$

By EEA, $\exists x, y \in \mathbb{Z}$ s.t $ax + by = 1$

Let $\gcd(a+b, a-b) = d$. Then d divides both $a+b$ and $a-b$.

Since $2a = (a+b) + (a-b)$ and

$2b = (a+b) - (a-b)$

By DIC, $d \mid 2a$ and $d \mid 2b$

On the other hand, $2 = 2(ax+by) = (2a)x + (2b)y$

Invoking DIC reversely we see that $d \mid 2$. Since 2 is prime and $d \in \mathbb{Z}^+$ we conclude $d = 1$ or 2

(Note: $d = 2$ iff $a \equiv b \pmod 2$)

7. Prove that for any $n \in \mathbb{Z}$, $\gcd(21n-5, 6n-2)$ can be either 1, 2 or 4. More precisely, show that $\gcd(21n-5, 6n-2) = \gcd(n-1, 4)$

Soln: We want to know $d = \gcd(21n-5, 6n-2)$ equals 1, 2 or 4, simply observe that

$$2(21n-5) - 7(6n-2) = 42n - 10 - 42n + 14 = 4$$

Then, by DIC we see $d \mid 4$. Hence, $d = 1, 2,$ or 4.

To prove that $d = \gcd(n-1, 4)$ we apply GCD WR

$$\begin{aligned}
d &= \gcd(21n-5, 6n-2) \\
&= \gcd(21n-5-3(6n-2), 6n-2) = \gcd(3n+1, 6n-2) \\
&= \gcd(3n+1, (6n-2)-2(3n+1)) = \gcd(3n+1, -4) \\
&= \gcd((3n+1)-4n, -4) = \gcd(-n+1, -4) \\
&= \gcd(n-1, 4)
\end{aligned}$$

8. Find general solution to LDE if it has any solutions

a) $42x - 56y = 21$; $\gcd(42, -56) = 2 \nmid 21$ DNE

b) $27x + 78y = 12$; $\gcd(27, 78) = 3 \mid 12$ ✓

c) $81x - 24y = 6$; $\gcd(81, -24) = 3 \mid 6$ ✓

Sol :

b) Apply EEA : $27(3) + 78(-1) = 3$

Multiply by $\frac{12}{3} = 4$: $27(12) + 78(-4) = 12$

Hence, $(x_0, y_0) = (12, -4)$ is initial solution

By LDET2 the complete solution is

$$\begin{cases} x = 12 + \frac{\overset{b}{78}}{3} n \Rightarrow 12 + 26n \\ y = -4 - \frac{\overset{a}{27}}{3} n \Rightarrow -4 - 9n \end{cases} \quad n \in \mathbb{Z}$$

Find positive solutions ?

$$\begin{cases} 12 + 26n > 0 \\ -4 - 9n > 0 \end{cases} \Leftrightarrow \begin{cases} 26n > -12 \\ 9n < -4 \end{cases} \Leftrightarrow \begin{cases} n > -\frac{6}{13} \\ n < -\frac{4}{9} \end{cases}$$

There are no integers n that satisfy both conditions

Hence, LDE has no positive solutions