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for Statics and Dynamics of Plane Structures (with Matlab)

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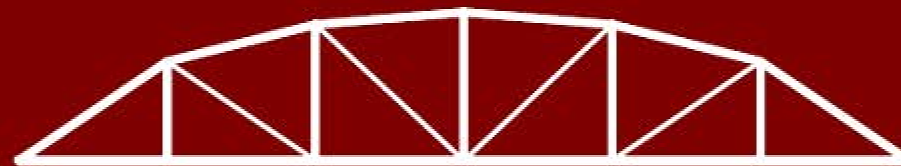

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Malaysian Society for Numerical Methods

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Dedication

To our families and students…

to the future!

Preface

Finite Element Method (FEM) has become a compulsory knowledge for present day engineers as it allows (what used to be) very complex behavior of physical phenomenon to be known (approximately) and exploited. However, the teaching and learning of the subject are still difficult, as usually described by the learners. In the authors' opinion, the difficulties can be blamed on the fragmentation (of the discussions) between mathematics, engineering fundamentals and the basic concepts of numerical method. Realizing this, the authors are promoting a new approach in this book by insisting for a "close-loop" type of discussion in each topic or chapter. A topic always begins with the derivation of the differential equation/s (of the problem). It is followed by the conversion of the equation/s into matrix forms through finite element argument. A worked example is then immediately given (in a very detailed manner) before it is closed by a MATLAB source code.

This book is neither designed to be a complete book on FEM nor intended to dwell on the practice of FEM modelling (using on-shelf software). Instead, it is prepared with a specific idea in mind; the book is about easy tracing of the evolution of the finite element formulation and thus has the following features:

1. A complete loop in each formulation (from the derivation of the partial/ordinary differential equations to the discretization of the equations into matrix system to the computer programming)

2. Increasing complexity from one formulation to another (that is, from bar element to beam element to truss element to frame to free vibration and buckling problems and finally to forced vibration of the structures)

For the above reasons, this book does not have abundant worked examples but focusing on a few examples, detailing every step so as to make obvious what has been discussed in the preceding text and what awaits in succeeding source code. Also (with the specific example per chapter), the evolution and the continuity of arguments can be clearly established from one chapter to another (It is the authors' opinion that too many examples per chapter would make the relationship between examples in different chapters less obvious). Nevertheless, there are plenty of solved exercises provided.

This book has evolved from a series of lecture notes of the first author refined over the period of ten years with the co-authors. It revolves around frame structural analysis, both statics and dynamics. In Chapter 1, the book begins with the basic concepts of numerical methods before introducing the concept of Galerkin weighted residual method towards the end. Chapter 2 focuses on bar finite element. The formulation of beam element is discussed in Chapter 3. Chapter 4 discusses the concept of space orientation and the assembly of elements for plane structures (truss and frame). Chapter 5 discusses two classes of eigenvalue problems; free vibration and buckling of structures. Chapter 6 details the formulation of forced vibration of bar, beam and plane frame. In this final chapter, time discretization by finite difference method is introduced.

Airil Yasreen Mohd Yassin
October, 2018
Putrajaya, Malaysia

Contents

1 Basic Concept of Numerical Techniques

1.1 Introduction: What is Finite Element Method?

Finite Element Method or FEM can be both "everything" and "nothing". At one end, FEM is everything when it allows engineers to get information (i.e. stresses, displacements, forces) of complex physical phenomenon for design purposes. At the other end, FEM is nothing because the information obtained is actually nothing but a solution to a partial differential equation (PDE) or ordinary differential equations (ODE). In other words, FEM is nothing but another numerical method to solve PDE or ODE.

Realizing how FEM can be "everything" is important as it can motivate the study. But realizing how FEM can be "nothing" is just as important as it can guide the proper learning of FEM that is, any discussion must begin from the first principle (i.e. PDE or ODE) if strong understanding is desired.

To note, since ODE is a special case of PDE, from now on, PDE will be quoted when references to both class of equations are made.

A formal description of FEM can be given as follows. FEM is a numerical method that approximates the solution of a PDE by breaking up the physical domain into smaller elements where adjacent elements are connected at nodes to form a mesh. Such a mesh formation process is technically termed as element assembly. In FEM, the dependent variables at nodal locations (referred as the degree of freedoms) are interpolated by shape functions. Insertion of these interpolation functions into the PDE produces a residual error function which, when forced to zero with the employment of weighted residual method, in turn, produces a matrix system. Imposition of boundary conditions can be done directly before the unknown degree of freedoms be solved.

1.2 Basic Concept of Numerical Techniques

Having said how FEM is just another numerical method, below is the list of established numerical methods.

i. Finite Element Method (FEM)
ii. Finite Difference Method (FDM)
iii. Boundary Element Method (BEM)
iv. Meshless or Meshfree Methods (Meshfree)

However, despite their variations, all the methods share similar concept that is;

> *"to convert the continuous nature of PDE (or ODE) into 'equivalent' simultaneous algebraic equations in the form of a matrix system".*

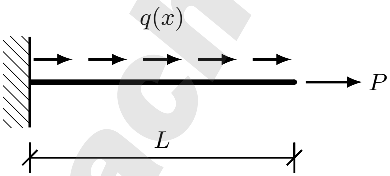

Figure 1.1: Bar element / structure.

In elaborating the concept, we discuss herein the solution of the simplest forms of ODE, that is of a bar element. By leaving the derivation for later, the ODE of a bar element (as shown in Fig. 1.1) can be given as:

Domain equation

$$EA\frac{d^2u}{dx^2} = -q \tag{1.1}$$

where u and q are the axial displacement of the bar and the external distributed load acting on the bar, respectively. E and A are the Young's modulus and the cross-sectional area of the bar respectively which both are constant for a linear problem. L is the length of the bar and P is an external point load acting at the end of the bar as shown in the Fig. 1.1.

Complementing the domain ODE are the boundary conditions given as follows (which detailed derivation and discussion are delayed until Chapter 2)

Boundary conditions (equations)

$$EA \left. \frac{du}{dx} \right|_{x=L} = P \tag{1.2}$$

$$\left. u \right|_{x=0} = 0 \tag{1.3}$$

The ODE is considered solved when a solution, $u = f(x)$ is found which satisfies all the equations above (i.e. Eqs. (1.2) and (1.3)).

In fact, the exact (closed-formed) solution of the problem can already be obtained by direct integration, thus:

$$u = \left(-\frac{q}{2EA}\right) x^2 + \left(\frac{P + ql}{EA}\right) x \tag{1.4}$$

However, despite the availability of Eq. (1.4), the ODEs of the problem (Eqs. (1.1) to (1.3)) are still discretised numerically herein so as to demonstrate the basic concept of numerical techniques. The problem is chosen due to its simplicity allowing for easy tracing of the discussion.

To convert the ODEs (i.e. Eqs. (1.1) to (1.3)) to its 'equivalent' simultaneous algebraic equations, we start by assuming a guessed solution in the forms of polynomials. In our case, we guess:

$$u = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 \tag{1.5}$$

Then, we satisfy Eq. (1.3) by inserting Eq. (1.5) into the equation to give:

$$\left. u \right|_{x=0} = a_1 + a_2(0) + a_3(0)^2 + a_4(0)^3 + a_5(0)^4 = 0 \tag{1.6}$$

which gives:

$$a_1 = 0 \tag{1.7}$$

Next we satisfy Eq. (1.2) by inserting Eq. (1.5) into the equation to obtain:

$$EA \left.\frac{du}{dx}\right|_{x=L} = EA\left(a_2 + 2a_3L + 3a_4L^2 + 4a_5L^3\right) = P \tag{1.8}$$

Finally, by inserting Eq. (1.5) into Eq. (1.1), the following is obtained:

$$EA\frac{d^2u}{dx^2} + q = EA\left(2a_3 + 6a_4x + 12a_5x^2\right) + q \neq 0 \tag{1.9}$$

Observing Eq. (1.9), it must be noted that, whilst each of Eqs. (1.6) to (1.8) is an act of forcing the equation to a certain values (i.e 0 and P, respectively), hence the "satisfaction" of the equations, the insertion of the guess function (Eq. (1.5)) into the domain equation (Eq. (1.1)) is yet a satisfaction of the original equation hence the use of the inequality symbol ($\neq$). As we are going to see, the forcing of Eq. (1.8) at several locations within the domain to a null value is what satisfies the equation and what creates sufficient number of equations.

By grouping Eqs. (1.7) to (1.9) together, we can see that, so far, we have established three simultaneous equations as follows:

$$a_1 = 0 \tag{1.10a}$$

$$EA\left(a_2 + 2a_3L + 3a_4L^2 + 4a_5L^3\right) = P \tag{1.10b}$$

$$EA\left(2a_3 + 6a_4x + 12a_5x^2\right) + q \neq 0 \tag{1.10c}$$

However, observing Eq. (1.10) we can notice that:

1. We have five (5) unknown constants, a_1, a_2, a_3, a_4 and a_5 but with only three (3) simultaneous equations.

2. The last equation that is Eq. (1.10c) (previously Eq. (1.9)) is still not algebraic but continuous in x. Also, the left hand side of the equation is not equal to ($\neq$) the right hand side because the guessed function is yet the solution of the ODE, as mentioned previously.

So to get the sufficient number of equations (and to convert Eq. (1.10c) into algebraic) we argue that, since Eq. (1.10c) is obtained from domain equation, the equation must hold (must be true) throughout the domain thus we can evaluate Eq. (1.10c) everywhere in the domain as much as we need. In our case, to complement Eqs. (1.10a) and (1.10b), we evaluate Eq. (1.10c) at three (3) locations in the bar, says at $x = L/3, L/2, 2L/3$ to obtain:

$$2EAa_3 + 6EAa_4\left(\frac{L}{3}\right) + 12EAa_5\left(\frac{L}{3}\right)^2 + q = 0 \tag{1.11a}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{2}\right) + 12EAa_5\left(\frac{L}{2}\right)^2 + q = 0 \tag{1.11b}$$

$$2EAa_3 + 6EAa_4\left(\frac{2L}{3}\right) + 12EAa_5\left(\frac{2L}{3}\right)^2 + q = 0 \tag{1.11c}$$

Eq. (1.11) are the results of forcing Eq. (1.10c) to a null value at several locations within the domain as mentioned previously. Also, as can be seen, such an act does not only represent the satisfaction of the original ODE, but also convert the ODE into a set of algebraic equations.

Now, by re-grouping Eqs. (1.10a), (1.10b) and (1.11), we then have a sufficient number of algebraic equations, given as:

$$a_1 = 0 \tag{1.12a}$$

$$EAa_2 + 2EAa_3L + 3EAa_4L^2 + 4EAa_5L^3 = P \tag{1.12b}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{3}\right) + 12EAa_5\left(\frac{L}{3}\right)^2 + q = 0 \tag{1.12c}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{2}\right) + 12EAa_5\left(\frac{L}{2}\right)^2 + q = 0 \tag{1.12d}$$

$$2EAa_3 + 6EAa_4\left(\frac{2L}{3}\right) + 12EAa_5\left(\frac{2L}{3}\right)^2 + q = 0 \tag{1.12e}$$

Eq. (1.12) is thus the 'equivalent' simultaneous algebraic equations of the ODE of the problem which are given originally in Eqs. (1.1) to (1.3). In other words, we can say that:

> *"Eq.* (1.12) *are the 'equivalent' algebraic forms of Eqs.* (1.1) *to* (1.3)*".*

So this is basically the main concept shared by all numerical techniques such as FEM, FDM, BEM and Meshfree. But it is also the character of a numerical technique to treat the equations in matrix forms as this what suits computer programming. In this context, Eq. (1.12) can be arranged in matrix forms as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & EA & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{3}\right) & 12EA\left(\frac{L}{3}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{2}\right) & 12EA\left(\frac{L}{2}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{2L}{3}\right) & 12EA\left(\frac{2L}{3}\right)^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ p \\ -q \\ -q \\ -q \end{Bmatrix} \quad (1.13)$$

or

$$\left[k\right]\left\{u\right\} = \left\{r\right\} \quad (1.14)$$

where

$$[k] = k_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & EA & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{3}\right) & 12EA\left(\frac{L}{3}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{2}\right) & 12EA\left(\frac{L}{2}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{2L}{3}\right) & 12EA\left(\frac{2L}{3}\right)^2 \end{bmatrix} \tag{1.15a}$$

$$\{u\} = u_j = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{1.15b}$$

$$\{r\} = r_j = \begin{Bmatrix} 0 \\ P \\ -q \\ -q \\ -q \end{Bmatrix} \tag{1.15c}$$

Note: k_{ij}, u_j, and r_j are the alternative notations known as indicial/tensorial notations. We are going to use this notation system and the matrix forms interchangeably.

By solving Eq. (1.12) or Eq. (1.13), we would then obtain the numerical values of a_1, a_2, a_3, a_4 and a_5, and by inserting these values into the original guessed function, Eq. (1.5) we thus obtain the numerical solution of the problem.

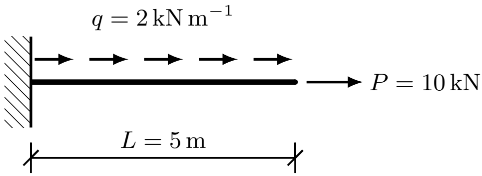

Figure 1.2: Cantilever bar with uniformly distributed load, q and a single point load, P. ($E = 200{\times}10^6\,\mathrm{kN\,m^{-2}}$ and $A = 0.04\,\mathrm{m^2}$).

1.2.1 Worked Example 1.1

Let's put what we have learned so far into practice. We are going to compare the result obtained from Eq. (1.13) with the closed-form solution of Eq. (1.4) for the particular problem below (Fig. 1.2).

For this problem, the equation in the form of Eq. (1.13) can be given as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 8.00{\times}10^6 & 8.00{\times}10^7 & 6.00{\times}10^8 & 4.00{\times}10^9 \\ 0 & 0 & 1.60{\times}10^7 & 8.00{\times}10^7 & 2.67{\times}10^8 \\ 0 & 0 & 1.60{\times}10^7 & 1.20{\times}10^8 & 6.00{\times}10^8 \\ 0 & 0 & 1.60{\times}10^7 & 1.60{\times}10^8 & 1.07{\times}10^9 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \\ -2 \\ -2 \\ -2 \end{Bmatrix} \tag{1.16}$$

By solving Eq. (1.16) using Matlab command "\", (refer source codes in Section 1.5.1), the values of a_i are thus as given in Table 1.1.

Table 1.1: Values of guessed solution's constant

a_i	a_1	a_2	a_3	a_4	a_5
Value	0	$2.5{\times}10^{-6}$	$-1.25{\times}10^{-7}$	0	0

And by inserting the values into Eq. (1.5) we thus obtained the numerical solution of the problem as:

$$u_\mathrm{n} = \left(2.5{\times}10^{-6}\right) x + \left(-1.25{\times}10^{-7}\right) x^2 \tag{1.17}$$

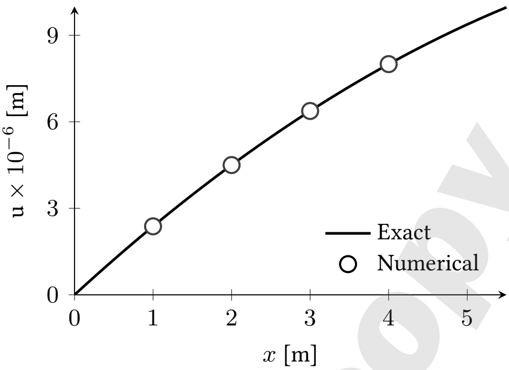

Figure 1.3: Axial displacement, u, along the bar between numerical and exact solutions.

The accuracy of this numerical solution can be assessed by comparing its value with the exact/closed-form solution which is obtained from Eq. (1.4) as:

$$u_{\mathrm{e}} = \left(\frac{P + ql}{EA}\right) x + \left(-\frac{q}{2EA}\right) x^2 \tag{1.18}$$

The plot of numerical, u_{n} and exact, u_{e} solutions throughout the length, L of the bar and their respective values at several locations are given in Fig. 1.3 and Table 1.2. Based on the plot and the table, it can be concluded that the basic concept of the numerical technique is able to provide similar results to the ones provided by the closed-form solution. However, it must be noted that this is generally not the case. Generally, a numerical solution would not be accurate but converge to the "accurate" solution with the increase of polynomial order of the guessed functions (or degree of freedoms and mesh density as discussed later on).

Table 1.2: Axial displacement, u, along the bar between numerical and exact solutions.

x [m]	1	2	3	4
u_{n} [m]	2.38×10^{-6}	4.5×10^{-6}	6.38×10^{-6}	8×10^{-6}
u_{e} [m]	2.38×10^{-6}	4.5×10^{-6}	6.38×10^{-6}	8×10^{-6}

1.3 Basic Concept of FEM: Weighted Residual Method

In previous section, we have discussed the basic concept shared by all numerical techniques that is, to convert the PDE or ODE of a problem into its equivalent algebraic equations. Herein, we are going to focus our discussion on Finite Element Method (FEM).

FEM is a numerical method that is widely used in engineering field of today. The theoretical argument for FEM can be done in three different approaches which are;

i. Direct Method
ii. Variational Method
iii. Weighted Residual Method (WRM)

In our present discussion, our main interest is on WRM where we are going to leave the discussion on Variational Method for later. And also, due to the simplicity of Direct Method, we are not going to discuss herein.

Actually, the specific form of WRM, which is employed in the present day of FEM, is known as Galerkin WRM formulations. However, before we get into that, we are going to discuss first the basic idea (or concept) of WRM, as given next.

1.3.1 Basic Concept of WRM

The basic concept of WRM can be better argued by observing the evolution of the discussion from the previous basic concept of numerical technique. We start by re-observing Eqs. (1.1) to (1.3). Since we are going to discuss the basic concept of WRM by using these equations as our starting point, we re-write the equations herein as:

$$EA\frac{d^2u}{dx^2} = -q \tag{1.19a}$$

$$EA\left.\frac{du}{dx}\right|_{x=L} = P \tag{1.19b}$$

$$u|_{x=0} = 0 \tag{1.19c}$$

The treatment on Eq. (1.19a) is what differentiates WRM from other numerical techniques. Whilst in previous discussion, Eq. (1.19a) is evaluated at three different locations along the domain (of the bar) so as to yield the necessary three algebraic equations. In WRM, we have quite a different argument for this.

In WRM, Eqn. (1.19a) is written as:

$$EA\frac{d^2u}{dx^2} \neq -q = R \text{ (residual error)} \tag{1.20}$$

Residual error, R is a function emerges due to the fact that the guessed solution is not the actual solution of the domain ODE. For the guessed solution of Eq. (1.5), the residual error function is as given by Eq. (1.10c), or can be re-written as:

$$R(x) = 2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q \tag{1.21}$$

Plot of Eq. (1.21) would give us the graphical view of the residual error function, where a typical plot is as shown in Fig. 1.4.

Since this (residual) error function should not exist, integrating the function and setting it equal to zero will also mean forcing the area beneath the curve to become zero, as also illustrated in Fig. 1.4.

This action of forcing the area under the residual curve would give us one algebraic equation in a similar way if we evaluate Eq. (1.10c) at one location within the domain as previously discussed. However, the present approach is more effective because the integration somehow disperses the error in a more smooth and 'overall' manner.

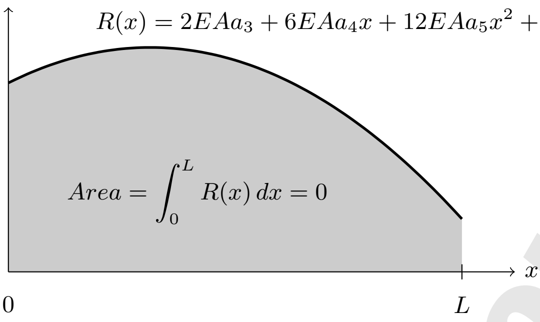

Figure 1.4: The "hypothetical" plot of the residual error function and the act of forcing the area underneath the function to be zero.

But so far, this only gives us one equation, while the fact is, we need as much equations as the number of unknowns. In previous discussion, we evaluate at three locations to provide the three required algebraic equations, remember?

Herein, to provide the sufficient number of algebraic equations, what we do is to firstly create three (3) 'independent' residual error functions before integrating these resulting functions over the domain and setting them to zero. In order to create the three independent residual functions, we simply multiply Eq. (1.21) with different independent function, i.e. 1, x, x^2, which gives:

$$R_1 = 1\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right) \tag{1.22a}$$

$$R_2 = x\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right) \tag{1.22b}$$

$$R_3 = x^2\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right) \tag{1.22c}$$

where R_1, R_2 and R_3 are the three 'new' residual error functions (of x), where all of them can be typically plotted as in Fig. 1.5.

Now by integrating separately each of the new residual functions and setting them to zero, the action would force the area under each curve to 'disappear', thus results in the three required algebraic equations, given as:

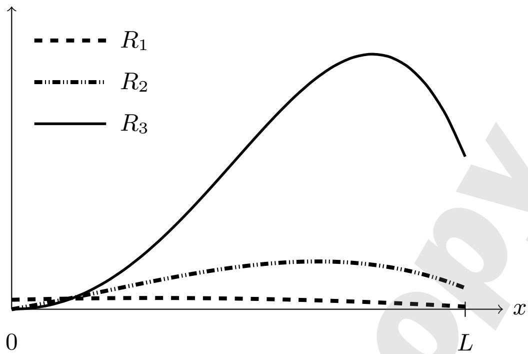

Figure 1.5: Plots of residual error functions, R_1, R_2 and R_3.

$$\int_0^L R_1\,dx = \int_0^L 1\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right)dx = 0 \quad (1.23a)$$

$$\int_0^L R_2\,dx = \int_0^L x\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right)dx = 0 \quad (1.23b)$$

$$\int_0^L R_3\,dx = \int_0^L x^2\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right)dx = 0 \quad (1.23c)$$

By conducting the integration we obtain:

$$2EALa_3 + 3EAL^2a_4 + 4EAL^3a_5 + qL = 0 \quad (1.24a)$$

$$EAL^2a_3 + 2EAL^3a_4 + 3EAL^4a_5 + \frac{1}{2}qL^2 = 0 \quad (1.24b)$$

$$\frac{2}{3}EAL^3a_3 + \frac{3}{2}EAL^4a_4 + \frac{12}{5}EAL^5a_5 + \frac{1}{3}qL^3 = 0 \quad (1.24c)$$

Now, as we did previously, by grouping Eqs. (1.24a) to (1.24c) and the previously two equations of Eqs. (1.7) and (1.8) (which are actually the satisfaction of Eqs. (1.19b) and (1.19c)) together, we get:

$$a_1 = 0 \tag{1.25a}$$

$$EAa_2 + 2EALa_3 + 3EAL^2a_4 + 4EAL^3a_5 = P \tag{1.25b}$$

$$2EALa_3 + 3EAL^2a_4 + 4EAL^3a_5 + qL = 0 \tag{1.25c}$$

$$EAL^2a_3 + 2EAL^3a_4 + 3EAL^4a_5 + \frac{1}{2}qL^2 = 0 \tag{1.25d}$$

$$\frac{2}{3}EAL^3a_3 + \frac{3}{2}EAL^4a_4 + \frac{12}{5}EAL^5a_5 + \frac{1}{3}qL^3 = 0 \tag{1.25e}$$

In matrix forms, Eq. (1.25) can be given as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & EA & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & EAL^2 & 2EAL^3 & 3EAL^4 \\ 0 & 0 & \frac{2}{3}EAL^3 & \frac{3}{2}EAL^4 & \frac{12}{5}EAL^5 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ -qL \\ -\frac{1}{2}qL^2 \\ -\frac{1}{3}qL^3 \end{Bmatrix} \tag{1.26}$$

Note that, Eq. (1.26) is the discretised equation of the bar ODE obtained from the basic WRM.

1.3.2 Worked Example 1.2

Let's solve the previous problem of Worked Example 1.1, but this time with the employment of WRM. We are going to compare the result obtained from this approach with those obtained using basic numerical technique given by Eq. (1.17) and the closed-form solution of Eq. (1.4).

For the given problem, the WRM discretised equation in the form of

Eq. (1.26) can be given as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 8\times10^{6} & 8\times10^{7} & 6\times10^{8} & 4\times10^{9} \\ 0 & 0 & 8\times10^{7} & 6\times10^{8} & 4\times10^{9} \\ 0 & 0 & 2\times10^{8} & 2\times10^{9} & 1.5\times10^{10} \\ 0 & 0 & 6.67\times10^{8} & 7.5\times10^{9} & 6\times10^{10} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \\ -10 \\ -25 \\ -83.33 \end{Bmatrix} \tag{1.27}$$

By solving Eq. (1.27) using Matlab command “\”, (refer source codes in Section 1.5), the values of a_i are thus as given in Table 1.3.

Table 1.3: Values of guessed solution’s constant

a_i	a_1	a_2	a_3	a_4	a_5
Value	0	2.5×10^{-6}	-1.25×10^{-7}	0	0

And by inserting the values into Eq. (1.5) we thus obtained the numerical solution, u_{w} of the problem as:

$$u_{\mathrm{w}} = \left(2.5\times10^{-6}\right) x + \left(-1.25\times10^{-7}\right) x^2 \tag{1.28}$$

The accuracy of WRM can be assessed by comparing its numerical value with those previously obtained in Worked Example 1.1. The plot of WRM results, u_w together with numerical, u_n and exact, u_e solutions are given in Fig. 1.6, and their numerical values at several locations along the bar are given in Table 1.4.

Based on the plot and the table, it can be concluded that WRM gives similar results to previous approaches hence validates its concepts. However, as mentioned previously, such accurate result obtained herein is generally not the case. Generally, like any other numerical techniques, WRM converges to the “accurate” solution with the increase in the order of polynomial of the guessed functions (or degree of freedom and mesh density later on).

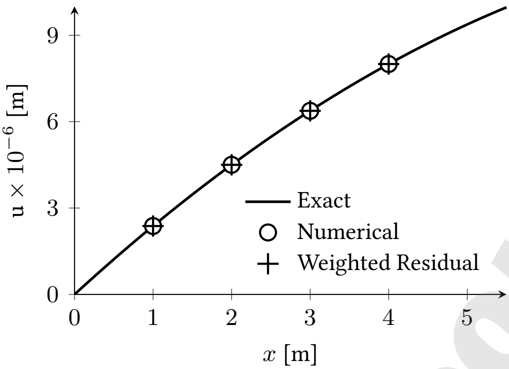

Figure 1.6: Axial displacement, u, along the bar between numerical and exact solutions.

Table 1.4: Axial displacement, u, along the bar between numerical and exact solutions.

x [m]	1	2	3	4
u_{w} [m]	2.38×10^{-6}	4.5×10^{-6}	6.38×10^{-6}	8×10^{-6}
u_{n} [m]	2.38×10^{-6}	4.5×10^{-6}	6.38×10^{-6}	8×10^{-6}
u_{e} [m]	2.38×10^{-6}	4.5×10^{-6}	6.38×10^{-6}	8×10^{-6}

1.4 Closing Remarks

So far we have discussed about the basic concept shared by all numerical techniques. Then we discussed about the basic concept of weighted residual method (WRM). However, it has also been mentioned that the specific form of WRM employed in present day FEM is Galerkin WRM. Since this is the actual discussion on FEM, we are going to discuss this in the next chapter.

1.5 Matlab Source Code

1.5.1 General Numerical Technique

```
% Clear data
clc; clear; close all

% Input
E = 200e6;  % Young's modulus (kN/m^2)
A = 0.04;   % Area (m^2)
L = 5;      % Bar length [m]
P = 10;     % Point load [kN]
q = 2;      % Distributed load [kN/m]

% ---------------------------------------------
% Exact solution
% ---------------------------------------------

x   = 0:L/20:L;
ue  = ((P+q*L)/(E*A))*x + (-q/(2*E*A))*x.^2;
plot(x, ue, '-k'); hold on

% ---------------------------------------------
% General numerical solution
% ---------------------------------------------

% Matrix k & force r
k = [1 0    0       0             0;
     0 E*A  2*E*A*L 3*E*A*L^2       4*E*A*L^3;
     0 0    2*E*A   6*E*A*(L/3)   12*E*A*(L/3)^2;
     0 0    2*E*A   6*E*A*(L/2)   12*E*A*(L/2)^2;
     0 0    2*E*A   6*E*A*(2*L/3) 12*E*A*(2*L/3)^2;
    ];
r = [0;P;-q;-q;-q];

% Solve for constant a
a = k\r;

% Plot the solution
x   = 1:4;
un  = a(1) + a(2).*x + a(3)*x.^2 + a(4)*x.^3 + a(5)*x.^4;
plot(x, un, 'ok')

% Label and legends
xlabel('x [m]')
ylabel('u(x) [m]')
legend('Exact', 'Numerical', 'location', 'SouthEast');
```

1.5.2 Weighted Residual Method

```
% Clear data
clc; clear; close all

% Input
E = 200e6;  % Young's modulus (kN/m^2)
A = 0.04;   % Area (m^2)
L = 5;      % Bar length [m]
P = 10;     % Point load [kN]
q = 2;      % Distributed load [kN/m]

% ----------------------------------------------
% Exact solution
% ----------------------------------------------

x   = 0:L/20:L;
ue  = ((P+q*L)/(E*A))*x + (-q/(2*E*A))*x.^2;
plot(x, ue, '-k'); hold on

% ----------------------------------------------
% Weighted residual solution
% ----------------------------------------------

% Matrix k & force r
k = [1 0   0           0           0;
     0 E*A 2*E*A*L     3*E*A*L^2   4*E*A*L^3;
     0 0   2*E*A*L     3*E*A*L^2   4*E*A*L^3;
     0 0   E*A*L^2     2*E*A*L^3   3*E*A*L^4;
     0 0   2/3*E*A*L^3 3/2*E*A*L^4 12/5*E*A*L^5;
    ];
r = [0;P;-q*L;-1/2*q*L^2;-1/3*q*L^3];

% Solve for constant a
a = k\r;

% Plot the solution
x   = 1:4;
un  = a(1) + a(2).*x + a(3)*x.^2 + a(4)*x.^3 + a(5)*x.^4;
plot(x, un, 'ok')

% Label and legends
xlabel('x [m]')
ylabel('u(x) [m]')
legend('Exact', 'WRM', 'location', 'SouthEast');
```

1.6 Exercises

1. Describe what you understand about

   i. Differential equation
   ii. Numerical methods
   iii. Finite Element Method

iv. Shape functions and degree of freedom
v. Force (natural) and displacement (essential) boundary conditions

2. The ordinary differential equation (ODE) of bar is given as:

$$EA\frac{d^2u}{dx^2} = -q$$

with the boundary conditions:

$$EA\left.\frac{du}{dx}\right|_{x=L} = P$$

$$u|_{x=0} = 0$$

where E is the Young's Modulus, A is the cross-sectional area, $u(x)$ is the displacements field, L is the bar length, P and q is the external forces. If the approximate solution for the ODE is chosen as:

$$u(x) = a_1 + a_2 \sin(x) + a_3 \cos(x)$$

determine the 'equivalent' simultaneous algebraic equations of the ODE using:

i. Basic Numerical Technique (Collocation)
ii. Weighted Residual Method

2 Galerkin Formulation: Bar Element

2.1 Introduction

In the previous chapter, we have discussed the basic concept of numerical technique and the basic concept of WRM. In this chapter, we are going to discuss on the specific form of WRM that is employed in the present day of FEM that is Galerkin WRM. However, since it has also been mentioned earlier that FEM is nothing but a numerical solution to a PDE, it is important in any FEM endeavour for the analyst to be familiar with the relevant PDE including its derivation and its closed-form solution (if available). This way, when the analyst is required to embark on a new project or study, he or she is already being trained to look at the problem from the first principle, and identify all the relevant aspects or concerns, before he or she employs FEM in getting the solution of the problem.

2.2 Ordinary Differential Equation of Bar Problem

As mentioned, it is vital for an analyst to get into the problem from the first principle and in many cases; this would mean from the derivation of the relevant PDE (or ODE). Since in the previous chapter, we have been introduced to the ODE of bar problem, herein we are going to show the derivation of the ODE. Fig. 2.1(a) shows a bar element subjected to an external distributed load, $q(x)$ and an end load, P and its differential element.

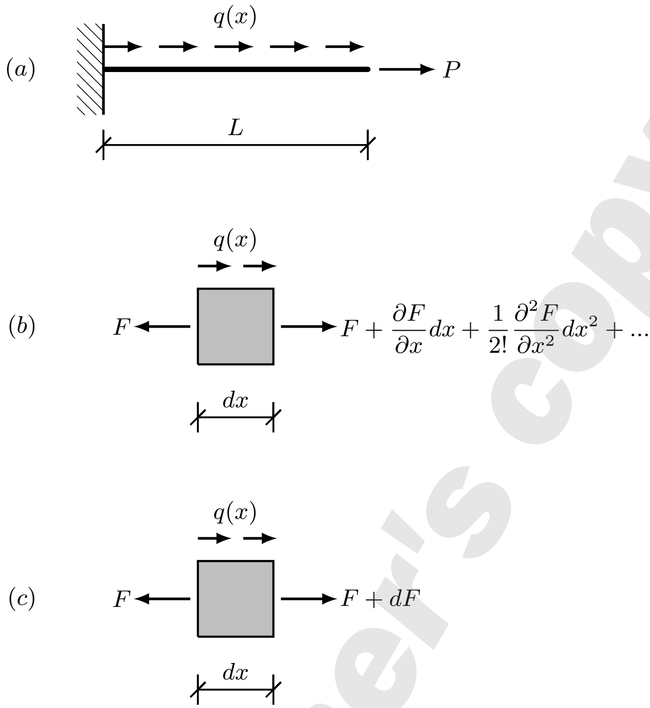

Figure 2.1: Bar structure and its differential element.

It can be argued that, although we have an axial force, F at the left side of the differential element, due to the 'disturbance' along the differential length, dx (due to external load, for example) the magnitude of the axial force at the right side of the differential element must change. However we don't know the exact change of this force, else we would not have this problem in the first place, would we? But by assuming the change is continuous, we can say that the force at the right-side of the differential element can be represented by Taylor series, as shown in Fig. 2.1(b).

But by assuming higher order terms as insignificant and since this is a 1D problem (i.e. $\partial(\quad) = d(\quad)$), a state as shown in Fig. 2.1(c) is considered. This is an important argument thus must be grasped by the readers because as we going to see later, the derivation of PDE (or ODE) for other problems will be based on the same argument as well.

Having established the differential element and the corresponding forces acting on it, we are in the position to derive the ODE for the bar problem. Since the bar only deform in axial direction, only equilibrium in x-direction need to be considered thus:

$$\sum F_x = 0 = -F + (F + dF) + q\,dx \tag{2.1}$$

By rearranging gives:

$$\frac{dF}{dx} = -q \tag{2.2}$$

From Hooke's Law, we know that:

$$\sigma = E\epsilon \tag{2.3}$$

where σ is the axial stress, E is the Modulus Young and ϵ is the axial strain. Since

From Hooke's Law, we know that:

$$\sigma = \frac{F}{A} \tag{2.4}$$

and

$$\epsilon = \frac{du}{dx} \tag{2.5}$$

By inserting Eq. (2.4) and Eq. (2.5) into Eq. (2.3) gives:

$$F = EA\frac{du}{dx} \tag{2.6}$$

By differentiating Eq. (2.6) once gives:

$$\frac{dF}{dx} = EA\frac{d^2u}{dx^2} \tag{2.7}$$

By inserting Eq. (2.7) into Eq. (2.2) would then give the ODE which we have previously encountered in Chapter 1 (Eq. (1.1)), that is:

$$EA\frac{d^2u}{dx^2} = -q \tag{2.8}$$

In obtaining a unique solution, for every ODE (or PDE for that matter), the domain equation/s must be supplemented by boundary equations, and for this particular case, the equations are given as below:

Natural/force boundary conditions

$$EA\,\frac{du(x)}{dx}\bigg|_{x=0} = F_0 \tag{2.9a}$$

$$EA\,\frac{du(x)}{dx}\bigg|_{x=L} = -F_L \tag{2.9b}$$

Essential/displacement boundary conditions

$$u|_{x=0} = u_0 \tag{2.9c}$$

$$u|_{x=L} = u_L \tag{2.9d}$$

Note that the sign convention for the natural (force) boundary conditions above is based on the assumption that the boundary conditions are the reactions at support hence the opposite directions to the internal forces. Since Eq. (2.8) is a 2nd order ODE, two boundary conditions out of the four given above must be known in prior so as to have a well-posed problem.

Having established the bar ODE, we are now all set to discuss the basic concept of Galerkin WRM. This is given next.

2.3 Fundamental of Galerkin WRM

We have discussed the basic concept of WRM that is, the integration of a function consists of a product of the residual function and a weight function and forcing this integrated value to zero in getting an algebraic function.

The Galerkin WRM differs in the following aspects:

1. The guessed solution is expressed in terms of shape functions, N_i and degree of freedoms, u_i instead of interpolation functions and its

constant. In other words and in bar problem in particular, whilst the basic WRM uses the following guessed function:

$$u = a_1 + a_2 x$$

In Galerkin WRM, the following guessed function are used instead:

$$u = N_1 u_1 + N_2 u_2$$

where N_1 and N_2 are the shape functions and u_1 and u_2 are the degree of freedoms (dofs). We are going to discuss in detail on shape functions and degree of freedoms in the next sub-section.

2. Galerkin WRM involves integration by part (IBP), of which the purposes are to:

    i. reduce the order of the original derivative terms so as to relax the continuity requirement

    ii. introduce the natural (force) boundary conditions explicitly into the formulation

    Points i and ii deserve further elaboration. For point i, as we are going to see later, the IBP will reduce the order of derivation of bar problem from $\frac{d^2u}{dx^2}$ to $\frac{du}{dx}$. Such a reduction would mean that a guessed function, say a linear polynomial function such $(A + Bx)$, once inserted into the former would vanish, but a constant if inserted into the latter. This is what we mean by "to relax the continuity requirement". Due to this, the final equation of Galerkin WRM is known as weak statement because it has been "weakened" from the original PDE (or ODE). Accordingly, the original form of the PDE (or ODE) is known as the strong statement of the problem.

    For point ii, as we are going to see, the IBP will induce boundary terms which exactly in the forms of Eq. (1.2) (or Eq. (1.19b)). Having these terms induced rather naturally, these terms are called natural boundary conditions hence the interchangeable use of the term with force boundary conditions . Due to this natural induction of the force (or natural) boundary condition, we are not going to see the separate satisfaction of the force boundary conditions as we saw previously in the basic WRM (i.e Eq. (1.25b)).

3. Due to the use of dofs (hence the advantage of Galerkin WRM), we are also not going to see the separate satisfaction of the displacement boundary conditions as previously seen in Eq. (1.25a). Instead, the displacement boundary conditions will be imposed only after the complete establishment of the algebraic equation and prior to the solution of the equation. Due to this, what is known as displacement is also interchangeably known as essential boundary condition in Galerkin WRM and FEM for that matter.

Actually, there are two types of Galerkin WRM, which are different depending on the type of weight function. If the shape functions are used as the weight functions, the method is called Bubnov-Galerkin but if other functions is used (for example, interpolation function as in the basic WRM) the method is called Petrov-Galerkin. However, herein, our main interest will be on the former, that is Bubnov-Galerkin method. So the above is the elaboration about Galerkin WRM (on how it differs from the basic WRM, discussed previously in Chapter 1). Observing such an evolution is important as it can allow for easier yet deeper understanding of FEM.

Having established all these, we are now all set to discuss in detail the Galerkin WRM and FEM for that matter and we are going to do this by solving the same bar ODE which we have previously discussed in Chapter 1 and which derivation have been made earlier in this chapter. To do this, we start by discussing the derivation of shape functions and the definition of dofs for a bar problem. Before that, let's note that Galerkin WRM is referred to FEM from now on.

2.4 Degree of Freedom and Shape Functions for Bar Element

FEM bar element can usually be of two types; linear element and quadratic element. These elements are shown in Fig. 2.2; the former has two nodes and the latter has three nodes per element.

Herein, we are going to derive the shape functions for both elements and we start by assuming an interpolation function, $u(x)$ in the forms of polynomial, thus:

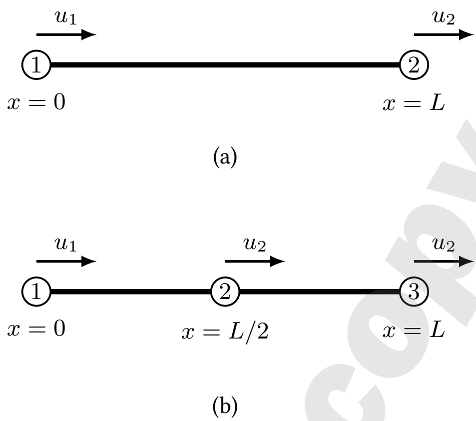

Figure 2.2: Degree of freedoms of (a) linear (b) quadratic bar elements.

For linear bar element

$$u = a_1 + a_2 x \tag{2.10a}$$

For quadratic bar element

$$u = a_1 + a_2 x + a_3 x^2 \tag{2.10b}$$

Despite the familiar use of polynomial functions, it is preferable to deal with a special set of functions, known as shape functions, N_i and a special sets of coefficient, known as degree of freedoms (dof), u_i. This means, Eq. (2.10a) and Eq. (2.10b) must be equivalently expressed as:

For linear bar element

$$u = N_1 u_1 + N_2 u_2 \tag{2.11a}$$

For quadratic bar element

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \tag{2.11b}$$

The equations can be expressed compactly in components form as:

$$u(x) = N_i u_i \tag{2.12}$$

where $i = 1, 2$ and $i = 1, 2, 3$ for linear and quadratic bar elements respectively.

Being referred as the degree of freedoms or dofs, u_1, u_2, and u_3 in Eqs. (2.11a) and (2.11b) (or u_i in Eq. (2.12)), are actually the nodal values of the axial displacement, $u(x)$. These dofs are graphically described in Fig. 2.2.

The conversion from Eqs. (2.10a) and (2.10b) to Eqs. (2.11a) and (2.11b), hence the derivation of the shape functions, N_i can be done as follows.

2.4.1 Derivation of Linear Bar Shape Function

Evaluating Eq. (2.10a) at the location of the nodes (i.e. at both ends, $x = 0$ and $x = L$) and equating them according to the dofs will give:

$$\begin{aligned} u(x)|_{x=0} &= a_1 + a_2(0) = u_1 \\ u(x)|_{x=L} &= a_1 + a_2(L) = u_2 \end{aligned} \tag{2.13}$$

Solving for the values of a_1 and a_2 then gives:

$$\begin{aligned} a_1 &= u_1 \\ a_2 &= \frac{u_2 - u_1}{L} \end{aligned} \tag{2.14}$$

Inserting Eq. (2.14) into Eq. (2.10a) and by factorizing gives:

$$u(x) = \frac{L - x}{L} u_1 + \frac{x}{L} u_2 \tag{2.15}$$

By comparing Eq. (2.15) and Eq. (2.11a) it can be concluded that:

$$\begin{aligned} N_1 &= \frac{L - x}{L} \\ N_2 &= \frac{x}{L} \end{aligned} \tag{2.16}$$

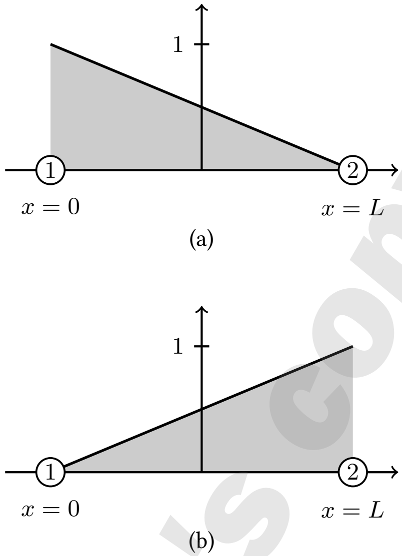

Figure 2.3: Linear shape functions (a) N_1 (b) N_2. The distance, x is measured from node 1.

Both N_1 and N_2 take the shapes as shown in Fig. 2.3. Take note of the shapes as they will be referred later in the discussion of integration by parts and the resulting boundary terms. For example, note that in both cases the shape functions, N_i has a value of unity at node i and zero at other nodes.

2.4.2 Derivation of Quadratic Bar Shape Function

Evaluate Eq. (2.10b) at the location of the nodes (i.e. at both ends, $x = 0$, $x = L/2$ and $x = L$) and equate them according to the dof give:

$$
\begin{aligned}
u(x)|_{x=0} &= a_1 + a_2(0) + a_3(0)^2 = u_1 \\
u(x)|_{x=\frac{L}{2}} &= a_1 + a_2(\frac{L}{2}) + a_3(\frac{L}{2})^2 = u_2 \\
u(x)|_{x=L} &= a_1 + a_2(L) + a_3(L)^2 = u_3
\end{aligned}
\tag{2.17}
$$

Solving for the values of a_1, a_2 and a_3 then gives:

$$\begin{aligned} a_1 &= u_1 \\ a_2 &= -\frac{3}{L}u_1 + \frac{4}{L}u_2 - \frac{1}{L}u_3 \\ a_3 &= \frac{2}{L^2}u_1 - \frac{4}{L^2}u_2 + \frac{2}{L^2}u_3 \end{aligned} \tag{2.18}$$

Inserting Eq. (2.18) into Eq. (2.10b) and by factorizing gives:

$$\begin{aligned} u(x) = &\left(\frac{L-2x}{L}\right)\left(\frac{L-x}{L}\right)u_1 + \frac{4x}{L}\left(\frac{L-x}{L}\right)u_2 \\ &+ \frac{x}{L}\left(\frac{L-2x}{L}\right)u_3 \end{aligned} \tag{2.19}$$

By comparing Eq. (2.19) and Eq. (2.11b), it can be concluded that:

$$\begin{aligned} N_1 &= \left(\frac{L-2x}{L}\right)\left(\frac{L-x}{L}\right) \\ N_2 &= \frac{4x}{L}\left(\frac{L-x}{L}\right) \\ N_3 &= \frac{x}{L}\left(\frac{L-2x}{L}\right) \end{aligned} \tag{2.20}$$

N_1, N_2 and N_3 take the shapes as shown in Fig. 2.4. Take note of the shapes as they will be referred later in the discussion of integration by parts and the resulting boundary terms. For example, note that in both cases the shape functions, N_i has a value of unity at node i and zero at other nodes.

2.5 Discretization by Galerkin Method

Having established the shape functions and the degree of freedoms of the bar element, we are all set to proceed with the discretization of the ODE.

For linear element, inserting Eq. (2.11a) into Eq. (2.8) gives:

$$EA\frac{d^2(N_1u_1 + N_2u_2)}{dx^2} \neq -q \tag{2.21}$$

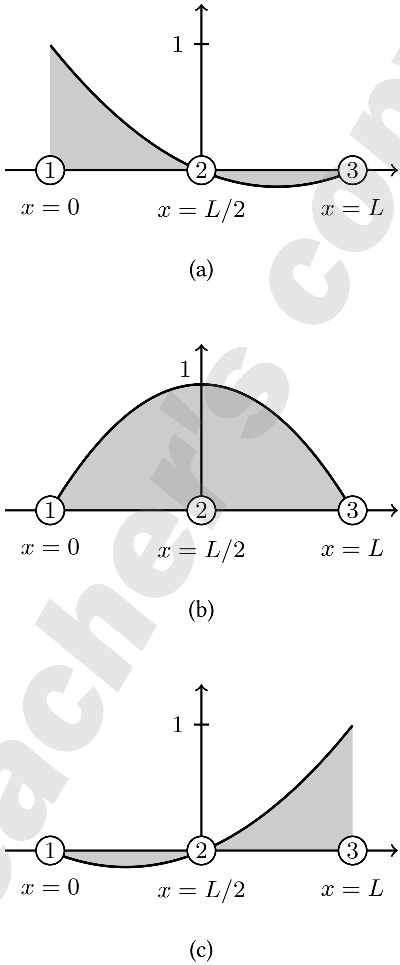

Figure 2.4: Quadratic shape functions (a) N_1 (b) N_2 (c) N_3. The distance, x is measured from node 1.

whilst for quadratic element, inserting Eq. (2.11b) into Eq. (2.8) gives:

$$EA\frac{d^2(N_1u_1 + N_2u_2 + N_3u_3)}{dx^2} \neq -q \tag{2.22}$$

Take note that, from now on, we are going to deal frequently with the component forms as this would eliminate the necessity to distinguish between linear and quadratic elements. With this, it should also demonstrate the advantage of such notation system. The component forms of Eq. (2.21) is given as:

$$EA\frac{d^2(N_ju_j)}{dx^2} \neq -q, \quad \text{for } j = 1, 2 \text{ (linear) and } j = 1, 2, 3 \text{ (quadratic)} \tag{2.23}$$

Since we have j number of unknowns (because we want to determine the values of the dofs, u_j) we thus need j number of simultaneous equations (i.e. 2 for linear and 3 for quadratic). These required equations can be obtained by multiplying Eq. (2.23) subsequently with a set of weight functions. This act is similar to the one conducted to Eq. (1.22) during the discussion of the basic concept of WRM.

Also, as mentioned earlier in this chapter, for Bubnov-Galerkin formulation, the weight functions are also the shape functions. For example, for a linear bar element, the two simultaneous equations can be obtained as follows:

$$N_1\left(EA\frac{d^2(N_jd_j)}{dx^2} - q\right) \neq 0 = R_1 \tag{2.24a}$$

$$N_2\left(EA\frac{d^2(N_jd_j)}{dx^2} - q\right) \neq 0 = R_2 \tag{2.24b}$$

So, in component forms, the required j simultaneous equations can be given as:

$$N_i\left(EA\frac{d^2(N_ju_j)}{dx^2} + q\right) \neq 0 = R_i, \quad \text{for } i = j = 1, 2 \text{ (linear bar element)} \tag{2.25}$$

The continuous nature of Eq. (2.24) (since it is still in ODE forms) can be converted into the equivalent algebraic forms by integrating over the length of the bar. This action is referred to as 'weighting the residual'; an action from which an approximate solution can be obtained by forcing the residual, R_i to zero. The algebraic forms of Eq. (2.25) are thus given as:

$$\begin{aligned} &\int_0^L N_i \left(EA \frac{d^2(N_j u_j)}{dx^2} + q \right) dx = 0 \\ \Rightarrow &\int_0^L N_i EA \frac{d^2(N_j u_j)}{dx^2}\, dx + \int_0^L N_i q \, dx = 0 \end{aligned} \tag{2.26}$$

It can be seen that Eq. (2.26) involves the integration of a product of functions, thus integration by parts (IBP) can be employed. The reasons for employing the IBP, as mentioned earlier in this chapter, are:

i. to reduce the continuity requirement since only first and lower derivatives will be dealt with

ii. to induce (make explicit) the natural boundary conditions as boundary terms resulting from the IBP

Employing IBP to Eq. (2.26) gives:

$$-\int_0^L \left(\frac{dN_i}{dx} EA \frac{dN_j}{dx} \right) dx \, u_j = -\int_0^L qN_i \, dx - N_i EA \left. \frac{d(N_j u_j)}{dx} \right|_0^L \tag{2.27}$$

In Eq. (2.27) above, the dofs, u_j are taken out from the integral sign because they are constants thus do not involve in both differentiation and integration. The last terms on the right hand side of Eq. (2.27), are called boundary terms resulting from the IBP.

By considering Eq. (2.12), the boundary terms can be written as:

$$N_i EA \left. \frac{d(N_j u_j)}{dx} \right|_0^L = N_i EA \left. \frac{du(x)}{dx} \right|_0^L \tag{2.28}$$

And further comparison with Eqs. (2.9a) and (2.9b), one can identify that Eq. (2.28) is actually the natural boundary conditions of the bar; hence it can be given that:

$$
\begin{aligned}
EA\,\frac{d(N_j u_j)}{dx}\bigg|_0^L &= N_i EA\,\frac{du(x)}{dx}\bigg|_{x=L} - N_i EA\,\frac{du(x)}{dx}\bigg|_{x=0} \\
&= -\,N_i|_{x=L}\,F_L - N_i|_{x=0}\,F_0
\end{aligned}
\tag{2.29}
$$

Eq. (2.29) deserves further explanation. The prevalence of the natural/force boundary conditions , i.e. F_0 and F_L within the equation depends on the accompanying value of the shape functions, being evaluated at the location of the respective node i.e. $N_i|_{x=L}$ and $N_i|_{x=0}$. As can be seen, say from Fig. 2.3 (for linear element), the value of the shape functions at locations $x = 0$ and $x = L$ can be given as:

$$
\begin{aligned}
&\text{at } x = 0: \quad N_1|_{x=0} = 1, \quad N_2|_{x=0} = 0 \\
&\text{at } x = L: \quad N_1|_{x=L} = 0, \quad N_2|_{x=L} = 1
\end{aligned}
\tag{2.30}
$$

As a result, for linear bar element, the boundary terms of Eq. (2.29) can be specifically given as:

$$
\begin{aligned}
N_1|_{x=L}\,F_L - N_1|_{x=0}\,F_0 &= -(0)F_L - (1)F_0 = -F_0 \\
N_2|_{x=L}\,F_L - N_2|_{x=0}\,F_0 &= -(1)F_L - (0)F_0 = -F_L
\end{aligned}
\tag{2.31}
$$

But, for quadratic bar element, there is one particular feature worth of attention, that is, the evaluated value of N_2 always null at the location of the edge nodes (refer Fig. 2.4). By taking this into consideration, the boundary terms of Eq. (2.29) for a quadratic bar element are thus given as:

$$
\begin{aligned}
N_1|_{x=L}\,F_L - N_1|_{x=0}\,F_0 &= -(0)F_L - (1)F_0 = -F_0 \\
N_2|_{x=L}\,F_L - N_2|_{x=0}\,F_0 &= -(0)F_L - (0)F_0 = 0 \\
N_3|_{x=L}\,F_L - N_3|_{x=0}\,F_0 &= -(1)F_L - (0)F_0 = -F_L
\end{aligned}
\tag{2.32}
$$

Due to these various conditions and combination, it is common in the discussion of finite element formulation to write Eq. (2.27) simply as:

$$
\int_0^L \left(\frac{dN_i}{dx} EA \frac{dN_j}{dx} \right) dx\, u_j = \int_0^L q N_i\, dx + b_i
\tag{2.33}
$$

where b_i refers to boundary terms which can be nodal loads or reaction forces at support. Eq. (2.33) completes the discretization of the bar's ODE by Galerkin Method.

2.6 Local Load Vector and Stiffness Matrix

Now, let's discuss the other term on the right hand side of Eq. (2.33), that is $\int_0^L qN_i dx$. Furthermore, let's call this term as equivalent nodal loads and denote it as q_i, thus:

$$q_i = \int_0^L qN_i \, dx \tag{2.34}$$

It is termed as equivalent nodal loads because these loads are the equivalent effects of a load, q that is acting on the domain; effects as perceived at the nodes. To better describe such equivalence, let's refer to Fig. 2.5. As shown in the figure, the distributed loading, $q(x)$ which is acting on the domain of the bar would deform the bar in the same manner as if the bar is subjected to the two nodal loads (or three for quadratic bar) which values are calculated by Eq. (2.34) as follows

For linear bar element

$$q_1 = \int_0^L N_1 q(x) \, dx$$
$$q_2 = \int_0^L N_2 q(x) \, dx$$

For quadratic bar element

$$q_1 = \int_0^L N_1 q(x) \, dx$$
$$q_2 = \int_0^L N_2 q(x) \, dx$$
$$q_3 = \int_0^L N_3 q(x) \, dx$$

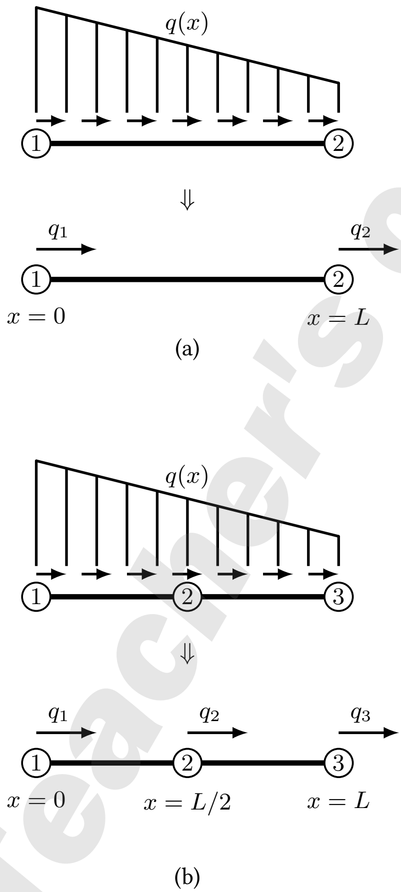

Figure 2.5: Equivalent nodal loads for (a) linear and (b) quadratic bar elements

Note that, despite the illustration that displays as if the distributed loading $q(x)$ is in the vertical direction, it must be understood that the load acts in the axial direction of the bar. Such an illustration is for convenience purposes only.

Having defined Eq. (2.34), Eq. (2.33) can be rewritten as:

$$\int_0^L \left(\frac{dN_i}{dx} EA \frac{dN_j}{dx} \right) dx \, u_j = q_i + b_i \tag{2.35}$$

Eq. (2.35) has been derived for one particular bar element, thus it is called local equilibrium equation. For a more compact statement, Eq. (2.35) is usually written in component forms as:

$$k_{ij} u_j = r_i \tag{2.36}$$

where k_{ij} is termed as the local stiffness matrix of the bar element. u_j and r_i, on the other hand, are the local vector of dofs and the load vector of the bar element, respectively. Thus, the stiffness matrix and the load vector of the bar element can be given as:

$$\begin{aligned} k_{ij} &= \int_0^L \left(\frac{dN_i}{dx} EA \frac{dN_j}{dx} \right) dx \\ r_i &= q_i + b_i \end{aligned} \tag{2.37}$$

Alternatively, in matrix forms, the local equilibrium of Eq. (2.36) can also be represented as:

$$[k]\,\{u\} = \{r\} \tag{2.38}$$

In an expanded matrix forms, the stiffness matrix of linear and quadratic elements can be respectively given as:

For linear bar element

$$\begin{aligned} [k] &= \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{12} \end{bmatrix} \\ &= \begin{bmatrix} \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_1}{dx} \right) dx & \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_2}{dx} \right) dx \\ \int_0^L \left(\frac{dN_2}{dx} EA \frac{dN_1}{dx} \right) dx & \int_0^L \left(\frac{dN_2}{dx} EA \frac{dN_2}{dx} \right) dx \end{bmatrix} \end{aligned} \tag{2.39a}$$

For quadratic bar element

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{12} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_1}{dx} \right) dx & k_{12} & k_{13} \\ k_{21} & k_{12} & k_{23} \\ \int_0^L \left(\frac{dN_3}{dx} EA \frac{dN_1}{dx} \right) dx & k_{32} & k_{33} \end{bmatrix} \tag{2.39b}$$

(Note: since the procedure is standard and repetitive, only elements k_{11} and k_{31} are expanded for demonstration).

Stiffness matrices which are given in Eqs. (2.39a) and (2.39b) are still in integral forms. Generally, to carry out the integration, numerical integration is used due to the complexity. However, since those in Eqs. (2.39a) and (2.39b) are simple, direct integration is used herein. The main purpose is that, with direct (analytical) integration, the evolution of the procedure can easier be followed since all variables remained to be seen.

For demonstration purposes, the integration of k_{12} are shown below for both linear and quadratic elements. The fully integrated stiffness matrix of both elements is then given.

For linear bar element

$$\begin{aligned} k_{12} &= \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_2}{dx} \right) dx \\ &= \int_0^L \left(-\frac{1}{L} \right) EA \left(\frac{1}{L} \right) dx \\ &= -\frac{EA}{L} \end{aligned} \tag{2.40a}$$

For quadratic bar element

$$\begin{aligned} k_{12} &= \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_2}{dx} \right) dx \\ &= \int_0^L \left(-\frac{3}{L} + \frac{4x}{L^2} \right) EA \left(\frac{4}{L} - \frac{8x}{L^2} \right) dx \\ &= -\frac{8EA}{3L} \end{aligned} \tag{2.40b}$$

The fully (or analytically) integrated stiffness matrix for both elements can be given as:

For linear bar element

$$\left[k\right] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2.41a}$$

For quadratic bar element

$$\left[k\right] = \frac{EA}{L} \begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix} \tag{2.41b}$$

The load vectors are respectively given as:

For linear bar element

$$\left\{r\right\} = r_i = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} q_1 + b_1 \\ q_2 + b_2 \end{Bmatrix} = \begin{Bmatrix} \int_0^L qN_1\, dx + b_1 \\ \int_0^L qN_2\, dx + b_2 \end{Bmatrix} \tag{2.42a}$$

for a specific case of uniform distributed load (where q is constant), it can be given that:

$$\{r\} = r_i = \begin{Bmatrix} \dfrac{qL}{2} + b_1 \\ \dfrac{qL}{2} + b_2 \end{Bmatrix} \tag{2.42b}$$

For quadratic bar element

$$\{r\} = r_i = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} q_1 + b_1 \\ q_2 + b_2 \\ q_3 + b_3 \end{Bmatrix} = \begin{Bmatrix} \displaystyle\int_0^L qN_1\,dx + b_1 \\ \displaystyle\int_0^L qN_2\,dx + b_2 \\ \displaystyle\int_0^L qN_3\,dx + b_3 \end{Bmatrix} \tag{2.43a}$$

for a specific case of uniform distributed load (where q is constant), it can be given that:

$$\{r\} = r_i = \begin{Bmatrix} \dfrac{qL}{6} + b_1 \\ \dfrac{2qL}{3} + b_2 \\ \dfrac{qL}{6} + b_3 \end{Bmatrix} \tag{2.43b}$$

Note that, component forms and matrix forms will be used interchangeably in our forthcoming discussion.

2.7 Assembly of Elements: Global Stiffness Matrix and Load Vector

A body or a structure is meshed by many elements in a finite element analysis. This is required so that the interpolation of the dependent variables can approximate well the actual distribution of the variables. The mesh is constructed by breaking up the physical domain into smaller elements which are then connected at the nodes to ensure appropriate interaction. The meshing requires the assembly of local stiffness matrix, $[K]$ and local

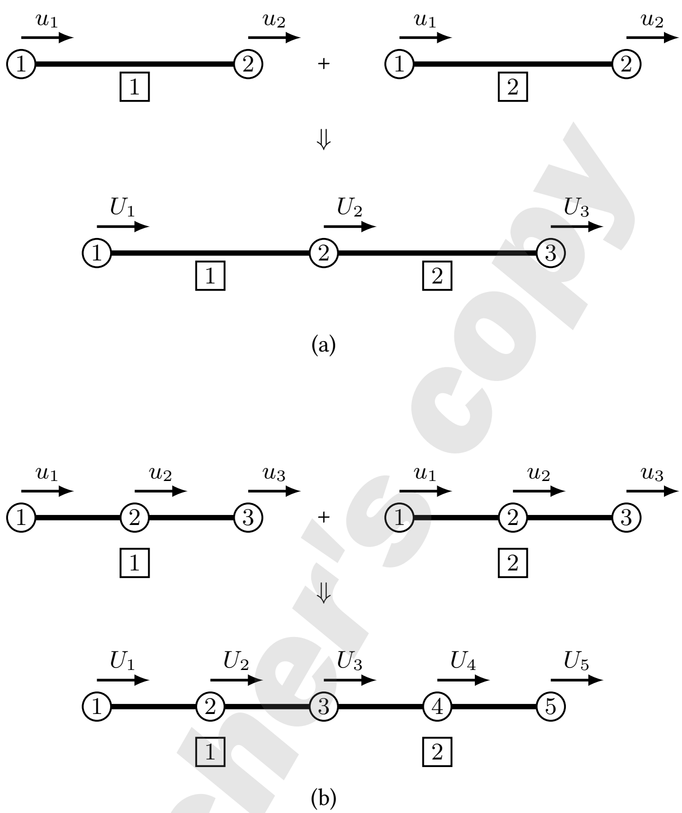

Figure 2.6: Local and global numbering of (a) linear (b) quadratic bar elements.

load vector, $\{R\}$ of individual element to form the global stiffness matrix, $[K]$ and load vector, $\{R\}$. Although in practice, hundreds and thousands of elements might be assembled, herein, only the assembly of two bar elements is demonstrated for easy tracing of the procedure. Fig. 2.6(a) and (b) show the assembly of two bar elements (for linear and quadratic elements respectively), together with the numbering of the local and global nodes, elements and also the dof. Note that, node numbering is given in circle whilst element numbering is given in square box.

Based on the above, it is obvious that we need to have extra notation to distinguish between different elements. In doing this, we introduce the use of superscript as follows:

i. The local stiffness matrix of element N is denoted as k_{ij}^N or $\left[k^N\right]$
ii. The local load vector of element N is denoted as r_i^N or $\left\{r^N\right\}$
iii. The vector of dof of element N is denoted as u_j^N or $\left\{u^N\right\}$

Having introduced the new superscript system, the local equilibrium for each element can thus be given as:

For linear bar element

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 \end{Bmatrix} \quad \Leftarrow \text{Element 1}$$

$$\begin{bmatrix} k_{11}^2 & k_{12}^2 \\ k_{21}^2 & k_{22}^2 \end{bmatrix} \begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = \begin{Bmatrix} r_1^2 \\ r_2^2 \end{Bmatrix} \quad \Leftarrow \text{Element 2} \tag{2.44a}$$

Now the assembly of the elements can be done as follows:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 = u_1^2 \\ u_2^2 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 + r_1^2 \\ r_2^2 \end{Bmatrix} \tag{2.44b}$$

or in global numbering as:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} \tag{2.44c}$$

which can be simplified as:

$$\left[K\right]\left\{U\right\} = \left\{R\right\} \tag{2.44d}$$

For quadratic bar element

$$\begin{aligned}
&\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 \end{Bmatrix} \quad \Leftarrow \text{Element 1} \\
&\begin{bmatrix} k_{11}^2 & k_{12}^2 & k_{13}^2 \\ k_{21}^2 & k_{22}^2 & k_{23}^2 \\ k_{31}^2 & k_{32}^2 & k_{33}^2 \end{bmatrix} \begin{Bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{Bmatrix} = \begin{Bmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \end{Bmatrix} \quad \Leftarrow \text{Element 2}
\end{aligned} \tag{2.45a}$$

The assembly of quadratic bar elements can be done as follows:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{12}^2 & k_{13}^2 \\ 0 & 0 & k_{21}^2 & k_{22}^2 & k_{23}^2 \\ 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 = u_1^2 \\ u_2^2 \\ u_3^2 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 + r_1^2 \\ r_2^2 \\ r_3^2 \end{Bmatrix} \tag{2.45b}$$

or in global numbering as:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} \tag{2.45c}$$

which can be simplified as:

$$[K]\{U\} = \{R\} \tag{2.45d}$$

Based on Eq. (2.45d), it can be deduced that $[K]$ is the global stiffness matrix, $\{U\}$ is the global vector of dof and $\{R\}$ is the global load vector.

2.8 Imposition of Essential Boundary Conditions

As mentioned earlier, to have a well-posed problem, essential (or displacement) boundary conditions must be imposed prior to the solution of the equilibrium equation. There are various ways to impose boundary conditions such as row-column condensation , penalty method, Lagrange multiplier method etc. Herein, row-column condensation is employed.

The basic concept of the row-column condensation can be explained as follows. Assume that we have a set of three simultaneous equations given as:

$$\begin{aligned} a_1U + a_2V + a_3W &= d \\ b_1U + b_2V + b_3W &= e \\ c_1U + c_2V + c_3W &= f \end{aligned} \tag{2.46}$$

Now, let's assume that V is known (the boundary conditions) i.e. $V = \hat{V}$ By inserting this known value into Eq. (2.46) gives:

$$\begin{aligned} a_1U + a_2\hat{V} + a_3W &= d \\ b_1U + b_2\hat{V} + b_3W &= e \\ c_1U + c_2\hat{V} + c_3W &= f \end{aligned} \tag{2.47}$$

Since $\hat{V}$ is known, all terms associated with it can be moved to the right hand side of the equation, thus:

$$\begin{aligned} a_1U + a_3W &= d - a_2\hat{V} \\ b_1U + b_3W &= e - b_2\hat{V} \\ c_1U + c_3W &= f - c_2\hat{V} \end{aligned} \tag{2.48}$$

By examining Eq. (2.48), one can realize that we have two unknowns with three equations. Since we need only two equations for the solution, we can just use, for example, the first and the third equations. (We can also use the second equation together with either first or third equation), thus:

$$\begin{aligned} a_1 U + a_3 W &= d - a_2 \hat{V} \\ c_1 U + c_3 W &= f - c_2 \hat{V} \end{aligned} \tag{2.49}$$

Now, the condensation can be made more obvious if we discuss the above in matrix forms. Eq. (2.49) can be given in matrix forms as:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \begin{Bmatrix} d \\ e \\ f \end{Bmatrix} \tag{2.50}$$

If $\hat{V}$ is known, Eq. (2.50) can be condensed (thus the employment of the boundary conditions) as follows:

$$\begin{bmatrix} a_1 & \square & a_3 \\ \square & \square & \square \\ c_1 & \square & c_3 \end{bmatrix} \begin{Bmatrix} U \\ \square \\ W \end{Bmatrix} = \begin{Bmatrix} d - a_2 \hat{V} \\ \square \\ f - c_2 \hat{V} \end{Bmatrix} \tag{2.51}$$

The above explains the basic concept of row-column condensation where it can be concluded that, an imposition of n number of boundary conditions to a stiffness matrix of $N \times N$ size and to a load vector of $N \times 1$, would condense the former to a size of $(N - n) \times (N - n)$ and the latter to a size of $(N - n) \times 1$. Once condensed, the solution of the matrix system can proceed.

Now, let's employ this method directly to our global equilibrium equation as given by Eqs. (2.44c) and (2.45c) for linear and quadratic element, respectively. For the purpose of discussion, let's assume that the right end of the bar is having a specified displacement. Such a condition leads to the following essential boundary conditions:

- For linear element: $U_3 = \hat{U}_3$
- For quadratic element: $U_5 = \hat{U}_5$

The imposition of the above boundary conditions to the global equilibrium equation can be done as follows.

For linear bar element

$$
\begin{aligned}
&\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} \\
\Rightarrow &\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} R_1 - K_{13}\hat{U}_3 \\ R_2 - K_{23}\hat{U}_3 \end{Bmatrix}
\end{aligned} \tag{2.52}
$$

For quadratic bar element

$$
\begin{aligned}
&\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} \\
\Rightarrow &\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} R_1 - K_{15}\hat{U}_5 \\ R_2 - K_{25}\hat{U}_5 \\ R_3 - K_{35}\hat{U}_5 \\ R_4 - K_{45}\hat{U}_5 \end{Bmatrix}
\end{aligned} \tag{2.53}
$$

Eqs. (2.52) and (2.53) are now ready to be solved. There are various ways to solve a matrix equation such as direct inversion, Gauss Elimination, Cholesky Decomposition, etc. In the accompanying Matlab source codes, Gauss Elimination is adopted. Symbolically, the solution of the matrix equation is represented as:

$$
\{U\} = [K]^{-1}\{R\} \tag{2.54}
$$

2.9 Worked Example 2.1

Now, we are going to solve the previous problem of Worked Example 1.1 by considering FEM as illustrated in Fig. 2.7. The problem is solved by the assembly of two bar elements. Both linear and quadratic bars are considered. The results are then verified against the closed-form solution of Eq. (1.4).

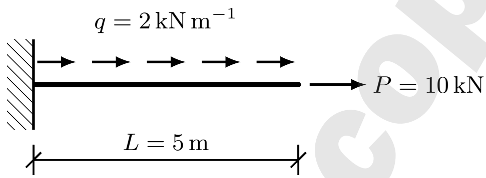

Figure 2.7: Cantilever bar with uniformly distributed load, q and a single point load, P. ($E = 200\times10^6\,\mathrm{kN\,m^{-2}}$ and $A = 0.04\,\mathrm{m^2}$).

2.9.1 Linear Bar Element

Due to the symmetricality, element 1 and element 2 would have a similar local stiffness matrix and load vector, thus:

$$k_{ij}^1 = k_{ij}^2 = \begin{bmatrix} 3.2 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \times 10^6 \tag{2.55}$$

and

$$r_i^1 = \begin{Bmatrix} 2.5 + b_1^1 \\ 2.5 + 0 \end{Bmatrix} \tag{2.56}$$

$$r_i^2 = \begin{Bmatrix} 2.5 + 0 \\ 2.5 + 10 \end{Bmatrix} \tag{2.57}$$

Note that b_1^1 is the local reaction at the support of element 1.

The assembled global stiffness matrix, $[K]$ is given as:

$$
\begin{aligned}
[K] &= \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \\
&= \begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix} \\
&= \begin{bmatrix} 3.2 & -3.2 & 0 \\ -3.2 & 6.4 & -3.2 \\ 0 & -3.2 & 3.2 \end{bmatrix} \times 10^6
\end{aligned}
\tag{2.58}
$$

and the assembled load vector, $\{R\}$ is given as:

$$
\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 + r_1^2 \\ r_2^2 \end{Bmatrix} = \begin{Bmatrix} 2.5 + b_1^1 \\ 2.5 + 2.5 \\ 2.5 + 10 \end{Bmatrix} = \begin{Bmatrix} 2.5 + B_1 \\ 5.0 \\ 12.5 \end{Bmatrix} \tag{2.59}
$$

To emphasize the global reactions forces, b_1^1 is expressed as B_1. This is unknown variable because its corresponding dof is the essential boundary condition which in turn, is a known value. The whole equilibrium equation can thus be given as:

$$
\begin{bmatrix} 3.2 & -3.2 & 0 \\ -3.2 & 6.4 & -3.2 \\ 0 & -3.2 & 3.2 \end{bmatrix} \times 10^6 \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 2.5 + B_1 \\ 5.0 \\ 12.5 \end{Bmatrix} \tag{2.60}
$$

By imposing the essential boundary conditions ($U_1 = 0$), the assembled global equilibrium equation is reduced to:

$$
\begin{bmatrix} 6.4 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \times 10^6 \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 5.0 - 0 \\ 12.5 - 0 \end{Bmatrix} \tag{2.61}
$$

By solving Eq. (2.61) using Matlab command “\”, the values of the assembled global dof U_i are thus obtained as:

$$\{U\} = \begin{Bmatrix} 0 \\ 5.47\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} \tag{2.62}$$

or in terms of global dofs as summarised in Table 2.1.

Table 2.1: Values of linear bar global degree of freedom, $\{U\}$

U_i	U_1	U_2	U_3
Deflection	0	5.47×10^{-6}	9.38×10^{-6}

Eq. (2.62) is the assembled global solution. By inserting these values accordingly into Eq. (2.38), the local forces can then be determined. However, before that, lets determine first the global reaction forces, B_1 which can be done by inserting Eq. (2.62) into Eq. (2.60) which yields:

$$\begin{Bmatrix} B_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 3.2 & -3.2 & 0 \\ -3.2 & 6.4 & -3.2 \\ 0 & -3.2 & 3.2 \end{bmatrix} \times 10^6 \begin{Bmatrix} 0 \\ 5.47\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 5.0 \\ 12.5 \end{Bmatrix} \tag{2.63}$$

The global reaction force can thus be given as:

$$\{B_1\} = \{-20\} \tag{2.64}$$

Now, let’s proceed with the determination of the local values. The elemental local solution for each element is thus:

$$\{u^1\} = \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5.47\times10^{-6} \end{Bmatrix} \tag{2.65}$$

$$\{u^2\} = \begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = \begin{Bmatrix} 5.47\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} \tag{2.66}$$

Eqs. (2.65) and (2.66) are the elemental local solutions.

Element 1

By inserting Eq. (2.65) back into the local equilibrium as given in Eq. (2.38), the local force (or reaction if at support) can be determined as follows:

$$\begin{aligned}\left\{b^1\right\} &= \left[k^1\right]\left\{u^1\right\} - \left\{q^1\right\} \\ &= \begin{bmatrix} 3.2 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \times 10^6 \begin{Bmatrix} 0 \\ 5.47\times10^{-6} \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 2.5 \end{Bmatrix} \\ &= \begin{Bmatrix} -17.5 \\ 17.5 \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 2.5 \end{Bmatrix} \\ &= \begin{Bmatrix} -20 \\ 15 \end{Bmatrix}\end{aligned} \tag{2.67}$$

Element 2

By inserting Eq. (2.66) into Eq. (2.38), the local force (or reaction if at support) can be determined as follows:

$$\begin{aligned}\left\{b^2\right\} &= \left[k^2\right]\left\{u^2\right\} - \left\{q^2\right\} \\ &= \begin{bmatrix} 3.2 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \times 10^6 \begin{Bmatrix} 5.47\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 2.5 \end{Bmatrix} \\ &= \begin{Bmatrix} -12.5 \\ 12.5 \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 2.5 \end{Bmatrix} \\ &= \begin{Bmatrix} -15 \\ 10 \end{Bmatrix}\end{aligned} \tag{2.68}$$

Finally, the elemental stress, σ^1 and σ^2 can be determined as follows:

$$\begin{aligned}
\{\sigma^1\} &= \frac{\{b^1\}}{A} \\
&= \frac{1}{0.04}\begin{Bmatrix} -20 \\ 15 \end{Bmatrix} \\
&= \begin{Bmatrix} -500 \\ 375 \end{Bmatrix}
\end{aligned} \tag{2.69}$$

$$\begin{aligned}
\{\sigma^2\} &= \frac{\{b^2\}}{A} \\
&= \frac{1}{0.04}\begin{Bmatrix} -15 \\ 10 \end{Bmatrix} \\
&= \begin{Bmatrix} -375 \\ 250 \end{Bmatrix}
\end{aligned} \tag{2.70}$$

2.9.2 Quadratic Bar Element

Now, we are going to see the calculation of the same bar problem using quadratic bar element. Again, due to the symmetrically, element 1 and element 2 would have a similar local stiffness matrix and load vector, thus:

$$k_{ij}^1 = k_{ij}^2 = \begin{bmatrix} 7.47 & -8.53 & 1.07 \\ -8.53 & 17.1 & -8.53 \\ 1.07 & -8.53 & 7.47 \end{bmatrix} \times 10^6 \tag{2.71}$$

and

$$r_i^1 = \begin{Bmatrix} 0.83 + b_1^1 \\ 3.33 + 0 \\ 0.83 + 0 \end{Bmatrix} \tag{2.72}$$

$$r_i^2 = \begin{Bmatrix} 0.83 + 0 \\ 3.33 + 0 \\ 0.83 + 10 \end{Bmatrix} \tag{2.73}$$

The assembled global stiffness matrix, $[K]$ is given as:

$$\begin{aligned}
\begin{bmatrix} K \end{bmatrix} &= \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \\
&= \begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{12}^2 & k_{13}^2 \\ 0 & 0 & k_{21}^2 & k_{22}^2 & k_{23}^2 \\ 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 \end{bmatrix} \\
&= \begin{bmatrix} 7.47 & -8.53 & 1.07 & 0 & 0 \\ -8.53 & 17.1 & -8.53 & 0 & 0 \\ 1.07 & -8.53 & 14.9 & -8.53 & 1.07 \\ 0 & 0 & -8.53 & 17.1 & -8.53 \\ 0 & 0 & 1.07 & -8.53 & 7.47 \end{bmatrix} \times 10^6
\end{aligned} \tag{2.74}$$

and the assembled load vector, $\{R\}$ is given as:

$$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 + r_1^2 \\ r_2^2 \\ r_3^2 \end{Bmatrix} = \begin{Bmatrix} 0.83 + b_1^1 \\ 3.33 + 0 \\ 0.83 + 0 + 0.83 + 0 \\ 3.33 + 0 \\ 0.83 + 10 \end{Bmatrix} = \begin{Bmatrix} 0.83 + B_1 \\ 3.33 \\ 1.67 \\ 3.33 \\ 10.83 \end{Bmatrix} \tag{2.75}$$

To emphasize the global reactions forces, b_1^1 is expressed as B_1. This is unknown variable because its corresponding dof is the essential boundary condition which in turn, is a known value. Having established $[K]$ and , the whole equilibrium equation in the form of Eq. (2.45b) can thus be given as:

$$\begin{bmatrix} 7.47 & -8.53 & 1.07 & 0 & 0 \\ -8.53 & 17.1 & -8.53 & 0 & 0 \\ 1.07 & -8.53 & 14.9 & -8.53 & 1.07 \\ 0 & 0 & -8.53 & 17.1 & -8.53 \\ 0 & 0 & 1.07 & -8.53 & 7.47 \end{bmatrix} \times 10^6 \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0.83 + B_1 \\ 3.33 \\ 1.67 \\ 3.33 \\ 10.83 \end{Bmatrix} \tag{2.76}$$

By imposing the essential boundary conditions ($U_1 = 0$), the assembled global equilibrium equation is reduced to:

$$\begin{bmatrix} 17.1 & -8.53 & 0 & 0 \\ -8.53 & 14.9 & -8.53 & 1.07 \\ 0 & -8.53 & 17.1 & -8.53 \\ 0 & 1.07 & -8.53 & 7.47 \end{bmatrix} \times 10^6 \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 3.33 \\ 1.67 \\ 3.33 \\ 10.83 \end{Bmatrix} \tag{2.77}$$

By solving Eq. (2.77) using Matlab command "\", the values of the assembled global dof U_i are thus obtained as:

$$\{U\} = \begin{Bmatrix} 0 \\ 2.93 \times 10^{-6} \\ 5.47 \times 10^{-6} \\ 7.62 \times 10^{-6} \\ 9.38 \times 10^{-6} \end{Bmatrix} \tag{2.78}$$

or in terms of global dofs as summarised in Table 2.2.

Table 2.2: Values of quadratic bar global degree of freedom, $\{U\}$

	U_1	U_2	U_3	U_4	U_5
Deflection	0	2.93×10^{-6}	5.47×10^{-6}	7.62×10^{-6}	9.38×10^{-6}

Eq. (2.78) is the assembled global solution. As in the linear bar element example, the local forces can simply be determined by inserting these values accordingly into the local equilibrium Eq. (2.38). Before that, lets determine first the global reaction forces, B_1 which can be done by inserting Eqn. (E2.18a) into Eqn. (E2.16) which yields:

$$\begin{Bmatrix} B_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 7.47 & -8.53 & 1.07 & 0 & 0 \\ -8.53 & 17.1 & -8.53 & 0 & 0 \\ 1.07 & -8.53 & 14.9 & -8.53 & 1.07 \\ 0 & 0 & -8.53 & 17.1 & -8.53 \\ 0 & 0 & 1.07 & -8.53 & 7.47 \end{bmatrix} \times 10^6 \begin{Bmatrix} 0 \\ 2.93\times10^{-6} \\ 5.47\times10^{-6} \\ 7.61\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} - \begin{Bmatrix} 0.83 \\ 3.33 \\ 1.67 \\ 3.33 \\ 10.83 \end{Bmatrix} \tag{2.79}$$

The global reaction force can thus be given as:

$$\{B_1\} = \{-20\} \tag{2.80}$$

Now, let's proceed with determination of the local values. The elemental local solution for each element is thus:

$$\{u^1\} = \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{Bmatrix} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2.93\times10^{-6} \\ 5.47\times10^{-6} \end{Bmatrix} \tag{2.81}$$

$$\{u^2\} = \begin{Bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{Bmatrix} = \begin{Bmatrix} U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 5.47\times10^{-6} \\ 7.62\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} \tag{2.82}$$

Eqs. (2.81) and (2.82) are the elemental local solutions.

Element 1 By inserting Eq. (2.81) into Eq. (2.38), the local force (or reaction if at support) can be determined as follows:

$$\begin{aligned} \{b^1\} &= [k^1]\{u^1\} - \{q^1\} \\ &= \begin{bmatrix} 7.47 & -8.53 & 1.07 \\ -8.53 & 17.1 & -8.53 \\ 1.07 & -8.53 & 7.47 \end{bmatrix} \times 10^6 \begin{Bmatrix} 0 \\ 2.93\times10^{-6} \\ 5.47\times10^{-6} \end{Bmatrix} \\ &\quad - \begin{Bmatrix} 0.83 \\ 3.33 \\ 0.83 \end{Bmatrix} \\ &= \begin{Bmatrix} -19.17 \\ 3.33 \\ 15.83 \end{Bmatrix} - \begin{Bmatrix} 0.83 \\ 3.33 \\ 0.83 \end{Bmatrix} \\ &= \begin{Bmatrix} -20 \\ 0 \\ 15 \end{Bmatrix} \end{aligned} \tag{2.83}$$

Element 2 By inserting Eq. (2.82) into Eq. (2.38), the local force (or reaction at support) can be determined as follows:

$$
\begin{aligned}
\left\{b^2\right\} &= \left[k^2\right]\left\{u^2\right\} - \left\{q^2\right\} \\
&= \begin{bmatrix} 7.47 & -8.53 & 1.07 \\ -8.53 & 17.1 & -8.53 \\ 1.07 & -8.53 & 7.47 \end{bmatrix} \times 10^6 \begin{Bmatrix} 5.47\times10^{-6} \\ 7.62\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} \\
&\quad - \begin{Bmatrix} 0.83 \\ 3.33 \\ 0.83 \end{Bmatrix} \\
&= \begin{Bmatrix} -14.17 \\ 3.33 \\ 10.83 \end{Bmatrix} - \begin{Bmatrix} 0.83 \\ 3.33 \\ 0.83 \end{Bmatrix} \\
&= \begin{Bmatrix} -15 \\ 0 \\ 10 \end{Bmatrix}
\end{aligned}
\tag{2.84}
$$

Finally, the elemental stress, σ_1 and σ_2 can be determined as follows:

$$
\begin{aligned}
\left\{\sigma^1\right\} &= \frac{\{b^1\}}{A} \\
&= \frac{1}{0.04}\begin{Bmatrix} -20 \\ 0 \\ 15 \end{Bmatrix} \\
&= \begin{Bmatrix} -500 \\ 0 \\ 375 \end{Bmatrix}
\end{aligned}
\tag{2.85}
$$

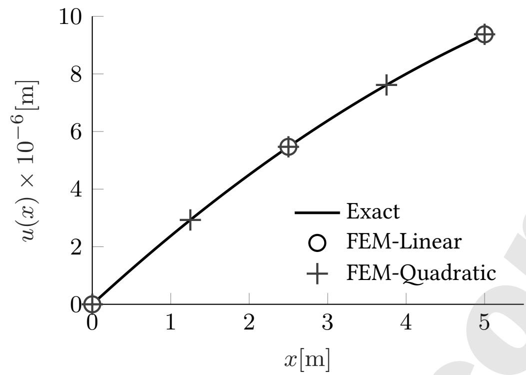

Figure 2.8: Comparison between u_{exact}, u_{linear} and $u_{\text{quadratic}}$

$$\begin{aligned}\{\sigma^2\} &= \frac{\{b^2\}}{A} \\ &= \frac{1}{0.04}\begin{Bmatrix}-15\\0\\10\end{Bmatrix} \\ &= \begin{Bmatrix}-375\\0\\250\end{Bmatrix}\end{aligned} \tag{2.86}$$

The validity of both results thus formulation can be assessed by comparing their values with those previously obtained in Worked Example 1.2. The plot of FEM results against the closed-form solution (Eq. (1.4)) are shown in Fig. 2.8 which numerical values at several locations along the bar are given in Table 2.3.

Based on the plot and the table, for the first time the convergence nature of a numerical analysis becomes obvious. As can be seen, the assembled two linear elements provide quite a poor approximation except at the location of the nodes where the results agree with the closed-form solution. A better hence a converged solution is provided by the quadratic elements thus an immediate demonstration of the beneficial effect of the use of higher or-

Table 2.3: Comparison of values between u_{exact}, u_{linear} and $u_{\text{quadratic}}$

x	0 m	1.25 m	2.5 m	3.75 m	5 m
u_{linear}	-		5.47×10^{-6}	-	9.38×10^{-6}
$u_{\text{quadratic}}$	0	2.93×10^{-6}	5.47×10^{-6}	7.62×10^{-6}	9.38×10^{-6}
u_{exact}	0	2.93×10^{-6}	5.47×10^{-6}	7.62×10^{-6}	9.38×10^{-6}

der elements. This is an immediate demonstration to the statement *"FEM converges to the 'accurate' solution with the increase in the order of polynomial of the guessed functions (or degree of freedom and mesh density later on)"* which was mentioned in the previous chapter.

2.10 Matlab Source Codes

2.10.1 Linear Bar Element

```
% Clear data
clc; clear; close all

% Input
E = 200e6;   % Young's modulus (kN/m)
A = 0.04;    % Area (m^2)
L = 5;       % Bar length [m]
P = 10;      % Point load [kN]
q = 2;       % Distributed load [kN/m]

% ------------------------------------------
% FEM solution - Displacement
% ------------------------------------------

% Elements length
L1 = L/2; L2 = L/2;

% Matrix k & force r
k1 = A*E/L1*[ 1 -1;
             -1  1];
k2 = A*E/L2*[ 1 -1;
             -1  1];

r1 = [q*L1/2; q*L1/2];
r2 = [q*L1/2; q*L1/2];

% Assemble global matrix, K and vector, R
K = zeros(3);
K(1:2, 1:2) = K(1:2, 1:2) + k1;
K(2:3, 2:3) = K(2:3, 2:3) + k2;
```

```
R = zeros(3,1);
R(1:2) = R(1:2) + r1;
R(2:3) = R(2:3) + r2;

% Point load, P at the bar end
R(3) = R(3)+P;

% Solve for global displacement
U      = zeros(3,1);
U(2:3) = K(2:3,2:3)\R(2:3);

% ----------------------------------------------
% FEM solution - Reaction, b & stress, sig
% ----------------------------------------------

% Internal force or reaction
b1 = k1*[U(1); U(2)] - r1;
b2 = k2*[U(2); U(3)] - r2;

% Internal stress
sig1 = b1/A;
sig2 = b2/A;
```

2.10.2 Quadratic Bar Element

```
% Clear data
clc; clear; close all

% Input
E = 200e6;  % Young's modulus (kN/m)
A = 0.04;   % Area (m^2)
L = 5;      % Bar length [m]
P = 10;     % Point load [kN]
q = 2;      % Distributed load [kN/m]

% ----------------------------------------------
% FEM solution - Displacement
% ----------------------------------------------

% Elements length
L1 = L/2; L2 = L/2;

% Matrix k & force r
k1 = A*E*[ 7/(3*L1) -8/(3*L1)  1/(3*L1);
          -8/(3*L1) 16/(3*L1) -8/(3*L1);
           1/(3*L1) -8/(3*L1)  7/(3*L1)];
k2 = A*E*[ 7/(3*L2) -8/(3*L2)  1/(3*L2);
          -8/(3*L2) 16/(3*L2) -8/(3*L2);
           1/(3*L2) -8/(3*L2)  7/(3*L2)];

r1 = [q*L1/6; 2*q*L1/3; q*L1/6];
r2 = [q*L2/6; 2*q*L2/3; q*L2/6];

% Assemble global matrix, K and vector, R
K = zeros(5);
K(1:3, 1:3) = K(1:3, 1:3) + k1;
K(3:5, 3:5) = K(3:5, 3:5) + k2;
```

```
R = zeros(5,1);
R(1:3) = R(1:3) + r1;
R(3:5) = R(3:5) + r2;

% Point load, P at the bar end
R(5) = R(5)+P;

% Solve for global displacement
U      = zeros(5,1);
U(2:5) = K(2:5,2:5)\R(2:5);

% ---------------------------------------------
% FEM solution - Reaction, b & stress, sig
% ---------------------------------------------

% Internal force or reaction
b1 = k1*[U(1); U(2); U(3)] - r1;
b2 = k2*[U(3); U(4); U(5)] - r2;

% Internal stress
sig1 = b1/A;
sig2 = b2/A;
```

2.11 Excercises

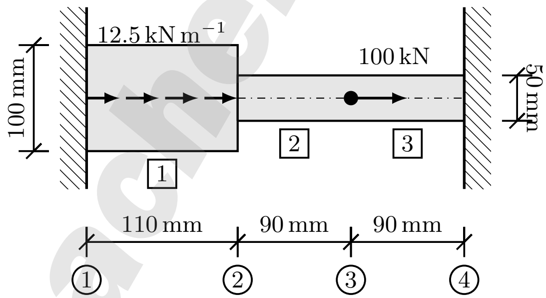

Figure 2.9

1. The rod structure of circular cross section shown in Fig. 2.9 is subjected to a concentrated force of 100 kN and a traction force of $12.5\,\mathrm{kN\,m^{-1}}$. The material for all rods is aluminium with $E = 70\,\mathrm{GPa}$. By using three 2-node bar elements, determine;

   i. Displacement at node 2 and 3.
   ii. Stresses in each element.
   iii. Reaction forces at the supports.
   (Note: $1\,\mathrm{GPa} = 1{\times}10^9\,\mathrm{N\,m^{-2}}$)

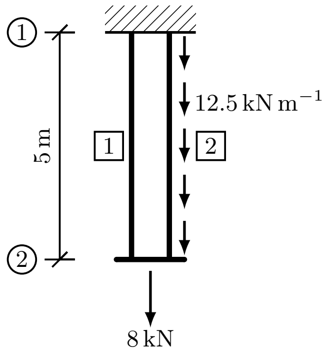

Figure 2.10

2. Fig. 2.10 shows two bars hanged at node 1 and rigidly linked together at node 2 to take a point load of $8\,\mathrm{kN}$. Bar 2 on the right is subjected to a uniformly distributed load of $12.5\,\mathrm{kN\,m^{-1}}$ applied in the direction of the bar. The material for all rods is aluminium with $E = 70\,\mathrm{GPa}$ and $A = 50\,\mathrm{mm}^2$.

   i. Calculate the vertical displacements at node 2.
   ii. Calculate the stresses in both bars and plot them along the length of the bars.
   iii. If the two bars are modelled with 3-nodes element, will the stress values in both bars change? Explain your answer.

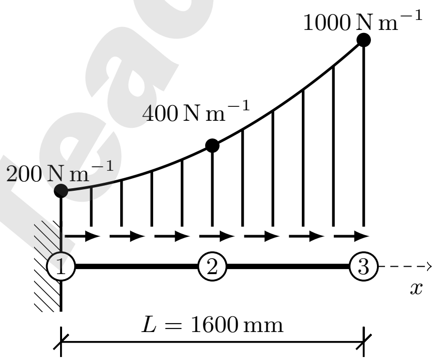

Figure 2.11

3. A quadratic load is exerted on a cantilever bar as shown in Fig. 2.11.

The bar Young's modulus is $185\,\mathrm{GPa}$. The cross sectional area is $A = 1200\,\mathrm{mm}^2$. Using a 3-nodes element, determine:

i. The stiffness matrix, **K** and the force vector, **r**.
ii. Maximum stress in the bar.

4. Fig. 2.12 shows a $1\,\mathrm{mm}$ thick plate that is subjected to force $100\,\mathrm{N}$ at its midpoint in addition to its self weight. The Young's modulus and weight density of the plate are $E = 210\,\mathrm{GPa}$ and $\rho = 7850\,\mathrm{kg\,m^{-2}}$, respectively. By modelling the plate as two linear, one-dimensional bar elements, determine:

   i. The element stiffness matrices.
   ii. The assembled global stiffness matrix and global load vector.
   iii. The global axial displacements .
   iv. The axial stresses in each element.

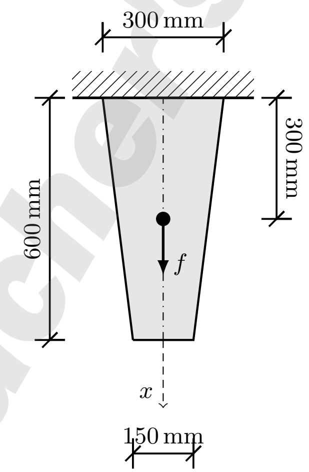

Figure 2.12

3 Galerkin Formulation: Beam Element

3.1 Introduction

In the previous chapter, we have discussed the concept of Galerkin WRM hence FEM and formulated the discretised equation for bar element. Herein, we are going to continue our discussion by formulating the discretised equation for beam problem. We begin by deriving the Euler-Bernoulli differential equation of beam which is given next.

3.2 Ordinary Differential Equation of Euler-Bernoulli Beam

An Euler-Bernoulli beam is a structural member that resists loads by bending and shearing. The corresponding deformation would be rotation and translation. Consider a structural beam which is subjected to a distributed load, $q(x)$ as shown in Fig. 3.1(a). The differential equation for such a beam can be derived for static loading by considering its differential element as shown in Fig. 3.1(b). Also shown is the typical arrangement of a beam structure.

As argued previously for bar element (in Chapter 2), although we have moment force, M and shear force, V at the left side of the differential element, due to the 'disturbance' along the differential length, dx (due to external load, for example) the magnitude of these forces at the right side of the differential element must change. However we don't know the exact change of these forces, else we would not have the problem in the first place would we? But by assuming the change is continuous, we can say that the forces at the right-side of the differential element can be represented by a Taylor series, as shown in Fig. 3.1(b).

By assuming higher order terms as insignificant and since this is a 1D problem (i.e.$\partial(\quad) = d(\quad)$), a state as shown in Fig. 3.1(c) is considered. This is an important argument thus must be grasped by the readers because as we are going to see later, the derivation of PDE (or ODE) for other problems will be based on the same argument as well.

Having established the differential element and the corresponding forces acting on it, we are in the position to derive the ODE for beam problem. However, it must be noted again that the following differential equation does not consider axial deformation thus the absent of axial forces. Beams allowing such forces are called beam-column , of which the FEM formulation is given in the next chapter. Also, present formulation assumes slope is equalled to rotation. A more general formulation would be the Timoshenko Beam Theory as it allows different values for the two entities, but it is not included in our discussion.

Based on Fig. 3.1(c), the following equilibrium of forces can be employed:

$$\sum F_x = 0 \tag{3.1}$$

$$\sum F_y = 0 \tag{3.2}$$

$$\sum M_z = 0 \tag{3.3}$$

which yield

$$V - (V + dV) - q\,dx = 0 \tag{3.4}$$

$$-M + (M + dM) - V\,dx - q\frac{dx^2}{2} = 0 \tag{3.5}$$

where q is the distributed transverse external loading and w the deflection of the beam.

Eqs. (3.4) and (3.5) can be simplified into:

$$\frac{dV}{dx} = -q \tag{3.6}$$

$$\frac{dM}{dx} = V \tag{3.7}$$

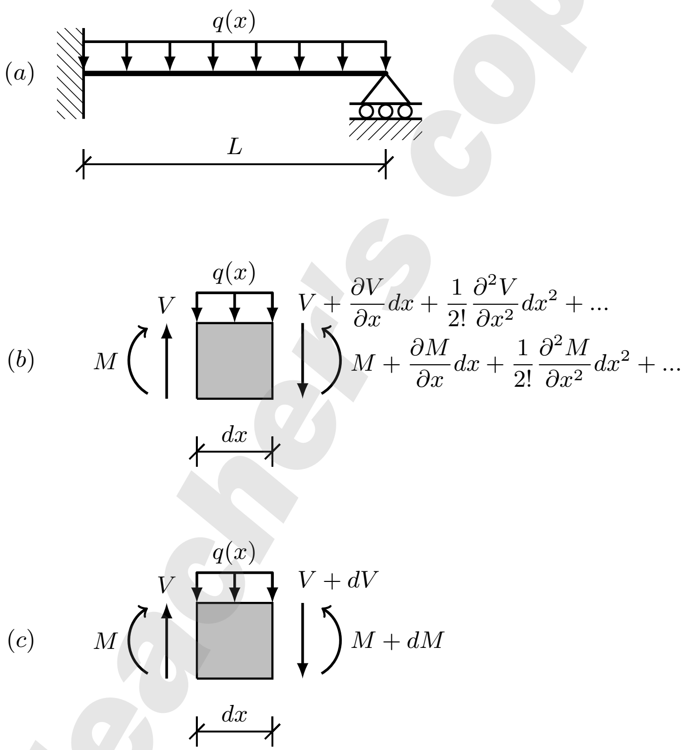

Figure 3.1: Beam element and its differential element.

To arrive at Eq. (3.7), higher order term $\frac{dx^2}{2}$ has been assumed as insignificant thus omitted. By differentiating Eq. (3.7) once then inserting it into Eq. (3.6) gives:

$$\frac{d^2M}{dx^2} = -q \tag{3.8}$$

Eq. (3.8) is the differential equation of the beam in terms of moment force. But since our FEM discussion focuses on displacement-based formulation, we need to express the ODE in terms of displacement. This can be done by employing the constitutive equation which relates the curvature of the beam with the moment force given as:

$$EI\frac{d^2w}{dx^2} = -M \tag{3.9}$$

where E is the Young's modulus of the material and I is the second moment of area of the beam's cross-section. Together, they are known as the flexural stiffness of the beam, EI.

By differentiating Eq. (3.9) twice, and inserting into Eq. (3.8), we obtain:

$$EI\frac{d^4w}{dx^4} = q \tag{3.10}$$

Eq. (3.10) is the ODE of the beam in terms of displacement variable as required. Like any other differential equations, Eq. (3.10) must be complemented by boundary condition equations. There are two types of boundary conditions ; force (natural) and displacement (essential).

The force (natural) boundary conditions of the beam can be a single or a combination of the following:

$$EI\left.\frac{d^3w}{dx^3}\right|_{x=0} = V_0 \tag{3.11a}$$

$$EI\left.\frac{d^3w}{dx^3}\right|_{x=L} = -V_L \tag{3.11b}$$

$$EI\left.\frac{d^2w}{dx^2}\right|_{x=0} = M_0 \tag{3.11c}$$

$$EI\left.\frac{d^2w}{dx^2}\right|_{x=L} = -M_L \tag{3.11d}$$

where V_0, V_L are the specified end shear forces and M_0, M_L are the specified end moment forces. Note that the sign convention above is based on the assumption that the boundary conditions are the reactions at support hence the opposite directions to the internal forces.

On the other hand, the displacement (essential) boundary conditions of the beam can be a single or a combination of the following:

$$\left.\frac{dw(x)}{dx}\right|_{x=0} = \theta_0 \tag{3.12a}$$

$$\left.\frac{dw(x)}{dx}\right|_{x=L} = \theta_L \tag{3.12b}$$

$$w(x)|_{x=0} = w_0 \tag{3.12c}$$

$$w(x)|_{x=L} = w_L \tag{3.12d}$$

where θ_0, θ_L are the specified end rotations and w_0, w_L are the specified end transverse displacements.

Since the beam ODE is of 4^{th} order respectively, four boundary conditions out of the six given above (Eqns. (3.11) and (3.12)) must be known in prior so as to have a well-posed problem.

Having established the beam ODE, we are now all set to discuss its discretization by FEM. We begin with the derivation the shape functions and the beam's degree of freedoms.

3.3 Degree of Freedom and Shape Functions for Beam Element

Fig. 3.2 shows a beam element with two end nodes. At each node, two degree of freedoms are introduced; a translational dof and a rotational dof. However to avoid the use of separate symbols for two types of dof (e.g. w, θ), d_i is used as the notation system for dof, as shown in Fig. 3.2. Note the sign conventions. Directions of the arrows shown in Fig. 3.2 are taken as positive (+).

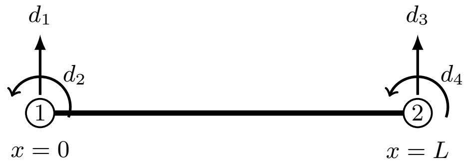

Figure 3.2: Degree of freedoms of beam element.

We start the derivation of the shape function by assuming an interpolation function for the transverse displacement, $w(x)$ of the beam in the forms of polynomial. Thus;

$$w(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \tag{3.13}$$

As frequently mentioned, it is preferable in finite element formulation, to deal with a special set of functions, known as shape functions, N_i and a special sets of coefficient, known as degree of freedoms (dofs), d_i. This means, Eq. (3.13) must be equivalently expressed as:

$$w(x) = N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4 \tag{3.14}$$

The above equations can be expressed compactly in component forms as:

$$w(x) = N_i d_i \quad \text{for } i = 1, 2, 3, 4 \tag{3.15}$$

Being referred as the degree of freedoms, d_1, d_2, d_3 and d_4 in Eq. (3.14) (or d_i in Eq. (3.15), are actually the nodal values of the transverse displacement, $w(x)$ as graphically described in Fig. 3.2.

The conversion from Eq. (3.14) into Eq. (3.15), hence the derivation of the shape functions, N_i can be done as follows:

1. Evaluate Eq. (3.13) and its first derivative at the location of the nodes

(i.e. at both ends, $x = 0$, $x = L$) and equate them to the dof to give:

$$
\begin{aligned}
w(x)|_{x=0} &= a_1 = d_1 \\
\left.\frac{dw(x)}{dx}\right|_{x=0} &= a_2 = d_2 \\
w(x)|_{x=L} &= a_1 + a_2 L + a_3 L^2 + a_4 L^3 = d_3 \\
\left.\frac{dw(x)}{dx}\right|_{x=L} &= a_2 + 2a_3 L + 3a_4 L^2 = d_4
\end{aligned}
\tag{3.16}
$$

2. Solving for the values of a_1, a_2, a_3 and a_4 then gives:

$$
\begin{aligned}
a_1 &= d_1 \\
a_2 &= d_2 \\
a_3 &= -\frac{2d_2 L + 3d_1 - 3d_3 + Ld_4}{L^2} \\
a_4 &= \frac{d_2 L + 2d_1 - 2d_3 + Ld_4}{L^3}
\end{aligned}
\tag{3.17}
$$

3. Inserting Eq. (3.17) into Eq. (3.13) and by factorizing give:

$$
\begin{aligned}
w(x) = \quad & \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right) d_1 \\
& + \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right) d_2 \\
& + \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right) d_3 \\
& + \left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right) d_4
\end{aligned}
\tag{3.18}
$$

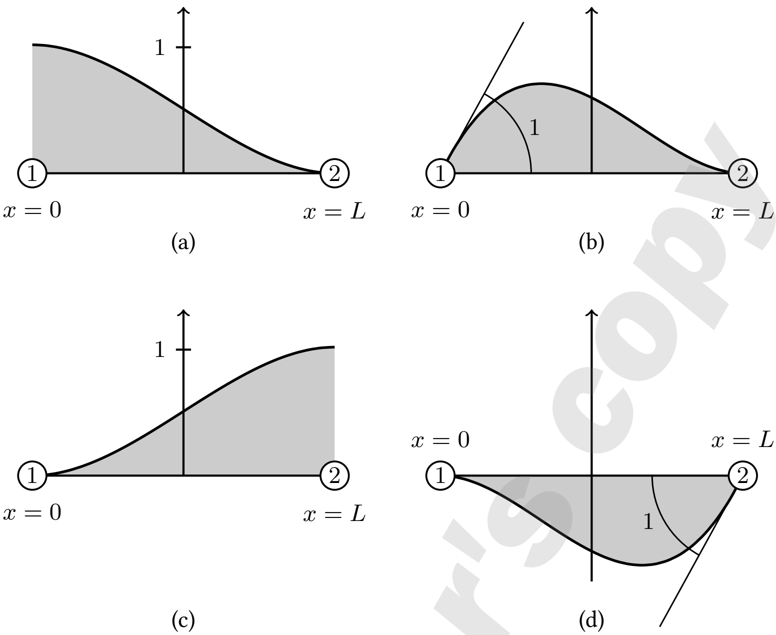

Figure 3.3: Hermite shape functions of beam element, (a)N_1, (b)N_2, (c)N_3, (d)N_4. The distance, x is measured from node 1.

4. By comparing Eq. (3.18) with Eq. (3.14), it can be concluded that:

$$
\begin{aligned}
N_1 &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\
N_2 &= x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\
N_3 &= \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\
N_4 &= -\frac{x^2}{L} + \frac{x^3}{L^2}
\end{aligned}
\tag{3.19}
$$

The shape functions are known as Hermite shape functions and they take the shapes as shown in Fig. 3.3. Take note of the shapes as they will be referred later in the discussion of integration by parts and the resulting boundary terms.

3.4 Discretization by Galerkin Method

Having established the shape functions and the degree of freedoms of the beam element, we are all set to proceed with the discretization of the ODE.

By inserting Eq. (3.14) into Eq. (3.10) gives:

$$EI\frac{d^4(N_1d_1 + N_2d_2 + N_3d_3 + N_4d_4)}{dx^4} \neq q \tag{3.20}$$

Take note that, from now on, we are going to deal with the component forms as this would eliminate the need to write (hence to read) long statement, thus Eq. (3.20) can be given in component forms as:

$$EI\frac{d^4(N_jd_j)}{dx^4} - q \neq 0 \quad \text{for } j = 1,2,3,4 \tag{3.21}$$

Since we have j number of unknowns (because we want to determine the values of the dofs, u_j) we thus need j number of simultaneous equation. In our present discussion, so we require four (4) simultaneous equations. These required equations can be obtained by multiplying Eq. (3.21) subsequently with a set of weight functions.

As mentioned before, since our discussion focuses on Bubnov-Galerkin formulation, the weight functions are also the shape functions, thus:

$$\begin{aligned}
N_1\left(EI\frac{d^4(N_jd_j)}{dx^4} - q\right) &\neq 0 \\
N_2\left(EI\frac{d^4(N_jd_j)}{dx^4} - q\right) &\neq 0 \\
N_3\left(EI\frac{d^4(N_jd_j)}{dx^4} - q\right) &\neq 0 \\
N_4\left(EI\frac{d^4(N_jd_j)}{dx^4} - q\right) &\neq 0
\end{aligned} \tag{3.22}$$

By observing Eq. (3.22) it can be deduced that, in component forms, the required j simultaneous equations can be given as:

$$N_i \left(EI \frac{d^4(N_j d_j)}{dx^4} - q \right) \neq 0 \quad \text{for } i = j = 1, 2, 3, 4 \tag{3.23}$$

The continuous nature of Eq. (3.23) (since it is still in ODE forms) can be converted into the equivalent algebraic forms by integrating over the length of the beam. This action is referred to as 'weighting the residual'; an action from which an approximate solution can be obtained by forcing the residual to zero. The algebraic forms of Eq. (3.23) are thus given as:

$$\begin{aligned} &\int_0^L N_i \left(EI \frac{d^4(N_j d_j)}{dx^4} - q \right) dx = 0 \\ \Rightarrow &\int_0^L N_i EI \frac{d^4(N_j d_j)}{dx^4}\, dx - \int_0^L N_i q\, dx = 0 \end{aligned} \tag{3.24}$$

It can be seen that Eq. (3.24) involves the integration of an inner product of functions, thus integration by parts (IBP) can be employed. The reasons for employing the IBP, as mentioned earlier in this chapter, are:

i. to reduce the continuity requirement since only second order derivatives will be dealt with
ii. to induce (make explicit) the natural boundary conditions as boundary terms resulting from the IBP

Employing the first IBP to Eq. (3.24) gives:

$$-\int_0^L \frac{dN_i}{dx} EI \frac{d^3 N_j}{dx^3}\, dx\, d_j = \int_0^L q N_i\, dx - N_i EI\, \left.\frac{d^3(N_j d_j)}{dx^3}\right|_0^L \tag{3.25}$$

In Eq. (3.25) above, the dofs, d_j is taken out from the integral sign because it represents constants thus does not involve in both differentiation and integration. The last terms on the right hand side of Eq. (3.25), are called boundary terms resulting from the first IBP. A careful examination reveals that this boundary term is also the force (natural) boundary conditions given by Eqs. (3.11a) and (3.11b), which are specifically related to end shear force. Inserting these equations into Eq. (3.25) modifies the latter into:

$$
\begin{aligned}
&-\int_0^L \frac{dN_i}{dx} EI \frac{d^3 N_j}{dx^3}\, dx\, d_j \\
&\qquad = \int_0^L q N_i\, dx - \left(-\left. N_i\right|_{x=L} V_L - \left. N_i\right|_{x=0} V_0\right)
\end{aligned}
\tag{3.26}
$$

Next, we conduct IBP one more time on Eq. (3.26). But why we want to do this? The reasons are:

i. We want to optimize the continuity relaxation and since the order of derivatives between the terms in the right hand side integral is still "unbalanced"(i.e. $\frac{dN_i}{dx}$, $\frac{d^3 N_j}{dx^3}$), we have the choice to balance them by another IBP.
ii. Another IBP, as we are going to see, would induce another type of force (natural) boundary conditions of the beam which are the end moments.

By conducting another IBP on Eq. (3.26), we obtain:

$$
\begin{aligned}
&\int_0^L \frac{d^2 N_i}{dx^2} EI \frac{d^2 N_j}{dx^2}\, dx\, d_j \\
&\qquad = \int_0^L q N_i\, dx - \left(\left. N_i\right|_{x=L} V_L - \left. N_i\right|_{x=0} V_0\right) \\
&\qquad\quad + \left. \frac{dN_i}{dx} EI \frac{d^2 (N_j d_j)}{dx^2}\right|_0^L
\end{aligned}
\tag{3.27}
$$

The last terms on the right hand side of Eq. (3.27), are boundary terms resulting from the second IBP. Again, one can show that this boundary term is the force (natural) boundary conditions related to end moment forces given by Eqs. (3.11c) and (3.11d). Inserting these equations into Eq. (3.27) modifies the latter into:

$$\begin{aligned}\int_0^L \frac{d^2N_i}{dx^2} EI \frac{d^2N_j}{dx^2}\,dx\,d_j \\ &= \int_0^L qN_i\,dx - \left(N_i|_{x=L}\,V_L - \,N_i|_{x=0}\,V_0 \right) \\ &\quad + \left(\left.\frac{dN_i}{dx}\right|_{x=L} M_L - \left.\frac{dN_i}{dx}\right|_{x=0} M_0 \right)\end{aligned} \tag{3.28}$$

Eq. (3.28) is the final weak statement of the beam problem but the last two terms on the right hand side of the equation deserves further explanation. The prevalence of the natural/force boundary conditions (whether they remain within the equation or not) depends on the accompanying value of the shape functions when evaluated at the location of the respective node i.e. $N_i|_{x=L}$, $N_i|_{x=0}$, $\left.\frac{dN_i}{dx}\right|_{x=L}$, $\left.\frac{dN_i}{dx}\right|_{x=0}$. As can be seen in Fig. 3.3 the value of the shape functions and its derivatives at the node's location can be given as:

At $x = 0$

$$\begin{aligned}&N_1|_{x=0} = 1, \quad N_2|_{x=0} = 0, \quad N_3|_{x=0} = 0, \quad N_4|_{x=0} = 0, \\ &\left.\frac{dN_1}{dx}\right|_{x=0} = 0, \quad \left.\frac{dN_2}{dx}\right|_{x=0} = 1, \quad \left.\frac{dN_3}{dx}\right|_{x=0} = 0, \quad \left.\frac{dN_4}{dx}\right|_{x=0} = 0.\end{aligned} \tag{3.29}$$

At $x = L$

$$\begin{aligned}&N_1|_{x=L} = 0, \quad N_2|_{x=L} = 0, \quad N_3|_{x=L} = 1, \quad N_4|_{x=L} = 0, \\ &\left.\frac{dN_1}{dx}\right|_{x=L} = 0, \quad \left.\frac{dN_2}{dx}\right|_{x=L} = 0, \quad \left.\frac{dN_3}{dx}\right|_{x=L} = 0, \quad \left.\frac{dN_4}{dx}\right|_{x=L} = 1.\end{aligned} \tag{3.30}$$

Based on the above characteristic of shape functions, we can observe what remains on the right side of Eq. (3.30), say for $i = 4$:

$$\begin{aligned} \int_0^L \frac{d^2 N_4}{dx^2} EI \frac{d^2 N_j}{dx^2}\, dx\, d_j & \\ &= \int_0^L q N_4\, dx - \left(N_4|_{x=L} V_L - N_4|_{x=0} V_0 \right) \\ &\quad + \left(- \left.\frac{dN_4}{dx}\right|_{x=L} M_L - \left.\frac{dN_4}{dx}\right|_{x=0} M_0 \right) \end{aligned} \tag{3.31}$$

Employing Eq. (3.30) to Eq. (3.31), we obtained:

$$\begin{aligned} \int_0^L \frac{d^2 N_4}{dx^2} EI \frac{d^2 N_j}{dx^2}\, dx\, d_j & \\ &= \int_0^L q N_4\, dx - (0V_L - 0V_0) \quad + (-1M_L - 0M_0) \end{aligned} \tag{3.32}$$

Simplifying Eq. (3.32), it becomes:

$$\int_0^L \frac{d^2 N_4}{dx^2} EI \frac{d^2 N_j}{dx^2}\, dx\, d_j = \int_0^L q N_4\, dx - M_L \tag{3.33}$$

By observing the evolution of discussion for other values of i (i.e. $i = 1, 2$ or 3), the general formulation in the forms of Eq. (3.33) can thus be given as:

$$\int_0^L \frac{d^2 N_i}{dx^2} EI \frac{d^2 N_j}{dx^2}\, dx\, d_j = \int_0^L q N_i\, dx + b_i \tag{3.34}$$

where b_i refers to boundary terms or nodal loads. Eq. (3.34) completes the discretization of the beam's ODE by Galerkin Method.

3.5 Local Load Vector and Stiffness Matrix

Now, let's discuss the other term on the right hand side of Eq. (3.34), that is $\int_0^L q N_i\, dx$. Furthermore, let's call this term as equivalent nodal loads and

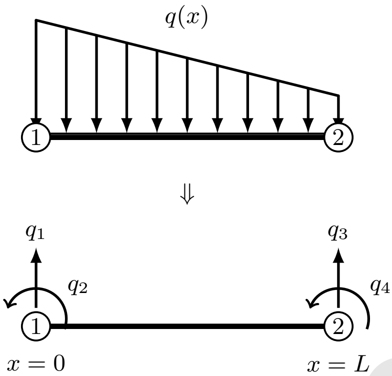

Figure 3.4: Equivalent nodal loads of beam element.

denote it as q_i, thus:

$$q_i = \int_0^L qN_i \, dx \tag{3.35}$$

It is termed as equivalent nodal loads because these loads are the equivalent effects of a load, q that is acting on the domain; effects as perceived at the nodes. To better describe such equivalence, let's refer to Fig. 3.4. As shown in the figure, the distributed loading, $q(x)$ which is acting on the beam domain would deform the beam in the same manner as if the beam is subjected to the four nodal loads, of which the values can be calculated using Eq. (3.35) as follows

$$q_1 = \int_0^L N_1 q(x) \, dx$$

$$q_2 = \int_0^L N_2 q(x) \, dx$$

$$q_3 = \int_0^L N_3 q(x) \, dx$$

$$q_4 = \int_0^L N_4 q(x) \, dx$$

Having defined Eq. (3.35), Eq. (3.34) can be rewritten as:

$$\int_0^L \frac{d^2 N_i}{dx^2} EI \frac{d^2 N_j}{dx^2}\, dx\, d_j = q_i + b_i \tag{3.36}$$

Eq. (3.36) has been derived for one particular bar element, thus it is called local equilibrium equation. For a more compact statement, Eq. (3.36) is usually written in component forms as:

$$k_{ij} d_j = r_i \tag{3.37}$$

where k_{ij} is termed as the local stiffness matrix of the beam element. d_j and r_i, on the other hand, are the local vector of dofs and the load vector of the beam element, respectively.

Thus, the stiffness matrix and the load vector of the beam element can be given as:

$$\begin{aligned} k_{ij} &= \int_0^L \frac{d^2 N_i}{dx^2} EI \frac{d^2 N_j}{dx^2}\, dx \\ r_i &= q_i + b_i \end{aligned} \tag{3.38}$$

Alternatively, in matrix forms, the local equilibrium can also be represented as:

$$[k]\{d\} = \{r\} \tag{3.39}$$

In an expanded matrix forms, the stiffness matrix of a beam element can be respectively given as:

$$
\begin{aligned}
\left[k\right] &= \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \\
&= \begin{bmatrix} \displaystyle\int_0^L \frac{d^2N_1}{dx^2} EI \frac{d^2N_1}{dx^2}\,dx & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ \displaystyle\int_0^L \frac{d^2N_4}{dx^2} EI \frac{d^2N_1}{dx^2}\,dx & k_{42} & k_{43} & k_{44} \end{bmatrix}
\end{aligned}
\tag{3.40}
$$

(Note: since the procedure is standard and repetitive, only elements k_{11} and k_{41} are expanded for demonstration).

The stiffness matrix which is given by Eq. (3.40) is still in integral forms. Generally, to carry out the integration, numerical integration is used due to the complexity. However, since Eq. (3.40) involves simple integration functions, direct integration is used herein. The main purpose is that, with direct (analytical) integration, the evolution of the procedure can easier be followed since all variables remained to be seen. For demonstration purposes, the integration of k_{12} and k_{34} are shown below:

$$
\begin{aligned}
k_{12} &= \int_0^L \frac{d^2N_1}{dx^2} EI \frac{d^2N_2}{dx^2}\,dx \\
&= \int_0^L \left(-\frac{6}{L^2} + \frac{12x}{L^3}\right) EI \left(\frac{6x}{L^2} + \frac{4}{L}\right)\,dx \\
&= \frac{6EI}{L^2}
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
k_{34} &= \int_0^L \frac{d^2 N_3}{dx^2} EI \frac{d^2 N_4}{dx^2}\, dx \\
&= \int_0^L \left(-\frac{12x}{L^3} + \frac{6}{L^2}\right) EI \left(\frac{6x}{L^2} + \frac{2}{L}\right) dx \\
&= -\frac{6EI}{L^2}
\end{aligned}
\tag{3.42}
$$

The fully integrated stiffness matrix of the beam element is then given as:

$$
\left[k\right] = k_{ij} = \begin{bmatrix}
\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} & -\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} \\[2ex]
\dfrac{6EI}{L^2} & \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{2EI}{L} \\[2ex]
-\dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\[2ex]
\dfrac{6EI}{L^2} & \dfrac{2EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{4EI}{L}
\end{bmatrix}
\tag{3.43}
$$

The load vectors are respectively given as:

$$
\left\{r\right\} = r_i = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} q_1 + b_1 \\ q_2 + b_2 \\ q_3 + b_3 \\ q_4 + b_4 \end{Bmatrix} = \begin{Bmatrix} \int_0^L qN_1\, dx + b_1 \\ \int_0^L qN_2\, dx + b_2 \\ \int_0^L qN_3\, dx + b_3 \\ \int_0^L qN_4\, dx + b_4 \end{Bmatrix}
\tag{3.44}
$$

It is worth to note again that component forms and matrix forms will be used interchangeably in our forthcoming discussion.

For a uniform distributed load, the load vector can be given as: The load vectors are respectively given as:

$$\{r\} = r_i = \begin{Bmatrix} \dfrac{qL}{2} + b_1 \\ \dfrac{qL^2}{12} + b_2 \\ \dfrac{qL}{2} + b_3 \\ \dfrac{-qL^2}{12} + b_4 \end{Bmatrix} \tag{3.45}$$

3.6 Assembly of Elements: Global Stiffness Matrix and Load Vector

A body or a structure is meshed by many elements in a finite element analysis. The meshing requires the assembly of local stiffness matrix, $[k]$ and local load vector, $\{r\}$ of individual element to form the global stiffness matrix, $[K]$ and the global load vector, $\{R\}$. Although in practice, hundreds and thousands of elements might be assembled, herein, only the assembly of two bar elements is demonstrated for easy tracing of the procedure. Fig. 3.5 shows the two assembled beam elements together with the local and global numbering of nodes and dofs.

Based on the above and to distinguish different elements we introduce the following superscript notation:

i. The local stiffness matrix of element N is denoted as k_{ij}^N or $[k^N]$
ii. The local load vector of element N is denoted as r_i^N or $\{r^N\}$
iii. The vector of dofs of element N is denoted as d_j^N or $\{d^N\}$

Having introduced the new superscript system, the local equilibrium for each element can thus be given as:

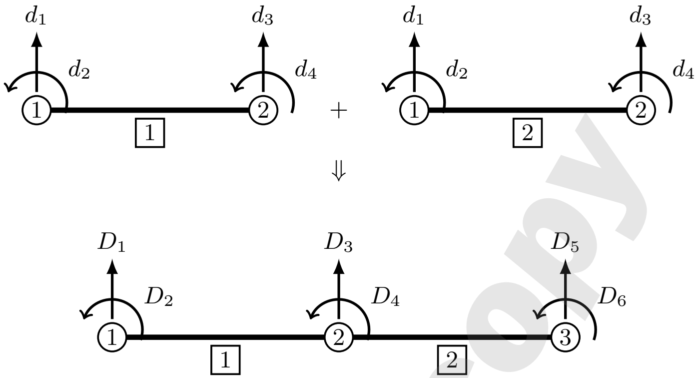

Figure 3.5: Local and global numbering of beam element.

For element 1

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 & k_{34}^1 \\ k_{41}^1 & k_{42}^1 & k_{43}^1 & k_{44}^1 \end{bmatrix} \begin{Bmatrix} d_1^1 \\ d_2^1 \\ d_3^1 \\ d_4^1 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 \\ r_4^1 \end{Bmatrix} \tag{3.46}$$

For element 2

$$\begin{bmatrix} k_{11}^2 & k_{12}^2 & k_{13}^2 & k_{14}^2 \\ k_{21}^2 & k_{22}^2 & k_{23}^2 & k_{24}^2 \\ k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\ k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 \end{bmatrix} \begin{Bmatrix} d_1^2 \\ d_2^2 \\ d_3^2 \\ d_4^2 \end{Bmatrix} = \begin{Bmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \\ r_4^2 \end{Bmatrix} \tag{3.47}$$

Having established the local matrices and their notation system, the assembly of the two elements hence the global equilibrium equation is given as:

$$
\begin{bmatrix}
k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 \\
k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^2 & 0 & 0 \\
k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 \\
k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 \\
0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\
0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2
\end{bmatrix}
\begin{Bmatrix}
d_1^1 \\ d_2^1 \\ d_3^1 = d_1^2 \\ d_4^1 = d_2^2 \\ d_3^2 \\ d_4^2
\end{Bmatrix}
= \begin{Bmatrix}
r_1^1 \\ r_2^1 \\ r_3^1 + r_1^2 \\ r_4^1 + r_2^2 \\ r_3^2 \\ r_4^2
\end{Bmatrix}
\tag{3.48}
$$

or in global numbering as

$$
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66}
\end{bmatrix}
\begin{Bmatrix}
D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6
\end{Bmatrix}
=
\begin{Bmatrix}
R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6
\end{Bmatrix}
\tag{3.49}
$$

where $\{D\}$ and $\{R\}$ are the global vectors of degree of freedoms and loads, respectively.

3.7 Imposition of Essential Boundary Conditions

Essential (or displacement) boundary conditions must be imposed before the equilibrium equation of the beam can be solved. Basic concept of direct imposition of boundary conditions has been described in detail in previous chapter.

Eq. (3.12) give all the possible essential boundary conditions of a beam, reproduced herein for ease of reading as:

$$\left.\frac{dw}{dx}(x)\right|_{x=0} = \theta_0 \tag{3.50a}$$

$$\left.\frac{dw}{dx}(x)\right|_{x=L} = \theta_L \tag{3.50b}$$

$$w(x)|_{x=0} = w_0 \tag{3.50c}$$

$$w(x)|_{x=L} = w_L \tag{3.50d}$$

Graphical representations of these boundary conditions are given in Fig. 3.6.

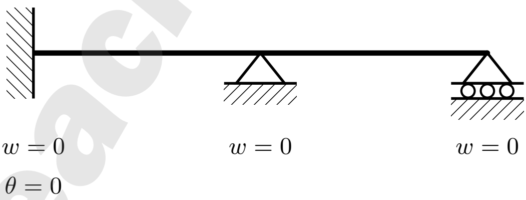

Figure 3.6: Various beam's essential boundary conditions.

3.8 Worked Example 3.1

Let's put what we have learned so far into practice. Fig. 3.7 shows a cantilever beam subjected to a distributed load of $2\,\mathrm{kN\,m^{-1}}$ and an edge point load of $-5\,\mathrm{kN}$.

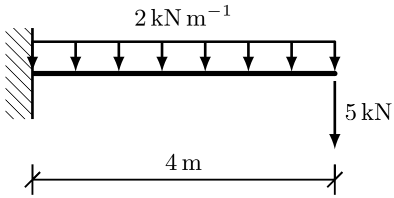

Figure 3.7: Beam with distributed and point loads. ($E = 200{\times}10^6\,\mathrm{kN\,m^{-1}}$ and $I = 1.333{\times}10^{-4}\,\mathrm{m^4}$).

An analytical solution for the beam can be obtained by conducting integration directly on the ODE given by Eq. (3.10) and satisfying the all the relevant boundary conditions (both force and displacement in Eqs. (3.11) and (3.12), which are:

$$EI\,\frac{d^3w}{dx^3}\bigg|_{x=4} = -V_L = -(-5\,\mathrm{kN}) \tag{3.51a}$$

$$EI\,\frac{d^2w}{dx^2}\bigg|_{x=4} = -M_L = 0 \tag{3.51b}$$

$$\frac{dw}{dx}(x)\bigg|_{x=0} = \theta_0 = 0 \tag{3.51c}$$

$$w(x)|_{x=0} = w_0 = 0 \tag{3.51d}$$

The analytical or the closed form solution of the beam can be given as:

$$w_{\text{exact}} = \frac{qx^4}{24EI} + \frac{(-P-qL)x^3}{6EI} - \frac{qL^2x^2}{4EI} - \frac{(-P-qL)Lx^2}{2EI} \tag{3.52}$$

Herein, we are going to solve the problem by the assembly of two beam elements of equal length and compare the results against Eq. (3.52) to validate the FEM formulation. Due to the symmetrically, element 1 and element 2 would have a similar local stiffness matrix thus:

$$k_{ij}^1 = k_{ij}^2 = \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 \\ 4.00 & 5.33 & -4.00 & 2.67 \\ -4.00 & -4.00 & 4.00 & -4.00 \\ 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \tag{3.53}$$

and the load vectors are given as:

$$r_i^1 = \begin{Bmatrix} -2 + b_1^1 \\ -0.67 + b_2^1 \\ -2 + 0 \\ 0.67 + 0 \end{Bmatrix} \tag{3.54}$$

$$r_i^2 = \begin{Bmatrix} -2 + 0 \\ -0.67 + 0 \\ -2 - 5 \\ 0.67 + 0 \end{Bmatrix} \tag{3.55}$$

The assembled global stiffness matrix and load vector can be given as (based on Eq. (3.48)):

$$[K] = \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 & 0 & 0 \\ 4.00 & 5.33 & -4.00 & 2.67 & 0 & 0 \\ -4.00 & -4.00 & 8.00 & 0 & -4.00 & 4.00 \\ 4.00 & 2.67 & 0 & 10.7 & -4.00 & 2.67 \\ 0 & 0 & -4.00 & -4.00 & 4.00 & -4.00 \\ 0 & 0 & 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \tag{3.56}$$

$$R = \begin{Bmatrix} -2 + b_1^1 \\ -0.67 + b_2^1 \\ -2 - 2 \\ 0.67 - 0.67 \\ -2 - 5 \\ 0.67 \end{Bmatrix} = \begin{Bmatrix} -2 + B_1 \\ -0.67 + B_2 \\ -4 \\ 0 \\ -7 \\ 0.67 \end{Bmatrix} \tag{3.57}$$

To emphasize the global reactions forces, b_1^1 and b_2^1 are expressed as B_1 and B_2, respectively. These are unknown because their corresponding dofs are the essential boundary conditions, which in turn, are known values. The whole equilibrium equation can thus be given as:

$$\begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 & 0 & 0 \\ 4.00 & 5.33 & -4.00 & 2.67 & 0 & 0 \\ -4.00 & -4.00 & 8.00 & 0 & -4.00 & 4.00 \\ 4.00 & 2.67 & 0 & 10.7 & -4.00 & 2.67 \\ 0 & 0 & -4.00 & -4.00 & 4.00 & -4.00 \\ 0 & 0 & 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \tag{3.58}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -2 + B_1 \\ -0.67 + B_2 \\ -4 \\ 0 \\ -7 \\ 0.67 \end{Bmatrix}$$

By imposing the essential boundary conditions (i.e. $D_1 = D_2 = 0$), Eq. (3.58) is reduced to:

$$\begin{bmatrix} 8.00 & 0 & -4.00 & 4.00 \\ 0 & 10.7 & -4.00 & 2.67 \\ -4.00 & -4.00 & 4.00 & -4.00 \\ 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -4 \\ 0 \\ -7 \\ 0.67 \end{Bmatrix} \tag{3.59}$$

By solving Eq. (3.59) using Matlab command “\”, the values of the assembled global dof D_i are thus obtained as:

$$\begin{Bmatrix} 0 \\ 0 \\ -2.10\times10^{-3} \\ -1.83\times10^{-3} \\ -6.40\times10^{-3} \\ -2.30\times10^{-3} \end{Bmatrix} \tag{3.60}$$

Eq. (3.60) is the assembled global solution. By inserting these values accordingly into Eq. (3.39) the local forces can then be determined. However, before that, let's determine first the global reaction forces, B_1 and B_2 which can be done by inserting Eq. (3.60) back into Eq. (3.58), which yields:

$$\begin{Bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 & 0 & 0 \\ 4.00 & 5.33 & -4.00 & 2.67 & 0 & 0 \\ -4.00 & -4.00 & 8.00 & 0 & -4.00 & 4.00 \\ 4.00 & 2.67 & 0 & 10.7 & -4.00 & 2.67 \\ 0 & 0 & -4.00 & -4.00 & 4.00 & -4.00 \\ 0 & 0 & 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4$$

$$\begin{Bmatrix} 0 \\ 0 \\ -2.10\times10^{-3} \\ -1.83\times10^{-3} \\ -6.40\times10^{-3} \\ -2.30\times10^{-3} \end{Bmatrix} - \begin{Bmatrix} -2 \\ -0.67 \\ -4 \\ 0 \\ -7 \\ 0.67 \end{Bmatrix} \tag{3.61}$$

The global reaction forces can thus be given as

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 36 \end{Bmatrix} \tag{3.62}$$

Now, let's proceed with determination of the local values. The elemental local solution for each element is thus:

$$\left\{d^1\right\} = \begin{Bmatrix} d_1^1 \\ d_2^1 \\ d_3^1 \\ d_4^1 \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -2.10\times10^{-3} \\ -1.83\times10^{-3} \end{Bmatrix} \tag{3.63}$$

$$\left\{d^2\right\} = \begin{Bmatrix} d_1^2 \\ d_2^2 \\ d_3^2 \\ d_4^2 \end{Bmatrix} = \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -2.10\times10^{-3} \\ -1.83\times10^{-3} \\ -6.40\times10^{-3} \\ -2.30\times10^{-3} \end{Bmatrix} \tag{3.64}$$

Eqs. (3.63) and (3.64) are the elemental local solutions. Next, we determine the local forces.

By inserting Eqs. (3.63) and (3.64) into Eq. (3.39), the local forces can be determined as follows.

Element 1

$$\{b^1\} = [k^1]\{d^1\} - \{q^1\}$$

$$= \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 \\ 4.00 & 5.33 & -4.00 & 2.67 \\ -4.00 & -4.00 & 4.00 & -4.00 \\ 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \begin{Bmatrix} 0 \\ 0 \\ -2.10\times10^{-3} \\ -1.83\times10^{-3} \end{Bmatrix} - \begin{Bmatrix} -2 \\ -0.67 \\ -2 \\ 0.67 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 36 \\ -9 \\ -14 \end{Bmatrix} \tag{3.65}$$

Element 2

$$
\begin{aligned}
\left\{b^2\right\} &= \left[k^2\right]\left\{d^2\right\} - \left\{q^2\right\} \\
&= \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 \\ 4.00 & 5.33 & -4.00 & 2.67 \\ -4.00 & -4.00 & 4.00 & -4.00 \\ 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \begin{Bmatrix} -2.10\times10^{-3} \\ -1.83\times10^{-3} \\ -6.40\times10^{-3} \\ -2.30\times10^{-3} \end{Bmatrix} \\
&\quad - \begin{Bmatrix} -2 \\ -0.67 \\ -2 \\ 0.67 \end{Bmatrix} \\
&= \begin{Bmatrix} 9 \\ 14 \\ -5 \\ 0 \end{Bmatrix}
\end{aligned} \tag{3.66}
$$

The validity of the formulation is assessed by comparing its result against the exact solution as given by Eq. (3.52). Whilst the reactions can also be obtained analytically, herein reactions obtained from commercial software are used for validation so as to have diversity in data sources. The plot of FEM results against the exact solution (Eq. (3.52)) is shown in Fig. 3.8 which numerical values at several locations along the beam are given in Table 3.1. Table 3.2 gives the comparison of the reaction forces at support.

Table 3.1: Comparison of deflection values (in m) between w_{exact} and w_{FEM}

x	0 m	1 m	2 m	3 m	4 m
w_{exact}	0	-6.0×10^{-4}	-2.1×10^{-3}	-4.1×10^{-3}	-6.4×10^{-3}
w_{FEM}	0	-	-2.1×10^{-3}	-	-6.4×10^{-3}

Based on the plot and the tables, it can be seen, the assembled two beam elements provide quite a poor approximation except at the location of the

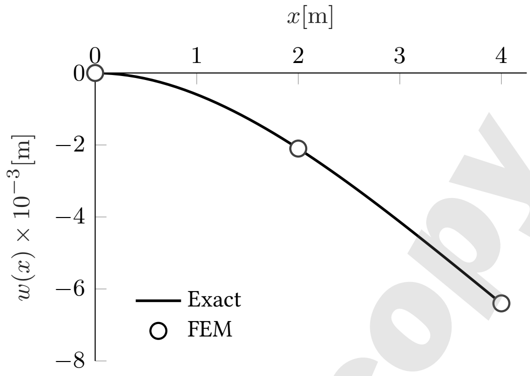

Figure 3.8: Plot of w_{exact} and w_{FEM}.

Table 3.2: Comparison of values of reaction forces (in kN) at fix support.

	B_1	B_2
w_{FEM}	13	36
Software (converged)	13	36

nodes where the results agree with the closed-form solution and the commercial software. A better hence a converged solution is provided by the use of more elements. Fig. 3.9, Table 3.3 and also Table 3.4 show the convergence if four elements are used.

Table 3.3: Comparison of deflection values (in m) between $w_{\text{FEM-2 elements}}$, $w_{\text{FEM-4 elements}}$ and w_{exact}

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
w_{exact}	0	-6.0×10^{-4}	-2.1×10^{-3}	-4.1×10^{-3}	-6.4×10^{-3}
$w_{\text{FEM-2 elements}}$	0	-	-2.1×10^{-3}	-	-6.4×10^{-3}
$w_{\text{FEM-4 elements}}$	0	-6.0×10^{-4}	-2.1×10^{-3}	-4.1×10^{-3}	-6.4×10^{-3}

The convergence of the results above is an immediate demonstration of the beneficial effect of the use of higher order elements. This is an immediate demonstration to the statement: "*FEM converges to the 'accurate' solution*

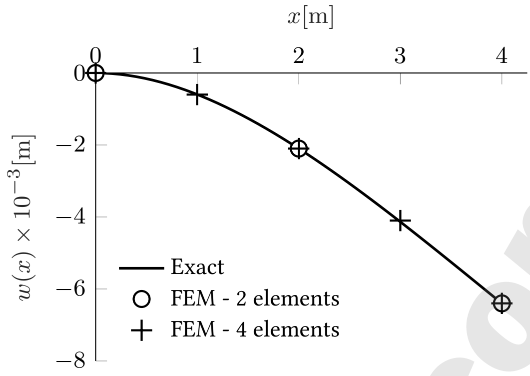

Figure 3.9: Plot of w_{FEM} and w_{exact}.

Table 3.4: Comparison of values of reaction forces (in kN) at fix support.

	B_1	B_2
$w_{\text{FEM-2 elements}}$	13	36
$w_{\text{FEM-4 elements}}$	13	36
Software (converged)	13	36

with the increase in the order of polynomial of the guessed functions (or degree of freedom and mesh density later on)" which was stated in previous chapter.

3.9 Matlab Source Codes

```
% Clear data
clc; clear; close all

%Input
E = 200e6;      % Young's modulus [kN/m]
I = 1.333e-4;   % Moment of inertia [m^4]
L = 4;          % Bar length [m]
P = -5;         % Point load [kN]
q = -2;         % Distributed load [kN/m]


% ---------------------------------------------
% FEM solution - Displacement
```

```
% -------------------------------------------

% Elements length
L1 = L/2; L2 = L/2;

% Matrix k & force r
k1 = [12*E*I/L1^3  6*E*I/L1^2 -12*E*I/L1^3  6*E*I/L1^2;
       6*E*I/L1^2  4*E*I/L1    -6*E*I/L1^2  2*E*I/L1;
     -12*E*I/L1^3 -6*E*I/L1^2  12*E*I/L1^3 -6*E*I/L1^2;
       6*E*I/L1^2  2*E*I/L1    -6*E*I/L1^2  4*E*I/L1];
k2 = [12*E*I/L1^3  6*E*I/L1^2 -12*E*I/L1^3  6*E*I/L1^2;
       6*E*I/L1^2  4*E*I/L1    -6*E*I/L1^2  2*E*I/L1;
     -12*E*I/L1^3 -6*E*I/L1^2  12*E*I/L1^3 -6*E*I/L1^2;
       6*E*I/L1^2  2*E*I/L1    -6*E*I/L1^2  4*E*I/L1];

r1 = [q*L1/2; q*L1^2/12; q*L1/2; -q*L1^2/12];
r2 = [q*L2/2; q*L2^2/12; q*L2/2; -q*L2^2/12];

% Assemble global matrix, K and vector, R
K = zeros(6);
K(1:4,1:4) = K(1:4,1:4) + k1;
K(3:6,3:6) = K(3:6,3:6) + k2;

R = zeros(6,1);
R(1:4) = R(1:4) + r1;
R(3:6) = R(3:6) + r1;

% Point load, P
R(5) = R(5) + P;

% Solve for global displacement
D      = zeros(6,1);
D(3:6) = K(3:6,3:6)\R(3:6);

% -------------------------------------------
% FEM solution - Internal reaction, b
% -------------------------------------------

% Internal force or reaction
b1 = k1*[D(1); D(2); D(3); D(4)] - r1;
b2 = k2*[D(3); D(4); D(5); D(6)] - r2;
```

3.10 Exercises

1. The displacement function for beam element as shown in Fig. 3.10(a) is assumed as:

$$v = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

   i. Explain the steps that need to be taken to determine the values of a_1, a_2, a_3 and a_4.
   ii. If a_1, a_2, a_3 and a_4 are as given below, construct the equation

for the shape functions, N_1, N_2, N_3 and N_4.

$$a_1 = q_1$$

$$a_2 = q_2$$

$$a_3 = -\frac{2q_2L + 3q_1 - 3q_3 + Lq_4}{L^2}$$

$$a_4 = \frac{q_2L + 2q_1 - 2q_3 + Lq_4}{L^3}$$

iii. Calculate the value of equivalent moments, M_1 and M_2 for the beam due to a partial uniform distributed load (UDL) of $10\,\mathrm{kN\,m^{-1}}$ as shown in Fig. 3.10(b). (Hint: Use the shape functions obtained in ii. above).

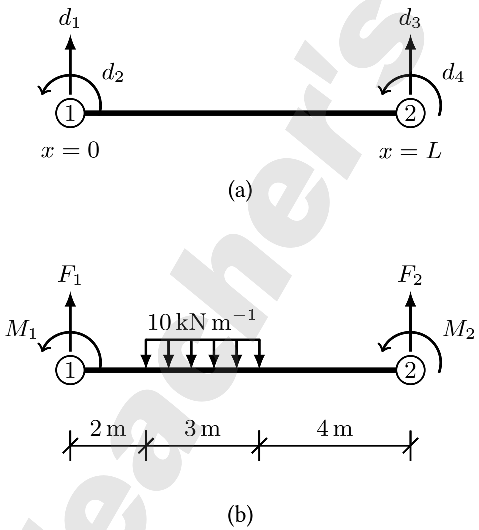

Figure 3.10

2. The beam structure shown in Fig. 3.11 is subjected to concentrated force of $8\,\mathrm{kN}$ and uniform distributed load of $12.5\,\mathrm{kN\,m^{-1}}$. The beam is made of timber with $20\,\mathrm{kN\,mm^{-2}}$. By using two beam elements, determine:

   i. Deflection and rotations at nodes 2 and 3,

ii. Reaction forces at the supports and bending moment at node 2. (Take $I = 2.1\times10^{7}\,\mathrm{mm}^4$)

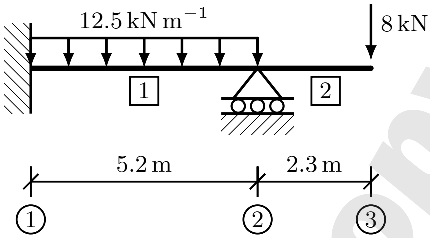

Figure 3.11

3. A simply supported beam loaded with a uniformly distributed load of $28.5\,\mathrm{kN\,m^{-1}}$ and a moment of $5\,\mathrm{kN\,m}$ at the left end is shown in Fig. 3.12. The Young's modulus and the moment of inertia of the beam are $E = 200\,\mathrm{GPa}$ and $I = 1\times10^{-4}\,\mathrm{m}^4$ respectively.

   i. Using one beam element, construct a system model and solve for the nodal rotations.
   ii. Then using two beam elements, reconstruct a system model and solve for the nodal rotations.
   iii. Compare the results and explain the differences (if any).

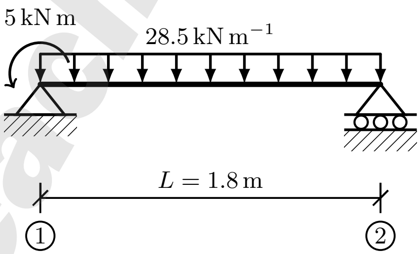

Figure 3.12

4. A beam shown in Fig. 3.13 is having fixed support at left end and roller in vertical direction at right end. The beam is loaded with a uniformly distributed load of $25.4\,\mathrm{kN\,m^{-1}}$. The Young's modulus and the moment of inertia of the beam are $E = 70\,\mathrm{GPa}$ and $I = 600\,\mathrm{mm}^4$ respectively. By considering two beam elements, determine:

   i. the vertical displacement at the right end.

ii. the bending moment at the two ends of the beam.

If the beam was analysed using more than two beam elements, will the nodal displacement results will be improved or similar? Discuss your answer.

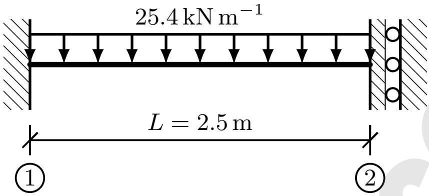

Figure 3.13

5. A simply supported beam is subjected to a quadratic distributed load, $q(x)$ as shown in Fig. 3.14. The length of the beam is $3\,\mathrm{m}$ whilst the Young's modulus and the moment of inertia are $E = 185\,\mathrm{GPa}$ and $I = 1\times10^{-4}\,\mathrm{m}^4$ respectively.

   i. Determine the numerical values for the equivalent nodal load vector, $\{R\}$ where the general formulation is given as:

   $$\{R\} = \int_0^L N_i q(x)\,dx$$

   with the shape functions are given as:

   $$\begin{aligned} N_1 &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ N_2 &= x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ N_3 &= \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ N_4 &= -\frac{x^2}{L} + \frac{x^3}{L^2} \end{aligned} \tag{3.67}$$

   ii. By taking the value of the flexural stiffness of the beam and its length as shown in the figure, determine the numerical values of the degree of freedom vector, $\{D\}$ where $[K]\{D\} = \{R\}$.

iii. If the beam is subjected to an additional applied moment at point 1, $M_1 = -10\,\mathrm{kN\,m}$, determine the new values of the degree of freedom vector, $\{D\}$ where $[K]\{D\} = \{R_{\mathrm{new}}\}$. Note that, $\{R_{\mathrm{new}}\}$ is now the total effect of distributed loads, $\{R\}$ and point loads, $\{B\}$ so that $\{R_{\mathrm{new}}\} = \{R\} + \{B\}$.

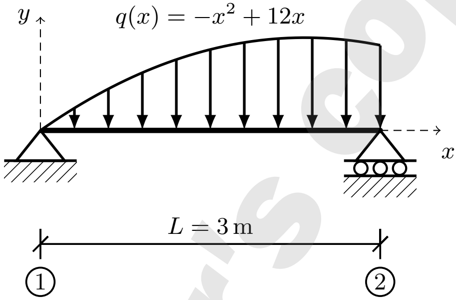

Figure 3.14

6. The ordinary differential equation (ODE) of beam can be given as:

$$EI\frac{d^4w}{dx^4} = q \tag{3.68}$$

with the boundary conditions:

$$EI\left.\frac{d^3w}{dx^3}\right|_{x=L} = -V_L = P \tag{3.69a}$$

$$EI\left.\frac{d^2w}{dx^2}\right|_{x=L} = -M_L = 0 \tag{3.69b}$$

$$\left.\frac{dw}{dx}(x)\right|_{x=0} = \theta_0 = 0 \tag{3.69c}$$

$$w(x)|_{x=0} = w_0 \tag{3.69d}$$

where E is the Young's Modulus, I is the second moment of area, $w(x)$ is the beam deflection, L is the beam length, P and q is the external forces. By considering the Finite Element Method, provide the solution for the deflection of the beam, $w(x)$ in terms of the values of the solved degree of freedoms.

4 Plane Structures: Truss and Frame

4.1 Introduction

In the previous chapters, we have discussed the derivation of FEM formulation for bar and beam elements. However, these elements were arranged and assembled in a line. A more general arrangement would require the elements to be arbitrarily oriented and assembled. Assembly of such oriented elements would make up a truss system and a frame system, respectively. As we are going to see, the orientation process requires the introduction of the transformation matrix. The use of this matrix is to transform local entities into global entities. Another point to emphasize is the introduction of two degree of freedoms and two load components into the bar element's global representation. Also, since the construction of a frame would require the transfer of axial load/force, a beam element is supplied with extra degree of freedoms in the axial direction. As can be seen, this will involve the combination of previously derived bar and beam elements to form what is called beam-column element.

4.2 Truss System

A truss system is an assembly of inclined bar elements. Figure 4.1 shows a typical arrangement of a plane truss system.

To allow for the inclination of the truss members and the corresponding assembly, a bar element formulation must be re-expressed in a global manner. In a global axis, a bar element would have two degree of freedoms per node. Such a transformation requires us to establish what is known as a

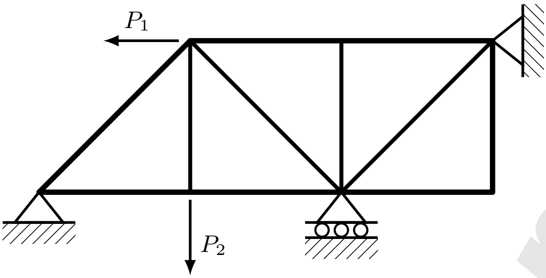

Figure 4.1: A typical truss system.

transformation matrix as discussed next.

4.2.1 Bar Transformation Matrix

Consider an inclined bar as shown in Fig. 4.2, together with the newly introduced elemental global direction dofs and previously defined local dofs. Note that, to distinguish between local and elemental global direction dofs, the former is primed.

For a linear bar (two-nodes bar), by considering the geometry of Fig. 4.2 the relationship between the local and the elemental global dofs can be given as:

$$u_1' = u_1 \cos\beta + u_2 \sin\beta \tag{4.1a}$$

$$u_2' = u_3 \cos\beta + u_4 \sin\beta \tag{4.1b}$$

In matrix forms, Eq. (4.1a) can be given as:

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{4.2}$$

or

$$\{u'\} = [T]\,\{u\} \tag{4.3}$$

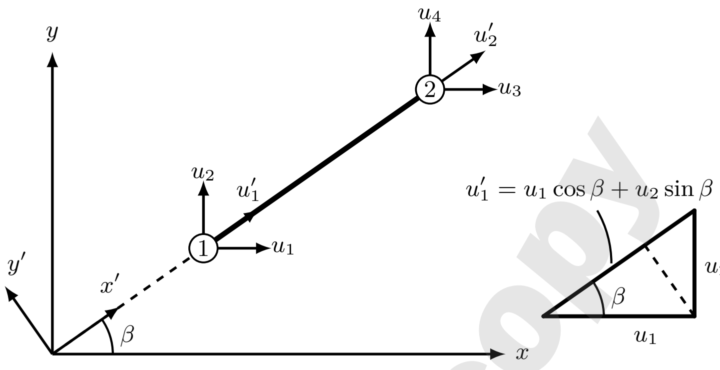

Figure 4.2: Degree of freedoms of inclined bar element

where $\{u'\}$ is the vector of local dofs, $[T]$ is from now on is called the bar transformation matrix and $\{u\}$ is the vector of the elemental global dofs of the bar element.

To simplify the expression, $[T]$s is expressed as:

$$\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \tag{4.4}$$

where

$$c = \cos\beta$$

$$s = \sin\beta$$

So far, we have established the relationship between local dofs of bar and its elemental global dofs. Now we are going to establish the relationship between local forces and the elemental global forces. By referring to Fig. 4.3, the relationship can be given as:

$$r_1 = r_1' \cos\beta \tag{4.5a}$$

$$r_2 = r_1' \sin\beta \tag{4.5b}$$

$$r_3 = r_2' \cos\beta \tag{4.5c}$$

$$r_4 = r_2' \sin\beta \tag{4.5d}$$

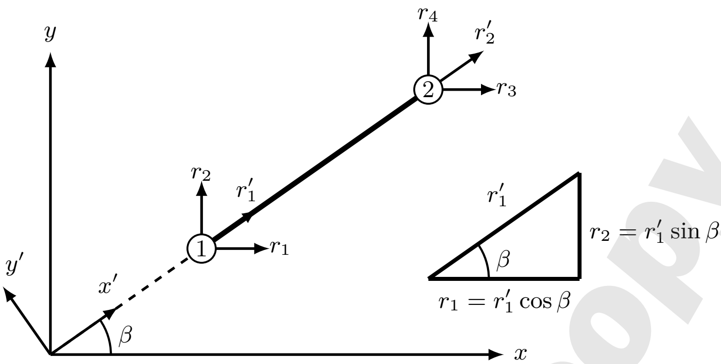

Figure 4.3: Nodal forces of Inclined bar element

In matrix forms Eq. (4.5) can be given as:

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{bmatrix} \cos\beta & 0 \\ \sin\beta & 0 \\ 0 & \cos\beta \\ 0 & \sin\beta \end{bmatrix} \begin{Bmatrix} r_1' \\ r_2' \end{Bmatrix} \tag{4.6}$$

A comparison with Eqs. (4.2) and (4.3) would show that the 4 by 2 matrix on the right hand side of Eq. (4.6) is actually the transpose of the transformation matrix, $[T]^T$. Realizing this, Eq. (4.6) can be given as:

$$\{r\} = [T]^T \{r'\} \tag{4.7}$$

where $\{r\}$ is the vector of elemental global forces and $\{r'\}$ is the vector of the local forces of the bar. Eq. (4.7) is therefore the relationship between global and local forces of an element.

Having established the relationship between elemental global and local entities for both dofs and forces, we are now in the position to establish the elemental global equilibrium equation for the inclined bar element from its local equation. First, we write the local equation of the bar (previously given as Eq. (2.38) but with the prime superscript to distinguish them from their corresponding elemental global entities):

$$[k'] \{u'\} = \{r'\} \tag{4.8}$$

Note that, as given in Eq. (2.37), $\{r'\} = \{q' + b'\}$. Inserting Eq. (4.3) into Eq. (4.8) would then give:

$$\left[k'\right]\left[T\right]\left\{u\right\} = \left\{r'\right\} \tag{4.9}$$

Now, we multiply the transpose of transformation matrix, $[T]^T$ to both sides of Eq. (4.9) to obtain:

$$\left[T\right]^T\left[k'\right]\left[T\right]\left\{u\right\} = \left[T\right]^T\left\{r'\right\} \tag{4.10}$$

Observing Eq. (4.10) we should notice that the right hand side terms is also the previously given Eq. (4.7). So, by inserting Eq. (4.7) into Eq. (4.10), we obtain:

$$\left[T\right]^T\left[k'\right]\left[T\right]\left\{u\right\} = \left\{r\right\} \tag{4.11}$$

Based on Eq. (4.11) we can conclude that the elemental global stiffness matrix of the bar, $[k]$ can be defined as:

$$\left[k\right] = \left[T\right]^T\left[k'\right]\left[T\right] \tag{4.12}$$

which for a linear bar element, $[k]$ can explicitly be given as:

$$\begin{aligned}\left[k\right] &= \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \\ &= \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}\end{aligned} \tag{4.13}$$

Our discussion on the transformation of discretised equilibrium equation from local to element global condition is completed by considering Eq. (4.12) in Eq. (4.11) to obtain:

$$\left[k\right]\left\{u\right\} = \left\{r\right\} \tag{4.14}$$

Once the elemental global equilibrium equation is established, the assembly, the imposition of essential boundary conditions and the solution process follow the same as outlined previously. However once a solution is obtained, the corresponding local values becomes of interest. The determination of the internal axial forces, $\{r'\}$ in the bar requires the solved dofs, $\{u\}$ to be inserted back into Eq. (4.9), and the axial displacement, $\{u'\}$, on the other hand, can be obtained by inserting $\{u\}$ into Eq. (4.3). However before such a procedure can be executed, the truss elements must be assembled first.

4.2.2 Assembly of Truss System

What we have derived so far are the elemental global entities. To solve the problem, these entities must be assembled. The discussion on the assembly of a truss system can be best done by considering a typical arrangement as shown in Fig. 4.4

Fig. 4.4 shows both elemental global dofs $\{u\} = \{u_1 \ u_2 \ u_3 \ u_4\}$ for each element and the assembled global $\{U\} = \{U_1 \ U_2 \ U_3 \ U_4 \ U_5 \ U_6\}$ for the whole structure. The assembly of the elemental values into assembled matrix and vectors can be done by correlating the elemental numbering subscript to the assembled numbering. For example, by observing Fig. 4.4, we can correlate that, say, k^2_{31} of elemental 2 to K_{53} in the assembled matrix because it can be traced that elemental subscript 3 corresponds to assembled subscript 5 (i.e. $3 \rightarrow 5$) and elemental subscript 1 corresponds to assembled subscript 3 (i.e. $1 \rightarrow 3$). Based on this, the following is the correlation between elemental entities to assembled entities for typical truss system of Fig. 4.4:

For element 1

$$k^1_{11} \rightarrow K_{11} \quad k^1_{12} \rightarrow K_{12} \quad k^1_{13} \rightarrow K_{13} \quad k^1_{14} \rightarrow K_{14}$$

$$k^1_{21} \rightarrow K_{21} \quad k^1_{22} \rightarrow K_{22} \quad k^1_{23} \rightarrow K_{23} \quad k^1_{24} \rightarrow K_{24}$$

$$k^1_{31} \rightarrow K_{31} \quad k^1_{32} \rightarrow K_{32} \quad k^1_{33} \rightarrow K_{33} \quad k^1_{34} \rightarrow K_{34}$$

$$k^1_{41} \rightarrow K_{41} \quad k^1_{42} \rightarrow K_{42} \quad k^1_{43} \rightarrow K_{43} \quad k^1_{14} \rightarrow K_{44}$$

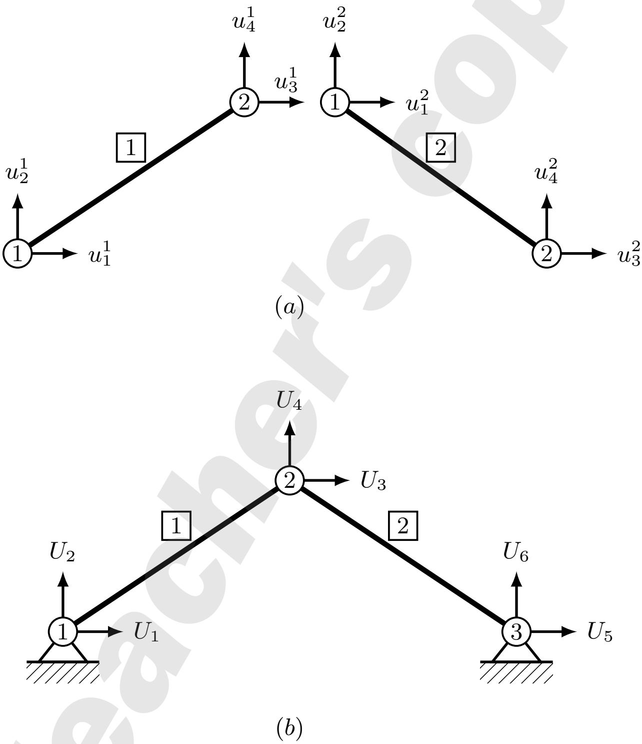

Figure 4.4: Assembly of a typical truss system from (a) local to (b) global.

Note that the correlation of element 1 seems to match perfectly with the assembled numbering because their numbering is counted from the same starting point. Such a direct correlation is not the case for element 2 as we can see next.

For element 2

$$
\begin{array}{llll}
k^2_{11} \rightarrow K_{33} & k^2_{12} \rightarrow K_{34} & k^2_{13} \rightarrow K_{35} & k^2_{14} \rightarrow K_{36} \\
k^2_{21} \rightarrow K_{43} & k^2_{22} \rightarrow K_{44} & k^2_{23} \rightarrow K_{45} & k^2_{24} \rightarrow K_{46} \\
k^2_{31} \rightarrow K_{53} & k^2_{32} \rightarrow K_{54} & k^2_{33} \rightarrow K_{55} & k^2_{34} \rightarrow K_{56} \\
k^2_{41} \rightarrow K_{63} & k^2_{42} \rightarrow K_{64} & k^2_{43} \rightarrow K_{65} & k^2_{44} \rightarrow K_{66}
\end{array}
$$

Having established the above correlation, we thus can build the assembled global stiffness matrix, K_{ij} as:

$$
\left[K\right] = K_{ij} = \begin{bmatrix}
k^1_{11} & k^1_{12} & k^1_{13} & k^1_{14} & 0 & 0 \\
k^1_{21} & k^1_{22} & k^1_{23} & k^1_{24} & 0 & 0 \\
k^1_{31} & k^1_{32} & k^1_{33}+k^2_{11} & k^1_{34}+k^2_{12} & k^2_{13} & k^2_{14} \\
k^1_{41} & k^1_{42} & k^1_{43}+k^2_{21} & k^1_{44}+k^2_{22} & k^2_{23} & k^2_{24} \\
0 & 0 & k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\
0 & 0 & k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44}
\end{bmatrix} \tag{4.15}
$$

Based on the same correlation argument, the assembled load vector, $\{R\}$ can thus be given as:

$$
\left\{R\right\} = R_i = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = \begin{Bmatrix} r^1_1 \\ r^1_2 \\ r^1_3 + r^2_1 \\ r^1_4 + r^2_2 \\ r^2_3 \\ r^2_4 \end{Bmatrix} \tag{4.16}
$$

Finally, we complete our discussion on truss system formulation by writing down the assembled global equilibrium equation as:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 \\ k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 \\ 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\ 0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 + r_1^2 \\ r_4^1 + r_2^2 \\ r_3^2 \\ r_4^2 \end{Bmatrix} \tag{4.17}$$

or

$$\left[K\right]\left\{U\right\} = \left\{R\right\} \tag{4.18}$$

By imposing the essential boundary conditions using the same concept as in Section 2.8, $\{U\}$ can then be solved.

Now, let's nail our understanding with an example and the corresponding Matlab source code.

4.2.3 Worked Example 4.1

In this example, we are required to determine the internal axial forces, reaction forces at supports and displacements of a truss system as shown in Fig. 4.5.

Based on the data given in the figure, the local stiffness matrix for each element, determined based on Eq. (2.41a), is given below. Note that, the external point load is inserted directly into the assembled global load vector once everything is assembled; an act which omits the requirement for it to be considered at a local level.

Element 1

$$\left[k'^1\right] = \begin{bmatrix} 2.00 & -2.00 \\ -2.00 & 2.00 \end{bmatrix} \times 10^6 \tag{4.19}$$

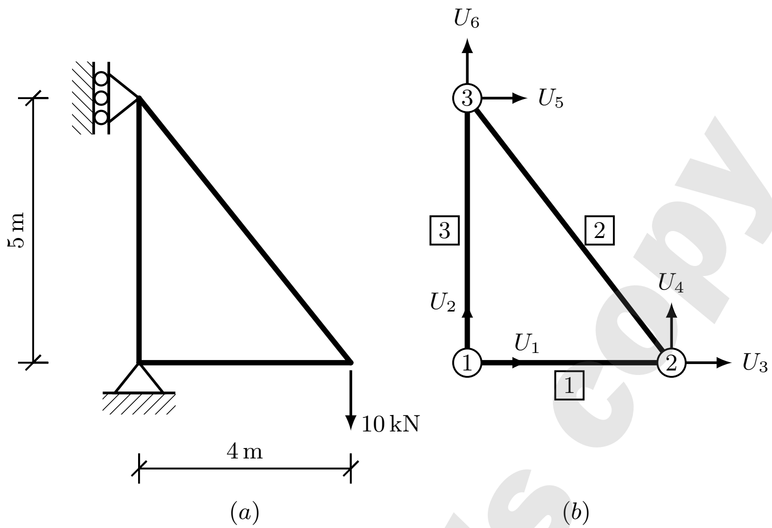

Figure 4.5: (a) Truss with single point load and (b) its elements. All elements have similar properties ($A = 0.04\,\mathrm{m}^2$, $E = 200\times10^6\,\mathrm{kN\,m^{-2}}$)

Element 2

$$\left[k'^2\right] = \begin{bmatrix} 1.25 & -1.25 \\ -1.25 & 1.25 \end{bmatrix} \times 10^6 \tag{4.20}$$

Element 3

$$\left[k'^3\right] = \begin{bmatrix} 1.60 & -1.60 \\ -1.60 & 1.60 \end{bmatrix} \times 10^6 \tag{4.21}$$

Their corresponding element global entities can be determined using Eq. (4.12) as follows.

Element 1

$$
\begin{aligned}
\left\{k^1\right\} &= \left\{T\right\}^T \left[k'^1\right] \left\{T\right\} \\
&= \begin{bmatrix} \cos 0^\circ & 0 \\ \sin 0^\circ & 0 \\ 0 & \cos 0^\circ \\ 0 & \sin 0^\circ \end{bmatrix} \left[k'^1\right] \begin{bmatrix} \cos 0^\circ & \sin 0^\circ & 0 & 0 \\ 0 & 0 & \cos 0^\circ & \sin 0^\circ \end{bmatrix} \\
&= \begin{bmatrix} 2.00 & 0 & -2.00 & 0 \\ 0 & 0 & 0 & 0 \\ -2.00 & 0 & 2.00 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^6
\end{aligned}
\tag{4.22}
$$

Element 2

$$
\begin{aligned}
\left\{k^2\right\} &= \left\{T\right\}^T \left[k'^2\right] \left\{T\right\} \\
&= \begin{bmatrix} \cos 128.7^\circ & 0 \\ \sin 128.7^\circ & 0 \\ 0 & \cos 128.7^\circ \\ 0 & \sin 128.7^\circ \end{bmatrix} \left[k'^2\right] \\
&\quad \begin{bmatrix} \cos 128.7^\circ & \sin 128.7^\circ & 0 & 0 \\ 0 & 0 & \cos 128.7^\circ & \sin 128.7^\circ \end{bmatrix} \\
&= \begin{bmatrix} 4.89 & -6.10 & -4.89 & 6.10 \\ -6.10 & 7.61 & 6.10 & -7.61 \\ -4.89 & 6.10 & 4.89 & -6.10 \\ 6.10 & -7.61 & -6.10 & 7.61 \end{bmatrix} \times 10^5
\end{aligned}
\tag{4.23}
$$

Element 3

$$\left\{k^3\right\} = \left\{T\right\}^T \left[k'^3\right] \left\{T\right\}$$

$$= \begin{bmatrix} \cos 90^\circ & 0 \\ \sin 90^\circ & 0 \\ 0 & \cos 90^\circ \\ 0 & \sin 90^\circ \end{bmatrix} \left[k'^2\right] \begin{bmatrix} \cos 90^\circ & \sin 90^\circ & 0 & 0 \\ 0 & 0 & \cos 90^\circ & \sin 90^\circ \end{bmatrix} \tag{4.24}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.60 & 0 & -1.60 \\ 0 & 0 & 0 & 0 \\ 0 & -1.60 & 0 & 1.60 \end{bmatrix} \times 10^6$$

Having established the elemental global stiffness for each element, the assembled global stiffness matrix, $[K]$ (refer Eq. (4.15)) is thus given as:

$$\left[K\right] = \begin{bmatrix} 20 & 0 & -20 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & -16 \\ -20 & 0 & 24.9 & -6.10 & -4.89 & 6.10 \\ 0 & 0 & -6.10 & 7.61 & 6.10 & -7.61 \\ 0 & 0 & -4.89 & 6.10 & 4.89 & -6.10 \\ 0 & -16 & 6.10 & -7.61 & -6.10 & 23.6 \end{bmatrix} \times 10^5 \tag{4.25}$$

The assembled global load vector is given as:

$$\left\{R\right\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = \begin{Bmatrix} 0 + B_1 \\ 0 + B_2 \\ 0 + 0 \\ 0 - 10 \\ 0 + B_5 \\ 0 + 0 \end{Bmatrix} \tag{4.26}$$

and thus, the complete equilibrium equation can be given as:

$$\begin{bmatrix} 20 & 0 & -20 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & -16 \\ -20 & 0 & 24.9 & -6.10 & -4.89 & 6.10 \\ 0 & 0 & -6.10 & 7.61 & 6.10 & -7.61 \\ 0 & 0 & -4.89 & 6.10 & 4.89 & -6.10 \\ 0 & -16 & 6.10 & -7.61 & -6.10 & 23.6 \end{bmatrix} \times 10^5 \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \\ 0 \\ -10 \\ B_5 \\ 0 \end{Bmatrix} \tag{4.27}$$

To note that B_1, B_2, and B_5 are the global reactions. These are unknown because their corresponding dofs are the essential boundary conditions which are known values. By imposing the essential boundary conditions ($U_1 = U_2 = U_5 = 0$), the assembled global equilibrium equation is reduced to:

$$\begin{bmatrix} 24.9 & -6.10 & 6.10 \\ -6.10 & 7.61 & -7.61 \\ 6.10 & -7.61 & 23.6 \end{bmatrix} \times 10^5 \begin{Bmatrix} U_3 \\ U_4 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix} \tag{4.28}$$

By solving Eq. (4.28) using Matlab command “\”, the values of the assembled global dofs U_i are thus obtained as:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -4.01\times10^{-6} \\ -2.26\times10^{-5} \\ 0 \\ -6.25\times10^{-6} \end{Bmatrix} \tag{4.29}$$

Eq. (4.29) is the assembled global solution. By inserting these values accordingly into Eqs. (4.3) and (4.9), the local values can then be determined (i.e. axial displacements and local forces). However, before that, let’s determine first the global reaction forces, B_1, B_2, and B_5 which can be done by inserting Eq. (4.29) into Eq. (4.27) which yields:

$$\begin{Bmatrix} B_1 \\ B_2 \\ 0 \\ -10 \\ B_5 \\ 0 \end{Bmatrix} = \begin{bmatrix} 20 & 0 & -20 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & -16 \\ -20 & 0 & 24.9 & -6.10 & -4.89 & 6.10 \\ 0 & 0 & -6.10 & 7.61 & 6.10 & -7.61 \\ 0 & 0 & -4.89 & 6.10 & 4.89 & -6.10 \\ 0 & -16 & 6.10 & -7.61 & -6.10 & 23.6 \end{bmatrix} \times 10^5$$
$$\begin{Bmatrix} 0 \\ 0 \\ -4.01\times10^{-6} \\ -2.26\times10^{-5} \\ 0 \\ -6.25\times10^{-6} \end{Bmatrix} = \begin{Bmatrix} 8 \\ 10 \\ 0 \\ -10 \\ -8 \\ 0 \end{Bmatrix} \tag{4.30}$$

The reaction forces at support can thus be obtained as:

$$\begin{Bmatrix} B_1 \\ B_2 \\ B_5 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 10 \\ -8 \end{Bmatrix} \tag{4.31}$$

Now, let's proceed with determination of the local values. The elemental global solution for each element is thus:

$$\left\{u^1\right\} = \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \end{Bmatrix} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -4.01\times10^{-6} \\ -2.26\times10^{-5} \end{Bmatrix} \tag{4.32a}$$

$$\left\{u^2\right\} = \begin{Bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \\ u_4^2 \end{Bmatrix} = \begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} -4.01\times10^{-6} \\ -2.26\times10^{-5} \\ 0 \\ -6.25\times10^{-6} \end{Bmatrix} \tag{4.32b}$$

$$\left\{u^3\right\} = \begin{Bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \\ u_4^3 \end{Bmatrix} = \begin{Bmatrix} U_1 \\ U_2 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -6.25\times10^{-6} \end{Bmatrix} \tag{4.32c}$$

Eq. (4.32) are the elemental global solutions. In practice, local values are of interest. To obtain local axial displacements and local internal forces, Eqs. (4.3) and (4.9) and are respectively reemployed.

Element 1 By inserting Eq. (4.32a) into Eq. (4.3), the local dofs (axial displacements) are determined as follows:

$$\left\{u'^1\right\} = \left[T\right]\left\{u^1\right\}$$

$$\begin{Bmatrix} u_1{'}^1 \\ u_2{'}^1 \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \end{Bmatrix}$$

$$= \begin{bmatrix} \cos 0^\circ & \sin 0^\circ & 0 & 0 \\ 0 & 0 & \cos 0^\circ & \sin 0^\circ \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -4.01\times10^{-6} \\ -2.26\times10^{-5} \end{Bmatrix} \tag{4.33}$$

$$= \begin{Bmatrix} 0 \\ -4.01\times10^{-6} \end{Bmatrix}$$

Eq. (4.33) give the local dofs of element 1. By inserting this equation into Eq. (4.8), the local forces can be determined as follows:

$$\left\{b'^1\right\} = \left\{k'^1\right\}\left[u'^1\right] - \left\{q_1{'}^1\right\}$$

$$\begin{Bmatrix} b_1{'}^1 \\ b_2{'}^1 \end{Bmatrix} = \begin{bmatrix} 2.00 & -2.00 \\ -2.00 & 2.00 \end{bmatrix} \times 10^6 \begin{Bmatrix} 0 \\ -4.01\times10^{-6} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.34}$$

$$= \begin{Bmatrix} 8 \\ -8 \end{Bmatrix}$$

Element 2 By inserting Eq. (4.32b) into Eq. (4.3), the local dofs (axial displacements) are determined as follows:

$$\begin{Bmatrix} {u_1}'^2 \\ {u_2}'^2 \end{Bmatrix} = \begin{bmatrix} \cos 128.7^\circ & \sin 128.7^\circ & 0 & 0 \\ 0 & 0 & \cos 128.7^\circ & \sin 128.7^\circ \end{bmatrix} \begin{Bmatrix} -4.01\times 10^{-6} \\ -2.26\times 10^{-5} \\ 0 \\ -6.25\times 10^{-6} \end{Bmatrix}$$
$$= \begin{Bmatrix} -1.51\times 10^{5} \\ -4.9\times 10^{-6} \end{Bmatrix} \tag{4.35}$$

Eq. (4.35) give the local dofs of element 2. By inserting this equation into Eq. (4.8), the local forces can be determined as follows:

$$\left\{ b'^2 \right\} = \left\{ k'^2 \right\} \left[u'^2 \right] - \left\{ q'^2 \right\}$$
$$\begin{Bmatrix} {b_1}'^2 \\ {b_2}'^2 \end{Bmatrix} = \begin{bmatrix} 1.25 & -1.25 \\ -1.25 & 1.25 \end{bmatrix} \times 10^6 \begin{Bmatrix} -1.51\times 10^{5} \\ -4.9\times 10^{-6} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.36}$$
$$= \begin{Bmatrix} -12.8 \\ 12.8 \end{Bmatrix}$$

Element 3 By inserting Eq. (4.32c) into Eq. (4.3), the local dofs (axial displacements) are determined as follows:

$$\begin{Bmatrix} {u_1}'^3 \\ {u_2}'^3 \end{Bmatrix} = \begin{bmatrix} \cos 90^\circ & \sin 90^\circ & 0 & 0 \\ 0 & 0 & \cos 90^\circ & \sin 90^\circ \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -6.25\times 10^{-6} \end{Bmatrix} \tag{4.37}$$
$$= \begin{Bmatrix} 0 \\ -6.25\times 10^{-6} \end{Bmatrix}$$

Eq. (4.37) give the local dofs of element 3. By inserting this equation into

Eq. (4.8), the local forces can be determined as follows:

$$\left\{b'^3\right\} = \left\{k'^3\right\}\left[u'^3\right] - \left\{q'^3\right\}$$

$$\begin{Bmatrix} b_1{}'^3 \\ b_2{}'^3 \end{Bmatrix} = \begin{bmatrix} 1.60 & -1.60 \\ -1.60 & 1.60 \end{bmatrix} \times 10^6 \begin{Bmatrix} 0 \\ -6.25\times10^{-6} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 10 \\ -10 \end{Bmatrix} \tag{4.38}$$

Finally, the elemental stress, σ^1, σ^2, and σ^3 can be determined as follows:

$$\left\{\sigma^1\right\} = \frac{\{b^1\}}{A} = \frac{1}{0.04}\begin{Bmatrix} 8 \\ -8 \end{Bmatrix} = \begin{Bmatrix} 200 \\ -200 \end{Bmatrix} \tag{4.39}$$

$$\left\{\sigma^2\right\} = \frac{\{b^2\}}{A} = \frac{1}{0.04}\begin{Bmatrix} -12.8 \\ 12.8 \end{Bmatrix} = \begin{Bmatrix} -320.2 \\ 320.2 \end{Bmatrix} \tag{4.40}$$

$$
\begin{aligned}
\{\sigma^3\} &= \frac{\{b^3\}}{A} \\
&= \frac{1}{0.04}\begin{Bmatrix} 10 \\ -10 \end{Bmatrix} \\
&= \begin{Bmatrix} 250 \\ -250 \end{Bmatrix}
\end{aligned}
\tag{4.41}
$$

The obtained results are validated herein against those obtained with commercial software. Table 4.1 compares the results of local dofs whilst Tables 4.2 and 4.3 compare the results of the global reactions and internal forces, respectively.

Table 4.1: Validation of local dofs

Node			Present (m)	Software (m)
1	${u_1}'^1$	U_1	0	0
	${u_2}'^1$	U_2	0	0
2	${u_1}'^2$	U_3	-4.01×10^{-6}	-4.01×10^{-6}
	${u_2}'^2$	U_4	-2.26×10^{-5}	-2.26×10^{-5}
3	${u_1}'^3$	U_5	0	0
	${u_2}'^3$	U_6	-6.25×10^{-6}	-6.25×10^{-6}

Table 4.2: Validation of reaction forces at support

Global reaction	Present (kN)	Software (kN)
B_1	8	8
B_2	10	10
B_5	-8	-8

Based on Tables 4.1 to 4.3, it can be concluded that the formulation agrees well with the results produced by the commercial software. Fig. 4.6 shows the deformation of the truss system, plotted based on the obtained results (magnified for easy viewing).

Table 4.3: Validation of truss internal forces

Element	Nodal forces	Present		Software	
		Internal force (kN)	State	Internal force (kN)	State
1	${b_1'}^1$ ${b_2'}^1$	8	compression	8	compression
2	${b_1'}^2$ ${b_2'}^2$	12.8	tension	12.8	tension
3	${b_1'}^3$ ${b_2'}^3$	10	compression	10	compression

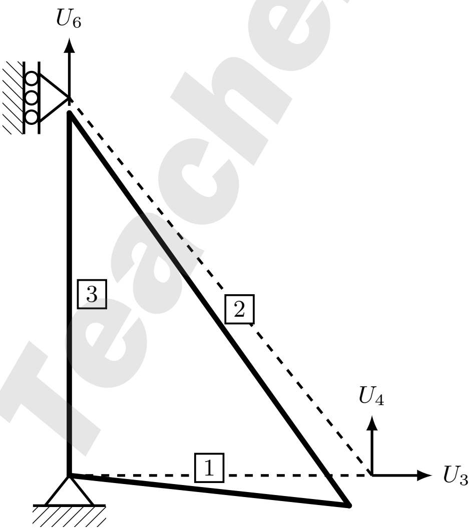

Figure 4.6: Plot of deformation of plane truss.

4.3 Frame System

In the same line of argument as in a truss system, a frame system is an assembly of inclined beam-column elements as shown in Fig. 4.7. As mentioned in the introduction of this chapter, such an element is used because in addition to the bending and shearing, a transfer of axial forces and displacement also occur in the structure. Therefore, before we proceed, we must first establish the formulation of beam-column element. Fig. 4.7 shows a typical frame structure subjected to a distributed load, $q(x)$ and lateral point loads, P_1 and P_2.

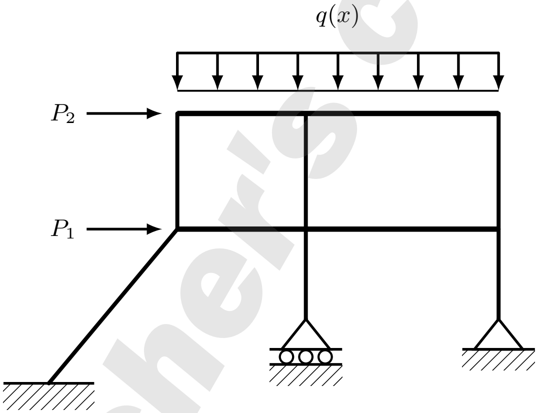

Figure 4.7: A plane-frame system.

4.3.1 Beam-Column Elements

Due to the need to cater for the axial forces and displacement, a plane beam-column element can be formulated by combining bar element and Euler-Bernoulli beam element together. Thus an arrangement of a beam-column element can be given as in Fig. 4.8.

Please note the numbering system set up for the beam-column element. This is for convenience purposes which to be seen later. Based on such an

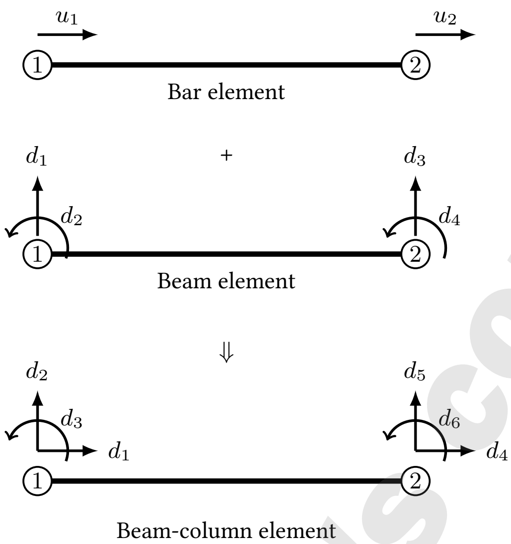

Figure 4.8: Beam-column element as a combination of bar and beam elements.

arrangement, the local dof vector of the beam is thus given as:

$$\{d'\} = \begin{Bmatrix} d'_1 \\ d'_2 \\ d'_3 \\ d'_4 \\ d'_5 \\ d'_6 \end{Bmatrix} \tag{4.42}$$

Note also that in Eq. (4.42), the variables are primed since they represent local values. Based on the arrangement of the dofs, the stiffness matrix and the load vector of the beam-column are thus given as:

$$\left[k'\right] = \begin{bmatrix} \dfrac{EA}{L} & 0 & 0 & -\dfrac{EA}{L} & 0 & 0 \\ 0 & \dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} & 0 & -\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} \\ 0 & \dfrac{6EI}{L^2} & \dfrac{4EI}{L} & 0 & -\dfrac{6EI}{L^2} & \dfrac{2EI}{L} \\ -\dfrac{EA}{L} & 0 & 0 & \dfrac{EA}{L} & 0 & 0 \\ 0 & -\dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} & 0 & \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\ 0 & \dfrac{6EI}{L^2} & \dfrac{2EI}{L} & 0 & -\dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix} \tag{4.43}$$

and the load vector is given as:

$$\left\{r'\right\} = \begin{Bmatrix} r'_1 \\ r'_2 \\ r'_3 \\ r'_4 \\ r'_5 \\ r'_6 \end{Bmatrix} \tag{4.44}$$

where r'_1 and r'_4 are basically the axial (or bar) related load values as given by Eq. (2.42b), while r'_2, r'_3, r'_5 and r'_6 are the bending as well shearing (Euler beam) related load values as given by Eq. (3.45). Note that $\{r'\} = \{q' + b'\}$.

As in truss system, the assembly of the frame system requires the local entities i.e. $[k']$, $\{d'\}$, $\{r'\}$ of the beam-column element to be transformed into elemental global entities. Fortunately, the general formulations previously derived for bar elements are still applied for beam-column element thus the elemental global dofs, $\{d\}$ of beam-column is given as:

$$\left\{d'\right\} = \left[T\right]\left\{d\right\} \tag{4.45}$$

the elemental global load vector $\{r\}$ as:

$$\left\{r\right\} = \left[T\right]^T\left\{r'\right\} \tag{4.46}$$

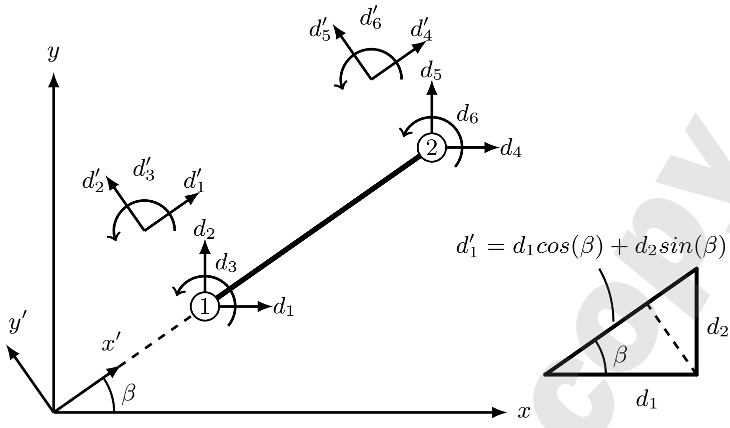

Figure 4.9: Degree of freedoms of Inclined beam-column element.

and the elemental global stiffness matrix

$$\big[k\big] = \big[T\big]^T \big[k'\big] \big[T\big] \tag{4.47}$$

All we need now is to establish the transformation matrix for the beam-column element. This is done next.

4.3.2 Beam-column Transformation Matrix

The transformation matrix $[T]$ of the beam-column can be derived by considering an inclined beam-column as shown in Fig. 4.9. It can be noticed that such a discussion is just an extension of previously given for bar.

Based on Fig. 4.9, the relationship between local dofs $\{d'\}$ and its elemental global counterpart $\{d\}$ can be given as:

$$\begin{aligned}
d_1' &= d_1 \cos\beta + d_2 \sin\beta \\
d_2' &= -d_1 \sin\beta + d_2 \cos\beta \\
d_3' &= d_3 \\
d_4' &= d_4 \cos\beta + d_5 \sin\beta \\
d_5' &= -d_4 \sin\beta + d_5 \cos\beta \\
d_6' &= d_6
\end{aligned} \tag{4.48}$$

Eq. (4.48) can be arranged in matrix form as:

$$\begin{Bmatrix} d_1' \\ d_2' \\ d_3' \\ d_4' \\ d_5' \\ d_6' \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} \tag{4.49}$$

or

$$\{d'\} = [T]\{d\} \tag{4.50}$$

hence the attainment of the equation previously given as Eq. (4.45). The attainment of Eq. (4.50) also yields the transformation matrix for beam-column element, that is:

$$[T] = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.51}$$

Having established the transformation matrix, $[T]$ of the beam-column element, further discussion on frame system analysis, particularly on the assembly process and the imposition of boundary conditions , follows the same procedure previously outlined for truss system in Section 4.2.

4.3.3 Worked Example 4.2

In this example, a plane frame as shown in Fig. 4.10 is going to be analysed. Both local and global displacements (dofs) and nodal forces of the structure are going to be determined.

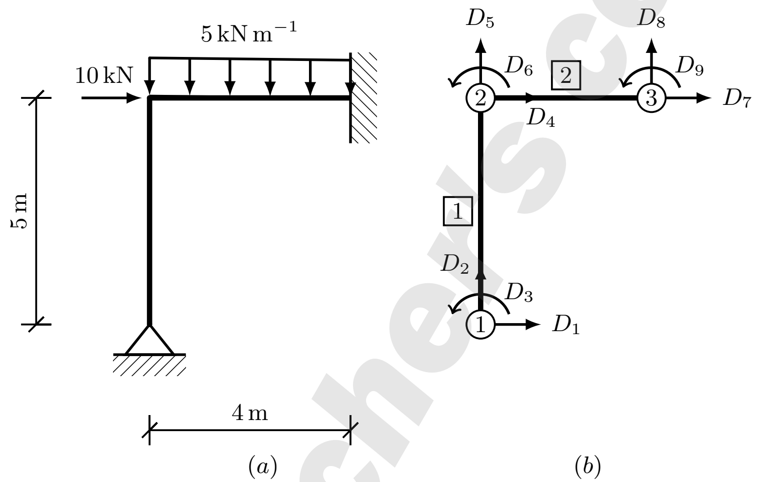

Figure 4.10: (a) Plane frame with a single point load and uniform load, and (b) its elements. All elements have similar properties ($E = 200{\times}10^6\,\mathrm{kN\,m^{-2}}$, $A = 0.04\,\mathrm{m^2}$, $I = 1{\times}10^{-4}\,\mathrm{m^4}$)

Based on the data given in, the local stiffness matrix and local load vector for each element, determined based on Eqs. (4.43) and (4.44) are as given below. Note that, the external point load is inserted directly into the assembled global load vector once everything is assembled; an act which omits the requirement for it to be considered at a local level.

Element 1

$$
\left[k'^{1}\right]=\begin{bmatrix} 1600 & 0 & 0 & -1600 & 0 & 0 \\ 0 & 2.56 & 6.40 & 0 & -2.56 & 6.40 \\ 0 & 6.40 & 21.3 & 0 & -6.40 & 10.7 \\ -1600 & 0 & 0 & 1600 & 0 & 0 \\ 0 & -2.56 & -6.40 & 0 & 2.56 & -6.40 \\ 0 & 6.40 & 10.7 & 0 & -6.40 & 21.3 \end{bmatrix}\times 10^{3} \quad (4.52)
$$

and

$$
\left\{r'^{1}\right\}=\begin{Bmatrix} 0+b_1'^{1} \\ 0+b_2'^{1} \\ 0+0 \\ 0+0 \\ 0+0 \\ 0+0 \end{Bmatrix} \quad (4.53)
$$

To note, $b_1'^{1}$, and $b_2'^{1}$ are the local reactions at support for element 1 and unknown variables (to be solved) since the corresponding displacement (essential boundary conditions) are known.

Element 2

$$
\left[k'^{2}\right]=\begin{bmatrix} 2000 & 0 & 0 & -2000 & 0 & 0 \\ 0 & 5.00 & 10 & 0 & -5.00 & 10 \\ 0 & 10 & 26.7 & 0 & -10 & 13.3 \\ -2000 & 0 & 0 & 2000 & 0 & 0 \\ 0 & -5.00 & -10 & 0 & 5.00 & -10 \\ 0 & 10 & 13.3 & 0 & -10 & 26.7 \end{bmatrix}\times 10^{3} \quad (4.54)
$$

and

$$\left\{r'^2\right\} = \begin{Bmatrix} 0+0 \\ -10+0 \\ -6.67+0 \\ 0+b_4'^2 \\ -10+b_5'^2 \\ 6.67+b_6'^2 \end{Bmatrix} \tag{4.55}$$

To note, $b_4'^2$, $b_5'^2$ and $b_6'^2$ are the local reactions at support for element 2 and unknown variables (to be solved) since the corresponding displacement (essential boundary conditions) are known.

The corresponding elemental global stiffness matrix and elemental global load vector can be determined using Eqs. (4.46) and (4.47) given as follows:

Element 1

$$
\begin{aligned}
\left[k^1\right] &= \{T\}^T \left[k'^1\right] \{T\} \\
&= \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \left\{k'^1\right\} \{T\} \\
&= \begin{bmatrix} \cos 90^\circ & \sin 90^\circ & 0 & 0 & 0 & 0 \\ -\sin 90^\circ & \cos 90^\circ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 90^\circ & \sin 90^\circ & 0 \\ 0 & 0 & 0 & -\sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \left\{k'^1\right\} \{T\} \\
&= \begin{bmatrix} 2.56 & 0 & -6.40 & -2.56 & 0 & -6.40 \\ 0 & 1600 & 0 & 0 & -1600 & 0 \\ -6.40 & 0 & 21.3 & 6.40 & 0 & 10.7 \\ -2.56 & 0 & 6.40 & 2.56 & 0 & 6.40 \\ 0 & -1600 & 0 & 0 & 1600 & 0 \\ -6.40 & 0 & 10.7 & 6.40 & 0 & 21.3 \end{bmatrix} \times 10^3
\end{aligned} \tag{4.56}
$$

and

$$
\begin{aligned}
\left\{r^1\right\} &= \left\{T\right\}^T \left\{r'^1\right\} \\
&= \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{Bmatrix} 0 + b_1'^1 \\ 0 + b_2'^1 \\ 0+0 \\ 0+0 \\ 0+0 \\ 0+0 \end{Bmatrix} \\
&= \begin{Bmatrix} 0 + \cos 90° b_1'^1 + \sin 90° b_2'^1 \\ 0 - \sin 90° b_1'^1 + \cos 90° b_2'^1 \\ 0+0 \\ 0+0 \\ 0+0 \\ 0+0 \end{Bmatrix} = \begin{Bmatrix} b_2'^1 \\ -b_1'^1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} b_1^1 \\ b_2^1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
\end{aligned}
\tag{4.57}
$$

Take note that the 'unprimed' i.e. b_1^1 and b_2^1 represent the elemental reaction forces. Due to the transformation, it can be seen that, there are possibilities for what used to be in the x-direction to be the y-direction and vice versa and what to be in one direction to be in the other direction.

Element 2

$$
\begin{aligned}
\left[k^2\right] &= \{T\}^T \left[k'^2\right] \{T\} \\
&= \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \left\{k'^2\right\} \{T\} \\
&= \begin{bmatrix} 2000 & 0 & 0 & -2000 & 0 & 0 \\ 0 & 5.00 & 10 & 0 & -5.00 & 10 \\ 0 & 10 & 26.7 & 0 & -10 & 13.3 \\ -2000 & 0 & 0 & 2000 & 0 & 0 \\ 0 & -5.00 & -10 & 0 & 5.00 & -10 \\ 0 & 10 & 13.3 & 0 & -10 & 26.7 \end{bmatrix} \times 10^3
\end{aligned}
\tag{4.58}
$$

and

$$\left\{r^2\right\} = \left[T\right]^T \left\{r'^2\right\}$$
$$= \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{Bmatrix} 0+0 \\ -10+0 \\ -6.67+0 \\ 0+b_4'^2 \\ -10+b_5'^2 \\ 6.67+b_6'^2 \end{Bmatrix} \quad (4.59)$$
$$= \begin{Bmatrix} 0 \\ -10 \\ -6.67 \\ b_4^2 \\ -10+b_5^2 \\ 6.67+b_6^2 \end{Bmatrix}$$

Take note that the unprimed i.e. b_4^2, b_5^2 and b_6^2 represent the element reaction forces.

Having established the elemental global stiffness for each element, the assembled global stiffness matrix, $[K]$ is thus given as:

$$
\left[K\right] = \begin{bmatrix}
2.56 & 0 & -6.40 & -2.56 & 0 & -6.40 & 0 & 0 & 0 \\
0 & 1600 & 0 & 0 & -1600 & 0 & 0 & 0 & 0 \\
-6.40 & 0 & 21.3 & 6.40 & 0 & 10.7 & 0 & 0 & 0 \\
-2.56 & 0 & 6.40 & 2003 & 0 & 6.40 & -2000 & 0 & 0 \\
0 & -1600 & 0 & 0 & 1605 & 10 & 0 & -5 & 10 \\
-6.40 & 0 & 10.7 & 6.40 & 10 & 48 & 0 & -10 & 13.3 \\
0 & 0 & 0 & -2000 & 0 & 0 & 2000 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & -10 & 0 & 5 & -10 \\
0 & 0 & 0 & 0 & 10 & 13.3 & 0 & -10 & 26.7
\end{bmatrix} \times 10^3 \tag{4.60}
$$

The assembled global load vector is given as:

$$
\left\{R\right\} = \begin{Bmatrix}
0 + B_1 \\
0 + B_2 \\
0 + 0 \\
0 + 10 \\
-10 + 0 \\
-6.67 + 0 \\
0 + B_7 \\
-10 + B_8 \\
6.67 + B_9
\end{Bmatrix} \tag{4.61}
$$

The complete global equilibrium equation can be given as:

$$
\begin{bmatrix}
2.56 & 0 & -6.40 & -2.56 & 0 & -6.40 & 0 & 0 & 0 \\
0 & 1600 & 0 & 0 & -1600 & 0 & 0 & 0 & 0 \\
-6.40 & 0 & 21.3 & 6.40 & 0 & 10.7 & 0 & 0 & 0 \\
-2.56 & 0 & 6.40 & 2003 & 0 & 6.40 & -2000 & 0 & 0 \\
0 & -1600 & 0 & 0 & 1605 & 10 & 0 & -5 & 10 \\
-6.40 & 0 & 10.7 & 6.40 & 10 & 48 & 0 & -10 & 13.3 \\
0 & 0 & 0 & -2000 & 0 & 0 & 2000 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & -10 & 0 & 5 & -10 \\
0 & 0 & 0 & 0 & 10 & 13.3 & 0 & -10 & 26.7
\end{bmatrix} \times 10^3
\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix}
=
\begin{Bmatrix} 0 + B_1 \\ 0 + B_2 \\ 0 + 0 \\ 0 + 10 \\ -10 + 0 \\ -6.67 + 0 \\ 0 + B_7 \\ -10 + B_8 \\ 6.67 + B_9 \end{Bmatrix}
\tag{4.62}
$$

To note, B_1, B_2, B_7, B_8 and B_9 are the global reactions. These are unknown because their corresponding dofs are the essential boundary conditions which are known values. By imposing the essential boundary conditions ($D_1 = D_2 = D_7 = D_8 = D_9 = 0$), the assembled global equilibrium equation is reduced to:

$$\begin{bmatrix} 21.3 & 6.40 & 0 & 10.7 \\ 6.40 & 2003 & 0 & 6.40 \\ 0 & 0 & 1605 & 10 \\ 10.7 & 6.40 & 10 & 48 \end{bmatrix} \times 10^3 \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \\ -10 \\ -6.67 \end{Bmatrix} \tag{4.63}$$

By solving Eq. (4.63) using Matlab command "\", the values of the assembled global dofs D_i are thus obtained as:

$$\begin{Bmatrix} D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 7.62\times10^{-5} \\ 5.25\times10^{-6} \\ -5.26\times10^{-6} \\ -1.55\times10^{-4} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.64}$$

Eq. (4.64) is the assembled global solution. By inserting these values accordingly into Eqs. (4.45) and (4.48), the local values can then be determined (i.e. local displacements and local forces). However, before let's determine first the global reaction forces, B_1, B_2, B_7, B_8,and B_9 which can be done by inserting Eq. (4.64) into Eq. (4.65) which yields:

$$
\begin{Bmatrix} 0+B_1 \\ 0+B_2 \\ 0+0 \\ 0+10 \\ -10+0 \\ -6.67+0 \\ 0+B_7 \\ -10+B_8 \\ 6.67+B_9 \end{Bmatrix}
= \begin{bmatrix}
2.56 & 0 & -6.40 & -2.56 & 0 & -6.40 & 0 & 0 & 0 \\
0 & 1600 & 0 & 0 & -1600 & 0 & 0 & 0 & 0 \\
-6.40 & 0 & 21.3 & 6.40 & 0 & 10.7 & 0 & 0 & 0 \\
-2.56 & 0 & 6.40 & 2003 & 0 & 6.40 & -2000 & 0 & 0 \\
0 & -1600 & 0 & 0 & 1605 & 10 & 0 & -5 & 10 \\
-6.40 & 0 & 10.7 & 6.40 & 10 & 48 & 0 & -10 & 13.3 \\
0 & 0 & 0 & -2000 & 0 & 0 & 2000 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & -10 & 0 & 5 & -10 \\
0 & 0 & 0 & 0 & 10 & 13.3 & 0 & -10 & 26.7
\end{bmatrix} \times 10^3
\begin{Bmatrix} 0 \\ 0 \\ 7.62\times10^{-5} \\ 5.25\times10^{-6} \\ -5.26\times10^{-6} \\ -1.55\times10^{-4} \\ 0 \\ 0 \\ 0 \end{Bmatrix}
- \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10 \\ -6.67 \\ 0 \\ -10 \\ 6.67 \end{Bmatrix}
\tag{4.65}
$$

The global reaction forces can thus be given as:

$$\begin{Bmatrix} B_1 \\ B_2 \\ B_7 \\ B_8 \\ B_9 \end{Bmatrix} = \begin{Bmatrix} 0.49 \\ 8.42 \\ -10.49 \\ 11.58 \\ -8.79 \end{Bmatrix} \tag{4.66}$$

Now, let's proceed with determination of the local values. The elemental global solution for each element is thus:

$$\{d^1\} = \begin{Bmatrix} d_1^1 \\ d_2^1 \\ d_3^1 \\ d_4^1 \\ d_5^1 \\ d_6^1 \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 7.62\times10^{-5} \\ 5.25\times10^{-6} \\ -5.26\times10^{-6} \\ -1.55\times10^{-4} \end{Bmatrix} \tag{4.67}$$

$$\{d^2\} = \begin{Bmatrix} d_1^2 \\ d_2^2 \\ d_3^2 \\ d_4^2 \\ d_5^2 \\ d_6^2 \end{Bmatrix} = \begin{Bmatrix} D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix} = \begin{Bmatrix} 5.25\times10^{-6} \\ -5.26\times10^{-6} \\ -1.55\times10^{-4} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.68}$$

Eqs. (4.67) and (4.68) are the elemental global solutions. In practice, local values are of interest. To obtain local internal forces and local displacements, Eqs. (4.8) and (4.45) are respectively reemployed.

Element 1 By inserting Eq. (4.67) into Eq. (4.45), the local dofs are determined as follows.

$$
\begin{aligned}
&\left\{d'^1\right\} = \left[T\right]\left\{d^1\right\} \\
&\begin{Bmatrix} d'^1_1 \\ d'^1_2 \\ d'^1_3 \\ d'^1_4 \\ d'^1_5 \\ d'^1_6 \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d^1_1 \\ d^1_2 \\ d^1_3 \\ d^1_4 \\ d^1_5 \\ d^1_6 \end{Bmatrix} \\
&\qquad = \begin{bmatrix} \cos 90^\circ & \sin 90^\circ & 0 & 0 & 0 & 0 \\ -\sin 90^\circ & \cos 90^\circ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 90^\circ & \sin 90^\circ & 0 \\ 0 & 0 & 0 & -\sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 7.62\times 10^{-5} \\ 5.25\times 10^{-6} \\ -5.26\times 10^{-6} \\ -1.55\times 10^{-4} \end{Bmatrix} \\
&\qquad = \begin{Bmatrix} 0 \\ 0 \\ 7.62\times 10^{-5} \\ -5.26\times 10^{-6} \\ -5.25\times 10^{-6} \\ -1.55\times 10^{-4} \end{Bmatrix}
\end{aligned}
\tag{4.69}
$$

Eq. (4.69) gives the local dofs of element 1. By inserting this equation into Eq. (4.45), the local force (or reaction if at support) can be determined as follows.

$$
\begin{aligned}
\left[b'^{1}\right] &= \left\{k'^{1}\right\}\left[d'^{1}\right] - \left\{q'^{1}\right\} \\
&= \begin{bmatrix}
1600 & 0 & 0 & -1600 & 0 & 0 \\
0 & 2.56 & 6.40 & 0 & -2.56 & 6.40 \\
0 & 6.40 & 21.3 & 0 & -6.40 & 10.7 \\
-1600 & 0 & 0 & 1600 & 0 & 0 \\
0 & -2.56 & -6.40 & 0 & 2.56 & -6.40 \\
0 & 6.40 & 10.7 & 0 & -6.40 & 21.3
\end{bmatrix} \times 10^{3} \\
&\quad \begin{Bmatrix} 0 \\ 0 \\ 7.62\times10^{-5} \\ -5.26\times10^{-6} \\ -5.25\times10^{-6} \\ -1.55\times10^{-4} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \\
&= \begin{Bmatrix} 8.42 \\ -0.49 \\ 0.00 \\ -8.42 \\ 0.49 \\ -2.47 \end{Bmatrix}
\end{aligned} \tag{4.70}
$$

Note that in the above, $\{q'^{1}\}$ is equalled to zero because no distributed load acts on the element.

Element 2 By inserting Eq. (4.68) into Eq. (4.45), the local dofs are determined as follows.

$$
\begin{Bmatrix} d_1'^2 \\ d_2'^2 \\ d_3'^2 \\ d_4'^2 \\ d_5'^2 \\ d_6'^2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 5.25\times10^{-6} \\ -5.26\times10^{-6} \\ -1.55\times10^{-4} \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

$$
= \begin{Bmatrix} 5.25\times10^{-6} \\ -5.26\times10^{-6} \\ -1.55\times10^{-4} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.71}
$$

Eq. (4.71) gives the local dofs of element 2. By inserting this equation into Eq. (4.45), the local force (or reaction if at support) can be determined as follows.

$$
\begin{aligned}
\left[b'^2\right] &= \left\{k'^2\right\}\left[d'^2\right] - \left\{q'^2\right\} \\
&= \begin{bmatrix} 2000 & 0 & 0 & -2000 & 0 & 0 \\ 0 & 5.00 & 10 & 0 & -5.00 & 10 \\ 0 & 10 & 26.7 & 0 & -10 & 13.3 \\ -2000 & 0 & 0 & 2000 & 0 & 0 \\ 0 & -5.00 & -10 & 0 & 5.00 & -10 \\ 0 & 10 & 13.3 & 0 & -10 & 26.7 \end{bmatrix} \times 10^3 \\
&\quad \begin{Bmatrix} 5.25\times10^{-6} \\ -5.26\times10^{-6} \\ -1.55\times10^{-4} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -10 \\ -6.67 \\ 0 \\ -10 \\ 6.67 \end{Bmatrix} \\
&= \begin{Bmatrix} 10.49 \\ 8.42 \\ 2.47 \\ -10.49 \\ 11.58 \\ -8.79 \end{Bmatrix}
\end{aligned}
\tag{4.72}
$$

The obtained results are validated herein against those obtained from commercial software. Table 4.4 compares the results of local dofs whilst Table 4.5 and Table 4.6 compare the results of global reactions and internal forces, respectively.

Based on Table 4.4, Table 4.5 and Table 4.6, it can be concluded that the formulation agrees well with the results produced by the commercial software. However, it must be noted that the tabulated values are those taken from nodal locations. To have better accuracy elsewhere, normally denser

Table 4.4: Validation of local dofs.

Node			Present	Software
1	$d_1'^1$	D_1	0	0
	$d_3'^1$	D_2	0	0
	$d_4'^1$	D_3	7.62×10^{-5}	0
2	$d_1'^1$	D_4	5.25×10^{-6}	5.00×10^{-6}
	$d_2'^1$	D_5	-5.26×10^{-6}	-5.00×10^{-6}
	$d_3'^1$	D_6	-1.55×10^{-4}	0
3	$d_4'^1$	D_7	0	0
	$d_5'^1$	D_8	0	0
	$d_6'^1$	D_9	0	0

Table 4.5: Validation of reaction forces at support.

Reaction	Present	Software
B_1(kN)	0.49	0.50
B_2(kN)	8.42	8.43
B_7(kN)	−10.49	−10.50
B_8(kN)	11.58	11.58
B_9(kN m)	−8.79	−8.78

Table 4.6: Validation of internal forces (only element 1 is compared)

Element	Nodal forces	Present	Software
1	$b_1'^1$	8.42	8.43
	$b_2'^1$	−0.49	−0.50
	$b_3'^1$	0	0
	$b_4'^1$	−8.42	−4.43
	$b_5'^1$	0.49	0.50
	$b_6'^1$	−2.47	−2.48

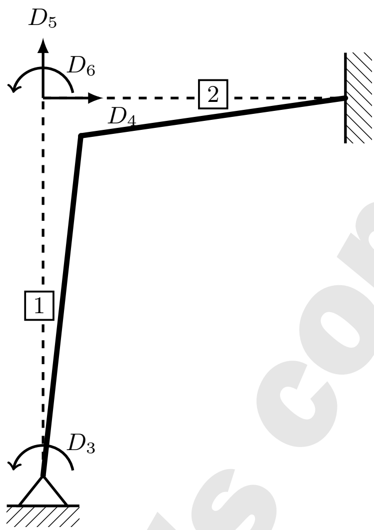

Figure 4.11: Plot of deformed plane frame.

mesh (more elements) is needed for convergence.

Fig. 4.11 shows the deformation of the frame system, plotted based on the obtained results (magnified for easy viewing).

4.4 Matlab Source Codes

4.4.1 Worked Example 4.1 – Truss

```
% Clear data
clc; clear; close all

%Input
E = 200e6;        % Young's modulus [kN/m]
A = 0.04;         % Area [m^2]
P = -10;          % Point load [kN]

% -----------------------------------------------
% FEM solution - Displacement
% -----------------------------------------------

% Elements length [m]
L1 = 4;
```

```
L2 = 6.4;
L3 = 5;

% Elements angle [degree]
th1 = 0;
th2 = 128.7;
th3 = 90;

% Transformation matrix
T1 = [cosd(th1) sind(th1) 0 0;0 0 cosd(th1) sind(th1)];
T2 = [cosd(th2) sind(th2) 0 0;0 0 cosd(th2) sind(th2)];
T3 = [cosd(th3) sind(th3) 0 0;0 0 cosd(th3) sind(th3)];

% Local stiffness matrix, k & force vector, r
k1 = A*E/L1*[1 -1;-1 1];
k2 = A*E/L2*[1 -1;-1 1];
k3 = A*E/L3*[1 -1;-1 1];

% Assemble global matrix, K and vector, R
K = zeros(6);
K(1:4,1:4) = K(1:4,1:4) + T1' * k1 * T1;
K(3:6,3:6) = K(3:6,3:6) + T2' * k2 * T2;
K([1 2 5 6],[1 2 5 6]) = K([1 2 5 6],[1 2 5 6]) + T3' * k3 * T3;

R    = zeros(6,1);
R(4) = R(4) + P;

% Free variables
fv  = [3 4 6];

%Solve for global displacement
D     = zeros(6,1);
D(fv) = K(fv,fv)\R(fv);
```

4.4.2 Worked Example 4.2 – Plane Frame

```
% Clear data
clc; clear; close all

%Input
E = 200e6;      % Young's modulus [kN/m]
A = 0.04;       % Area [m^2]
I = 1.333e-4;   % Moment of inertia [m^4]
P = 10;         % Point load [kN]
q = -5;         % Distrobuted load [kN/m]

% ---------------------------------------------
% FEM solution - Displacement
% ---------------------------------------------

% Elements length [m]
L1 = 5;
L2 = 4;

% Elements angle [degree]
th1 = 90;
th2 = 0;
```

```
%Transformation matrix
c  = cosd(th1);
s  = sind(th1);
T1 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

c  = cosd(th2);
s  = sind(th2);
T2 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

% Local stiffness matrix
k1 = [A*E/L1  0           0          -A*E/L1  0            0;
      0       12*E*I/L1^3 6*E*I/L1^2  0      -12*E*I/L1^3  6*E*I/L1^2;
      0       6*E*I/L1^2  4*E*I/L1    0      -6*E*I/L1^2   2*E*I/L1;
     -A*E/L1  0           0           A*E/L1  0            0;
      0      -12*E*I/L1^3 -6*E*I/L1^2 0       12*E*I/L1^3 -6*E*I/L1^2;
      0       6*E*I/L1^2  2*E*I/L1    0      -6*E*I/L1^2   4*E*I/L1];

k2 = [A*E/L2  0           0          -A*E/L2  0            0;
      0       12*E*I/L2^3 6*E*I/L2^2  0      -12*E*I/L2^3  6*E*I/L2^2;
      0       6*E*I/L2^2  4*E*I/L2    0      -6*E*I/L2^2   2*E*I/L2;
     -A*E/L2  0           0           A*E/L2  0            0;
      0      -12*E*I/L2^3 -6*E*I/L2^2 0       12*E*I/L2^3 -6*E*I/L2^2;
      0       6*E*I/L2^2  2*E*I/L2    0      -6*E*I/L2^2   4*E*I/L2];

% Local load vector
f1 = [0; 0; 0; 0; 0; 0];
f2 = [0; q*L2/2; q*L2^2/12; 0; q*L2/2; -q*L2^2/12];

% Assemble global matrix, K and load vector, F
K = zeros(9);
K(1:6,1:6) = K(1:6,1:6) + T1'*k1* T1;
K(4:9,4:9) = K(4:9,4:9) + T2'*k2* T2;

F = zeros(9,1);
F(1:6,1) = F(1:6,1) + T1'*f1;
F(4:9,1) = F(4:9,1) + T2'*f2;

% Point load
F(4) = F(4) + P;

% Free variables
fv  = [3 4 5 6];

%Solve for global displacement
D     = zeros(9,1);
D(fv) = K(fv,fv)\F(fv);
```

4.5 Excercises

1. Fig. 4.12 shows a truss structure which is subjected to a 8 kN vertical joint load at B. The truss is being pinned at joint A and C. The element number and global degree of freedom numbering are shown in the figure. The axial stiffness for all truss members are given as constant i.e. $EA =$ constant. Determine:

   i. Displacement at joint B.
   ii. Horizontal reaction at joint A and joint C.

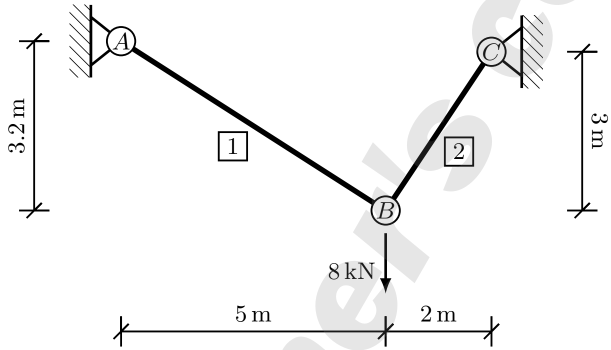

Figure 4.12

2. Fig. 4.13 demonstrates a simply supported truss structure subjected to several point loads at Node A and Node D.

   a. Redraw the truss system and provide an appropriate degree-of-freedoms at every node.
   b. Based on the given value of point loads, determine the nodal forces vector, R for the system.
   c. After assembling all the local stiffness matrices (k^1, k^2, k^3, k^4, k^5, k^6 and k^7) into the global matrix (K), the matrix element k^3_{22} of element 3, for example, is located at row 4 and column 4 in the global matrix.

   Determine the location (in terms of row and column) for all matrix elements below:

   i. k^2_{11} of element 2.

ii. k^4_{24} of element 4.
iii. k^7_{43} of element 7.

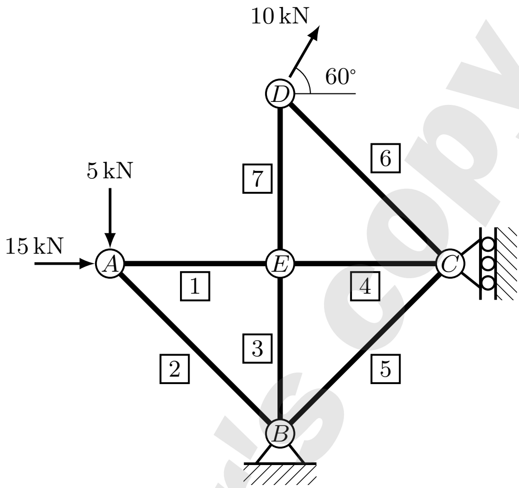

Figure 4.13

3. Fig. 4.14 shows a truss structure being pin jointed at joint A and supported by a roller at joint B. A 25 kN horizontal joint load acts at joint C. The element number and global degree of freedom numbering are shown in the figure. The axial stiffness for all truss members are given as constant i.e. $EA =$ constant. Determine:

   i. Displacement at joint C.
   ii. Internal force in element 2 and 3.

4. Fig. 4.15 shows a simply supported frame subjected to a uniform distributed load, q. If the problem is solved using two elements as depicted in the figure, calculate the numerical values of the individual local stiffness matrix, k^1 and k^2. Use the following properties.

$$E = 200\,\text{GPa}$$
$$A_1 = 0.03\,\text{m}^2 \quad I_1 = 1\times10^{-4}\,\text{m}^4$$
$$A_2 = 0.06\,\text{m}^2 \quad I_2 = 2\times10^{-4}\,\text{m}^4$$

Calculate the global stiffness matrix, K by assembling k^1 and k^2, and by considering the transformation matrix, T. Can the problem

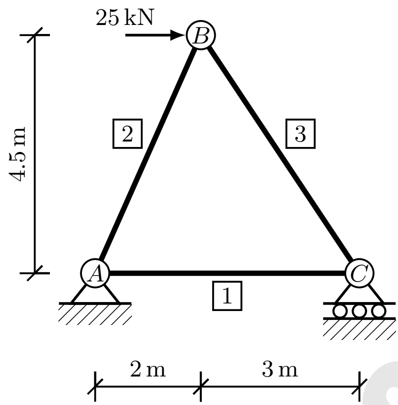

Figure 4.14

be solved using more than two elements? Discuss your answer.

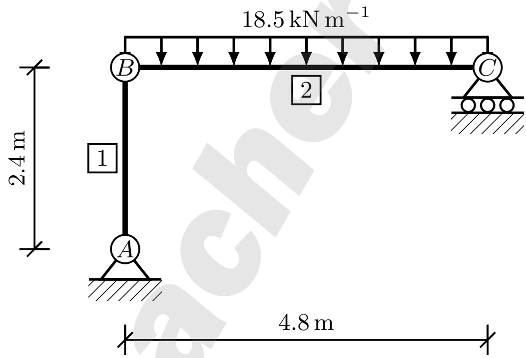

Figure 4.15

5. Fig. 4.16 shows a frame structure that is fixed at support A, B and C. The frame is subjected to a uniform distributed load of $9.4\,\mathrm{kN\,m^{-1}}$ along span BD and a vertical point load of $20\,\mathrm{kN}$ at joint D. Cross-sectional area, A for all members is taken as $0.03\,\mathrm{m^2}$, second moment of area, I as $1{\times}10^{-4}\,\mathrm{m^4}$ and the Young's modulus, E as $210{\times}10^9\,\mathrm{N\,m^{-2}}$. Determine:

   i. Displacement (i.e. degrees of freedom) at joint D.

ii. Support reactions at A, B and C.

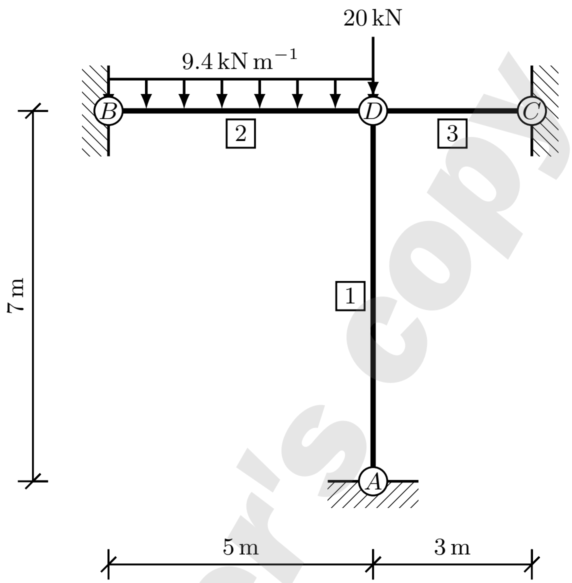

Figure 4.16

5 Eigenproblems: Free Vibration and Buckling

5.1 Introduction

In previous chapters, we have been dealing with structures which are loaded in the direction of the degree of freedoms; point loads (either equivalent or nodal loads) in the direction of the translational dof or applied moment in the direction of the rotational dof. Herein, we are going to discuss a quite different situation that is, the deformation which is 'ungoverned' by loads that are acting on the structure.

5.2 "Ungoverned by the Loads" and Eigenproblems

The word "ungoverned", however, requires further elaboration. The word refers to the statement that the load vector, $\{r\}$ or $\{R\}$ is set to zero. Mathematically this means solving only the homogenous part of the differential equation of the problem. The resulting values would be some properties of the structure and their corresponding deformation.

For example, free vibration refers to the "vibration" of a structure which is described by the natural frequencies and the corresponding mode shapes, without any explicit consideration on the type of external loading. Only during the discussion of resonance, these frequencies would then be compared with the incoming frequencies (external loads).

Same goes to the discussion of buckling of a structure. The load that would cause the buckling will be specific critical values of compressive axial force termed as buckling loads . As will be seen, since Euler-Bernoulli beam formulation able to capture such a phenomenon and determine the buckling loads and their corresponding buckling modes without the need to introduce axial dofs, it should be obvious that the bucking loads and their modes are not governed by the external applied loads.

Since both values (natural frequencies, buckling loads) are not governed by the loads, they must be some properties of the structure hence the name “eigen” which means “inherent” or “characteristic” in German. Then, what governs their values? Something must affect their values, must not they? As some properties, they are governed by other properties of the structures; material and geometrical properties. In our discussion on bar, beam and their inclined elements, these would be Young’s modulus, E, cross-sectional area, A, second moment of area, I and element’s length, L.

Also, since these values are “ungoverned” by the loads, we will see that the FEM formulation for both problems will involve with the discretization of their differential equations without the forcing terms hence the setting up of load vector, $\{r\}$ or $\{R\}$ to zero.

Accordingly, all physical problems that fall under the same argument are called eigenproblems as their equilibrium equations can all be arranged into a standard mathematical statement. If $[A]$ and $[B]$ are square matrices with known coefficients and λ is an unknown constant, an eigenproblem is a problem that can be described by the following typical mathematical statement:

$$([A] + \lambda[B])\{d\} = 0 \tag{5.1}$$

where λ is termed as eigenvalue and $\{d\}$ is termed as eigenvector.

For free vibration analysis, matrix $[A]$ represents the stiffness matrix, $[K]$, matrix $[B]$ represents the mass matrix, $[M]$, constant λ represents the square of natural frequencies and vector $\{d\}$ represents the vector of mode shape dofs $\{\hat{d}\}$.

For buckling problem, matrix $[A]$ represents the stiffness matrix, $[K]$, matrix $[B]$ represents the stress stiffness matrix $[K_G]$, constant λ represents the buckling load, P and vector $\{d\}$ represents the buckling modes . Table 5.1 summarizes these.

Table 5.1: Eigenproblem grouping

	Free vibration	Buckling
$[A]$	$[K]$	$[K]$
$[B]$	$[M]$	$[K_G]$
λ	λ	P
$\{d\}$	$\{d\}$	$\{d\}$
$([A]+\lambda[B])\{d\}=0$	$([K]-\lambda[M])\{d\}=0$	$([K]+P[K_G])\{d\}=0$

Besides having a typical statement, eigenproblem also has a typical argument for the solution that is;

> "*The solution of Eq.* (5.1) *has two solutions, the trivial solution when* $\{d\}=0$ *and the non-trivial solution when the determinant of the coefficient matrices is zero i.e.* $[A]+\lambda[B]=0$. *Since a trivial solution would also mean that there will be no deformation, a non-trivial solution is therefore the solution of interest.*"

Since the non-trivial solution is of interest, the setting up of

$$|[A]+\lambda[B]|=0 \tag{5.2}$$

will result in a characteristic polynomial. If the coefficient matrices are 4×4 matrix, the resulting characteristic polynomial will be in the forms:

$$\lambda^4+\Phi_4\lambda^3+\Phi_3\lambda^2+\Phi_2\lambda+\Phi_1=0 \tag{5.3}$$

where Φ_1, Φ_2, Φ_3, and Φ_4 are constants resulted from the process of setting up the determinant to zero. For $n\times n$ matrices, the resulting polynomial would be

$$\lambda^n+\Phi_n\lambda^{n-1}+...+\Phi_1\lambda^0=0 \tag{5.4}$$

The four roots of Eq. (5.3) and the n roots of Eq. (5.4) are thus the eigenvalues for the problem. For free vibration, these would be the natural frequencies and for buckling problem, these would be the buckling loads. Inserting one of the roots (or eigenvalues) back into Eq. (5.1) will allow the eigenvector for that particular eigenvalue to be solved hence determined. For free vibration, the eigenvector is the mode shape associated with the inserted natural frequency and for buckling problem, the eigenvector is the buckling mode.

However it must be noted that the whole process is lacking one equation. Say, coefficient matrices of Eq. (5.1) are 4×4 matrix. This means vector $\{d\}$ would consist of four unknowns i.e. d_1, d_2, d_3 and d_4 and with λ as another unknown. In this situation, we have five unknowns in total with only four simultaneous equations. There are two ways to proceed;

i. Employ continuity equation in providing the extra equation.
ii. Solve $\{d\}$ by setting any of the dofs as unity.

In the next section, we are going to discuss all these in more detail by focusing on the FEM formulation of free vibration of bar and beam followed by a discussion on the buckling analysis of beam. Formulation for plane frame is given last.

5.3 Free Vibration of Bar

Free vibration analysis of a bar concerns the determination of the value of the natural frequencies of the bar and their corresponding mode shapes. Such information is important especially in the consideration of resonance; a large deformation induced by an external load (which can be small in magnitude if compared to the 'strength' of the bar) which frequency is close to the natural frequency of the bar.

The discussion on free vibration of bar requires the derivation of the bar's partial differential equation subjected to time-dependent external loading. Whilst the homogenous part of the PDE will be solved herein hence the term "free vibration", the complete PDE will be solved in the next chapter where we consider FEM formulation for dynamic problem hence the form "forced vibration" therein.

5.3.1 Derivation of Bar's Partial Differential Equation of Motion

The derivation of the bar's PDE of motion is basically similar to the one outlined in Chapter 2, except that:

i. The external loading is time-dependent.
ii. Due to the time dependency of the load, it would set the bar in motion. Based on D'Alembert principle, this motion is resisted by an intensity of inertial force of $\rho A \frac{\partial^2 u}{\partial t^2}$ where $\frac{\partial^2 u}{\partial t^2}$ represents the acceleration of the bar and ρA is the mass per unit length of the bar since ρ is the density and A is the cross-sectional of the bar respectively.

Having established the above, a typical bar configuration with time dependent distributed loading, $q(x,t)$ is given in Fig. 5.1(a) while Fig. 5.1(b) shows the corresponding differential element.

Based on Fig. 5.1(c), the equilibrium of forces in the axial direction is employed:

$$\sum F_x = 0 \tag{5.5}$$

which yield:

$$-F + F + dF + q(x,t)\,dx - \rho A \frac{\partial^2 u}{\partial t^2} = 0 \tag{5.6}$$

where $q(x,t)$ is the time-varying axially distributed external loading and u is the axial displacement.

Eq. (5.6) can be arranged into:

$$\frac{dF}{dx} - \rho A \frac{\partial^2 u}{\partial t^2} = -q(x,t) \tag{5.7}$$

From Hooke's Law, we know that:

$$\sigma = E\epsilon = E\frac{du}{dx} \tag{5.8}$$

and since:

$$F = \sigma A = EA\frac{du}{dx} \tag{5.9}$$

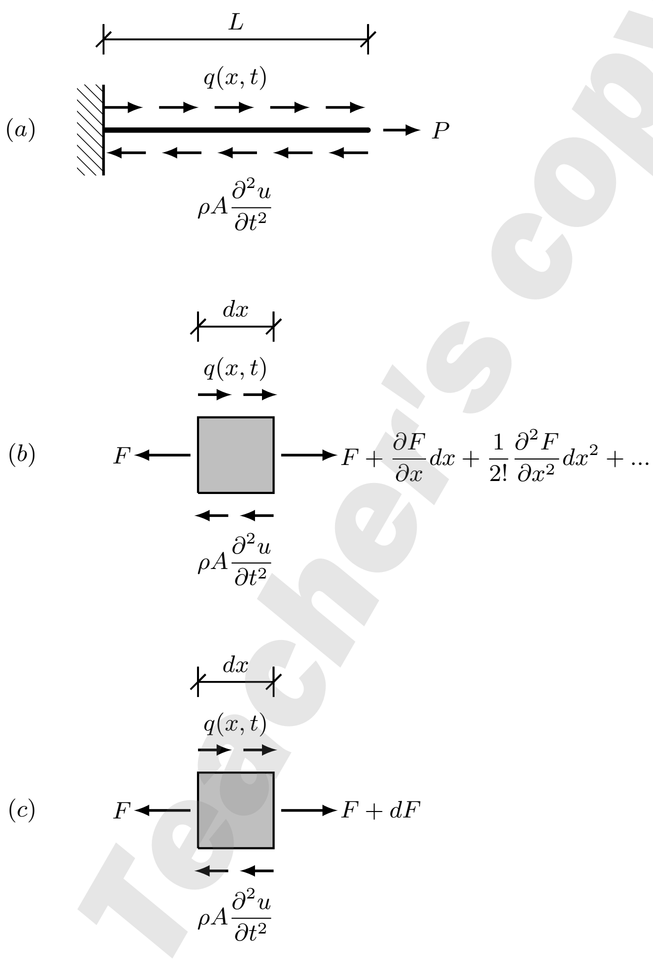

Figure 5.1: Bar differential element with inertial force.

By differentiating Eq. (5.9) once and inserting into Eq. (5.7), we obtain

$$EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} = -q(x,t) \tag{5.10}$$

Eq. (5.10) is the PDE of motion of the bar in terms of axial displacement as required. It describes the behaviour of the bar when subjected to time-varying loading (or time-varying force boundary conditions). The discussion on its solution will be given in the next chapter where therein, we will discuss the topic of dynamic. However, herein, as mentioned, since we are dealing with eigenproblem, we will discuss the solution of the homogenous part of the equation thus:

$$EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} = 0 \tag{5.11}$$

Since an eigenproblem does not involve "loading", Eq. (5.11) does not require any specification of the force (natural) boundary conditions . All it needs are the essential boundary conditions which are given as:

$$u|_{x=0} = u_0 \tag{5.12a}$$

$$u|_{x=L} = u_L \tag{5.12b}$$

Having established the bar's vibration PDE, we are now all set to discretize the equation by FEM.

5.3.2 Discretization of Bar's Eigenproblem by Galerkin Method

Like previous elements, the discretization process begins with the provision of the interpolation function (for linear bar element):

$$u(x) = N_1 u_1 + N_2 u_2 \tag{5.13a}$$

or in component forms as:

$$u(x) = N_j u_j \tag{5.13b}$$

where N_j and u_j are the shape functions and degree of freedoms as previously given in Chapter 2 for linear bar element (Eq. (2.16)).

By inserting Eq. (5.13a) into Eq. (5.10) gives:

$$EA\frac{\partial^2(N_1u_1+N_2u_2)}{\partial x^2}-\rho A\frac{\partial^2(N_1u_1+N_2u_2)}{\partial t^2}\neq 0 \tag{5.14a}$$

or in component forms as:

$$EA\frac{\partial^2(N_ju_j)}{\partial x^2}-\rho A\frac{\partial^2(N_ju_j)}{\partial t^2}\neq 0 \tag{5.14b}$$

Having established the PDE in terms of shape functions and dofs, we then multiply Eq. (5.14a) with weight functions, N_i consecutively and integrate the inner product so as to obtain the discretised equation.

Thus:

$$\int_0^L N_i\left(EA\frac{\partial^2(N_ju_j)}{\partial x^2}-\rho A\frac{\partial^2(N_ju_j)}{\partial t^2}\right)dx=0 \tag{5.15}$$

Next, we conduct integration by parts (IBP) to the first term. It must be noted that such integration is no longer conducted to induce naturally the force boundary conditions as this is an eigenproblem but to optimize on the continuity relaxation only. Also, no IBP is conducted to the second term because the term is not a spatial x derivative but a time derivative. In fact, the time derivative term requires further elaboration which will be given next. But until then, let's conduct IBP to Eq. (5.15) so as to obtain:

$$\int_0^L \frac{\partial N_i}{\partial x}EA\frac{\partial(N_ju_j)}{\partial x}\,dx+\rho A\int_0^L N_iN_j\frac{\partial^2 u_j}{\partial t^2}\,dx=0 \tag{5.16}$$

Whilst the first integral term is familiar to us from the discussion of bar in Chapter 2, the second term, as mentioned, deserves further explanation. In this term, the shape functions, N_j is taken out from the derivative since it will not involve with the differentiation due to the fact that the shape functions are not function of time, t but function of x only. But something must be a function of time; else the second term would vanish. This leaves us with nothing but the degree of freedoms, u_j. So, u_j must be a function of time, t thus must involve in the differentiation as shown in Eq. (5.16).

Now, by expressing the time second derivative by a double dot, Eq. (5.16) can be given in matrix forms as:

$$[\,k\,]\{u\} + [m]\{\ddot{u}\} = 0 \tag{5.17}$$

where $[\,k\,]$ is the stiffness matrix of the bar (as previously given in Eq. (2.41)) and $[m]$ is termed as the equivalent mass matrix of the beam. These matrices can be given as:

$$\begin{aligned} [\,k\,] = k_{ij} &= \int_0^L \frac{\partial N_i}{\partial x} EA \frac{\partial N_j}{\partial x}\, dx \\ &= \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned} \tag{5.18}$$

$$[m] = m_{ij} = \rho A \int_0^L N_i N_j \, dx \tag{5.19}$$

To note, $[m]$ given by Eq. (5.19) is termed as equivalent mass matrix as it is derived by the integration of shape functions in the same sense as the equivalent nodal loads hence the term "equivalent". An alternative is to simply lump the mass at the nodal location to result in a lumped mass matrix. Due to its simplicity, lumped mass matrix is not discussed herein.

For demonstration purpose the derivation of m_{12} is shown below:

$$m_{12} = \rho A \int_0^L N_1 N_2 \, dx \tag{5.20}$$

By inserting N_1 and N_2 of Eq. (2.16), Eq. (5.20) will become:

$$m_{12} = \rho A \int_0^L \left(\frac{L-x}{L}\right)\left(\frac{x}{L}\right) dx \tag{5.21}$$

By conducting the integration to Eq. (5.21), m_{12} is obtained as:

$$m_{12} = \frac{\rho A L}{6} \tag{5.22}$$

The complete integrated values of $[m]$ for a bar element can thus be given as:

$$[m] = m_{ij} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{5.23}$$

5.3.3 Eigenproblem Statement for Free Vibration of Bar

Now, as mentioned, to derive the eigenproblem statement for the bar problem, we must express $\{u\}$ as a function of time and we can do this by employing separation of variable which gives:

$$\{u\} = \{\hat{u}\}\sin(\omega t) \tag{5.24}$$

Based on Eq. (5.24) we can say that in time dimension, our $\{u\}$ vary in a harmonic manner with amplitudes $\{\hat{u}\}$ and frequencies, ω. Inserting Eq. (5.24) into Eq. (5.17) and by conducting the differentiation in time twice would give:

$$[\,k\,]\{\hat{u}\}\sin(\omega t) - \omega^2[m]\{\hat{u}\}\sin(\omega t) = 0 \tag{5.25}$$

By collecting and by expressing $\omega^2 = \lambda$ (the latter is to suit earlier definition), Eq. (5.25) can be given as:

$$([\,k\,] - \lambda[m])\{u\} = 0 \tag{5.26}$$

which is in a similar form as Eq. (5.1) and as given in Table 5.1.

As mentioned, Eq. (5.26) has two solutions; trivial and non-trivial. The latter is of interest to us and this is when the determinant of the multiplying matrix is set to zero:

$$|[\,k\,] - \lambda[m]| = 0 \tag{5.27}$$

Once essential boundary conditions are imposed, Eq. (5.28) will yield a polynomial characteristic equation in the forms of Eq. (5.4) which roots would be the eigenvalues, mathematically and square of the natural frequencies, physically. Consecutive insertion of these roots back into

Eq. (5.28) gives the corresponding eigenvector or mode shape; the former is a mathematical terms whilst the latter is a physical terms.

However, with Matlab, this process can automatically be carried out using the built-in function *eig()*. All we need to do is to establish stiffness matrix, $[\,k\,]$ and mass matrix, $[m]$, assemble them and impose the essential boundary conditions . The rest of the process will be taken care by the built-in function.

An assembled equation would be:

$$|[K] - \lambda[M]| = 0 \tag{5.28}$$

which the assembled nature is highlighted by the use of capital letters. The assembly process is similar to the one outlined in Chapter 4.

5.4 Free Vibration of Beam

Free vibration analysis of a beam concerns the determination of the value of the natural frequencies of the beam and their corresponding mode shapes. Such information is important especially in the consideration of resonance; a large deformation induced by an external load (which can be small in magnitude if compared to the 'strength' of the beam) which frequency is close to the natural frequency of the beam.

The discussion on free vibration of beam requires the derivation of the beam's partial differential equation subjected to time-dependent external loading. Whilst the homogenous part of the PDE will be solved herein hence the term "free vibration", the complete PDE will be solved in the next chapter where we consider FEM formulation for dynamic problem hence the form "forced vibration" therein.

5.4.1 Derivation of Beam's Partial Differential Equation of Motion

The derivation of the beam's vibration partial differential equation (PDE) is basically similar to the one outlined in Chapter 3, except that

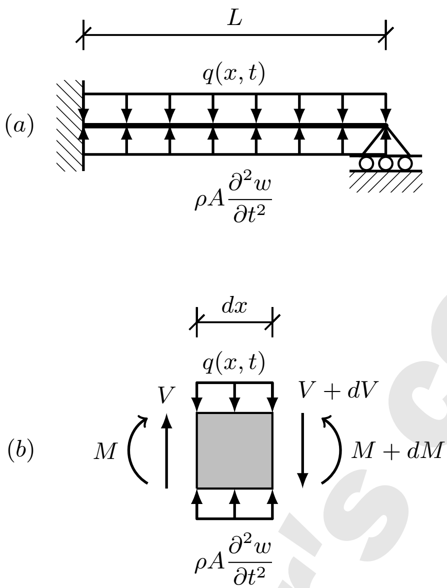

Figure 5.2: Beam differential element with inertial force.

i. The external loading is time-dependent.
ii. Due to the time dependency of the load, it would set the beam in motion. Based on D'Alembert principle, this motion is resisted by an intensity of inertial force of $\rho A \frac{\partial^2 w}{\partial t^2}$ where $\frac{\partial^2 w}{\partial t^2}$ represents the acceleration of the beam and ρA is the mass per unit length of the beam since ρ is the density and A is the cross-sectional of the beam respectively.

Having established the above, a typical beam configuration with time dependent loading is given in Fig. 5.2(a) while Fig. 5.2(b) shows the corresponding differential element.

Based on Fig. 5.2(b), the following equilibrium of forces are employed:

$$\sum F_x = 0$$
$$\sum F_y = 0 \tag{5.29}$$
$$\sum M_z = 0$$

which yield:

$$V - \left(V + \frac{\partial V}{\partial x}\, dx\right) - q(x,t)\, dx + \rho A \frac{\partial^2 w}{\partial t^2}\, dx = 0 \tag{5.30}$$

$$-M + \left(M + \frac{\partial M}{\partial x}\, dx\right) + q(x,t)\frac{dx^2}{2} - \rho A \frac{\partial^2 w}{\partial t^2}\frac{dx^2}{2} = 0 \tag{5.31}$$

where $q(x,t)$ is the time dependent distributed transverse external loading, and w is the deflection of the beam.

Eqs. (5.30) and (5.31) can be simplified into:

$$\frac{\partial V}{\partial x} - \rho A \frac{\partial^2 w}{\partial t^2} = -q(x,t) \tag{5.32}$$

$$\frac{\partial M}{\partial x} = V \tag{5.33}$$

To arrive at Eq. (5.33), higher order term $\dfrac{dx^2}{2}$ have been assumed as insignificant thus omitted. By differentiating Eq. (5.33) once and then inserting it into Eq. (5.32) give

$$\frac{\partial^2 M}{\partial x^2} - \rho A \frac{\partial^2 w}{\partial t^2} = -q(x,t) \tag{5.34}$$

Eq. (5.34) is the differential equation of the beam in terms of both flexural moment and displacement. But since our FEM discussion focuses on displacement-based formulation, we need to express the PDE completely in terms of displacement. This can be done by employing the constitutive equation which relates the curvature of the beam with the flexural moment given as

$$EI\frac{\partial^2 w}{\partial x^2} = -M \tag{5.35}$$

where E is the Young's modulus of the material and I is the second moment of area of the beam's cross-section. Together, they are known as the flexural stiffness of the beam, EI.

By differentiating Eq. (5.35) twice and inserting into Eq. (5.34), we obtain

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = q(x,t) \tag{5.36}$$

Eq. (5.36) is the PDE of the beam in terms of displacement as required. It describes the behaviour of the beam when subjected to time-varying loading (or time-varying force boundary conditions). The discussion on its solution will be given in the next chapter where therein, we will discuss the topic of dynamic. However herein, as mentioned, since we are dealing with eigenvalue problems, we will discuss the solution of the homogenous part of the equation only thus

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{5.37}$$

Since eigenproblems do not involve with "loading", Eq. (5.37) does not require any specification of the force (natural) boundary condition. All it needs are the essential boundary conditions which can be a single or a combination of the following:

$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = \theta_0 \tag{5.38a}$$

$$\left.\frac{\partial w}{\partial x}\right|_{x=L} = \theta_L \tag{5.38b}$$

$$w|_{x=0} = w_0 \tag{5.38c}$$

$$w|_{x=L} = w_L \tag{5.38d}$$

where θ_0, θ_L are the specified end rotations and w_0, w_L are the specified end transverse displacements. Having established the beam's vibration PDE, we are now all set to discuss its discretization by FEM.

5.4.2 Discretization Beam's Eigenproblem by Galerkin Method

Like previous elements, the discretization process begins with the provision of the interpolation function:

$$w(x) = N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4 \tag{5.39}$$

or in component forms as:

$$w(x) = N_j d_j$$

where N_j and d_j are the shape functions and degree of freedoms as previously given in Chapter 3 for Euler-Bernoulli beam.

By inserting Eq. (5.39) into Eq. (5.37) gives:

$$\begin{aligned} &EI \frac{\partial^4 \left(N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4\right)}{\partial x^4} \\ &+\rho A \frac{\partial^2 \left(N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4\right)}{\partial t^2} \neq 0 \end{aligned} \tag{5.40a}$$

or in component forms as

$$EI \frac{\partial^4 \left(N_j d_j\right)}{\partial x^4} + \rho A \frac{\partial^2 \left(N_j d_j\right)}{\partial t^2} \neq 0 \tag{5.40b}$$

Having established the PDE in terms of shape functions and dofs, we then multiply Eq. (5.40b) with weight functions, N_i consecutively and integrate the inner product so as to obtain the discretised equation.

Thus

$$\int_0^L N_i \left(EI \frac{\partial^4 \left(N_j d_j\right)}{\partial x^4} + \rho A \frac{\partial^2 \left(N_j d_j\right)}{\partial t^2} \right) dx = 0 \tag{5.41}$$

Next, we conduct integration by parts (IBP) to the first term. It must be noted that such integration is no longer conducted to induce naturally the force boundary conditions as this is an eigenproblem but to optimize on the continuity relaxation only. Also, no IBP is conducted to the second term because the term is not a spatial x derivative but a time derivative.

In fact, the time derivative term requires further elaboration which will be given next. But until then, let's conduct IBP to Eq. (5.41) so as to obtain

$$\int_0^L \frac{\partial^2 N_i}{\partial x^2} EI \frac{\partial^2 \left(N_j d_j\right)}{\partial x^2}\, dx + \rho A \int_0^L N_i N_j \frac{\partial^2 d_j}{\partial t^2}\, dx = 0 \tag{5.42}$$

Whilst the first integral term is familiar to us from the discussion of beam bending in Chapter 3, the second term, as mentioned, deserves further explanation. In this term, the shape functions, N_j is taken out from the derivative since it would not involve with the differentiation due to the fact that the shape functions are not function of time, t but function of x only. But something must be a function of time else the second term would vanish. This leaves us with nothing but the degree of freedoms, d_j. So, d_j must be a function of time, t thus must involve in the differentiation as shown in Eq. (5.42). Now, by expressing the time second derivative by a double dot, Eq. (5.42) can be given in matrix forms as:

$$[\,k\,]\{d\} + [m]\{\ddot{d}\} = 0 \tag{5.43}$$

where $[\,k\,]$ is the stiffness matrix of the beam (similar to Eq. (3.43)) and $[m]$ is termed as the equivalent mass matrix of the beam. These matrices can be given as:

$$[\,k\,] = k_{ij} = \int_0^L \frac{\partial^2 N_i}{\partial x^2} EI \frac{\partial^2 N_j}{\partial x^2}\, dx \tag{5.44}$$

$$[m] = m_{ij} = \rho A \int_0^L N_i N_j\, dx \tag{5.45}$$

To note, $[m]$ as given by Eq. (5.45) is termed as equivalent mass matrix as it is derived by the integration of shape functions in the same sense as the equivalent nodal loads hence the term "equivalent". An alternative is to simply lump the mass at the nodal location to result in a lumped mass matrix. Due to its simplicity, lumped mass matrix is not discussed herein.

For demonstration purpose the derivation, of m_{13} is shown below:

$$m_{13} = \rho A \int_0^L N_1 N_3\, dx \tag{5.46}$$

By inserting N_1 and N_3 of Eq. (3.19), Eq. (5.46) will become:

$$m_{13} = \rho A \int_0^L \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right)\left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right) dx \tag{5.47}$$

By conducting the integration to Eq. (5.47), m_{13} is obtained as:

$$m_{13} = \rho AL\frac{54}{420} \tag{5.48}$$

The complete integrated values of $[m]$ for a beam element can thus be given as:

$$[m] = m_{ij} = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -13L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \tag{5.49}$$

5.4.3 Eigenproblem Statement for Free Vibration of Beam

Now, as mentioned, to derive the eigenproblem statement for the problem, we must express $\{d\}$ as a function of time and we can do this by employing separation of variable which gives:

$$\{d\} = \{\hat{d}\}\sin(\omega t) \tag{5.50}$$

Based on Eq. (5.50) we can say that in time dimension, our $\{d\}$ vary in a harmonic manner with amplitudes $\{\hat{d}\}$ and frequencies, ω. Inserting Eq. (5.50) into Eq. (5.43) and by conducting the differentiation in time twice would give:

$$[\,k\,]\{\hat{d}\}\sin(\omega t) - \omega^2[m]\{\hat{d}\}\sin(\omega t) = 0 \tag{5.51}$$

By collecting and by expressing $\omega^2 = \lambda$ (the latter is to suit earlier definition), Eq. (5.51) can be given as:

$$([\,k\,] - \lambda[m])\{d\} = 0 \tag{5.52}$$

which is in a similar form as Eq. (5.1) and as given in Table 5.1.

As mentioned, Eq. (5.52) has two solutions; trivial and non-trivial. The latter is of interest to us and this is when the determinant of the multiplying matrix is set to zero;

$$|[\,k\,] - \lambda[m]| = 0 \tag{5.53}$$

Once essential boundary conditions are imposed, Eq. (5.54) will yield a polynomial characteristic equation in the forms of Eq. (5.4) which roots would be the eigenvalues, mathematically and square of the natural frequencies, physically. Consecutive insertion of these roots back into Eq. (5.52) gives the corresponding eigenvector or mode shape; the former is a mathematical terms whilst the latter is a physical terms.

However, with Matlab, this process can automatically be carried out using the built-in function eig(). All we need to do is to establish stiffness matrix, $[\,k\,]$ and mass matrix, $[m]$, assemble them and impose the essential boundary conditions . The rest of the process will be taken care by the built-in function.

An assembled equation would be:

$$|[K] - \lambda[M]| = 0 \tag{5.54}$$

which the assembled nature is highlighted by the use of capital letters. The assembly process is similar to the one outlined in Chapter 4. We are going to employ the whole procedure hence the formulation in the next example.

5.4.4 Worked Example 5.1

Fig. 5.3 shows a cantilever beam and its properties. Herein, the beam's natural frequencies and the corresponding mode shapes are of interest. The results are validated against a closed form solution.

Due to the symmetrically, element 1 and element 2 would have a similar local stiffness matrix and local mass matrix, thus:

$$[k^1_{ij}] = [k^2_{ij}] = \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 \\ 4.00 & 5.33 & -4.00 & 2.67 \\ -4.00 & -4.00 & 4.00 & -4.00 \\ 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \tag{5.55}$$

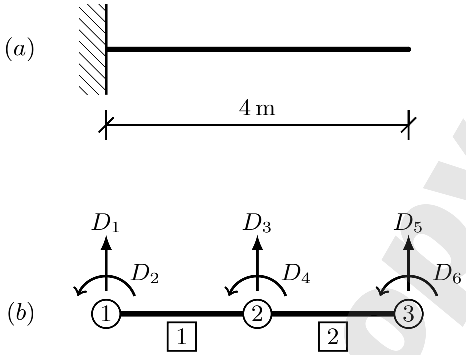

Figure 5.3: Cantilever beam and its corresponding degree of freedoms ($E = 200\times10^6\,\mathrm{kN\,m^{-2}}$, $I = 1.33\times10^{-4}\,\mathrm{m^4}$, $A = 0.04\,\mathrm{m^2}$, $\rho = 76\,\mathrm{kg\,m^{-3}}$).

and

$$[m_{ij}^1] = [m_{ij}^2] = \begin{bmatrix} 2.258 & 0.637 & 0.782 & -0.376 \\ 0.637 & 0.232 & 0.376 & -0.174 \\ 0.782 & 0.376 & 2.258 & -0.637 \\ -0.376 & -0.174 & -0.637 & 0.232 \end{bmatrix} \tag{5.56}$$

The assembled global mass matrix can be given as:

$$[K] = \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 & 0 & 0 \\ 4.00 & 5.33 & -4.00 & 2.67 & 0 & 0 \\ -4.00 & -4.00 & 8.00 & 0 & -4.00 & 4.00 \\ 4.00 & 2.67 & 0 & 10.7 & -4.00 & 2.67 \\ 0 & 0 & -4.00 & -4.00 & 4.00 & -4.00 \\ 0 & 0 & 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \tag{5.57}$$

and

$$[M] = \begin{bmatrix} 2.258 & 0.637 & 0.782 & -0.376 & 0 & 0 \\ 0.637 & 0.232 & 0.376 & -0.174 & 0 & 0 \\ 0.782 & 0.376 & 4.517 & 0 & 0.782 & -0.376 \\ -0.376 & -0.174 & 0 & 0.463 & 0.376 & -0.174 \\ 0 & 0 & 0.782 & 0.376 & 2.258 & -0.637 \\ 0 & 0 & -0.376 & -0.174 & -0.637 & 0.232 \end{bmatrix} \tag{5.58}$$

Using Eq. (5.54), the eigenproblem statement can be given as:

$$\left| \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 & 0 & 0 \\ 4.00 & 5.33 & -4.00 & 2.67 & 0 & 0 \\ -4.00 & -4.00 & 8.00 & 0 & -4.00 & 4.00 \\ 4.00 & 2.67 & 0 & 10.7 & -4.00 & 2.67 \\ 0 & 0 & -4.00 & -4.00 & 4.00 & -4.00 \\ 0 & 0 & 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \right.$$
$$\left. -\lambda \begin{bmatrix} 2.258 & 0.637 & 0.782 & -0.376 & 0 & 0 \\ 0.637 & 0.232 & 0.376 & -0.174 & 0 & 0 \\ 0.782 & 0.376 & 4.517 & 0 & 0.782 & -0.376 \\ -0.376 & -0.174 & 0 & 0.463 & 0.376 & -0.174 \\ 0 & 0 & 0.782 & 0.376 & 2.258 & -0.637 \\ 0 & 0 & -0.376 & -0.174 & -0.637 & 0.232 \end{bmatrix} \right| = 0 \tag{5.59}$$

By imposing the essential boundary conditions (i.e. $D_1 = D_2 = 0$), Eq. (5.59) is reduced to

$$\left| \begin{bmatrix} 8.00 & 0 & -4.00 & 4.00 \\ 0 & 10.7 & -4.00 & 2.67 \\ -4.00 & -4.00 & 4.00 & -4.00 \\ 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 - \lambda \begin{bmatrix} 4.517 & 0 & 0.782 & -0.376 \\ 0 & 0.463 & 0.376 & -0.174 \\ 0.782 & 0.376 & 2.258 & -0.637 \\ -0.376 & -0.174 & -0.637 & 0.232 \end{bmatrix} \right| = 0 \tag{5.60}$$

By solving Eq. (5.60) using Matlab command "eig($[K]$,$[M]$)", (source codes in Section 5.7.1), the values of λ are thus obtained as:

$$\lambda = \begin{Bmatrix} 4.24\times10^2 \\ 1.69\times10^4 \\ 1.94\times10^5 \\ 1.63\times10^6 \end{Bmatrix} \tag{5.61}$$

Since $\omega^2 = \lambda$, thus the frequency, ω can be obtained by

$$\omega = \sqrt{\lambda} = \begin{Bmatrix} 2.06\times10^1 \\ 1.30\times10^2 \\ 4.40\times10^2 \\ 1.28\times10^3 \end{Bmatrix} \tag{5.62}$$

The corresponding mode shapes and frequency are presented in Table 5.2 and Fig. 5.4.

Table 5.2: Comparison of ω values between FEM and exact solution

	FEM			Exact
	2 elements	4 elements	8 elements	
1^{st} mode	20.59	20.58	20.58	20.58
2^{nd} mode	130.08	129.13	128.99	128.96

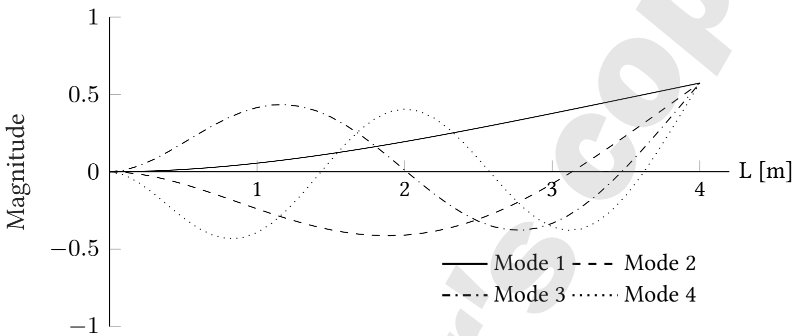

Figure 5.4: Mode shapes of beam's free vibration

5.5 Buckling of Beam

Beam buckling is a phenomenon of instability. The phenomenon is characterized by the out-of-plane deformation; a deformation that is away from the plane in which the buckling load is acting.

The analysis of buckling of structures, on the other hand, refers to the determination of buckling (critical) load, P and its corresponding mode shape, $\{d\}$. If the compressive load within a structure equals this buckling load, buckling will occur. In the context of ultimate limit state design, occurrence of buckling is considered as premature failure as it prevents the attainment of the ultimate resistances of the structure.

The determination of buckling load and its corresponding mode will be done in the same manner as outlined previously for free vibration of beam as the analysis falls under the same category of eigenproblem. By referring to Table 5.1, it can be seen that the process requires the derivation of the stiffness matrix $[k]$ and stress stiffness matrix $[k_G]$ which can be done by discretising the governing differential equation. Therefore, for better understanding, it is best to start our discussion by deriving first the ODE of the problem; a philosophy which has been and will always be upheld throughout this book.

Before we proceed with derivation of the ODE, it is worth to note that the stiffness matrix $[k]$ will be the same as the one previously derived in Chapter 3 and Eq. (5.44).

5.5.1 Derivation of Beam's Differential Equation with Constant Axial Load

The derivation of the ODE is basically similar to the one outlined in Chapter 3, except that this time, we introduce a couple of constant axial loads at each side of the differential element as can be seen in Fig. 5.5(b). Take note of the constant nature of the axial load, evident by the fact that they are not affected by the disturbance within the differential element (or by the varying boundary conditions). This is in contrast to the bar element formulation as described in Chapter 2 where the axial load on the right side of the element is expanded by the Taylor expansion.

Another point to consider is the deformed shape of the differential element as shown in Fig. 5.5(b) which was absent in our previous discussions. This consideration is necessary to allow for the moment contributed by the axial load. By considering the geometry, if the total moment is taken about point n and if the angle is assumed small, the axial load on the right side of the differential element would have a lever arm of dw and would be contributing a moment of $P\,dw$.

Having established all the relevant differences, we are all set to derive the ODE for the problem.

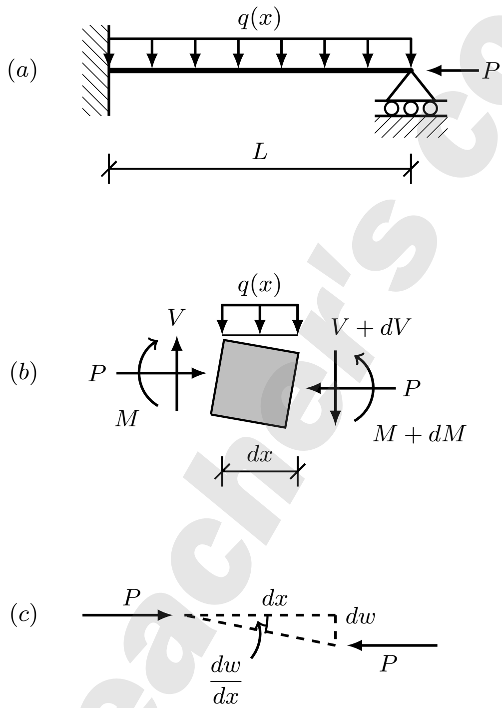

Figure 5.5: Beam differential element with constant axial load.

Based on Fig. 5.5(b), the following equilibriums of forces are employed:

$$\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum M_z &= 0 \end{aligned} \tag{5.63}$$

which yield:

$$V - \left(V + \frac{\partial V}{\partial x}\, dx\right) - q(x,t)\, dx = 0 \tag{5.64}$$

$$-M + \left(M + \frac{\partial M}{\partial x}\, dx\right) - V\, dx + q(x,t)\frac{dx^2}{2} - P\, dw = 0 \tag{5.65}$$

Eqs. (5.64) and (5.65) can be simplified into:

$$\frac{\partial V}{\partial x} = -q \tag{5.66}$$

$$\frac{\partial M}{\partial x} - P\frac{dw}{dx} = V \tag{5.67}$$

To arrive at Eq. (5.67), higher order term $\frac{dx^2}{2}$ have been assumed as insignificant thus omitted. By differentiating Eq. (5.67) once and then inserting it into Eq. (5.66) give

$$\frac{\partial^2 M}{\partial x^2} - P\frac{\partial^2 w}{\partial x^2} = -q \tag{5.68}$$

Eq. (5.68) is the differential equation of the beam in terms of both flexural moment and displacement. But since our FEM discussion focuses on displacement-based formulation, we need to express the PDE completely in terms of displacement. This can be done by employing the constitutive equation which relates the curvature of the beam with the moment given as

$$EI\frac{\partial^2 w}{\partial x^2} = -M \tag{5.69}$$

where E is the Young's modulus of the material and I is the second moment of area of the beam's cross-section. Together, they are known as the flexural stiffness of the beam, EI.

By differentiating Eq. (5.69) twice and inserting into Eq. (5.68), we obtain:

$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} = q \tag{5.70}$$

Eq. (5.70) is the ODE for the bending problem of a beam with constant axial load and is yet a buckling statement. To get the latter, under the argument of eigenproblem, we will discuss the solution of the homogenous part of the equation only, thus:

$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} = 0 \tag{5.71}$$

where now P is treated as unknown and termed as the buckling load. Similar to free vibration of beam, since eigenproblems do not involve with "loading", Eq. (5.71) does not require any specification of the force (natural) boundary condition. All it needs are the essential boundary conditions which similar to those described for free vibration of beam Eq. (5.38).

Having established the beam's bucking ODE, we are now all set to discuss its discretization by FEM.

5.5.2 Discretization of Beam's Buckling Problem by Galerkin Method

Like in free vibration of beam, the discretization process begins with the provision of the interpolation function:

$$w(x) = N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4 \tag{5.72}$$

or in component forms as:

$$w(x) = N_j d_j$$

By inserting Eq. (5.72) into Eq. (5.71) gives:

$$\begin{aligned} &EI\frac{\partial^4 \left(N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4\right)}{\partial x^4} \\ &+P\frac{\partial^2 \left(N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4\right)}{\partial x^2} \neq 0 \end{aligned} \tag{5.73a}$$

or in component forms as

$$EI\frac{\partial^4\left(N_j d_j\right)}{\partial x^4}+P\frac{\partial^2\left(N_j d_j\right)}{\partial x^2}\neq 0 \tag{5.73b}$$

Having established the PDE in terms of shape functions and dofs, we then multiply Eq. (5.73b) with weight functions, N_i consecutively and integrate the inner product so as to obtain the discretised equation.

Thus

$$\int_0^L N_i\left(EI\frac{\partial^4\left(N_j d_j\right)}{\partial x^4}+P\frac{\partial^2\left(N_j d_j\right)}{\partial x^2}\right)dx=0 \tag{5.74}$$

Next, we conduct integration by parts (IBP) to both terms in Eq. (5.74); twice to the first and once to the second. As argued for free vibration, such integrations are no longer conducted to induce natural (force) boundary conditions as this is an eigenproblem but to optimize on the continuity relaxation only. By conducting the IBP gives:

$$\int_0^L \frac{\partial^2 N_i}{\partial x^2}EI\frac{\partial^2\left(N_j d_j\right)}{\partial x^2}\,dx+P\int_0^L \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\,dx\,d_j=0 \tag{5.75}$$

As can be seen, while the first integral term is familiar to us from the discussion of beam bending in Chapter 3 which is the stiffness matrix, the second integral term is herein termed as stress stiffness matrix.

By collecting the dofs, d_j , Eq. (5.75) can be given in matrix forms as:

$$([\,k\,]+P[k_G])\,\{d\}=0 \tag{5.76}$$

where $[\,k\,]$ is the stiffness matrix of the beam (similar to Eq. (3.43)) and $[k_G]$ is termed as the stress stiffness matrix of the beam. These matrices can be given as:

$$[\,k\,]=k_{ij}=\int_0^L \frac{\partial^2 N_i}{\partial x^2}EI\frac{\partial^2 N_j}{\partial x^2}\,dx \tag{5.77}$$

$$[k_G]=k_{Gij}=\int_0^L \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\,dx \tag{5.78}$$

For demonstration purpose the derivation, of $k_G 13$ is shown below.

By inserting N_1 and N_3 (as previously given in Eq. (3.19)), Eq. (5.78) will become:

$$\begin{aligned} k_{G13} &= \int_0^L \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x}\,dx \\ &= \int_0^L \frac{\partial}{\partial x}\left(\frac{1-3x^2}{L^2}+\frac{2x^3}{L^3}\right)\frac{\partial}{\partial x}\left(\frac{3x^2}{L^2}-\frac{2x^3}{L^3}\right)dx \end{aligned} \tag{5.79}$$

By conducting the differentiation, Eq. (5.79) becomes:

$$k_{G13} = \int_0^L \left(\frac{-6x}{L^2}+\frac{6x^2}{L^3}\right)\left(\frac{6x}{L^2}-\frac{6x^2}{L^3}\right)dx \tag{5.80}$$

Finally, by conducting the integration, k_{G13} is numerically obtained as:

$$k_{G13} = -\frac{36}{30L} \tag{5.81}$$

The complete integrated value of the stress stiffness matrix, $[k_G]$ for a beam element can thus be given as:

$$[k_G] = k_{Gij} = \frac{1}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \tag{5.82}$$

5.5.3 Eigenproblem Statement for Buckling of Beam

There will be two solutions for Eq. (5.76) either $\{d\} = 0$ which is the trivial solution or the following determinant is equal to zero, which is the non-trivial solution hence the solution of interest.

$$|[\,k\,] + P[k_G]| = 0 \tag{5.83}$$

Eq. (5.83) is the eigenproblem for the buckling problem of beam. Once essential boundary conditions are imposed, Eq. (5.83) will yield a polynomial characteristic equation in the forms of Eq. (5.4) which roots would be the eigenvalues, mathematically and the buckling load, P physically. Consecutive insertion of these roots back into Eq. (5.76) gives the corresponding eigenvector or buckling mode $\{d\}$; the former is a mathematical term whilst the latter is a physical term.

However, with Matlab, this process can automatically be carried out using the built-in function eig(). All we need to do is to establish stiffness matrix, $[\,k\,]$ and stress stiffness matrix, $[k_G]$, assemble them and impose the essential boundary conditions . The rest of the process will be taken care by the built-function. An assembled equation would be

$$|[K] + P[K_G]| = 0 \tag{5.84}$$

which is highlighted by the use of capital letters. The assembly process is similar to the one outlined in Chapter 4. We are going to employ the whole procedure hence the FEM buckling formulation in the next example.

5.5.4 Worked Example 5.2

Fig. 5.6 shows a beam-column with pinned at both ends. The material properties and the length of the element are similar with Worked Example 5.1. The buckling loads and the buckling modes are of interest. Results are validated against closed form solution.

Since the material and the length of element is similar, $[K]$ is obtain from Eq. (5.57). All we need to provide afresh is the global stress stiffness matrix. We begin by providing the local stress stiffness matrix for each element. However like in Worked Example 5.1, due to the symmetrically, element 1 and element 2 would have a similar local stress stiffness matrix, thus:

$$[k^1_{Gij}] = [k^2_{Gij}] = \begin{bmatrix} 0.600 & 0.100 & -0.600 & 0.100 \\ 0.100 & 0.267 & -0.100 & -0.067 \\ -0.600 & -0.100 & 0.600 & -0.100 \\ 0.100 & -0.067 & -0.100 & 0.267 \end{bmatrix} \tag{5.85}$$

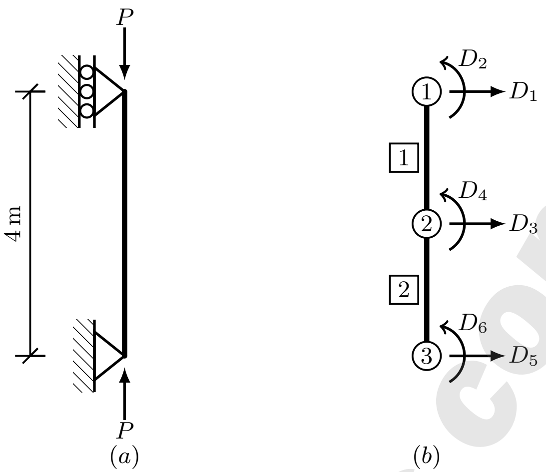

Figure 5.6: a) Column with pinned at both ends and b) the beam elements

The assembled global stress stiffness matrix can be given as:

$$[K_G] = \begin{bmatrix} 0.600 & 0.100 & -0.600 & 0.100 & 0 & 0 \\ 0.100 & 0.267 & -0.100 & -0.067 & 0 & 0 \\ -0.600 & -0.100 & 1.200 & 0 & -0.600 & 0.100 \\ 0.100 & -0.067 & 0 & 0.533 & -0.100 & -0.067 \\ 0 & 0 & -0.600 & -0.100 & 0.600 & -0.100 \\ 0 & 0 & 0.100 & -0.067 & -0.100 & 0.267 \end{bmatrix} \tag{5.86}$$

Using Eq. (5.84) the eigenproblem statement can be given as:

$$
\left|\begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 & 0 & 0 \\ 4.00 & 5.33 & -4.00 & 2.67 & 0 & 0 \\ -4.00 & -4.00 & 8.00 & 0 & -4.00 & 4.00 \\ 4.00 & 2.67 & 0 & 10.7 & -4.00 & 2.67 \\ 0 & 0 & -4.00 & -4.00 & 4.00 & -4.00 \\ 0 & 0 & 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \right.
$$
$$
\left. +P \begin{bmatrix} 0.600 & 0.100 & -0.600 & 0.100 & 0 & 0 \\ 0.100 & 0.267 & -0.100 & -0.067 & 0 & 0 \\ -0.600 & -0.100 & 1.200 & 0 & -0.600 & 0.100 \\ 0.100 & -0.067 & 0 & 0.533 & -0.100 & -0.067 \\ 0 & 0 & -0.600 & -0.100 & 0.600 & -0.100 \\ 0 & 0 & 0.100 & -0.067 & -0.100 & 0.267 \end{bmatrix} \right| = 0 \tag{5.87}
$$

By imposing the essential boundary conditions (i.e. $D_1 = D_5 = 0$), Eq. (5.87) is reduced to:

$$
\left|\begin{bmatrix} 5.33 & -4.00 & 2.67 & 0 \\ -4.00 & 8.00 & 0 & 4.00 \\ 2.67 & 0 & 10.7 & 2.67 \\ 0 & 4.00 & 2.67 & 5.33 \end{bmatrix} \times 10^4 \right.
$$
$$
\left. +P \begin{bmatrix} 0.267 & -0.100 & -0.067 & 0 \\ -0.100 & 1.200 & 0 & 0.100 \\ -0.067 & 0 & 0.533 & -0.067 \\ 0 & 0.100 & -0.067 & 0.267 \end{bmatrix} \right| = 0 \tag{5.88}
$$

By solving Eq. (5.88) using Matlab command "eig([K],[K_G])", (source codes in Section 5.7.2), the values of P are thus obtained as:

$$P = \begin{Bmatrix} 1.66\times10^4 \\ 8.00\times10^4 \\ 2.15\times10^5 \\ 4.00\times10^5 \end{Bmatrix} \tag{5.89}$$

Table 5.3: Comparison of P values between FEM and exact solution

P (N)	FEM			Exact
	2 elements	4 elements	8 elements	
1^{st} mode	1.66×10^4	1.65×10^4	1.65×10^4	1.65×10^4
2^{nd} mode	8.00×10^4	6.63×10^4	6.58×10^4	6.58×10^4
3^{rd} mode	2.15×10^5	1.53×10^5	1.48×10^5	1.48×10^5
4^{th} mode	4.00×10^5	3.2×10^5	2.65×10^5	2.63×10^5

5.6 Eigenproblem for Plane Frame (Free Vibration)

Previous sub-sections detailed the eigenproblem formulation for free vibration of bar and beam as well as buckling of beam. For a frame system, similar to the construction of the stiffness matrix of beam-column element (as detailed in Section 4.3), the mass matrix of the beam-column can also be obtained by superposing the beam's mass matrix (Eq. (5.49)) and the bar's mass matrix (Eq. (5.23)). This can be given as (note that, the matrix

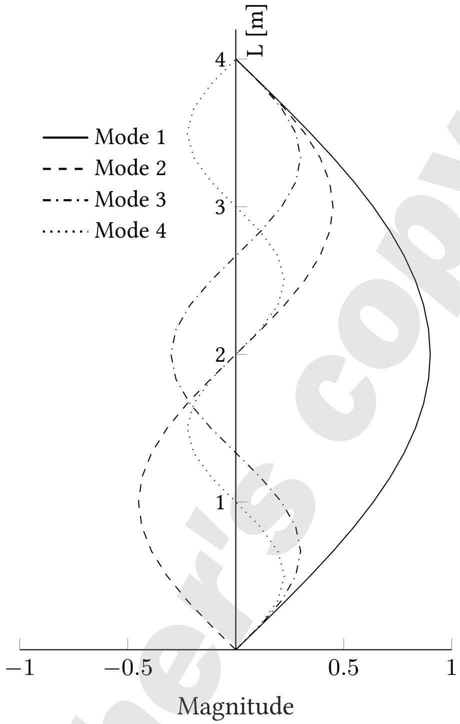

Figure 5.7: The first four buckling modes of a column with pinned at both ends.

is primed so as to distinguish them from their elemental):

$$[m'] = \frac{\rho AL}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \tag{5.90}$$

Having established the beam-column's mass matrix, $[m']$ as given by Eq. (5.90), whereas the beam-column's stiffness matrix, $[k']$ is already known as given by Eq. (4.43), the discussion on free vibration of a frame system would just be an extension of what have been discussed; all we need to do is to insert $[m']$ and $[k']$ into Eq. (5.54), which is derived based on a local system, and then, transform them into their corresponding global equation. Similar process as outlined in Chapter 4 applies where each member is multiplied by the transformation matrix $[T]$ of Eq. (4.51) accordingly. Once transformed, the elemental matrices are assembled and after the imposition of the essential boundary conditions , the equation can be solved simply by using Matlab command eig() to obtain the eigenvalues (i.e. natural frequencies) and the eigenvectors (i.e. mode shapes) for the plane frame.

For easy tracing, the elemental global eigenproblem statement for free vibration of beam (Eq. (5.54)) is re-written herein:

$$|[k] - \lambda[m]| = 0 \tag{5.91}$$

where

$$[k] = [T]^T [k'][T] \tag{5.92}$$

$$[m] = [T]^T [m'][T] \tag{5.93}$$

The primed matrices, $[m']$ and $[k']$ are the matrices given by Eq. (5.90) and Eq. (4.43), respectively (the prime symbol is used so as to distinguish them from their elemental global counterparts). The global assembled eigenproblem statement for free vibration of the frame can thus respectively given as:

$$|[K] - \lambda[M]| = 0 \tag{5.94}$$

which is highlighted by the use of capital letters. The assembly process is similar to the one outlined in Chapter 4. Note that, for buckling of frame, similar procedure applies.

We are going to employ the whole procedure of FEM free vibration for plane frame analysis in the next example.

5.6.1 Worked Example 5.3

In this example, a plane frame as shown in Fig. 5.8 is going to be analysed. The natural frequencies and the corresponding mode shapes are of interest.

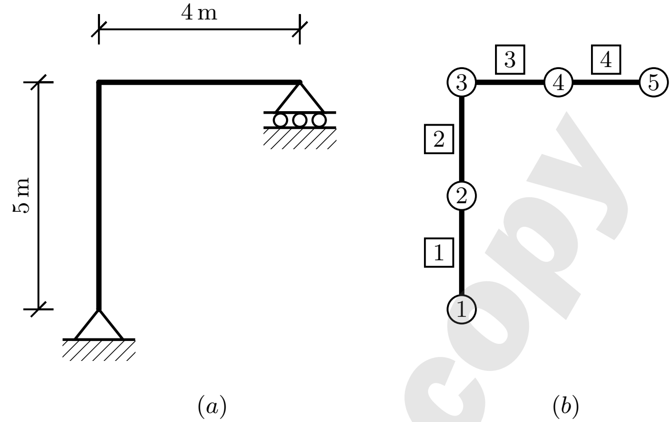

Figure 5.8: a) Plane Frame structure and b) the beam-column elements for free vibration. All elements have square cross section ($0.1\,\mathrm{m} \times 0.1\,\mathrm{m}$) with a Young's modulus of $E = 200{\times}10^6\,\mathrm{kN\,m^{-2}}$.

Results are validated against Cook et. al. (2002)[†].

Based on the data given in, due to the symmetrically, element 1 and element 2 would have a similar local mass matrix and local mass matrix, thus:

Element 1 and 2

$$[k'^{1}] = [k'^{2}] =$$

$$\begin{bmatrix} 1333 & 0 & 0 & -1333 & 0 & 0 \\ 0 & 5.92 & 4.44 & 0 & -5.92 & 4.44 \\ 0 & 4.44 & 4.44 & 0 & -4.44 & 2.22 \\ -1333 & 0 & 0 & 1333 & 0 & 0 \\ 0 & -5.92 & -4.44 & 0 & 5.92 & -4.44 \\ 0 & 4.44 & 2.22 & 0 & -4.44 & 4.44 \end{bmatrix} \times 10^6 \tag{5.95}$$

[†] Cook, R. D., Malkus, D. S., Plesha, M. E., & Witt, R. J. (1974). *Concepts and applications of finite element analysis* (Vol. 4). New York: Wiley. (Section 11.17, page 436)

$$[m'^1] = [m'^2] =$$

$$\begin{bmatrix} 39.300 & 0 & 0 & 19.650 & 0 & 0 \\ 0 & 43.791 & 9.264 & 0 & 15.159 & -5.474 \\ 0 & 9.264 & 2.526 & 0 & 5.474 & -1.895 \\ 19.650 & 0 & 0 & 39.300 & 0 & 0 \\ 0 & 15.159 & 5.474 & 0 & 43.791 & -9.264 \\ 0 & -5.474 & -1.895 & 0 & -9.264 & 2.526 \end{bmatrix} \tag{5.96}$$

Element 3 and 4

$$[k'^3] = [k'^4] =$$

$$\begin{bmatrix} 2000 & 0 & 0 & -2000 & 0 & 0 \\ 0 & 20 & 10 & 0 & -20 & 10 \\ 0 & 10 & 6.67 & 0 & -10 & 3.33 \\ -2000 & 0 & 0 & 200 & 0 & 0 \\ 0 & -20 & -10 & 0 & 20 & -10 \\ 0 & 10 & 3.33 & 0 & -10 & 6.67 \end{bmatrix} \times 10^6 \tag{5.97}$$

$$[m'^3] = [m'^4] =$$

$$\begin{bmatrix} 26.200 & 0 & 0 & 13.100 & 0 & 0 \\ 0 & 29.194 & 4.117 & 0 & 10.106 & -2.433 \\ 0 & 4.117 & 0.749 & 0 & 2.433 & -0.561 \\ 13.100 & 0 & 0 & 26.200 & 0 & 0 \\ 0 & 10.106 & 2.433 & 0 & 29.194 & -4.117 \\ 0 & -2.433 & -0.561 & 0 & -4.117 & 0.749 \end{bmatrix} \tag{5.98}$$

The corresponding elemental global stiffness matrix and global mass matrix can be determined using Eqs. (5.92) and (5.93), given as follows:

Element 1 and 2

$$[k^1] = [k^2] = [T]^T [k'^1][T]$$

$$= \begin{bmatrix} \cos 90^\circ & \sin 90^\circ & 0 & 0 & 0 & 0 \\ -\sin 90^\circ & \cos 90^\circ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 90^\circ & \sin 90^\circ & 0 \\ 0 & 0 & 0 & -\sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \{k'^1\}\{T\}$$

$$= \begin{bmatrix} 5.92 & 0 & -4.44 & -5.92 & 0 & -4.44 \\ 0 & 1333 & 0 & 0 & -1333 & 0 \\ -4.44 & 0 & 4.44 & 4.44 & 0 & 2.22 \\ -5.92 & 0 & 4.44 & 5.92 & 0 & 4.44 \\ 0 & -1333 & 0 & 0 & 1333 & 0 \\ -4.44 & 0 & 2.22 & 4.44 & 0 & 4.44 \end{bmatrix} \times 10^6 \tag{5.99}$$

$$
\begin{aligned}
[m^1] =&[m^2] = [T]^T[m'^1][T] \\
=& \begin{bmatrix}
\cos 90^\circ & \sin 90^\circ & 0 & 0 & 0 & 0 \\
-\sin 90^\circ & \cos 90^\circ & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos 90^\circ & \sin 90^\circ & 0 \\
0 & 0 & 0 & -\sin 90^\circ & \cos 90^\circ & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}^T \left\{m'^1\right\}\left\{T\right\} \\
=& \begin{bmatrix}
43.791 & 0 & -9.264 & 15.159 & 0 & 5.474 \\
0 & 39.300 & 0 & 0 & 19.650 & 0 \\
-9.264 & 0 & 2.526 & -5.474 & 0 & -1.895 \\
15.159 & 0 & -5.474 & 43.791 & 0 & 9.264 \\
0 & 19.650 & 0 & 0 & 39.300 & 0 \\
5.474 & 0 & -1.895 & 9.264 & 0 & 2.526
\end{bmatrix}
\end{aligned}
\tag{5.100}
$$

Element 3 and 4

$$[k^3] = [k^4] = [T]^T [k'^3][T]$$

$$= \begin{bmatrix} \cos 0^\circ & \sin 0^\circ & 0 & 0 & 0 & 0 \\ -\sin 0^\circ & \cos 0^\circ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 0^\circ & \sin 0^\circ & 0 \\ 0 & 0 & 0 & -\sin 0^\circ & \cos 0^\circ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \{k'^3\} \{T\}$$

$$= \begin{bmatrix} 2000 & 0 & 0 & -2000 & 0 & 0 \\ 0 & 20 & 10 & 0 & -20 & 10 \\ 0 & 10 & 6.67 & 0 & -10 & 3.33 \\ -2000 & 0 & 0 & 2000 & 0 & 0 \\ 0 & -20 & -10 & 0 & 20 & -10 \\ 0 & 10 & 3.33 & 0 & -10 & 6.67 \end{bmatrix} \times 10^6 \tag{5.101}$$

$$[m^3] = [m^4] = [T]^T [m'^3][T]$$

$$= \begin{bmatrix} \cos 0^\circ & \sin 0^\circ & 0 & 0 & 0 & 0 \\ -\sin 0^\circ & \cos 0^\circ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 0^\circ & \sin 0^\circ & 0 \\ 0 & 0 & 0 & -\sin 0^\circ & \cos 0^\circ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \{m'^3\}\{T\}$$

$$= \begin{bmatrix} 26.200 & 0 & 0 & 13.100 & 0 & 0 \\ 0 & 29.194 & 4.117 & 0 & 10.106 & -2.433 \\ 0 & 4.117 & 0.749 & 0 & 2.433 & -0.561 \\ 13.100 & 0 & 0 & 26.200 & 0 & 0 \\ 0 & 10.106 & 2.433 & 0 & 29.194 & -4.117 \\ 0 & -2.433 & -0.561 & 0 & -4.117 & 0.749 \end{bmatrix} \tag{5.102}$$

Having established the elemental global stiffness for each element, the assembled global stiffness matrix $[K]$, and global mass matrix, $[M]$ is thus given as:

$$
\begin{aligned}
&\left|[K] - \lambda[M]\right| = 0\\
&\left|\begin{bmatrix}
5.92 & 0 & -4.44 \ldots & 0 & 0 & 0\\
0 & 1333 & 0 \;\;\ldots & 0 & 0 & 0\\
-4.44 & 0 & 4.44 \ldots & 0 & 0 & 0\\
. & . & . \;\;\ldots & . & . & .\\
. & . & . \;\;\ldots & . & . & .\\
. & . & . \;\;\ldots & . & . & .\\
0 & 0 & 0 \;\;\ldots & 2000 & 0 & 0\\
0 & 0 & 0 \;\;\ldots & 0 & 20 & -10\\
0 & 0 & 0 \;\;\ldots & 0 & -10 & 6.66
\end{bmatrix} \times 10^6 \right. \qquad\qquad \dagger\\
&\left. -\,\lambda \begin{bmatrix}
43.791 & 0 & -9.264 \ldots & 0 & 0 & 0\\
0 & 39.300 & 0 \;\;\ldots & 0 & 0 & 0\\
-9.264 & 0 & 2.526 \ldots & 0 & 0 & 0\\
. & . & . \;\;\ldots & . & . & .\\
. & . & . \;\;\ldots & . & . & .\\
. & . & . \;\;\ldots & . & . & .\\
0 & 0 & 0 \;\;\ldots & 26.200 & 0 & 0\\
0 & 0 & 0 \;\;\ldots & 0 & 29.194 & -4.117\\
0 & 0 & 0 \;\;\ldots & 0 & -4.117 & 0.749
\end{bmatrix}\right| = 0
\end{aligned}
\tag{5.103}
$$

By imposing the essential boundary conditions ($D_1 = D_2 = D_{14} = 0$) to Eq. (5.103), the resulting global equilibrium equation can be solved using Matlab command "eig($[K]$, $[M]$", (source code in Section 5.7.3), the values

[†]Refer appendix for full description of the matrix.

of λ are thus obtained as:

$$\begin{Bmatrix} 4.32\times10^2 \\ 4.90\times10^4 \\ 2.03\times10^5 \\ 7.70\times10^5 \\ 2.69\times10^6 \end{Bmatrix} \tag{5.104}$$

Since $\omega^2 = \lambda$, thus the frequency, λ can be obtained by:

$$\omega\,(\mathrm{rad\,s^{-1}}) = \sqrt{\lambda} = \begin{Bmatrix} 20.78 \\ 221.46 \\ 450.90 \\ 877.70 \\ 1638.95 \end{Bmatrix} \tag{5.105}$$

or

$$f\,(\mathrm{Hz}) = \begin{Bmatrix} 3.31 \\ 35.25 \\ 71.76 \\ 139.69 \\ 260.85 \end{Bmatrix} \tag{5.106}$$

Table 5.4: Comparison of f values between FEM and Cook et. al. (2002) (50 beam elements)

	1st mode	2nd mode	3rd mode	4th mode	5th mode
FEM	3.31	35.25	71.78	139.72	260.90
Cook et. al.	3.32	35.08	70.78	122.7	226

The corresponding mode shapes of the plane frame are shown in Fig. 5.9.

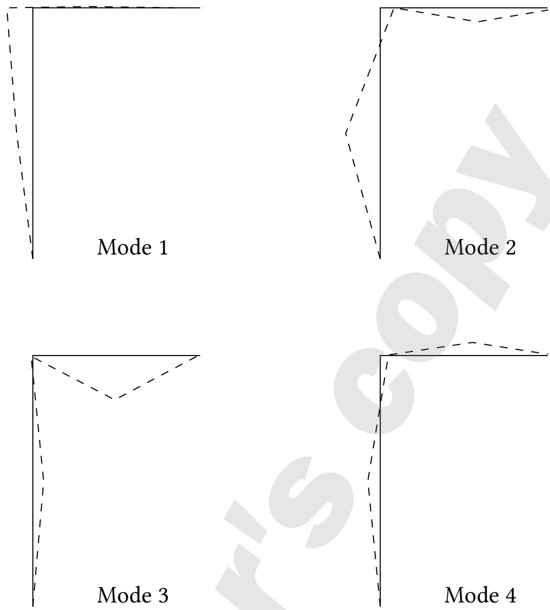

Figure 5.9: Mode shapes of plane frame

5.7 Matlab Source Codes

5.7.1 Worked Example 5.1 – Free Vibration (Beam)

```
% Clear data
clc; clear; close all

%Input
E   = 200e6;     % Young's modulus [kN/m]
A   = 0.04;      % Area [m^2]
I   = 1.333e-4;  % Moment of inertia [m^4]
rho = 76;        % Density [kg/m^3]

% --------------------------------------------
% FEM solution - Eigenvalue
% --------------------------------------------

% Elements length [m]
L1 = 2; L2 = 2;

% Local stiffness, k & mass, m matrices
k1 = [12*E*I/L1^3  6*E*I/L1^2  -12*E*I/L1^3   6*E*I/L1^2;
       6*E*I/L1^2  4*E*I/L1     -6*E*I/L1^2   2*E*I/L1;
```

```
     -12*E*I/L1^3 -6*E*I/L1^2   12*E*I/L1^3  -6*E*I/L1^2;
       6*E*I/L1^2  2*E*I/L1     -6*E*I/L1^2   4*E*I/L1];
k2 = [12*E*I/L2^3  6*E*I/L2^2  -12*E*I/L2^3   6*E*I/L2^2;
       6*E*I/L2^2  4*E*I/L2     -6*E*I/L2^2   2*E*I/L2;
     -12*E*I/L2^3 -6*E*I/L2^2   12*E*I/L2^3  -6*E*I/L2^2;
       6*E*I/L2^2  2*E*I/L2     -6*E*I/L2^2   4*E*I/L2];

m1 = rho*A*L1/420*[ 156     22*L1    54      -13*L1;
                    22*L1   4*L1^2   13*L1   -3*L1^2;
                    54      13*L1    156     -22*L1;
                   -13*L1  -3*L1^2  -22*L1   4*L1^2];
m2 = rho*A*L2/420*[ 156     22*L2    54      -13*L2;
                    22*L2   4*L2^2   13*L2   -3*L2^2;
                    54      13*L2    156     -22*L2;
                   -13*L2  -3*L2^2  -22*L2   4*L2^2];

% Assemble global matrix, K and M
K = zeros(6);
K(1:4,1:4) = K(1:4,1:4) + k1;
K(3:6,3:6) = K(3:6,3:6) + k2;

M = zeros(6);
M(1:4,1:4) = M(1:4,1:4) + m1;
M(3:6,3:6) = M(3:6,3:6) + m2;

% Solve for eigen value
[~, eigVal] = eig(K(3:end,3:end), M(3:end,3:end));

eigVal = diag(eigVal);   % Eigen-values
Freq   = sqrt(eigVal);   % Circular freq (rad/sec)
```

5.7.2 Worked Example 5.2 – Buckling (Beam)

```
% Clear data
clc; clear; close all

%Input
E   = 200e6;     % Young's modulus [kN/m]
A   = 0.04;      % Area [m^2]
I   = 1.333e-4;  % Moment of inertia [m^4]

% ------------------------------------------
% FEM solution - Buckling
% ------------------------------------------

% Elements length [m]
L1 = 2; L2 = 2;

% Local stiffness, k & mass, m matrices
k1 = [12*E*I/L1^3   6*E*I/L1^2 -12*E*I/L1^3   6*E*I/L1^2;
       6*E*I/L1^2   4*E*I/L1    -6*E*I/L1^2   2*E*I/L1;
     -12*E*I/L1^3  -6*E*I/L1^2  12*E*I/L1^3  -6*E*I/L1^2;
       6*E*I/L1^2   2*E*I/L1    -6*E*I/L1^2   4*E*I/L1];
k2 = [12*E*I/L2^3   6*E*I/L2^2 -12*E*I/L2^3   6*E*I/L2^2;
       6*E*I/L2^2   4*E*I/L2    -6*E*I/L2^2   2*E*I/L2;
     -12*E*I/L2^3  -6*E*I/L2^2  12*E*I/L2^3  -6*E*I/L2^2;
       6*E*I/L2^2   2*E*I/L2    -6*E*I/L2^2   4*E*I/L2];
```

```
%Calculate Consistent Mass Matrix
kg1 = 1/(30*L1)*[36     3*L1    -36      3*L1;
                 3*L1   4*L1^2  -3*L1   -L1^2;
                -36    -3*L1     36     -3*L1;
                 3*L1  -L1^2    -3*L1    4*L1^2];
kg2 = 1/(30*L2)*[36     3*L2    -36      3*L2;
                 3*L2   4*L2^2  -3*L2   -L2^2;
                -36    -3*L2     36     -3*L2;
                 3*L2  -L2^2    -3*L2    4*L2^2];

% Assemble global matrix, K and M
K = zeros(6);
K(1:4,1:4) = K(1:4,1:4) + k1;
K(3:6,3:6) = K(3:6,3:6) + k2;

KG = zeros(6);
KG(1:4,1:4) = KG(1:4,1:4) + kg1;
KG(3:6,3:6) = KG(3:6,3:6) + kg2;

% Free variables
fv = [2 3 4 6];

% Solve for eigen value
[~, eigVal] = eig(K(fv,fv), KG(fv,fv));
Pcr = eigVal(1,1);
```

5.7.3 Worked Example 5.3 – Frame Vibration (Plane Frame)

```
% Clear data
clc; clear; close all

%Input
E   = 200e9;     % Young's modulus [kN/m]
A   = 0.01;      % Area [m^2]
I   = 8.33e-6;   % Moment of inertia [m^4]
rho = 7860;      % Density [kg/m^3]

% ------------------------------------------
% FEM solution - Free vibration
% ------------------------------------------

% Elements length [m]
L1 = 1.5; L2 = 1.5; L3 = 1.0; L4 = 1.0;

% Elements angle [degree]
th1 = 90; th2 = 90; th3 = 0; th4 = 0;


% Calculate Stifness
k1 = [A*E/L1  0            0           -A*E/L1  0             0;
      0       12*E*I/L1^3  6*E*I/L1^2   0       -12*E*I/L1^3  6*E*I/L1^2;
      0       6*E*I/L1^2   4*E*I/L1     0       -6*E*I/L1^2   2*E*I/L1;
     -A*E/L1  0            0            A*E/L1  0             0;
      0      -12*E*I/L1^3 -6*E*I/L1^2   0        12*E*I/L1^3 -6*E*I/L1^2;
      0       6*E*I/L1^2   2*E*I/L1     0       -6*E*I/L1^2   4*E*I/L1];
k2 = [A*E/L2  0            0           -A*E/L2  0             0;
      0       12*E*I/L2^3  6*E*I/L2^2   0       -12*E*I/L2^3  6*E*I/L2^2;
```

```
      0       6*E*I/L2^2   4*E*I/L2    0       -6*E*I/L2^2   2*E*I/L2;
     -A*E/L2  0            0           A*E/L2  0             0;
      0      -12*E*I/L2^3 -6*E*I/L2^2  0        12*E*I/L2^3 -6*E*I/L2^2;
      0       6*E*I/L2^2   2*E*I/L2    0       -6*E*I/L2^2   4*E*I/L2];
k3 = [A*E/L3  0            0          -A*E/L3  0             0;
      0       12*E*I/L3^3  6*E*I/L3^2  0       -12*E*I/L3^3  6*E*I/L3^2;
      0       6*E*I/L3^2   4*E*I/L3    0       -6*E*I/L3^2   2*E*I/L3;
     -A*E/L3  0            0           A*E/L3  0             0;
      0      -12*E*I/L3^3 -6*E*I/L3^2  0        12*E*I/L3^3 -6*E*I/L3^2;
      0       6*E*I/L3^2   2*E*I/L3    0       -6*E*I/L3^2   4*E*I/L3];
k4 = [A*E/L4  0            0          -A*E/L4  0             0;
      0       12*E*I/L4^3  6*E*I/L4^2  0       -12*E*I/L4^3  6*E*I/L4^2;
      0       6*E*I/L4^2   4*E*I/L4    0       -6*E*I/L4^2   2*E*I/L4;
     -A*E/L4  0            0           A*E/L4  0             0;
      0      -12*E*I/L4^3 -6*E*I/L4^2  0        12*E*I/L4^3 -6*E*I/L4^2;
      0       6*E*I/L4^2   2*E*I/L4    0       -6*E*I/L4^2   4*E*I/L4];


%Calculate Consistent Mass Matrix
m1 = rho*A*L1/420* [140     0       0       70      0       0;
                    0       156     22*L1   0       54     -13*L1;
                    0       22*L1   4*L1^2  0       13*L1  -3*L1^2;
                    70      0       0       140     0       0;
                    0       54      13*L1   0       156    -22*L1;
                    0      -13*L1  -3*L1^2  0      -22*L1   4*L1^2];
m2 = rho*A*L2/420* [140     0       0       70      0       0;
                    0       156     22*L2   0       54     -13*L2;
                    0       22*L2   4*L2^2  0       13*L2  -3*L2^2;
                    70      0       0       140     0       0;
                    0       54      13*L2   0       156    -22*L2;
                    0      -13*L2  -3*L2^2  0      -22*L2   4*L2^2];
m3 = rho*A*L3/420* [140     0       0       70      0       0;
                    0       156     22*L3   0       54     -13*L3;
                    0       22*L3   4*L3^2  0       13*L3  -3*L3^2;
                    70      0       0       140     0       0;
                    0       54      13*L3   0       156    -22*L3;
                    0      -13*L3  -3*L3^2  0      -22*L3   4*L3^2];
m4 = rho*A*L4/420* [140     0       0       70      0       0;
                    0       156     22*L4   0       54     -13*L4;
                    0       22*L4   4*L4^2  0       13*L4  -3*L4^2;
                    70      0       0       140     0       0;
                    0       54      13*L4   0       156    -22*L4;
                    0      -13*L4  -3*L4^2  0      -22*L4   4*L4^2];

% Transformation matrix
c  = cosd(th1);
s  = sind(th1);
T1 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

c  = cosd(th2);
s  = sind(th2);
T2 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
```

```
     0 0 0  0 0 1];

c  = cosd(th3);
s  = sind(th3);
T3 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

c  = cosd(th4);
s  = sind(th4);
T4 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

% Assemble global stiffness, K and mass, M matrices
K = zeros(15);
K(1:6,1:6)       = K(1:6,1:6) + T1'*k1* T1;
K(4:9,4:9)       = K(4:9,4:9) + T2'*k2* T2;
K(7:12,7:12)     = K(7:12,7:12) + T3'*k3* T3;
K(10:15,10:15)   = K(10:15,10:15) + T4'*k4* T4;

M = zeros(15);
M(1:6,1:6)       = M(1:6,1:6) + T1'*m1* T1;
M(4:9,4:9)       = M(4:9,4:9) + T2'*m2* T2;
M(7:12,7:12)     = M(7:12,7:12) + T3'*m3* T3;
M(10:15,10:15)   = M(10:15,10:15) + T4'*m4* T4;

% Free variables
fv = [3:13 15];

% Solve for eigen value
[~, eigVal] = eig(K(fv,fv), M(fv,fv));
eigVal = diag(eigVal);
```

5.8 Exercises

1. Buckling of a beam is described by the following differential equation:

$$\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} = 0$$

where w is the displacement (deflection) and EI is the flexural stiffness of the beam. P is the buckling load to be determined.

   i. Show (by derivation) that the eigenproblem statement for the differential equation is:

$$(\mathbf{k} + P\mathbf{k}_G)\mathbf{d} = 0$$

where $\mathbf{k}$ is the stiffness matrix, $\mathbf{k}_G$ is the stress stiffness matrix and $\mathbf{d}$ is the buckling mode vector of the beam.

ii. Explain why the determinant $|\mathbf{k}+P\mathbf{k}_G|$ is taken as the solution.

2. Fig. 5.10 illustrates a structure with fixed supports at top and bottom ends. It is made of two elements. Both elements are symmetrical, and therefore the elemental stiffness matrix ($\mathbf{k}$) and the elemental stress stiffness matrix ($\mathbf{k}_G$) are similar. If $\mathbf{k}$ and $\mathbf{k}_G$ for each element are given as:

$$\mathbf{k} = \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 4 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$\mathbf{k}_G = \begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & 1 \\ -36 & -3 & 36 & -3 \\ 3 & -12 & -3 & 4 \end{bmatrix}$$

i. Develop the global stiffness matrix ($\mathbf{K}$) and the global stress stiffness matrix ($\mathbf{K}_G$)

ii. By considering appropriate boundary conditions, determine the buckling loads of the structure.

iii. If the elemental mass matrix ($\mathbf{m}$) for each element is given as:

$$\mathbf{m} = \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}$$

Determine the natural frequencies of the structure.

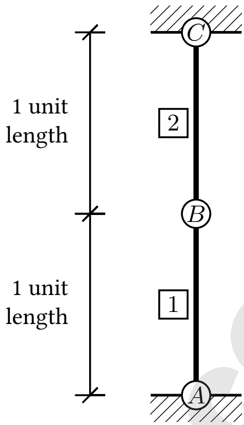

Figure 5.10

3. Fig. 5.11 shows a frame that linked a column and a beam having a fixed support at the two ends. If the elemental stiffness matrix (**k**) and the elemental mass matrix (**m**) are given as in the table below, determine the natural frequency of the frame.

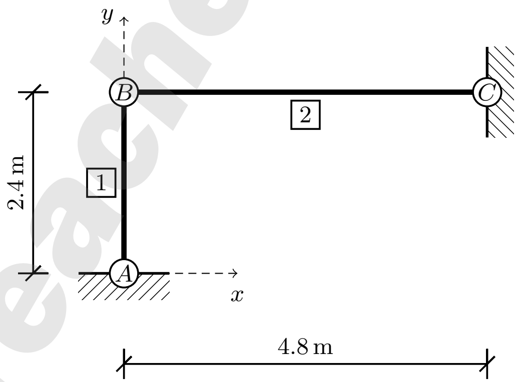

Figure 5.11

Element 1

$$\mathbf{k}_1 = \begin{bmatrix} 67.5 & 0 & -67.5 & -67.5 & 0 & -67.5 \\ 0 & 30.0 & 0 & 0 & -30.0 & 0 \\ -67.5 & 0 & 90.0 & 67.5 & 0 & 45.0 \\ -67.5 & 0 & 67.5 & 67.5 & 0 & 67.5 \\ 0 & -30.0 & 0 & 0 & 30.0 & 0 \\ -67.5 & 0 & 45.0 & 67.5 & 0 & 90.0 \end{bmatrix}$$

$$\mathbf{m}_1 = \begin{bmatrix} 93.6 & 0 & -26.4 & 32.4 & 0 & 15.6 \\ 0 & 84.0 & 0 & 0 & 42.0 & 0 \\ -26.4 & 0 & 9.6 & -15.6 & 0 & -7.2 \\ 32.4 & 0 & -15.6 & 93.6 & 0 & 26.4 \\ 0 & 42.0 & 0 & 0 & 84.0 & 0 \\ 15.6 & 0 & -7.2 & 26.4 & 0 & 9.6 \end{bmatrix}$$

Element 2

$$\mathbf{k}_2 = \begin{bmatrix} 10.0 & 0 & 0 & -10.0 & 0 & 0 \\ 0 & 2.5 & 7.5 & 0 & -2.5 & 7.5 \\ 0 & 7.5 & 30.0 & 0 & -7.5 & 15.0 \\ -10.0 & 0 & 0 & 10.0 & 0 & 0 \\ 0 & -2.5 & -7.5 & 0 & 2.5 & -7.5 \\ 0 & 7.5 & 15.0 & 0 & -7.5 & 30.0 \end{bmatrix}$$

$$\mathbf{m}_2 = \begin{bmatrix} 252.0 & 0 & 0 & 126.0 & 0 & 0 \\ 0 & 280.8 & 237.6 & 0 & 97.2 & -140.4 \\ 0 & 237.6 & 259.2 & 0 & 140.4 & -194.4 \\ 126.0 & 0 & 0 & 252.0 & 0 & 0 \\ 0 & 97.2 & 140.4 & 0 & 280.8 & -237.6 \\ 0 & -140.4 & -194.4 & 0 & -237.6 & 259.2 \end{bmatrix}$$

6 Dynamic: Forced Vibration

6.1 Introduction

Dynamic analysis is a topic where the main concern is to determine structure's responses or motions when subjected to time-varying loading (or/and time-varying boundary conditions). It is an act of solving the equation of motion of a problem. Accordingly, in this chapter, we are going to focus our discussion on the solution of forced vibration problems for bar and beam elements, as well as frame structures. Although we have derived the discretised equation of motion of beam element in Chapter 5, for completeness, we are going to begin this chapter with the derivation of the discretised equation for bar element, starting from its PDE derivation.

6.2 Bar's PDE of Motion

Derivation of the bar's PDE of motion has been given in Section 5.3.1. We are not going to repeat the procedure but to give directly the PDE as (previously given by Eq. (5.10)):

$$EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} = -q(x,t) \tag{6.1}$$

Although we have discretised this PDE in Section 5.3.2, but for completeness, we are going to detail the discretization all over again, this time with the existence of the forcing terms i.e. $q(x,t)$. As a reminder, in previous discretization, this forcing term was omitted because we were considering the homogenous solution only (eigenproblem). Before we proceed, below are the natural and essential boundary conditions of the problem (previously given by Eq. (2.9a)). However, since this is a dynamic analysis, there is a possibility for the boundary conditions to vary with time, thus:

Natural/force boundary conditions

$$EA\frac{du}{dx}\bigg|_{x=0} = F_0(t) \tag{6.2a}$$

$$EA\frac{du}{dx}\bigg|_{x=L} = -F_L(t) \tag{6.2b}$$

Essential/displacement boundary conditions

$$u|_{x=0} = u_0(t) \tag{6.3a}$$

$$u|_{x=L} = u_L(t) \tag{6.3b}$$

Note that the sign convention for the natural (force) boundary conditions above is based on the assumption that the boundary conditions are the reactions at support hence the opposite directions to the internal forces. Since Eq. (6.1) is a second order ODE in x, two boundary conditions out of the four given above must be known in prior so as to have a well-posed problem. Also, since Eq. (6.1) consists of a second order time derivative, two initial conditions must be known. We are going to detail on this later during the discussion of time integration by finite difference method.

6.2.1 Discretization of Bar's PDE of Motion by Galerkin Method

Like previous elements, the discretization process begins with the provision of the interpolation function (for linear bar element):

$$u(x) = N_1 u_1 + N_2 u_2 \tag{6.4a}$$

or in component forms as:

$$u(x) = N_j u_j \tag{6.4b}$$

where $N_j$and u_j are the shape functions and degree of freedoms as already discussed in previous chapter for linear bar element (Eq. (2.16)).

By inserting Eq. (6.4a) into Eq. (6.1) gives:

$$EA\frac{\partial^2(N_1u_1+N_2u_2)}{\partial x^2}-\rho A\frac{\partial^2(N_1u_1+N_2u_2)}{\partial t^2}\neq -q(x,t) \tag{6.5a}$$

or in component forms as:

$$EA\frac{\partial^2(N_ju_j)}{\partial x^2}-\rho A\frac{\partial^2(N_ju_j)}{\partial t^2}\neq -q(x,t) \tag{6.5b}$$

Having established the PDE in terms of shape functions and dofs, we then multiply Eq. (6.5a) with weight functions, N_i consecutively and integrate the inner product so as to obtain the discretised equation.

Thus:

$$\begin{aligned}&\int_0^L N_i\left(EA\frac{\partial^2(N_ju_j)}{\partial x^2}-\rho A\frac{\partial^2(N_ju_j)}{\partial t^2}+q(x,t)\right)\,dx=0\\ \Rightarrow &\int_0^L N_iEA\frac{\partial^2(N_ju_j)}{\partial x^2}-\rho A\int_0^L N_i\frac{\partial^2(N_ju_j)}{\partial t^2}+\int_0^L N_iq(x,t)\,dx=0\end{aligned} \tag{6.6}$$

Next, we conduct integration by parts (IBP) to the first term. It must be noted that, no IBP is conducted to the second term because the term is not a spatial x derivative, but a time, t derivative instead. By conducting the IBP, we obtain:

$$\int_0^L \frac{\partial N_i}{\partial x}EA\frac{\partial(N_ju_j)}{\partial x}+\rho A\int_0^L N_iN_j\frac{\partial^2u_j}{\partial t^2}=\int_0^L N_iq(x,t)\,dx+b_i(t) \tag{6.7}$$

where $b_i(t)$ is the boundary terms (conditions) which can be either Eqs. (6.2) and (6.3) as already explained in detailed in Section 2.5 (Eqs. (2.27) and (2.33)) except that, this time, they can vary with time, t.

While the first integral term of Eq. (6.7) is familiar to us from the discussion of bar element (static problem) in Chapter 2, the second term, as mentioned, deserves further explanation. In this term, the shape functions, N_j is taken out from the derivative since it would not involve with the differentiation because the shape functions are not a function of time, t but instead, a function of x only. But something must be a function of time else the second term would vanish. This leaves us with nothing but the degree of freedoms, u_j. Therefore, u_j must be a function of time thus must involve in the differentiation as shown in Eqn. Eq. (6.7). Now, by expressing the time second derivative by a double dot, Eq. (6.7) can be given in matrix forms as:

$$[k]\{u\} + [m]\{\ddot{u}\} = \{r(t)\} \tag{6.8}$$

where $\{r(t)\}$ is the load vector given as:

$$\{r(t)\} = \{q(t) + b(t)\} = \int_0^L N_i q(x,t)\,dx + b_i(t) \tag{6.9}$$

and $[k]$ is the stiffness matrix of the bar and $[m]$ is termed as the equivalent mass matrix of the bar, as already given by Eqs. (2.41a) and (5.23), respectively. Discussion on time-integration to solve Eq. (6.8) will be discussed later. Next, we are going to discuss the forced vibration analysis of beam.

6.3 Beam's PDE of Motion

Derivation of the beam's PDE of motion has been given in Section 5.4.1. Again, we are not going to repeat the procedure but to give directly the PDE as (given by Eq. (5.36)):

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = q(x,t) \tag{6.10}$$

Similar to the discussion of bar problem, we are going to detail the discretization of Eq. (6.10) all over again for completeness, although we have discretised this PDE in Section 5.4.2. This time, the existence of the forcing terms i.e. $q(x,t)$ is considered. As a reminder, in previous discretization, this forcing term was omitted because we were considering the homogenous solution only (eigenproblem). Before we proceed, below are the natural and essential boundary conditions of the problem. There are similar to the ones in Eqs. (3.11) and (3.12), but since this is a dynamic analysis, there is a possibility for the boundary conditions to vary with time.

The force (natural) boundary conditions of the beam as function of time can be given as:

$$EI \left. \frac{d^3w}{dx^3} \right|_{x=0} = V_0(t) \tag{6.11a}$$

$$EI \left. \frac{d^3w}{dx^3} \right|_{x=L} = -V_L(t) \tag{6.11b}$$

$$EI \left. \frac{d^2w}{dx^2} \right|_{x=0} = M_0(t) \tag{6.11c}$$

$$EI \left. \frac{d^2w}{dx^2} \right|_{x=L} = -M_L(t) \tag{6.11d}$$

where $V_0(t)$, $V_L(t)$ are the specified end shear forces (edge transverse point loads) and $M_0(t)$, $M_L(t)$ are the specified end moment forces. Note that the sign convention above is based on the assumption that the boundary conditions are the reactions at support hence the opposite directions to the internal forces.

On the other hand, the displacement (essential) boundary conditions of the beam can be given as function of time as;

$$\left. \frac{dw}{dx} \right|_{x=0} = \theta_0(t) \tag{6.12a}$$

$$\left. \frac{dw}{dx} \right|_{x=L} = \theta_L(t) \tag{6.12b}$$

$$w|_{x=0} = w_0(t) \tag{6.12c}$$

$$w|_{x=L} = w_L(t) \tag{6.12d}$$

where $\theta_0(t)$, $\theta_L(t)$ are the specified end rotations and $w_0(t)$, $w_L(t)$ are the specified end transverse displacements. Also, since Eq. (6.10) consists of second order time derivatives, two initial conditions must be known. We are going to detail on this later during the discussion on time integration by finite difference method.

6.3.1 Discretization of Beam's PDE of Motion by Galerkin Method

Like previous elements, the discretization process begins with the provision of the beam's interpolation function which can be given as:

$$w(x) = N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4 \tag{6.13}$$

The above equations can be expressed compactly in component forms as:

$$w(x) = N_j d_j$$

where N_j and d_j are the shape functions and degree of freedoms as previously given in Section 3.3 for Euler-Bernoulli beam.

By inserting Eq. (6.13) into Eq. (6.10) gives:

$$\begin{aligned} &EI\frac{\partial^4 \left(N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4\right)}{\partial x^4} \\ &+\rho A \frac{\partial^2 \left(N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4\right)}{\partial t^2} \neq q(x,t) \end{aligned} \tag{6.14}$$

or in component forms as

$$EI\frac{\partial^4 \left(N_j d_j\right)}{\partial x^4} + \rho A \frac{\partial^2 \left(N_j d_j\right)}{\partial t^2} \neq q(x,t)$$

Having established the PDE in terms of shape functions and dofs, we then multiply Eq. (6.14) with weight functions, N_i consecutively and integrate the inner product so as to obtain the discretised equation.

Thus:

$$\begin{aligned} &\int_0^L N_i \left(EI\frac{\partial^4 \left(N_j d_j\right)}{\partial x^4} + \rho A \frac{\partial^2 \left(N_j d_j\right)}{\partial t^2} - q(x,t)\right) dx = 0 \\ &\Rightarrow \int_0^L N_i EI \frac{\partial^4 \left(N_j d_j\right)}{\partial x^4}\, dx + \rho A \int_0^L N_i \frac{\partial^2 \left(N_j d_j\right)}{\partial t^2}\, dx \\ &- \int_0^L N_i q(x,t)\, dx = 0 \end{aligned} \tag{6.15}$$

Next, we conduct integration by parts (IBP) to the first term. No IBP is conducted to the second term because the term is not a spatial x derivative but a time derivative. In fact, the time derivative term requires further elaboration which will be given next. But until then, let's conduct IBP to Eq. (6.15) so as to obtain:

$$\begin{aligned}\int_0^L \frac{\partial^2 N_i}{\partial x^2} EI \frac{\partial^2 \left(N_j d_j\right)}{\partial x^2}\, dx + \rho A \int_0^L N_i N_j \frac{\partial^2 d_j}{\partial t^2}\, dx \\ = \int_0^L N_i q(x,t)\, dx + b_i(t)\end{aligned} \tag{6.16}$$

where $b_i(t)$ is the boundary terms (conditions) which can be any two of the four of Eq. (6.12) as explained in detailed in Section 3.4 (Eqs. (3.25) and (3.34)) except that now they can vary with time, t.

Similar to the previous discussion, while the first integral term is familiar to us (from the discussion of beam bending in Chapter 3), the second term deserves further explanation. In this term, the shape functions, N_j is taken out from the derivative since it would not involve with the differentiation due to the fact that the shape functions are not a function of time but a function of x only. Therefore, something must be a function of time, else the second term would vanish. This leaves us with nothing but the degree of freedoms, d_j. So, d_j must be a function of time, t thus must involve in the differentiation as shown in Eq. (6.16). Now, by expressing the time second derivative by a double dot, Eq. (6.16) can be given in matrix forms as:

$$[k]\{d\} + [m]\{\ddot{d}\} = \{r(t)\} \tag{6.17}$$

where $\{r(t)\}$ is the load vector given as:

$$\{r(t)\} = \{q(t)\} + \{b(t)\} = \int_0^L N_i q(x,t)\, dx + b_i(t) \tag{6.18}$$

and $[k]$ is the stiffness matrix of the beam and $[m]$ is termed as the equivalent mass matrix of the beam. These matrices are already given by Eqs. (2.43b) and (5.49), respectively.

As can be noticed, Eq. (6.17) is exactly in a similar form as Eq. (6.8) derived for bar problem. This highlights that the same time-integration algorithm can be used for both problems, in fact for all problems, provided that they are arranged in a similar form. Discussion on time-integration to solve Eqs. (6.8) and (6.17) will be discussed next. But, before that, we will discuss first the analysis of forced vibration for a plane frame (and truss) system.

6.4 Forced Vibration of Plane Frame (and Truss)

The discussion on forced vibration of a frame (and truss) system would just be an extension of what have been discussed for bar and beam problems. All we need to do is to insert the beam-column's mass matrix, $[m']$ as given by Eq. (5.90) and the beam-column's stiffness matrix, $[k']$ as given by Eq. (4.43) into Eq. (6.17), which are derived based on local frame system. Then, transform the system into their corresponding global system by considering the transformation matrix $[T]$ of Eq. (4.51) as outlined in Section 4.3.2. Once transformed, the elemental matrices are assembled.

Accordingly, the elemental global discretised equation of forced vibration of plane frame can be given as:

$$[k]\{d\} + [m]\{\ddot{d}\} = \{r(t)\} \tag{6.19}$$

where

$$[k] = [T]^T[k'][T] \tag{6.20}$$

$$[m] = [T]^T[m'][T] \tag{6.21}$$

$$\{r\} = [T]^T\{r'(t)\} \tag{6.22}$$

As previously mentioned, the primed matrices, $[m']$, $[k']$ and $\{r'(t)\}$ are actually the local matrices and vector given by Eq. (5.90), Eq. (4.43) and Eq. (6.16), respectively. These are primed so as to distinguish them from their elemental global counterparts. As a reminder, detailed explanation on transformation are given in Section 4.3.2.

Having established the elemental global equation, the assembled global discretised equation of forced vibration of plane frame can thus be given as:

$$[M]\{\ddot{D}\} + [K]\{D\} = \{R(t)\} \tag{6.23}$$

which their assembled states are highlighted by the use of capital letters. The assembly process is similar to the one outlined in Section 4.2.

By observing Eq. (6.23), we should realize by now that the equation poses a new problem which we have never encountered before, that is, the time second derivative term symbolized by the double dots i.e. $\{\ddot{D}\}$. To solve this kind of problem, we need time integration, which is discussed next.

6.5 Time Integration and the Solution of Discretised Equation of Motion

Eq. (6.23) involves the second derivative of the dof's in time, expressed by $\{\ddot{D}\}$. As a derivative, it can be discretized, which the common practice would be, by finite difference method. Basically there are three ways to do this:

i. Forward difference
ii. Backward difference
iii. Central difference

Despite the three available techniques, herein, we will concern only with central difference technique. Readers who are interested to know more about the other techniques can refer to books on finite difference method and finite element method.

A central difference time discretization of the derivative terms, $\{\ddot{D}\}$ can be given as:

$$\{\ddot{D}\} = \left\{ \frac{D^{t+1} - 2D^t + D^{t-1}}{\Delta t^2} \right\} \tag{6.24}$$

where $t+1$ refers to the time at which the problem is solved, t is the time of the previous solution and $t-1$ is the time just before t. Δ_t refers to the time step taken.

Inserting Eq. (6.24) into Eq. (6.23) gives:

$$[M] \left\{ \frac{D^{t+1} - 2D^t + D^{t-1}}{\Delta t^2} \right\} + [K]\{D\}^t = \{R\}^t \tag{6.25}$$

Eq. (6.25) is the fully discretised equation of motion for the problem. As can be seen, spatial dimension terms are discretised by FEM whilst time derivative term is discretised by finite difference method (FDM).

It must be noted that, the vector of dof's associates with the stiffness matrix (i.e. $[K]\{D\}^t$ has been chosen as known values; solved values of previous time, t (i.e. $\{D\}^t$). Such a selection is actually a matter of choice. As will be seen, this will ease the solution process where the problem can be solved in an explicit manner (forward marching). However this comes at a cost.

Due to such a selection, despite the ease it provides, the process is not always stable. Therefore, the process requires that the time step, Δt to be smaller than the critical time step, Δt_{cr}, else the analysis will "blow-up". The critical time step, Δt_{cr} can be calculated as:

$$\Delta t_{cr} = \frac{2}{\omega_{\max}} \tag{6.26}$$

where $\omega_{\max}$ is simply the highest natural frequency of the system, obtained from an eigenproblem analysis discussed in Chapter 5.

By rearranging Eq. (6.25) we can obtain:

$$\frac{1}{\Delta t^2}[M]\{D\}^{t+1} = \{R\}^t - [K]\{D\}^t + \frac{2}{\Delta t^2}[M]\{D\}^t - \frac{1}{\Delta t^2}[M]\{D\}^{t-1} \tag{6.27}$$

It is a common practice to simplify Eq. (6.27) further as:

$$\begin{aligned} & a_0[M]\{D\}^{t+1} = \{R\}^t - [K]\{D\}^t + a_2[M]\{D\}^t - a_0[M]\{D\}^{t-1} \\ & \Rightarrow a_0[M]\{D\}^{t+1} = \{R\}^t - ([K] - a_2[M])\{D\}^t - a_0[M]\{D\}^{t-1} \end{aligned} \tag{6.28}$$

where $a_0 = \dfrac{1}{\Delta t^2}$ and $a_2 = \dfrac{2}{\Delta t^2}$.

Having established Eq. (6.28), we are now in the position to solve the problem. The straight forward nature of Eq. (6.28), should be obvious to us because as we can see, all the terms on the right hand side of the equation are known values, obtained from previous analysis (previous time step). Having everything known on the right hand side the equation, $\{D\}^{t+1}$ now can be solved by inversing the matrix $[M]$ (or by employing Gauss Elimination or any other solvers). This is best demonstrated by examples. However, before we do that, we must discuss the initial conditions for the problem; a compulsory discussion and provision for all dynamic (time-dependent) problems.

6.5.1 Initial Conditions

Examining Eq. (6.28), we can see that, to kick start the procedure, we must provide two known values of $\{D\}^t$ and $\{D\}^{t-1}$. To provide these values and since we have second order derivatives in time (i.e. $\frac{\partial D_j}{\partial t^2}$), two initial conditions must be known which can either be, displacement, $\{D\}^0$, velocity $\{\dot{D}\}^0$ or the acceleration $\{\ddot{D}\}^0$. Knowing any two of these is sufficient because the third one can be obtained consequently. As an example, for a system at rest, $\{D\}^0$ and $\{\dot{D}\}^0$ can be taken as zero (i.e. $\{D\}^0 = \{\dot{D}\}^0 = 0$).

With these conditions, $\{\ddot{D}\}^0$ can be determined by solving Eq. (6.23) as below:

$$
\begin{aligned}
&[M]\{\ddot{D}\}^0 + [K]\{0\} = \{R\}^0 \\
&\Rightarrow [M]\{\ddot{D}\}^0 = \{R\}^0 \\
&\Rightarrow \{\ddot{D}\}^0 = [M]^{-1}\{R\}^0
\end{aligned}
\tag{6.29}
$$

Based on these initial conditions, the values of $\{D\}^t$ and $\{D\}^{t-1}$ at the first time step, $t = 0$ (for the case of a system at rest prior to the imposition of loading) can be given as:

$$\{D\}^t = \{D\}^0 = 0 \tag{6.30}$$

while $\{D\}^{t-1}$ can be obtained from the following relationship:

$$\{D\}^{t-1} = \{D\}^0 - \Delta t\{\dot{D}\}^0 + \frac{\Delta t^2}{2}\{\ddot{D}\}^0 \tag{6.31}$$

For a system at rest, Eq. (6.31) means:

$$\{D\}^{t-1} = \frac{\Delta t^2}{2}\{\ddot{D}\}^0 \tag{6.32}$$

Having established the values of $\{D\}^t$ and $\{D\}^{t-1}$ for the first time step (to kick start the recursion), we are now all set for some examples.

6.6 Worked Example 6.1

Let's solve the plane frame problem of Worked Example 5.3 (as shown in Fig. 6.1). This time, the frame is subjected to a time-varying point load as shown in Fig. 6.2.

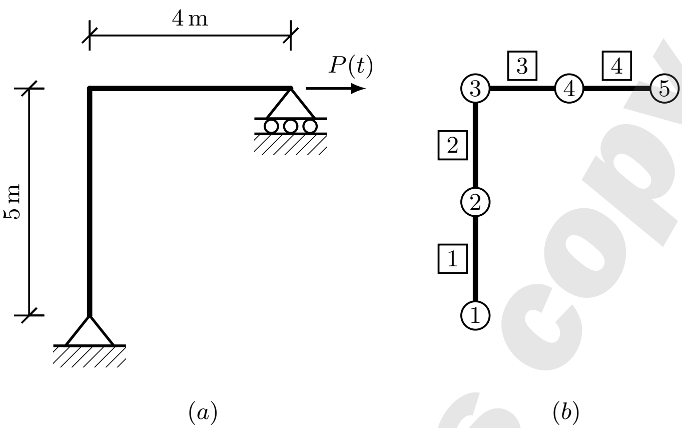

Figure 6.1: a) Plane Frame structure and b) the beam-column elements for free vibration. All elements have square cross section ($0.1\,\mathrm{m} \times 0.1\,\mathrm{m}$) with a Young's modulus of $E = 200\times10^6\,\mathrm{kN\,m^{-2}}$.

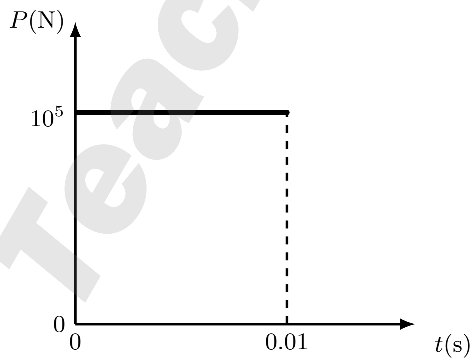

Figure 6.2: Time-dependent loading.

The global stiffness matrix, $[K]$ and the global mass matrix, $[M]$ are similar to Eq. (5.103), in which the local matrices together with the transformation matrices, and also the assembly process have already been discussed. All we need to do now is to provide the global displacement and load vector for the current problem.

For completeness, $[K]$ and $[M]$ are given herein as follows:

$$[K] = \begin{bmatrix} 5.92 & 0 & -4.44 \ldots & 0 & 0 & 0 \\ 0 & 1333 & 0 \quad \ldots & 0 & 0 & 0 \\ -4.44 & 0 & 4.44 \ldots & 0 & 0 & 0 \\ . & . & . \quad \ldots & . & . & . \\ . & . & . \quad \ldots & . & . & . \\ . & . & . \quad \ldots & . & . & . \\ 0 & 0 & 0 \quad \ldots & 2000 & 0 & 0 \\ 0 & 0 & 0 \quad \ldots & 0 & 20 & -10 \\ 0 & 0 & 0 \quad \ldots & 0 & -10 & 6.66 \end{bmatrix} \times 10^6 \tag{6.33}$$

and

$$[M] = \begin{bmatrix} 43.791 & 0 & -9.264 \ldots & 0 & 0 & 0 \\ 0 & 39.300 & 0 \quad \ldots & 0 & 0 & 0 \\ -9.264 & 0 & 2.526 \ldots & 0 & 0 & 0 \\ . & . & . \quad \ldots & . & . & . \\ . & . & . \quad \ldots & . & . & . \\ . & . & . \quad \ldots & . & . & . \\ 0 & 0 & 0 \quad \ldots & 26.200 & 0 & 0 \\ 0 & 0 & 0 \quad \ldots & 0 & 29.194 & -4.117 \\ 0 & 0 & 0 \quad \ldots & 0 & -4.117 & 0.749 \end{bmatrix} \tag{6.34}$$

Since the external point load is inserted directly into the assembled global load vector, the global load vector, $\{R(t)\}$ (which is a function of time) can be given as:

$$\{R(t)\} = \left\{ \begin{array}{c} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P(t) \\ R_{14} \\ 0 \end{array} \right\} \tag{6.35}$$

where R_1, R_2 and R_{14} are the reaction forces. Note that, $P(t)$ is a constant when $t \leq 0.01$ seconds as depicted in Fig. 6.2.

The complete global equilibrium can be given as:

$$[K]\{D\} + [M]\{\ddot{D}\} = \{R(t)\}$$

$$\begin{bmatrix} 5.92 & 0 & -4.44 \ldots & 0 & 0 & 0 \\ 0 & 1333 & 0 \quad \ldots & 0 & 0 & 0 \\ -4.44 & 0 & 4.44 \ldots & 0 & 0 & 0 \\ . & . & . \quad \ldots & . & . & . \\ . & . & . \quad \ldots & . & . & . \\ . & . & . \quad \ldots & . & . & . \\ 0 & 0 & 0 \quad \ldots & 2000 & 0 & 0 \\ 0 & 0 & 0 \quad \ldots & 0 & 20 & -10 \\ 0 & 0 & 0 \quad \ldots & 0 & -10 & 6.66 \end{bmatrix} \times 10^6 \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ . \\ . \\ . \\ D_{13} \\ D_{14} \\ D_{15} \end{Bmatrix}$$

$$+ \begin{bmatrix} 43.791 & 0 & -9.264 \ldots & 0 & 0 & 0 \\ 0 & 39.300 & 0 \quad \ldots & 0 & 0 & 0 \\ -9.264 & 0 & 2.526 \ldots & 0 & 0 & 0 \\ . & . & . \quad \ldots & . & . & . \\ . & . & . \quad \ldots & . & . & . \\ . & . & . \quad \ldots & . & . & . \\ 0 & 0 & 0 \quad \ldots & 26.200 & 0 & 0 \\ 0 & 0 & 0 \quad \ldots & 0 & 29.194 & -4.117 \\ 0 & 0 & 0 \quad \ldots & 0 & -4.117 & 0.749 \end{bmatrix} \begin{Bmatrix} \ddot{D}_1 \\ \ddot{D}_2 \\ \ddot{D}_3 \\ . \\ . \\ . \\ \ddot{D}_{13} \\ \ddot{D}_{14} \\ \ddot{D}_{15} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ P(t) \\ 0 \\ 0 \end{Bmatrix} \tag{6.36}$$

By imposing the essential boundary conditions ($D_1 = D_2 = D_{14} = 0$), the assembled global equilibrium equation is reduced to:

$$[K]\{D\} + [M]\{\ddot{D}\} = \{R(t)\}$$

$$\begin{bmatrix} 4.44 & 4.44 & 0 & \dots & 0 & 0 & 0 \\ 4.44 & 11.8 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 2667 & \dots & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . \\ . & . & . & \dots & . & . & . \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & 13.3 & 0 & 3.33 \\ 0 & 0 & 0 & \dots & 0 & 2000 & 0 \\ 0 & 0 & 0 & \dots & 3.33 & 0 & 6.66 \end{bmatrix} \times 10^6 \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ . \\ . \\ . \\ D_{12} \\ D_{13} \\ D_{15} \end{Bmatrix}$$

$$+ \begin{bmatrix} 2.526 & -5.474 & 0 & \dots & 0 & 0 & 0 \\ -5.474 & 87.583 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 78.600 & \dots & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . \\ . & . & . & \dots & . & . & . \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & 1.497 & 0 & -0.561 \\ 0 & 0 & 0 & \dots & 0 & 26.200 & 0 \\ 0 & 0 & 0 & \dots & -0.561 & 0 & 0.749 \end{bmatrix} \begin{Bmatrix} \ddot{D}_3 \\ \ddot{D}_4 \\ \ddot{D}_5 \\ . \\ . \\ . \\ \ddot{D}_{12} \\ \ddot{D}_{13} \\ \ddot{D}_{15} \end{Bmatrix} \tag{6.37}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ P(t) \\ 0 \\ 0 \end{Bmatrix}$$

6.6.1 Solution at initial time-step ($t = 0$)

1: Determine $\{\ddot{D}\}$ from Eq. (6.29)

Let's take $\Delta t = 0.000\,01\,\mathrm{s}$ (as mentioned earlier, Δt need to be smaller than Δt_{cr}, and assume that $\{D\}^0 = 0$. Thus $\{\ddot{D}\}^0$ can be solved using Matlab command "\" as:

$$[K]\{0\} + [M]\{\ddot{D}\}^0 = \{R\}^0$$

$$\begin{Bmatrix} \ddot{D}_3 \\ \ddot{D}_4 \\ \ddot{D}_5 \\ \ddot{D}_6 \\ \ddot{D}_7 \\ \ddot{D}_8 \\ \ddot{D}_9 \\ \ddot{D}_{10} \\ \ddot{D}_{11} \\ \ddot{D}_{12} \\ \ddot{D}_{13} \\ \ddot{D}_{15} \end{Bmatrix} = [M]\backslash\{R\}^0 = \begin{Bmatrix} -59.62 \\ 17.00 \\ -16.83 \\ -128.62 \\ 403.90 \\ 67.31 \\ -1450.19 \\ -1205.91 \\ 36.82 \\ -541.99 \\ 4419.75 \\ -286.82 \end{Bmatrix} \tag{6.38}$$

2: Solve for $\{D\}$ from Eq. (6.31)

$$\{D\}^{-1} = \{D\}^0 - \Delta t\{\dot{D}\}^0 + \frac{\Delta t^2}{2}\{\ddot{D}\}^0$$

$$\begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \\ D_{13} \\ D_{15} \end{Bmatrix}^{-1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \frac{\Delta t^2}{2} \begin{Bmatrix} -59.62 \\ 17.00 \\ -16.83 \\ -128.62 \\ 403.90 \\ 67.31 \\ -1450.19 \\ -1205.91 \\ 36.82 \\ -541.99 \\ 4419.75 \\ -286.82 \end{Bmatrix} = \begin{Bmatrix} -2.98\times10^{-9} \\ 8.50\times10^{-10} \\ -8.41\times10^{-10} \\ -6.43\times10^{-9} \\ 2.02\times10^{-8} \\ 3.37\times10^{-9} \\ -7.25\times10^{-8} \\ -6.03\times10^{-8} \\ 1.84\times10^{-9} \\ -2.71\times10^{-8} \\ 2.21\times10^{-7} \\ -1.43\times10^{-8} \end{Bmatrix} \tag{6.39}$$

3: Substitute $\{D\}^{-1}$ and $\{D\}^{0}$ into Eq. (6.27)

$$\frac{1}{\Delta t^2}[M]\{D\}^1 = \{R\}^0 - ([K] - \frac{2}{\Delta t^2}[M])\{D\}^0 - \frac{1}{\Delta t^2}[M]\{D\}^{-1}$$

simplifying

$$[LHS]\{D\}^1 = \{RHS\}$$

where

$$[LHS] = \frac{1}{\Delta t^2}[M]$$

$$\{RHS\} = \{R\}^0 - ([K] - \frac{2}{\Delta t^2}[M])\{D\}^0 - \frac{1}{\Delta t^2}[M]\{D\}^{-1}$$

4: Solve $\{D\}^1$ using Matlab command "\", thus:

$$\{D\}^1 = [LHS]\backslash\{RHS\} = \begin{Bmatrix} -2.98\times10^{-9} \\ 8.50\times10^{-10} \\ -8.41\times10^{-10} \\ -6.43\times10^{-9} \\ 2.02\times10^{-8} \\ 3.37\times10^{-9} \\ -7.25\times10^{-8} \\ -6.03\times10^{-8} \\ 1.84\times10^{-9} \\ -2.71\times10^{-8} \\ 2.21\times10^{-7} \\ -1.43\times10^{-8} \end{Bmatrix}$$

6.6.2 Solution at subsequent time-steps ($t > 0$)

Solution at $t = 1$

The next time step can be computed by replacing $\{D\}^0$ with $\{D\}^1$, $\{D\}^{-1}$ with $\{D\}^0$, and $\{R\}^0$ with $\{R\}^1$ to yield the following form:

$$\frac{1}{\Delta t^2}[M]\{D\}^2 = \{R\}^1 - ([K] - \frac{2}{\Delta t^2}[M])\{D\}^1 - \frac{1}{\Delta t^2}[M]\{D\}^0$$

and solve for $\{D\}^2$ gives:

$$\{D\}^2 = \begin{Bmatrix} -1.17\times10^{-8} \\ 3.34\times10^{-9} \\ -3.30\times10^{-9} \\ -2.53\times10^{-8} \\ 7.95\times10^{-8} \\ 1.32\times10^{-8} \\ -2.85\times10^{-7} \\ -2.39\times10^{-7} \\ 7.25\times10^{-9} \\ -1.07\times10^{-7} \\ 8.81\times10^{-7} \\ -5.64\times10^{-8} \end{Bmatrix}$$

Solution at $t = 2$

Again, by replacing $\{D\}^1$ with $\{D\}^2$, $\{D\}^0$ with $\{D\}^1$, and $\{R\}^1$ with $\{R\}^2$ to yield the following form:

$$\frac{1}{\Delta t^2}[M]\{D\}^3 = \{R\}^2 - ([K] - \frac{2}{\Delta t^2}[M])\{D\}^2 - \frac{1}{\Delta t^2}[M]\{D\}^1$$

and solve for $\{D\}^3$ gives:

$$\{D\}^3 = \begin{Bmatrix} -2.55\times10^{-8} \\ 7.31\times10^{-9} \\ -7.19\times10^{-9} \\ -5.52\times10^{-8} \\ 1.74\times10^{-7} \\ 2.89\times10^{-8} \\ -6.25\times10^{-7} \\ -5.27\times10^{-7} \\ 1.59\times10^{-8} \\ -2.33\times10^{-7} \\ 1.97\times10^{-6} \\ -1.23\times10^{-7} \end{Bmatrix}$$

Solution at $t = 3$

Again, by replacing $\{D\}^2$ with $\{D\}^3$, $\{D\}^1$ with $\{D\}^2$, and $\{R\}^2$ with $\{R\}^3$ to yield the following form:

$$\frac{1}{\Delta t^2}[M]\{D\}^4 = \{R\}^3 - ([K] - \frac{2}{\Delta t^2}[M])\{D\}^3 - \frac{1}{\Delta t^2}[M]\{D\}^2$$

and solve for $\{D\}^4$ gives:

$$\{D\}^4 = \begin{Bmatrix} -4.35\times10^{-8} \\ 1.25\times10^{-8} \\ -1.22\times10^{-8} \\ -9.40\times10^{-8} \\ 2.98\times10^{-7} \\ 4.94\times10^{-8} \\ -1.07\times10^{-6} \\ -9.14\times10^{-7} \\ 2.72\times10^{-8} \\ -3.98\times10^{-7} \\ 3.47\times10^{-6} \\ -2.10\times10^{-7} \end{Bmatrix}$$

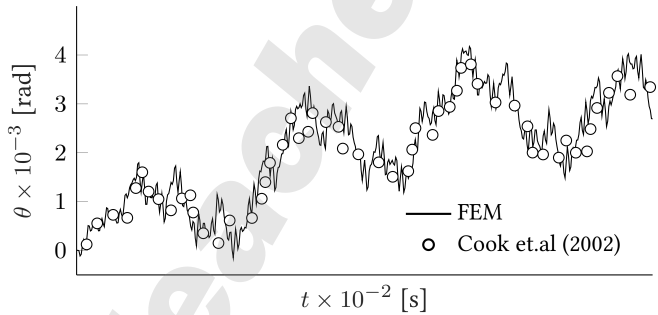

Figure 6.3: Vibration (rotation) of the frame.

These steps can be continued until a specific period of time by repeating the same procedure.

Fig. 6.3 shows the vibration (rotation) of the frame taken at $x = 1\,\mathrm{m}$ from the roller support. The results is compared with Cook et.al (2002)[†].

[†] Cook, R. D., Malkus, D. S., Plesha, M. E., & Witt, R. J. (1974). *Concepts and applications of finite element analysis* (Vol. 4). New York: Wiley. (Figure 11.18-2(d), page 441)

6.7 Matlab Source Code

```
% Clear data
clc; clear; close all

%Input
E   = 200e9;      % Young's modulus [N/m]
A   = 0.01;       % Area [m^2]
I   = 8.33e-6;    % Moment of inertia [m^4]
rho = 7860;       % Density [kg/m^3]
P   = 1e5;        % Point load [N]

dt  = 0.00001;    % Time step [s]
t   = 0:dt:0.1;   % Time vector [s]

% ---------------------------------------------
% FEM solution - Forced vibration
% ---------------------------------------------

% Elements length [m]
L1 = 1.5; L2 = 1.5; L3 = 1.0; L4 = 1.0;

% Elements angle [degree]
th1 = 90; th2 = 90; th3 = 0; th4 = 0;

% Calculate Stifness
k1 = [
 A*E/L1  0            0           -A*E/L1     0            0;
 0       12*E*I/L1^3  6*E*I/L1^2   0         -12*E*I/L1^3  6*E*I/L1^2;
 0       6*E*I/L1^2   4*E*I/L1     0         -6*E*I/L1^2   2*E*I/L1;
-A*E/L1  0            0            A*E/L1     0            0;
 0      -12*E*I/L1^3 -6*E*I/L1^2   0          12*E*I/L1^3 -6*E*I/L1^2;
 0       6*E*I/L1^2   2*E*I/L1     0         -6*E*I/L1^2   4*E*I/L1];
k2 = [
 A*E/L2  0            0           -A*E/L2     0            0;
 0       12*E*I/L2^3  6*E*I/L2^2   0         -12*E*I/L2^3  6*E*I/L2^2;
 0       6*E*I/L2^2   4*E*I/L2     0         -6*E*I/L2^2   2*E*I/L2;
-A*E/L2  0            0            A*E/L2     0            0;
 0      -12*E*I/L2^3 -6*E*I/L2^2   0          12*E*I/L2^3 -6*E*I/L2^2;
 0       6*E*I/L2^2   2*E*I/L2     0         -6*E*I/L2^2   4*E*I/L2];
k3 = [
 A*E/L3  0            0           -A*E/L3     0            0;
 0       12*E*I/L3^3  6*E*I/L3^2   0         -12*E*I/L3^3  6*E*I/L3^2;
 0       6*E*I/L3^2   4*E*I/L3     0         -6*E*I/L3^2   2*E*I/L3;
-A*E/L3  0            0            A*E/L3     0            0;
 0      -12*E*I/L3^3 -6*E*I/L3^2   0          12*E*I/L3^3 -6*E*I/L3^2;
 0       6*E*I/L3^2   2*E*I/L3     0         -6*E*I/L3^2   4*E*I/L3];
k4 = [
 A*E/L4  0            0           -A*E/L4     0            0;
 0       12*E*I/L4^3  6*E*I/L4^2   0         -12*E*I/L4^3  6*E*I/L4^2;
 0       6*E*I/L4^2   4*E*I/L4     0         -6*E*I/L4^2   2*E*I/L4;
-A*E/L4  0            0            A*E/L4     0            0;
 0      -12*E*I/L4^3 -6*E*I/L4^2   0          12*E*I/L4^3 -6*E*I/L4^2;
 0       6*E*I/L4^2   2*E*I/L4     0         -6*E*I/L4^2   4*E*I/L4];


%Calculate Consistent Mass Matrix
m1 = rho*A*L1/420* [140     0       0       70      0       0;
                    0       156     22*L1   0       54      -13*L1;
                    0       22*L1   4*L1^2  0       13*L1   -3*L1^2;
```

```
                        70      0       0       140     0        0;
                        0       54      13*L1   0       156     -22*L1;
                        0       -13*L1  -3*L1^2 0       -22*L1   4*L1^2];
m2 = rho*A*L2/420* [140     0       0       70      0        0;
                        0       156     22*L2   0       54      -13*L2;
                        0       22*L2   4*L2^2  0       13*L2   -3*L2^2;
                        70      0       0       140     0        0;
                        0       54      13*L2   0       156     -22*L2;
                        0       -13*L2  -3*L2^2 0       -22*L2   4*L2^2];
m3 = rho*A*L3/420* [140     0       0       70      0        0;
                        0       156     22*L3   0       54      -13*L3;
                        0       22*L3   4*L3^2  0       13*L3   -3*L3^2;
                        70      0       0       140     0        0;
                        0       54      13*L3   0       156     -22*L3;
                        0       -13*L3  -3*L3^2 0       -22*L3   4*L3^2];
m4 = rho*A*L4/420* [140     0       0       70      0        0;
                        0       156     22*L4   0       54      -13*L4;
                        0       22*L4   4*L4^2  0       13*L4   -3*L4^2;
                        70      0       0       140     0        0;
                        0       54      13*L4   0       156     -22*L4;
                        0       -13*L4  -3*L4^2 0       -22*L4   4*L4^2];

% Transformation matrix
c  = cosd(th1);
s  = sind(th1);
T1 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

c  = cosd(th2);
s  = sind(th2);
T2 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

c  = cosd(th3);
s  = sind(th3);
T3 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

c  = cosd(th4);
s  = sind(th4);
T4 =[c s 0  0 0 0;
    -s c 0  0 0 0;
     0 0 1  0 0 0;
     0 0 0  c s 0;
     0 0 0 -s c 0;
     0 0 0  0 0 1];

% Assemble global stiffness, K and mass, M matrices
K = zeros(15);
```

```
K(1:6,1:6)      = K(1:6,1:6) + T1'*k1* T1;
K(4:9,4:9)      = K(4:9,4:9) + T2'*k2* T2;
K(7:12,7:12)    = K(7:12,7:12) + T3'*k3* T3;
K(10:15,10:15)  = K(10:15,10:15) + T4'*k4* T4;

M = zeros(15);
M(1:6,1:6)      = M(1:6,1:6) + T1'*m1* T1;
M(4:9,4:9)      = M(4:9,4:9) + T2'*m2* T2;
M(7:12,7:12)    = M(7:12,7:12) + T3'*m3* T3;
M(10:15,10:15)  = M(10:15,10:15) + T4'*m4* T4;

% Free variables
fv = [3:13 15];

% Solve for eigen value
[~, eigVal] = eig(K(fv,fv), M(fv,fv));
eigVal  = diag(eigVal);
Freq    = sqrt(eigVal);   % circular freq (rad/sec)

% Check critical time step
dt_crit = 2/max(Freq);

if dt <= dt_crit
   fprintf('Time step  = %.2e s\n', dt)
   fprintf('Critical time step  = %.2e s\n', dt_crit)
   fprintf('Time step > Critical time step. Pass !\n')
else
   fprintf('Time step  = %.2e s\n', dt)
   fprintf('Critical time step  = %.2e s\n', dt_crit)
   fprintf('Time step < Critical time step. Fail !\n')
end

% ---------------------------------------------
% Solution at initial time-step (t=0)
% ---------------------------------------------

% Initialize solution vector
DT  = zeros(length(t), 15);

% Calculate Equation Constant
a0  = 1/dt^2;
a1  = dt^2/2;
a2  = 2/dt^2;

% Force vector
R       = zeros(15,1);
R(13)   = P;

% Acceleration at t = 0
A      = zeros(15,1);
A(fv)  = M(fv,fv)\R(fv);

% Displacement
D1  = zeros(15,1);  % Displacement at time, t
D0  = a1*A;         % Displacement at time, t-dt

% ---------------------------------------------
% Solution at subsequent time-step (t>0)
% ---------------------------------------------

% Loop over total time step t=t+1
```

```
for i = 2:length(t)

%   Apply load for 0.01s only
if t(i) <= 0.01
    R(13)   = P;
else
    R(13) = 0;
end

%   LHS & RHS of Eqn 1.28
lhs = a0*M;
rhs = R - (K-a2*M)*D1 - a0*M*D0;

%   Solve for displacement at t+1
D2      = zeros(15,1);
D2(fv)  = lhs(fv,fv)\rhs(fv);

%   Save result
DT(i,:) = D2;

%   Next time step
D0 = D1;
D1 = D2;

end

% Plot result (12th dof - rotation)
plot(t, DT(:,12), '-k')
xlabel('Time (sec)');
ylabel('Theta (rad)');
```


6.8 Excercises

1. Fig. 6.4(a) shows a beam subjected to a time dependent distributed load

   i. What is the function of the distributed load, in x direction?
   ii. If the flexural stiffness of the beam is EI and its density is ρ, show that (by derivation) the partial differential equation (PDE) of the problem is:

   $$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = q(x,t)$$

   iii. Show symbolically (by derivation) that the PDE above can be discretized into:

   $$[M]\{\ddot{d}\} + [K]\{d\} = \{R\}$$

   where $[M]$ is the mass matrix, $[K]$ is the stiffness matrix and

$\{R\}$ is the load vector. $\{d\}$ is the vector of degree of freedom, which are shown in Fig. 6.4(b).

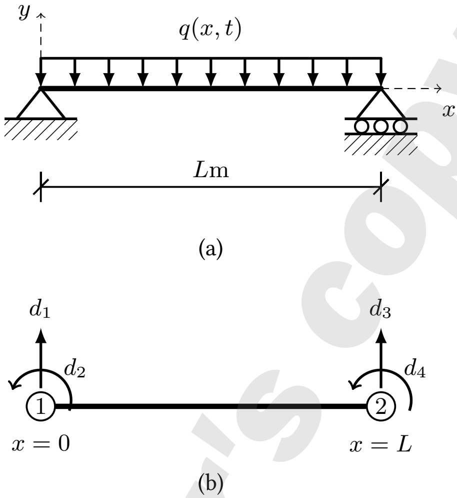

Figure 6.4

2. A fixed-free beam subjected to a time-dependent point loading of $10t^2$ at the free-edge (where t is time) is illustrated in Fig. 6.5. The mass matrix, $[m]$ and the stiffness matrix, $[k]$ of the beam after the imposition of the essential boundary conditions are given as:

$$[m] = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}$$

$$[k] = \begin{bmatrix} 12 & 3 \\ 3 & 4 \end{bmatrix}$$

   i. Determine the values of the point load at time $t = 1$ and $t = 2$.
   ii. Based on answer (i), what is the appropriate value for the time step, Δt?
   iii. If central difference is employed for the time integration of the

problem which equation can be given as:

$$\frac{1}{\Delta t^2}[m]\{d\}^{t+1} = \{r\}^t - [k]\{d\}^t + \frac{2}{\Delta t^2}[m]\{d\}^t - \frac{1}{\Delta t^2}[m]\{d\}^{t-1} \tag{6.40}$$

solve $\{d\}^{t+1}$ for the first three (3) time steps (i.e. $t = 1, = 2$ and $t = 3$). Take the initial conditions of the problem at $t = 0$, as $\{d\}^t = 0$ and $\{d\}^{t-1} = 0$.

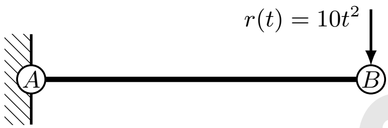

Figure 6.5

3. Fig. 6.6 shows a frame that linked a column and a beam having a fixed support at the two ends, and subjected to a time-dependant loading of moment, $r(t) = 12.5t$.

   If the elemental stiffness matrix $[k]$ and the elemental mass matrix $[m]$ are given as in the table below, solve the displacements $\{d\}$ for the first three (3) time steps (consider central difference for the time integration).

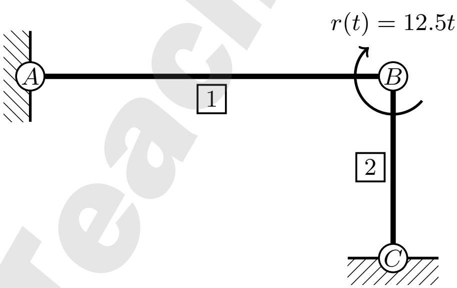

Figure 6.6

Element 1

$$
\mathbf{k}_2 = \begin{bmatrix}
10.0 & 0 & 0 & -10.0 & 0 & 0 \\
0 & 2.5 & 7.5 & 0 & -2.5 & 7.5 \\
0 & 7.5 & 30.0 & 0 & -7.5 & 15.0 \\
-10.0 & 0 & 0 & 10.0 & 0 & 0 \\
0 & -2.5 & -7.5 & 0 & 2.5 & -7.5 \\
0 & 7.5 & 15.0 & 0 & -7.5 & 30.0
\end{bmatrix}
$$

$$
\mathbf{m}_2 = \begin{bmatrix}
252.0 & 0 & 0 & 126.0 & 0 & 0 \\
0 & 280.8 & 237.6 & 0 & 97.2 & -140.4 \\
0 & 237.6 & 259.2 & 0 & 140.4 & -194.4 \\
126.0 & 0 & 0 & 252.0 & 0 & 0 \\
0 & 97.2 & 140.4 & 0 & 280.8 & -237.6 \\
0 & -140.4 & -194.4 & 0 & -237.6 & 259.2
\end{bmatrix}
$$

Element 2

$$
\mathbf{k}_1 = \begin{bmatrix}
67.5 & 0 & -67.5 & -67.5 & 0 & -67.5 \\
0 & 30.0 & 0 & 0 & -30.0 & 0 \\
-67.5 & 0 & 90.0 & 67.5 & 0 & 45.0 \\
-67.5 & 0 & 67.5 & 67.5 & 0 & 67.5 \\
0 & -30.0 & 0 & 0 & 30.0 & 0 \\
-67.5 & 0 & 45.0 & 67.5 & 0 & 90.0
\end{bmatrix}
$$

$$\mathbf{m}_1 = \begin{bmatrix} 93.6 & 0 & -26.4 & 32.4 & 0 & 15.6 \\ 0 & 84.0 & 0 & 0 & 42.0 & 0 \\ -26.4 & 0 & 9.6 & -15.6 & 0 & -7.2 \\ 32.4 & 0 & -15.6 & 93.6 & 0 & 26.4 \\ 0 & 42.0 & 0 & 0 & 84.0 & 0 \\ 15.6 & 0 & -7.2 & 26.4 & 0 & 9.6 \end{bmatrix}$$

Answers to Excercises

Chapter 1

Question 2

i. Basic numerical technique

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & EA\cos L & -EA\sin L \\ 0 & -EA\sin\dfrac{L}{2} & -EA\cos\dfrac{L}{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ -q \end{Bmatrix}$$

ii. Weighted Residual Method

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & EA\cos L & -EA\sin L \\ 0 & -EA\sin\dfrac{L}{2} & -EA\cos\dfrac{L}{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ -qL \end{Bmatrix}$$

Chapter 2

Question 1

i. Displacement at nodes 2 and 3

- Node 2: 8.8×10^{-6} m

- Node 3: $3.71{\times}10^{-5}\,\mathrm{m}$

ii. Stress in each element

- Element 1: $5.69{\times}10^{6}\,\mathrm{N\,m^{-2}}$ (tension)
- Element 2: $2.20{\times}10^{7}\,\mathrm{N\,m^{-2}}$ (tension)
- Element 3: $2.89{\times}10^{7}\,\mathrm{N\,m^{-2}}$ (compression)

iii. Reaction force at supports

- Node 1: $-4.47{\times}10^{4}\,\mathrm{N}$
- Node 4: $-5.67{\times}10^{4}\,\mathrm{N}$

Question 2

i. Vertical displacement at nodes 2

- Node 2: $0.028\,\mathrm{m}$

ii. Stress in both element

- Element 1: $3.93{\times}10^{8}\,\mathrm{N\,m^{-2}}$
- Element 2: $1.02{\times}10^{9}\,\mathrm{N\,m^{-2}}$ and $2.33{\times}10^{8}\,\mathrm{N\,m^{-2}}$

Question 3

i. The stiffness matrix, **K** and the force vector, **r**

$$\mathbf{K} = \begin{bmatrix} 32.4 & -37 & 4.63 \\ -37 & 74 & -37 \\ 4.63 & -37 & 32.4 \end{bmatrix} \times 10^{7}$$

$$\mathbf{r} = \begin{Bmatrix} 53.33 \\ 426.67 \\ 266.67 \end{Bmatrix}$$

ii. Maximum stress in the bar

- Node 1: $6.22{\times}10^5\,\mathrm{N\,m^{-2}}$ (tension)

Question 4

i. Elemental stiffness matrices and body force vectors

$$k_1 = \begin{bmatrix} 1.84{\times}10^8 & -1.84{\times}10^8 \\ -1.84{\times}10^8 & 1.84{\times}10^8 \end{bmatrix}$$

$$k_2 = \begin{bmatrix} 1.31{\times}10^8 & -1.31{\times}10^8 \\ -1.31{\times}10^8 & 1.31{\times}10^8 \end{bmatrix}$$

$$r_1 = \begin{Bmatrix} -3.03 \\ -3.03 \end{Bmatrix}$$

$$r_2 = \begin{Bmatrix} -2.17 \\ -2.17 \end{Bmatrix}$$

ii. Assemble global stiffness and load vector

$$\mathbf{K} = \begin{bmatrix} 1.84{\times}10^8 & -1.84{\times}10^8 & 0 \\ -1.84{\times}10^8 & 3.15{\times}10^8 & -1.31{\times}10^8 \\ 0 & -1.31{\times}10^8 & 1.31{\times}10^8 \end{bmatrix}$$

$$\mathbf{r} = \begin{Bmatrix} -3.03 \\ -5.30 \\ -2.17 \end{Bmatrix}$$

iii. Global axial displacement (m)

$$\mathbf{U} = \begin{Bmatrix} 0 \\ -5.84\times10^{-7} \\ -6.01\times10^{-7} \end{Bmatrix}$$

iv. Axial stress in each element

- Element 1: $4.21\times10^{5}\,\mathrm{N\,m^{-2}}$ and $-3.97\times10^{5}\,\mathrm{N\,m^{-2}}$
- Element 2: $2.31\times10^{4}\,\mathrm{N\,m^{-2}}$

Chapter 3

Question 1

ii. Shape function

$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$$

$$N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$

$$N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$$

$$N_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

iii. Equivalent moment

$$M_1 = 37.13$$

$$M_2 = -24.54$$

Question 2

i. Deflection and rotation at nodes 2 and 3

- At node 2: $D_3 = 0\,\mathrm{m}$, $D_4 = 3.02\times10^{-2}\,\mathrm{rad}$
- At node 3: $D_5 = -7.7\times10^{-3}\,\mathrm{m}$, $D_6 = -2.02\times10^{-2}\,\mathrm{rad}$

ii. Reaction forces at support and bending moment at node 2

- At node 1: $F_1 = 3.53\times10^{4}\,\mathrm{N}$, $F_2 = 3.31\times10^{4}\,\mathrm{N\,m}$
- At node 2: $F_3 = 3.77\times10^{4}\,\mathrm{N}$, $F_4 = 0\,\mathrm{N\,m}$

Question 3

i. System model and nodal rotation using one beam element

$$\begin{bmatrix} 4.12 & 3.70 & -4.12 & 3.70 \\ 3.70 & 4.44 & -3.70 & 2.22 \\ -4.12 & -3.70 & 4.12 & -3.70 \\ 3.70 & 2.22 & -3.70 & 4.44 \end{bmatrix} \times 10^7 \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} -25\,650 \\ -2695 \\ -25\,650 \\ 7695 \end{Bmatrix}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1.96\times10^{-4} \\ -1.44\times10^{-4} \\ 2.71\times10^{-4} \end{Bmatrix}$$

ii. System model and nodal rotation using two beam elements

$$\begin{bmatrix} 3.29 & 1.48 & -3.29 & 1.48 & 0 & 0 \\ 1.48 & 0.89 & -1.48 & 0.44 & 0 & 0 \\ -3.29 & -1.48 & 6.58 & 0 & -3.29 & 1.48 \\ 1.48 & 0.44 & 0 & 1.78 & -1.48 & 0.44 \\ 0 & 0 & -3.92 & -1.48 & 3.29 & -1.48 \\ 0 & 0 & 1.48 & 0.44 & -1.48 & 0.89 \end{bmatrix} \times 10^8 \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -1.28\times10^4 \\ 3.08\times10^3 \\ -2.57\times10^4 \\ 0 \\ -1.29\times10^4 \\ 1.92\times10^3 \end{Bmatrix}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1.96\times10^{-4} \\ -1.44\times10^{-4} \\ -1.88\times10^{-5} \\ 0 \\ 2.71\times10^{-4} \end{Bmatrix}$$

Question 4

i. The vertical displacement at the right end: $-4.72\,\mathrm{m}$

ii. The bending moment at the two ends of the beam:

- At node 1: $D_3 = 5.29\times10^4\,\mathrm{N\,m}$
- At node 2: $D_5 = 2.65\times10^4\,\mathrm{N\,m}$

Question 5

i. Equivalent nodal load vector

$$R = \begin{Bmatrix} -14.4 \\ -9.45 \\ -30.60 \\ 13.5 \end{Bmatrix}$$

ii. The degree of freedom vector

$$D = \begin{Bmatrix} 0 \\ -8.76\times10^{-7} \\ 0 \\ 9.85\times10^{-7} \end{Bmatrix}$$

iii. The new degree of freedom vector

$$D = \begin{Bmatrix} 0 \\ -5.41\times10^{-4} \\ 0 \\ 2.71\times10^{-4} \end{Bmatrix}$$

Question 6

$$\begin{aligned} w = &\left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)\left(-\frac{L^2(2PL+3M_L)}{6EI}\right) \\ &+ \left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right)\left(-\frac{L(PL+2M_L)}{2EI}\right) \end{aligned}$$

Chapter 4

Question 1

i. Displacement at joint B:

- U_3: $8.83\,\mathrm{m}$
- U_4: $-35.08\,\mathrm{m}$

ii. Horizontal reaction at joint A and C

- R_1: $-3.74\,\mathrm{N}$
- R_5: $3.74\,\mathrm{N}$

Question 2

a. Nodal forces vector

$$\begin{Bmatrix} 15.00 \\ -5.00 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5.00 \\ 8.66 \\ 0 \\ 0 \end{Bmatrix}$$

b. Location of global matrix element

Element	Row	Column
k^2_{11}	1	1
k^4_{24}	6	10
k^7_{43}	10	9

Question 3

i. Displacement at joint C

- $U_3 = 322.6\,\mathrm{m}$
- $U_4 = -10.70\,\mathrm{m}$

ii. Internal force in element 2 and 3

- Element 1: $24.62\,\mathrm{N}$
- Element 2: $27.04\,\mathrm{N}$

Question 4

$$k^1 = \begin{bmatrix} 2.500 & 0 & 0 & -2.500 & 0 & 0 \\ 0 & 0.017 & 0.021 & 0 & -0.017 & 0.021 \\ 0 & 0.021 & 0.033 & 0 & -0.021 & 0.017 \\ -2.500 & 0 & 0 & 2.500 & 0 & 0 \\ 0 & -0.017 & -0.021 & 0 & 0.017 & -0.021 \\ 0 & 0.021 & 0.017 & 0 & -0.021 & 0.033 \end{bmatrix} \times 10^9$$

$$k^1 = \begin{bmatrix} 2.500 & 0 & 0 & -2.500 & 0 & 0 \\ 0 & 0.004 & 0.010 & 0 & -0.004 & 0.010 \\ 0 & 0.010 & 0.033 & 0 & -0.010 & 0.017 \\ -2.500 & 0 & 0 & 2.500 & 0 & 0 \\ 0 & -0.004 & -0.010 & 0 & 0.004 & -0.010 \\ 0 & 0.010 & 0.017 & 0 & -0.010 & 0.033 \end{bmatrix} \times 10^9$$

$$K = \begin{bmatrix} 0.017 & 0 & -0.021 & -0.017 & 0 & -0.021 & 0 & 0 & 0 \\ 0 & 2.500 & 0 & 0 & -2.500 & 0 & 0 & 0 & 0 \\ -0.021 & 0 & 0.033 & 0.021 & 0 & 0.017 & 0 & 0 & 0 \\ -0.017 & 0 & 0.021 & 2.517 & 0 & 0.021 & -2.500 & 0 & 0 \\ 0 & -2.500 & 0 & 0 & 2.504 & 0.010 & 0 & -0.004 & 0.010 \\ -0.021 & 0 & 0.017 & 0.021 & 0.010 & 0.067 & 0 & -0.010 & 0.017 \\ 0 & 0 & 0 & -2.500 & 0 & 0 & 2.500 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.004 & -0.010 & 0 & 0.004 & -0.010 \\ 0 & 0 & 0 & 0 & 0.010 & 0.017 & 0 & -0.010 & 0.033 \end{bmatrix} \times 10^9$$

Question 5

i. Displacement at joint D

- x-direction: -3.00×10^{-7} m
- y-direction: -5.12×10^{-5} m

ii. Support reactions

Node	F_x (N)	F_y, (N)	M_z, (N m)
A	-9.07×10^2	4.6×10^4	2.12×10^3
B	3.40×10^2	2.54×10^4	2.28×10^4
C	5.67×10^2	4.46×10^3	4.22×10^3

Chapter 5

Question 2

i. Global stiffness matrix, K and global stress stiffness matrix, K_G

$$K = \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}$$

$$K_G = \begin{bmatrix} 36 & 3 & -36 & 3 & 0 & 0 \\ 3 & 4 & -3 & -1 & 0 & 0 \\ -36 & -3 & 72 & 0 & -36 & 3 \\ 3 & -1 & 0 & 8 & -3 & -1 \\ 0 & 0 & -36 & -3 & 36 & -3 \\ 0 & 0 & 3 & -1 & -3 & 4 \end{bmatrix}$$

ii. Buckling load, P

	P(N)
1st mode	0.33
2nd mode	1.00

iii. Natural frequencies

	$\omega(\mathrm{rad\,s^{-1}})$
1st mode	0.28
2nd mode	1.00

Question 3

	$\omega(\mathrm{rad\,s^{-1}})$
1st mode	0.27
2nd mode	0.36
3rd mode	1.06

Chapter 6

Question 2

i. Value of point load, $r(t)$ at time, t

$t(\mathrm{s})$	$r(t)$
1	10
2	40

ii. $\Delta t = 1\,\mathrm{s}$

iii. Solution of $\{d\}^{t+1}$

t(s)	$\{d\}^{t+1}$
1	$\{d\}^2 = \begin{Bmatrix} 1.03 \\ -0.77 \end{Bmatrix}$
2	$\{d\}^3 = \begin{Bmatrix} 6.23 \\ -4.83 \end{Bmatrix}$
3	$\{d\}^4 = \begin{Bmatrix} 21.13 \\ -17.12 \end{Bmatrix}$

Question 3

t(s)	$\{d\}^{t+1}$
1	$\{d\}^1 = \begin{Bmatrix} 0.009 \\ 0.073 \\ 0.112 \end{Bmatrix}$
2	$\{d\}^2 = \begin{Bmatrix} 0.045 \\ 0.213 \\ 0.333 \end{Bmatrix}$
3	$\{d\}^3 = \begin{Bmatrix} 0.137 \\ 0.345 \\ 0.561 \end{Bmatrix}$

Full Matrices

Full matrix for Eq. (5.103)

$$
\left|
\begin{bmatrix}
5.92\times10^{6} & 0 & -4.44\times10^{6} & -5.92\times10^{6} & 0 & -4.44\times10^{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.33\times10^{9} & 0 & 0 & -1.33\times10^{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4.44\times10^{6} & 0 & 4.44\times10^{6} & 4.44\times10^{6} & 0 & 2.22\times10^{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5.92\times10^{6} & 0 & 4.44\times10^{6} & 1.18\times10^{7} & 0 & 0 & -5.92\times10^{6} & 0 & -4.44\times10^{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.33\times10^{9} & 0 & 0 & 2.67\times10^{9} & 0 & 0 & -1.33\times10^{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4.44\times10^{6} & 0 & 2.22\times10^{6} & 0 & 0 & 8.89\times10^{6} & 4.44\times10^{6} & 0 & 2.22\times10^{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5.92\times10^{6} & 0 & 4.44\times10^{6} & 2.01\times10^{9} & 0 & 4.44\times10^{6} & -2.00\times10^{9} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1.33\times10^{9} & 0 & 0 & 1.35\times10^{9} & 1.00\times10^{7} & 0 & -2.00\times10^{7} & 1.00\times10^{7} & 0 & 0 & 0 \\
0 & 0 & 0 & -4.44\times10^{6} & 0 & 2.22\times10^{6} & 4.44\times10^{6} & 1.00\times10^{7} & 1.11\times10^{7} & 0 & -1.00\times10^{7} & 3.33\times10^{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2.00\times10^{9} & 0 & 0 & 4.00\times10^{9} & 0 & 0 & -2.00\times10^{9} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.00\times10^{7} & -1.00\times10^{7} & 0 & 4.00\times10^{7} & 0 & 0 & -2.00\times10^{7} & 1.00\times10^{7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00\times10^{7} & 3.33\times10^{6} & 0 & 0 & 1.33\times10^{7} & 0 & -1.00\times10^{7} & 3.33\times10^{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.00\times10^{9} & 0 & 0 & 2.00\times10^{9} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.00\times10^{7} & -1.00\times10^{7} & 0 & 2.00\times10^{7} & -1.00\times10^{7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00\times10^{7} & 3.33\times10^{6} & 0 & -1.00\times10^{7} & 6.66\times10^{6}
\end{bmatrix}
\right.
$$

$$
\left.
-\lambda
\begin{bmatrix}
43.791 & 0 & -9.264 & 15.159 & 0 & 5.474 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 39.300 & 0 & 0 & 19.650 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-9.264 & 0 & 2.526 & -5.474 & 0 & -1.895 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15.159 & 0 & -5.474 & 87.583 & 0 & 0 & 15.159 & 0 & 5.474 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 19.650 & 0 & 0 & 78.600 & 0 & 0 & 19.650 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5.474 & 0 & -1.895 & 0 & 0 & 5.053 & -5.474 & 0 & -1.895 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 15.159 & 0 & -5.474 & 69.991 & 0 & 9.264 & 13.100 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 19.650 & 0 & 0 & 68.494 & 4.117 & 0 & 10.106 & -2.433 & 0 & 0 & 0 \\
0 & 0 & 0 & 5.474 & 0 & -1.895 & 9.264 & 4.117 & 3.275 & 0 & 2.433 & -0.561 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 13.100 & 0 & 0 & 52.400 & 0 & 0 & 13.100 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 10.106 & 2.433 & 0 & 58.389 & 0 & 0 & 10.106 & -2.433 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.433 & -0.561 & 0 & 0 & 1.497 & 0 & 2.433 & -0.561 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13.100 & 0 & 0 & 26.200 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10.106 & 2.433 & 0 & 29.194 & -4.117 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.433 & -0.561 & 0 & -4.117 & 0.749
\end{bmatrix}
\right| = 0
$$

Full matrix for Eq. (6.36)

$$
\begin{bmatrix}
5.92\times10^6 & 0 & -4.44\times10^6 & -5.92\times10^6 & 0 & -4.44\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.33\times10^9 & 0 & 0 & -1.33\times10^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4.44\times10^6 & 0 & 4.44\times10^6 & 4.44\times10^6 & 0 & 2.22\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5.92\times10^6 & 0 & 4.44\times10^6 & 1.18\times10^7 & 0 & 0 & -5.92\times10^6 & 0 & -4.44\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.33\times10^9 & 0 & 0 & 2.67\times10^9 & 0 & 0 & -1.33\times10^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4.44\times10^6 & 0 & 2.22\times10^6 & 0 & 0 & 8.89\times10^6 & 4.44\times10^6 & 0 & 2.22\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5.92\times10^6 & 0 & 4.44\times10^6 & 2.01\times10^9 & 0 & 4.44\times10^6 & -2.00\times10^9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1.33\times10^9 & 0 & 0 & 1.35\times10^9 & 1.00\times10^7 & 0 & -2.00\times10^7 & 1.00\times10^7 & 0 & 0 & 0 \\
0 & 0 & 0 & -4.44\times10^6 & 0 & 2.22\times10^6 & 4.44\times10^6 & 1.00\times10^7 & 1.11\times10^7 & 0 & -1.00\times10^7 & 3.33\times10^6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2.00\times10^9 & 0 & 0 & 4.00\times10^9 & 0 & 0 & -2.00\times10^9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.00\times10^7 & -1.00\times10^7 & 0 & 4.00\times10^7 & 0 & 0 & -2.00\times10^7 & 1.00\times10^7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00\times10^7 & 3.33\times10^6 & 0 & 0 & 1.33\times10^7 & 0 & -1.00\times10^7 & 3.33\times10^6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.00\times10^9 & 0 & 0 & 2.00\times10^9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.00\times10^7 & -1.00\times10^7 & 0 & 2.00\times10^7 & -1.00\times10^7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00\times10^7 & 3.33\times10^6 & 0 & -1.00\times10^7 & 6.66\times10^6
\end{bmatrix}
\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \\ D_{13} \\ D_{14} \\ D_{15} \end{Bmatrix}
$$

$$
+\begin{bmatrix}
43.791 & 0 & -9.264 & 15.159 & 0 & 5.474 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 39.300 & 0 & 0 & 19.650 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-9.264 & 0 & 2.526 & -5.474 & 0 & -1.895 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15.159 & 0 & -5.474 & 87.583 & 0 & 0 & 15.159 & 0 & 5.474 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 19.650 & 0 & 0 & 78.600 & 0 & 0 & 19.650 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5.474 & 0 & -1.895 & 0 & 0 & 5.053 & -5.474 & 0 & -1.895 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 15.159 & 0 & -5.474 & 69.991 & 0 & 9.264 & 13.100 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 19.650 & 0 & 0 & 68.494 & 4.117 & 0 & 10.106 & -2.433 & 0 & 0 & 0 \\
0 & 0 & 0 & 5.474 & 0 & -1.895 & 9.264 & 4.117 & 3.275 & 0 & 2.433 & -0.561 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 13.100 & 0 & 0 & 52.400 & 0 & 0 & 13.100 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 10.106 & 2.433 & 0 & 58.389 & 0 & 0 & 10.106 & -2.433 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.433 & -0.561 & 0 & 0 & 1.497 & 0 & 2.433 & -0.561 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13.100 & 0 & 0 & 26.200 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10.106 & 2.433 & 0 & 29.194 & -4.117 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.433 & -0.561 & 0 & -4.117 & 0.749
\end{bmatrix}
\begin{Bmatrix} \ddot{D}_1 \\ \ddot{D}_2 \\ \ddot{D}_3 \\ \ddot{D}_4 \\ \ddot{D}_5 \\ \ddot{D}_6 \\ \ddot{D}_7 \\ \ddot{D}_8 \\ \ddot{D}_9 \\ \ddot{D}_{10} \\ \ddot{D}_{11} \\ \ddot{D}_{12} \\ \ddot{D}_{13} \\ \ddot{D}_{14} \\ \ddot{D}_{15} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P(t) \\ 0 \end{Bmatrix}
$$

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