Asset Pricing and Valuation

Lecture 3: Portfolio Theory

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Risk and Return

Investment Objective Function

- *Maximize expected return* for a given level of risk

Stock	Return	Risk
Stock A	10%	25%
Stock B	20%	25%

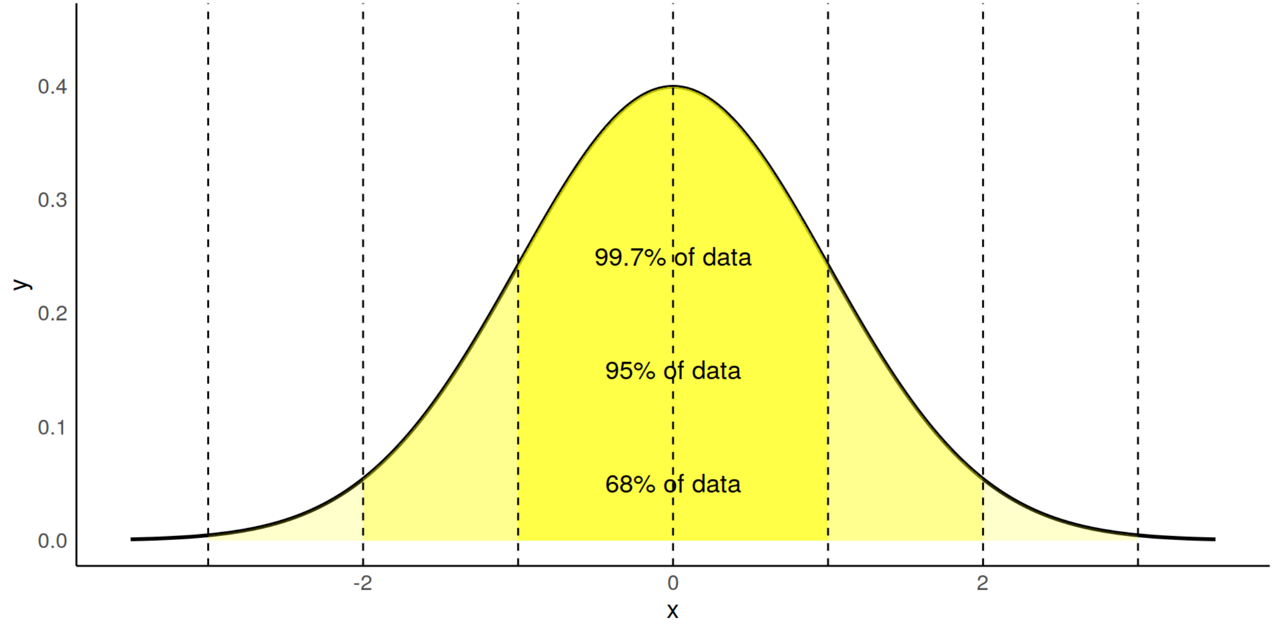
- Which stock should we choose?
  - Stock B has a higher expected return for the same level of risk
- *Minimize risk* for a given level of expected return

Stock	Return	Risk
Stock A	10%	15%
Stock B	10%	30%

- Which stock should we choose?
  - Stock A has a lower risk for the same level of expected return

Investor Assumptions

- Investors are risk averse
- Investors are rational
- Investment returns follow a *Normal Distribution*
  - Allows us to use just 2 parameters for evaluation investment performance
    - **Mean** (to measure return)
    - **Variance** or **Standard Deviation** (to measure risk) - greater the SD the greater the risk



Example (single stock)

Suppose we have the following historical data

Year	Return
1	0.05
2	0.04
3	-0.07
4	0.09

- Our future *expected* return would be the average of the historical returns.

$$R = \frac{\sum r}{n} = \frac{r_1 + r_2 + r_3 + r_4}{n}$$

$$r = \frac{0.05 + 0.04 + (-0.07) + 0.09}{4} = 0.0275 = 2.75\%$$

Example (single stock)

- Our future *expected* risk is determined by using the variable (s^2)

$$
\begin{aligned}
s^2 &= \frac{\sum(r-\bar{r})^2}{n-1} \\
&= \frac{(r_1-\bar{r})^2+(r_2-\bar{r})^2+(r_3-\bar{r})^2+(r_4-\bar{r})^2}{n-1} \\
&= \frac{(0.05-0.0275)^2+(0.04-0.0275)^2+(-0.07-0.0275)^2+(0.09-0.0275)^2}{4-1} \\
&= 0.0047
\end{aligned}
$$

- Can compare the variance of 2 or more investments - This is not equivalent to a percentage - its just a number - The larger it is, the more risky the investment

Standard Deviation

- To convert to a unit of measurement we can use the **Standard Deviation** (s)

$$s = \sqrt{s^2} = \sqrt{0.0047} = 0.0686 = 6.86\%$$

- Higher percentages in their SD suggest more risky investments

Coefficient of Variation (CV)

$$CV = \frac{s}{\bar{r}} = \frac{0.0686}{0.0275} = 2.50$$

- CV measures risk per unit of return - So for every unit of return goes with about 2.5 units of risk - The higher the CV the more risky the investment - Can compare investments with different expected returns

Portfolio Diversification

Evaluation of a portfolio

Stock	Expected.Return	Standard.Deviation	Amount.Invested	Investment.Proportion
Stock A	0.23	0.27	$50,000	0.5
Stock B	0.13	0.23	$30,000	0.3
Stock C	0.10	0.09	$20,000	0.2

- So we have 100,000 to invest and proportionally invest it.
- Expected return is simply the weighted average $\hat{r}_p = \sum w_i \hat{r}_i = w_1 \hat{r}_1 + w_2 \hat{r} 2 + \ldots + w_N \hat{r}_N$

$$\hat{r}_p = (0.50)(0.23) + (0.30)(0.13) + (0.20)(0.10) = 0.174 = 17.4\%$$

Portfolio Risk

- Variance of a portfolio (σ_p^2)
- Standard deviation of a portfolio (σ_p)
- Factors affecting portfolio risk
- Individual stock risk σ_i^2
- Investment proportion W_i
- Degree of diversification
- Measured by Covariance (σ_{ij}) or
- Correlation coefficient (ρ_{ij})

Portfolio Diversification

Suppose we had the following options

Option.A	Option.B	Option.C
Google	Google	Google
	Microsoft	Deutsche Bank
	Apple	American Airlines
	Intel	Target

- Which portfolio would you choose?
- Option A:
  - Single investment
  - No diversification
- Option B:
  - Portfolio
  - Not well diversified
  - High correlation
- Option C:
  - Portfolio
  - Well diversified
  - Low correlation

Portfolio Diversification example

- Suppose we have the following portfolio returns

Year	Return1	Return2
1	0.50	0.25
2	0.40	0.15
3	-0.07	0.32
4	0.09	-0.20

- The sample mean of Stock A $\bar{r_A} = 0.23$
- The sample mean of Stock B $\bar{r_B} = 0.13$
- The sample standard deviation of Stock A is $s_A = 0.2655$
- The sample standard deviation of Stock B is $s_B = 0.2308$

Sample Covariance: Cov(A, B)

$$Cov(A,B) = \frac{\sum(r_A - \bar{r_A})(r_B - \bar{r_B})}{n-1}$$

For securities A and B,

$$Cov(A,B) = \frac{(r_{A1} - \bar{r}_A)(r_{B1} - \bar{r}_B) + (r_{A2} - \bar{r}_A)(r_{B2} - \bar{r}_B) + (r_{A3} - \bar{r}_A)(r_{B3} - \bar{r}_B) + (r_{A4} - \bar{r}_A)(r_{B4} - \bar{r}_B)}{n-1}$$

- Substituting:

$$Cov(A,B) = \frac{(0.50 - 0.23)(0.25 - 0.13) + (0.40 - 0.23)(0.15 - 0.13) + (-0.07 - 0.23)(0.32 - 0.13) + (0.09 - 0}{4-1}$$

$$= \frac{0.0250}{3} = 0.00833$$

Covariance: Cov(A, B)

- We have the following: $Cov(A,B) = 0.00833$
  - If this was 0 then we conclude that these two portfolio have nothing in common
  - If this was negative then the securities are moving in opposite direction (when one does well the other does bad)
  - If this was positive then the securities are moving in the same direction (when one does well the other does well)

Correlation Coefficient: $r_{A,B}$

$$r = \frac{Cov(A,B)}{(S_A)(S_B)}$$

$$= \frac{0.00833}{(0.2655)(0.2308)} = 0.14,\ r = -1 \leq r \leq 1$$

We can solve for the $Cov(A,B)$ as:

$$Cov(A,B) = (r_{A,B})(S_A)(S_B)$$

- Where $r = 0$ that means that the $Cov(A,B)$ is 0 and the securities are not related at all.
- A covariance matrix value of -1 between two investments indicates a perfect negative correlation.
- A covariance matrix value of 1 between two investments indicates a perfect positive correlation.
- A covariance matrix value of 0 between two investments indicates no correlation.
  - This means that when one investment performs well, the other performs poorly, and vice versa. In practical terms, it suggests that the returns of these two investments move in exactly opposite directions.
  - By combining assets that are negatively correlated, you can potentially reduce the overall risk of the portfolio.
  - When one asset experiences a decline in value, the loss might be offset by an increase in the value of the other asset.
  - When its -1 then there is nothing to diversify, the two securities move in tandem.

Portfolio risk: N asset portfolio

Variance of a 2-Asset portfolio

$$\Sigma = \begin{bmatrix} \sigma_A\sigma_A & \sigma_A\sigma_B \\ \sigma_B\sigma_A & \sigma_B\sigma_B \end{bmatrix} = \begin{bmatrix} \sigma_A^2 & \sigma_A\sigma_B \\ \sigma_B\sigma_A & \sigma_B^2 \end{bmatrix}$$

- Portfolio variance is:

$$\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_A\sigma_B$$

- Which is essentially the *weighted average of the variance of security A*
  - the *weighted average of the variance of security B + 2 times the weighted average of the covariance of A and B*
  - The 2 reminds us that there are two observations

$$\sigma_2\sigma_1 \text{ and } \sigma_1\sigma_2$$

- Suppose we have the following:
  - Variance of A: $\sigma_A^2 = 0.03248$
  - Variance of B: $\sigma_B^2 = 0.0231$
  - $Cov_{A,B} = -0.0164$
  - Note $Cov_{A,B} = (\rho_{A,B})(\sigma_A)(\sigma_B)$
  - $w_A = 0.7$
  - $w_B = 0.3$ (weights, we choose these)

$$\sigma_p^2 = (0.7)^2(0.03248) + (0.3)^2(0.0231) + 2(0.7)(0.3)(-0.0164)$$
$$= 0.0111$$
$$\sigma_p = \sqrt{\sigma_p^2} = \sqrt{\text{Portfolio variance}} = \sqrt{0.0111} = 0.1054 = 10.54\%$$

Variance of an N-Asset portfolio

- The variance of a portfolio of N assets is given by:
  $\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$
- Where σ_{ij} is the covariance of asset i and j and w_i is the proportion of the portfolio invested in asset i.
- Its composed of the:
  - Individual security risks σ_i
  - Portfolio diversification σ_{ij}
  - Investment proportions w_i

$$\Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{bmatrix}$$

The Efficient Set

Basics

- The set of the possible portfolios a risk adverse investor will invest in.
- We need the *mean"* μ_P and standard deviation* σ_P of the portfolio, given as;

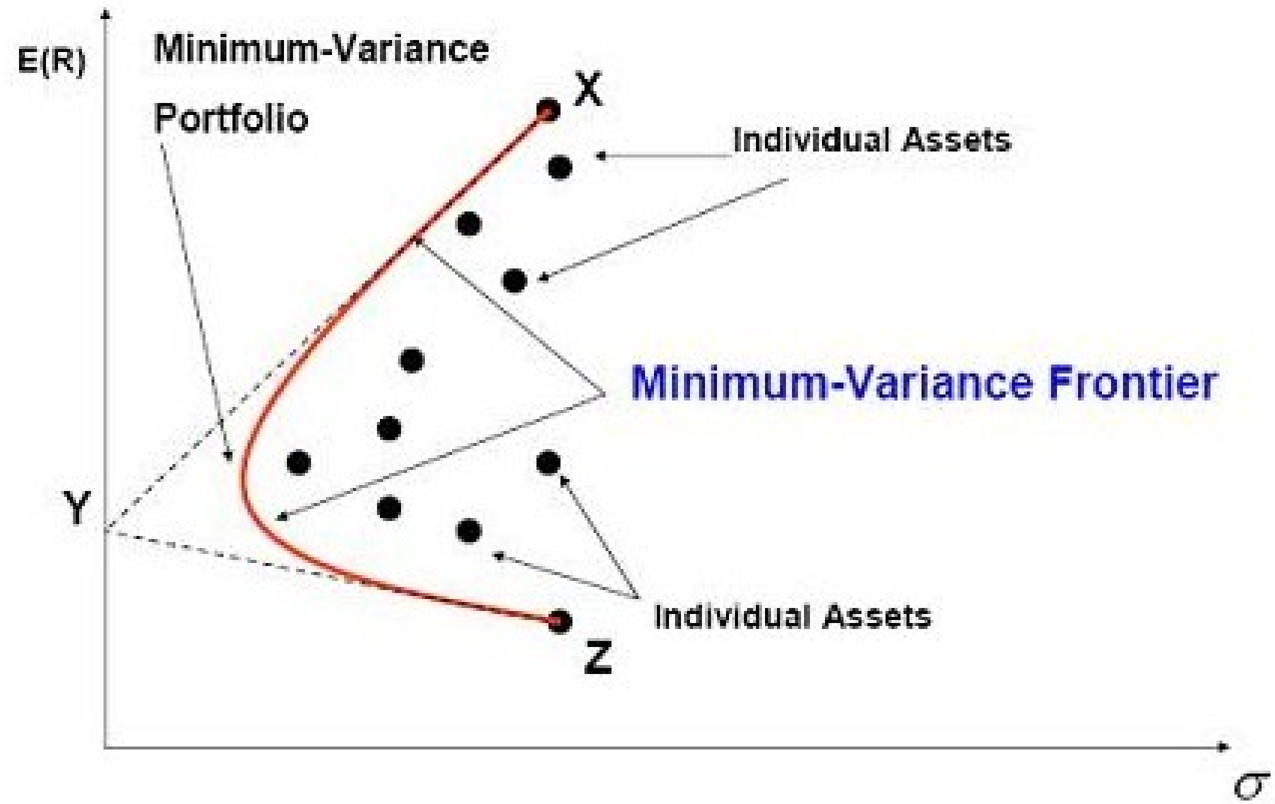
$$\mu_P = \hat{r}_P = w_A \hat{r}_A + w_B \hat{r}_B$$
$$\sigma_P = \sqrt{\sigma_P^2} = \sqrt{w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_A \sigma_B \rho_{A,B}}$$

-The mean being the weighted average of the returns in the portfolio

- The standard deviation being the weighted average of the variance of the securities in the portfolio + the weighted average of the covariance of the securities in the portfolio

Basics 2

- We plot the risk σ and return μ of the portfolio on a graph



- Y is our *minimum variance portfolio* - lowest risk (lowest σ)
- X is the maximum return portfolio (highest μ)
- Z is the worst performing portfolio
- (All portfolios below X are inefficient) - as we take the same risk but get less returns! for a different combination of asset weights.
- **So where do we belong on the efficient frontier**? - assume more risk for more return?
- Need to compare it to the “Market” - does our weight combination outperform the SYP500?

Portfolio Optimisation

- Markowitz (1958) set the basis for modern portfolio analysis. In this context, rational investors determine the optimal composition of their portfolio on the basis of expected returns (means) of individual assets and risk (covariance matrix)
- Objective:
  - Maximize expected return
  - Minimize the risk

Risk and Return

Returns

- Expected return for a group of n assets is calculated as;

$$E(R) = \sum_{i=1}^{n} p_i r_i$$

Risk

- Generally defined as the “dispersion of outcomes around the expected value”
- Variance and Standard Deviation measure dispersion
- The variance of a random variable X is

$$\sigma_X^2 = E(X - E(X)]^2 = \sum_{i=1}^{n} p_i [x_i - E(X)]^2$$

Where;

Risk and Return

Covariance of Returns

- Measures the association between two (or more) random variables
- The covariance is positive if the variables tend to move in the same direction
- THe covariance is negative if the variables tend to move in opposite directions

$$Cov(r_i, r_j) = E\{[r_i - E(r_i)][r_j - E(r_j)]\} = \sum_{s=1}^{n} p_s\{[r_{is} - E(r_i)][r_{js} - E(r_j)]\}$$

Correlation

- Standardized by standard deviation lies between -1 and 1

$$\rho_{XY} = \frac{Cov(r_i, r_j)}{\sigma_i \sigma_j}$$

Portfolio Risk and Return

Portfolio Return

$$R_p = \sum_{i=1}^{n} w_i r_i$$

Portfolio Risk

$$\sigma_p^2 = E[r_p - E(r_p)]^2$$

Where $r_p = (w_1 r_1 + w_2 r_2)$ for a 2 asset portfolio.

$$\sigma_p^2 = E\{w_1 r_1 + w_2 r_2 - [w_1 E(r_1) + w_2 E(r_2)]\}^2$$
$$\sigma_p^2 = E\{w_1^2 [r_1 - E(r_1)]^2 + w_2^2 [r_2 - E(r_2)]^2 + 2 w_1 w_2 [r_1 - E(r_1)][r_2 - E(r_2)]\}$$

Where $\sigma_{12} = Cov(r_1, r_2)$

Portfolio Risk and Return

Portfolio Risk (N Assets)

- Consider 3 assets
- Mean returns

$$\mu = \begin{pmatrix} \mu_a \\ \mu_b \\ \mu_c \end{pmatrix}$$

- Matrix of weights

$$w = \begin{pmatrix} w_a \\ w_b \\ w_c \end{pmatrix}$$
$$\sum w = 1$$

- Variance-covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} & \sigma_{ac} \\ \sigma_{ab} & \sigma_b^2 & \sigma_{bc} \\ \sigma_{ac} & \sigma_{bc} & \sigma_c^2 \end{pmatrix}$$

- Portfolio expected return

$$\mu_p = w'\mu$$

- Portfolio variance

$$\sigma_p^2 = w'\Sigma w$$

Markowitz Minimum Variance Portfolio

Markowitz

> Note
>
> **Objective:** Rational investors; maximize the expected return for a given risk, and they minimize the risk for a given return.

- A minimum variance set is the set of all portfolios that have the least volatility for each level of possible expected return.
- An efficient set (frontier) is the part of the minimum variance frontier that offers the highest expected return for each level of standard deviation (risk).
- Given the level of risk or standard deviation, investors prefer positions with higher expected return and given the expected return, they prefer the positions of lower risk.
- Taking this into account, we can determine the minimum variance set.
- The point where the standard deviation is at its lowest is the Global Minimum Variance portfolio (GMV).
- Portfolios that lie from the GMV portfolio upwards provide investors with the best risk–return combinations and thus are the candidates for the optimal portfolio.
- These portfolios are called the efficient set (frontier); and in order to be on the efficient frontier, the portfolios have to satisfy the following criterion:
- Given a particular level of standard deviation, the portfolios in the efficient set have the highest attainable expected rate of return.

Minimum variance portfolio

- A minimum variance (risk) portfolio objective is the following

$$
\begin{gathered}
Minimise\, \sigma_p^2 = w' \Sigma w \\
\text{subject to} \\
w' \mu = \mu_p \\
\text{and} \\
w' 1 = 1
\end{gathered}
$$

- For a portfolio with Two Risk Assets, the mean variance portfolio weights are calculated as

$$
\begin{gathered}
w_1 = \frac{\sigma_2^2 - Cov(r_1, r_2)}{\sigma_1^2 + \sigma_2^2 - 2Cov(r_1, r_2)} \\
w_2 = 1 - w_1
\end{gathered}
$$

- The above is derived after taking the first derivative of the portfolio variance w.r.t the weight of asset 1, w_1. Set that derivative equal to zero and solve for $w_{1}.

$$
\frac{\partial \sigma_p^2}{\partial w_1} = \frac{\partial}{\partial w_1}\left[w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2 w_1 (1 - w_1) \sigma_{12}\right]
$$

CAPM

- https://www.youtube.com/watch?v=3T2Gd133juk&list=PL6Y8SvWdPo0_H3V0e4NWuM8b5mjVL_9qm&index=

Security Market Line

- https://www.youtube.com/watch?v=WCaoTVnD42A&list=PL6Y8SvWdPo0_H3V0e4NWuM8b5mjVL_9qm&index

Alpha and Arbitrage Pricing Theory

- https://www.youtube.com/watch?v=LV9CS-0SxJ0&list=PL6Y8SvWdPo0_H3V0e4NWuM8b5mjVL_9qm&index=8&ab_cha