Asset Pricing and Valuation

Lecture 7: Options and Binomial Trees

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1 Options pricing

1.1 look at

https://financetrain.com/binomial-option-pricing-model-in-r

1.2 Binomial trees

- Used for pricing options
  - A stock price can follow many paths

  - Difficult to know with accuracy the path as T increases.

- Construct a binomial tree representing different possible paths that might be followed by the stock price over the life of an option
- Stock price follows a **random walk**
- process by which randomly-moving objects wander away from where they started
  - It has a probability of moving up or down a certain **percentage** point

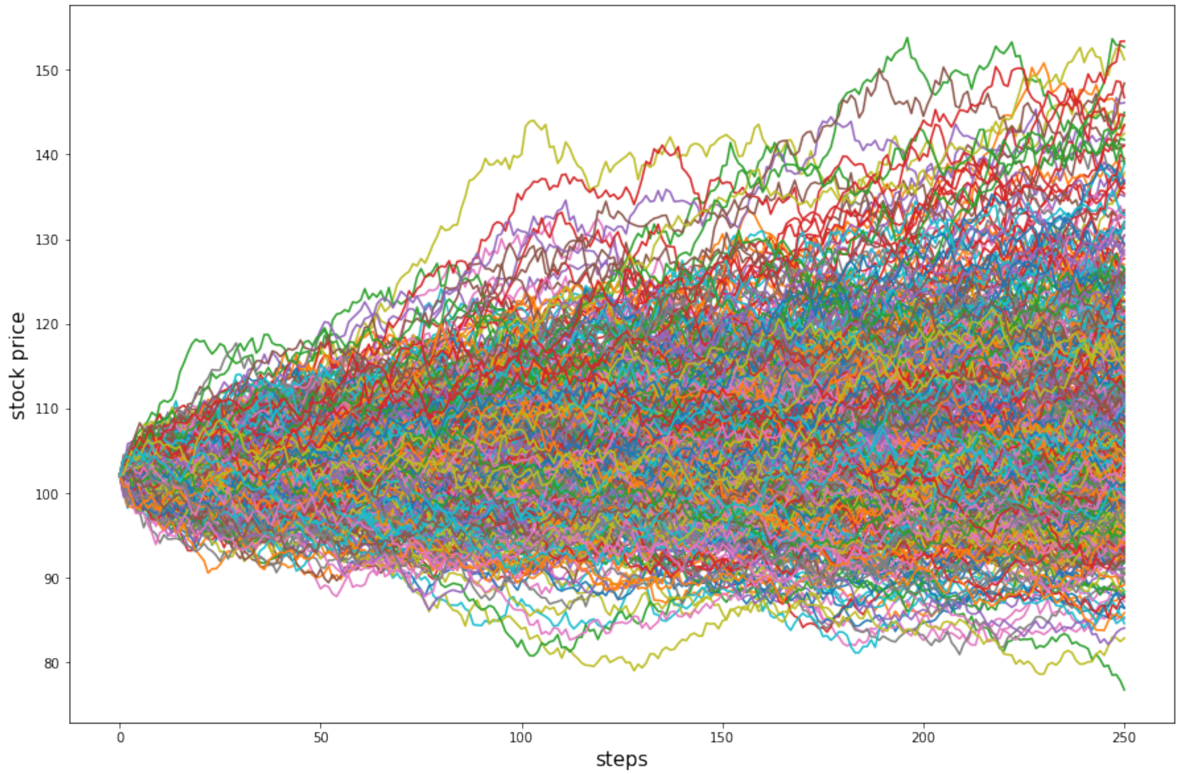


Figure 1: Stock price paths simulation

2 Overview

- Objective: Want to price an option
- Based on the concept of no-arbitrage
- **Binomial option pricing models** is a risk-free method for estimating the value of path-dependent alternatives.
- We can determinehow likely we are to **buy** or **sell** at a given price in the future.
- According to this model: The current option value is equal to the **present value** of the probability weighted **future** payoffs of the investment.

3 Basics of the Binomial Option Pricing Model

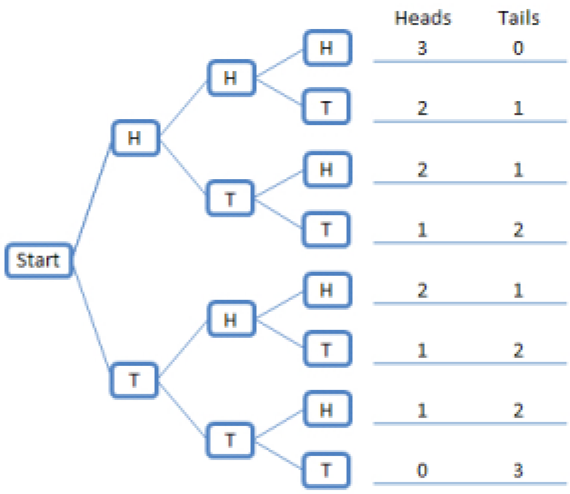
- As an investor, we are aware of the current stock price at any time.
- We want to predict future changes in stock prices
- We divide the time until option expiry into **equal** parts under this scenario (t = weeks, months, quarters)
- The binomial model uses an iterative process for each period to determine how likely the movement will be (up or down)
- We create a binomial distribution of stock prices.

4 Binomial distribution (model)

- The binomial model is usually used to price American options
- European options are usually priced using the Black Scholes Model
- William Sharpe (1978) first suggested binomial models for pricing derivatives
- In 1979 - 3 academics formalized a framework for pricing options using the binomial method
- Binomial model sometimes refered to as the Cox, Ross, Rubinstein model (John Cox, Stephen Ross and Mark Rubinstein in 1979)

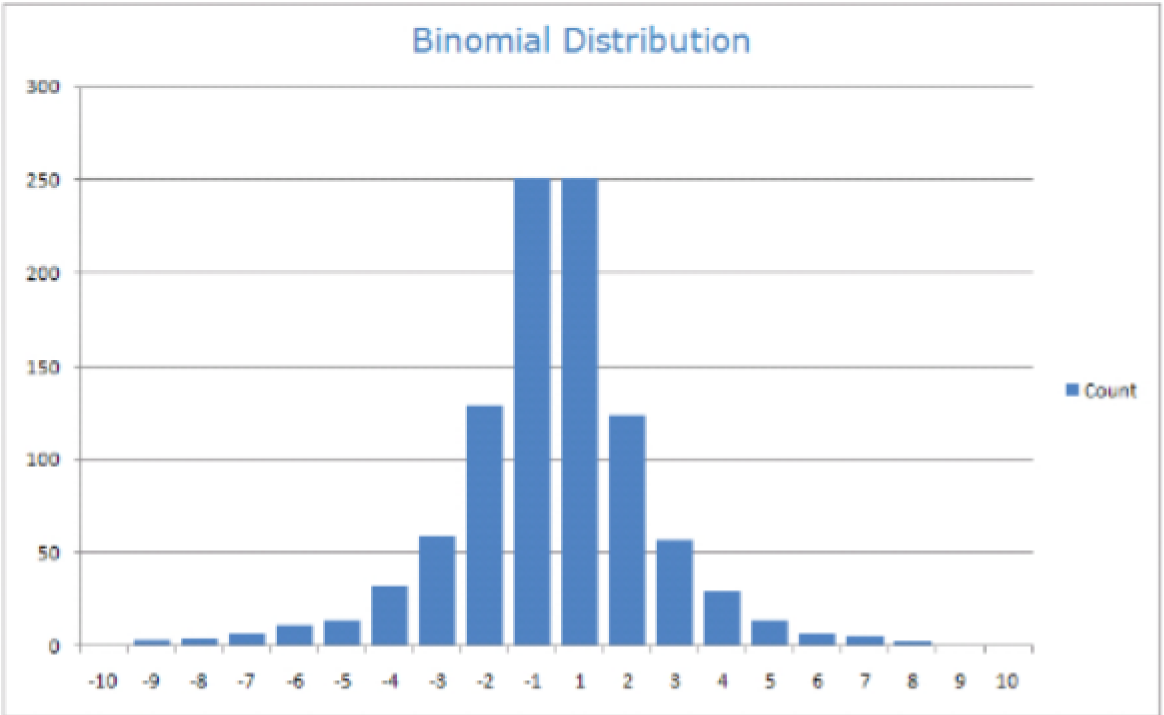
5 Binomial distribution (model)

- A binomial sequence is characterized by a series of Yes/No occurrences
- Consider a series of coin tosses (yes/no) - only two results (binomial)
- Trials (periods) are said to be independent of each other.
- Consider the following: Possible paths (coin toss) for 3 periods - each period has 50/50 probability



6 Binomial distribution (model)

- The iteration of yes/no occurrences creates a binomial distribution
- i.e. running a large series of yes/no occurrences with probability of 50/50 we obtain a distribution of outcomes.
  - 5 heads in a row, 4 heads in a row, 3 heads in a row



7 Binomial distribution (model)

- In stock investing, the binomial model is used for stock prices (up or down)
  - Period examined is broken into intervals - months, weeks, days, hours, minutes
  - At each point evaluated point the stock must therefore satisfy

$$S_u = S(1+u)$$

or

$$S_d = S(1+d)$$

- Where S_u is stock up, S_d is stock down, u is percentage increase and d is percentage decrease

8 Binomial distribution (model)

- The outcome of the up and down movements produces a binomial stock tree
- The tree describes all potential price paths that the **underlying** can take until expiry

American Call Option

	Trials			
Start	1	2	3	Payoff
			115	15
		110		
	105		105	5
100		100		
	95		95	0
		90		
			85	0

9 Binomial options pricing model

- To determine the accurate pricing for an asset is difficult
  - Which is why stock pricing change all the time (up/down)
  - Companies don't change their valuation on a daily basis -but their stock pricing fluctuate every second.
  - Question: What is the right current price today for an expected future payoff?
- Binomial options pricing model:
  - Values options using an *iterative* approach utilizing multiple periods to value American options.
  - Two possible outcomes in each iteration - **up** or **down**
  - From each iteration we obtain a **binomial tree**
  - Used often in practice (more than Black-Scholes model)

10 Binomial options pricing model

- In competitive markets, to avoid arbitrage opportunities, assets with identical payoff stuctures must have the same price
  - Valuing options is challenging with pricing variations leading to arbitrage opportunities.

    - Can use **Black-Scholes** model or **binomial options pricing**

- **Example**
    - Suppose we have $100 call options on a stock with a market price $100
        * i.e. An at the money strike price of $100 with 1 year expiry
        * Two traders A and B agree the stock price will be either $110 or $90 in one year
        * They agree on these prices but disagree on the probability of **up** or **down**
        * Tradeer A thinks the stock will go up to $110 with 60% probability
        * Trader B thinks it is 40% probability.
    - Who is willing to pay more for the **call** option?
        * Possibly A, since he is more confident the stock will go up to $110

11 Binomial options pricing model

- We have 2 assets which the valuation depends on
    - the **call option**
    - the **underlying stock**
- Both parties agree the stock price will move from $100 in one year (no other price moves possible)
    - A to $110
    - B to $90

12 Binomial options pricing model

- We create a portfolio of the two assets
    - A call option
    - Underlying stock
- **objective** - create a portfolio such that regardless of where the underlying price goes $110 or $90 the return on the portfolio is **always** the same.
- Suppose we buy d shares of the underlying and **short one call** to create this portfolio
    - If the price goes to $110 our shares are worth $\$110 * d$ and we lose $10 on the **short call** payoff
    - Portfolio net value is $(110d - 10)$ If the price goes to $90 our shares are worth $90 * d$ (option expires worthless)
    - Portfolio net value is $90 * d$

- We want the portfolio value to be the same regardless of the underlying stock price movement
  - Therefore we want $110d - 10 = 90d$
  - Therefore $d = 1/2$ (half a share of underlying)
  - Portfolio value is \$45 in one years time
- Need to compute the present value (discounting it by the risk-free rate)

$$\text{Present value} = 90d \times e^{-5\% \times 1Year} = 45 \times 0.9523 = 42.85$$

13 Binomial options pricing model

- At present, the portfolio has 1/2 a share of the underlying (priced at \$100 and one **short call**)
- Portfolio should equal the present valye

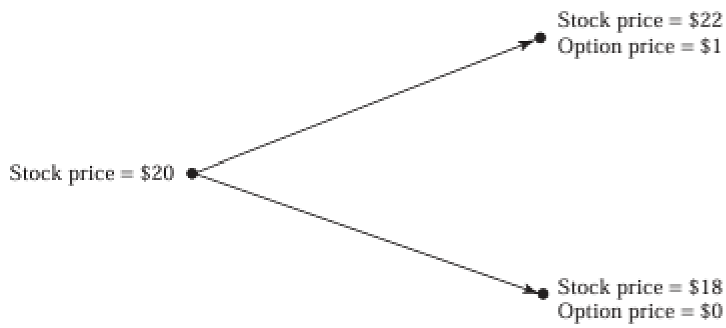
$$\frac{1}{2} \times 100 - 1 \times \text{Call price} = \$42.85$$

- Assuming that the portfolio value remains the same regardless of the underlying stock price fluctuations (risk free of the underlying movements)
- Our portfolio is neutral to price movement risk and earns the risk-free rate of return
- Both trader A and B are willing to pay the same \$7.14 for this call option (despite their different perception of probabilities 60% and 40%)
- The call option price is worth \$7.14 today
- Where is the **volatility** in the pricing of the option?
  - The volatility is included in the problems definition
  - Assuming 2 (binomial) states of price levels (\$110 and \$90) - volatility is implicit in the assumption of the two price levels

14 One step binomial model

14.1 One step binomial model

- Consider a stock is trading at \$20
- We assume the stock could be worth \$22 or \$18 in 3 months
- We want to value a European **call** option to buy the stock for \$21 in 3 months
- The option will have one of two values in 3 months
  - If stock price is \$22 then the option is valued at \$1
  - If stock price is \$18 then the option is valued at \$0



14.2 One step binomial model

- Definitions:
  - A European **call** option gives the holder the **right** but **not the obligation** to **buy** an asset at a **specified price** on a **specified date**
    - Therefore: **Writing a call**: We **sell** somebody (buyer of the call) the **right** to purchase an asset from us (writers of the call) for a **specified price** on a **specified date**.
- Consider a portfolio consisting of;
  - **A long position in Δ (Delta) shares of a stock** (own the shares)
  - **A short position in one call option** (sell the option for somebody to buy X shares)
- **We want to calculate the value of Δ which makes the portfolio riskless**
- **Case 1**
  - If stock price moves up from $20 to $22 then the value of shares is 22Δ (22 * # shares) and the value of the option is 1
  - The total value of the portfolio is $22\Delta - 1$
- **Case 2**
  - Stock goes down $20 to $18 then the value of shares is 18Δ and value of option is 0
  - Total value of the portfolio is 18Δ

14.3 One step binomial model

Set the portfolios equal to each other

$$22\Delta - 1 = 18\Delta \Delta = 0.25$$

- A riskless portfolio is therefore;
  - Long 0.25 shares
  - Short 1 option
- Stock price to \$22 then the portfolio is worth $22 \times 0.25 - 1 = \$4.5$
- Stock price to \$18 then the portfolio is worth $18 \times 0.25 - 0 = \$4.5$
- Regardless of stock price fluctuations the value is always \$4.5 (at option expiry)

14.4 One step binomial model

- A riskless portfolio must earn the risk free rate of interest
- Suppose the risk-free rate is 12%
- If a riskless portfolio is worth \$4.5 in 3 months time - we need to discount it back to the present value e^{-rT}

  $$\text{portfolio value today} 4.5 \times e^{-0.12 \times 3/12} = 4.367$$

  Therefore, if the value of the stock today is known to be \$20
- Denote the option price to be f the value of the portfolio is:

$$20\Delta - f = \text{portfolio value} 20 \times 0.25 - f = \text{portfolio value}$$

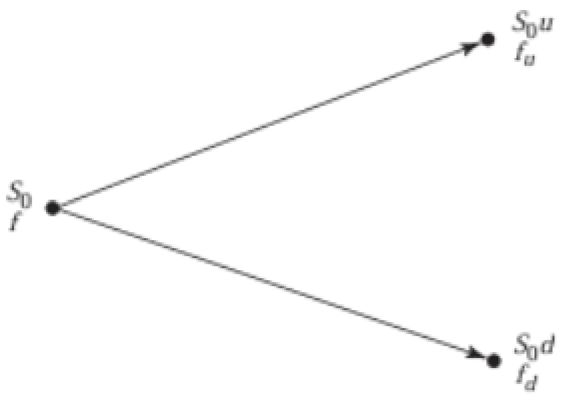
- We know the riskless portfolio is worth 4.367, so

$$5 - f = 4.367 f = 0.633$$

- Therefore the current value of the option must be 0.633
- If $f > 0.633$ the portfolio would cost less than 4.367 (earning more than the risk free rate)
- If $f < 0.633$ shorting the portfolio, we could borrow money at less than the risk free rate.

15 Generalization - One step model

- Consider a stock trading at S_0 and a stock option f, with expiry at T
- Stock can move **up** to S_0u ($u > 1$) or **down** to S_0d ($d < 1$) in one period
- The percentage increase in an upwards stock movement is $u - 1$
- The percentage decrease in a downwards stock movement is $1 - d$
- If stock moves up to S_0u the payoff from the option f_u
- If stock moves down to S_0d the payoff from the option f_d



16 Generalization - One step model

- We have a:
  - Portfolio with a long position in Δ shares
  - Write a call (short position in one option)
- **Calculate the value of Δ that makes the portfolio riskless**
- **Upwards movement** in stock $S_0u\Delta - f_u$ | (similar to $22\Delta - 1$)
- **Downwards movement** in stock $S_0d\Delta - f_d$ | (similar to $18\Delta - 0$)
- Set them equal:

$$S_0u\Delta - f_u = S_0d\Delta - f_d$$

- Solve for Δ

$$S_0u\Delta - S_0d\Delta = f_u - f_d \Delta(S_0u - S_0d) = f_u - f_d \frac{\Delta(S_0u - S_0d)}{S_0u - S_0d} = \frac{f_u - f_d}{S_0u - S_0d}\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

- So Δ is a ratio of the change in the option price to the change in the stock price

17 Generalization - One step model

- For a riskless portfolio it must earn the risk free rate of interest
- Suppose the risk-free rate is r
- If a riskless portfolio worth $S_0 u\Delta - f_u$ in T months time (future value) - we need to discount it back to the present value using e^{-rT}

$$(S_0 u\Delta - f_u)e^{-rT}$$

- So r is our risk free rate - then $(S_0 u\Delta - f_u)e^{-rT}$ is our riskless portfolio in PV terms.

- **NOTE: Similar to previous answer where we found our riskless portfolio value of $4.5 (future value) and had to discount it back to PV** $4.5 \times e^{-0.12\times 3/12} = 4.367$ **using the risk free rate of 12%**

$$S_0\Delta - f = \text{portfolio value today}$$

- **NOTE: Similar to previous example** $20\Delta - f$
- We know the riskless portfolio **today** is worth $(S_0 u\Delta - f_u)e^{-rT}$, so our portfolio value **today** equals the risk free portfolio value

$$S_0\Delta - f = (S_0 u\Delta - f_u)e^{-rT}$$

- **NOTE: Previously our PV of the riskless portfolio was worth 4.367, we plug in the same values Similar to our** $5 - f = 4.367$
- Solving for f to get our options price - of a risk-free interest portfolio

$$f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT}$$

18 Generalization - One step model

- We know that;
- Δ - ratio of the change in the option price to the change in the stock price
- f - current value of the options price

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT}$$

- Substituting Δ into f

$$f = S_0(\frac{f_u - f_d}{S_0 u - S_0 d})(1 - ue^{-rT}) + f_u e^{-rT}$$

- or;

$$f = \frac{f_u(1 - de^{-rT}) + f_d(ue^{-rT} - 1)}{u - d}$$

- or;

$$f = e^{-rT}[pf_u + (1+p)f_d]\text{Where; } p = \frac{e^{rT} - d}{u - d}$$

19 Generalization - One step model

We have:

$$f = e^{-rT}[pf_u + (1+p)f_d]\text{Where; } p = \frac{e^{rT} - d}{u - d}$$

- Which enable an option to be priced when stock price movements are given by a one-step binomial tree.
- Here p should be interpreted as the probability of an up movement and (1-p) a probability of a down movement
- The $[pf_u + (1+p)f_d]$ is the expected future payoff from the option with e^{-rT} discounting it back to today.

20 Generalization - One step model

- Consider a stock trading at S_0 and a stock option f, with expiry at T
- Stock can move **up** to $S_0 u$ ($u > 1$) or **down** to $S_0 d$ ($d < 1$) in one period
- The percentage increase in an upwards stock movement is $u - 1$
- The percentage decrease in a downwards stock movement is $1 - d$
- If stock moves up to $S_0 u$ the payoff from the option f_u
- If stock moves down to $S_0 d$ the payoff from the option f_d
- Consider the following example:
  - $u = 1.1$, $d = 0.9$, $r = 0.12$, $T = 0.25$, $f_u = 1$ and $f_d = 0$

- Recall,

$$p = \frac{e^{rT} - d}{u - d}$$

- So;

$$p = \frac{e^{0.12 \times 3/12} - 0.9}{1.1 - 0.9} = 0.6523$$

- Recall p is the probability of an up movement and $(1 - p)$ is a probability of a down movement.
- The options payout given by plugging p into:

$$f = e^{-rT}[pf_u + (1+p)f_d]f = e^{-0.12 \times 3/12}[0.6523 \times 1 + (1 + 0.3477) \times 0] = 0.633$$

- Which is exactly what we got in our previous answer.

21 Revisit example 1

- Consider a stock is trading at \$20
- We assume the stock could be worth \$22 or \$18 in 3 months
- European call with a strike of \$21
- Risk free rate of interest is 12% per annum.
- p is the probability of an upwards movement in stock \$(1-p) is the probability of a downwards movement in stock
- Recall probability of an upwards movement is;

$$p = \frac{e^{rT} - d}{u - d} p = \frac{e^{0.12 \times 3/12} - 0.9}{1.1 - 0.9} = 0.6523$$

- or p can be computed as

$$22p + 18(1 - p) = 20e^{0.12 \times 3/12} p = \frac{20e^{0.12*(3/12)} - 18}{4} p = 0.6523$$

- So at the end of the 3 months the call option has a 0.6523 probability of being worth \$1 and 0.3477 probability of being worth \$0
- The expected value of the option in the future is; (multiply probabilities to payoff);

$$0.6523 \times 1 + 0.3477 \times 0 = 0.6523$$

- Discount the FV back to the PV

$$0.6523e^{-0.12 \times 3/12} = 0.633$$

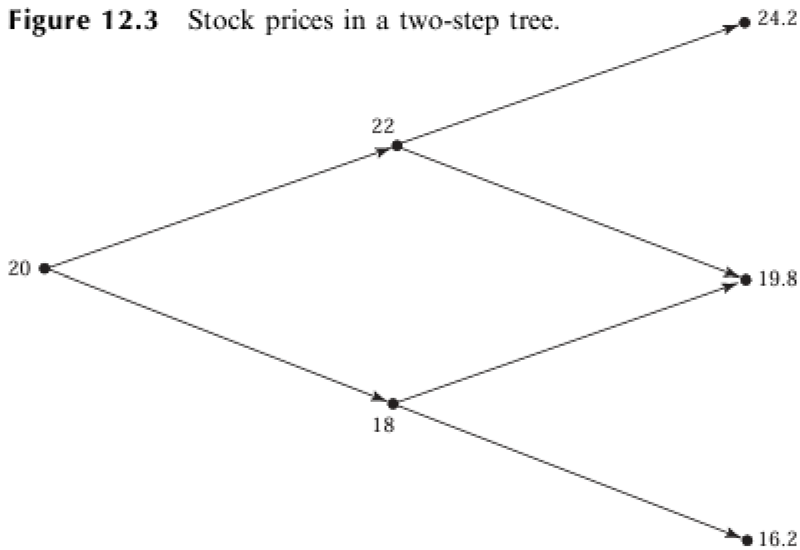
- This is the same value as we obtained in the binomial tree model (Indicates that there is no arbitrage opportunity when portfolio uses the risk-free rate)

22 Two step binomial tree

- Stock price is \$20
- May go up by 10% or down by 10% in both periods
- Each time step is 3 months
- risk free rate $r = 12\%$ per annum
- Consider a 6 month option with a strike of \$21
- We want to calculate the options price today

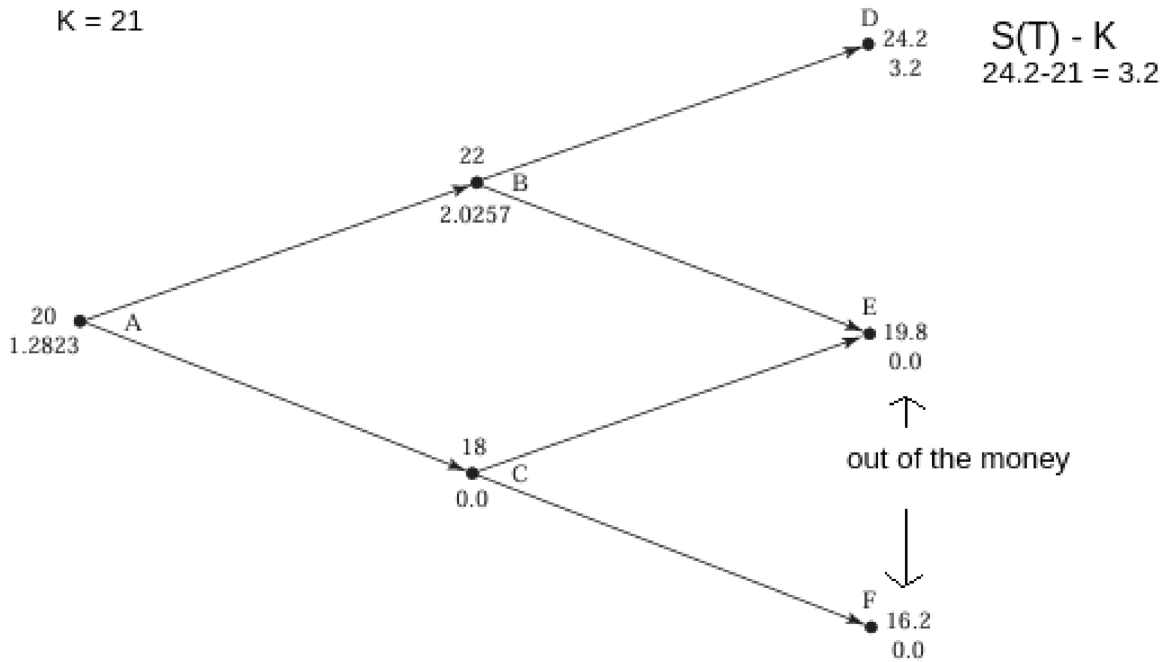
23 Two step binomial tree

Figure 12.3 Stock prices in a two-step tree.



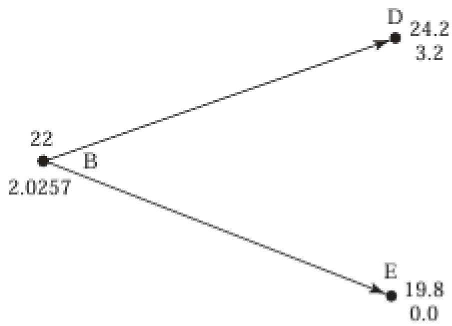
24 Two step binomial tree

Figure 12.4 Stock and option prices in a two-step tree. The upper number at each node is the stock price and the lower number is the option price.



25 Two step binomial tree

- Calculate the options price at node **B**
- Suppose $u = 1.1$, $d = 0.9$, $r = 0.12$, $T = 0.25$, we know $p = 0.6523$
- Recall; $f = e^{-rT}[pf_u + (1+p)f_d]$
- $e^{-0.12\times(3/12)}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$
- Now we know the options price at node **B** is 2.0257
- Value at node **A** is $e^{-0.12\times(3/12)}(0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823$



26 Delta

- A greek letter and used in pricing and headging of options
- The Δ of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.
- Its the number of units of stock we should hold for each option shorted in order to create a riskless portfolio.
- Delta hedging is the creation of a **riskless** portfolio
- Δ of a call option is **positive**
- Δ of a put option is **negative**

27 Delta

- Recall;

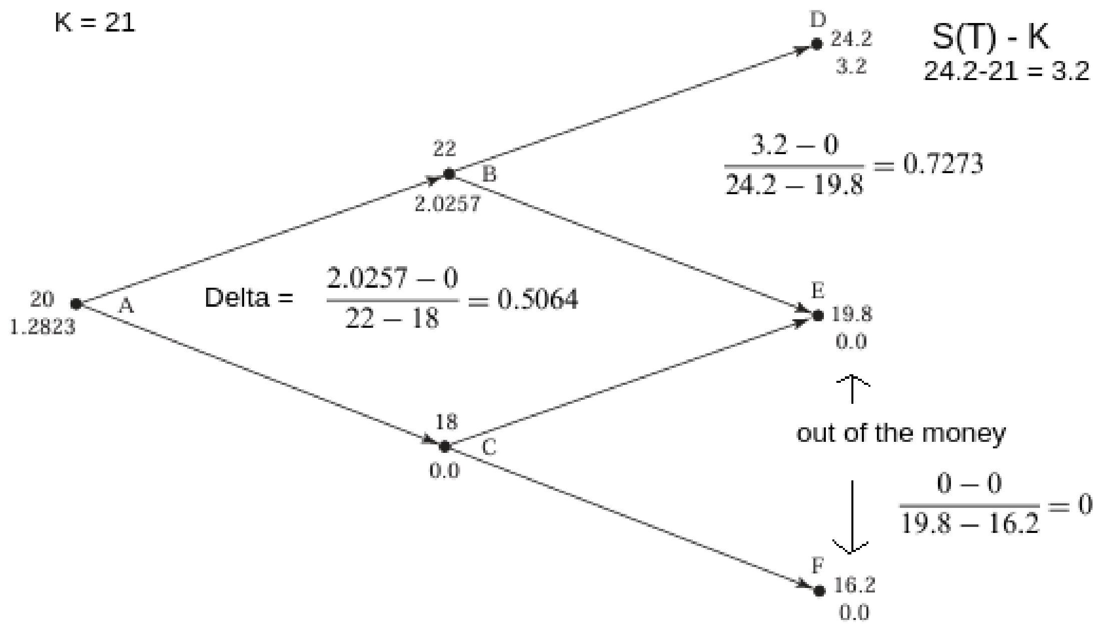
$$\frac{1-0}{22-18} = 0.25$$

- So when the stock price changes from 18 to 22, the option price changes from 0 to 1

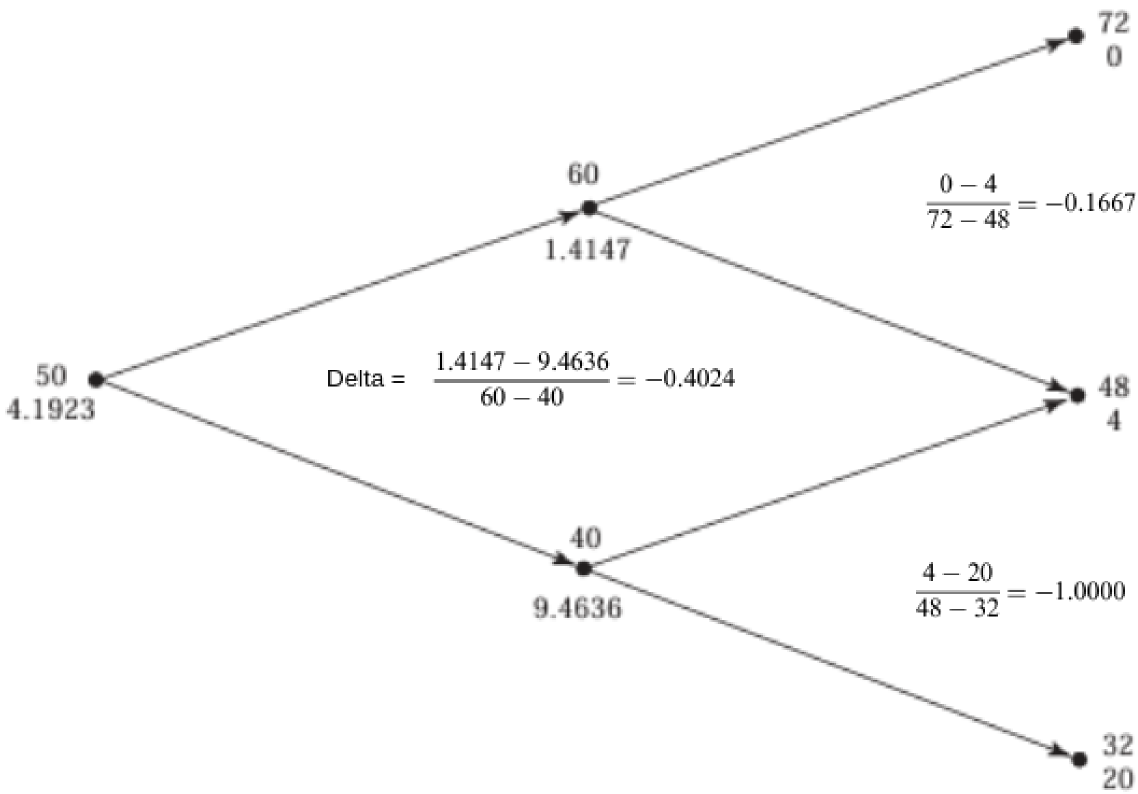
- Recall (in the 2 step binomial example)

- We can compute the Δ's for each time step for each movement (upwards or downwards)

Figure 12.4 Stock and option prices in a two-step tree. The upper number at each node is the stock price and the lower number is the option price.



28 Delta (Example 2)



29 Delta

- In both examples, Δ changes over time
- In the first, from 0.5064 to either 0.7273 or 0
- In the second, from -0.4024 to either -0.1667 or - 1.000