University of Waterloo

MATH 135 Final Examination

Algebra for Honours Mathematics
Fall 2018

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Duration of exam:	150 minutes
Number of exam pages:	14 (includes cover page)
Exam type:	Closed book
Materials allowed:	Only the provided reference sheet. No additional materials are allowed.

Additional instructions

1. Write your answers in the space provided. If you need more room, use the space available on the last page, and indicate this on the question page.
2. *Do NOT detach any paper from this exam.*
3. You must justify all of your answers. You can write in pen or pencil as long as your writing is legible. Your arguments must be logical, clear and easy to understand.
4. There is a Provided Reference sheet supplied separately from this exam booklet. There you will find some of the major propositions that were covered in class. You may use any result from the list without proof. Make sure to clearly state the name or the acronym of the result you are using.
5. There are 14 questions. They are ordered more for space rather than level of difficulty. The total marks available on this exam is 50.

For each of Questions 1 to 6, full marks will be given for a correct answer which is placed in the box provided. If the final answer in the box is incorrect, part marks may be awarded based on work shown in the space provided.

1. Determine the value of $\gcd(135^{2018}, 10!)$. (You may state your answer in factored form.) [**2 marks**]

2. Determine the remainder when 31^{66} is divided by 17. [**2 marks**]

3. Alice's RSA public key is $(7, 407)$. Note that $407 = 11 \times 37$. What is Alice's private key, (d, n)?
[3 marks]

4. Determine the complete solution to the following system of equations in $\mathbb{Z}_6$. **[3 marks]**

$$[4][x] + [3][y] = [2]$$
$$[2][x] + [4][y] = [2]$$

5. Write $f(x) = x^4 - 3x^3 + 4x^2 - 2x$ as a product of irreducible polynomials in $\mathbb{C}[x]$. **[3 marks]**

6. For how many integers a satisfying $1 \leq a \leq 21$ does the linear equation

$$ax + 14y = 2$$

have at least one integer solution (x, y)? **[3 marks]**

For each part below, in the box provided, indicate whether the given statement is true (T) or false (F). No justification is required. **[1 mark each]**

7. (a) For all statement variables P and Q, $\neg(P \implies Q)$ is logically equivalent to $\neg P \implies \neg Q$.

   (b) For all $x, y, z \in \mathbb{Z}$, if $5 \mid (xy + xz)$, then $5 \mid x$ or $5 \mid y$ or $5 \mid z$.

   (c) The number -32 has a complex fifth root which is purely imaginary.

   (d) The element $[5]$ has a multiplicative inverse in $\mathbb{Z}_9$.

   (e) For all sets A, B and C, $A - (B \cup C) = (A - B) \cup (A - C)$.

   (f) For all polynomials $f(x) \in \mathbb{R}[x]$, if $f(i) = 0$, then $f\left(\frac{1}{i}\right) = 0$.

   (g) If $f(z) = z^{12} - z + 65 + i$, then $1 + i$ is a complex root of $f(z)$.

The remaining questions require proofs. Write clearly and justify your steps. Do NOT use the amount of available space as an indication of how long your answer "should be".

8. Prove that for all integers a, b, c and d, if $a \mid b$ and $ac \mid d$, then $a \mid (3b - 5d)$. **[3 marks]**

9. Prove that for all $a, b \in \mathbb{Z}$, and all $c \in \mathbb{N}$, if $\gcd(a, b) = 1$ and $c \mid a$, then $\gcd(a, bc) = c$. **[4 marks]**

10. Prove that for all $z \in \mathbb{C}$, if $w = z^3 - 3z^2(\overline{z}) + 3z(\overline{z})^2 - (\overline{z})^3$, then w is purely imaginary. **[3 marks]**

11. A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 14$, $a_2 = 21$ and $a_m = 3a_{m-1} + a_{m-2}$ for all integers $m \geq 3$. Prove that $\gcd(a_n, a_{n+1}) = 7$ for all $n \in \mathbb{N}$. **[5 marks]**

12. Let $A = \{x \in \mathbb{Z} : x \equiv 4 \pmod{6} \wedge x \equiv 4 \pmod{10}\}$ and $B = \{y \in \mathbb{Z} : y \equiv 4 \pmod{60}\}$. For each of the following statements, clearly indicate whether it is true or false and then prove or disprove the statement. **[5 marks]**

(a) $A \subseteq B$.

Circle one of the following: This statement is TRUE. This statement is FALSE.

(b) $B \subseteq A$.

Circle one of the following: This statement is TRUE. This statement is FALSE.

13. Prove that for all $a \in \mathbb{Z}$, if $a^8 \not\equiv 1 \pmod{64}$, then $a \not\equiv 1 \pmod{8}$. **[4 marks]**

14. Prove that there exists a complex number z such that $|z| > 1$ and $z^{135} + (2 + 3i)z - 100 = 0$.

[3 marks]

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