

On the Nash equilibrium of moment-matching GANs for stationary Gaussian processes

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Abstract

Generative Adversarial Networks (GANs) learn an implicit generative model from data samples through a two-player game. In this paper, we study the existence of Nash equilibrium of the game which is consistent as the number of data samples grows to infinity. In a realizable setting where the goal is to estimate the ground-truth generator of a stationary Gaussian process, we show that the existence of consistent Nash equilibrium depends crucially on the choice of the discriminator family. The discriminator defined from second-order statistical moments can result in non-existence of Nash equilibrium, existence of consistent non-Nash equilibrium, or existence and uniqueness of consistent Nash equilibrium, depending on whether symmetry properties of the generator family are respected. We further study empirically the local stability and global convergence of gradient descent-ascent methods towards consistent equilibrium.

Keywords: GANs, Nash equilibrium, moment-matching, stationary process, statistical consistency

1. Introduction

Estimating the probability distribution of data from finite samples is a classical problem in statistics and machine learning. Unlike conventional models based on a probability density function, Generative Adversarial Networks (GANs) aim to learn a generator, which describes how to draw samples of model distributions, through a two-player game with a discriminator. A central question in GANs is what type of solutions of the game is suitable for learning the data distribution. The original formulation (Goodfellow et al., 2014) shows that Nash equilibrium exists in an ideal realizable setting, where infinite data samples can be generated from a ground-truth generator. However, in some unrealizable settings, Farnia and Ozdaglar (2020) show that GANs may have no Nash equilibrium. Indeed, many research works are devoted to extending the notion of Nash equilibrium to other types of solutions such as mixed Nash equilibrium (Arora et al., 2017), proximal equilibrium (Farnia and Ozdaglar, 2020), or local minimax (Jin et al., 2020). Nevertheless, in practice, GANs are often trained with gradient-based methods (Mescheder et al., 2017; Brock et al., 2019), and it is shown that Nash equilibrium is typically contained in the limiting points of such methods (Daskalakis and Panageas, 2018). This motivates one to study in what situations GANs have meaningful Nash equilibrium, and whether it is possible to find such solution using gradient-based methods.

To make the problem concrete, we focus on a special instance of GANs in the realizable setting with finite-samples. In particular, we study a notion of consistent Nash equilibrium, which allows one to estimate the ground-truth generator as the number of samples grows to infinity. The main contribution of this paper is to show that the existence of consistent Nash equilibrium depends crucially on the choice of the discriminator family. More precisely, in order to learn the ground-truth generator of a stationary Gaussian distribution, we find that a suitable discriminator family

based on second-order moments which respects the symmetry property of stationary processes, i.e. the translational invariance, can result in the existence and the uniqueness of consistent Nash equilibrium. When this is not the case, Nash equilibrium may not exist due to a finite number of samples. More surprisingly, we find that there can exist consistent non-Nash equilibrium around which gradient-descent-ascent methods are nearly stable as the number of samples goes to infinity. It indicates why in practice, GANs training may empirically converge to non-Nash equilibrium while still achieving good results with respect to standard evaluation metrics (Berard et al., 2020).

The paper is organized as follows: Section 2 introduces Moment-matching GANs for stationary Gaussian distributions, as well as the notion of consistent Nash equilibrium. The generator is parameterized by a linear convolutional network. To define the discriminator, we take a moment-matching perspective as in MMD GANs (Li et al., 2015; Dziugaite et al., 2015). Section 3 presents the main theoretical results about the existence of consistent Nash equilibrium on three different families of discriminators. Section 4 studies numerically how gradient-based methods for two-player games behave on the considered GANs from either local or global convergence point of view. Section 5 discusses the challenges to extend our results to non-Gaussian distributions.

Notations: We use $\|\cdot\|$ to denote the Euclidean metric in a finite dimensional space. The identity matrix in dimension d is I_d . For $x \in \mathbb{R}^d$, the discrete Fourier transform of x is written \hat{x} . It is defined by $\hat{x}(\omega) = \sum_u x(u)e^{-i\omega u}$ for $u \in \{0, \dots, d-1\}$ and $\omega \in \Omega_d = \{\frac{2\pi\ell}{d} | 0 \leq \ell \leq d-1\}$. For a complex number z , z^* denotes its complex conjugate. For $x \in \mathbb{C}^d$, we write $\tilde{x}(u) = x(-u)^*$. The inner product between $x \in \mathbb{C}^d$ and $y \in \mathbb{C}^d$ is written $\langle x, y \rangle = \sum_u x(u)^* y(u)$.

2. Preliminaries

2.1. Moment matching GANs

We consider a class of GANs defined by a generator $g_\alpha : \mathbb{R}^k \rightarrow \mathbb{R}^d$, and a discriminator $f_\beta : \mathbb{R}^d \rightarrow \mathbb{R}^m$. They are parameterized by $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ in finite dimensional Euclidean spaces. In this paper, we consider the realizable setting where there is a ground-truth model $g_{\bar{\alpha}}$ which generates the data $X = g_{\bar{\alpha}}(\bar{Z})$ from a random vector \bar{Z} for some $\bar{\alpha} \in \mathcal{A}$. Our goal is to find a solution $(g_{\alpha^\bullet}, f_{\beta^\bullet})$ of GANs such that g_{α^\bullet} is close to $g_{\bar{\alpha}}$ in terms of their probability distributions.

A moment matching GAN can be formalized as the following min-max problem,

$$\min_{\alpha \in \mathcal{A}} \max_{\beta \in \mathcal{B}} \|\mathbb{E}(f_\beta(X)) - \mathbb{E}(f_\beta(g_\alpha(Z)))\|^2. \quad (1)$$

We shall consider the min-max problem with n finite samples

$$\min_{\alpha \in \mathcal{A}} \max_{\beta \in \mathcal{B}} V_n(\alpha, \beta) = \|\mathbb{E}_n(f_\beta(X)) - \mathbb{E}_n(f_\beta(g_\alpha(Z)))\|^2. \quad (2)$$

The empirical expectation $\mathbb{E}_n(f(X))$ of a function f of a random variable X is computed from n i.i.d samples of X . The metric $\|\cdot\|$ in (1) and (2) is chosen such that this is an MMD GAN in the Euclidean space \mathbb{R}^m . It is related to various GANs, from the perspective of feature matching (Liu et al., 2017).

2.2. Stationary Gaussian processes

We specify the generator g_α to model stationary Gaussian processes observed in a finite and discrete-time interval. Without the loss of generality, we consider circular stationary process defined by

$$g_\alpha(Z) = \alpha \star Z, \quad \alpha \in \mathcal{A} = \mathbb{R}^d.$$

It computes a circular convolution between a filter α and a Gaussian white noise $Z \sim \mathcal{N}(0, I_d)$ on the interval $\{0, \dots, d-1\}$. As d is finite, $g_\alpha(Z)$ can be regarded as a zero-mean linear stationary process observed on this interval (Priestley, 1981).

To measure the closeness between $g_\alpha(Z)$ and X (two zero-mean Gaussian distributions), we compute the spectral norm of the difference of their covariance matrices $\Sigma_\alpha = \mathbb{E}(g_\alpha(Z)g_\alpha(Z)^\top)$ and $\Sigma = \mathbb{E}(XX^\top)$. This allows one to define an error to measure the quality of a generator. By definition, $X = g_{\bar{\alpha}}(\bar{Z})$ is generated by a Gaussian white noise \bar{Z} , which is independent of Z . Due to the stationarity of $g_\alpha(Z)$, the spectral norm can be computed in the Fourier domain by the following classical result.

Proposition 1 (Generator error) *For any $\alpha \in \mathcal{A}$, we have*

$$\|\Sigma_\alpha - \Sigma\| = \max_{\omega \in \Omega_d} ||\hat{\alpha}(\omega)|^2 - |\hat{\bar{\alpha}}(\omega)|^2|.$$

Proof For any $\alpha \in \mathcal{A}$, Σ_α is a Toeplitz and circulant matrix, thus it can be diagonalized by the discrete Fourier transform. As Z is a stationary Gaussian white noise, the eigenvalues of Σ_α are given by $|\hat{\alpha}(\omega)|^2$ for $\omega \in \Omega_d$ (Priestley, 1981). The spectral norm is thus the maximal absolute difference between the eigenvalues of Σ_α and $\Sigma = \Sigma_{\bar{\alpha}}$. ■

In this paper, we are interested in a particular set of generators defined by

$$\mathcal{A}_n = \{\alpha \in \mathbb{R}^d \mid |\hat{\alpha}(\omega)|^2 = \mathbb{E}_n(|\hat{X}(\omega)|^2)/\mathbb{E}_n(|\hat{Z}(\omega)|^2), \forall \omega \in \Omega_d\}.$$

In fact, every sequence of generators in \mathcal{A}_n is consistent in the following sense.

Proposition 2 (Consistent generator) *Assume $\alpha_n \in \mathcal{A}_n$, then $\|\Sigma_{\alpha_n} - \Sigma\| \rightarrow 0$ in probability as $n \rightarrow \infty$.*

Proof The set \mathcal{A}_n is well defined, as almost surely $\mathbb{E}_n(|\hat{Z}(\omega)|^2) \neq 0$ for all $\omega \in \Omega_d$. The law of large numbers implies that for any $\omega \in \Omega_d$,

$$\mathbb{E}_n(|\hat{Z}(\omega)|^2)/\mathbb{E}_n(|\hat{Z}(\omega)|^2) \rightarrow 1, \quad n \rightarrow \infty, \quad \text{in probability.}$$

As $\mathbb{E}_n(|\hat{X}(\omega)|^2) = \mathbb{E}_n(|\hat{Z}(\omega)|^2)|\hat{\alpha}(\omega)|^2$, it implies the convergence of $|\hat{\alpha}_n(\omega)|^2$ to $|\hat{\alpha}(\omega)|^2$ in probability for any $\omega \in \Omega_d$. From Proposition 1, we conclude that $\|\Sigma_{\alpha_n} - \Sigma\| \rightarrow 0$ in probability. ■

For some technical reasons in Section 3, we next introduce a working assumption regarding the number of samples n , and the ground-truth generator $g_{\bar{\alpha}}$.

Assumption 1 *Assume $n \geq 2$, d is even, and $\bar{\alpha} \notin \mathcal{A}_0$, where*

$$\mathcal{A}_0 = \{\alpha \in \mathcal{A} \mid \exists \omega \in \Omega_d, \hat{\alpha}(\omega) = 0\}.$$

The condition $\bar{\alpha} \notin \mathcal{A}_0$ is needed to avoid degeneracy (i.e. to avoid having zero eigenvalues in Σ). We are thus considering stationary processes whose power spectrum are supported on the Fourier domain Ω_d .¹

1. The power spectrum of a stationary process X observed over an interval of length d is defined as the limit of $\mathbb{E}(|\hat{X}(\omega)|^2)/d$ as $d \rightarrow \infty$. In this paper, we use the same name for the finite d case (without taking the limit).

2.3. Nash equilibrium and its consistency

To study the solution of moment matching GANs, we review the notion of Nash equilibrium in differentiable zero-sum games, and then discuss its statistical consistency property. This property allows one to estimate the ground-truth generator $g_{\bar{\alpha}}$ as $n \rightarrow \infty$.

In this paper, we assume that both \mathcal{A} and \mathcal{B} belong to finite-dimensional Euclidean spaces, and that V_n is everywhere twice-differentiable with respect to α and β .

Definition 1 (*Equilibrium*) Let \mathcal{A} and \mathcal{B} be open sets. We say that $(\alpha^\bullet, \beta^\bullet)$ is an equilibrium of a game $(\mathcal{A}, \mathcal{B}, V_n)$, or an equilibrium of V_n , if

$$\nabla_{\alpha} V_n(\alpha^\bullet, \beta^\bullet) = 0, \quad \nabla_{\beta} V_n(\alpha^\bullet, \beta^\bullet) = 0. \quad (3)$$

This notion of equilibrium is used in [Daskalakis and Panageas \(2019\)](#) to study fixed points of games. It is also called stationary or critical point ([Daskalakis and Panageas, 2018](#); [Jin et al., 2020](#)).

Definition 2 (*Nash equilibrium*) We say that $(\alpha^\bullet, \beta^\bullet)$ is a Nash equilibrium of $(\mathcal{A}, \mathcal{B}, V_n)$, or a Nash equilibrium of V_n , if

$$V_n(\alpha^\bullet, \beta) \leq V_n(\alpha^\bullet, \beta^\bullet) \leq V_n(\alpha, \beta^\bullet), \quad \forall \alpha \in \mathcal{A}, \forall \beta \in \mathcal{B}. \quad (4)$$

For open \mathcal{A} and \mathcal{B} , we say $(\alpha^\bullet, \beta^\bullet)$ is a non-Nash equilibrium when (3) holds and (4) does not hold,

Note that when \mathcal{A} and \mathcal{B} are open sets, a Nash equilibrium $(\alpha^\bullet, \beta^\bullet)$ of V_n is also an equilibrium because (4) implies (3). As the original GANs ([Goodfellow et al., 2014](#)), Nash equilibrium always exists for the game defined in (1). The existence is less clear for V_n as we have only finite samples: we are no longer in the situation where $(\bar{\alpha}, \beta)$ is a Nash equilibrium for any $\beta \in \mathcal{B}$. This is different to the unrealizable setting in [Farnia and Ozdaglar \(2020\)](#) which assumes in our context that $\bar{\alpha} \notin \mathcal{A}$ and studies (1) rather than V_n .

In finite-sample realizable GANs, we can introduce a notion of consistent equilibrium as in the classical estimation theory in statistics ([A. I. Ibragimov, 1981](#)). This can be formalized in our context using the following definition.

Definition 3 (*Consistent Nash equilibrium*) We say that $\{(\alpha_n^\bullet, \beta_n^\bullet)\}_{n \geq 1}$ is a sequence of consistent Nash equilibrium if $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $(\alpha_n^\bullet, \beta_n^\bullet)$ is almost surely a Nash equilibrium of V_n , and it satisfies

$$\|\Sigma_{\alpha_n^\bullet} - \Sigma\| \rightarrow 0, \quad n \rightarrow \infty \quad \text{in probability.} \quad (5)$$

When \mathcal{A} and \mathcal{B} are open sets, we say that $\{(\alpha_n^\bullet, \beta_n^\bullet)\}_{n \geq 1}$ is a sequence of consistent equilibrium if $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, (3) holds for $(\alpha_n^\bullet, \beta_n^\bullet)$ and (5) holds. It is a sequence of consistent non-Nash equilibrium if it is a sequence of consistent equilibrium, and $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, (4) does not hold almost surely for $(\alpha_n^\bullet, \beta_n^\bullet)$.

Remark 1 In the next, we say simply $(\alpha_n^\bullet, \beta_n^\bullet)$ is a consistent Nash (resp. non-Nash) equilibrium of V_n without mentioning the sequence. Note that in the above definition, we do not specify whether β_n^\bullet is a convergent sequence, because what matters most in our problem is the convergence of $g_{\alpha_n^\bullet}$. The convergence in terms of the covariance matrix in (5) may be generalized to measure certain distance or divergence between probability distributions.

In some situations, we are interested in consistent Nash equilibrium or consistent non-Nash equilibrium. We next review a necessary condition of Nash equilibrium obtained from the Jacobian matrix of a differentiable (continuous) game.

Definition 4 (*Jacobian matrix*) Let \mathcal{A} and \mathcal{B} be open sets. The gradient vector of V_n is

$$\begin{pmatrix} \nabla_{\alpha} V_n(\alpha, \beta) \\ -\nabla_{\beta} V_n(\alpha, \beta) \end{pmatrix}.$$

The Jacobian matrix of V_n is the derivative of the gradient vector, i.e.

$$J_n(\alpha, \beta) = \begin{pmatrix} \nabla_{\alpha\alpha}^2 V_n(\alpha, \beta) & \nabla_{\alpha\beta}^2 V_n(\alpha, \beta) \\ -\nabla_{\beta\alpha}^2 V_n(\alpha, \beta) & -\nabla_{\beta\beta}^2 V_n(\alpha, \beta) \end{pmatrix}.$$

The following result is known in differentiable games (Ratliff et al., 2013, Proposition 2).

Proposition 3 Let \mathcal{A} and \mathcal{B} be open sets. If $(\alpha^{\bullet}, \beta^{\bullet})$ is a Nash equilibrium of V_n , then both $\nabla_{\alpha\alpha}^2 V_n(\alpha^{\bullet}, \beta^{\bullet})$ and $-\nabla_{\beta\beta}^2 V_n(\alpha^{\bullet}, \beta^{\bullet})$ are semi-positive definite.

Remark 2 It follows that if $\nabla_{\beta\beta}^2 V_n(\alpha^{\bullet}, \beta^{\bullet})$ has at least one strictly positive eigenvalue, then the equilibrium $(\alpha^{\bullet}, \beta^{\bullet})$ is a non-Nash equilibrium.

3. Existence of consistent Nash equilibrium

In this section, we study the impact of the discriminator family $\{f_{\beta}, \beta \in \mathcal{B}\}$ on the existence of consistent Nash equilibrium of V_n . As we consider only Gaussian stationary processes, all the discriminators are constructed from second-order statistical moments. For non-Gaussian distributions such as those generated by a one-layer neural network, second-order moments are also used to construct the discriminator family (Lei et al., 2020, Section 5). However, the existence of Nash equilibrium has not been studied in these GANs.

3.1. Real discriminator

Consider

$$f_{\beta}(X) = |\langle \beta, X \rangle|^2, \quad \mathcal{B} = \{\beta \in \mathbb{R}^d \mid \|\beta\| \leq 1\}.$$

This discriminator has only one feature ($m = 1$), and it is called a real discriminator because β is a real-valued vector. The next result shows that it can completely capture the spectral properties of X due to the maximization of V_n with respect to β .

Proposition 4 Let $\Sigma_n = \mathbb{E}_n(XX^{\top})$, $\Sigma_{\alpha,n} = \mathbb{E}_n(g_{\alpha}(Z)g_{\alpha}(Z)^{\top})$, then for any $\alpha \in \mathcal{A}$,

$$\sup_{\beta \in \mathcal{B}} V_n(\alpha, \beta) = \|\Sigma_n - \Sigma_{\alpha,n}\|^2.$$

Moreover, if $\Sigma_n \neq \Sigma_{\alpha,n}$, then the optimal β is a unit-norm eigenvector of $\Sigma_n - \Sigma_{\alpha,n}$ which has the largest absolute eigenvalue.

The proof is given in Appendix A. We next show that in general, there is no generator which can achieve a zero error of $\|\Sigma_n - \Sigma_{\alpha,n}\|^2$, i.e. the empirical covariance of X and $g_\alpha(Z)$ can not be perfectly matched.

Lemma 1 *Under Assumption 1, $\forall \alpha \in \mathcal{A}$, we have almost surely $\|\Sigma_n - \Sigma_{\alpha,n}\| > 0$.*

The proof is given in Appendix B. It is due to the fact that n is finite, and the samples of $X = g_{\bar{\alpha}}(\bar{Z})$ are generated from \bar{Z} , which are independent of the samples from Z . Based on this result, we next show that there is no Nash equilibrium in V_n .

Non-existence of Nash equilibrium Assume that $(\alpha^\bullet, \beta^\bullet)$ is a Nash equilibrium of V_n , then it is a best response solution for each player, i.e.

$$\alpha^\bullet \in \arg \min_{\alpha \in \mathcal{A}} V_n(\alpha, \beta^\bullet) \quad (6)$$

$$\beta^\bullet \in \arg \max_{\beta \in \mathcal{B}} V_n(\alpha^\bullet, \beta) \quad (7)$$

The next result shows that such solution does not exist in general. It implies that no consistent Nash equilibrium exists in V_n .

Theorem 3 *Under Assumption 1, there is almost surely no Nash equilibrium in V_n .*

Proof From (7) and Proposition 4, it follows that $V_n(\alpha^\bullet, \beta^\bullet) = \|\Sigma_n - \Sigma_{\alpha^\bullet,n}\|^2$. We next show that (6) does not hold, i.e. there exists $\alpha \in \mathcal{A}$ such that

$$V_n(\alpha, \beta^\bullet) = \langle \beta^\bullet, (\Sigma_n - \Sigma_{\alpha,n})\beta^\bullet \rangle < V_n(\alpha^\bullet, \beta^\bullet).$$

We minimize $V_n(\alpha, \beta^\bullet)$ with respect to α . Note that

$$\begin{aligned} V_n(\alpha, \beta^\bullet) &= ((\beta^\bullet)^\top (\Sigma_n - \Sigma_{\alpha,n}) \beta^\bullet)^2 \\ &= ((\beta^\bullet)^\top \Sigma_n \beta^\bullet)^2 + ((\beta^\bullet)^\top \Sigma_{\alpha,n} \beta^\bullet)^2 - 2((\beta^\bullet)^\top \Sigma_n \beta^\bullet)((\beta^\bullet)^\top \Sigma_{\alpha,n} \beta^\bullet). \end{aligned} \quad (8)$$

Assume $\{z_i\}$ are the n i.i.d. samples to compute the empirical expectation of Z in V_n . We rewrite $V_n(\alpha, \beta^\bullet)$ more explicitly in terms of α , in the following equation

$$(\beta^\bullet)^\top \Sigma_{\alpha,n} \beta^\bullet = \mathbb{E}_n(|\langle \beta^\bullet, \alpha \star Z \rangle|^2) = \alpha^\top S_{\beta^\bullet} \alpha, \quad (9)$$

where $S_{\beta^\bullet}(v, v') = \frac{1}{n} \sum_{i=1}^n \beta^\bullet \star \tilde{z}_i(v) \beta^\bullet \star \tilde{z}_i(v')$ for $(v, v') \in \{0, \dots, d-1\}^2$.

It follows from (9) and (8) that

$$\nabla_\alpha V_n(\alpha, \beta^\bullet) = 4((\beta^\bullet)^\top \Sigma_{\alpha,n} \beta^\bullet) S_{\beta^\bullet} \alpha - 4((\beta^\bullet)^\top \Sigma_n \beta^\bullet) S_{\beta^\bullet} \alpha. \quad (10)$$

Setting $\nabla_\alpha V_n(\alpha, \beta^\bullet) = 0$ implies two situations

- $S_{\beta^\bullet} \alpha = 0$: from (8) and (9), it follows that $V_n(\alpha, \beta^\bullet) = ((\beta^\bullet)^\top \Sigma_n \beta^\bullet)^2$.
- $(\beta^\bullet)^\top \Sigma_{\alpha,n} \beta^\bullet = (\beta^\bullet)^\top \Sigma_n \beta^\bullet$: from (8), it follows that $V_n(\alpha, \beta^\bullet) = 0$.

The second situation implies that the minimum of $V_n(\alpha, \beta^\bullet)$ is zero, as long as one can find a solution α such that

$$(\beta^\bullet)^\top \Sigma_{\alpha,n} \beta^\bullet = (\beta^\bullet)^\top \Sigma_n \beta^\bullet. \quad (11)$$

We next show that almost surely, this is possible, i.e. $\min_{\alpha \in \mathcal{A}} V_n(\alpha, \beta^\bullet) = 0$. However, this contradicts to Lemma 1, which implies that almost surely $\Sigma_n - \Sigma_{\alpha^\bullet,n} \neq 0$, i.e. $V_n(\alpha^\bullet, \beta^\bullet) = \|\Sigma_n - \Sigma_{\alpha^\bullet,n}\|^2 > 0$. Thus $(\alpha^\bullet, \beta^\bullet)$ can not be a Nash equilibrium.

From (9) and (11), it remains to find $\alpha \in \mathcal{A}$ such that

$$\alpha^\top S_{\beta^\bullet} \alpha = (\beta^\bullet)^\top \Sigma_n \beta^\bullet. \quad (12)$$

From Proposition 4, we know that β^\bullet is a unit-norm eigenvector of $\Sigma_n - \Sigma_{\alpha^\bullet,n}$. As a consequence, almost surely

$$\forall i \leq n, \quad \beta^\bullet \star \tilde{z}_i \neq 0. \quad (13)$$

This is because \tilde{z}_i is sampled i.i.d. from $\mathcal{N}(0, I_d)$, thus almost surely $\widehat{\tilde{z}_i}(\omega) \neq 0$ for all $\omega \in \Omega_d$. As $\widehat{\beta^\bullet} \neq 0$, it follows that $\widehat{\beta^\bullet} \widehat{\tilde{z}_i} \neq 0$. This proves our claim of (13).

By definition, S_{β^\bullet} is a semi-definite positive matrix. It results from (13) that S_{β^\bullet} has at least one strictly positive eigenvalue (otherwise $S_{\beta^\bullet} = 0$). Let h be one of the eigenvectors of S_{β^\bullet} whose eigenvalue $\lambda > 0$. Assume $\|h\| = 1$. To construct a minimal solution of $V_n(\alpha, \beta^\bullet)$, it suffices to take

$$\alpha = ch, \quad c = \sqrt{\frac{(\beta^\bullet)^\top \Sigma_n \beta^\bullet}{\lambda}}.$$

■

Remark 4 *This result suggests that although β^\bullet is an optimal discriminator for the generator g_{α^\bullet} , it is not able to stabilize the minimization process of α because $\|\Sigma_n - \Sigma_{\alpha,n}\|^2$ is non-zero. This phenomenon is apparently due to the finite sample size n , but it is also related to the fact that the generator family is restricted to be stationary. A larger generator family which can achieve a zero generator error may remedy the issue of the non-existence of Nash equilibrium. However, such model is necessarily non-stationary since Σ_n is not a Toeplitz matrix.*

Although there is no Nash equilibrium in V_n , one can still solve the min-max problem (2) from a sequential game point of view. For each α , it first solves the maximization problem of β , and then modifies α in order to minimize $\max_\beta V_n(\alpha, \beta)$. There is a recent trend in the literature to extend the notion of Nash equilibrium towards the notion of Stackelberg equilibrium for sequential games, see e.g. Jin et al. (2020); Fiez and Ratliff (2020); Farnia and Ozdaglar (2020). However, the remaining optimization problem is challenging in practice, in terms of the landscape of solution sets (Sun et al., 2020), as well as the algorithm design and convergence analysis (Wang et al., 2020). In this paper, we take the simultaneous game point of view where the notion of Nash equilibrium is fundamental (Laraki et al., 2019). We next study the existence of consistent Nash equilibrium in V_n by looking for translational invariant features in the discriminator.

3.2. Complex discriminator

Consider

$$f_\beta(X) = (|\langle \beta_\ell, X \rangle|^2)_{0 \leq \ell < m}, \quad \mathcal{B} = \{(\beta_\ell)_{0 \leq \ell < m} | \beta_\ell \in \mathbb{C}^d\}.$$

According to the proof of Proposition 1, the discrete Fourier basis diagonalizes the covariance matrix Σ_α of $g_\alpha(Z)$ for any $\alpha \in \mathcal{A}$. This family of discriminator is constructed to contain this basis. We show that when using $m = d$ features, consistent non-Nash equilibrium exists, and it is getting closer to a consistent Nash equilibrium as $n \rightarrow \infty$.

Existence of consistent non-Nash equilibrium Proposition 2 shows that \mathcal{A}_n is a set of consistent generators. We construct a family of consistent equilibrium $(\alpha^\bullet, \beta^\bullet)$ where $\alpha^\bullet \in \mathcal{A}_n$ and $\beta^\bullet \in \mathcal{B}$. Note that in the definition of \mathcal{B} , we do not impose any norm constraint on each β_ℓ . This simplifies our analysis of the equilibrium.

Theorem 5 Assume $m = d$, $\beta_\ell^\bullet(u) = e^{i2\pi\ell u/d}$ for $0 \leq u < d$ and $0 \leq \ell < d$, then $\forall \alpha^\bullet \in \mathcal{A}_n$, we have that $(\alpha^\bullet, \beta^\bullet)$ is a consistent equilibrium of V_n such that

$$V_n(\alpha^\bullet, \beta^\bullet) = 0.$$

Moreover, under Assumption 1, $(\alpha^\bullet, \beta^\bullet)$ is a consistent non-Nash equilibrium.

Proof We write $V_n(\alpha, \beta) = \sum_{\ell < m} r_{n,\ell}^2(\alpha, \beta)$, where

$$r_{n,\ell}(\alpha, \beta) = \mathbb{E}_n(|\langle \beta_\ell, X \rangle|^2) - \mathbb{E}_n(|\langle \beta_\ell, g_\alpha(Z) \rangle|^2)$$

To show that $(\alpha^\bullet, \beta^\bullet)$ is an equilibrium of V_n , we evaluate the gradient vector of V_n , which is computed with

$$\nabla_\alpha V_n = 2 \sum_\ell r_{n,\ell} \nabla_\alpha r_{n,\ell}, \quad \nabla_\beta V_n = 2 \sum_\ell r_{n,\ell} \nabla_\beta r_{n,\ell}. \quad (14)$$

We verify that $\forall \alpha^\bullet \in \mathcal{A}_n$ and $\forall \ell < d$, $r_{n,\ell}(\alpha^\bullet, \beta^\bullet) = 0$. Indeed, by the definition of β^\bullet , $\langle \beta_\ell^\bullet, X \rangle$ is the Fourier transform of X at the frequency $\omega = 2\pi\ell/d \in \Omega_d$. Thus $\alpha^\bullet \in \mathcal{A}_n$ implies that

$$\forall \omega \in \Omega_d, \quad \mathbb{E}_n |\widehat{X}(\omega)|^2 = \mathbb{E}_n |\widehat{g_{\alpha^\bullet}(Z)}(\omega)|^2 = |\widehat{\alpha^\bullet}(\omega)|^2 \mathbb{E}_n (|\widehat{Z}(\omega)|^2) \quad (15)$$

This is equivalent to $r_{n,\ell}(\alpha^\bullet, \beta^\bullet) = 0$ for all ℓ . From (14), (15) and Proposition 2, $(\alpha^\bullet, \beta^\bullet)$ is a consistent equilibrium of V_n .

To show that it is not a Nash equilibrium, it is sufficient to verify that the Jacobian matrix at this equilibrium has a non-positive definite symmetric part, according to Remark 2. Indeed, From (14) and (15), the symmetric part of $J_n(\alpha^\bullet, \beta^\bullet)$ is

$$\begin{pmatrix} 2 \sum_\ell (\nabla_\alpha r_{n,\ell})(\nabla_\alpha r_{n,\ell})^\top & 0 \\ 0 & -2 \sum_\ell (\nabla_\beta r_{n,\ell})(\nabla_\beta r_{n,\ell})^\top \end{pmatrix}.$$

We next check that, under Assumption 1, there exists almost surely at least one $\ell \in \{0, \dots, d-1\}$ such that $\nabla_\beta r_{n,\ell} \neq 0$. Let $\beta_\ell = \beta_\ell^{re} + i\beta_\ell^{im}$, we verify that at the equilibrium $(\alpha^\bullet, \beta^\bullet)$,

$$\nabla_{\beta_\ell^{re}} r_{n,\ell} = 2(\Sigma_n - \Sigma_{\alpha^\bullet, n})(\beta_\ell^\bullet)^{re}, \quad \nabla_{\beta_\ell^{im}} r_{n,\ell} = 2(\Sigma_n - \Sigma_{\alpha^\bullet, n})(\beta_\ell^\bullet)^{im} \quad (16)$$

For $\ell' \neq \ell$, by the definition of $r_{n,\ell}$ we have

$$\nabla_{\beta_{\ell'}^{re}} r_{n,\ell} = 0, \quad \nabla_{\beta_{\ell'}^{im}} r_{n,\ell} = 0.$$

As $\nabla_{\beta} r_{n,\ell}$ is a vector which concatenates $\nabla_{\beta_{\ell'}} r_{n,\ell} = (\nabla_{\beta_{\ell'}^{re}} r_{n,\ell}, \nabla_{\beta_{\ell'}^{im}} r_{n,\ell})$ for $\ell' < d$, we conclude that $\sum_{\ell} (\nabla_{\beta} r_{n,\ell})(\nabla_{\beta} r_{n,\ell})^T$ is a block diagonal matrix with d blocks. The ℓ -th block equals to a rank-one matrix

$$(\nabla_{\beta_{\ell}} r_{n,\ell})(\nabla_{\beta_{\ell}} r_{n,\ell})^T.$$

We next claim that at least for one ℓ , $\nabla_{\beta_{\ell}} r_{n,\ell}$ is non-zero. Otherwise, setting the gradients in (16) to zero for all ℓ implies that

$$\Sigma_n - \Sigma_{n,\alpha^{\bullet}} = 0,$$

since $(\beta_{\ell}^{\bullet})_{\ell < d}$ forms an orthogonal basis in \mathbb{C}^d . However, this is contradictory to Lemma 1 that, under Assumption 1, we have almost surely $\|\Sigma_n - \Sigma_{n,\alpha^{\bullet}}\| > 0$. We conclude that the symmetric part of $J_n(\alpha^{\bullet}, \beta^{\bullet})$ is not semi-positive definite, and $(\alpha^{\bullet}, \beta^{\bullet})$ is a consistent non-Nash equilibrium. \blacksquare

Remark 6 This result shows that $g_{\alpha^{\bullet}}(Z)$ matches empirically the power spectrum of X because of the perfect moment matching: $V_n(\alpha^{\bullet}, \beta^{\bullet}) = 0$. However, the fact that $\Sigma_n \neq \Sigma_{\alpha^{\bullet},n}$ still does not allow one to obtain a Nash equilibrium due to the loss of the semi-positive definiteness in the Jacobian matrix of V_n . Nevertheless, one can show that (a proof is given in Appendix C),

$$\sup_{\alpha^{\bullet} \in \mathcal{A}_n} \|\Sigma_n - \Sigma_{\alpha^{\bullet},n}\| \rightarrow 0, \quad n \rightarrow \infty, \quad \text{in probability.} \quad (17)$$

From (16), (17) we deduce that when n is large, the symmetric part of the Jacobian matrix of V_n at the equilibrium $(\alpha^{\bullet}, \beta^{\bullet})$ has vanishing negative eigenvalues. Therefore, the non-Nash equilibrium becomes closer to a Nash equilibrium as n grows.

3.3. Convolutional discriminator

Consider

$$f_{\beta}(X) = (\|X \star \beta_{\ell}\|^2)_{0 \leq \ell \leq m}, \quad \mathcal{B} = \{(\beta_{\ell})_{0 \leq \ell \leq m} | \beta_{\ell} \in \mathbb{R}^d\}.$$

Unlike the real and complex discriminator, this discriminator family uses features that are always invariant to the translations of X on the grid of u .² Under appropriate assumptions, we show that V_n admits infinite many consistent Nash equilibria, and in some sense they are unique.

Existence of consistent Nash equilibrium We write $V_n(\alpha, \beta) = \sum_{\ell < m} r_{n,\ell}^2(\alpha, \beta)$, where

$$r_{n,\ell}(\alpha, \beta) = \mathbb{E}_n(\|X \star \beta_{\ell}\|^2) - \mathbb{E}_n(\|g_{\alpha}(Z) \star \beta_{\ell}\|^2).$$

Proposition 5 Let $\alpha^{\bullet} \in \mathcal{A}_n$, then

$$V_n(\alpha^{\bullet}, \beta) = 0, \quad \forall \beta \in \mathcal{B}.$$

Moreover, $\forall \beta^{\bullet} \in \mathcal{B}$, $(\alpha^{\bullet}, \beta^{\bullet})$ is a consistent Nash equilibrium.

2. The translations are defined with periodic boundary conditions.

The proof is deferred to Appendix D. Unlike the previous two discriminators, the consistent Nash equilibrium can be easily found among the generator $\alpha \in \mathcal{A}_n$, and among all $\beta \in \mathcal{B}$. Indeed, due to Parseval's identity, this family of discriminator captures only the power spectrum of stationary processes through the empirical expectations $(\mathbb{E}_n(|\widehat{X}(\omega)|^2))_{\omega \in \Omega_d}$. These expectations are the diagonal values of the matrix $F\Sigma_n F^*$ using the discrete Fourier transform F on \mathbb{C}^d . Therefore the convolutional discriminator measures only a limited amount of information of Σ_n .

Uniqueness of consistent Nash equilibrium We next provide sufficient conditions for any equilibrium of V_n to be a consistent Nash equilibrium when $m = d$. It allows one to define a subset of \mathcal{B} such that every equilibrium of the game is a consistent Nash equilibrium.

For $x \in \mathbb{C}^d$, we write $|x|^{\circ 2}(u) = |x(u)|^2$. We need the following result,

Lemma 2 *Assume $m = d$ and $(|\widehat{\beta}_\ell|^{\circ 2})_{0 \leq \ell < m}$ is a basis on \mathbb{R}^d , then $V_n(\alpha, \beta) = 0$ is equivalent to $\alpha \in \mathcal{A}_n$ almost surely.*

It is proved in Appendix E. Next we make the same assumption on $\bar{\alpha}$ as in Assumption 1, but the other technical assumptions regarding n and d are not needed.

Theorem 7 *Assume $(\alpha^\bullet, \beta^\bullet)$ is an equilibrium of V_n . If the following assumptions are satisfied,*

- $\bar{\alpha} \notin \mathcal{A}_0$ and $m = d$,
- $(|\widehat{\beta}_\ell^\bullet|^{\circ 2})_{\ell < d}$ is a basis on \mathbb{R}^d ,
- $\forall \ell < d, \beta_\ell^\bullet \notin \mathcal{A}_0$,

then $\alpha^\bullet \in \mathcal{A}_n$ almost surely and $(\alpha^\bullet, \beta^\bullet)$ is a consistent Nash equilibrium.

The proof is deferred to Appendix F. To satisfy the conditions on the β^\bullet in Theorem 7, we can consider a game using an open subset of \mathcal{B} , e.g.

$$\mathcal{B}_0 = \{(\beta_\ell)_{\ell < m} \mid \beta_\ell \in \mathbb{R}^d, \det((|\widehat{\beta}_\ell|^{\circ 2})_{\ell < m}) \neq 0, \min((|\widehat{\beta}_\ell|^{\circ 2})_{\ell < m}) > 0\} \quad (18)$$

By following the proof of Theorem 7, one can verify that

Corollary 1 *If $\bar{\alpha} \notin \mathcal{A}_0$ and $m = d$, then any equilibrium of the game $(\mathcal{A}, \mathcal{B}_0, V_n)$ is a consistent Nash equilibrium.*

Note that the set \mathcal{B}_0 is not restrictive because Proposition 5 shows that the choice of β^\bullet can be arbitrary once $\alpha \in \mathcal{A}_n$. In the next section, we study numerically whether these conditions are implicitly satisfied along the dynamics of gradient-based methods to solve $(\mathcal{A}, \mathcal{B}, V_n)$.

4. Numerical results

In this section, we study gradient-descent-ascent optimization methods to find a consistent equilibrium of V_n . For the complex discriminator, we study the local stability of such methods near the consistent non-Nash equilibrium in Theorem 5. When the sample size n is large enough, we find that gradient-descent-ascent methods are nearly stable in the sense that the generator error remains almost constant. For the convolutional discriminator, we study their global convergence towards consistent Nash equilibrium from random initialization.

Table 1: The generator error difference $\|\Sigma - \Sigma_{\alpha^\bullet}\| - \|\Sigma - \Sigma_{\alpha_0}\|$ computed from the real discriminator using different sample size n and α_0 . The mean and standard deviation of the difference are computed from multiple simulations.

α_0 random			$\alpha_0 = \bar{\alpha}$		
n	Mean	Std. Dev	n	Mean	Std. Dev
10	-0.0656	2.0738	10	2.1073	1.8541
100	-0.8060	1.6804	100	0.5887	0.2476
1000	-0.6907	1.1686	1000	0.1843	0.0611
10^4	-0.6143	0.9407	10^4	0.0568	0.0144
10^5	-0.6393	0.9723	10^5	0.0183	0.0053

4.1. Real discriminator

For the real discriminator, we provide further discussions about the non-existence of Nash equilibrium. The proof of Theorem 3 suggests that for any $\beta^\bullet \in \mathcal{B}$, one can expect to find an optimal $\alpha \in \mathcal{A}$ such that $V_n(\alpha, \beta^\bullet) = 0$. We verify that such solution can be computed, and then we evaluate the generator error $\|\Sigma - \Sigma_\alpha\|$ at these solutions.

We consider β^\bullet which maximizes $V_n(\alpha_0, \beta)$, with two different α_0 . One is a random α_0 , whose elements are sampled i.i.d from $\mathcal{N}(0, 1/d)$. The other α_0 is set to be the ground-truth parameter $\bar{\alpha}$. From the obtained β^\bullet , we compute an optimal α^\bullet which solves $\min_\alpha V_n(\alpha, \beta^\bullet)$ using gradient descent starting from α_0 . The optimal value $V_n(\alpha^\bullet, \beta^\bullet)$ is expected to be very close to zero. If the solution α^\bullet is very different from α_0 , then one should be able to detect such difference in the generator error.

Table 1 shows generator error differences for the case $\bar{\alpha} = (1, 0, 0, 0)$ and $d = 4$. For the case of random α_0 , we find that $\|\Sigma - \Sigma_{\alpha^\bullet}\| < \|\Sigma - \Sigma_{\alpha_0}\|$, while for the case of $\alpha_0 = \bar{\alpha}$, we find that $\|\Sigma - \Sigma_{\alpha^\bullet}\| > \|\Sigma - \Sigma_{\alpha_0}\|$. This result agrees with our theoretical analysis which shows that there is no Nash equilibrium using the real discriminator. In the second case, $\|\Sigma - \Sigma_{\alpha_0}\| = 0$, thus the generator error difference measures directly $\|\Sigma - \Sigma_{\alpha^\bullet}\|$. We find that $\|\Sigma - \Sigma_{\alpha^\bullet}\|$ decreases with n at a rate of $1/\sqrt{n}$. This is similar to the rate of the empirical estimator $\|\Sigma - \Sigma_n\|$ computed later in Table 2. The reason that in the second case, the generator error of g_{α^\bullet} is decreasing is likely due to the initialization of the gradient-descent method (which is α_0). This is because according to (12), the solution set $\{\alpha \in \mathcal{A} : V_n(\alpha, \beta^\bullet) = 0\}$ is likely to be very large, containing generators with both a small error and a large error.

To compute $\max_{\beta \in \mathcal{B}} V_n(\alpha_0, \beta)$, we apply the classical power method (Golub and Van Loan, 2013) to the matrix $(\Sigma_n - \Sigma_{\alpha_0, n})^2$, according to Proposition 4. In order to compute β^\bullet with a good precision, we use a total number of 200 simulations, where V_n is computed from independent random realizations of X and Z . We keep only the first 100 simulations where the relative difference between $V_n(\alpha_0, \beta^\bullet)$ and $\|\Sigma_n - \Sigma_{\alpha_0, n}\|^2$ (computed from SVD) is smaller than 10^{-4} . We then compute α^\bullet by the gradient descent method with a constant step-size. The method is run for at most 10000 iterations and it stops when the relative loss decrease is smaller than 10^{-8} . We verify that all the optimal values $V_n(\alpha^\bullet, \beta^\bullet)$ are very close to zero (around 10^{-10}).

4.2. Complex discriminator

Theorem 5 shows that there exists consistent non-Nash equilibrium in V_n . We study how gradient-descent-ascent (GDA) methods (Nagarajan and Kolter, 2017) behave near such equilibrium from both a discrete-time and continuous-time perspective. The continuous-time GDA is interesting as it is known that it can converge towards a non-Nash equilibrium (Mazumdar et al., 2019).

Discrete-time and continuous-time GDA For an initial $(\alpha^{(0)}, \beta^{(0)})$, the discrete-time GDA method iteratively updates (α, β) by using the gradient vector of V_n . At iteration $t \geq 1$, it takes the form

$$\alpha^{(t)} = \alpha^{(t-1)} - \eta \nabla_{\alpha} V_n(\alpha^{(t-1)}, \beta^{(t-1)}), \quad \beta^{(t)} = \beta^{(t-1)} + \eta \nabla_{\beta} V_n(\alpha^{(t-1)}, \beta^{(t-1)})$$

where the step-size $\eta > 0$. The GDA method in the discrete-time differs from its continuous-time version, which is

$$\frac{d\alpha^{(t)}}{dt} = -\nabla_{\alpha} V_n(\alpha^{(t)}, \beta^{(t)}), \quad \frac{d\beta^{(t)}}{dt} = \nabla_{\beta} V_n(\alpha^{(t)}, \beta^{(t)})$$

For example, in bilinear games, where the symmetric part of the Jacobian matrix J_n is zero, $(\alpha^{(t)}, \beta^{(t)})$ diverges from the Nash equilibrium no matter how small η is, whereas the continuous-time solution (α, β) turns around the equilibrium (Balduzzi et al., 2018).

The continuous-time GDA is simulated by the classical Runge-Kutta 4 method as in Qin et al. (2020), with the same step-size η as the discrete-time GDA. In general, the step-size of α can be different from that of β (Heusel et al., 2017).

Local stability of GDA We study the stability of GDA near the non-Nash equilibrium $(\alpha^{\bullet}, \beta^{\bullet})$ in Theorem 5. The initial point of GDA $(\alpha^{(0)}, \beta^{(0)})$ is taken to be a small perturbation of $(\alpha^{\bullet}, \beta^{\bullet})$ by an additive white Gaussian noise $\mathcal{N}(0, \sigma^2 I_{d+2dm})$. To measure the closeness of $(\alpha^{(t)}, \beta^{(t)})$ to \mathcal{A}_n and β^{\bullet} respectively, we compute

$$\epsilon_{\alpha}^{(t)} = \max_{\omega \in \Omega_d} \left| |\widehat{\alpha^{(t)}}(\omega)|^2 - \mathbb{E}_n(|\widehat{X}(\omega)|^2) / \mathbb{E}_n(|\widehat{Z}(\omega)|^2) \right|, \quad \epsilon_{\beta}^{(t)} = \sqrt{\sum_{\ell} \|\beta_{\ell}^{(t)} - \beta_{\ell}^{\bullet}\|^2}.$$

We consider the same case as above, $\bar{\alpha} = (1, 0, 0, 0)$ and $d = 4$, with $m = d$ features. We set α^{\bullet} to be a special element in the set \mathcal{A}_n , and β^{\bullet} to be the discrete Fourier basis.³ Figure 1 shows how $\epsilon_{\alpha}^{(t)}$ and $\epsilon_{\beta}^{(t)}$ evolves over the discrete-time GDA dynamics with $\eta = 10^{-3}$ and $\sigma = 10^{-3}$. We find that when $n = 100$, GDA does not converge to the set \mathcal{A}_n in most simulations due to a large $\epsilon_{\alpha}^{(t)}$. We find that quite often it stopped when V_n becomes too big (NaN). As $n \rightarrow \infty$, the algorithm becomes more stable. When $n = 10000$, GDA seems to converge towards the set \mathcal{A}_n with a decreasing $\epsilon_{\alpha}^{(t)}$. It does not converge to β^{\bullet} as $\epsilon_{\beta}^{(t)}$ does not change much from the initialization. This may be explained according to Remark 6, which shows that the gradient $\nabla_{\beta} V_n$ tends to be very small at $\alpha^{\bullet} \in \mathcal{A}_n$ as n is big. This result shows that using a small number of iterations (10^4), the discrete-time GDA method is nearly stable around the set \mathcal{A}_n as long as n is large enough.

However, as we run the discrete-time GDA for 10^6 iterations, we observe that even at $n = 10000$, V_n slowly increases with t in most of the simulations. The unstable behavior of GDA

3. The special element is defined by $\widehat{\alpha^{\bullet}}(\omega) = \sqrt{\mathbb{E}_n(|\widehat{X}(\omega)|^2) / \mathbb{E}_n(|\widehat{Z}(\omega)|^2)}, \forall \omega \in \Omega_d$.

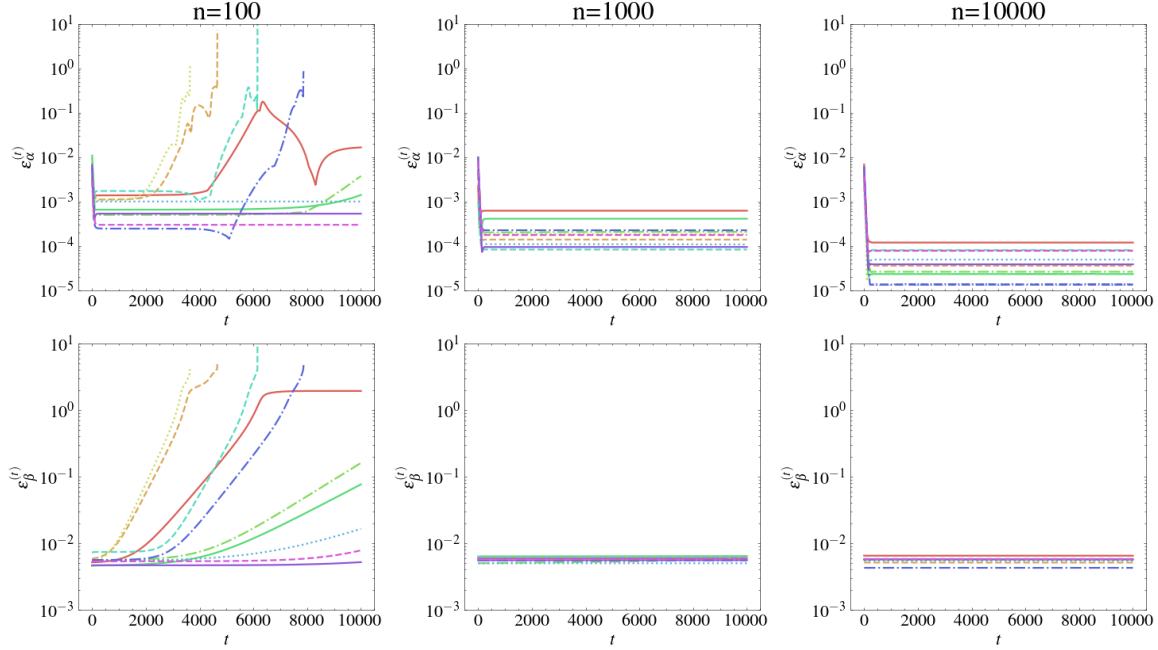


Figure 1: The dynamics of $\epsilon_\alpha^{(t)}$ and $\epsilon_\beta^{(t)}$ as a function of the number of iterations (10^4 in total) of the discrete-time GDA on the complex discriminator. Each of the 10 curves corresponds to a simulation from independent samples of X and Z . The GDA stops if V_n becomes nan.

suggests that the Jacobian matrix $J_n(\alpha^\bullet, \beta^\bullet)$ has eigenvalues with negative real-part. In order to verify this, we simulate the continuous-time GDA for a large number of iterations (10^6) with various $\sigma \in \{10^{-3}, 10^{-4}, 10^{-5}\}$. The result with a small $\sigma = 10^{-5}$ and large $n = 10000$ is given in Figure 3 in Appendix G. We find that in some simulations, the continuous-time GDA also has a decreasing V_n and then it remains constant over t . However in some simulations, it is unstable due to a decreasing and then slowly increasing V_n . It is remarkable that in most simulations, we find that the generator error $\|\Sigma - \Sigma_{\alpha(t)}\|$ remains constant over t . Therefore the instability is mostly due to the discriminator and it has a small impact on the generator quality.

4.3. Convolutional discriminator

Theorem 7 provides sufficient conditions for an equilibrium of V_n to be a consistent Nash equilibrium. This motivates us to study whether such equilibrium is the most likely limiting points of the continuous-time GDA method starting from random initialization.

To study such global convergence, the initial point of GDA $(\alpha^{(0)}, \beta^{(0)})$ is sampled randomly from Gaussian white noise.⁴ To verify the sufficient conditions about β in Theorem 7, we check that all the $\beta^{(t)}$ belongs to the set \mathcal{B}_0 defined in (18). It amounts to compute the following quantities

$$d_\beta^{(t)} = |\det(\widehat{|\beta^{(t)}|^{\circ 2}})|, \quad m_\beta^{(t)} = \min(\widehat{|\beta^{(t)}|^{\circ 2}}).$$

4. Each element of α and β is sampled i.i.d from $\mathcal{N}(0, 1/d)$. Each vector β_ℓ is further normalized to have a unit norm.

Table 2: Evaluation of the generator computed by the continuous-time GDA on the convolutional discriminator. Left: the generator error $\|\Sigma - \Sigma_{\alpha(T)}\|$ with $T = 10^4$. Right: the empirical error $\|\Sigma - \Sigma_n\|$. The mean and standard deviation of these errors are computed from multiple simulations and for various sample size n .

$\ \Sigma - \Sigma_{\alpha(T)}\ $			$\ \Sigma - \Sigma_n\ $		
n	Mean	Std. Dev	n	Mean	Std. Dev
10	0.9705	1.1089	10	1.0446	0.4060
100	0.2874	0.1583	100	0.3450	0.0965
1000	0.0790	0.0352	1000	0.1051	0.0234
10^4	0.0233	0.0106	10^4	0.0338	0.0076
10^5	0.0089	0.0097	10^5	0.0104	0.0026

By definition, $d_\beta^{(t)}$ is the absolute value of the determinant of a matrix whose columns are formed by $\widehat{|\beta_\ell^{(t)}|^{\circ 2}}$. A non-zero $d_\beta^{(t)}$ implies that $(\widehat{|\beta_\ell^{(t)}|^{\circ 2}})_{\ell < d}$ is a basis on \mathbb{R}^d . Similarly, if the minimal value of this matrix $m_\beta^{(t)}$ is non-zero then $\forall \ell < d, \beta_\ell^{(t)} \notin \mathcal{A}_0$.

The result for the case $\bar{\alpha} = (1, 0, 0, 0)$ and $d = 4$, with $m = d$ is given in Figure 2. We observe that the conditions of Theorem 7 are well respected in all the simulations. We also see that in almost all simulations, the continuous-time GDA converges to small values of V_n and $\epsilon_\alpha^{(t)}$. When $n \leq 1000$, there are several simulations which have a slow convergence. The slowness seems to be related to the smallest value of $d_\beta^{(t)}$ over t . To verify the convergence of the continuous-time GDA, a longer run at $n = 10000$ with 10^6 iterations is given in Figure 3 in Appendix G. We find that the GDA always has a convergent V_n . This suggests that the algorithm has found a consistent Nash equilibrium (α_n, β_n) in the set $\mathcal{A}_n \times \mathcal{B}_0$.

Some values of $\epsilon_\alpha^{(t)}$ in Figure 2 are quite close to zero, suggesting that at the last iteration of GDA, the generator is close to \mathcal{A}_n . As a consequence, the generator error should be close to the empirical error $\|\Sigma - \Sigma_n\|$. This is because when n is large, Σ_n is very close to $\Sigma_{\alpha^\bullet, n}$ for $\alpha^\bullet \in \mathcal{A}_n$, according to (17). To validate this, we report the generator error at the last iteration of GDA in Table 2, estimated from 100 simulations. This error is compared to the empirical estimator of $\Sigma = \mathbb{E}(XX^\top)$ using the same number of samples of X . The decreasing mean and standard deviation as n grows show that the GDA converges well to the generators whose errors are close to those of \mathcal{A}_n . We also find that when n is large, the generator error (mean) is slightly smaller on average than the empirical estimator. This is not so surprising as $\Sigma_{\alpha(T)}$ is a Toeplitz and circulant covariance matrix, as Σ . But the empirical covariance Σ_n does not satisfy these properties. However, the standard deviation of the generator error is slightly larger than the empirical estimator. This should be related to the global convergence of GDA. Indeed, when n is small ($n = 10$), we find that in a few simulations (four among a hundred), the GDA diverges to NaN in V_n (these cases are excluded in the error computation). No such divergence happens when $n \geq 100$. In this case, the larger standard deviation may be related to the fluctuations of the limiting values of $\epsilon_\alpha^{(t)}$.

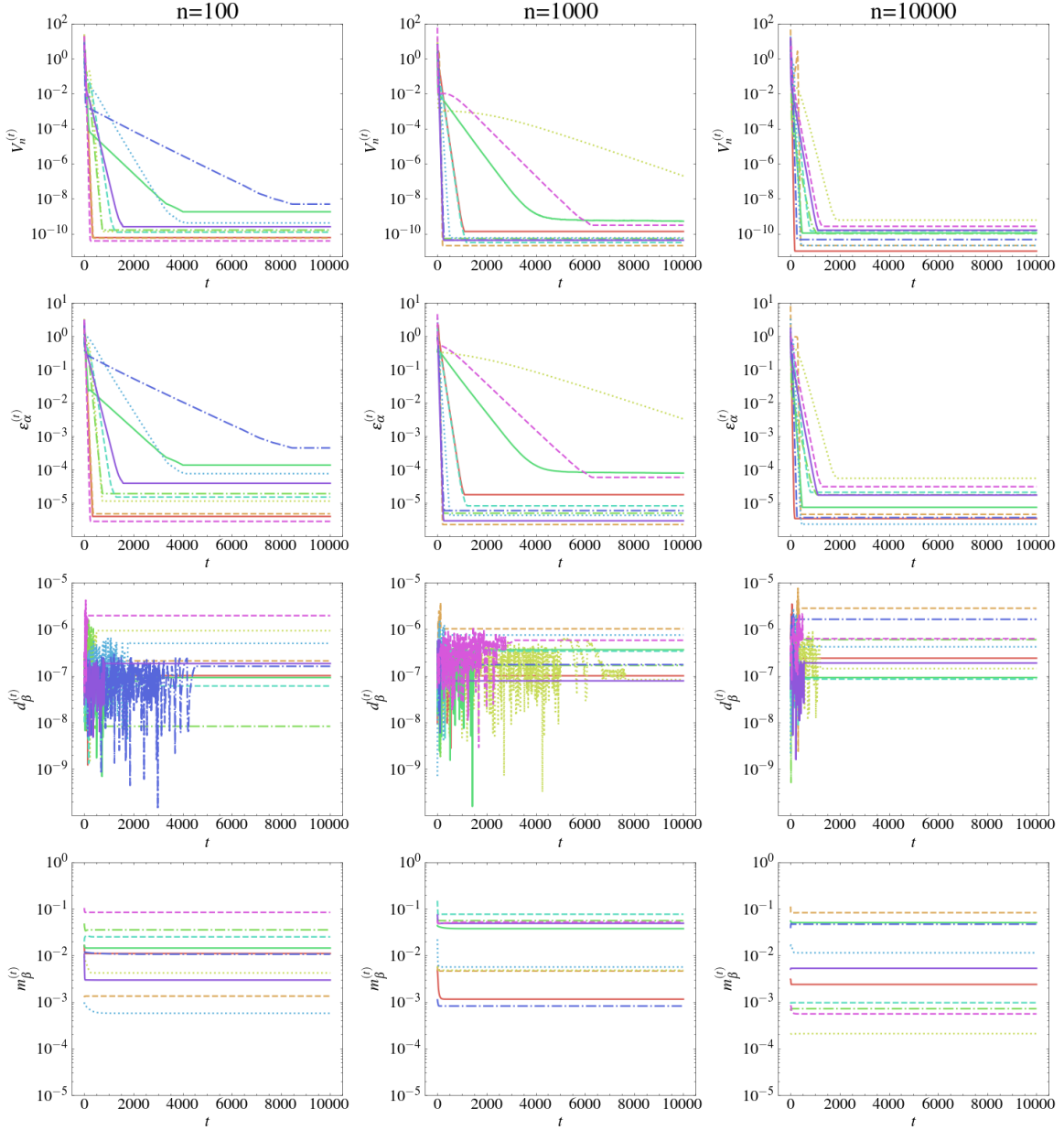


Figure 2: The convergence of the continuous-time GDA as a function of number of iterations (10^4 in total) on the convolutional discriminator. From top to bottom: $V_n^{(t)} = V_n(\alpha^{(t)}, \beta^{(t)})$, $\epsilon_\alpha^{(t)}$, $d_\beta^{(t)}$ and $m_\beta^{(t)}$. From left to right: $n \in \{100, 1000, 10000\}$. Each of the 10 curves corresponds to one simulation.

5. Conclusion

In this paper, we take a moment matching and simultaneous game perspective to study the existence of Nash equilibrium in the finite-sample realizable setting. By focusing on a particular generator family of Gaussian stationary processes, we show that GANs have a rich variability of equilibrium properties. Although these properties vary greatly with the choice of the discriminator family, our results suggest that a suitable discriminator family can result in the existence of consistent Nash or non-Nash equilibrium which are both meaningful solutions. We also find that GDA methods have nearly stable local convergence or global convergence properties when the number of samples is large enough.

To extend our work to non-Gaussian distributions remains a challenging problem, as one would have to construct discriminator families beyond second-order moments. In order to extract non-Gaussian information, one example is to use the rectifier non-linearity to define a discriminator family so as to match higher-order moments between two distributions (Li and Dou, 2021). As we can see from the convolutional discriminator, the existence of consistent Nash equilibrium depends on a careful balance between the discriminator family and the generator family. As a consequence, symmetry properties of a distribution also need to be considered in order to capture invariant non-Gaussian information (Zhang and Mallat, 2021). The difficulty is to define what it means in general to achieve a balance. Recent advances in designing GANs to avoid the curse of dimensionality in high dimensional density estimation (Bai et al., 2019; Liang, 2021; Feizi et al., 2020) could provide some insights into this problem. On the other hand, the existence of consistent non-Nash equilibrium seems to be more relevant to practical GANs. In our case, its existence depends on a perfect moment-matching condition (i.e. $V_n(\alpha^\bullet, \beta^\bullet) = 0$ in Theorem 5) which allows one to extract sufficient information of stationary Gaussian processes. The sufficiency is important for GDA to be stable along α , as it constraints the solution set $\{\alpha \in \mathcal{A}, V_n(\alpha, \beta^\bullet) = 0\}$ to consistent generators. Whether such condition holds for other moments remains open.

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Appendix A. Proof of Proposition 4

Proof We apply the KKT necessary condition (Nocedal and Wright, 2006, Theorem 12.1) to the following Lagrangian function for $\beta \in \mathcal{B}$ and $\lambda \geq 0$,

$$L(\beta, \lambda) = -(\beta^\top (\Sigma_n - \Sigma_{\alpha,n}) \beta)^2 - \lambda(1 - \|\beta\|^2)$$

If β^\bullet is an optimal solution of V_n , then there exists λ^\bullet , such that:

$$\nabla_\beta L(\beta^\bullet, \lambda^\bullet) = 0, \quad \lambda^\bullet(1 - \|\beta^\bullet\|^2) = 0, \quad \|\beta^\bullet\|^2 \leq 1, \quad \lambda^\bullet \geq 0. \quad (19)$$

Let $A = \Sigma_n - \Sigma_{\alpha,n}$, then $\nabla_\beta L(\beta^\bullet, \lambda^\bullet) = -4(\beta^\top A \beta) A \beta + 2\lambda \beta$. From (19), we have two situations:

- Case $\lambda^\bullet > 0$ and $\|\beta^\bullet\| = 1$: $\nabla_\beta L(\beta^\bullet, \lambda^\bullet) = 0$ implies that $\lambda^\bullet \beta^\bullet = 2((\beta^\bullet)^\top A \beta^\bullet) A \beta^\bullet$. This means that β^\bullet is an eigenvector of A , whose eigenvalue $\mu = \frac{\lambda^\bullet}{2((\beta^\bullet)^\top A \beta^\bullet)}$ is non-zero because $(\beta^\bullet)^\top A \beta^\bullet$ must be strictly positive. Therefore the value $V_n(\beta^\bullet, \lambda^\bullet) = \mu^2 > 0$. Note that this situation can not happen if $A = 0$.
- Case $\lambda^\bullet = 0$: $\nabla_\beta L(\beta^\bullet, \lambda^\bullet) = 0$ implies that $((\beta^\bullet)^\top A \beta^\bullet) A \beta^\bullet = 0$. Therefore either $(\beta^\bullet)^\top A \beta^\bullet = 0$ or $A \beta^\bullet = 0$, and $V_n(\beta^\bullet, \lambda^\bullet) = ((\beta^\bullet)^\top A \beta^\bullet)^2 = 0$.

When $A \neq 0$, the above analysis shows that the optimal solution of V_n is attained at β^\bullet . Moreover, it is a unit-norm eigenvector of A which has the maximal absolute eigenvalue so that μ^2 is maximal. The maximal value of V_n thus coincides with $\|\Sigma_n - \Sigma_{\alpha,n}\|^2$. \blacksquare

Appendix B. Proof of Lemma 1

Proof For any $\alpha \in \mathcal{A}$, we show that the event $B_1 = \{\|\Sigma_n - \Sigma_{\alpha,n}\| = 0\}$ has a zero probability under Assumption 1. Indeed, under B_1 , $\Sigma_n = \Sigma_{\alpha,n}$ and therefore $\forall(u, u')$,

$$\frac{1}{n} \sum_{i=1}^n \bar{\alpha} \star \bar{z}_i(u) \bar{\alpha} \star \bar{z}_i(u') = \frac{1}{n} \sum_{i=1}^n \alpha \star z_i(u) \alpha \star z_i(u').$$

where $\{z_i\}$ and $\{\bar{z}_i\}$ are the n i.i.d samples of Z and \bar{Z} .

Applying the Fourier transform along both u and u' , we have equivalently $\forall(\omega, \omega') \in \Omega_d^2$

$$\frac{1}{n} \sum_{i=1}^n \widehat{\alpha}(\omega) \widehat{\bar{z}}_i(\omega) \widehat{\alpha}(\omega')^* \widehat{\bar{z}}_i(\omega')^* = \frac{1}{n} \sum_{i=1}^n \widehat{\alpha}(\omega) \widehat{z}_i(\omega) \widehat{\alpha}(\omega')^* \widehat{z}_i(\omega')^*$$

As d is even, we take $\omega = \omega' = 0$ or $\omega = \omega' = \pi$, and it follows that

$$|\hat{\alpha}(0)|^2 \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(0)|^2 = |\hat{\alpha}(0)|^2 \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(0)|^2 \quad (20)$$

$$|\hat{\alpha}(\pi)|^2 \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(\pi)|^2 = |\hat{\alpha}(\pi)|^2 \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(\pi)|^2 \quad (21)$$

Taking $\omega = 0$ and $\omega' = \pi$, we have

$$\hat{\alpha}(0)\hat{\alpha}(\pi) \frac{1}{n} \sum_{i=1}^n \hat{z}_i(0)\hat{z}_i(\pi) = \hat{\alpha}(0)\hat{\alpha}(\pi) \frac{1}{n} \sum_{i=1}^n \hat{z}_i(0)\hat{z}_i(\pi) \quad (22)$$

To arrange the above terms (20),(21),(22), we need the following event to avoid division by zero

$$B_2 = \left\{ \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(0)|^2 \neq 0, \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(0)|^2 \neq 0, \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(\pi)|^2 \neq 0, \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(\pi)|^2 \neq 0 \right\}$$

As $\bar{\alpha} \notin \mathcal{A}_0$, we have $\hat{\alpha}(0) \neq 0$ and $\hat{\alpha}(\pi) \neq 0$. Then under $B_1 \cap B_2$, it follows from (20),(21) and (22) that the following event holds,

$$B_3 = \left\{ \frac{|\frac{1}{n} \sum_{i=1}^n \hat{z}_i(0)\hat{z}_i(\pi)|^2}{\frac{1}{n} \sum_{i=1}^n |\hat{z}_i(0)|^2 \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(\pi)|^2} = \frac{|\frac{1}{n} \sum_{i=1}^n \hat{z}_i(0)\hat{z}_i(\pi)|^2}{\frac{1}{n} \sum_{i=1}^n |\hat{z}_i(0)|^2 \frac{1}{n} \sum_{i=1}^n |\hat{z}_i(\pi)|^2} \right\}$$

This means that $B_1 \cap B_2 \subset B_3$.

We claim that

$$\text{Prob}(B_2) = 1, \quad \text{Prob}(B_3) = 0 \quad (23)$$

From (23), the statement of this lemma holds, because we will have $\text{Prob}(B_1) = 0$ by using the fact that $\text{Prob}(B_1 \cap B_2) \leq \text{Prob}(B_3) = 0$ and $\text{Prob}(B_1) = \text{Prob}(B_1 \cap B_2) - \text{Prob}(B_2) + \text{Prob}(B_1 \cup B_2) = \text{Prob}(B_1 \cap B_2)$.

To show (23), we denote $y(\omega) = (\hat{z}_i(\omega))_{i \leq n} \in \mathbb{C}^n$, then both $y(0)$ and $y(\pi)$ follow $\mathcal{N}(0, \sigma^2 I_n)$ for $\sigma > 0$. Moreover $y(0)$ is independent of $y(\pi)$ since for any $i \leq n$, $\hat{z}_i(0)$ is independent of $\hat{z}_i(\pi)$. Similarly for $\bar{y}(\omega) = (\hat{z}_i(\omega))_{i \leq n} \in \mathbb{C}^n$ at $\omega = 0$ or $\omega = \pi$. Therefore we have $\text{Prob}(B_2) = 1$ because the distribution of $\|y(0)\|^2$ (and $\|y(\pi)\|^2$, $\|\bar{y}(0)\|^2$, $\|\bar{y}(\pi)\|^2$) is Chi-square of degree n (up to some constant normalization).

To show $\text{Prob}(B_3) = 0$, we notice that the event B_3 is equivalent to

$$\frac{|\langle \bar{y}(0), \bar{y}(\pi) \rangle|^2}{\|\bar{y}(0)\|^2 \|\bar{y}(\pi)\|^2} = \frac{|\langle y(0), y(\pi) \rangle|^2}{\|y(0)\|^2 \|y(\pi)\|^2} \quad (24)$$

We next argue that for $n \geq 2$, the distribution of LHS or RHS of (24) has a continuous density on $[0, 1]$. As the LHS and RHS are sampled from the same distribution independently, the chance that they are the same is thus zero. To verify this, it is sufficient to derive the distribution of

$$\frac{\langle Y_0, Y_1 \rangle}{\|Y_0\| \|Y_1\|}$$

where $Y_0 \sim \mathcal{N}(0, I_n)$ and $Y_1 \sim \mathcal{N}(0, I_n)$ are independent Gaussian white noise in \mathbb{R}^n .

Assume $F(\theta) = \text{Prob} \left(\frac{\langle Y_0, Y_1 \rangle}{\|Y_0\| \|Y_1\|} \leq \sin(\theta) \right)$ for $\theta \in [-\pi/2, \pi/2]$. To compute $F(\theta)$, we denote the coordinates of $Y_0 \in \mathbb{R}^n$ by $x = (x_1, \dots, x_n)$. Then by the rotational invariant property of the distribution of Y_0 and Y_1 , we fix $Y_1 = (1, 0, \dots, 0)$ and obtain

$$F(\theta) = \frac{1}{(\sqrt{2\pi})^n} \int_{x_1 \leq \sin(\theta) \|x\|} e^{-\|x\|^2/2} dx$$

We first focus on the case $\theta \in [0, \pi/2]$. The condition $x_1 \leq \sin(\theta) \|x\|$ is equivalent to

$$x_1 \leq \sqrt{\frac{\sin^2(\theta)}{1 - \sin^2(\theta)}} (x_2^2 + \dots + x_n^2)$$

Denote the cumulative function of the standard normal distribution by Φ . By writing (x_2, \dots, x_n) in the spherical coordinate for $r > 0$, $\psi_2 \in [0, \pi]$, \dots , $\psi_{n-2} \in [0, \pi]$, and $\psi_{n-1} \in [0, 2\pi]$,

$$(r \cos(\psi_2), r \sin(\psi_2) \cos(\psi_3), \dots, r \sin(\psi_2) \dots \sin(\psi_{n-2}) \sin(\psi_{n-1})),$$

we have that

$$\begin{aligned} F(\theta) &= \frac{1}{(\sqrt{2\pi})^n} \int e^{-(x_2^2 + \dots + x_n^2)/2} \int_{-\infty}^{\tan(\theta) \sqrt{x_2^2 + \dots + x_n^2}} e^{-x_1^2/2} dx_1 dx_2 \dots dx_n \\ &= \frac{1}{(\sqrt{2\pi})^{n-1}} \int e^{-(x_2^2 + \dots + x_n^2)/2} \Phi \left(\tan(\theta) \sqrt{x_2^2 + \dots + x_n^2} \right) dx_2 \dots dx_n \\ &= \frac{1}{(\sqrt{2\pi})^{n-1}} \int e^{-r^2/2} \Phi(\tan(\theta)r) r^{n-2} \sin^{n-3}(\psi_2) \dots \sin(\psi_{n-2}) dr d\psi_2 \dots d\psi_{n-1} \\ &= c_n \int_0^\infty e^{-r^2/2} \Phi(\tan(\theta)r) r^{n-2} dr \end{aligned}$$

with a normalization constant c_n . It follows that the density

$$\begin{aligned} F'(\theta) &= c_n \int_0^\infty e^{-r^2/2} \Phi'(\tan(\theta)r) \frac{r}{\cos^2(\theta)} r^{n-2} dr \\ &= \frac{c_n}{\sqrt{2\pi}} \int_0^\infty e^{-(1+\tan^2(\theta))r^2/2} \frac{r^{n-1}}{\cos^2(\theta)} dr \\ &\propto \frac{1}{\cos^2(\theta)} \int_0^\infty e^{-(1+\tan^2(\theta))t/2} t^{(n-2)/2} dt \end{aligned}$$

This integral can be computed from the Gamma distribution which gives for $\alpha > 0, \beta > 0$,

$$\int_0^\infty t^{\alpha-1} e^{-\beta t} dt = \Gamma(\alpha)/\beta^\alpha.$$

Taking $\alpha = n/2$, and $\beta = (1 + \tan^2(\theta))/2$, we conclude that for $\theta \in [0, \pi/2]$,

$$F'(\theta) \propto \frac{1}{\cos^2(\theta)} \frac{\Gamma(n/2)}{(\frac{1+\tan^2(\theta)}{2})^{n/2}} \propto \cos^{n-2}(\theta). \quad (25)$$

As the distribution of $\langle Y_0, Y_1 \rangle$ is symmetric around zero, the density $F'(\theta)$ is a symmetric function around $\theta = 0$. Thus (25) holds also for $\theta \in [-\pi/2, 0]$. As $\sin(\theta)$ is differentiable and monotone increasing on $[-\pi/2, \pi/2]$, a change of variable shows that the density of $\frac{\langle Y_0, Y_1 \rangle}{\|Y_0\| \|Y_1\|}$ exists and it is supported on $[-1, 1]$. \blacksquare

Appendix C. Proof of Remark 6

We show that

$$\sup_{\alpha^\bullet \in \mathcal{A}_n} \|\Sigma_n - \Sigma_{\alpha^\bullet, n}\| \rightarrow 0, \quad n \rightarrow \infty, \quad \text{in probability}$$

Proof The key idea is to show that \mathcal{A}_n is a bounded set with high probability when n is large enough, and to use classical results about the convergence of empirical covariance matrices through a uniform upper bound of $\|\Sigma_n - \Sigma_{\alpha^\bullet, n}\|$.

Using the triangular inequality of the norm $\|\cdot\|$, we have

$$\sup_{\alpha^\bullet \in \mathcal{A}_n} \|\Sigma_{\alpha^\bullet, n} - \Sigma_n\| \leq \sup_{\alpha^\bullet \in \mathcal{A}_n} \|\Sigma_{\alpha^\bullet, n} - \Sigma_{\alpha^\bullet}\| + \sup_{\alpha^\bullet \in \mathcal{A}_n} \|\Sigma_{\alpha^\bullet} - \Sigma\| + \|\Sigma - \Sigma_n\| \quad (26)$$

It is sufficient to show that each of the three terms on the RHS of (26) converges to zero in probability. Since X is a Gaussian distribution on \mathbb{R}^d , the convergence of $\|\Sigma - \Sigma_n\|$ follows immediately from the classical results established for sub-Gaussian distributions (Vershynin, 2018, Theorem 4.7.1).

Recall that

$$\mathcal{A}_n = \{\alpha \in \mathbb{R}^d \mid |\hat{\alpha}(\omega)|^2 = \mathbb{E}_n(|\hat{X}(\omega)|^2)/\mathbb{E}_n(|\hat{Z}(\omega)|^2), \forall \omega \in \Omega_d\}.$$

Proposition 1 implies that, $\forall \alpha^\bullet \in \mathcal{A}_n$

$$\begin{aligned} \|\Sigma_{\alpha^\bullet} - \Sigma\| &= \max_{\omega \in \Omega_d} |\hat{\alpha}^\bullet(\omega)|^2 - |\hat{\alpha}(\omega)|^2 \\ &= \max_{\omega \in \Omega_d} |\mathbb{E}_n(|\hat{X}(\omega)|^2)/\mathbb{E}_n(|\hat{Z}(\omega)|^2) - |\hat{\alpha}(\omega)|^2| \\ &= \max_{\omega \in \Omega_d} |(\mathbb{E}_n(|\hat{Z}(\omega)|^2)/\mathbb{E}_n(|\hat{Z}(\omega)|^2) - 1)|\hat{\alpha}(\omega)|^2| \end{aligned}$$

Applying the law of large numbers to the n samples of \bar{Z} and the n samples of Z , we have

$$\sup_{\alpha^\bullet \in \mathcal{A}_n} \|\Sigma_{\alpha^\bullet} - \Sigma\| \rightarrow 0, \quad n \rightarrow \infty, \quad \text{in probability}$$

For the convergence of $\sup_{\alpha^\bullet \in \mathcal{A}_n} \|\Sigma_{\alpha^\bullet, n} - \Sigma_{\alpha^\bullet}\|$, we note that for any $\alpha \in \mathcal{A}$, $\forall (u, u')$,

$$(\Sigma_{\alpha, n} - \Sigma_\alpha)(u, u') = \sum_{v, v'} \alpha(u - v)\alpha(u' - v')(\Sigma_n^0(v, v') - \Sigma^0(v, v')),$$

where $\Sigma_n^0(v, v') = \mathbb{E}_n(Z(v)Z(v'))$ and $\Sigma^0(v, v') = \mathbb{E}(Z(v)Z(v'))$. Let $\alpha_u(v) = \alpha(u - v)$, then

$$(\Sigma_{\alpha, n} - \Sigma_\alpha)(u, u') = \alpha_u^\top (\Sigma_n^0 - \Sigma^0) \alpha_{u'}$$

Denote $M_\alpha = d\|\alpha\|^2$, we next check that

$$\|\Sigma_{\alpha,n} - \Sigma_\alpha\|^2 \leq M_\alpha^2 \|\Sigma_n^0 - \Sigma^0\|^2 \quad (27)$$

To verify (27), we use the Cauchy-Schwartz inequality,

$$\begin{aligned} \|\Sigma_{\alpha,n} - \Sigma_\alpha\|^2 &= \sup_{\|w\| \leq 1} \|(\Sigma_{\alpha,n} - \Sigma_\alpha)w\|^2 \\ &= \sup_{\|w\| \leq 1} \sum_u \left| \sum_{u'} \alpha_u^\top (\Sigma_n^0 - \Sigma^0) \alpha_{u'} w(u') \right|^2 \\ &\leq \sup_{\|w\| \leq 1} \sum_u \|\alpha_u\|^2 \cdot \left\| \sum_{u'} (\Sigma_n^0 - \Sigma^0) \alpha_{u'} w(u') \right\|^2 \\ &\leq \sup_{\|w\| \leq 1} M_\alpha \|\Sigma_n^0 - \Sigma^0\|^2 \left\| \sum_{u'} \alpha_{u'} w(u') \right\|^2 \\ &= M_\alpha \|\Sigma_n^0 - \Sigma^0\|^2 \sup_{\|w\| \leq 1} \sum_{v'} \left| \sum_{u'} \alpha_{u'}(v') w(u') \right|^2 \\ &\leq M_\alpha^2 \|\Sigma_n^0 - \Sigma^0\|^2. \end{aligned}$$

From (27), we conclude that

$$\sup_{\alpha^\bullet \in \mathcal{A}_n} \|\Sigma_{\alpha^\bullet} - \Sigma\| \leq \left(\sup_{\alpha^\bullet \in \mathcal{A}_n} M_{\alpha^\bullet} \right) \|\Sigma_n^0 - \Sigma^0\|$$

For any $\alpha^\bullet \in \mathcal{A}_n$,

$$\|\alpha^\bullet\|^2 = d^{-1} \sum_{\omega \in \Omega_d} |\hat{\alpha}^\bullet(\omega)|^2 = d^{-1} \sum_{\omega \in \Omega_d} \frac{\mathbb{E}_n(|\hat{Z}(\omega)|^2)}{\mathbb{E}_n(|\hat{Z}(\omega)|^2)} |\hat{\alpha}(\omega)|^2$$

Therefore $M_{\alpha^\bullet} = d\|\alpha^\bullet\|^2$ does not vary in the set \mathcal{A}_n .

According to (Vershynin, 2018, Theorem 4.7.1), $\|\Sigma_n^0 - \Sigma^0\|$ converges to zero in probability when $n \rightarrow \infty$. Therefore, it remains to show that $\exists C > 0, \forall \delta > 0, \exists N \in \mathbb{Z}$ such that if $n \geq N$, then

$$\text{Prob}\left(\sup_{\alpha^\bullet \in \mathcal{A}_n} M_{\alpha^\bullet} \leq C\right) > 1 - \delta.$$

This is true because the law of large numbers, applied to Z and \bar{Z} , implies that

$$\max_{\omega \in \Omega_d} \left| \frac{\mathbb{E}_n(|\hat{Z}(\omega)|^2)}{\mathbb{E}_n(|\hat{Z}(\omega)|^2)} - 1 \right| \rightarrow 0, \quad n \rightarrow \infty, \quad \text{in probability,}$$

i.e. $\sup_{\alpha^\bullet \in \mathcal{A}_n} M_{\alpha^\bullet} \rightarrow d\|\bar{\alpha}\|^2$ as $n \rightarrow \infty$ in probability. ■

Appendix D. Proof of Proposition 5

Proof For $\alpha^\bullet \in \mathcal{A}_n$, we are going to show that $r_{n,\ell}(\alpha^\bullet, \beta) = 0$ for all ℓ . This implies that $V_n(\alpha^\bullet, \beta) = \sum_\ell r_{n,\ell}^2(\alpha^\bullet, \beta) = 0$ for $\forall \beta \in \mathcal{B}$. Indeed, using Parseval's identity, $\forall x \in \mathbb{R}^d$

$$\|x \star \beta_\ell\|^2 = d^{-1} \sum_{\omega \in \Omega_d} |\widehat{x}(\omega)|^2 |\widehat{\beta}_\ell(\omega)|^2$$

This implies that for $\forall \ell < m, \forall \beta \in \mathcal{B}$,

$$r_{n,\ell}(\alpha^\bullet, \beta) \propto \sum_{\omega} \left(\mathbb{E}_n(|\widehat{X}(\omega)|^2) - \mathbb{E}_n(|\widehat{\alpha^\bullet}(\omega) \widehat{Z}(\omega)|^2) \right) |\widehat{\beta}_\ell(\omega)|^2 = 0$$

To show that $(\alpha^\bullet, \beta^\bullet)$ is a Nash equilibrium for any $\beta^\bullet \in \mathcal{B}$, we verify that for any $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$,

$$V_n(\alpha^\bullet, \beta) \leq V_n(\alpha^\bullet, \beta^\bullet) \leq V_n(\alpha, \beta^\bullet)$$

The consistency of $(\alpha^\bullet, \beta^\bullet)$ follows from Proposition 2. ■

Appendix E. Proof of Lemma 2

Proof If $\alpha \in \mathcal{A}_n$, then Proposition 5 implies that $V_n(\alpha, \beta) = 0$ for any $\beta \in \mathcal{B}$. We next show that if $(|\widehat{\beta}_\ell|^{\circ 2})_{\ell < d}$ is a basis, then $V_n(\alpha, \beta) = 0$ implies that $\alpha \in \mathcal{A}_n$ almost surely.

Using Parseval's identity, we write

$$\begin{aligned} r_{n,\ell}(\alpha, \beta) &= (\mathbb{E}_n(\|X \star \beta_\ell\|^2) - \mathbb{E}_n(\|g_\alpha(Z) \star \beta_\ell\|^2)) \\ &= d^{-1} \left(\sum_{\omega} \mathbb{E}_n(|\widehat{X}(\omega) \widehat{\beta}_\ell(\omega)|^2) - \sum_{\omega} \mathbb{E}_n(|\widehat{g_\alpha(Z)}(\omega) \widehat{\beta}_\ell(\omega)|^2) \right) \\ &= d^{-1} \sum_{\omega} (\mathbb{E}_n(|\widehat{X}(\omega)|^2) - \mathbb{E}_n(|\widehat{\alpha}(\omega) \widehat{Z}(\omega)|^2)) |\widehat{\beta}_\ell(\omega)|^2 \\ &= \langle h, |\widehat{\beta}_\ell|^{\circ 2} \rangle, \end{aligned}$$

where $h(\omega) = d^{-1}(\mathbb{E}_n(|\widehat{X}(\omega)|^2) - \mathbb{E}_n(|\widehat{\alpha}(\omega) \widehat{Z}(\omega)|^2))$ for $\omega \in \Omega_d$.

By definition, $V_n(\alpha, \beta) = 0$ is equivalent to $r_{n,\ell}(\alpha, \beta) = 0$ for all ℓ , i.e.

$$\langle h, |\widehat{\beta}_\ell|^{\circ 2} \rangle = 0, \quad \forall \ell < d$$

It implies that $h = 0$ because $|\widehat{\beta}_\ell|^{\circ 2}$ is a basis of \mathbb{R}^d . Therefore $\alpha \in \mathcal{A}_n$ almost surely. ■

Appendix F. Proof of Theorem 7

Proof

Assume $(\alpha^\bullet, \beta^\bullet)$ is an equilibrium of V_n . Consider two cases,

- $\alpha^\bullet \notin \mathcal{A}_0$: we show that $\alpha^\bullet \in \mathcal{A}_n$, and therefore by Proposition 5, $(\alpha^\bullet, \beta^\bullet)$ is a consistent Nash equilibrium.
- $\alpha^\bullet \in \mathcal{A}_0$: we show that $(\alpha^\bullet, \beta^\bullet)$ can not be an equilibrium, because $\nabla_\beta V_n(\alpha^\bullet, \beta^\bullet) \neq 0$.

Case $\alpha^\bullet \notin \mathcal{A}_0$: From Lemma 2, it is sufficient to check that $V_n(\alpha^\bullet, \beta^\bullet) = 0$ almost surely. The gradient of V_n with respect to α and β are computed as in (14), with

$$\nabla_\alpha r_{n,\ell}(\alpha, \beta) = -2(\beta_\ell \star \tilde{\beta}_\ell) \star \mathbb{E}_n(Z \star \tilde{Z}) \star \alpha \quad (28)$$

$$\nabla_{\beta_\ell} r_{n,\ell}(\alpha, \beta) = 2(\mathbb{E}_n(X \star \tilde{X} \star \beta_\ell) - \mathbb{E}_n(g_\alpha(Z) \star \widetilde{g_\alpha(Z)} \star \beta_\ell)) \quad (29)$$

As $\nabla_\alpha V_n(\alpha^\bullet, \beta^\bullet) = 0$, (28) implies that

$$\sum_\ell r_{n,\ell}(\alpha^\bullet, \beta^\bullet)(\beta_\ell^\bullet \star \tilde{\beta}_\ell^\bullet) \star \mathbb{E}_n(Z \star \tilde{Z}) \star \alpha^\bullet = 0 \quad (30)$$

We take the Fourier transform on the LHD of (30). It results in an equivalent equation

$$\sum_\ell r_{n,\ell}(\alpha^\bullet, \beta^\bullet) |\widehat{\beta}_\ell^\bullet(\omega)|^2 \mathbb{E}_n(|\widehat{Z}(\omega)|^2) \widehat{\alpha}^\bullet(\omega) = 0, \quad \forall \omega \in \Omega_d$$

Since for any $\omega \in \Omega_d$, $\mathbb{E}_n(|\widehat{Z}(\omega)|^2) \widehat{\alpha}^\bullet(\omega) \neq 0$ almost surely, it follows that almost surely

$$\sum_\ell r_{n,\ell}(\alpha^\bullet, \beta^\bullet) |\widehat{\beta}_\ell^\bullet(\omega)|^2 = 0, \quad \forall \omega \in \Omega_d. \quad (31)$$

As $(|\widehat{\beta}_\ell^\bullet|^2)_{\ell < d}$ is a basis on \mathbb{R}^d , (31) implies that $r_{n,\ell}(\alpha^\bullet, \beta^\bullet) = 0$ for all ℓ , i.e. $V_n(\alpha^\bullet, \beta^\bullet) = 0$.

Case $\alpha^\bullet \in \mathcal{A}_0$: by the definition of \mathcal{A}_0 , there exists $\omega_0 \in \Omega_d$ such that $\widehat{\alpha}^\bullet(\omega_0) = 0$.

Firstly, we have almost surely

$$\mathcal{A}_n \cap \mathcal{A}_0 = \emptyset$$

because $\alpha \in \mathcal{A}_n$ implies that almost surely

$$|\widehat{\alpha}(\omega)|^2 = |\widehat{\alpha}(\omega)|^2 \mathbb{E}_n(|\widehat{Z}(\omega)|^2) / \mathbb{E}_n(|\widehat{Z}(\omega)|^2) \neq 0, \forall \omega \in \Omega_d$$

As $\alpha^\bullet \in \mathcal{A}_0$, this implies that almost surely $\alpha^\bullet \notin \mathcal{A}_n$. Then from Lemma 2, $V_n(\alpha^\bullet, \beta^\bullet) \neq 0$ almost surely, so that there exists $\ell_0 < d$ such that $r_{n,\ell_0}(\alpha^\bullet, \beta^\bullet) \neq 0$. We next verify that the gradient of V_n with respect to β_{ℓ_0} is non-zero. From (14) and (29)

$$\begin{aligned} \nabla_{\beta_{\ell_0}} V_n(\alpha^\bullet, \beta^\bullet) &= 2r_{n,\ell_0}(\alpha^\bullet, \beta^\bullet) \nabla_{\beta_{\ell_0}} r_{n,\ell_0}(\alpha^\bullet, \beta^\bullet) \\ &= r_{n,\ell_0}(\alpha^\bullet, \beta^\bullet) (\mathbb{E}_n(X \star \tilde{X} \star \beta_{\ell_0}^\bullet) - \mathbb{E}_n(g_{\alpha^\bullet}(Z) \star \widetilde{g_{\alpha^\bullet}(Z)} \star \beta_{\ell_0}^\bullet)) \end{aligned} \quad (32)$$

As $r_{n,\ell_0}(\alpha^\bullet, \beta^\bullet) \neq 0$, taking the Fourier transform of (32), $\nabla_{\beta_{\ell_0}} V_n(\alpha^\bullet, \beta^\bullet) = 0$ implies that

$$\mathbb{E}_n(|\widehat{X}(\omega)|^2) \widehat{\beta}_{\ell_0}(\omega) = \mathbb{E}_n(|\widehat{g_{\alpha^\bullet}(Z)}(\omega)|^2) \widehat{\beta}_{\ell_0}(\omega), \forall \omega \in \Omega_d$$

As $\beta_{\ell_0} \notin \mathcal{A}_0$, it implies that almost surely $\mathbb{E}_n(|\widehat{X}(\omega_0)|^2) = \mathbb{E}_n(|\widehat{g_{\alpha^\bullet}(Z)}(\omega_0)|^2) = 0$ because $\widehat{\alpha}^\bullet(\omega_0) = 0$. This is contradictory because $\alpha^\bullet \notin \mathcal{A}_0$ implies that $\mathbb{E}_n(|\widehat{X}(\omega_0)|^2)$ is almost surely non-zero. ■

Appendix G. Additional numerical results

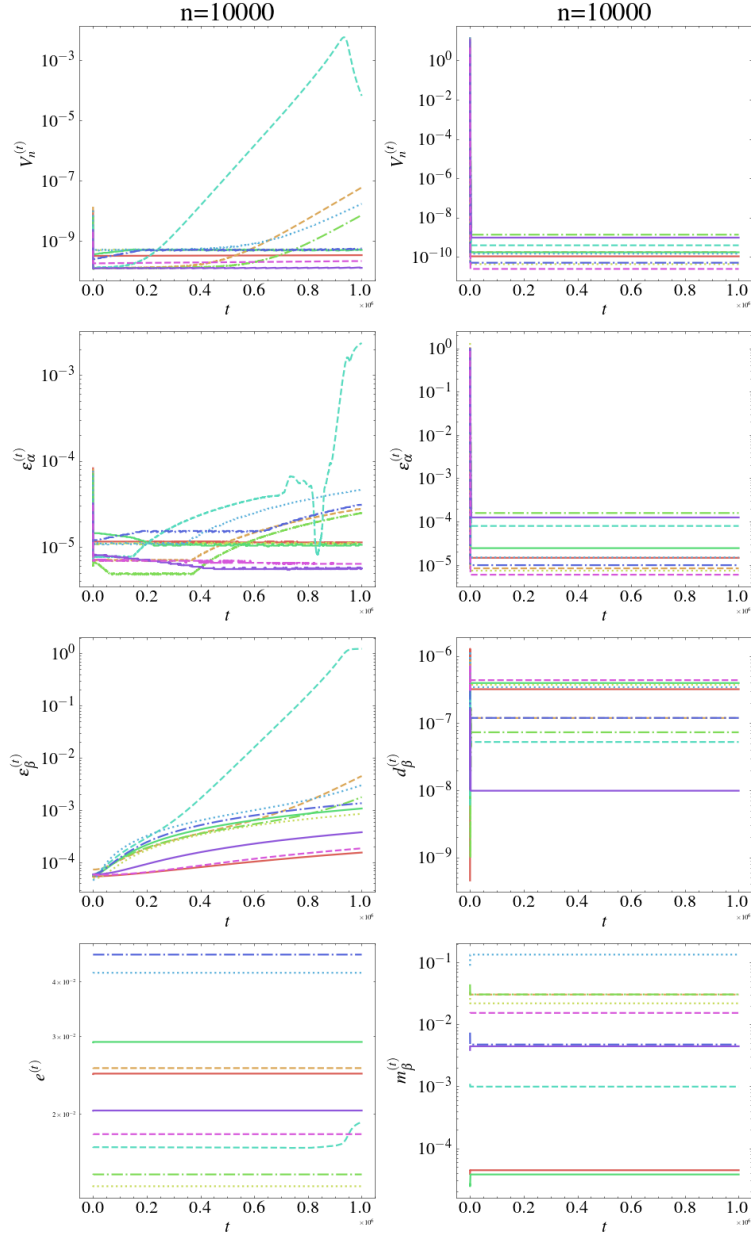


Figure 3: The local stability (left) and global convergence (right) of the continuous-time GDA as a function of the number of iterations (10^6 in total). Left: the dynamics of $V_n^{(t)} = V_n(\alpha^{(t)}, \beta^{(t)})$, $\epsilon_\alpha^{(t)}$, $\epsilon_\beta^{(t)}$ and $e^{(t)} = \|\Sigma - \Sigma_{\alpha(t)}\|$ on the complex discriminator. Right: the dynamics of $V_n^{(t)} = V_n(\alpha^{(t)}, \beta^{(t)})$, $\epsilon_\alpha^{(t)}$, $d_\beta^{(t)}$ and $m_\beta^{(t)}$ on the convolutional discriminator.