University of Waterloo

Math 137 Review Session (11:00-11:45), up to Limits at Infinity and Asymptotes

Ross Evans

Question 1

Consider a sequence $\{a_n\}$, which satisfies $a_{n+1} = \sqrt{2 + a_n}$ Suppose that there exists a natural number N such that $0 < a_N < a_{N+1} < 2$. Prove that $1 < a_{N+1} < a_{N+2} < 2$.

Question 2

Consider a sequence $\{a_n\}$ and $b, c, d \in R$. Assume that a_n is defined recursively such that $a_{n+1} = \frac{b}{c+da_n}$ for all positive integers n. Assume also that the limit of a_n exists. What equation must b, c, and d satisfy such that a_n has exactly 1 possible limit?

Question 3

Evaluate the limit of

$$a_n = \left(\frac{1 + 2(-1)^n(n + 4n^2)}{5 + 7n^3}\right)\left(\sin\left(\frac{3n\pi}{7}\right) + \cos\left(\frac{1}{n}\right)\right)$$

Question 4

Spot the mistake in the following proof:

We wish to prove that $\Sigma_{n=1}^{\infty} \frac{1}{n}$ converges. Consider the limit $\lim_{n\to\infty} \frac{1}{n}$. Since the degree of the denominator is higher than the degree of the numerator, we have that $\lim_{n\to\infty} \frac{1}{n} = 0$. Hence, by the divergence test, we have that $\Sigma_{n=1}^{\infty} \frac{1}{n}$ converges.

Question 5

Consider some sequence $\{a_n\}$ where $\{a_n\}$ is monotonically increasing and $lub\{a_n\} \neq 0$. Prove that $\sum_{i=1}^{\infty} a_n$ diverges.

Question 6

Prove using the formal definition of a limit that $\lim_{n\to\infty}((4 + \frac{1}{n^2})^3) = 64$

Question 7

Prove using epsilon-delta that $\lim_{x \to 4} \frac{1}{x^2} = \frac{1}{16}$

Question 8

Prove that $\lim_{x \to 2} F(x)$ does not exist where

$$F(x) = \begin{cases} \frac{e^x}{x^{137}} & x \in \mathbb{Q} \\ \sin^2(\pi \cdot \cos(\pi \cdot x)) & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Question 9

Find (using any method) the value of $\frac{\tan(x)}{\cos^2(x)\sin(-2x)}$

Question 10

Determine (if it exists) $\lim_{x \to 0} F(x)$ where

$$F(x) = \begin{cases} |x| + 1 & x < 0 \\ 1 & x = 0 \\ \cos(x) & x > 0 \end{cases}$$

1 Continuity

1. Where are each of the following functions discontinuous?

   (a) $f(x) = \dfrac{x^2 - 4}{x + 2}$

   (b) $f(x) = \dfrac{|x|}{x}$

   (c) $f(x) = \dfrac{1}{x}$

   (d) $f(x) = \begin{cases} 2^x & \text{if } x \leq 1 \\ 3 - x & \text{if } 1 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

   (e) $f(x) = \sin(\dfrac{1}{x})$

2. Use the definition of continuity and the definition of limits to show that the function is continuous at the given number a.

   (a) $g(t) = \dfrac{t^2 + 5^t}{2t + 1}$, a = 2

   (b) $h(t) = 2\sqrt{3t^2 + 1}$, a = 1

3. Explain why the function is continuous at every number in its domain. State the domain.

   (a) $A(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$

   (b) $B(x) = \dfrac{x^2 + 1}{2x^2 - x - 1}$

   (c) $C(x) = \dfrac{e^{\sin(x)}}{2 + \cos(\pi x)}$

   (d) $D(x) = \sqrt{1 + \dfrac{1}{x}}$

4. What is an example of a function that is not continuous on some [a,b] but is continuous on (a,b)?

5. Show that there is a root of the equation $x^4 - 2x^3 + 3x - 4$ between 1 and 2 using IVT.

6. For all functions f defined on a non-empty interval I, do there exist points $c_1, c_2 \in I$ such that $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in I$? That is does f always have both a global maximum and minimum and if not why?

2 Curve Sketching

7. Sketch the curve of $f(x) = \dfrac{2e^x}{e^x - 5}$ without using the derivative.

8. Sketch the curve of $f(x) = \dfrac{(x-2)(x-1)}{(x+1)^2}$ without using the derivative.

3 Derivatives - Pt. 1

9. Let f be a continuous function at $t = a$. Is f differentiable at $t = a$?

10. Determine the value of $f'(a)$ using the definition of the derivative.

    (a) $f(x) = 4x^5$, a = 3

    (b) $f(x) = sin(x)$, a = 0

11. Determine the equation of the tangent line to the graph of $f(x) = 12x^4 - 2x + 1$ at $x = -1$ using the definition of the derivative.

MATH 135 Final Review: Session 3 Questions

1. Compute the derivative of $ln(x^3+7)+\frac{7x}{x^2-1}$.

2. Prove the product rule:
   If f and g are differentiable, then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

3. Compute the derivatives of arctan, arccos, arcsin.

4. Evaluate $\lim_{x\to 0} \frac{e^{x^2} - \cos(x)}{x^2}$.

5. Classify the critical points of the following function:

$$f(x) = \frac{x^3}{3} - x + 4$$

6. Let $f(x)$ be continuos and differentiable. Prove that if $f(x)$ has two roots, then $f'(x)$ has at least one root.

7. Determine all values of c which satisfy MVT for $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$.

8. Determine the concavity of the following function:

$$f(x) = \frac{2}{x^2} - \frac{3}{x^3}$$

Math 137 Taylor Series Review Updated

November 2019

1 Questions

1. a) Find a linear approximation of

$$y(x) = x^3 - 4x^2 + 11$$

at $x = 5$
b) Find $T_{7,0}$ in order to approximate $y(x)$. Compare the two approximations.

2. What is the maximum possible error if Taylor polynomial of degree 5 for the function

$$f(x) = \sin(x)$$

centred at $x = 0$, if used to approximate $\sin(1/2)$.

3. Use Taylor series to evaluate

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2}$$

***4. Find the following limit using Taylor's Approximation Theorem 1,

$$\lim_{x\to 0} \frac{\arctan x - x}{x^2}$$

5. Using Big-O notation, evaluate

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x\sin x}{(\ln(1+x))^4}$$

6. Let $f(x) = \ln(\cos x)$. Find $T_{7,0}$ of $f(x)$

7.Let $f(x) = (\cos x)^2$. Find the first 3 terms of the Taylor polynomial centered at $x = 0$.

8. Find the first three terms of the Taylor polynomial of the following functions

a) $f(x) = \cosh x$

b) $f(x) = \frac{1}{\sqrt{1+x^4}} - \cos x^2$