Brownian Motion and Random Walks

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1 Introduction

This independent inquiry paper explores random walks, Brownian Motion, and their applications. We explain random walks with a computer generated depiction, discuss how it relates to Brownian Motion, and then explore two common applications.

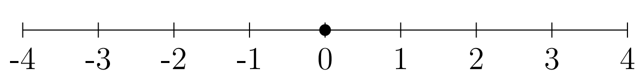
2 Random Walks

2.1 A Brief History

A random walk can be defined as a series of discrete steps an object takes in some direction. While the idea feels abstract, random walks are in fact present in the world around us, from the movement of molecules in a gas to the behavior of the stock market over time. The concept of Random Walks was popularized as a mathematical model of the stock market. This theory was known as "Random Walk Theory" and was the belief that the prices of the securities in the stock market evolve according to a random walk, explored later in this paper. This means that changes in asset price are random and move unpredictably. Economist Burton Malkiel clearly lined out this theory in his 1973 book, *A Random Walk Down Wall Street*.

2.2 The Simplest Random Walk

The simplest random walk involves a black dot positioned on a one-dimensional line, capable of moving either forward or backward with equal probability, seen on the number line below. The steps are denoted as a_1 for the first forward step, a_2 for the second, and so forth, where each a_i is either $+1$ for forward steps or -1 for backward steps. To assess how far the black dot travels after taking N steps, where N is any number, we aim to determine the expected value of the distance traveled, given that the distance (d) varies with each repetition.



Thus, the total distance is represented as

$$d = a_1 + a_2 + \cdots + a_N$$

And expected value of the distance is

$$< d >=< (a_1 + a_2 + \cdots + a_N) >$$

$$< d >=< \sum_{i=1}^{N} a_i >$$

given each $< a_i >= 0$ since a_i is either 1 or -1 with equal probability, therefore:

$$< d >= 0$$

Since forward and backward steps are equally likely at all times, the expected value of d (average ending position) would be back at the origin or 0. To calculate the dispersion about the origin of the distance after N steps, we will find the average values of the distance squared (d^2). This is similar to taking the sums of squares for data points against a regression. Thus,

$$< d^2 >=< (a_1 + a_2 + a_3 + ... + a_N)^2 >$$

Upon squaring the sum of each step results in N^2 terms. The equation is comprised of either the step squared (ex. $< a_1^2 >$) or the step times another step (ex. $< a_1 * a_2 >$. In the context of a simple random walk where each step can either be forward (+1) or backwards (-1) denoted in the a_n term, any a_n^2 term equals to 1. The product of the two steps can equal 1 (where $a_1 = a_2 = 1$ or -1) or -1 (where $a_1 \neq a_2$, and each term is -1 or 1). Because the steps are equally likely to be forwards or backwards, the average value of these product terms ($< a_1 a_2 >$) is 0. As a result, these terms do not contribute to the sum because they are 0. Thus,

$$\begin{aligned} < d^2 >=< (a_1 + a_2 + a_3 + \cdots + a_N)^2 > &= N \\ \sqrt{< d^2 >} &= \sqrt{N} \end{aligned}$$

This means that after N steps, the black dot will have a dispersion about the origin of $\sqrt{N}$.

2.3 Multiple Random Walks and The Normal Distribution

To illustrate the behavior of a random walk, we performed Monte Carlo simulations in MATLAB. Specifically, we simulated a 1-dimensional unbiased random walk with a step size equal to one. Unbiased here refers to an equal probability of making a positive and a negative step. We assumed that the initial position was at the origin.

Fig. 1 shows 1000 simulations of a random walk simulation after 100 steps. As the number of steps increases from 1 to 100, the positions are moving away from the origin. The figure on the right shows the histogram (approximate probability density function) of a position after 100 steps. Figs. 2 and 3 show an equivalent simulation for 300 steps and 500 steps, respectively. All histograms resemble Gaussian densities, albeit the spread of the distribution (variance) increases with the number of steps taken. In the context of a very large number of steps, these histograms would approach a uniform probability density function because of the law of large numbers. For a smaller step size, random walk approaches a Wiener process (if working with continuous values instead of integer steps like here).

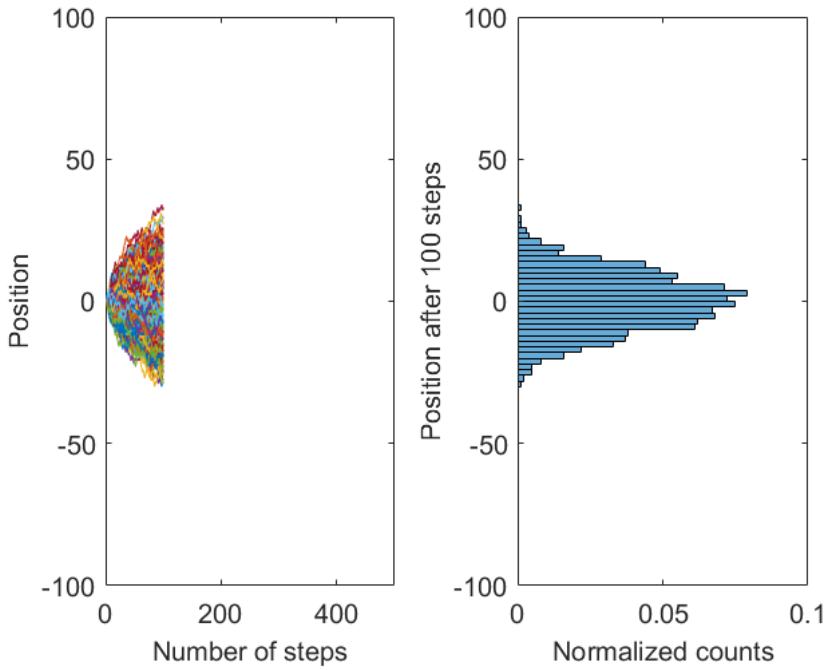


Figure 1: (Left) 1000 realizations of an unbiased random walk with 100 steps. (Right) The histogram of a position after 100 steps.

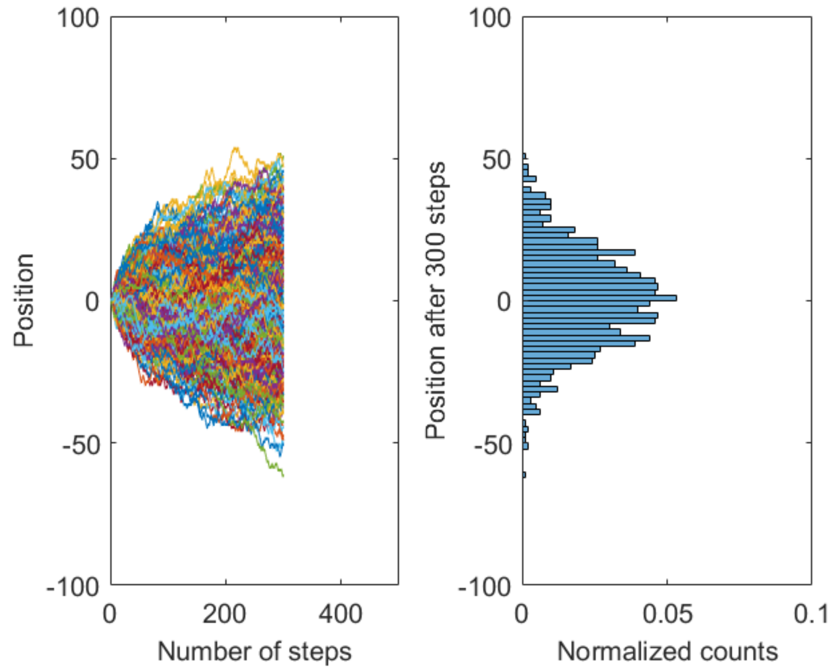


Figure 2: (Left) 1000 realizations of an unbiased random walk with 300 steps. (Right) The histogram of a position after 300 steps.

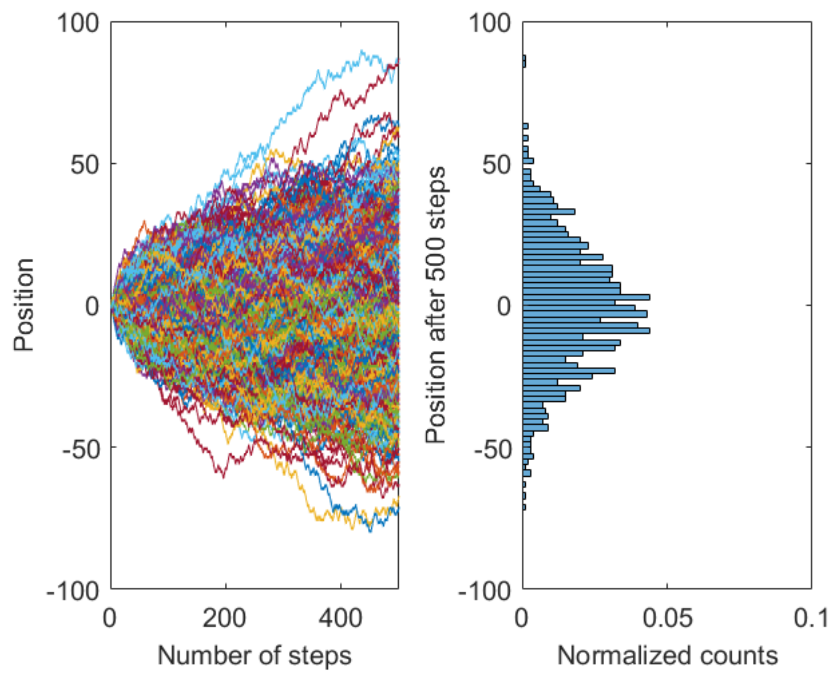


Figure 3: (Left) 1000 realizations of an unbiased random walk with 500 steps. (Right) The histogram of a position after 500 steps.

In conclusion, the distribution of d would be represented by a **Gaussian** or **Normal** distribution with a mean 0 and a variance proportional to the number of steps as shown above ($\mu = 0$, $\sigma \propto \sqrt{N}$). Another important thing to note is that as we increase the number of steps the distribution appears to be flatter and flatter, until eventually it appears almost uniform (happens as $N \to \infty$). This is a model of a diffusion process (discussed in section 4.2) where after a long period of time, the concentration of a chemical equalizes. For example, a drop of ink in a water cup will initially be concentrated at the point of delivery. The longer we wait, the ink molecules are going to become more dispersed (as shown below). Eventually, the ink concentration will become uniform and our ability to perceive its color will be diminished.

3 Brownian Motion

3.1 Brownian Motion from Random Walks

Brownian motion is a specific type of random walk that was first discovered by the botanist Robert Brown in the early 19th century. He was observing pollen particles suspended in water when he noticed that these particles moved in a seemingly random zigzag motion. Brownian Motion, therefore, describes the random movement of particles suspended in a fluid (like water or air) due to collision with molecules of the surrounding medium.

The 1-dimensional random walk derived above can be generalized to 2 or 3 dimensions. A random walk in 3 dimensions then becomes a suitable model for Brownian Motion. Specifically, as the step size goes to 0 and the number of steps tends to ∞, 3-dimensional random walk models a Brownian Motion in 3-dimensional space, such as the movement of the pollen particle suspended in water. Here is another illustrative example:

You are dirty and never clean your house. You find a clear box that has many dust particles inside, and when held up in the light, you can see the dust particles continuously moving around. You're curious about the random movement of each particle and if it can be modeled by an equation.

Let parameter t be time and B_t be for Brownian motion (the motion of the particles). The length of the interval must be $t_2 - t_1$, and thus can extrapolate that the difference in Brownian Motion is $B_{t2} - B_{t1}$

Choosing a **non-random model for** $B_{t2} - B_{t1}$ **is what defines Brownian Motion**. Using the central limit theorem, it can be deduced that $B_{t2} - B_{t1} \sim \ Normal(\mu = 0, \sigma^2 = t_2 - t_1)$.

3.2 Types of Brownian Motion

3.2.1 Arithmetic Brownian Motion (ABM)

This is modeled by

$$X_t = X_0 + \mu t + \sigma B_t$$

$$dX_t = \mu dt + \sigma dB_t$$

where:
X_t = position at time t
X_0 = initial position
μ = drift coefficient, representing the trend or growth rate per unit time
σ = volatility, representing variation or spread of the increments
t = time
B_t = standard Brownian Motion

Some applications of ABM are in environmental science to model the dispersion of pollutants in air or water. The linear growth aspect is relevant when considering the gradual spread of pollutants over time. The arithmetic Brownian Motion equation is also unique because volatility (σ) stays constant. Lastly, this model has a mean reverting behavior.

3.2.2 Geometric Brownian Motion (GBM)

This is modeled by

$$X_t = X_0 exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$$
$$dB_t = \mu dt + \sigma B_t dW_t$$

where:
X_t = position at time t
X_0 = initial position
μ = drift coefficient, representing the trend or growth rate per unit time
σ = volatility, representing variation or spread of the increments
t = time
B_t = standard Brownian Motion

Some applications of geometric Brownian Motion are to population growth and options pricing through the Black-Scholes model. This will be expanded upon in section 4.3. GBM is unique because it is always positive (because of the exponential), making it especially crucial to finance because asset prices are always positive. Because of the exponential nature of GBM, this model is also useful when dealing with compounding.

In general, predicting the drift coefficient (μ) is useful to represent rate of growth to determine trends. Variation (or volatility, σ) is important to determine risk/uncertainty and short-term movements that are reliant on volatility.

4 Applications of Brownian Motion

4.1 Creating a Brownian Motion in R Studio

Below is a normal distribution generated by using the Monte-Carlo simulation 1000 times to estimate the drift coefficient (μ) of the geometric Brownian Motion.

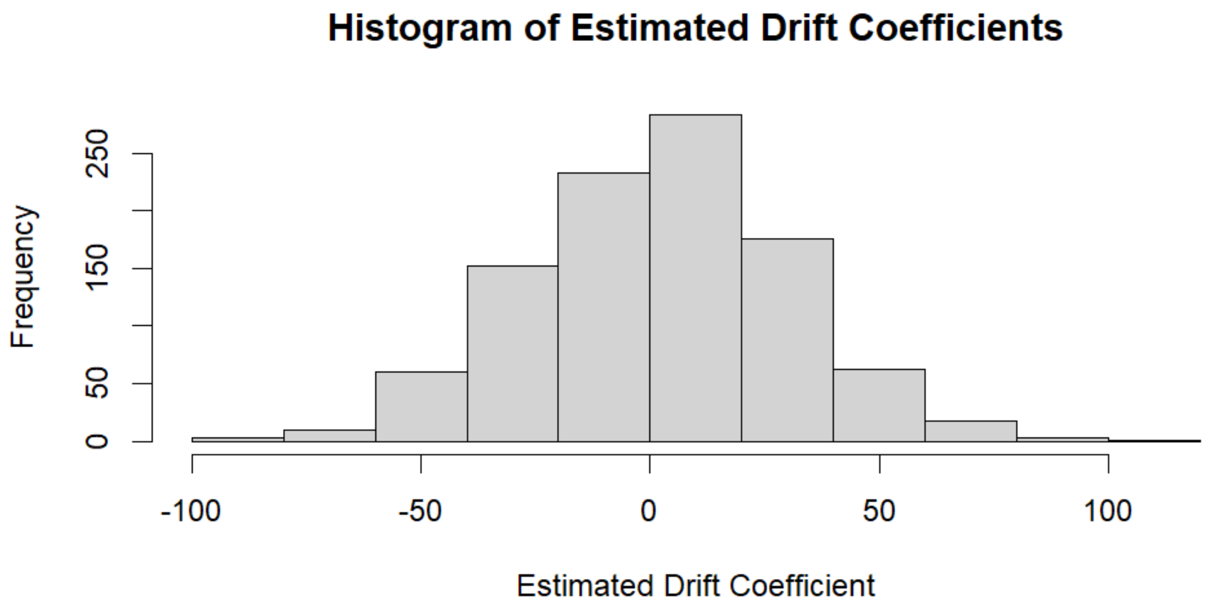


Figure 4: Monte Carlo Simulation to predict μ, or drift parameter

The drift coefficient (μ) represents the average rate of return per unit of time. In the context of a stock's performance, the code is choosing a random value that adjusts for the volatility parameter (σ). Using this volatility value, the stock's performance is modeled using GBM and change in stock price (average rate of return) is calculated, yielding our drift component μ. The distribution is normal because of the central limit theorem.

4.2 Diffusion

By utilizing Brownian Motion, Albert Einstein was able to measure the density ρ of gas molecules by using a non-random equation to ascribe random movements - the epitome of Brownian Motion. The density satisfies this differential equation:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2}$$

D is the diffusion coefficient that can be calculated. This equation provides some certainty from a random model and shows us that diffusion processes are caused by Brownian motion.

4.3 Deriving the Black-Scholes Model for Options Pricing

One of the most famous uses of Brownian Motion is for the Black-Scholes model to determine European options pricing. Geometric Brownian Motion is utilized because it is an accurate model for the change in stock price, and is always positive (stock price is also always positive). Here is a very simplified derivation of the Black-Scholes model.

Let us have a European options contract called $V(S,T)$ that is a stochastic process of current stock price (S) and time (T). We want to determine the theoretical fair price of the option and determine if there is a discrepancy for any arbitrage trading opportunities for us to take advantage of.

Let's say our portfolio is modeled by

$$P = V(S,t) - \sigma * S$$

where σS represents the number of shares of the underlying asset that you would buy or sell to offset the changes in the option's value due to changes in the asset price. How does our portfolio change?

$$dP = dV - \Delta dS$$

where dV is the change in the option's value and dS is change in the stock price. dS can be modeled by

$$dS = \mu S dt + \sigma S dW$$

where dW represents Brownian Motion. $\mu S dt$ is the drift coefficient and $\sigma S dW$ is the volatility.

Using Ito's Lemma, it can be determined that if V is a function of a stochastic process, we can get a dV. With Ito's Lemma,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt$$

Substituting for d^2S, we get

$$dV = (\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \frac{\partial V}{\partial S}\sigma S dW$$

Plugging dV into dP simplifies to:

$$dP = (\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + ((\frac{\partial V}{\partial S} - \Delta)\sigma S dW$$

such that $\Delta = \partial V/\partial S$, making the second half of the equation 0. The second term was our volatility or risk value. Now that this is 0, it has essentially eliminated our risk. Now, our portfolio is just a dP term and doesn't carry any risk, yielding a risk free rate.

We know our portfolio grows at a risk free rate now, modeled by

$$dP = (rV - rS\frac{\partial V}{\partial S})dt$$

where r is the risk-free rate. Equating our two dP equations yields the Black-Scholes equation,

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

5 Sources

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