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2D Finite Element Formulations for Heat, Solids And Fluids (with Matlab)

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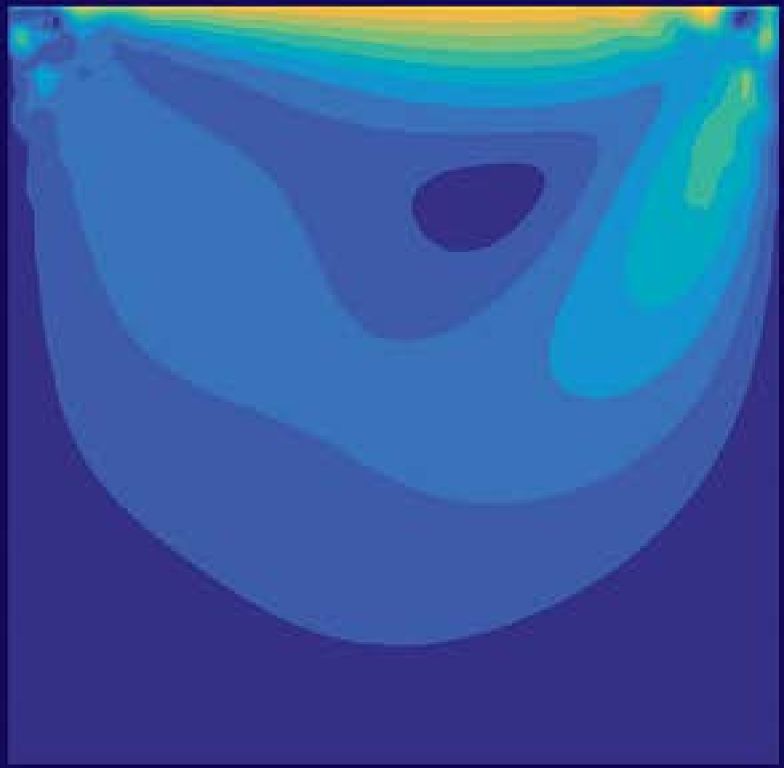

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Mohd Al-Akhbar Mohd Noor
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Malaysian Society for Numerical Methods

Two-Dimensional Finite Element Formulations

for Heat, Solids and Fluids (with Matlab)

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Dedication

To our families and students…

to the future!

Preface

This book can be considered as an extension to the previous publication (Finite Element Formulations for Statics and Dynamics of Frame Structures) as it dwells on two-dimensional formulation of continuum. Also, the discussion has been extended to nonlinear formulation to cater for the nonlinearity of Navier-Stokes equations. However, in ensuring it to be self-contained, the discussions on the basic concept of numerical method from the previous publication have been combined, shortened and included in the introduction chapter of this book (Chapter 1). The book is still prepared based on the similar approach that a topic always begins with the derivation of the partial differential equation/s of the problem and followed by the discretization into matrix forms using Galerkin Weighted Residual method hence FEM. At the end of a chapter, worked example and Matlab source code are provided.

Chapter 2 is dedicated to heat transfer. The general unsteady partial differential equation is derived based on energy balance but later reduced to steady during the discretization as the main purpose of the chapter is to highlight the extension of 1D formulation in Chapter 1 to 2D. The unsteady formulation is reserved for Chapter 5 where fluid flow is discussed. The scalar formulation of heat transfer is extended to vector-valued formulation of plane stress in Chapter 3. The partial differential equations are derived based on the conservation of momentum and matrix representation is employed as early as at the derivation stage. As preparation for the nonlinear nature of Navier-Stokes, an introduction to nonlinear formulation is given in Chapter 4 where a hypothetical 1D nonlinear differential equation is discretized and iteratively solved by employing two schemes; Picard and Newton-Raphson. In Chapter 5, the unsteady Navier-Stokes equations for fluid flow are derived and solved using Picard method. In all aforementioned chapters, the integrations are done analytically as the intention is not to distract readers from observing smoothly the conversion of the partial differential equation/s into matrix forms. However, since the actual practice is to integrate numerically, discussion on numerical integration based on Gauss quadrature is given in the last chapter (Chapter 6) where the determination of the plane stress matrices is demonstrated.

Similar to the previous publication, this book is not intended to be a complete book on FEM but to introduce the concept in the way which the authors believe as more effective. In fact, it is the authors' intention for this book to be the first book on FEM that prepares the readers for the more comprehensive texts on FEM and hence the rather straightforward manner of the discussion and the lack of worked examples and the very minimal discussion on the wider practice of finite element modelling. The idea is this, the 'thinner' the book, the quicker it to be finished and thus the quicker the reading of the next more comprehensive books on FEM.

Airil Yasreen Mohd Yassin
January, 2020
Putrajaya, Malaysia

Contents

1 The Basic Idea

1.1 Introduction

One of the roles of an engineer is to determine the magnitude and the distribution of physical variables of interest such as forces, displacements, temperature and their derivatives for design and construction purposes. In a simple arrangement, this can be done in a close-formed hand-calculation manner. However, of late, such a degree of simplicity is no longer suffice as engineers are striving for better yet cheaper, safer and more sustainable performance of their technologies,. The need to meet such ideals, however, is what causing the increase in the complexity of the mathematical representation of the problem we are seeing in present days where the equations are becoming more difficult to solve.

In short;

> *"The more we want to know (or to produce), the more complex the maths and the equations will become".*

To highlight this, let's take a structural beam as an example. In the simplest manner, a beam, as shown in Fig. 1.1(a), can be treated as a line element, that is, spatially one-dimensional. However, as we zoom-in (in knowing more), we will realize that the incorporation of the effect of Poison ratio would require the beam to be modelled, at the least, as two-dimensional continuum. If the beam is a steel section, due to its "thinness", this can be accomplished by combining plane stress element (for the web) and plate elements (for the flanges) together, as shown in Fig. 1.1(b). However, if the beam is a reinforced concrete, due to the "bulkiness", it is best to model it spatially in three-dimension (e.g. Fig. 1.1(c)).

What we have just discussed is just one of the possible causes for complex-

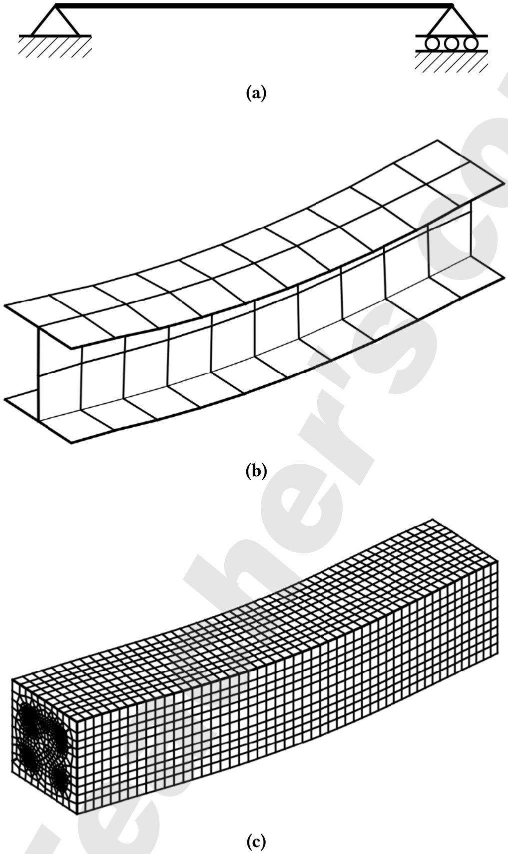

Figure 1.1: Increasing complexity of FEM modelling; (a) 1D beam line element model, (b) Combination of 2D elements of plane stress and plate and (c) 3D element for reinforced concrete modelling

ity that is, the increase in the order of the spatial dimensions; from 1D to 2D then 3D. We are yet to discuss time dependency, coupling of equations due to various laws and interactions, nonlinearity and the stability of solutions, complex prescription of boundary and initial conditions as well as material properties and many more numerical challenges. Too complex, modern engineering problems are no longer solvable except approximately and not without the help of computers. Finite element method (FEM) is one of the numerical technique and the most written procedure in today's engineering analysis and design software.

It is therefore the intention of this manuscript to introduce the readers to the basic concepts of finite element method for continuum, as an extension to the one-dimensional formulations of previous publication [1]. Specifically, we will focus on the two-dimensional formulations of heat, solid and fluid. Although this manuscript can be seen and treated as an extension, it is important for it to be self-contained. Therefore, some few important topics and subtopics from previous publication are repeated here and there in this manuscript. For readers who have read them before, feel free to skip these parts.

1.2 Basic Concept of Numerical Techniques

The basic idea of any numerical methods can be summarized as;

> *"To covert the partial or ordinary differential equation/s (PDE or ODE) of a physical problem into 'equivalent' simultaneous algebraic equations in the form of a matrix system".*

In elaborating the concept, we discuss the solution of the simplest forms of ODE that is of a bar element. We start by deriving the governing equation. Fig. 1.2 shows a bar structure and its differential element.

By assuming any changes as continuous, the axial force, F on the left-side of the differential element can be expanded by Taylor series on the right-side, as shown in Fig. 1.2(b). Next, by assuming higher order terms as insignificant thus can be ignored, and since this is a 1D problem (i.e. $\partial(\quad) = d(\quad)$), the state as shown in Fig. 1.2(c) is established.

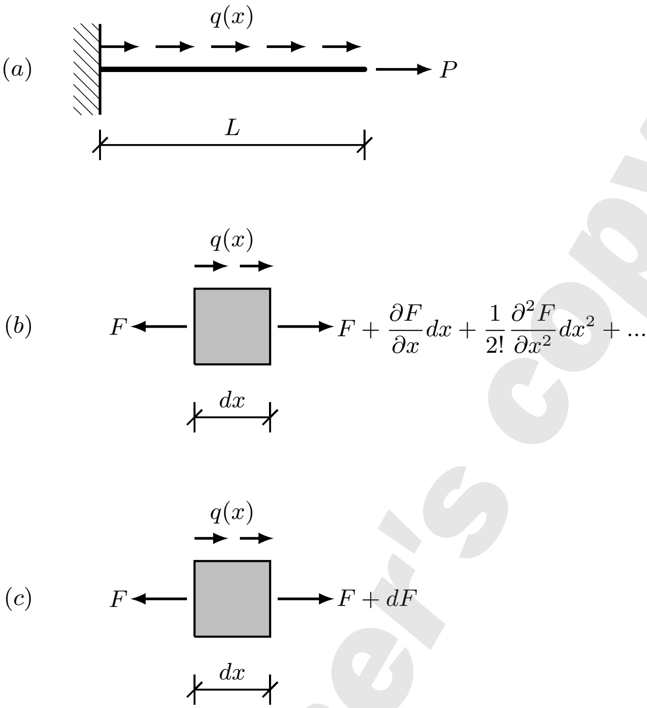

Figure 1.2: Bar structure and its differential element.

Having established the differential element and the corresponding forces acting on it, we are ready to derive the ODE for the bar problem. Since the bar deforms in the axial direction, only equilibrium in x-direction needed to be considered thus;

$$\sum F_x = 0 = -F + (F + dF) + q\,dx \tag{1.1}$$

where q is the axially distributed load. By rearranging gives:

$$\frac{dF}{dx} = -q \tag{1.2}$$

From Hooke's Law, we know that

$$\sigma = E\epsilon \tag{1.3}$$

where σ is the axial stress, E is the Young's modulus and ϵ is the axial strain. Since

$$\sigma = \frac{F}{A} \tag{1.4}$$

and

$$\epsilon = \frac{du}{dx} \tag{1.5}$$

where A is the cross-sectional area of the bar and u is the axial displacement. Inserting Eq. (1.4) and Eq. (1.5) into Eq. (1.3) gives

$$F = EA\frac{du}{dx} \tag{1.6}$$

By differentiating Eq. (1.6) once gives

$$\frac{dF}{dx} = EA\frac{d^2u}{dx^2} \tag{1.7}$$

By inserting Eq. (1.7) into Eq. (1.2) would then give

$$EA\frac{d^2u}{dx^2} = -q \tag{1.8}$$

Eq. (1.8) is the ODE for the bar problem or also known as the domain equation. For every domain differential equation such as Eq. (1.8), there must be boundary equations to complete it and for our particular case, all possible boundary equations are given as below:

Natural/force boundary conditions

$$EA\left.\frac{du(x)}{dx}\right|_{x=0} = F_0 \tag{1.9a}$$

$$EA\left.\frac{du(x)}{dx}\right|_{x=L} = -F_L \tag{1.9b}$$

Essential/displacement boundary conditions

$$u|_{x=0} = u_0 \tag{1.9c}$$

$$u|_{x=L} = u_L \tag{1.9d}$$

Note that the sign convention for the natural (force) boundary conditions above is based on the assumption that the boundary conditions are the reactions at support hence the opposite directions to the internal forces. Since Eq. (1.8) is a second-order ODE, two out of the four given boundary conditions must be known in prior so as to have a well-posed problem.

Having established the bar ODE, we are now all set to discuss the basic concept of numerical methods.

1.2.1 Collocation Method

The collocation method can be considered as the most basic approach to numerical techniques. To demonstrate the application of the method, we re-write the domain equation of the problem and the accompanying boundary conditions that are specific for the case of a bar fixed at the left end and loaded by a point load, P at the right end as in Fig. 1.2(a).

Domain equation

$$EA\frac{d^2u}{dx^2} = -q \tag{1.10}$$

Boundary conditions (equations)

$$EA\left.\frac{du}{dx}\right|_{x=L} = P \tag{1.11}$$

$$u|_{x=0} = 0 \tag{1.12}$$

The ODE is considered solved when a solution, $u = f(x)$ is found which satisfies all the equations above. Approximately, this can be done by first converting all the equations (i.e. Eqs. (1.10) to (1.12)) to 'equivalent' simultaneous algebraic equations. We start by assuming a solution in the forms of polynomials. Let's assume

$$u = a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4 \tag{1.13}$$

Then, we satisfy Eq. (1.12) by inserting Eq. (1.13) into the equation to give;

$$u|_{x=0} = a_1 + a_2(0) + a_3(0)^2 + a_4(0)^3 + a_5(0)^4 = 0 \tag{1.14}$$

which gives

$$a_1 = 0 \tag{1.15}$$

Next we satisfy Eq. (1.11) by inserting Eq. (1.13) into the equation to obtain:

$$EA\ \frac{du}{dx}\bigg|_{x=L} = EA\left(a_2 + 2a_3L + 3a_4L^2 + 4a_5L^3\right) = P \tag{1.16}$$

Finally, by inserting Eq. (1.13) into Eq. (1.10), the following is obtained:

$$EA\frac{d^2u}{dx^2} + q = EA\left(2a_3 + 6a_4x + 12a_5x^2\right) + q \neq 0 \tag{1.17}$$

Observing Eq. (1.17), one can notice that, whilst each of Eqs. (1.14) to (1.17) is an act of forcing the equation to a certain values (i.e 0 and P, respectively) hence the "satisfaction" of the equations, the insertion of the guessed function into the domain equation is yet a satisfaction of the original equation hence the use of the inequality symbol ($\neq$) in Eq. (1.17). As we are going to see, the forcing of Eq. (1.17) later, at several locations within the domain to a null value is what satisfies the equation and what creates sufficient number of equations.

By grouping Eqs. (1.15) to (1.17) together, we can see that, so far, we have established three simultaneous equations as follows

$$a_1 = 0 \tag{1.18}$$

$$EAa_2 + 2EAa_3L + 3EAa_4L^2 + 4EAa_5L^3 = P \tag{1.19}$$

$$2EAa_3 + 6EAa_4x + 12EAa_5x^2 \neq q \tag{1.20}$$

However, we can notice that:

- We have five (5) unknown constant, a_1, a_2, a_3, a_4, and a_5 but with only three simultaneous equations.
- The last equation (Eq. (1.20)) is still not algebraic but continuous in x, instead. Also, the left hand side of the equation is not equal to the right hand side because the guessed function is yet the solution of the ODE.

So to get the sufficient number of equations (and to convert Eq. (1.20) into algebraic) we argue that, since Eq. (1.20) is obtained from the domain equation (Eq. (1.10)), the equation must hold (must be true) throughout the domain thus we can evaluate Eq. (1.20) everywhere in the domain and force it to zero as much as we need. In our case, to complement Eqs. (1.18) and (1.19), we evaluate Eq. (1.20) at three locations in the bar says at $x = L/3$, $L/2$, $2L/3$ to obtain

$$2EAa_3 + 6EAa_4\left(\frac{L}{3}\right) + 12EAa_5\left(\frac{L}{3}\right)^2 + q = 0 \tag{1.21a}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{2}\right) + 12EAa_5\left(\frac{L}{2}\right)^2 + q = 0 \tag{1.21b}$$

$$2EAa_3 + 6EAa_4\left(\frac{2L}{3}\right) + 12EAa_5\left(\frac{2L}{3}\right)^2 + q = 0 \tag{1.21c}$$

Eq. (1.21) are the results of forcing Eq. (1.20) to a null value at several locations within the domain. Also, as can be seen, such an act not only represents the attempt to satisfy the original ODE but also converts the ODE into a set of algebraic equations.

Now, by re-grouping Eq. (1.21) and Eqs. (1.18) and (1.19) together, we now have the sufficient number of algebraic equations for the unknowns, given as;

$$a_1 = 0 \tag{1.22a}$$

$$EAa_2 + 2EAa_3L + 3EAa_4L^2 + 4EAa_5L^3 = P \tag{1.22b}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{3}\right) + 12EAa_5\left(\frac{L}{3}\right)^2 + q = 0 \tag{1.22c}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{2}\right) + 12EAa_5\left(\frac{L}{2}\right)^2 + q = 0 \tag{1.22d}$$

$$2EAa_3 + 6EAa_4\left(\frac{2L}{3}\right) + 12EAa_5\left(\frac{2L}{3}\right)^2 + q = 0 \tag{1.22e}$$

Eq. (1.22) is thus the ‘equivalent’ simultaneous algebraic equations of the ODE of the problem. In other words, we can say that;

> *“Eq.* (1.22a) *are the ‘equivalent’ algebraic forms of Eqs.* (1.10) *to* (1.12)*”.*

So this is basically the main concept shared by all numerical techniques such as FEM, Finite Difference Method (FDM), Boundary Element Method (BEM) and many more. But it is also the main character of a numerical technique to treat the equations in matrix forms as this what suits computer programming. In this context, Eq. (1.22) can be arranged in matrix forms as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & EA & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{3}\right) & 12EA\left(\frac{L}{3}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{2}\right) & 12EA\left(\frac{L}{2}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{2L}{3}\right) & 12EA\left(\frac{2L}{3}\right)^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ p \\ -q \\ -q \\ -q \end{Bmatrix} \tag{1.23}$$

or

$$[k]\{u\} = \{r\} \tag{1.24}$$

where

$$[k] = k_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & EA & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{3}\right) & 12EA\left(\frac{L}{3}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{2}\right) & 12EA\left(\frac{L}{2}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{2L}{3}\right) & 12EA\left(\frac{2L}{3}\right)^2 \end{bmatrix}$$

$$\{u\} = u_j = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix}$$

$$\{r\} = r_j = \begin{Bmatrix} 0 \\ P \\ -q \\ -q \\ -q \end{Bmatrix}$$

Note: k_{ij}, d_j, and r_j are the alternative notations known as indicial/tensorial notations. We are going to use this notation system and the matrix notation interchangeably.

By solving Eq. (1.22) or Eq. (1.23), we then obtain the numerical values of a_1, a_2, a_3, a_4 and a_5 and by inserting these values back into the original guessed function (Eq. (1.13)), we thus obtain the numerical solution to the problem.

Worked Example 1.1

Let's put what we have learned so far into practice. We are going to compare the result obtained from Eq. (1.23) with the closed-form solution for the problem given in Fig. 1.3. The figure shows a bar structure fixed onto a wall at its left end while subjected to a pulling force of $10\,\mathrm{kN}$ at the right end and a distributed load of $2\,\mathrm{kN\,m^{-1}}$ throughout its span.

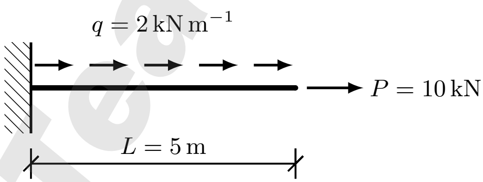

Figure 1.3: Cantilever bar with uniformly distributed load, q and a single point load, P. ($E = 200\times10^6\,\mathrm{kN\,m^{-2}}$ and $A = 0.04\,\mathrm{m^2}$).

For this problem, the equation in the form of Eq. (1.23) can be given as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 8.00\times10^{6} & 8.00\times10^{7} & 6.00\times10^{8} & 4.00\times10^{9} \\ 0 & 0 & 1.60\times10^{7} & 8.00\times10^{7} & 2.67\times10^{8} \\ 0 & 0 & 1.60\times10^{7} & 1.20\times10^{8} & 6.00\times10^{8} \\ 0 & 0 & 1.60\times10^{7} & 1.60\times10^{8} & 1.07\times10^{9} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \\ -2 \\ -2 \\ -2 \end{Bmatrix} \tag{1.25}$$

By solving Eq. (1.25) using Matlab command "\", (refer source codes in Section 1.5.1), the values of a_i are thus as given in Table 1.1.

Table 1.1: Values of guessed solution's constant

a_i	a_1	a_2	a_3	a_4	a_5
Value	0	2.5×10^{-6}	-1.25×10^{-7}	0	0

And by inserting the values into Eq. (1.13) we thus obtained the numerical solution of the problem as:

$$u_{\mathrm{n}} = \left(2.5\times10^{-6}\right) x + \left(-1.25\times10^{-7}\right) x^2 \tag{1.26}$$

The accuracy of this numerical solution can be assessed by comparing its value with the exact/closed-form solution below

$$u_{\mathrm{e}} = \left(\frac{P + ql}{EA}\right) x + \left(-\frac{q}{2EA}\right) x^2 \tag{1.27}$$

The plot of numerical, u_{n} and exact, u_{e} solutions throughout the length, L of the bar and their respective values at several locations are given in Fig. 1.4 and Table 1.2. Based on the plot and the table, it can be concluded that the basic concept of the numerical technique is able to provide similar results to the ones provided by the closed-form solution. However, it must be noted that this is generally not the case. Generally, a numerical solution would not be accurate but converge to the "accurate" solution with the increase of polynomial order of the guessed functions (or degree of freedoms and mesh density as discussed later on).

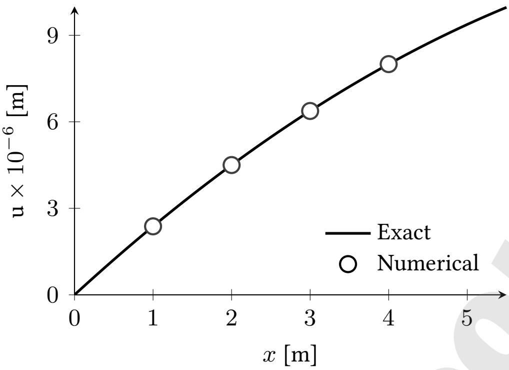

Figure 1.4: Axial displacement, u, along the bar between numerical and exact solutions.

Table 1.2: Axial displacement, u, along the bar between numerical, u_n, and exact solutions, u_e.

x [m]	1	2	3	4
u_n [m]	$2.38{\times}10^{-6}$	$4.5{\times}10^{-6}$	$6.38{\times}10^{-6}$	$8{\times}10^{-6}$
u_e [m]	$2.38{\times}10^{-6}$	$4.5{\times}10^{-6}$	$6.38{\times}10^{-6}$	$8{\times}10^{-6}$

1.2.2 Weighted Residual Method

In previous section, we have discussed the basic concept of converting the PDE or ODE of a problem into equivalent algebraic equations (matrix form). Herein, we are going to focus our discussion on Finite Element Method (FEM). FEM is a numerical method that widely used in engineering field of today. The theoretical argument for FEM can be done in three different approaches which are;

i. Direct Method
ii. Variational Method
iii. Weighted Residual Method (WRM)

In this book, our main interest is on WRM only. Readers who are interested on the other two approaches are encouraged to read them elsewhere.

Actually, the specific form of WRM that is employed in present day FEM is known as Galerkin-WRM formulations. However, before we get into Galerkin, we are going to establish our understanding on the basic idea (or concept) of WRM first.

The basic concept of WRM can be better discussed by picking-up what we have left off in the previous discussion. We start by re-observing Eqs. (1.10) to (1.12)). Since we are going to use these equations as our starting point, we re-write the equations herein as

$$EA\frac{d^2u}{dx^2} = -q \tag{1.28a}$$

$$EA\left.\frac{du}{dx}\right|_{x=L} = P \tag{1.28b}$$

$$u|_{x=0} = 0 \tag{1.28c}$$

The treatment on Eq. (1.28a) is what differentiates WRM from the previous numerical technique. Whilst in the previous discussion, Eq. (1.28a) is evaluated at three different locations in the domain so as to yield the necessary three algebraic equations, in WRM, we have quite a different argument for it.

In WRM, Eq. (1.28a) is written as

$$EA\frac{d^2u}{dx^2} \neq -q = R \quad \text{(residual error)} \tag{1.29}$$

Residual error is a function emerges from the fact that the guessed solution is not the actual solution of the domain ODE. For the guessed solution of Eq. (1.13), the residual error function, R is as given by Eq. (1.20), re-given herein for easy viewing as

$$R(x) = 2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q \tag{1.30}$$

A hypothetical plot of Eq. (1.30) would give us the graphical view of the residual error function as shown in Fig. 1.5.

Since this (residual) error function should not exist, integrating the function and setting it equal to zero will also mean forcing the area beneath the curve to become zero.

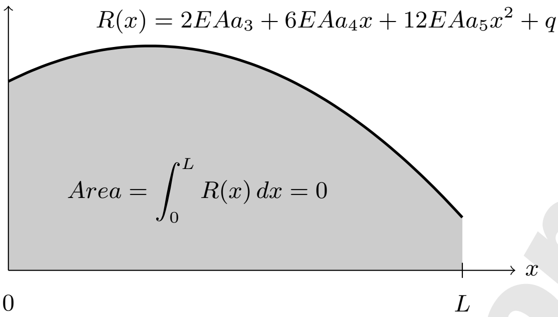

Figure 1.5: Typical plot of residual error function, R.

This action of forcing the area under the residual curve would give us one algebraic equation in a similar essence we evaluated Eq. (1.20) at one location within the domain as previously discussed. However, our present approach is more effective because the integration somehow disperses the error in a more smooth and 'overall' manner.

But so far, this only gives us one equation, when the fact is, we need as much equations as necessary. In previous discussion, we evaluate at three locations to provide the three required algebraic equations, remember?

Herein, to provide the sufficient number of algebraic equations, what we do is to firstly create three independent residual error functions and to integrate these resulting functions over the domain before setting them to zero. In accordance, to create the three independent residual functions, we multiply Eq. (1.30) separately with independent function, i.e. 1, x, x^2 to give

$$R_1 = 1\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right) \tag{1.31a}$$

$$R_2 = x\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right) \tag{1.31b}$$

$$R_3 = x^2\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right) \tag{1.31c}$$

where $R_1(x)$, $R_2(x)$, and $R_3(x)$ are the three 'new' residual error functions which all can be hypothetically plotted as in Fig. 1.6.

Now by integrating separately each of the residual functions and setting it to zero, the action would force the area under each curve to 'disappear'

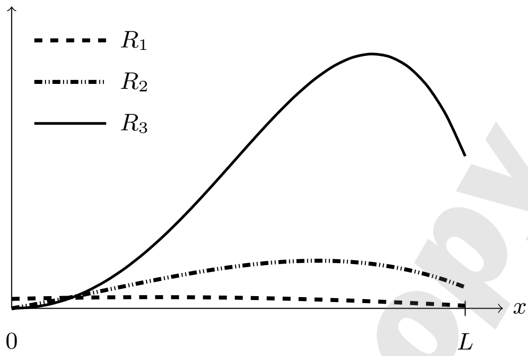

Figure 1.6: Forcing of the area of the residual error function to zero.

thus results in the three required algebraic equations, given as

$$\int_0^L R_1\,dx = \int_0^L 1\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right)dx = 0 \quad (1.32a)$$

$$\int_0^L R_2\,dx = \int_0^L x\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right)dx = 0 \quad (1.32b)$$

$$\int_0^L R_3\,dx = \int_0^L x^2\left(2EAa_3 + 6EAa_4x + 12EAa_5x^2 + q\right)dx = 0 \quad (1.32c)$$

By conducting the integration we obtain:

$$2EALa_3 + 3EAL^2a_4 + 4EAL^3a_5 + qL = 0 \quad (1.33a)$$

$$EAL^2a_3 + 2EAL^3a_4 + 3EAL^4a_5 + \frac{1}{2}qL^2 = 0 \quad (1.33b)$$

$$\frac{2}{3}EAL^3a_3 + \frac{3}{2}EAL^4a_4 + \frac{12}{5}EAL^5a_5 + \frac{1}{3}qL^3 = 0 \quad (1.33c)$$

Now, as we did previously, by grouping Eqs. (1.18) and (1.19) together with the above, we get:

$$a_1 = 0 \tag{1.34a}$$

$$EAa_2 + 2EALa_3 + 3EAL^2a_4 + 4EAL^3a_5 = P \tag{1.34b}$$

$$2EALa_3 + 3EAL^2a_4 + 4EAL^3a_5 + qL = 0 \tag{1.34c}$$

$$EAL^2a_3 + 2EAL^3a_4 + 3EAL^4a_5 + \frac{1}{2}qL^2 = 0 \tag{1.34d}$$

$$\frac{2}{3}EAL^3a_3 + \frac{3}{2}EAL^4a_4 + \frac{12}{5}EAL^5a_5 + \frac{1}{3}qL^3 = 0 \tag{1.34e}$$

In matrix forms, Eq. (1.34) can be given as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & EA & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & EAL^2 & 2EAL^3 & 3EAL^4 \\ 0 & 0 & \frac{2}{3}EAL^3 & \frac{3}{2}EAL^4 & \frac{12}{5}EAL^5 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ -qL \\ -\frac{1}{2}qL^2 \\ -\frac{1}{3}qL^3 \end{Bmatrix} \tag{1.35}$$

Eq. (1.35) is the discretised equation of the bar ODE obtained from the basic WRM.

Worked Example 1.2

Let's solve the previous problem of Worked Example 1.1 this time by employing WRM. We are going to compare the result obtained from this approach with those obtained using basic numerical technique given by Eq. (1.25) and the closed-form solution.

For the given problem, the WRM discretised equation in the form of

Eq. (1.35) can be given as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 8\times10^6 & 8\times10^7 & 6\times10^8 & 4\times10^9 \\ 0 & 0 & 8\times10^7 & 6\times10^8 & 4\times10^9 \\ 0 & 0 & 2\times10^8 & 2\times10^9 & 1.5\times10^{10} \\ 0 & 0 & 6.67\times10^8 & 7.5\times10^9 & 6\times10^{10} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \\ -10 \\ -25 \\ -83.33 \end{Bmatrix} \tag{1.36}$$

By solving Eq. (1.36) using Matlab command "\", (refer source codes in Section 1.5), the values of a_i are thus as given in Table 1.3.

Table 1.3: Values of guessed solution's constant

a_i	a_1	a_2	a_3	a_4	a_5
Value	0	2.5×10^{-6}	-1.25×10^{-7}	0	0

And by inserting the values into Eq. (1.13) we thus obtained the numerical solution, u_{w} of the problem as:

$$u_{\mathrm{w}} = \left(2.5\times10^{-6}\right) x + \left(-1.25\times10^{-7}\right) x^2 \tag{1.37}$$

The accuracy of WRM can be assessed by comparing its numerical value with those previously obtained in Worked Example 1.1. The plot of WRM results, u_w together with numerical, u_n and exact, u_e solutions are given in Fig. 1.7, and their numerical values at several locations along the bar are given in Table 1.4.

Based on the plot and the table, it can be concluded that WRM gives similar results to previous approaches hence validates its concepts. However, as mentioned previously, such accurate result obtained herein is generally not the case. Generally, like any other numerical techniques, WRM converges to the "accurate" solution with the increase in the order of polynomial of the guessed functions (or degree of freedom and mesh density later on).

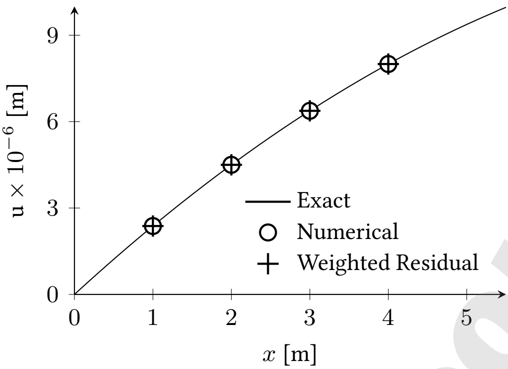

Figure 1.7: Axial displacement, u, along the bar between numerical and exact solutions.

Table 1.4: Axial displacement, u, along the bar between numerical and exact solutions.

x [m]	1	2	3	4
u_{w} [m]	2.38×10^{-6}	4.5×10^{-6}	6.38×10^{-6}	8×10^{-6}
u_{n} [m]	2.38×10^{-6}	4.5×10^{-6}	6.38×10^{-6}	8×10^{-6}
u_{e} [m]	2.38×10^{-6}	4.5×10^{-6}	6.38×10^{-6}	8×10^{-6}

1.3 Fundamental of Galerkin-WRM

We have discussed the basic concept of WRM that is, the integration of a function consists of a product of the residual function and a weight function and forcing this integrated value to zero in getting an algebraic function.

The Galerkin-WRM differs in the following aspects:

1. The guessed solution is expressed in terms of shape of functions, N_i and degree of freedoms, u_i instead of polynomial functions and their constants. In other words and in bar problem in particular, whilst the basic WRM uses the following guessed function

$$u = a_1 + a_2 x$$

in Galerkin-WRM, the following guessed function are used instead

$$u = N_1 u_1 + N_2 u_2$$

where N_1 and N_2 are the shape functions and u_1 and u_2 are the degree of freedoms (dofs). We are going to discuss in detail on shape functions and degree of freedoms in the next sub-section.

2. Galerkin WRM involves integration by part (IBP) which purposes are to:

   i. reduce the order of the original derivative terms so as to relax the continuity requirement
   ii. induce the natural (force) boundary conditions explicitly into the formulation

   Point i. and ii. deserve further elaboration. For point i., as we going to see later, the IBP will reduce the order of derivation of bar problem from $\frac{d^2u}{dx^2}$ to $\frac{du}{dx}$. Such a reduction would mean that a guessed function, say a linear polynomial, $(a + bx)$ once inserted into the former would vanish but a constant if inserted into the latter. This is what we mean by "to relax the continuity requiremen". Due to this, the final equation of Galerkin WRM is known as weak statement because it has been "weakened" from the original PDE (or ODE). Accordingly, the original form of the PDE (or ODE) is known as the strong statement of the problem.

   For point ii., as we are going to see, the IBP will induce boundary terms which exactly in the forms of Eqs. (1.9a) and (1.9b) (or Eq. (1.28b)). Having these terms induced rather naturally, these terms are called natural boundary conditions hence the interchangeable use of the term with force boundary conditions. Due to this natural induction of the force (or natural) boundary condition, we are not going to see the separate satisfaction of the force boundary conditions as we saw in the previous two methods.

3. Due to the use of dofs (hence the advantage of Galerkin-WRM), we are also not going to see the separate satisfaction of the displacement boundary conditions as previously seen in Eqs. (1.9c) and (1.9d) (or Eq. (1.28c)). Instead, the displacement boundary conditions will be imposed only after the complete establishment of the algebraic equation and prior to the solution of the equation. Due to this, what is

known as displacement boundary condition is also interchangeably known as essential boundary condition in Galerkin WRM and FEM for that matter.

Actually, there are two types of Galerkin-WRM which difference depends on the type of weight function. If the shape functions are used as the weight functions, the method is called Bubnov–Galerkin but if other functions are used (for example, interpolation function as in the basic WRM) the method is called Petrov–Galerkin. However, herein, our main interest will be on the former, that is Bubnov–Galerkin method.

The above is the description about Galerkin WRM (and how it differs from the basic WRM). Observing such an evolution is important as it can allow easier but deeper understanding of FEM.

Having established all these, we are now all set to discuss in detail the Galerkin-WRM and FEM for that matter and we are going to do this by solving the same bar ODE which we have previously discussed. To do this, we start by discussing the derivation of shape functions and the definition of dofs for a bar problem. Before that, let's note that Galerkin WRM is referred to FEM from now on.

1.3.1 Degree of Freedom and Shape Functions For Bar Element

FEM bar element can usually be of two types; linear element or quadratic element. These elements are shown in Fig. 1.8; the former has two nodes and the latter has three nodes per element. At each node, there is one translational degree of freedom (dof) which is the value of the axial displacement, u evaluated at that node.

Herein, we are going to derive the shape functions for both elements and we start by assuming an interpolation function, $u(x)$ in the forms of polynomial, thus

For linear bar element

$$u = a_1 + a_2 x \tag{1.38a}$$

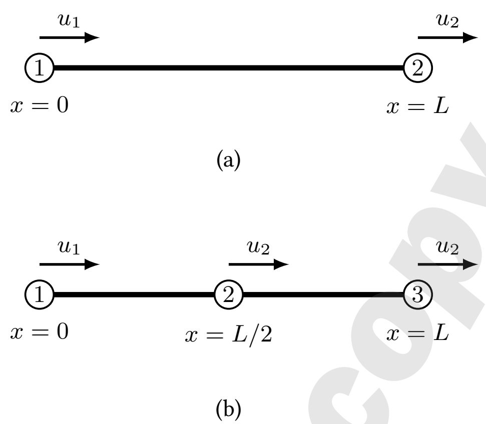

Figure 1.8: Degree of freedoms of (a) linear (b) quadratic bar elements.

For quadratic bar element

$$u = a_1 + a_2x + a_3x^2 \tag{1.38b}$$

Despite the familiar use of polynomial functions, it is preferable to deal with a special set of functions, known as shape functions, N_i and a special sets of coefficient, known as translational degree of freedoms (dof), u_i. This means, Eq. (1.38) must be equivalently expressed as:

For linear bar element

$$u = N_1u_1 + N_2u_2 \tag{1.39a}$$

For quadratic bar element

$$u = N_1u_1 + N_2u_2 + N_3u_3 \tag{1.39b}$$

The equations can be expressed compactly in components forms as:

$$u(x) = N_iu_i \tag{1.40}$$

where $i = 1, 2$ and $i = 1, 2, 3$ for linear and quadratic bar elements respectively.

Being referred as the translational degree of freedoms or dofs, u_1, u_2, and u_3 or u_i, are actually the nodal values of the axial displacement, $u(x)$. These dofs are graphically shown in Fig. 1.8.

The conversion from Eq. (1.38) to Eq. (1.39) or Eq. (1.40), hence the derivation of the shape functions, N_i can be done as follows.

For linear bar element

i. Evaluating Eq. (1.38a) at the location of the nodes (i.e. at both ends, $x = 0$, $x = L$) and equating them to the dofs will give:

$$\begin{aligned} u(x)|_{x=0} &= a_1 + a_2(0) = u_1 \\ u(x)|_{x=L} &= a_1 + a_2(L) = u_2 \end{aligned} \tag{1.41}$$

ii. Solving for the values of a_1, a_2 then gives:

$$\begin{aligned} a_1 &= u_1 \\ a_2 &= \frac{u_2 - u_1}{L} \end{aligned} \tag{1.42}$$

iii. Inserting Eq. (1.42) into Eq. (1.38a) and by factorizing give:

$$u(x) = \frac{L-x}{L}u_1 + \frac{x}{L}u_2 \tag{1.43}$$

By comparing Eq. (1.43) and Eq. (1.39a), it can be concluded that:

$$\begin{aligned} N_1 &= \frac{L-x}{L} \\ N_2 &= \frac{x}{L} \end{aligned} \tag{1.44}$$

Both N_1 and N_2 take the shapes as shown in Fig. 1.9. Take note of the shapes as they will be referred later in the discussion of integration by parts and the resulting boundary terms. For example, in both cases the shape functions, N_i has the value of unity at node i and zero at the other node.

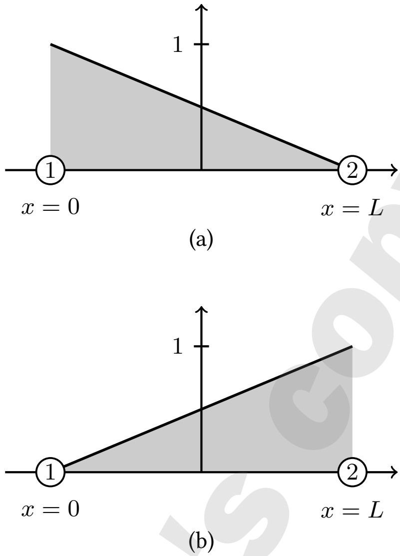

Figure 1.9: Linear shape functions (a) N_1 (b) N_2. The distance, x is measured from node 1.

For quadratic bar element

i. Evaluating Eq. (1.38b) at the location of the nodes (i.e. at both ends, $x = 0$, $x = L/2$, $x = L$) and equating them to the dofs will give:

$$\begin{aligned} u(x)|_{x=0} &= a_1 + a_2(0) + a_3(0)^2 = u_1 \\ u(x)|_{x=\frac{L}{2}} &= a_1 + a_2(\frac{L}{2}) + a_3(\frac{L}{2})^2 = u_2 \\ u(x)|_{x=L} &= a_1 + a_2(L) + a_3(L)^2 = u_3 \end{aligned} \tag{1.45}$$

ii. Solving for the values of a_1, a_2, a_3 then gives:

$$\begin{aligned} a_1 &= u_1 \\ a_2 &= -\frac{3}{L}u_1 + \frac{4}{L}u_2 - \frac{1}{L}u_3 \\ a_3 &= \frac{2}{L^2}u_1 - \frac{4}{L^2}u_2 + \frac{2}{L^2}u_3 \end{aligned} \tag{1.46}$$

iii. Inserting Eq. (1.46) into Eq. (1.38b) and by factorizing give:

$$u(x) = \left(\frac{L-2x}{L}\right)\left(\frac{L-x}{L}\right)u_1 + \frac{4x}{L}\left(\frac{L-x}{L}\right)u_2 + \frac{x}{L}\left(\frac{L-2x}{L}\right)u_3 \tag{1.47}$$

By comparing Eq. (1.47) and Eq. (1.39b), it can be concluded that:

$$\begin{aligned} N_1 &= \left(\frac{L-2x}{L}\right)\left(\frac{L-x}{L}\right) \\ N_2 &= \frac{4x}{L}\left(\frac{L-x}{L}\right) \\ N_3 &= \frac{x}{L}\left(\frac{L-2x}{L}\right) \end{aligned} \tag{1.48}$$

Both N_1, N_2 and N_3 take the shapes as shown in Fig. 1.10. Take note of the shapes as they will be referred later in the discussion of integration by parts and the resulting boundary terms. For example, in both cases the shape functions, N_i has the value of unity at node i and zero at the other node.

1.3.2 Discretization by Galerkin Method

Having established the shape functions and the degree of freedoms of the bar element, we are all set to proceed with the discretization of the ODE.

For linear element, inserting Eq. (1.43) into Eq. (1.8) gives:

$$EA\frac{d^2(N_1u_1 + N_2u_2)}{dx^2} \neq -q \tag{1.49a}$$

whilst for quadratic element, inserting Eq. (1.47) into Eq. (1.8) gives:

$$EA\frac{d^2(N_1u_1 + N_2u_2 + N_3u_3)}{dx^2} \neq -q \tag{1.49b}$$

Take note that, from now on, we are going to deal frequently with the component forms as this would eliminate the necessity to distinguish between linear and quadratic elements. With this, it should also demonstrate the

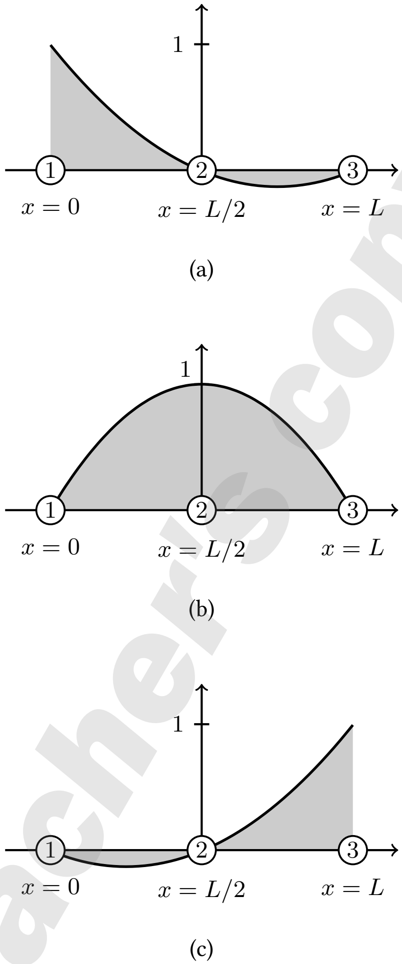

Figure 1.10: Quadratic shape functions (a) N_1 (b) N_2 (c) N_3. The distance, x is measured from node 1.

advantage of such notation system. The component forms of Eq. (1.49) is given as

$$EA\frac{d^2(N_j u_j)}{dx^2} \neq -q, \tag{1.50}$$
$$\Rightarrow \text{for } j = 1, 2 \text{ (linear) and } j = 1, 2, 3 \text{ (quadratic)}$$

Since we have j number of unknowns (because we want to determine the values of the dofs, u_j) we thus need j number of simultaneous equation. These required equations can be obtained by multiplying Eq. (1.50) subsequently with a set of weight functions. This act is similar to the one conducted to Eq. (1.31) during the discussion of the basic concept of WRM.

Also, as mentioned earlier in this chapter, for Bubnov-Galerkin formulation, the weight functions are also the shape functions. For example, for a linear bar element, the two simultaneous equations can be obtained as follows:

$$N_1 \left(EA \frac{d^2(N_j d_j)}{dx^2} - q \right) \neq 0 = R_1 \tag{1.51a}$$

$$N_2 \left(EA \frac{d^2(N_j d_j)}{dx^2} - q \right) \neq 0 = R_2 \tag{1.51b}$$

In component forms;

$$\begin{aligned} &N_i \left(EA \frac{d^2(N_j u_j)}{dx^2} + q \right) \neq 0 = R_i, \\ &\Rightarrow \text{for } i, j = 1, 2 \text{ (linear) and for } i, j = 1, 2, 3 \text{ (quadratic)} \end{aligned} \tag{1.52}$$

The continuous nature of Eq. (1.52) (since it is still in ODE forms) can be converted into the equivalent algebraic forms by integrating over the length of the bar. This action is referred to as 'weighting the residual'; an action from which an approximate solution can be obtained by forcing the residual to zero. The algebraic forms of Eq. (1.52) are thus given as:

$$\begin{aligned} &\int_0^L N_i \left(EA \frac{d^2(N_j u_j)}{dx^2} + q \right) dx = 0 \\ &\Rightarrow \int_0^L N_i EA \frac{d^2(N_j u_j)}{dx^2} \, dx + \int_0^L N_i q \, dx = 0 \end{aligned} \tag{1.53}$$

It can be seen that Eq. (1.53) involves the integration of a product of functions, thus integration by parts (IBP) can be employed. The reasons for employing the IBP, as mentioned earlier in this chapter, are:

i. to reduce the continuity requirement since only first and lower derivatives will be dealt with

ii. to induce (make explicit) the natural boundary conditions as boundary terms resulting from the IBP

Employing IBP to the first term on the LHS of Eq. (1.53) gives:

$$-\int_0^L \left(\frac{dN_i}{dx} EA \frac{dN_j}{dx} \right) dx\, u_j = -\int_0^L qN_i\, dx - N_i EA\, \frac{d(N_j u_j)}{dx}\bigg|_0^L \tag{1.54}$$

In Eq. (1.54) above, the dofs, u_j are taken out from the integral because they are constants thus do not involve in both differentiation and integration. The last terms on the right hand side of Eq. (1.54), are called boundary terms resulting from the IBP.

By inserting back Eq. (1.40), the boundary terms can be given as:

$$N_i EA\, \frac{d(N_j u_j)}{dx}\bigg|_0^L = N_i EA\, \frac{du(x)}{dx}\bigg|_0^L \tag{1.55}$$

And further comparison with Eqs. (1.9a) and (1.9b), one can identify that Eq. (1.55) is actually the natural boundary conditions of the bar; hence it can be given that:

$$\begin{aligned} EA\, \frac{d(N_j u_j)}{dx}\bigg|_0^L &= N_i EA\, \frac{du(x)}{dx}\bigg|_{x=L} - N_i EA\, \frac{du(x)}{dx}\bigg|_{x=0} \\ &= -\, N_i|_{x=L}\, F_L - N_i|_{x=0}\, F_0 \end{aligned} \tag{1.56}$$

Eq. (1.56) deserves further explanation. The prevalence of the natural/force boundary conditions, i.e. F_0 and F_L inside the equation depends on the accompanying value of the shape functions when evaluated at the location of the respective node i.e. $N_i|_{x=L}$ and $N_i|_{x=0}$. As can be seen, say from Fig. 1.9 (linear element), the value of the shape functions at the location can be given as:

$$\begin{aligned} &\text{at } x = 0: \quad N_1|_{x=0} = 1, \quad N_2|_{x=0} = 0 \\ &\text{at } x = L: \quad N_1|_{x=L} = 0, \quad N_2|_{x=L} = 1 \end{aligned} \tag{1.57}$$

As a result, for linear bar element, the boundary terms of Eq. (1.56) can be specifically given as:

$$\begin{aligned} N_1|_{x=L}\,F_L - N_1|_{x=0}\,F_0 &= -(0)F_L - (1)F_0 = -F_0 \\ N_2|_{x=L}\,F_L - N_2|_{x=0}\,F_0 &= -(1)F_L - (0)F_0 = -F_L \end{aligned} \tag{1.58}$$

But, for quadratic bar element, there is one particular feature worth of attention, that is, the evaluated value of $N2$ always null at the location of the edge nodes (refer Fig. 1.10). By taking this into consideration, the boundary terms of Eq. (1.9) for a quadratic bar element are thus given as:

$$\begin{aligned} N_1|_{x=L}\,F_L - N_1|_{x=0}\,F_0 &= -(0)F_L - (1)F_0 = -F_0 \\ N_2|_{x=L}\,F_L - N_2|_{x=0}\,F_0 &= -(0)F_L - (0)F_0 = 0 \\ N_3|_{x=L}\,F_L - N_3|_{x=0}\,F_0 &= -(1)F_L - (0)F_0 = -F_L \end{aligned} \tag{1.59}$$

Due to these various conditions and combination, it is common in the discussion of finite element formulation to write Eq. (1.54) simply as:

$$\int_0^L \left(\frac{dN_i}{dx} EA \frac{dN_j}{dx} \right) dx\, u_j = \int_0^L qN_i\, dx + b_i \tag{1.60}$$

where b_i refers to boundary terms which can be nodal loads or reaction forces at support. Eq. (1.60) completes the discretization of the bar's ODE by Galerkin-WRM.

1.3.3 Local Load Vector and Stiffness Matrix

Now, let's discuss the other term on the right hand side of Eq. (1.60). Furthermore, let's call this term as equivalent nodal loads and denote it as q_i, thus;

$$q_i = \int_0^L qN_i\, dx \tag{1.61}$$

It is termed as equivalent nodal loads because these conversion has an equivalent effects of a distributed load, q that is acting on the domain; effects as perceived at the nodes. To better describe such equivalence, let's refer to Fig. 1.11. As shown in the figure, the distributed load, $q(x)$ which is acting on the domain of the bar would deform the bar in the same manner as if the bar is subjected to the two nodal loads (or three for quadratic bar) which values are calculated by Eq. (1.61). Note that, despite the illustration that displays as if the distributed load, $q(x)$ is in the vertical direction, it must be understood that the load acts in the axial direction of the bar. Such an illustration is for convenience purposes only.

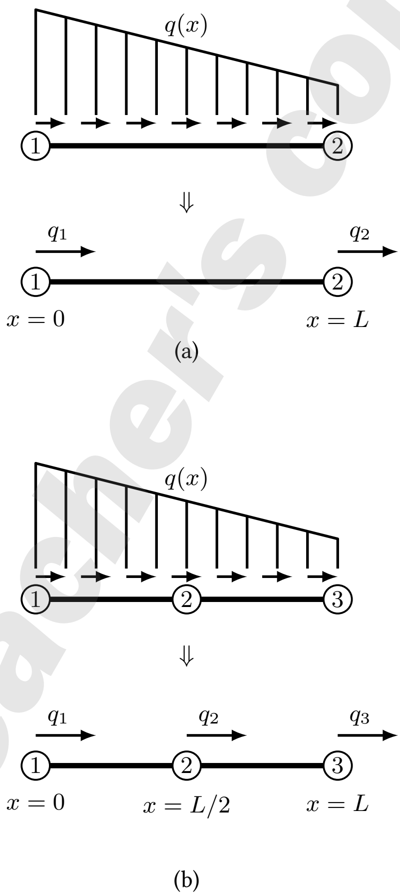

Figure 1.11: Equivalent nodal loads for (a) linear and (b) quadratic bar elements.

Having defined Eq. (1.61), Eq. (1.60) can be rewritten as:

$$\int_0^L \left(\frac{dN_i}{dx} EA \frac{dN_j}{dx} \right) dx \, u_j = q_i + b_i \tag{1.62}$$

Eq. (1.62) was derived for one particular bar element, thus it is called local equilibrium equation. For a more compact statement, Eq. (1.62) is usually written in component forms as:

$$k_{ij} u_j = r_i \tag{1.63}$$

where k_{ij} is termed as the local stiffness matrix of the bar element. u_j and r_i, on the other hand, are the local vector of dofs and the load vector of the bar element, respectively. Thus, the stiffness matrix and the load vector of the bar element can be given as:

$$\begin{aligned} k_{ij} &= \int_0^L \left(\frac{dN_i}{dx} EA \frac{dN_j}{dx} \right) dx \\ r_i &= q_i + b_i \end{aligned} \tag{1.64}$$

Alternatively, in matrix forms, the local equilibrium can also be represented as:

$$[k] \{u\} = \{r\} \tag{1.65}$$

In an expanded matrix forms, the stiffness matrix of linear and quadratic elements can be respectively given as:

For linear bar element

$$\begin{aligned} [k] &= \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{12} \end{bmatrix} \\ &= \begin{bmatrix} \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_1}{dx} \right) dx & \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_2}{dx} \right) dx \\ \int_0^L \left(\frac{dN_2}{dx} EA \frac{dN_1}{dx} \right) dx & \int_0^L \left(\frac{dN_2}{dx} EA \frac{dN_2}{dx} \right) dx \end{bmatrix} \end{aligned} \tag{1.66a}$$

For quadratic bar element

$$
\begin{aligned}
\left[k\right] &= \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{12} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \\
&= \begin{bmatrix} \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_1}{dx}\right) dx & k_{12} & k_{13} \\ k_{21} & k_{12} & k_{23} \\ \int_0^L \left(\frac{dN_3}{dx} EA \frac{dN_1}{dx}\right) dx & k_{32} & k_{33} \end{bmatrix}
\end{aligned} \tag{1.66b}
$$

(Note: since the procedure is standard and repetitive, as well due to the limited spacing, only k_{11} and k_{31} are expanded in Eq. (1.66b) for demonstration).

Stiffness matrices which are given in Eq. (1.66) are still in integral forms. Generally, to carry out the integration, numerical integration is employed. However, since those in Eq. (1.66) are simple, direct integration is used herein. The main purpose is that, with direct (analytical) integration, the evolution of the procedure can easier be observed since all variables remained to be seen. For readers who are interested to know about the numerical integration of the stiffness matrix, please refer to Chapter 6.

For demonstration purposes, the integration of k_{12} are shown below for both linear and quadratic elements. The fully integrated stiffness matrix for both elements is then given.

For linear bar element

$$
\begin{aligned}
k_{12} &= \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_2}{dx}\right) dx \\
&= \int_0^L \left(-\frac{1}{L}\right) EA \left(\frac{1}{L}\right) dx \\
&= -\frac{EA}{L}
\end{aligned} \tag{1.67a}
$$

For quadratic bar element

$$k_{12} = \int_0^L \left(\frac{dN_1}{dx} EA \frac{dN_2}{dx}\right) dx$$
$$= \int_0^L \left(-\frac{3}{L} + \frac{4x}{L^2}\right) EA \left(\frac{4}{L} - \frac{8x}{L^2}\right) dx \tag{1.67b}$$
$$= -\frac{8EA}{3L}$$

The fully (or analytically) integrated stiffness matrix for both elements can be given as:

For linear bar element

$$\left[k\right] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.68a}$$

For quadratic bar element

$$\left[k\right] = \frac{EA}{L}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix} \tag{1.68b}$$

The load vectors are respectively given as:

For linear bar element

$$\left\{r\right\} = r_i = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} q_1 + b_1 \\ q_2 + b_2 \end{Bmatrix} = \begin{Bmatrix} \int_0^L qN_1\, dx + b_1 \\ \int_0^L qN_2\, dx + b_2 \end{Bmatrix} \tag{1.69a}$$

for a specific case of uniform distributed load (where q is constant), it can be given that:

$$\{r\} = r_i = \begin{Bmatrix} \dfrac{qL}{2} + b_1 \\ \dfrac{qL}{2} + b_2 \end{Bmatrix} \tag{1.69b}$$

For quadratic bar element

$$\{r\} = r_i = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} q_1 + b_1 \\ q_2 + b_2 \\ q_3 + b_3 \end{Bmatrix} = \begin{Bmatrix} \displaystyle\int_0^L qN_1\,dx + b_1 \\ \displaystyle\int_0^L qN_2\,dx + b_2 \\ \displaystyle\int_0^L qN_3\,dx + b_3 \end{Bmatrix} \tag{1.70a}$$

for a specific case of uniform distributed load (where q is constant), it can be given that:

$$\{r\} = r_i = \begin{Bmatrix} \dfrac{qL}{6} + b_1 \\ \dfrac{2qL}{3} + b_2 \\ \dfrac{qL}{6} + b_3 \end{Bmatrix} \tag{1.70b}$$

Note that, component forms and matrix forms will be used interchangeably in our forthcoming discussion.

1.4 Assembly of Elements: Global Stiffness Matrix, $[K]$ and Load Vector, $\{R\}$

A body or a structure is meshed by many elements in a finite element analysis. This is required so that interpolation of the dependent variables can approximate well the actual distribution of the variables. The mesh is constructed by breaking up the physical domain into smaller elements which are then connected at the nodes to ensure appropriate interaction. The meshing requires the assembly of local stiffness matrix, $[K]$ and local load

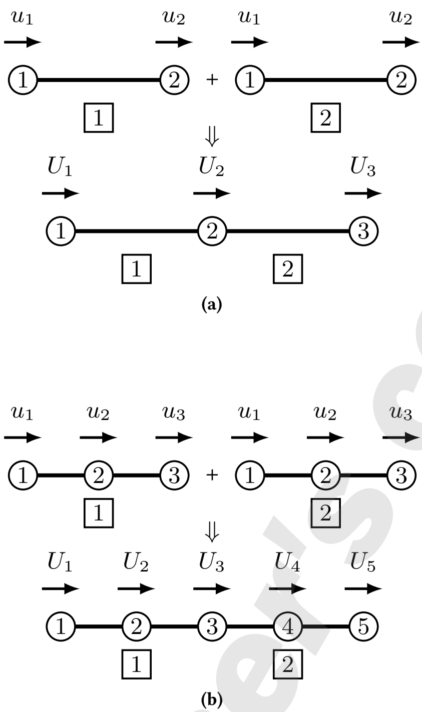

Figure 1.12: Local and global numbering of (a) linear (b) quadratic bar elements.

vector, $\{R\}$ of individual element to form the global stiffness matrix, $[K]$ and load vector, $\{R\}$. Although in practice, hundreds and thousands of elements might be assembled, herein, only the assembly of two bar elements is demonstrated for easy tracing of the procedure. Fig. 1.12a and Fig. 1.12b show the two assembled bar elements together with the local and global numbering of nodes and dof. Note that, node numbering is given in circle whilst element numbering is given in square box.

Based on the above, it is obvious that we need to have extra notation to distinguish different elements. In doing this, we introduce the use of superscript as follows:

i. The local stiffness matrix of element e is denoted as k_{ij}^N or $[k]$
ii. The local load vector of element e is denoted as r_i^N or $\{r\}$
iii. The vector of dof of element e is denoted as u_j^N or $\{u\}$

Having introduced the new superscript system, the local equilibrium for each element can thus be given as:

For linear bar element

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 \end{Bmatrix} \quad \Leftarrow \text{Element 1}$$

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 \end{Bmatrix} \quad \Leftarrow \text{Element 2} \tag{1.71}$$

Now the assembly of the elements can be done as follows:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \equiv u_1^2 \\ u_2^2 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 + r_1^2 \\ r_2^2 \end{Bmatrix} \tag{1.72}$$

or in global numbering as:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} \tag{1.73}$$

which can be simplified as:

$$[K]\{U\} = \{R\} \tag{1.74}$$

For quadratic bar element

$$
\begin{aligned}
&\begin{bmatrix} k^1_{11} & k^1_{12} & k^1_{13} \\ k^1_{21} & k^1_{22} & k^1_{23} \\ k^1_{31} & k^1_{32} & k^1_{33} \end{bmatrix} \begin{Bmatrix} u^1_1 \\ u^1_2 \\ u^1_3 \end{Bmatrix} = \begin{Bmatrix} r^1_1 \\ r^1_2 \\ r^1_3 \end{Bmatrix} \quad \Leftarrow \text{Element 1} \\
&\begin{bmatrix} k^2_{11} & k^2_{12} & k^2_{13} \\ k^2_{21} & k^2_{22} & k^2_{23} \\ k^2_{31} & k^2_{32} & k^2_{33} \end{bmatrix} \begin{Bmatrix} u^2_1 \\ u^2_2 \\ u^2_3 \end{Bmatrix} = \begin{Bmatrix} r^2_1 \\ r^2_2 \\ r^2_3 \end{Bmatrix} \quad \Leftarrow \text{Element 2}
\end{aligned}
\tag{1.75}
$$

The assembly of quadratic bar elements can be done as follows:

$$
\begin{bmatrix} k^1_{11} & k^1_{12} & k^1_{13} & 0 & 0 \\ k^1_{21} & k^1_{22} & k^1_{23} & 0 & 0 \\ k^1_{31} & k^1_{32} & k^1_{33} + k^2_{11} & k^2_{12} & k^2_{13} \\ 0 & 0 & k^2_{21} & k^2_{22} & k^2_{23} \\ 0 & 0 & k^2_{31} & k^2_{32} & k^2_{33} \end{bmatrix} \begin{Bmatrix} u^1_1 \\ u^1_2 \\ u^1_3 \equiv u^2_1 \\ u^2_2 \\ u^2_3 \end{Bmatrix} = \begin{Bmatrix} r^1_1 \\ r^1_2 \\ r^1_3 + r^2_1 \\ r^2_2 \\ r^2_3 \end{Bmatrix} \tag{1.76}
$$

or in global numbering as:

$$
\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} \tag{1.77}
$$

which can be simplified as:

$$
[K]\,\{U\} = \{R\} \tag{1.78}
$$

Based on Eq. (1.78), it can be deduced that $[K]$ is the global stiffness matrix, $\{U\}$ is the global vector of dof and $\{R\}$ is the global load vector.

1.5 Imposition of Essential Boundary Conditions

As mentioned earlier, to have a well-posed problem, essential (or displacement) boundary conditions must be imposed prior to the solution of the equilibrium equation. There are various ways to impose boundary conditions such as row-column condensation, penalty method, Lagrange multiplier method etc. Herein, row-column condensation is employed.

The basic concept of the row-column condensation can be explained as follows. Assume that we have a set of three simultaneous equations given as:

$$\begin{aligned} a_1U + a_2V + a_3W &= d \\ b_1U + b_2V + b_3W &= e \\ c_1U + c_2V + c_3W &= f \end{aligned} \tag{1.79}$$

Now, let's assume that V is known (the boundary conditions) i.e. $V = \bar{V}$ By inserting this known value into Eq. (1.79) gives:

$$\begin{aligned} a_1U + a_2\bar{V} + a_3W &= d \\ b_1U + b_2\bar{V} + b_3W &= e \\ c_1U + c_2\bar{V} + c_3W &= f \end{aligned} \tag{1.80}$$

Since $\bar{V}$ is known, all terms associated with it can be moved to the right hand side of the equation, thus:

$$\begin{aligned} a_1U + a_3W &= d - a_2\bar{V} \\ b_1U + b_3W &= e - b_2\bar{V} \\ c_1U + c_3W &= f - c_2\bar{V} \end{aligned} \tag{1.81}$$

By examining Eq. (1.81), one can realize that we have two unknowns with three equations. Since we need only two equations for the solution, we can just use say, the first and the third equations. (We can also decide to use the second equation together with either first or third equation), thus:

$$
\begin{aligned}
a_1 U + a_3 W &= d - a_2 \bar{V} \\
c_1 U + c_3 W &= f - c_2 \bar{V}
\end{aligned}
\tag{1.82}
$$

Now, the condensation can be made more obvious if we discuss the above in matrix forms. Eq. (1.82) can be given in matrix forms as:

$$
\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}
\begin{Bmatrix} U \\ V \\ W \end{Bmatrix}
=
\begin{Bmatrix} d \\ e \\ f \end{Bmatrix}
\tag{1.83}
$$

If $\bar{V}$ is known, Eq. (1.83) can be condensed (thus the employment of the boundary conditions) as follows:

$$
\begin{bmatrix} a_1 & \square & a_3 \\ \square & \square & \square \\ c_1 & \square & c_3 \end{bmatrix}
\begin{Bmatrix} U \\ \square \\ W \end{Bmatrix}
=
\begin{Bmatrix} d - a_2 \bar{V} \\ \square \\ f - c_2 \bar{V} \end{Bmatrix}
\tag{1.84}
$$

The above explains the basic concept of row-column condensation where it can be concluded that, an imposition of n number of boundary conditions to a stiffness matrix of $N \times N$ size and to a load vector of $N \times 1$, would condense the former to a size of $(N - n) \times (N - n)$ and the latter to a size of $(N - n) \times 1$. Once condensed, the solution of the matrix system can proceed.

Now, let's employ this method directly to our global equilibrium equation as given by Eqs. (1.73) and (1.77) for linear and quadratic element, respectively. For the purpose of discussion, let's assume that the right end of the bar is having a specified displacement. Such a condition leads to the following essential boundary conditions:

- For linear element: $U_3 = \bar{U_3}$
- For quadratic element: $U_5 = \bar{U_5}$

The imposition of the above boundary conditions to the global equilibrium equation can be done as follows.

For linear bar element

$$
\begin{aligned}
&\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} \\
\Rightarrow &\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} R_1 - K_{13}\bar{U_3} \\ R_2 - K_{23}\bar{U_3} \end{Bmatrix}
\end{aligned}
\tag{1.85}
$$

For quadratic bar element

$$
\begin{aligned}
&\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} \\
\Rightarrow &\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} R_1 - K_{15}\bar{U_5} \\ R_2 - K_{25}\bar{U_5} \\ R_3 - K_{35}\bar{U_5} \\ R_4 - K_{45}\bar{U_5} \end{Bmatrix}
\end{aligned}
\tag{1.86}
$$

Eqs. (1.85) and (1.86) are now ready to be solved. There are various ways to solve a matrix equation such as direct inversion, Gauss Elimination, Cholesky Decomposition, etc. In the accompanying MATLAB source codes, Gauss Elimination is adopted. Symbolically, the solution of the matrix equation is represented as:

$$
\{U\} = [K]^{-1}\{R\}
\tag{1.87}
$$

1.6 Worked Example

Let's solve the previous problem of Worked Example 1.1 this time by employing FEM. The problem is solved by the assembly of two bar elements. Both linear and quadratic bars are considered. The results are then verified against a closed-form solution.

1.6.1 Linear Bar Element

Due to the symmetrically, element 1 and element 2 would have a similar local stiffness matrix and load vector, thus:

$$k_{ij}^{1} = k_{ij}^{2} = \begin{bmatrix} 3.2\times 10^{6} & -3.2\times 10^{6} \\ -3.2\times 10^{6} & 3.2\times 10^{6} \end{bmatrix} \tag{1.88}$$

and

$$r_i^1 = \begin{Bmatrix} 2.5 + b_1^1 \\ 2.5 + 0 \end{Bmatrix} \tag{1.89}$$

$$r_i^2 = \begin{Bmatrix} 2.5 + 0 \\ 2.5 + 10 \end{Bmatrix} \tag{1.90}$$

Note that b_1^1 is the local reaction at the support of element 1.

The assembled global stiffness matrix, $[K]$ is given as:

$$\begin{aligned}
[K] &= \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \\
&= \begin{bmatrix} k^1_{11} & k^1_{12} & 0 \\ k^1_{21} & k^1_{22} + k^2_{11} & k^2_{12} \\ 0 & k^2_{21} & k^2_{22} \end{bmatrix} \\
&= \begin{bmatrix} 3.2\times10^6 & -3.2\times10^6 & 0 \\ -3.2\times10^6 & 6.4\times10^6 & -3.2\times10^6 \\ 0 & -3.2\times10^6 & 3.2\times10^6 \end{bmatrix}
\end{aligned} \tag{1.91}$$

and the assembled load vector, $\{R\}$ is given as:

$$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} r^1_1 \\ r^1_2 + r^2_1 \\ r^2_2 \end{Bmatrix} = \begin{Bmatrix} 2.5 + b^1_1 \\ 2.5 + 2.5 \\ 2.5 + 10 \end{Bmatrix} = \begin{Bmatrix} 2.5 + B_1 \\ 5.0 \\ 12.5 \end{Bmatrix} \tag{1.92}$$

To emphasize the global reactions forces, b^1_1 is expressed as B_1. This is unknown variable because its corresponding dof is the essential boundary condition which in turn, is a known value. The whole equilibrium equation can thus be given as :

$$\begin{bmatrix} 3.2\times10^6 & -3.2\times10^6 & 0 \\ -3.2\times10^6 & 6.4\times10^6 & -3.2\times10^6 \\ 0 & -3.2\times10^6 & 3.2\times10^6 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 2.5 + B_1 \\ 5.0 \\ 12.5 \end{Bmatrix} \tag{1.93}$$

By imposing the essential boundary conditions ($U_1 = 0$), the assembled global equilibrium equation is reduced to:

$$\begin{bmatrix} 6.4\times10^6 & -3.2\times10^6 \\ -3.2\times10^6 & 3.2\times10^6 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 5.0 - 0 \\ 12.5 - 0 \end{Bmatrix} \tag{1.94}$$

By solving Eq. (1.94) using Matlab command "\", the values of the assembled global dof U_i are thus obtained as:

$$\{U\} = \begin{Bmatrix} 0 \\ 5.47{\times}10^{-6} \\ 9.38{\times}10^{-6} \end{Bmatrix} \tag{1.95}$$

or in terms of global dofs as summarised in Table 1.5.

Table 1.5: Values of linear bar global degree of freedom, $\{U\}$

U_i	U_1	U_2	U_3
Deflection	0	$5.47{\times}10^{-6}$	$9.38{\times}10^{-6}$

Eq. (1.95) is the assembled global solution. By inserting these values accordingly into Eq. (1.65), the local forces can then be determined. However, let's determine first the global reaction forces, B_1 which can be done by inserting Eq. (1.95) into Eq. (1.93) which yields:

$$\begin{Bmatrix} B_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 3.2{\times}10^{6} & -3.2{\times}10^{6} & 0 \\ -3.2{\times}10^{6} & 6.4{\times}10^{6} & -3.2{\times}10^{6} \\ 0 & -3.2{\times}10^{6} & 3.2{\times}10^{6} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 5.0 \\ 12.5 \end{Bmatrix} \tag{1.96}$$

The global reaction force can thus be given as:

$$\{B_1\} = \{-20\} \tag{1.97}$$

Now, let's proceed with determination of the local values. The elemental local solution for each element is thus:

$$\{u^1\} = \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5.47{\times}10^{-6} \end{Bmatrix} \tag{1.98}$$

$$\{u^2\} = \begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = \begin{Bmatrix} 5.47{\times}10^{-6} \\ 9.38{\times}10^{-6} \end{Bmatrix} \tag{1.99}$$

Eqs. (1.98) and (1.99) are the elemental local solutions.

Element 1

By inserting Eq. (1.98) into Eq. (1.65), the local force (or reaction if at support) can be determined as follows:

$$\begin{aligned}\left\{b^1\right\} &= \left[k^1\right]\left\{u^1\right\} - \left\{q^1\right\}\\ &= \begin{bmatrix} 3.2\times10^6 & -3.2\times10^6 \\ -3.2\times10^6 & 3.2\times10^6 \end{bmatrix} \begin{Bmatrix} 0 \\ 5.47\times10^{-6} \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 2.5 \end{Bmatrix}\\ &= \begin{Bmatrix} -17.5 \\ 17.5 \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 2.5 \end{Bmatrix}\\ &= \begin{Bmatrix} -20 \\ 15 \end{Bmatrix}\end{aligned} \tag{1.100}$$

Element 2

By inserting Eq. (1.99) into Eq. (1.65), the local force (or reaction if at support) can be determined as follows:

$$\begin{aligned}\left\{b^2\right\} &= \left[k^2\right]\left\{u^2\right\} - \left\{q^2\right\}\\ &= \begin{bmatrix} 3.2\times10^6 & -3.2\times10^6 \\ -3.2\times10^6 & 3.2\times10^6 \end{bmatrix} \begin{Bmatrix} 5.47\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 2.5 \end{Bmatrix}\\ &= \begin{Bmatrix} -12.5 \\ 12.5 \end{Bmatrix} - \begin{Bmatrix} 2.5 \\ 2.5 \end{Bmatrix}\\ &= \begin{Bmatrix} -15 \\ 10 \end{Bmatrix}\end{aligned} \tag{1.101}$$

1.6.2 Quadratic Bar Element

Now, we are going to see the calculation of the same bar problem using quadratic bar element. Due to the symmetrically, element 1 and element 2 would have a similar local stiffness matrix and load vector, thus:

$$k_{ij}^1 = k_{ij}^2 = \begin{bmatrix} 0.75\times10^7 & -0.85\times10^7 & 0.11\times10^7 \\ -0.85\times10^7 & 1.71\times10^7 & -0.85\times10^7 \\ 0.11\times10^7 & -0.85\times10^7 & 0.75\times10^7 \end{bmatrix} \tag{1.102}$$

and

$$r_i^1 = \begin{Bmatrix} 0.83 + b_1^1 \\ 3.33 + 0 \\ 0.83 + 0 \end{Bmatrix} \tag{1.103}$$

$$r_i^2 = \begin{Bmatrix} 0.83 + 0 \\ 3.33 + 0 \\ 0.83 + 10 \end{Bmatrix} \tag{1.104}$$

The assembled global stiffness matrix, $[K]$ is given as:

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix}$$
$$= \begin{bmatrix} k^1_{11} & k^1_{12} & k^1_{13} & 0 & 0 \\ k^1_{21} & k^1_{22} & k^1_{23} & 0 & 0 \\ k^1_{31} & k^1_{32} & k^1_{33} + k^2_{11} & k^2_{12} & k^2_{13} \\ 0 & 0 & k^2_{21} & k^2_{22} & k^2_{23} \\ 0 & 0 & k^2_{31} & k^2_{32} & k^2_{33} \end{bmatrix}$$
$$= \begin{bmatrix} 0.75\times10^7 & -0.85\times10^7 & 0.11\times10^7 & 0 & 0 \\ -0.85\times10^7 & 1.71\times10^7 & -0.85\times10^7 & 0 & 0 \\ 0.11\times10^7 & -0.85\times10^7 & 1.49\times10^7 & -0.85\times10^7 & 0.11\times10^7 \\ 0 & 0 & -0.85\times10^7 & 1.71\times10^7 & -0.85\times10^7 \\ 0 & 0 & 0.11\times10^7 & -0.85\times10^7 & 0.75\times10^7 \end{bmatrix} \tag{1.105}$$

and the assembled load vector, $\{R\}$ is given as:

$$
\begin{aligned}
\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} &= \begin{Bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 + r_1^2 \\ r_2^2 \\ r_3^2 \end{Bmatrix} \\
&= \begin{Bmatrix} 0.83 + b_1^1 \\ 3.33 + 0 \\ 0.83 + 0 + 0.83 + 0 \\ 3.33 + 0 \\ 0.83 + 10 \end{Bmatrix} \\
&= \begin{Bmatrix} 0.83 + B_1 \\ 3.33 \\ 1.67 \\ 3.33 \\ 10.83 \end{Bmatrix}
\end{aligned}
\tag{1.106}
$$

To emphasize the global reactions forces, b_1^1 is expressed as B_1. This is unknown variable because its corresponding dof is the essential boundary condition which in turn, is a known value. Having established $[K]$ and , the whole equilibrium equation in the form of Eq. (1.76) can thus be given as:

$$\begin{bmatrix} 0.75\times10^7 & -0.85\times10^7 & 0.11\times10^7 & 0 & 0 \\ -0.85\times10^7 & 1.71\times10^7 & -0.85\times10^7 & 0 & 0 \\ 0.11\times10^7 & -0.85\times10^7 & 1.49\times10^7 & -0.85\times10^7 & 0.11\times10^7 \\ 0 & 0 & -0.85\times10^7 & 1.71\times10^7 & -0.85\times10^7 \\ 0 & 0 & 0.11\times10^7 & -0.85\times10^7 & 0.75\times10^7 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0.83 + B_1 \\ 3.33 \\ 1.67 \\ 3.33 \\ 10.83 \end{Bmatrix} \tag{1.107}$$

By imposing the essential boundary conditions ($U_1 = 0$), the assembled global equilibrium equation is reduced to:

$$\begin{bmatrix} 1.71\times10^7 & -0.85\times10^7 & 0 & 0 \\ -0.85\times10^7 & 1.49\times10^7 & -0.85\times10^7 & 0.11\times10^7 \\ 0 & -0.85\times10^7 & 1.71\times10^7 & -0.85\times10^7 \\ 0 & 0.11\times10^7 & -0.85\times10^7 & 0.75\times10^7 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 3.33 - 0 \\ 1.67 - 0 \\ 3.33 - 0 \\ 10.83 - 0 \end{Bmatrix} \tag{1.108}$$

By solving Eq. (1.108) using Matlab command "\", the values of the assembled global dof U_i are thus obtained as:

$$\{U\} = \begin{Bmatrix} 0 \\ 2.93\times10^{-6} \\ 5.47\times10^{-6} \\ 7.62\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} \tag{1.109}$$

or in terms of global dofs as summarised in Table 1.6.

Table 1.6: Values of quadratic bar global degree of freedom, $\{U\}$

	U_1	U_2	U_3	U_4	U_5
Deflection	0	2.93×10^{-6}	5.47×10^{-6}	7.62×10^{-6}	9.38×10^{-6}

Eq. (1.109) is the assembled global solution. As in the linear bar element example, the local forces can simply be determined by inserting these values accordingly into the local equilibrium Eq. (1.65). Before that, lets determine first the global reaction forces, B_1 which can be done by inserting Eq. (1.109) into Eq. (1.107) which yields:

$$\begin{Bmatrix} B_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0.75\times10^7 & -0.85\times10^7 & 0.11\times10^7 & 0 & 0 \\ -0.85\times10^7 & 1.71\times10^7 & -0.85\times10^7 & 0 & 0 \\ 0.11\times10^7 & -0.85\times10^7 & 1.49\times10^7 & -0.85\times10^7 & 0.11\times10^7 \\ 0 & 0 & -0.85\times10^7 & 1.71\times10^7 & -0.85\times10^7 \\ 0 & 0 & 0.11\times10^7 & -0.85\times10^7 & 0.75\times10^7 \end{bmatrix}$$
$$\begin{Bmatrix} 0 \\ 2.93\times10^{-6} \\ 5.47\times10^{-6} \\ 7.61\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} - \begin{bmatrix} 0.83 \\ 3.33 \\ 1.67 \\ 3.33 \\ 10.83 \end{bmatrix} \tag{1.110}$$

The global reaction force can thus be given as:

$$\{B_1\} = \{-20\} \tag{1.111}$$

Now, let's proceed with determination of the local values. The elemental local solution for each element is thus:

$$\{u_1\} = \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{Bmatrix} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2.93\times10^{-6} \\ 5.47\times10^{-6} \end{Bmatrix} \tag{1.112}$$

$$\{u_1\} = \begin{Bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{Bmatrix} = \begin{Bmatrix} U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 5.47\times10^{-6} \\ 7.62\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} \tag{1.113}$$

Eqs. (1.112) and (1.113) are the elemental local solutions.

Element 1

By inserting Eq. (1.112) into Eq. (1.65), the local force (or reaction if at support) can be determined as follows:

$$
\begin{aligned}
\left\{b^1\right\} &= \left[k^1\right]\left\{u^1\right\} - \left\{q^1\right\} \\
&= \begin{bmatrix} 7.47\times10^6 & -8.53\times10^6 & 1.07\times10^6 \\ -8.53\times10^6 & 1.71\times10^7 & -8.53\times10^6 \\ 1.07\times10^6 & -8.53\times10^6 & 7.47\times10^6 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.93\times10^{-6} \\ 5.47\times10^{-6} \end{Bmatrix} \\
&\quad - \begin{Bmatrix} 8.33\times10^{-1} \\ 3.33 \\ 8.33\times10^{-1} \end{Bmatrix} \\
&= \begin{Bmatrix} -19.17 \\ 3.33 \\ 15.83 \end{Bmatrix} - \begin{Bmatrix} 0.83 \\ 3.33 \\ 0.83 \end{Bmatrix} \\
&= \begin{Bmatrix} -20 \\ 0 \\ 15 \end{Bmatrix}
\end{aligned}
\tag{1.114}
$$

Element 2

By inserting Eq. (1.113) into Eq. (1.65), the local force (or reaction if at support) can be determined as follows:

$$
\begin{aligned}
\left\{b^2\right\} &= \left[k^2\right]\left\{u^2\right\} - \left\{q^2\right\} \\
&= \begin{bmatrix} 7.47\times10^6 & -8.53\times10^6 & 1.07\times10^6 \\ -8.53\times10^6 & 1.71\times10^7 & -8.53\times10^6 \\ 1.07\times10^6 & -8.53\times10^6 & 7.47\times10^6 \end{bmatrix} \begin{Bmatrix} 5.47\times10^{-6} \\ 7.62\times10^{-6} \\ 9.38\times10^{-6} \end{Bmatrix} \\
&\quad - \begin{Bmatrix} 0.83 \\ 3.33 \\ 0.83 \end{Bmatrix} \\
&= \begin{Bmatrix} -14.17 \\ 3.33 \\ 10.83 \end{Bmatrix} - \begin{Bmatrix} 0.83 \\ 3.33 \\ 0.83 \end{Bmatrix} \\
&= \begin{Bmatrix} -15 \\ 0 \\ 10 \end{Bmatrix}
\end{aligned}
\tag{1.115}
$$

The validity of both results thus formulation can be assessed by comparing their values with those previously obtained. The plot of FEM results against the closed-form solution (Eq. (1.27)) are shown in Fig. 1.13 which numerical values at several locations along the bar are given in Table 1.7.

Table 1.7: Comparison of values between u_{linear}, $u_{\text{quadratic}}$, u_{exact}

	$x = 0\,\text{m}$	$x = 1.25\,\text{m}$	$x = 2.5\,\text{m}$	$x = 3.75\,\text{m}$	$x = 5\,\text{m}$
u_{linear}	0	-	5.47×10^{-6}	-	9.38×10^{-6}
$u_{\text{quadratic}}$	0	2.93×10^{-6}	5.47×10^{-6}	7.62×10^{-6}	9.38×10^{-6}
u_{exact}	0	2.93×10^{-6}	5.47×10^{-6}	7.62×10^{-6}	9.38×10^{-6}

Based on the plot and the tabulated values, for the first time the convergence nature of a numerical analysis becomes obvious. As can be seen, the assembled two linear elements provide quite a poor approximation except at the location of the nodes where the results agree with the closed-form

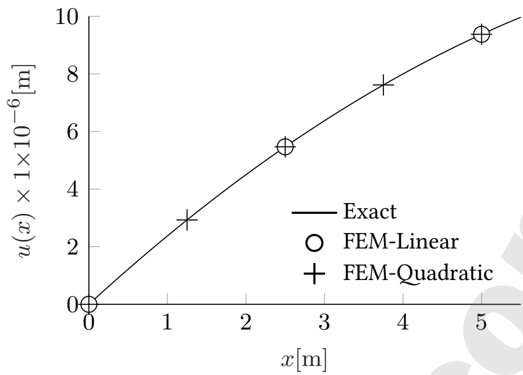

Figure 1.13: Axial displacement, u, along the bar between numerical and exact solutions.

solution. A better hence a converged solution is provided by the quadratic elements thus an immediate demonstration of the beneficial effect of the use of higher order elements. This is an immediate demonstration to the statement *"FEM converges to the 'accurate' solution with the increase in the order of polynomial of the guessed functions (or degree of freedom and mesh density later on)"* which was stated in previous discussions.

1.7 Matlab Source Codes

1.7.1 Linear Bar Element

```
% Clear data
clc; clear; close all

% Input
E = 200e6;  % Young's modulus (kN/m)
A = 0.04;   % Area (m^2)
L = 5;      % Bar length [m]
P = 10;     % Point load [kN]
q = 2;      % Distributed load [kN/m]

% ---------------------------------------------
% FEM solution - Displacement
% ---------------------------------------------

% Elements length
L1 = L/2; L2 = L/2;
```

```
% Matrix k & force r
k1 = A*E/L1*[ 1 -1;
             -1  1];
k2 = A*E/L2*[ 1 -1;
             -1  1];

r1 = [q*L1/2; q*L1/2];
r2 = [q*L1/2; q*L1/2];

% Assemble global matrix, K and vector, R
K = zeros(3);
K(1:2, 1:2) = K(1:2, 1:2) + k1;
K(2:3, 2:3) = K(2:3, 2:3) + k2;

R = zeros(3,1);
R(1:2) = R(1:2) + r1;
R(2:3) = R(2:3) + r2;

% Point load, P at the bar end
R(3) = R(3)+P;

% Solve for global displacement
U      = zeros(3,1);
U(2:3) = K(2:3,2:3)\R(2:3);

% -------------------------------------------
% FEM solution - Reaction, b & stress, sig
% -------------------------------------------

% Internal force or reaction
b1 = k1*[U(1); U(2)] - r1;
b2 = k2*[U(2); U(3)] - r2;

% Internal stress
sig1 = b1/A;
sig2 = b2/A;
```

1.7.2 Quadratic Bar Element

```
% Clear data
clc; clear; close all

% Input
E = 200e6;  % Young's modulus (kN/m)
A = 0.04;   % Area (m^2)
L = 5;      % Bar length [m]
P = 10;     % Point load [kN]
q = 2;      % Distributed load [kN/m]

% -------------------------------------------
% FEM solution - Displacement
% -------------------------------------------

% Elements length
L1 = L/2; L2 = L/2;

% Matrix k & force r
```

```
k1 = A*E*[ 7/(3*L1) -8/(3*L1)  1/(3*L1);
          -8/(3*L1) 16/(3*L1) -8/(3*L1);
           1/(3*L1) -8/(3*L1)  7/(3*L1)];
k2 = A*E*[ 7/(3*L2) -8/(3*L2)  1/(3*L2);
          -8/(3*L2) 16/(3*L2) -8/(3*L2);
           1/(3*L2) -8/(3*L2)  7/(3*L2)];

r1 = [q*L1/6; 2*q*L1/3; q*L1/6];
r2 = [q*L2/6; 2*q*L2/3; q*L2/6];

% Assemble global matrix, K and vector, R
K = zeros(5);
K(1:3, 1:3) = K(1:3, 1:3) + k1;
K(3:5, 3:5) = K(3:5, 3:5) + k2;

R = zeros(5,1);
R(1:3) = R(1:3) + r1;
R(3:5) = R(3:5) + r2;

% Point load, P at the bar end
R(5) = R(5)+P;

% Solve for global displacement
U      = zeros(5,1);
U(2:5) = K(2:5,2:5)\R(2:5);

% --------------------------------------------
% FEM solution - Reaction, b & stress, sig
% --------------------------------------------

% Internal force or reaction
b1 = k1*[U(1); U(2); U(3)] - r1;
b2 = k2*[U(3); U(4); U(5)] - r2;

% Internal stress
sig1 = b1/A;
sig2 = b2/A;
```

2 Scalar Element: Heat Transfer

2.1 Introduction

This chapter marks the beginning of the discussion on two-dimensional elements hence continuum mechanics. Whilst the previous chapter mainly discussed line element, herein we are going to discuss the first type of continuum element that is scalar element. Scalar element is an element which node has only one degree of freedom. It is employed in various physical problems such as heat transfer, potential flow, groundwater flow, electrostatics and magnetostatics. Despite its various applications, herein, for demonstration purposes on the use of scalar element and its corresponding FEM formulation, heat transfer problem is considered.

2.1.1 Derivation of Heat Transfer Partial Differential Equation

Heat transfer analysis concerns with the determination of temperature distribution within a body due to internal heat generation and temperature differences (as well external flux, convection and radiation) at the boundaries.

Since it is customary in this book to start a discussion from first principle, we begin our discussion by deriving the partial differential equation (PDE) of the problem. We limit our derivation to two-dimensional only.

The PDE can be derived by employing the conservation of energy principle to the differential element as shown in Fig. 2.1.

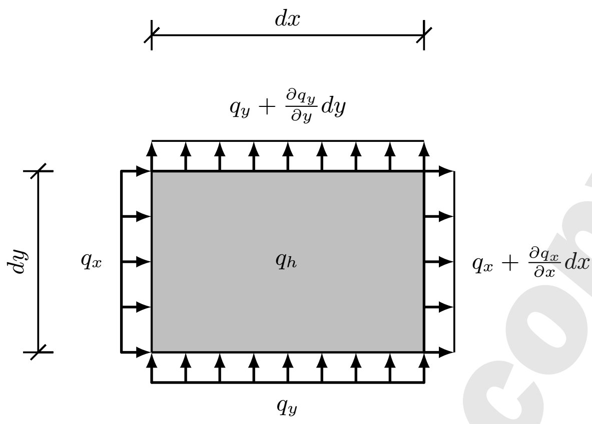

Figure 2.1: Heat flow differential element.

As shown in the figure, the dependent variables to be considered are the heat flux in x-direction, q_x and in y-direction, q_y. Similar to the previous chapter (i.e. Fig. 1.1), the variables are expanded by Taylor series at all sides (surfaces).

The conservation of energy principle requires that the time rate of change of internal energy inside the differential element must be equal to the net heat flowing into the differential element due to conduction plus the heat generated inside the differential element.

The first can be given as the time rate of change of internal energy:

$$Q_E = \rho C \frac{\partial T}{\partial t} dxdy \tag{2.1}$$

where ρ is the density, C is the specific heat and T is temperature.

Next, the determination of the net heat flow can be done by considering first the net heat flow in each direction.

In x-direction, by balancing terms on the right and left sides of the differential element, we obtain;

$$Q_x = q_x dy - \left(q_x + \frac{\partial q_x}{\partial x} dx \right) dy = -\frac{\partial q_x}{\partial x} dxdy \tag{2.2}$$

By similar argument, the net heat flow in y-direction is given as;

$$Q_y = q_y dx - \left(q_y + \frac{\partial q_y}{\partial y} dy \right) dx = -\frac{\partial q_y}{\partial y} dxdy \tag{2.3}$$

Having established the net heat flow in both directions, the net heat flowing into the differential element due to conduction through its surface can thus be given as

$$Q_{xy} = -\frac{\partial q_x}{\partial x} dxdy - \frac{\partial q_y}{\partial y} dxdy \tag{2.4}$$

Finally, if q_H is the rate of heat generation per unit volume, for the differential element, the heat generated inside the element can be given as

$$Q_h = q_h dxdy \tag{2.5}$$

The principle of energy conservation requires Eq. (2.1) to be balanced Eqs. (2.4) and (2.5) thus;

$$\rho C \frac{\partial T}{\partial t} dxdy = -\frac{\partial q_x}{\partial x} dxdy - \frac{\partial q_y}{\partial y} dxdy + q_h dxdy \tag{2.6}$$

which, by cancelling terms gives

$$\rho C \frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + q_h \tag{2.7}$$

Eq. (2.7) describes the problem in terms of heat fluxes (q_x, q_y) as the dependent variables as well as the temperature, T. To completely describe the problem in terms of temperature alone, we employ Fourier's law of heat conduction where

$$q_x = -k_x \frac{\partial T}{\partial x} \tag{2.8}$$

and

$$q_y = -k_y \frac{\partial T}{\partial y} \tag{2.9}$$

where k_x and k_y are thermal conductivity in x-direction and in y-direction, respectively. By differentiating Eqs. (2.8) and (2.9) once and inserting into Eq. (2.7), we obtain;

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial T}{\partial y} \right) + q_h \tag{2.10}$$

2.1.2 Steady Heat Transfer

For time independent problem, hence steady heat transfer, the time derivative terms on the left hand side of Eq. (2.10) can be omitted thus reducing the equation to

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) = -q_h \tag{2.11}$$

It is convenient to express Eq. (2.11) in matrix forms especially for the later FEM formulation, as follows.

$$\left\{\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right\}\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}\left\{\begin{matrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{matrix}\right\} T = -q_h \tag{2.12}$$

or

$$\{\partial\}\,[\,E\,]\,\{\partial\}^T T = -q_h \tag{2.13}$$

where

$$\{\partial\} = \left\{\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right\} \text{ and } [\,E\,] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{2.14}$$

2.1.3 Boundary Condition Equations

Like other PDEs, the heat transfer as given by Eq. (2.11) (or Eq. (2.13)) must be supplemented by boundary condition equations so as to have a well-posed problem. There are two types of boundary conditions; Neumann (natural) and Dirichlet (essential) boundary conditions.

Neumann (natural) boundary conditions

For general heat transfer problem, Neumann (natural) boundary conditions can be either (or combination of) specified flux, convection or/and radiation. However, for simplicity, only specified flux is considered herein.

$$\left(k_x \left.\frac{\partial T}{\partial x}\right|_b\right) n_x + \left(k_y \left.\frac{\partial T}{\partial y}\right|_b\right) n_y = q_b \tag{2.15}$$

where, q_b is the specified (known) flux at the boundary whilst n_x, and n_y are the components of the unit normal vector of the surface of the boundary. Symbol $\quad |_b$ means evaluated the boundary.

It worth to note that, for rectangular domain hence rectangular elements (meshes), simplification to the above equations can be obtained as for this case, in x-direction, $n_x = 1$ or -1 and $n_y = 0$ whilst in y-direction, $n_y = 1$ or -1 and $n_x = 0$.

It can be shown that the secondary terms produced by the integration by parts conducted on Eq. (2.11) (or Eq. (2.13)) will be in similar forms as in Eq. (2.15). Due to this, boundary equation above is also known as natural boundary condition

Dirichlet (essential) boundary conditions

Dirichlet boundary condition requires that the temperature at the boundary where natural conditions are unknown (to be solved) must be known and specified. Thus;

$$T|_b = \bar{T} \tag{2.16}$$

where $\bar{T}$ is the specified value of the temperature.

2.2 FEM Formulation for Steady Heat Transfer

Having established the PDE of the problem, we are all set to discretize Eq. (2.11) so as to obtain the FEM algebraic formulation. To do this, we must first derive the shape functions.

2.2.1 Degree of Freedoms and Shape Functions for Scalar Element

Herein, we are going to derive shape function for scalar element (heat transfer) having 4 nodes and 8 nodes.

4-Nodes Element

Fig. 2.2 shows a rectangular element with four nodes;

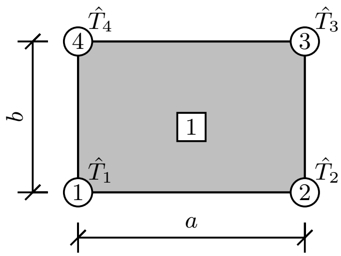

Figure 2.2: 4-nodes scalar element.

For this element, we take a polynomial interpolation function as

$$T = a_1 + a_2 x + a_3 y + a_4 xy \tag{2.17}$$

Next we evaluate the polynomial function above at the location of the nodes and equate the evaluated values to the corresponding degree of freedom, T_i, as follows.

$$T|_{x=0,y=0} = a_1 + a_2(0) + a_3(0) + a_4(0)(0) = \hat{T}_1 \tag{2.18a}$$

$$T|_{x=a,y=0} = a_1 + a_2(a) + a_3(0) + a_4(a)(0) = \hat{T}_2 \tag{2.18b}$$

$$T|_{x=a,y=b} = a_1 + a_2(a) + a_3(b) + a_4(a)(b) = \hat{T}_3 \tag{2.18c}$$

$$T|_{x=0,y=b} = a_1 + a_2(0) + a_3(b) + a_4(0)(b) = \hat{T}_4 \tag{2.18d}$$

By solving the simultaneous equations above, the polynomial coefficients can be obtained as

$$a_1 = \hat{T}_1 \tag{2.19a}$$

$$a_2 = \frac{-\hat{T}_1 + \hat{T}_2}{a} \tag{2.19b}$$

$$a_3 = \frac{-\hat{T}_1 + \hat{T}_4}{b} \tag{2.19c}$$

$$a_4 = \frac{\hat{T}_1 - \hat{T}_2 + \hat{T}_3 - \hat{T}_4}{a\,b} \tag{2.19d}$$

By inserting Eq. (2.19) into Eq. (2.17) and by collecting, we obtain;

$$\begin{aligned} T(x,y) = &\left(\frac{(a-x)\ (b-y)}{a\,b}\right)\hat{T}_1 + \left(\frac{x\ (b-y)}{a\,b}\right)\hat{T}_2 + \\ &\left(\frac{x\,y}{a\,b}\right)\hat{T}_3 + \left(\frac{y\ (a-x)}{a\,b}\right)\hat{T}_4 \end{aligned} \tag{2.20}$$

Based on the above, the shape functions can thus be given as

$$N_1 = \frac{(a-x)\ (b-y)}{a\,b} \tag{2.21a}$$

$$N_2 = \frac{x\ (b-y)}{a\,b} \tag{2.21b}$$

$$N_3 = \frac{x\,y}{a\,b} \tag{2.21c}$$

$$N_4 = \frac{y\ (a-x)}{a\,b} \tag{2.21d}$$

Therefore, Eq. (2.20) can also be given as

$$T = N_1\hat{T}_1 + N_2\hat{T}_2 + N_3\hat{T}_3 + N_4\hat{T}_4 \tag{2.22}$$

8-Nodes Element

Fig. 2.3 shows a rectangular element with 8 nodes.

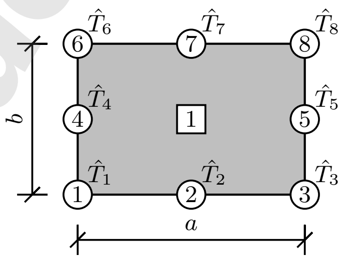

Figure 2.3: 8-nodes scalar element.

For this element, we take the polynomial interpolation function as

$$T = a_1 + a_2x + a_3y + a_4x^2 + a_5y^2 + a_6xy + a_7x^2y + a_8xy^2 \tag{2.23}$$

where the monomials are taken from the Pascal triangle as shown in Fig. 2.4.

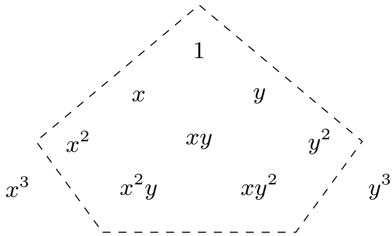

Figure 2.4: Pascal triangle of cubic terms.

Since Eq. (2.23) is relatively of high order, it would be efficient to discuss its shape function derivation from a more general approach which applies to all other cases. To do this, it is convenient to deal with matrix and vector representations. Thus, in vector forms, Eq. (2.23) can be given as

$$T = \begin{Bmatrix} 1 & x & y & x^2 & y^2 & xy & x^2y & xy^2 \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} \tag{2.24}$$

or

$$T = \{x\}\{a\}^T \tag{2.25}$$

Next we evaluate Eq. (2.25) at the location of the nodes and equate the evaluated values to the corresponding dof, T_i. This can be represented in matrix forms as

$$[\,x|\,]\,\{a\}^T = \{\hat{T}\}^T \tag{2.26}$$

where $[\,x|\,]$ is a 8×8 matrix which contains the evaluated values of the monomials and $\{\hat{T}\}^T$ is the vector of dof given as

$$\{\hat{T}\} = \begin{Bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 & \hat{T}_4 & \hat{T}_5 & \hat{T}_6 & \hat{T}_7 & \hat{T}_8 \end{Bmatrix} \tag{2.27}$$

Solving Eq. (2.26) for $\{a\}^T$ gives

$$\{a\}^T = [\,x|\,]^{-1}\,\{\hat{T}\}^T \tag{2.28}$$

Inserting Eq. (2.28) into Eq. (2.25) gives

$$T = \{x\}\,[\,x|\,]^{-1}\,\{\hat{T}\}^{T} \tag{2.29}$$

Based on Eq. (2.29), the shape functions can be given in vector forms as

$$\{N\} = \{x\}\,[\,x|\,]^{-1} \tag{2.30}$$

By conducting the multiplication in Eq. (2.30), individual shape function can thus be given as

$$N_1 = -\frac{(a-x)\,(b-y)\,(2\,a\,y - a\,b + 2\,b\,x)}{a^2\,b^2} \tag{2.31a}$$

$$N_2 = -\frac{x\,(b-y)\,(a\,b + 2\,a\,y - 2\,b\,x)}{a^2\,b^2} \tag{2.31b}$$

$$N_3 = \frac{x\,y\,(2\,a\,y - 3\,a\,b + 2\,b\,x)}{a^2\,b^2} \tag{2.31c}$$

$$N_4 = -\frac{y\,(a-x)\,(a\,b - 2\,a\,y + 2\,b\,x)}{a^2\,b^2} \tag{2.31d}$$

$$N_5 = \frac{4\,x\,(a-x)\,(b-y)}{a^2\,b} \tag{2.31e}$$

$$N_6 = \frac{4\,x\,y\,(b-y)}{a\,b^2} \tag{2.31f}$$

$$N_7 = \frac{4\,x\,y\,(a-x)}{a^2\,b} \tag{2.31g}$$

$$N_8 = \frac{4\,y\,(a-x)\,(b-y)}{a\,b^2} \tag{2.31h}$$

Therefore, the temperature distribution, T can also be given as

$$T = N_1\hat{T}_1 + N_2\hat{T}_2 + N_3\hat{T}_3 + N_4\hat{T}_4 + N_5\hat{T}_5 + N_6\hat{T}_6 + N_7\hat{T}_7 + N_8\hat{T}_8 \tag{2.32}$$

Referring to Eqs. (2.23) and (2.32), it can be observed that, for both elements, a general representation can be given in vector forms as;

$$T = \{N\}\{\hat{T}\}^{T} \tag{2.33}$$

where for 4-nodes element, $\{N\} = \begin{Bmatrix} N_1 & N_2 & N_3 & N_4 \end{Bmatrix}$ and $\{\hat{T}\} = \begin{Bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 & \hat{T}_4 \end{Bmatrix}$ whilst for 8-nodes element, $\{N\} = \begin{Bmatrix} N_1 & N_2 & \cdots & N_8 \end{Bmatrix}$ and $\{\hat{T}\} = \begin{Bmatrix} \hat{T}_1 & \hat{T}_2 & \cdots & \hat{T}_8 \end{Bmatrix}$.

2.2.2 Discretization by Galerkin WRM

By having established the shape functions, we are all set to discretize the PDE into algebraic equations. By inserting Eq. (2.33) into Eq. (2.13) we obtain

$$\{\partial\}\,[\,E\,]\,\{\partial\}^T\{N\}\{\hat{T}\}^T \neq -q_h \tag{2.34}$$

By observing Fig. 2.2 and Eq. (2.34), we can identify that, for a 4-nodes element, we have 4 unknown dofs, i.e. $\left\{\hat{T}_1 \quad \hat{T}_2 \quad \hat{T}_3 \quad \hat{T}_4\right\}$ with only 1 equation. To obtain sufficient number of equations, we will multiply Eq. (2.34) with the vector of shape function $\{N\}$ then integrate to produce four independent equations as required. Similar argument applies to 8-nodes element, resulting in eight independent equations as required.

Multiplying Eqn. (2.34) with the vector of shape functions $\{N\}$ and integrating give

$$\int_x\int_y \{N\}^T\left(\{\partial\}\,[\,E\,]\,\{\partial\}^T\{N\}\{\hat{T}\}^T + q_h\right)dy\,dx = 0 \tag{2.35}$$

Integration by Parts (IBP)

As mentioned in Chapter 1, the intentions of IBP are:

- To relax the continuity requirement
- To induce explicit the natural boundary conditions

Therefore, by conducting IBP to Eq. (2.35), we obtain

$$\begin{aligned}-\int_x\int_y &\{N\}^T\{\partial\}\,[\,E\,]\{\partial\}^T\{N\}\{\hat{T}\}^T dy\,dx \\ &= -\int_x\int_y \{N\}^T q_h\,dy\,dx - \int_s \{N\}^T q_n\,ds\end{aligned} \tag{2.36}$$

Rearranging gives

$$\begin{aligned}\int_x \int_y \{N\}^T \{\partial\} [\,E\,] \{\partial\}^T \{N\} dy\, dx\; \{\hat{T}\}^T \\ = \int_x \int_y \{N\}^T q_h \, dy\, dx + \int_s \{N\}^T q_n \, ds\end{aligned} \tag{2.37}$$

where q_n is the flux at the boundary (natural) as given by **??**. Now, let's introduce the following matrix and vectors.

Matrix, Vectors and Discretized Equation of Heat Transfer

Let's define the term on the LHS of Eq. (2.37) as the conductance matrix $[\,k\,]$ given as

$$[\,k\,] = \int_x \int_y \{N\}^T \{\partial\} [\,E\,] \{\partial\}^T \{N\} \, dy\, dx \tag{2.38}$$

Next, let's introduce a vector $\{q\}$ to represent the heat generation as

$$\{q\} = \int_x \int_y \{N\}^T q_h \, dy\, dx \tag{2.39}$$

Finally, let's introduce a vector for the boundary terms induced by the IBP and call it as natural boundary condition vector, $\{b\}$ where

$$\{b\} = \int_s \{N\}^T q_n \, ds \tag{2.40}$$

It can be shown that, for rectangular domain hence rectangular element (mesh), Eq. (3.37) (depending on the direction) can be given as

$$-\left(k_x \left.\frac{\partial T}{\partial x}\right|_x \right) n_x = q_x \tag{2.41a}$$

$$-\left(k_y \left.\frac{\partial T}{\partial y}\right|_y \right) n_y = q_y \tag{2.41b}$$

Having established these, the discretized equation of steady heat transfer can thus be given as

$$[\,k\,]\{\hat{T}\}^T = \{q\} + \{b\} = \{r\} \tag{2.42}$$

It must be noted that, since IBP is conducted, the multiplication in Eq. (3.35) between terms inside the integral must be carried out from left to right. To demonstrate this and to provide example on how to build the above matrix, the determination of k_{34} for 4-nodes scalar element is shown below. Although in practice, numerical integration is employed which detailed procedure is given in Chapter 6, herein analytical integration is employed so as to allow for easy tracing of the procedure and to allow for its immediate application. Such analytical integrations are possible herein because we limit our discussion to rectangular domain hence rectangular elements only.

Step-by-step procedure for the determination of k_{34}

Step 1

$$\ldots\{N_3\}\left\{\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right\}\ldots \tag{2.43}$$

Step 2

$$\ldots\left\{\frac{\partial N_3}{\partial x} \quad \frac{\partial N_3}{\partial y}\right\}\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}\ldots \tag{2.44}$$

Step 3

$$\ldots\left\{k_x\frac{\partial N_3}{\partial x} \quad k_y\frac{\partial N_3}{\partial y}\right\}\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix}\ldots \tag{2.45}$$

Step 4

$$\ldots\left\{k_x\frac{\partial N_3}{\partial x}\frac{\partial}{\partial x} \quad k_y\frac{\partial N_3}{\partial y}\frac{\partial}{\partial y}\right\}\{N_4\}\ldots \tag{2.46}$$

Step 5 (analytical integration)
Now by inserting N_3 and N_4 as given by Eq. (2.21) into Eq. (2.46), k_{34} is given as

$$k_{34} = \int_x \int_y \left(k_x \frac{\partial N_3}{\partial x} \frac{\partial N_4}{\partial x} + k_y \frac{\partial N_3}{\partial y} \frac{\partial N_4}{\partial y} \right) dy\, dx \tag{2.47}$$

For a rectangular element with a width of a and a height of b, as shown in Fig. 2.2, k_{34} can be re-given as

$$k_{34} = \int_0^a \int_0^b \left(k_x \frac{\partial N_3}{\partial x} \frac{\partial N_4}{\partial x} + k_y \frac{\partial N_3}{\partial y} \frac{\partial N_4}{\partial y} \right) dy\, dx = -\frac{b\,k_x}{3\,a} + \frac{a\,k_y}{6\,b} \tag{2.48}$$

The complete integrated values of $[\,k\,]$ for 4-nodes element can be given as

$$[\,k\,] = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & s_4 & s_3 \\ s_3 & s_4 & s_1 & s_2 \\ s_4 & s_3 & s_2 & s_1 \end{bmatrix} \tag{2.49}$$

where

$$s_1 = \frac{b\,k_x}{3\,a} + \frac{a\,k_y}{3\,b} \qquad s_3 = -\frac{b\,k_x}{6\,a} - \frac{a\,k_y}{6\,b}$$

$$s_2 = -\frac{b\,k_x}{3\,a} + \frac{a\,k_y}{6\,b} \qquad s_4 = \frac{b\,k_x}{6\,a} - \frac{a\,k_y}{3\,b}$$

Repeating the procedure above (Step 1 to Step 5), the complete integrated values of $[\,k\,]$ for 8-nodes element can be given as

$$[\,k\,] = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ s_2 & s_1 & s_4 & s_3 & s_5 & s_8 & s_7 & s_6 \\ s_3 & s_4 & s_1 & s_2 & s_7 & s_8 & s_5 & s_6 \\ s_4 & s_3 & s_2 & s_1 & s_7 & s_6 & s_5 & s_8 \\ s_5 & s_5 & s_7 & s_7 & s_9 & 0 & s_{10} & 0 \\ s_6 & s_8 & s_8 & s_6 & 0 & s_{11} & 0 & s_{12} \\ s_7 & s_7 & s_5 & s_5 & s_{10} & 0 & s_9 & 0 \\ s_8 & s_6 & s_6 & s_8 & 0 & s_{12} & 0 & s_{11} \end{bmatrix} \tag{2.50}$$

where:

$$s_1 = \frac{26\,b\,k_x}{45\,a} + \frac{26\,a\,k_y}{45\,b} \qquad s_7 = -\frac{4\,b\,k_x}{9\,a} - \frac{a\,k_y}{15\,b}$$

$$s_2 = \frac{14\,b\,k_x}{45\,a} + \frac{17\,a\,k_y}{90\,b} \qquad s_8 = \frac{b\,k_x}{15\,a} - \frac{8\,a\,k_y}{9\,b}$$

$$s_3 = \frac{23\,b\,k_x}{90\,a} + \frac{23\,a\,k_y}{90\,b} \qquad s_9 = \frac{16\,b\,k_x}{9\,a} + \frac{8\,a\,k_y}{15\,b}$$

$$s_4 = \frac{17\,b\,k_x}{90\,a} + \frac{14\,a\,k_y}{45\,b} \qquad s_{10} = \frac{8\,b\,k_x}{9\,a} - \frac{8\,a\,k_y}{15\,b}$$

$$s_5 = -\frac{8\,b\,k_x}{9\,a} + \frac{a\,k_y}{15\,b} \qquad s_{11} = \frac{8\,b\,k_x}{15\,a} + \frac{16\,a\,k_y}{9\,b}$$

$$s_6 = -\frac{b\,k_x}{15\,a} - \frac{4\,a\,k_y}{9\,b} \qquad s_{12} = -\frac{8\,b\,k_x}{15\,a} + \frac{8\,a\,k_y}{9\,b}$$

2.3 Worked Example

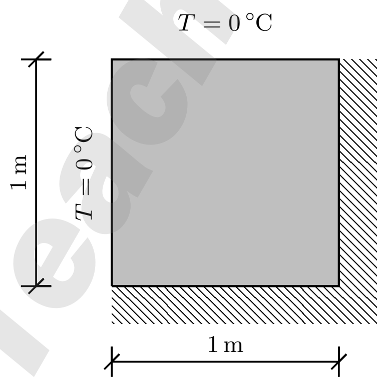

Figure 2.5: Plate with heat source, $Q = 600\,\mathrm{W\,m^{-3}}$ and thermal conductivity, $k_x = k_y = 400\,\mathrm{W\,m^{-1}\,{}^\circ C^{-1}}$. Shaded region is the insulated boundary.

Herein, we are going to demonstrate the FEM formulation for heat transfer equations by solving the problem as shown in Fig. 2.5. For a step-by-step demonstration (hand-calculation), only four assembled 4-nodes elements and two 8-nodes elements are used, so as to allow for easy tracing of the procedure. However, it must be noted that such an arrangement is for demonstration purpose only as the result can be poor. After the step-by-step calculation, the procedure is run using a sufficient number of elements (denser mesh). The results are then validated against results obtained from commercial software for proof of correctness.

4-nodes element

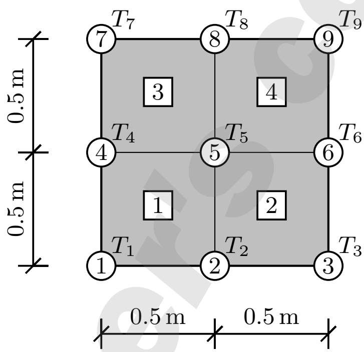

Figure 2.6: Arrangement of element and global dofs numbering for 4-nodes element

Due to the symmetricality (refer to Fig. 2.6), each element would have similar conductance matrix and internal heat generation forces. Based on Eq. (2.49), the conductance matrix is given as

$$[k^1] = [k^2] = [k^3] = [k^4] = \begin{bmatrix} 200 & -50 & -100 & -50 \\ -50 & 200 & -50 & -100 \\ -100 & -50 & 200 & -50 \\ -50 & -100 & -50 & 200 \end{bmatrix} \tag{2.51}$$

and the internal heat generation is given as

$$\{r^1\} = \{r^2\} = \{r^3\} = \{r^4\} = \begin{Bmatrix} \dfrac{Qab}{4} \\ \dfrac{Qab}{4} \\ \dfrac{Qab}{4} \\ \dfrac{Qab}{4} \end{Bmatrix} = \begin{Bmatrix} 37.5 \\ 37.5 \\ 37.5 \\ 37.5 \end{Bmatrix} \tag{2.52}$$

The complete global equilibrium equation can be given as

$$\begin{bmatrix} 200 & -50 & 0 & -50 & -100 & 0 & 0 & 0 & 0 \\ -50 & 400 & -50 & -100 & -100 & -100 & 0 & 0 & 0 \\ 0 & -50 & 200 & 0 & -100 & -50 & 0 & 0 & 0 \\ -50 & -100 & 0 & 400 & -100 & 0 & -50 & -100 & 0 \\ -100 & -100 & -100 & -100 & 800 & -100 & -100 & -100 & -100 \\ 0 & -100 & -50 & 0 & -100 & 400 & 0 & -100 & -50 \\ 0 & 0 & 0 & -50 & -100 & 0 & 200 & -50 & 0 \\ 0 & 0 & 0 & -100 & -100 & -100 & -50 & 400 & -50 \\ 0 & 0 & 0 & 0 & -100 & -50 & 0 & -50 & 200 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{Bmatrix} = \begin{Bmatrix} 37.5 \\ 75.0 \\ 37.5 \\ 75.0 \\ 150.0 \\ 75.0 \\ 37.5 \\ 75.0 \\ 37.5 \end{Bmatrix} \tag{2.53}$$

By imposing the essential boundary conditions ($T_1 = T_4 = T_7 = T_8 = T_9 = 0$), the assembled global equilibrium equation is reduced to

$$\begin{bmatrix} 400 & -50 & -100 & -100 \\ -50 & 200 & -100 & -50 \\ -100 & -100 & 800 & -100 \\ -100 & -50 & -100 & 400 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 75.0 \\ 37.5 \\ 150.0 \\ 75.0 \end{Bmatrix} \tag{2.54}$$

By solving Eq. (2.54) using Matlab command “\”,the values of the temperature, are thus obtained as:

$$\begin{Bmatrix} T_2 \\ T_3 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 0.482 \\ 0.621 \\ 0.386 \\ 0.482 \end{Bmatrix} \tag{2.55}$$

8-nodes element

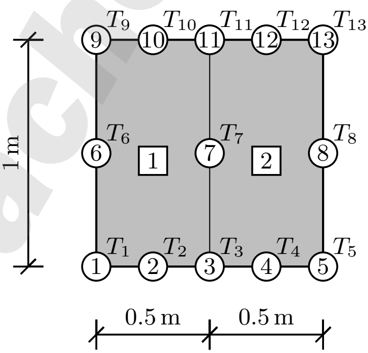

Figure 2.7: Arrangement of element and global dofs numbering for 8-nodes element

Due to the symmetrically (refer to Fig. 2.7), each element would have similar conductance matrix and internal heat generation forces. Based on Eq. (2.50), the conductance matrix (8×8) is given as

$$
[k^1] = [k^2]
$$

$$
= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & \cdots & 7 & 8 \end{array} \\
\begin{bmatrix}
4.333 & 2.150 & 1.917 & \dots & -2.767 & -0.933 \\
2.150 & 4.333 & 1.600 & \dots & -2.767 & -1.067 \\
1.917 & 1.600 & 4.333 & \dots & -5.233 & -1.067 \\
1.600 & 1.917 & 2.150 & \dots & -5.233 & -0.933 \\
-5.233 & -5.233 & -2.767 & \dots & 4.533 & 0.000 \\
-1.067 & -0.933 & -0.933 & \dots & 0.000 & -1.867 \\
-2.767 & -2.767 & -5.233 & \dots & 11.467 & 0.000 \\
-0.933 & -1.067 & -1.067 & \dots & 0.000 & 5.867
\end{bmatrix}
\end{array} \times 10^2 \tag{2.56}
$$

and the internal heat generation is given as

$$
\{r^1\} = \{r^2\} = \begin{Bmatrix} -\dfrac{Q}{12} \\ -\dfrac{Q}{12} \\ -\dfrac{Q}{12} \\ -\dfrac{Q}{12} \\ \dfrac{Q}{3} \\ \dfrac{Q}{3} \\ \dfrac{Q}{3} \\ \dfrac{Q}{3} \end{Bmatrix} = \begin{Bmatrix} -25 \\ -25 \\ -25 \\ -25 \\ 100 \\ 100 \\ 100 \\ 100 \end{Bmatrix} \tag{2.57}
$$

Since the size of an assembled stiffness matrix would be very large (13×13) when 8-nodes elements are used, herein, reduced size of complete global equilibrium equation with boundary conditions have been imposed (i.e. $T_1 = T_6 = T_9 = T_{10} = T_{11} = T_{12} = T_{13} = 0$) employing the row-column elimination as in Chapter 1, are presented. The size of the resulting stiffness matrix is (6×6).

$$
\begin{bmatrix}
11.467 & -5.233 & 0.000 & 0.000 & 0.000 & 0.000 \\
-5.233 & 8.667 & -5.233 & 2.150 & -1.867 & -1.067 \\
0.000 & -5.233 & 11.467 & -5.233 & 0.000 & 0.000 \\
0.000 & 2.150 & -5.233 & 4.333 & -1.067 & -0.933 \\
0.000 & -1.867 & 0.000 & -1.067 & 11.733 & -1.867 \\
0.000 & -1.067 & 0.000 & -0.933 & -1.867 & 5.867
\end{bmatrix} \times 10^2
\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_7 \\ T_8 \end{Bmatrix} = \begin{Bmatrix} 100 \\ -50 \\ 100 \\ -25 \\ 200 \\ 100 \end{Bmatrix} \tag{2.58}
$$

By solving Eq. (2.58) using Matlab command "\", the values of the temperature, T_i are thus obtained as:

$$
\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_7 \\ T_8 \end{Bmatrix} = \begin{Bmatrix} 0.290 \\ 0.445 \\ 0.557 \\ 0.584 \\ 0.368 \\ 0.461 \end{Bmatrix} \tag{2.59}
$$

The obtained results for 4-nodes and 8-nodes elements are validated herein against those obtained from commercial software, as shown in Table 2.1.

Fig. 2.8 show the contour of the temperature distribution obtained from 4-nodes elements, 8-nodes elements respectively. Both are plotted in Matlab

Table 2.1: Validation of global temperature (°C) at the bottom of the domain

	$x = \frac{L}{2}$	$x = L$
4-noded (2x2)	0.4821	0.6214
8-noded (2x1)	0.4448	0.5836
Software	0.4589	0.5897

environment.

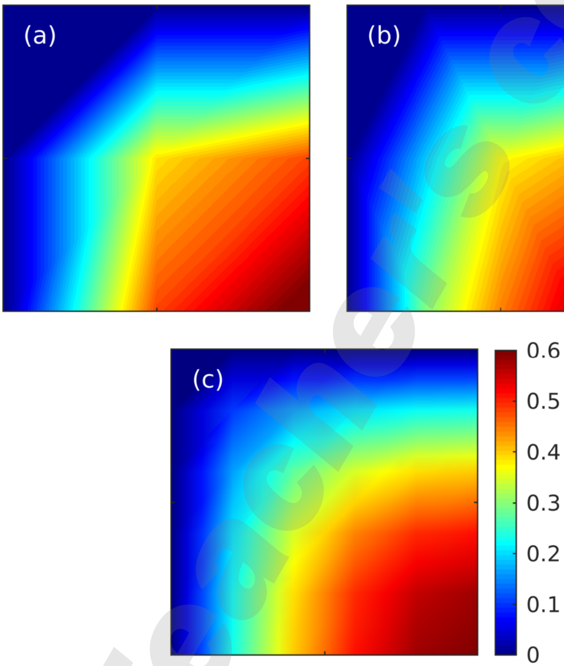

Figure 2.8: Temperature variation over the whole domain for (a)4-nodes element (2×2), (b)8-nodes element (1×2), and (c)software (5×5)

2.3.1 Source Code for Heat Transfer with 4-Noded Elements

```
% Clear data
clc; clear; close all

% --------------------
% Input parameters
% --------------------

% Domain and material properties
L  = 1;     % Plate length [m]
H  = 1;     % Plate height [m]
kx = 300;   % Thermal conductivity in x direction [W/(m.degC)]
ky = 300;   % Thermal conductivity in y direction [W/(m.degC)]
Q  = 600;   % Rate of heat generation [W/m^3]

% Dirichlet boundary condition
To = 0;     % Temperature at left & top wall [degC]

% Meshing
Nx   = 2;   % Mesh in x direction
Ny   = 2;   % Mesh in y direction
ndof = 9;   % Total number of degree of freedom (dof)

% --------------------
% Calculate the local matrix and force vector
% --------------------

% Elemental lengths
a = L/Nx;
b = H/Ny;

% Local conductivity matrix, k
s3 =  -b*kx/(6*a) - a*ky/(6*b);
s2 =   a*ky/(6*b) - b*kx/(3*a);
s4 =   b*kx/(6*a) - a*ky/(3*b);
s1 =   b*kx/(3*a) + a*ky/(3*b);
k  = [ s1 s2 s3 s4;
       s2 s1 s4 s3;
       s3 s4 s1 s2;
       s4 s3 s2 s1];

% Local load vector due to heat generation, Q
fQ = a*b*Q/4*[1; 1; 1; 1];

% --------------------
% Assemble the global matrix & vector
% --------------------

% Initialize matrix and vector
K = zeros(ndof,ndof);
F = zeros(ndof,1);

% Manual assembly of global matrix
K([1 2 5 4],[1 2 5 4]) = K([1 2 5 4],[1 2 5 4]) + k;    % Element 1
K([2 3 6 5],[2 3 6 5]) = K([2 3 6 5],[2 3 6 5]) + k;    % Element 2
K([4 5 8 7],[4 5 8 7]) = K([4 5 8 7],[4 5 8 7]) + k;    % Element 3
K([5 6 9 8],[5 6 9 8]) = K([5 6 9 8],[5 6 9 8]) + k;    % Element 4

% Manual assembly of global vector
F([1 2 5 4]) = F([1 2 5 4]) + fQ;    % Element 1
F([2 3 6 5]) = F([2 3 6 5]) + fQ;    % Element 2
F([4 5 8 7]) = F([4 5 8 7]) + fQ;    % Element 3
```

```
F([5 6 9 8]) = F([5 6 9 8]) + fQ;    % Element 4

% --------------------
% Impose boundary conditions
% --------------------

% Initialize dof
T = zeros(ndof,1);

% Manual identification of known dof index
dof_k = [1 4 7 8 9];

% Manual application of boundary condition for known dof
T([1 4 7 8 9]) = To;

% Manual modification of F vector for known boundary condition
F = F - K(:,1)*To - K(:,4)*To - K(:,7)*To ...
      - K(:,8)*To - K(:,9)*To;

% --------------------
% Solve the matrix system
% --------------------

% Unknown degree of freedom
dof_u = setdiff(1:ndof,dof_k);

% Solve for the unknown
T(dof_u) = K(dof_u,dof_u)\F(dof_u);
```

2.3.2 Source Code for Heat Transfer with 8-Noded Elements

```
% Clear data
clc; clear; close all

% --------------------
% Input parameters
% --------------------

% Domain and material properties
L    = 1;     % Plate length [m]
H    = 1;     % Plate height [m]
kx   = 300;   % Thermal conductivity in x direction [W/(m.degC)]
ky   = 300;   % Thermal conductivity in y direction [W/(m.degC)]
Q    = 600;   % Rate of heat generation [W/m^3]

% Dirichlet boundary condition
To   = 0;     % Temperature at left & top wall [degC]

% Meshing
Nx   = 2;   % Mesh in x direction
Ny   = 1;   % Mesh in y direction
ndof = 13;  % Total number of degree of freedom (dof)

% --------------------
% Calculate the local matrix and force vector
% --------------------
```

```
% Elemental lengths
a = L/Nx;
b = H/Ny;

% Local conductivity matrix, k
s1  =   (26*b*kx)/(45*a) + (26*a*ky)/(45*b);
s2  =   (14*b*kx)/(45*a) + (17*a*ky)/(90*b);
s3  =   (23*b*kx)/(90*a) + (23*a*ky)/(90*b);
s4  =   (17*b*kx)/(90*a) + (14*a*ky)/(45*b);
s5  =   (a*ky)/(15*b)    - (8*b*kx)/(9*a);
s6  = - (b*kx)/(15*a)    - (4*a*ky)/(9*b);
s7  = - (4*b*kx)/(9*a)   - (a*ky)/(15*b);
s8  =   (b*kx)/(15*a)    - (8*a*ky)/(9*b);
s9  =   (16*b*kx)/(9*a)  + (8*a*ky)/(15*b);
s10 =   (8*b*kx)/(9*a)   - (8*a*ky)/(15*b);
s11 =   (8*b*kx)/(15*a)  + (16*a*ky)/(9*b);
s12 =   (8*a*ky)/(9*b)   - (8*b*kx)/(15*a);
k=[ s1, s2, s3, s4, s5, s6, s7,  s8;
    s2, s1, s4, s3, s5, s8, s7,  s6;
    s3, s4, s1, s2, s7, s8, s5,  s6;
    s4, s3, s2, s1, s7, s6, s5,  s8;
    s5, s5, s7, s7, s9,  0,s10,   0;
    s6, s8, s8, s6,  0,s11,  0, s12;
    s7, s7, s5, s5,s10,  0, s9,   0;
    s8, s6, s6, s8,  0,s12,  0, s11];

% Local load vector due to heat generation, Q
fQ = a*b*Q/12*[-1;-1;-1;-1; 4; 4; 4; 4];

% --------------------
% Assemble the global matrix & vector
% --------------------

% Initialize matrix and vector
K = zeros(ndof,ndof);
F = zeros(ndof,1);

% Manual assembly of global matrix
K([1 3 11 9 2 7 10 6],[1 3 11 9 2 7 10 6]) = ...
K([1 3 11 9 2 7 10 6],[1 3 11 9 2 7 10 6]) + k;   % Element 1
K([3 5 13 11 4 8 12 7],[3 5 13 11 4 8 12 7]) = ...
K([3 5 13 11 4 8 12 7],[3 5 13 11 4 8 12 7]) + k; % Element 2

% Manual assembly of global vector
F([1 3 11 9 2 7 10 6])  = F([1 3 11 9 2 7 10 6])  + fQ;    % Element 1
F([3 5 13 11 4 8 12 7]) = F([3 5 13 11 4 8 12 7]) + fQ;    % Element 2

% --------------------
% Impose boundary conditions
% --------------------

% Initialize dof
T = zeros(ndof,1);

% Manual identification of known dof index
dof_k = [1 6 9 10 11 12 13];

% Manual application of boundary condition for known dof
T(dof_k) = To;

% Manual modification of F vector for known boundary condition
```

```
F = F - K(:,1)*To  - K(:,6)*To  - K(:,9)*To - K(:,10)*To ...
      - K(:,11)*To - K(:,12)*To - K(:,13)*To;

% --------------------
% Solve the matrix system
% --------------------

% Unknown degree of freedom
dof_u = setdiff(1:ndof,dof_k);

% Solve for unknown
T(dof_u) = K(dof_u,dof_u)\F(dof_u);
```

2.4 Exercises

1. Determine the conductance matrix for a rectangular element having $0.005\,\mathrm{m}$ thickness and equal sides of $0.025\,\mathrm{m}$ as shown in Fig. 2.9. The thermal conductivity of the material is $450\,\mathrm{W\,m^{-1}\,K^{-1}}$.

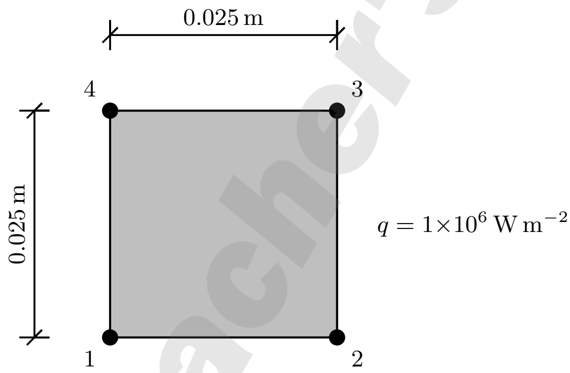

Figure 2.9: Node numbering of element

2. The right vertical side of element in Fig. 2.9 is subjected to an inward heat flux of $1\times10^{6}\,\mathrm{W\,m^{-2}}$. Determine the equivalent load vector for the inward heat flux.

3. The internal heat generation for element in Fig. 2.9 is $1\times10^{7}\,\mathrm{W\,m^{-3}}$. Determine the equivalent load vector for the internal heat generation.

4. Fig. 2.10 shows node numbering of four equal size four node rectangular element having $0.005\,\mathrm{m}$ thickness. The top and the bottom side of the plate are insulated. The left vertical side is subjected to a temperature of $400\,\mathrm{K}$ and the right vertical side is subjected to an inward heat flux of $1\times10^{6}\,\mathrm{W\,m^{-2}}$. The thermal conductivity is $450\,\mathrm{W\,m^{-1}\,K^{-1}}$. Determine the temperature at node 5.

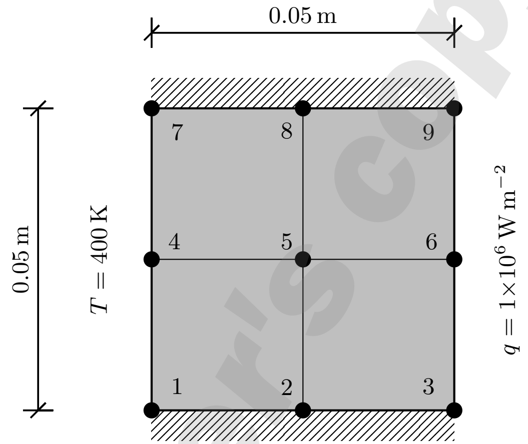

Figure 2.10: Node numbering of the elements

5. Determine the temperature in a rectangular plate of a unit thickness shown in Fig. 2.11. The bottom side of the plate is insulated, the top side is subjected to a temperature of $300\,\mathrm{K}$, the left vertical side is subjected to a temperature of $400\,\mathrm{K}$ and the right vertical side is subjected to an inward heat flux of $1\times10^{6}\,\mathrm{W\,m^{-2}}$. The internal heat generation is $1\times10^{7}\,\mathrm{W\,m^{-3}}$ and the thermal conductivity is $450\,\mathrm{W\,m^{-1}\,K^{-1}}$. (Plot the distribution of the temperature).

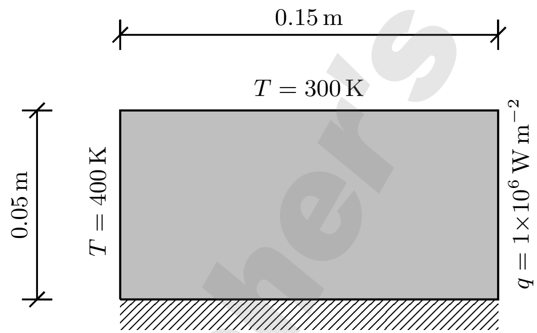

Figure 2.11: Rectangular plate with difference boundary conditions

3 Plane Elasticity: Plane Stress

3.1 Introduction

Plane stress element is used to model thin body or structure that is subjected to in plane loading (or boundary stresses) thus correspondingly stressed only in the plane direction. In practice, the element is employed to model load bearing wall and the web of steel beam, amongst others.

The study of plane stress FEM formulation is important, not only for its direct application to physical problems but also to further the study of FEM itself. For example, in the discussion of shell elements, the formulation can be viewed as combining the plane stress element and the plate element together. Also, in fluid dynamics, with some modifications, the plane stress element can be used to model fluid flow.

Herein, partial differential equation (PDE) of plane stress elasticity will be first derived followed by the FEM formulation.

3.1.1 Derivation of Plane Stress Partial Differential Equation

The PDE can be derived by employing the law of the conservation of linear momentum to the differential element shown in Fig. 3.1.

As shown in the figure, the dependent variables to be considered are the normal stresses (i.e. σ_{xx}, σ_{yy}) and the shear stresses (i.e. σ_{yx}, σ_{xy}). f_x and f_y, are the known body forces in x-direction and y-direction, respectively. Similar to previous chapters, the dependent variables are expanded by Taylor series at all sides (surfaces).

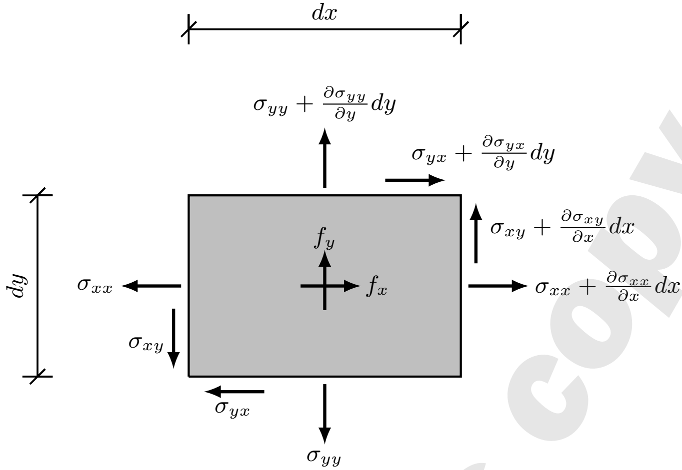

Figure 3.1: Plane stress differential element.

The conservation of linear momentum principle requires that, for a static condition, the total forces should be summed to zero; thus

$$\sum F_x = 0 \tag{3.1}$$

$$\sum F_y = 0 \tag{3.2}$$

In x-direction, by taking equilibrium of forces, Eq. (3.1) gives

$$\begin{aligned} \sum F_x = &\left(\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\, dx\right) - \sigma_{xx}\right) dy + \\ &\left(\left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\, dy\right) - \sigma_{yx}\right) dx + f_x\, dx\, dy = 0 \end{aligned} \tag{3.3}$$

By expanding and cancelling, the following is obtained;

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0 \tag{3.4}$$

In y-direction, by taking equilibrium of forces, Eq. (3.2) gives

$$\sum F_y = \left(\left(\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}\,dy\right) - \sigma_{yy}\right)dx + \\ \left(\left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial\text{‘}x}\,dx\right) - \sigma_{xy}\right)dy + f_y\,dx\,dy = 0 \tag{3.5}$$

By expanding and cancelling, the following is obtained;

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0 \tag{3.6}$$

Eqs. (3.4) and (3.6) are the PDEs for the plane stress problem, described in terms of stresses as the dependent variables. In matrix forms, the equations can be given as

$$\left\{\begin{matrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} \end{matrix}\right\} = \left\{\begin{matrix} -f_x \\ -f_y \end{matrix}\right\} \tag{3.7}$$

Based on conservation of angular momentum, it can be shown that $\sigma_{xy} = \sigma_{yx}$. Thus, Eq. (3.7) can further be expressed as

$$\left\{\begin{matrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} \end{matrix}\right\} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \left\{\begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{matrix}\right\} = \left\{\begin{matrix} -f_x \\ -f_y \end{matrix}\right\} \tag{3.8}$$

or

$$[\,\partial\,]\,\{\sigma\}^T = -\{f\} \tag{3.9}$$

where $[\,\partial\,]$ is termed from now on as differential operator matrix.

Eq. (3.9) (or Eqs. (3.4) and (3.6)) describes the problem in terms of stresses as the dependent variables. Since we are focusing on the displacement-based formulation, it would be necessary to express the equation in terms of displacement variables. This can be done by considering the following constitutive relationship for plane stress,

$$\sigma_{xx} = \frac{E}{1-\nu^2}\,\epsilon_{xx} + \frac{E\,\nu}{1-\nu^2}\,\epsilon_{yy} \tag{3.10a}$$

$$\sigma_{yy} = \frac{E\,\nu}{1-\nu^2}\,\epsilon_{xx} + \frac{E}{1-\nu^2}\,\epsilon_{yy} \tag{3.10b}$$

$$\sigma_{xy} = \frac{E\,(1-\nu)}{2\,(1-\nu^2)}\,\epsilon_{xy} \tag{3.10c}$$

which can be arranged in matrix forms as

$$\{\sigma\}^T = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix}\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} \tag{3.11}$$

or

$$\{\sigma\}^T = [\,E\,]\,\{\epsilon\}^T \tag{3.12}$$

where E is the Young's modulus and ν is the Poisson's ratio of the material. σ_{xx} and σ_{yy} are the axial strain in x and y directions, respectively whilst σ_{xy} is the shear strain. The strains can, on the other hand, be expressed in terms of displacements by employing the following strain-displacement relationship;

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \tag{3.13a}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \tag{3.13b}$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.13c}$$

which can be arranged in matrix forms as

$$\{\epsilon\}^T = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} \tag{3.14}$$

where u and v are the displacement components in x and y-direction, respectively.

Observe that the matrix on the right hand side of the equation above is also the transpose of the differential operator matrix, $[\partial]$ previously defined in Eq. (3.8). Based on this Eq. (3.14) can be given as;

$$\{\epsilon\}^T = [\,\partial\,]^T \{u\}^T \tag{3.15}$$

where $\{u\}$ is the vector which contains both displacement components, u and v. By inserting Eq. (3.15) into Eq. (3.12), the stresses can thus be given in terms of displacements as

$$\{\sigma\}^T = [\,E\,]\,[\,\partial\,]^T \{u\}^T \tag{3.16}$$

Finally, by inserting Eq. (3.16) into Eq. (3.9) the PDEs can be given in terms of displacement variables as

$$[\,\partial\,]\,[\,E\,]\,[\,\partial\,]^T \{u\}^T = -\{f\} \tag{3.17}$$

3.1.2 Boundary Condition Equations

Like other PDEs, the plane stress PDE given by Eqn. (3.17) (or (3.4) and (3.6)) must be supplemented by boundary condition equations so as to have a well-posed problem. There are two types of boundary conditions; Neumann (natural) and Dirichlet (essential) boundary conditions.

Neumann (natural) boundary conditions
In the x-direction, the Neumann boundary condition can be expressed as

$$b_x = \sigma_{xx}|_b\, n_x + \sigma_{yx}|_b\, n_y \tag{3.18a}$$

Inserting Eqs. (3.11) and (3.13), Eq. (3.18) can be given in displacement variables as

$$\begin{aligned} b_x = \left(\frac{E}{1-\nu^2}\, \frac{\partial u}{\partial x}\bigg|_b + \frac{E\nu}{1-\nu^2}\, \frac{\partial v}{\partial y}\bigg|_b \right) n_x +& \\ \frac{E\,(1-\nu)}{2\,(1-\nu^2)} \left(\frac{\partial u}{\partial y}\bigg|_b + \frac{\partial v}{\partial x}\bigg|_b \right) n_y& \end{aligned} \tag{3.18b}$$

In the y-direction, the Neumann boundary condition is given by

$$b_y = \sigma_{xy}|_b\, n_x + \sigma_{yy}|_b\, n_y \tag{3.19a}$$

Inserting Eqs. (3.11) and (3.13), Eq. (3.19) can be given in displacement variables as

$$b_y = \frac{E(1-\nu)}{2(1-\nu^2)} \left(\left.\frac{\partial u}{\partial y}\right|_b + \left.\frac{\partial v}{\partial x}\right|_b \right) n_x + \left(\frac{E\nu}{1-\nu^2} \left.\frac{\partial u}{\partial x}\right|_b + \frac{E}{1-\nu^2} \left.\frac{\partial v}{\partial y}\right|_b \right) n_y \tag{3.19b}$$

where, n_x, and n_y are the components of the unit normal vector of the surface under consideration. It worth to note that, for rectangular domain hence rectangular elements (meshes), major simplification to the above equations can be obtained because for this case, n_x, is either 1 or -1 whilst $n_y = 0$ in x-direction and $n_x = 0$, n_y is either 1 or -1 in y-direction, thus;

i. at $n_x = 1$ or -1 and $n_y = 0$

$$\begin{aligned} b_x &= \sigma_{xx}|_b \quad \text{or} \; -\sigma_{xx}|_b \\ b_y &= \sigma_{xy}|_b \quad \text{or} \; -\sigma_{xy}|_b \end{aligned} \tag{3.20}$$

ii. at $n_y = 1$ or -1 and $n_x = 0$

$$\begin{aligned} b_x &= \sigma_{yx}|_b \quad \text{or} \; -\sigma_{yx}|_b \\ b_y &= \sigma_{yy}|_b \quad \text{or} \; -\sigma_{yy}|_b \end{aligned} \tag{3.21}$$

It can be shown that the secondary terms produced by the integration by parts on the weighted PDEs will be in similar forms to Eqs. (3.20) and (3.21). Due to this, equations above are also known as natural boundary conditions.

Dirichlet (essential) boundary conditions

Dirichlet boundary condition requires that the displacement components at the boundary where natural conditions are unknown (to be solved) must be known thus specified.

$$u|_b = \bar{u} \tag{3.22}$$

$$v|_b = \bar{v} \tag{3.23}$$

where $\bar{u}$ and $\bar{v}$ are the specified displacement components in x and y-direction, respectively.

3.2 FEM Formulation for Plane Stress

Having established the PDE of the problem, we are all set to discretize the equations so as to obtain the FEM algebraic formulation.

3.2.1 Degree of Freedoms and Shape Functions for Plane Stress Element

In a plane stress element, two translational degree of freedoms exist at each node; u_i represents the nodal displacement in x-direction whilst v_i represents the nodal displacement in y-direction. Their arrangements in an element are shown in Fig. 3.2 for 4-nodes element and in Fig. 3.3 for 8-nodes element.

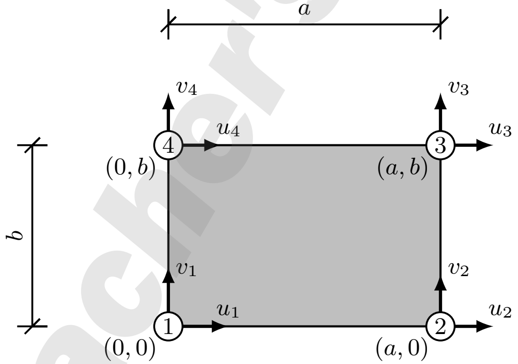

Figure 3.2: 4-nodes plane stress element.

For 4-nodes element, u and v are interpolated by the shape functions in such a way that

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 \tag{3.24}$$

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4 \tag{3.25}$$

whilst for 8-nodes element, they can be given as

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5u_5 + N_6u_6 + N_7u_7 + N_8u_8 \tag{3.26}$$

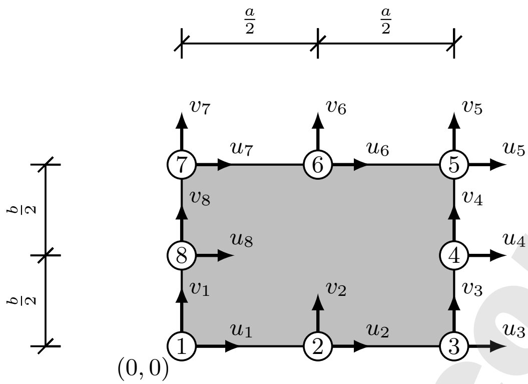

Figure 3.3: 8-nodes plane stress element.

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4 + N_5 v_5 + N_6 v_6 + N_7 v_7 + N_8 v_8 \quad (3.27)$$

For 4-nodes element, the shape functions, N_i are similar to those given by Eq. (2.21) whilst for 8-nodes element, they are similar to those given by Eq. (2.31). Observing Eqs. (3.24) to (3.27), it can be noticed that they can be given in vector forms as follows;

4-nodes element

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad (3.28)$$

8-nodes element

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix} \tag{3.29}$$

Observing Eqs. (4.9) and (5.56), it can be noticed that, for both elements, a general representation can be provided which is

$$\{u\}^T = [N]\,\{\hat{u}\}^T \tag{3.30}$$

3.2.2 Discretization by Galerkin WRM

By having established the shape functions, we are all set to discretize the PDE into algebraic equations. By inserting Eq. (3.30) into Eq. (3.17) we obtain

$$[\,\partial\,]\,[\,E\,]\,[\,\partial\,]^T\,[N]\,\{\hat{u}\}^T = -\{f\} \tag{3.31}$$

By observing Fig. 3.2 and Eqs. (3.24) and (3.25), we can identify that, for a 4-nodes element, we have 8 unknown dofs, i.e. $\begin{Bmatrix} \hat{u}_1 & \hat{v}_1 & \dots & \hat{u}_4 & \hat{v}_4 \end{Bmatrix}$ with only 1 equation, whilst for 8-nodes element we have 16 unknown dofs i.e. $\begin{Bmatrix} \hat{u}_1 & \hat{v}_1 & \dots & \hat{u}_8 & \hat{v}_8 \end{Bmatrix}$. To obtain sufficient number of equations, we will multiply Eq. (3.31) with the matrix of shape function $[N]$ then integrate to produce eight independent equations as required for 4-nodes element. Similar argument applies to 8-nodes element, resulting in sixteen independent equations as required.

Multiplying Eq. (3.31) with the matrix of shape functions $[N]$ and integrating give

$$\int_x \int_y [N]^T \left([\,\partial\,]\,[\,E\,]\,[\,\partial\,]^T\,[N]\,\{\hat{u}\}^T + \{f\} \right) dy\,dx = 0 \tag{3.32}$$

3.2.3 Integration By Parts (IBP)

As mentioned in Chapter 2, the intentions of IBP are:

- To relax the continuity requirement
- To induce explicit the natural boundary conditions

Therefore, by conducting IBP to Eq. (3.32), we obtain

$$\begin{aligned} -\int_x \int_y [N]^T [\partial] [E] [\partial]^T [N] \{\hat{u}\}^T dy\, dx \\ = -\int_x \int_y [N]^T \{f\}\, dy\, dx - \int_s [N]^T \{b\}^T ds \end{aligned} \tag{3.33}$$

Rearranging gives

$$\begin{aligned} \int_x \int_y [N]^T [\partial] [E] [\partial]^T [N]\, dy\, dx\, \{\hat{u}\}^T \\ = \int_x \int_y [N]^T \{f\}\, dy\, dx + \int_s [N]^T \{b\}^T ds \end{aligned} \tag{3.34}$$

Now, let's introduce the following matrix and vectors.

3.3 Matrix, Vectors and Discretized Equation of Plane Stress

The left hand side terms of Eq. (3.34) can be given in matrix forms. Let's call the corresponding matrix as stiffness matrix $[k]$ where

$$[\,k\,] = \int_x \int_y [N]^T [\partial] [E] [\partial]^T [N]\, dy\, dx \tag{3.35}$$

Next, let's introduce a vector for the body force, q where

$$\{q\} = \int_x \int_y [N]^T \{f\}\, dy\, dx \tag{3.36}$$

Finally, let's introduce a vector for the boundary terms induced by the IBP and call it as natural boundary condition vector, $\{b\}$ where

$$\{b\} = \int_s [N]^T \{b\}^T ds \tag{3.37}$$

It can be shown that, for rectangular domain hence rectangular element (mesh), Eq. (3.37) is given by Eqs. (3.20) and (3.21), respectively depending on the direction and location of evaluation.

Having established these, the discretized equation of plane stress can thus be given as In matrix form:

$$[\,k\,]\{\hat{u}\}^T = \{q\} + \{b\} = \{r\} \tag{3.38}$$

It must be noted that, since IBP is conducted, the multiplication in Eq. (3.35) between terms inside the integral must be carried out from left to right. To demonstrate this and to provide example on how to build the above matrix, the determination of k_{34} for 4-nodes element is shown below. Although in practice, numerical integration is employed which detailed procedure is given in Chapter 6, herein analytical integration is employed so as to allow for easy tracing of the procedure and to allow for its immediate application. Such analytical integrations are possible herein because we limit our discussion to rectangular domain hence rectangular elements only.

Step-by-step procedure for the determination of k_{34}

Step 1

$$\begin{gathered}
\cdots \begin{bmatrix} \vdots & \vdots & \vdots \\ \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_2}{\partial y} \\ \vdots & \vdots & \vdots \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \cdots \\
\Rightarrow \cdots \begin{bmatrix} \dfrac{E}{1-\nu^2}\dfrac{\partial N_2}{\partial x} & \dfrac{E\,\nu}{1-\nu^2}\dfrac{\partial N_2}{\partial x} & \dfrac{E\,(1-\nu)}{2\,(1-\nu^2)}\dfrac{\partial N_2}{\partial y} \end{bmatrix} \cdots
\end{gathered} \tag{3.39}$$

Step 2

$$\cdots\left[\frac{E}{1-\nu^2}\frac{\partial N_2}{\partial x} \quad \frac{E\,\nu}{1-\nu^2}\frac{\partial N_2}{\partial x} \quad \frac{E\,(1-\nu)}{2\,(1-\nu^2)}\frac{\partial N_2}{\partial y}\right]\begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & \dfrac{\partial N_2}{\partial y} & \cdots \\ \cdots & \dfrac{\partial N_2}{\partial x} & \cdots \end{bmatrix}\cdots$$

$$\Rightarrow \cdots\left[\frac{E\,\nu}{1-\nu^2}\frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial y}+\frac{E\,(1-\nu)}{2\,(1-\nu^2)}\frac{\partial N_2}{\partial y}\frac{\partial N_2}{\partial x}\right]\cdots \tag{3.40}$$

Step 3

$$\begin{aligned} t\int_0^a\int_0^b\left[\frac{E\,\nu}{1-\nu^2}\frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial y}+\frac{E\,(1-\nu)}{2\,(1-\nu^2)}\frac{\partial N_2}{\partial y}\frac{\partial N_2}{\partial x}\right]dy\,dx \\ = \frac{E\,t}{8\;(v-1)} \end{aligned} \tag{3.41}$$

The complete integrated values of $[k]$ for 4-nodes element can be given as

$$[\,k\,]=\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ s_2 & s_9 & s_8 & s_{10} & s_6 & s_{11} & s_4 & s_{12} \\ s_3 & s_8 & s_1 & s_6 & s_7 & s_4 & s_5 & s_2 \\ s_4 & s_{10} & s_6 & s_9 & s_8 & s_{12} & s_2 & s_{11} \\ s_5 & s_6 & s_7 & s_8 & s_1 & s_2 & s_3 & s_4 \\ s_6 & s_{11} & s_4 & s_{12} & s_2 & s_9 & s_8 & s_{10} \\ s_7 & s_4 & s_5 & s_2 & s_3 & s_8 & s_1 & s_6 \\ s_8 & s_{12} & s_2 & s_{11} & s_4 & s_{10} & s_6 & s_9 \end{bmatrix} \tag{3.42}$$

where

$$s_1=\frac{E\,t\,\left((1-\nu)\,a^2+2\,b^2\right)}{6\,(1-\nu^2)\,a\,b}$$

$$s_2=\frac{E\,t\,\left((1-\nu)+2\,\nu\right)}{8\,(1-\nu^2)}$$

$$s_3=\frac{E\,t\,\left((1-\nu)\,a^2-4\,b^2\right)}{12\,(1-\nu^2)\,a\,b}$$

$$s_4 = -\frac{E\,t\,\left((1-\nu)-2\,\nu\right)}{8\,(1-\nu^2)}$$

$$s_5 = -\frac{E\,t\,\left((1-\nu)\,a^2+2\,b^2\right)}{12\,(1-\nu^2)\,a\,b}$$

$$s_6 = -\frac{E\,t\,\left((1-\nu)+2\,\nu\right)}{8\,(1-\nu^2)}$$

$$s_7 = -\frac{E\,t\,\left((1-\nu)\,a^2-b^2\right)}{6\,(1-\nu^2)\,a\,b}$$

$$s_8 = \frac{E\,t\,\left((1-\nu)-2\,\nu\right)}{8\,(1-\nu^2)}$$

$$s_9 = \frac{E\,t\,\left(2\,a^2+(1-\nu)\,b^2\right)}{6\,(1-\nu^2)\,a\,b}$$

$$s_{10} = -\frac{E\,t\,\left((1-\nu)\,b^2-a^2\right)}{6\,(1-\nu^2)\,a\,b}$$

$$s_{11} = -\frac{E\,t\,\left(2\,a^2+(1-\nu)\,b^2\right)}{12\,(1-\nu^2)\,a\,b}$$

$$s_{12} = \frac{E\,t\,\left((1-\nu)\,b^2-4\,a^2\right)}{12\,(1-\nu^2)\,a\,b}$$

Repeating the procedure above (Step 1 to Step 3) using quadratic shape function (Eq. (2.31)), the complete integrated values of $[\,k\,]$ for 8-nodes element can be given as

$$[\,k\,] = \begin{bmatrix} S_a & | & S_b \\ -- & -- & -- \\ S_b^T & | & S_c \end{bmatrix} \tag{3.43}$$

where

$$[\,S_a\,] = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ s_2 & s_{16} & s_8 & s_{17} & s_6 & s_{18} & s_4 & s_{19} \\ s_3 & s_8 & s_1 & s_{24} & s_7 & s_4 & s_5 & s_{25} \\ s_4 & s_{17} & s_{24} & s_{16} & s_8 & s_{19} & s_{25} & s_{18} \\ s_5 & s_6 & s_7 & s_8 & s_1 & s_2 & s_3 & s_4 \\ s_6 & s_{18} & s_4 & s_{19} & s_2 & s_{16} & s_8 & s_{17} \\ s_7 & s_4 & s_5 & s_{25} & s_3 & s_8 & s_1 & s_{24} \\ s_8 & s_{19} & s_{25} & s_{18} & s_4 & s_{17} & s_{24} & s_{16} \end{bmatrix}$$

$$[\,S_b\,] = \begin{bmatrix} s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{12} & s_{14} & s_{15} \\ s_{15} & s_{20} & s_{12} & s_{21} & s_{12} & s_{22} & s_{10} & s_{23} \\ s_9 & s_{26} & s_{14} & s_{27} & s_{13} & s_{28} & s_{11} & s_{28} \\ s_{27} & s_{20} & s_{26} & s_{23} & s_{28} & s_{22} & s_{28} & s_{21} \\ s_{13} & s_{12} & s_{14} & s_{15} & s_9 & s_{10} & s_{11} & s_{12} \\ s_{12} & s_{22} & s_{10} & s_{23} & s_{15} & s_{20} & s_{12} & s_{21} \\ s_{13} & s_{28} & s_{11} & s_{28} & s_9 & s_{26} & s_{14} & s_{27} \\ s_{28} & s_{22} & s_{28} & s_{21} & s_{27} & s_{20} & s_{26} & s_{23} \end{bmatrix}$$

$$[\,S_c\,] = \begin{bmatrix} s_{29} & 0 & 0 & s_{30} & s_{31} & 0 & 0 & s_{32} \\ 0 & s_{33} & s_{30} & 0 & 0 & s_{34} & s_{32} & 0 \\ 0 & s_{30} & s_{35} & 0 & 0 & s_{32} & s_{36} & 0 \\ s_{30} & 0 & 0 & s_{37} & s_{32} & 0 & 0 & s_{38} \\ s_{31} & 0 & 0 & s_{32} & s_{29} & 0 & 0 & s_{30} \\ 0 & s_{34} & s_{32} & 0 & 0 & s_{33} & s_{30} & 0 \\ 0 & s_{32} & s_{36} & 0 & 0 & s_{30} & s_{35} & 0 \\ s_{32} & 0 & 0 & s_{38} & s_{30} & 0 & 0 & s_{37} \end{bmatrix}$$

$$s_1 = \frac{13\,E\,t\,\left((1-\nu)\,a^2 + 2\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_2 = \frac{17\,E\,t\,((1-\nu) + 2\,\nu)}{72\,(1-\nu^2)}$$

$$s_3 = \frac{E\,t\,\left(17\,(1-\nu)\,a^2 + 56\,b^2\right)}{180\,(1-\nu^2)\,a\,b}$$

$$s_4 = \frac{E\,t\,((1-\nu) - 2\,\nu)}{24\,(1-\nu^2)}$$

$$s_5 = \frac{23\,E\,t\,\left((1-\nu)\,a^2 + 2\,b^2\right)}{180\,(1-\nu^2)\,a\,b}$$

$$s_6 = \frac{7\,E\,t\,((1-\nu) + 2\,\nu)}{72\,(1-\nu^2)}$$

$$s_7 = \frac{E\,t\,\left(14\,(1-\nu)\,a^2 + 17\,b^2\right)}{90\,(1-\nu^2)\,a\,b}$$

$$s_8 = -\frac{E\,t\,((1-\nu) - 2\,\nu)}{24\,(1-\nu^2)}$$

$$s_9 = \frac{E\,t\,\left(3\,(1-\nu)\,a^2 - 80\,b^2\right)}{90\,(1-\nu^2)\,a\,b}$$

$$s_{10} = -\frac{E\,t\,(5\,(1-\nu) - 2\,\nu)}{18\,(1-\nu^2)}$$

$$s_{11} = -\frac{E\,t\,\left(10\,(1-\nu)\,a^2 + 3\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{12} = -\frac{E\,t\,((1-\nu) + 2\,\nu)}{18\,(1-\nu^2)}$$

$$s_{13} = -\frac{E\,t\,\left(3\,(1-\nu)\,a^2 + 40\,b^2\right)}{90\,(1-\nu^2)\,a\,b}$$

$$s_{14} = -\frac{E\,t\,\left(20\,(1-\nu)\,a^2 - 3\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{15} = \frac{E\,t\,((1-\nu) - 10\,\nu)}{18\,(1-\nu^2)}$$

$$s_{16} = \frac{13\,E\,t\,\left(2\,a^2 + (1-\nu)\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{17} = \frac{E\,t\,\left(17\,a^2 + 14\,(1-\nu)\,b^2\right)}{90\,(1-\nu^2)\,a\,b}$$

$$s_{18} = \frac{23\,E\,t\,\left(2\,a^2 + (1-\nu)\,b^2\right)}{180\,(1-\nu^2)\,a\,b}$$

$$s_{19} = \frac{E\,t\,\left(56\,a^2 + 17\,(1-\nu)\,b^2\right)}{180\,(1-\nu^2)\,a\,b}$$

$$s_{20} = -\frac{E\,t\,\left(20\,(1-\nu)\,b^2 - 3\,a^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{21} = -\frac{E\,t\,\left(40\,a^2 + 3\,(1-\nu)\,b^2\right)}{90\,(1-\nu^2)\,a\,b}$$

$$s_{22} = -\frac{E\,t\,\left(3\,a^2 + 10\,(1-\nu)\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{23} = \frac{E\,t\,\left(3\,(1-\nu)\,b^2 - 80\,a^2\right)}{90\,(1-\nu^2)\,a\,b}$$

$$s_{24} = -\frac{17\,E\,t\,((1-\nu) + 2\,\nu)}{72\,(1-\nu^2)}$$

$$s_{25} = -\frac{7\,E\,t\,((1-\nu) + 2\,\nu)}{72\,(1-\nu^2)}$$

$$s_{26} = \frac{E\,t\,(5\,(1-\nu) - 2\,\nu)}{18\,(1-\nu^2)}$$

$$s_{27} = -\frac{E\,t\,((1-\nu) - 10\,\nu)}{18\,(1-\nu^2)}$$

$$s_{28} = \frac{E\,t\,((1-\nu) + 2\,\nu)}{18\,(1-\nu^2)}$$

$$s_{29} = \frac{4\,E\,t\,\left(3\,(1-\nu)\,a^2 + 20\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{30} = -\frac{2\,E\,t\,((1-\nu) + 2\,\nu)}{9\,(1-\nu^2)}$$

$$s_{31} = -\frac{4\,E\,t\,\left(3\,(1-\nu)\,a^2 - 10\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{32} = \frac{2\,E\,t\,((1-\nu) + 2\,\nu)}{9\,(1-\nu^2)}$$

$$s_{33} = \frac{8\,E\,t\,\left(3\,a^2 + 5\,(1-\nu)\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{34} = \frac{4\,E\,t\,\left(5\,(1-\nu)\,b^2 - 6\,a^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{35} = \frac{8\,E\,t\,\left(5\,(1-\nu)\,a^2 + 3\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{36} = \frac{4\,E\,t\,\left(5\,(1-\nu)\,a^2 - 6\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{37} = \frac{4\,E\,t\,\left(20\,a^2 + 3\,(1-\nu)\,b^2\right)}{45\,(1-\nu^2)\,a\,b}$$

$$s_{38} = -\frac{4\,E\,t\,\left(3\,(1-\nu)\,b^2 - 10\,a^2\right)}{45\,(1-\nu^2)\,a\,b}$$

3.4 Worked Example

Herein, we are going to demonstrate the FEM formulation for plane stress by solving the cantilever beam problem as shown in Fig. 3.4.

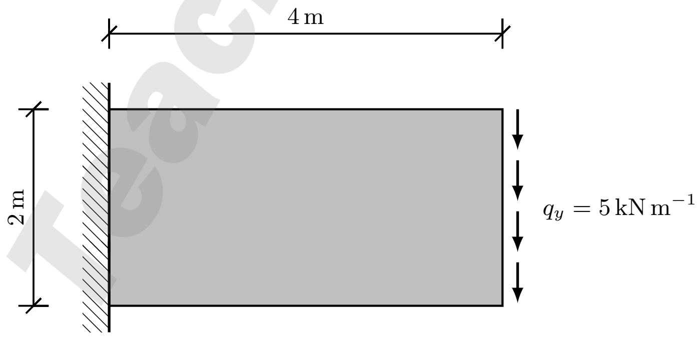

Figure 3.4: Plane stress and its properties. ($E = 200\times10^6\,\mathrm{N\,m^{-2}}$, $\upsilon = 0.33$ and $t = 0.01\,\mathrm{m}$)

For a step-by-step demonstration (hand-calculation), only four assembled

4-nodes elements and two 8-nodes elements are used, so as to allow for easy tracing of the procedure. However, it must be noted that such arrangements are for demonstration purpose only as the result can be poor. After the step-by-step calculation, the procedure is run using a sufficient number of elements (denser mesh). The results are then validated against results obtained from commercial software for proof of correctness.

4-nodes Element

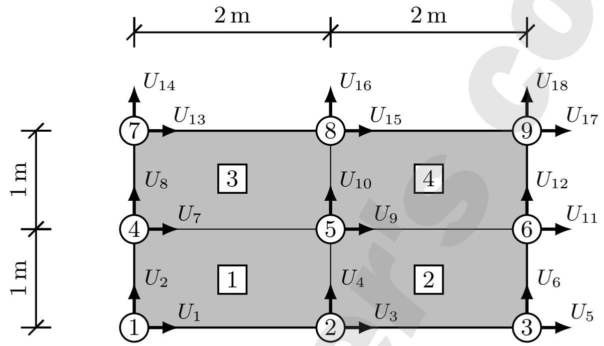

Figure 3.5: The arrangement of element and global dofs numbering for 4-nodes element

Due to the symmetricality, each element would have similar local stiffness matrix (8×8). Based on Eq. (3.42), the local stiffness matrix is given as

$$
[\,k\,] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & \cdots & 7 & 8 \end{array} \\
\begin{bmatrix}
0.875 & 0.373 & -0.123 & \dots & -0.314 & 0.003 \\
0.373 & 1.622 & 0.003 & \dots & -0.003 & -1.434 \\
-0.123 & 0.003 & 0.875 & \dots & -0.438 & 0.373 \\
-0.003 & 0.623 & -0.373 & \dots & 0.373 & -0.811 \\
-0.438 & -0.373 & -0.314 & \dots & -0.123 & -0.003 \\
-0.373 & -0.811 & -0.003 & \dots & 0.003 & 0.623 \\
-0.314 & -0.003 & -0.438 & \dots & 0.875 & -0.373 \\
0.003 & -1.434 & 0.373 & \dots & -0.373 & 1.622
\end{bmatrix}
\end{array} \times 10^6 \tag{3.44}
$$

and for the load vector for element 2 and element 4, they are given as follow

$$\{r^2\} = \{r^4\} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{Q_y b}{2} \\ 0 \\ \dfrac{Q_y b}{2} \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -2500 \\ 0 \\ -2500 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.45}$$

where $Q_y = -5000$ and $b = 1$. The assembled global stiffness matrix, $[K]$ (18×18) is thus given as

$$[K] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \\ 17 \\ 18 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 17 & 18 \end{array} \\ \begin{bmatrix} 0.875 & 0.373 & -0.123 & \dots & 0.000 & 0.000 \\ 0.373 & 1.622 & 0.003 & \dots & 0.000 & 0.000 \\ -0.123 & 0.003 & 1.751 & \dots & 0.000 & 0.000 \\ -0.003 & 0.623 & 0.000 & \dots & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.123 & \dots & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.003 & \dots & 0.000 & 0.000 \\ -0.314 & -0.003 & -0.438 & \dots & 0.000 & 0.000 \\ 0.003 & -1.434 & 0.373 & \dots & 0.000 & 0.000 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.000 & 0.000 & 0.000 & \dots & 0.875 & 0.373 \\ 0.000 & 0.000 & 0.000 & \dots & 0.373 & 1.622 \end{bmatrix} \end{array} \times 10^6 \tag{3.46}$$

The assembled global load vector is given as

$$\{R\} = \begin{Bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \\ 0 \\ -2500 \\ B_7 \\ B_8 \\ 0 \\ 0 \\ 0 \\ -5000 \\ B_{13} \\ B_{14} \\ 0 \\ 0 \\ 0 \\ -2500 \end{Bmatrix} \tag{3.47}$$

By imposing the essential boundary conditions ($U_1 = U_2 = U_7 = U_8 = U_{13} = U_{14} = 0$), the assembled global equilibrium equation is reduced to

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \quad 3 & \quad\quad 4 & \quad\quad 5 & \quad \cdots & \quad 17 & \quad\quad 18 \end{array} \\
\begin{array}{c} 3\\4\\5\\6\\9\\10\\11\\12\\15\\16\\17\\18 \end{array} &
\begin{bmatrix}
1.751 & 0.000 & -0.123 & \ldots & 0.000 & 0.000 \\
0.000 & 3.243 & 0.003 & \ldots & 0.000 & 0.000 \\
-0.123 & 0.003 & 0.875 & \ldots & 0.000 & 0.000 \\
-0.003 & 0.623 & -0.373 & \ldots & 0.000 & 0.000 \\
-0.628 & 0.000 & -0.438 & \ldots & -0.438 & -0.373 \\
0.000 & -2.867 & 0.373 & \ldots & -0.373 & -0.811 \\
-0.438 & -0.373 & -0.314 & \ldots & -0.314 & -0.003 \\
-0.373 & -0.811 & -0.003 & \ldots & 0.003 & -1.434 \\
0.000 & 0.000 & 0.000 & \ldots & -0.123 & 0.003 \\
0.000 & 0.000 & 0.000 & \ldots & -0.003 & 0.623 \\
0.000 & 0.000 & 0.000 & \ldots & 0.875 & 0.373 \\
0.000 & 0.000 & 0.000 & \ldots & 0.373 & 1.622
\end{bmatrix} \times 10^6
\end{array}
$$

$$
\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_9 \\ U_{10} \\ U_{11} \\ U_{12} \\ U_{15} \\ U_{16} \\ U_{17} \\ U_{18} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2500 \\ 0 \\ 0 \\ 0 \\ -5000 \\ 0 \\ 0 \\ 0 \\ -2500 \end{Bmatrix} \tag{3.48}
$$

By solving Eq. (3.48) using Matlab command "\", the values of the assembled global dofs $\{U\}$ are thus obtained as:

$$\{U\} = \begin{Bmatrix} 0 \\ 0 \\ -0.031 \\ -0.046 \\ -0.042 \\ -0.131 \\ 0 \\ 0 \\ 0.000 \\ -0.043 \\ 0.000 \\ -0.130 \\ 0 \\ 0 \\ 0.031 \\ -0.046 \\ 0.042 \\ -0.131 \end{Bmatrix} \tag{3.49}$$

The obtained results are validated herein against those obtained from commercial software, as shown in Table 3.1.

Table 3.1: Validation of global displacement

	U_5	U_6	U_{17}	U_{18}
Q4 (2x2)	-0.042	-0.131	0.042	-0.131
Q4 (16x16)	-0.061	-0.189	0.061	-0.189
Q4 (80x80)	-0.062	-0.192	0.062	-0.192
Software	-0.062	-0.192	0.062	-0.192

8-nodes Element

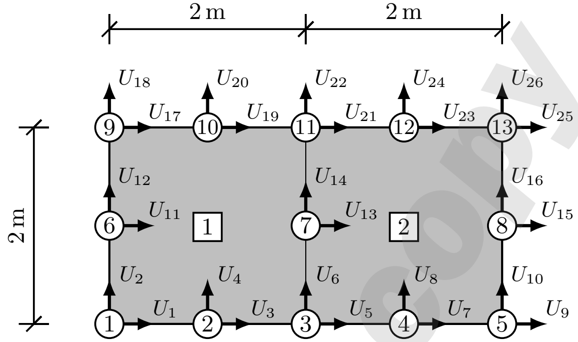

Figure 3.6: The arrangement of element and global dofs numbering for 8-nodes element

Due to the symmetricality, each element would have similar local stiffness matrix. Based on Eq. (3.43), the local stiffness matrix is given as

$$[k^1] = [k^2]$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \\ 15 \\ 16 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 15 & 16 \end{array} \\ \begin{bmatrix} 1.731 & 0.705 & 0.840 & \dots & -0.519 & -0.328 \\ 0.705 & 1.731 & -0.001 & \dots & -0.335 & -1.945 \\ 0.840 & -0.001 & 1.731 & \dots & -0.484 & 0.166 \\ 0.001 & 0.658 & -0.705 & \dots & 0.166 & -1.048 \\ 0.766 & 0.290 & 0.658 & \dots & -0.484 & -0.166 \\ 0.290 & 0.766 & 0.001 & \dots & -0.166 & -1.048 \\ 0.658 & 0.001 & 0.766 & \dots & -0.519 & 0.328 \\ -0.001 & 0.840 & -0.290 & \dots & 0.335 & -1.945 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -0.519 & -0.335 & -0.484 & \dots & 2.534 & 0.000 \\ -0.328 & -1.945 & 0.166 & \dots & 0.000 & 4.391 \end{bmatrix} \end{array} \times 10^6 \tag{3.50}$$

and for the load vector for element 2 is given as follow

$$\{r^2\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{Q_y b}{6} \\ 0 \\ \dfrac{Q_y b}{6} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{2Q_y b}{3} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.667x10^3 \\ 0 \\ -1.667x10^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -6.667x10^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.51}$$

where $Q_y = -5000$ and $b = 2$.

Since the size of an assembled stiffness matrix would be very large (26×26) when 8-nodes elements are used, herein, the reduced size of assembled global equilibrium equation with boundary conditions have been imposed (i.e. $U_1 = U_2 = U_{11} = U_{12} = U_{17} = U_{18} = 0$) are presented employing row-column elimination as discussed in Chapter 1.

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 3 & 4 & 5 & \cdots & 25 & 26 \end{array} \\
\begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ \vdots \\ 25 \\ 26 \end{array} &
\begin{bmatrix}
4.391 & 0.000 & -1.945 & \dots & 0.000 & 0.000 \\
0.000 & 2.534 & 0.335 & \dots & 0.000 & 0.000 \\
-1.945 & 0.335 & 3.462 & \dots & 0.766 & 0.290 \\
0.328 & -0.519 & 0.000 & \dots & 0.290 & 0.766 \\
0.000 & 0.000 & -1.945 & \dots & -1.048 & -0.166 \\
0.000 & 0.000 & -0.335 & \dots & -0.166 & -0.484 \\
0.000 & 0.000 & 0.840 & \dots & 0.658 & 0.001 \\
0.000 & 0.000 & 0.001 & \dots & -0.001 & 0.840 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0.000 & 0.000 & 0.766 & \dots & 1.731 & 0.705 \\
0.000 & 0.000 & 0.290 & \dots & 0.705 & 1.731
\end{bmatrix}
\end{array} \times 10^6
$$

$$
\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \\ U_{10} \\ \vdots \\ U_{25} \\ U_{26} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1.667 \end{Bmatrix} \tag{3.52}
$$

By solving Eq. (3.52) using Matlab command '\\", the values of the assembled global dof $\{U\}$ are thus obtained as:

$$\{U\} = \begin{Bmatrix} -0.024 \\ -0.020 \\ -0.043 \\ -0.062 \\ -0.054 \\ -0.116 \\ -0.058 \\ -0.178 \\ 0.000 \\ -0.058 \\ 0.000 \\ -0.178 \\ 0.024 \\ -0.020 \\ 0.043 \\ -0.062 \\ 0.054 \\ -0.116 \\ 0.058 \\ -0.178 \end{Bmatrix} \tag{3.53}$$

The obtained results are validated herein against those obtained from commercial software, as shown in Table 3.2.

Table 3.2: Validation of global displacement

	U_5	U_6	U_{17}	U_{18}
Q8 (2x1)	-0.058	-0.178	0.058	-0.131
Q8 (4x4)	-0.061	-0.190	0.061	-0.189
Q8 (30x30)	-0.062	-0.192	0.062	-0.192
Software	-0.062	-0.192	0.062	-0.192

Fig. 3.7 show the contour of the y-displacement distribution obtained from 4-nodes element, 8-nodes element and commercial software. As can be seen, close agreement are evident between all cases.

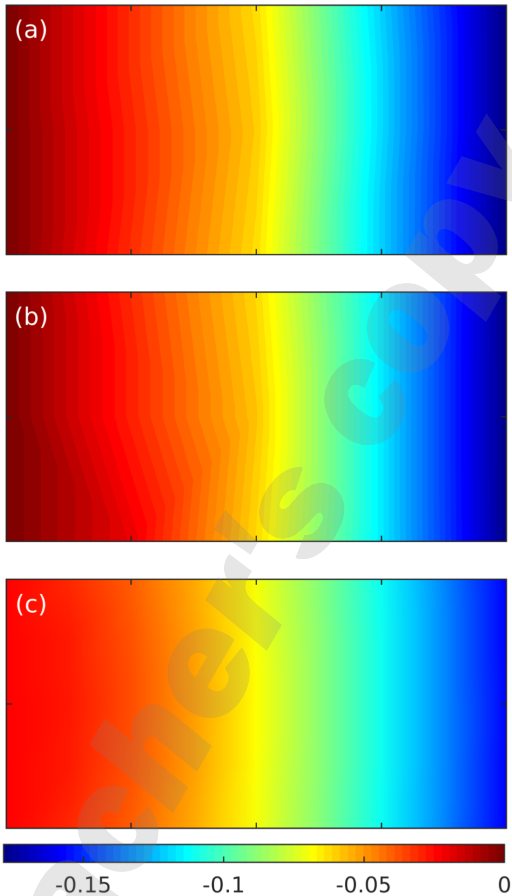

Figure 3.7: Displacement in y-direction over the whole domain for (a)4-nodes element (2×2), (b)8-nodes element (1×2), and (c)software (4×8)

3.4.1 Source Code for Plane Stress with 4-Noded Elements

```
% Clear data
clc; clear; close all

% --------------------
% Input parameters
% --------------------
```

```
% Domain and material properties
Lx   = 4;      % Plane length [m]
Ly   = 2;      % Plane height [m]
t    = 0.01;   % Plane thickness [m]
E    = 200e6;  % Young's Modulus [Pa]
nu   = 0.33;   % Poison ratio

% Force and boundary condition
qy   = 5000;   % Surface traction [N/m]
uo   = 0;      % Zero displacement [m]

% Meshing
Nx   = 2;      % Mesh in x direction
Ny   = 2;      % Mesh in y direction
ndof = 18;     % Total number of degree of freedom (dof)

% --------------------
% Calculate the local matrix and force vector
% --------------------

% Elemental lengths
a = Lx/Nx;
b = Ly/Ny;

% Stiffness matrix, k
s1  =  (E*t*((1-nu)*a^2 + 2*b^2))/(6*(1-nu^2)*a*b);
s2  =  (E*t*((1-nu) + 2*nu))/(8*(1-nu^2));
s3  =  (E*t*((1-nu)*a^2 - 4*b^2))/(12*(1-nu^2)*a*b);
s4  = -(E*t*((1-nu) - 2*nu))/(8*(1-nu^2));
s5  = -(E*t*((1-nu)*a^2 + 2*b^2))/(12*(1-nu^2)*a*b);
s6  = -(E*t*((1-nu) + 2*nu))/(8*(1-nu^2));
s7  = -(E*t*((1-nu)*a^2 - b^2))/(6*(1-nu^2)*a*b);
s8  =  (E*t*((1-nu) - 2*nu))/(8*(1-nu^2));
s9  =  (E*t*((1-nu)*b^2 + 2*a^2))/(6*(1-nu^2)*a*b);
s10 = -(E*t*((1-nu)*b^2 - a^2))/(6*(1-nu^2)*a*b);
s11 = -(E*t*((1-nu)*b^2 + 2*a^2))/(12*(1-nu^2)*a*b);
s12 =  (E*t*((1-nu)*b^2 - 4*a^2))/(12*(1-nu^2)*a*b);
k   = [s1  s2   s3  s4  s5  s6  s7  s8;
       s2  s9   s8  s10 s6  s11 s4  s12;
       s3  s8   s1  s6  s7  s4  s5  s2;
       s4  s10  s6  s9  s8  s12 s2  s11;
       s5  s6   s7  s8  s1  s2  s3  s4;
       s6  s11  s4  s12 s2  s9  s8  s10;
       s7  s4   s5  s2  s3  s8  s1  s6;
       s8  s12  s2  s11 s4  s10 s6  s9];

% --------------------
% Assemble the global matrix & vector
% --------------------

% Initialize matrix and vector
K = zeros(ndof,ndof);
F = zeros(ndof,1);

% Manual assembly of global matrix
K([1 2 3 4 9 10 7 8],[1 2 3 4 9 10 7 8]) = ...
    K([1 2 3 4 9 10 7 8],[1 2 3 4 9 10 7 8]) + k;             %
         Element 1
K([3 4 5 6 11 12 9 10],[3 4 5 6 11 12 9 10]) = ...
    K([3 4 5 6 11 12 9 10],[3 4 5 6 11 12 9 10]) + k;         %
         Element 2
```

```
K([7 8 9 10 15 16 13 14],[7 8 9 10 15 16 13 14]) = ...
    K([7 8 9 10 15 16 13 14],[7 8 9 10 15 16 13 14]) + k;       %
         Element 3
K([9 10 11 12 17 18 15 16],[9 10 11 12 17 18 15 16]) = ...
    K([9 10 11 12 17 18 15 16],[9 10 11 12 17 18 15 16]) + k;   %
         Element 4

% Manual assembly of global vector
F([6 12 18]) = [-2500 -5000 -2500];

% --------------------
% Impose boundary conditions
% --------------------

% Initialize dof
T = zeros(ndof,1);

% Manual identification of known dof index
dof_k = [1 2 7 8 13 14];

% Manual application of boundary condition (fixed) for known dof
T(dof_k) = 0;

% Manual modification of F vector for known boundary condition
F = F - K(:,1)*uo - K(:,2)*uo  - K(:,7)*uo ...
      - K(:,8)*uo - K(:,13)*uo - K(:,14)*uo;

% --------------------
% Solve the matrix system
% --------------------

% Unknown degree of freedom
dof_u = setdiff(1:ndof,dof_k);

% Solve for the unknown
T(dof_u) = K(dof_u,dof_u)\F(dof_u);
```

3.4.2 Source Code for Plane Stress with 8-Noded Elements

```
% Clear data
clc; clear; close all

% --------------------
% Input parameters
% --------------------

% Domain and material properties
Lx   = 4;       % Plane length [m]
Ly   = 2;       % Plane height [m]
t    = 0.01;    % Plane thickness [m]
E    = 200e6;   % Young's Modulus [Pa]
nu   = 0.33;    % Poison ratio

% Force and boundary condition
qy   = 5000;    % Surface traction [N/m]
uo   = 0;       % Zero displacement [m]
```

```
% Meshing
Nx   = 2;      % Mesh in x direction
Ny   = 1;      % Mesh in y direction
ndof = 26;     % Total number of degree of freedom (dof)

% --------------------
% Calculate the local matrix and force vector
% --------------------

% Elemental lengths
a = Lx/Nx;
b = Ly/Ny;

% Stiffness matrix, k
s1  =  (13*E*t*((1-nu)*a^2 + 2*b^2))/(45*(1-nu^2)*a*b);
s2  =  (17*E*t*((1-nu) + 2*nu))/(72*(1-nu^2));
s3  =  (E*t*(17*(1-nu)*a^2 + 56*b^2))/(180*(1-nu^2)*a*b);
s4  =  (E*t*((1-nu) - 2*nu))/(24*(1-nu^2));
s5  =  (23*E*t*((1-nu)*a^2 + 2*b^2))/(180*(1-nu^2)*a*b);
s6  =  (7*E*t*((1-nu) + 2*nu))/(72*(1-nu^2));
s7  =  (E*t*(14*(1-nu)*a^2 + 17*b^2))/(90*(1-nu^2)*a*b);
s8  = -(E*t*((1-nu) - 2*nu))/(24*(1-nu^2));
s9  =  (E*t*(3*(1-nu)*a^2 - 80*b^2))/(90*(1-nu^2)*a*b);
s10 = -(E*t*(5*(1-nu) - 2*nu))/(18*(1-nu^2));
s11 = -(E*t*(10*(1-nu)*a^2 + 3*b^2))/(45*(1-nu^2)*a*b);
s12 = -(E*t*((1-nu) + 2*nu))/(18*(1-nu^2));
s13 = -(E*t*(3*(1-nu)*a^2 + 40*b^2))/(90*(1-nu^2)*a*b);
s14 = -(E*t*(20*(1-nu)*a^2 - 3*b^2))/(45*(1-nu^2)*a*b);
s15 =  (E*t*((1-nu) - 10*nu))/(18*(1-nu^2));
s16 =  (13*E*t*((1-nu)*b^2 + 2*a^2))/(45*(1-nu^2)*a*b);
s17 =  (E*t*(14*(1-nu)*b^2 + 17*a^2))/(90*(1-nu^2)*a*b);
s18 =  (23*E*t*((1-nu)*b^2 + 2*a^2))/(180*(1-nu^2)*a*b);
s19 =  (E*t*(17*(1-nu)*b^2 + 56*a^2))/(180*(1-nu^2)*a*b);
s20 = -(E*t*(20*(1-nu)*b^2 - 3*a^2))/(45*(1-nu^2)*a*b);
s21 = -(E*t*(3*(1-nu)*b^2 + 40*a^2))/(90*(1-nu^2)*a*b);
s22 = -(E*t*(10*(1-nu)*b^2 + 3*a^2))/(45*(1-nu^2)*a*b);
s23 =  (E*t*(3*(1-nu)*b^2 - 80*a^2))/(90*(1-nu^2)*a*b);
s24 = -(17*E*t*((1-nu) + 2*nu))/(72*(1-nu^2));
s25 = -(7*E*t*((1-nu) + 2*nu))/(72*(1-nu^2));
s26 =  (E*t*(5*(1-nu) - 2*nu))/(18*(1-nu^2));
s27 = -(E*t*((1-nu) - 10*nu))/(18*(1-nu^2));
s28 =  (E*t*((1-nu) + 2*nu))/(18*(1-nu^2));
s29 =  (4*E*t*(3*(1-nu)*a^2 + 20*b^2))/(45*(1-nu^2)*a*b);
s30 = -(2*E*t*((1-nu) + 2*nu))/(9*(1-nu^2));
s31 = -(4*E*t*(3*(1-nu)*a^2 - 10*b^2))/(45*(1-nu^2)*a*b);
s32 =  (2*E*t*((1-nu) + 2*nu))/(9*(1-nu^2));
s33 =  (8*E*t*(5*(1-nu)*b^2 + 3*a^2))/(45*(1-nu^2)*a*b);
s34 =  (4*E*t*(5*(1-nu)*b^2 - 6*a^2))/(45*(1-nu^2)*a*b);
s35 =  (8*E*t*(5*(1-nu)*a^2 + 3*b^2))/(45*(1-nu^2)*a*b);
s36 =  (4*E*t*(5*(1-nu)*a^2 - 6*b^2))/(45*(1-nu^2)*a*b);
s37 =  (4*E*t*(3*(1-nu)*b^2 + 20*a^2))/(45*(1-nu^2)*a*b);
s38 = -(4*E*t*(3*(1-nu)*b^2 - 10*a^2))/(45*(1-nu^2)*a*b);
k  =[s1  s2  s3  s4  s5  s6  s7  s8  s9  s10 s11 s12 s13 s12 s14 s15;
      s2  s16 s8  s17 s6  s18 s4  s19 s15 s20 s12 s21 s12 s22 s10 s23;
      s3  s8  s1  s24 s7  s4  s5  s25 s9  s26 s14 s27 s13 s28 s11 s28;
      s4  s17 s24 s16 s8  s19 s25 s18 s27 s20 s26 s23 s28 s22 s28 s21;
      s5  s6  s7  s8  s1  s2  s3  s4  s13 s12 s14 s15  s9 s10 s11 s12;
      s6  s18 s4  s19 s2  s16 s8  s17 s12 s22 s10 s23 s15 s20 s12 s21;
      s7  s4  s5  s25 s3  s8  s1  s24 s13 s28 s11 s28  s9 s26 s14 s27;
      s8  s19 s25 s18 s4  s17 s24 s16 s28 s22 s28 s21 s27 s20 s26 s23;
      s9  s15 s9  s27 s13 s12 s13 s28 s29   0   0 s30 s31   0   0 s32;
```

```
      s10 s20 s26 s20 s12 s22 s28 s22   0 s33 s30   0   0 s34 s32   0;
      s11 s12 s14 s26 s14 s10 s11 s28   0 s30 s35   0   0 s32 s36   0;
      s12 s21 s27 s23 s15 s23 s28 s21 s30   0   0 s37 s32   0   0 s38;
      s13 s12 s13 s28 s9  s15 s9  s27 s31   0   0 s32 s29   0   0 s30;
      s12 s22 s28 s22 s10 s20 s26 s20   0 s34 s32   0   0 s33 s30   0;
      s14 s10 s11 s28 s11 s12 s14 s26   0 s32 s36   0   0 s30 s35   0;
      s15 s23 s28 s21 s12 s21 s27 s23 s32   0   0 s38 s30   0   0 s37];

% --------------------
% Assemble the global matrix & vector
% --------------------

% Initialize matrix and vector
K = zeros(ndof,ndof);
F = zeros(ndof,1);

% Manual assembly of global matrix
K([1 2 5 6 21 22 17 18 3 4 13 14 19 20 11 12],...
  [1 2 5 6 21 22 17 18 3 4 13 14 19 20 11 12]) = ...
K([1 2 5 6 21 22 17 18 3 4 13 14 19 20 11 12],...
  [1 2 5 6 21 22 17 18 3 4 13 14 19 20 11 12]) + k;    % Element 1
K([5 6 9 10 25 26 21 22 7 8 15 16 23 24 13 14],...
  [5 6 9 10 25 26 21 22 7 8 15 16 23 24 13 14]) = ...
K([5 6 9 10 25 26 21 22 7 8 15 16 23 24 13 14],...
  [5 6 9 10 25 26 21 22 7 8 15 16 23 24 13 14]) + k;    % Element 2

% Manual assembly of global vector
F([10 16 26]) = [-qy*b/6 -2*qy*b/3 -qy*b/6];

% --------------------
% Impose boundary conditions
% --------------------

% Initialize dof
T = zeros(ndof,1);

% Manual identification of known dof index
dof_k = [1 2 11 12 17 18];

% Manual application of boundary condition (fixed) for known dof
T(dof_k) = 0;

% Manual modification of F vector for known boundary condition
F = F - K(:,1)*uo - K(:,2)*uo  - K(:,7)*uo ...
      - K(:,8)*uo - K(:,13)*uo - K(:,14)*uo;

% --------------------
% Solve the matrix system
% --------------------

% Unknown degree of freedom
dof_u = setdiff(1:ndof,dof_k);

% Solve for the unknown
T(dof_u) = K(dof_u,dof_u)\F(dof_u);
```

3.5 Exercises

1. Consider a plane stress element having $0.05\,\mathrm{m}$ thickness and equal sides of $0.5\,\mathrm{m}$ as shown in Fig. 3.8(a). The modulus elasticity of the element is $2.05\times10^{8}\,\mathrm{kN\,m^{-2}}$ and the Poisson's ratio is 0.3. Determine the stiffness matrix for four-node and eight-node rectangular element shown in Fig. 3.8(b) and (c) respectively.

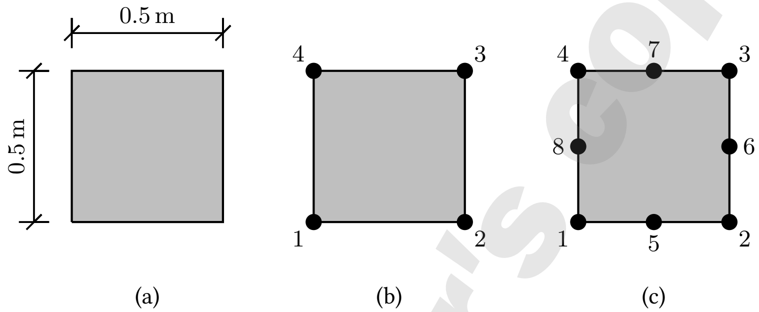

Figure 3.8: Node numbering for square plane stress element

2. The right side of the element shown in Fig. 3.8(a) is subjected to external load of $-25\,\mathrm{kN\,m^{-1}}$ in y-direction. Determine the equivalent nodal load for four-node and eight-node rectangular element shown in Fig. 3.8(b) and (c) respectively.

3. Consider a cantilever beam subjected to external load of $-25\,\mathrm{kN\,m^{-1}}$ as shown in Fig. 3.9. The thickness of the beam is $0.015\,\mathrm{m}$. The modulus elasticity of the beam is $2.05\times10^{8}\,\mathrm{kN\,m^{-2}}$ and the Poisson's ratio is 0.3. Determine the displacement at the centre of the beam using four-node and eight-node plane stress element.

4. The shear wall shown in Fig. 3.10 is carrying a vertical load of $-125\,\mathrm{kN\,m^{-1}}$. The thickness of the wall is $0.2\,\mathrm{m}$. The Young's modulus of the wall is $21\times10^{6}\,\mathrm{kN\,m^{-2}}$ and the Poisson's ratio is $0.15\,\mathrm{m}$. Compare the results of displacement at the top-right corner of both opening using four-node and eight-node plane stress element.

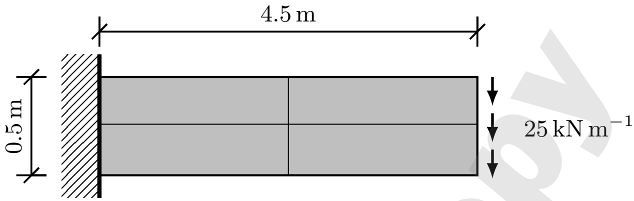

Figure 3.9: Cantilever beam subjected to vertical load at the end of the beam

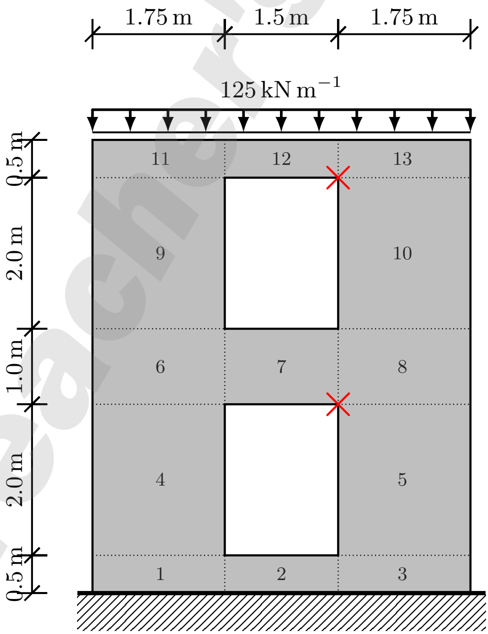

Figure 3.10: Numbering of element for shear wall with opening

4 Introduction to Nonlinear Formulation

4.1 Introduction

In previous chapters, discussions were made only for linear problems. However, in the next chapter, we will discuss the Navier-Stokes equations for fluid flow which are inherently nonlinear. It is, therefore, essential to establish our understanding on the concept of nonlinearity and the corresponding iterative solvers in preparing ourselves for the upcoming chapter. Being introductory, in this chapter, we establish our understanding by discussing 1D nonlinear formulation of a hypothetical bar ODE.

4.2 A Hypothetical Nonlinear Ordinary Differential Equation (ODE) of Bar Element

To start our discussion, the bar ODE previously given by Eq. (1.8) is rewritten here

$$EA\,\frac{d^2\,u}{dx^2} = -q \tag{4.1}$$

Eq. (4.1) is linear which can be made nonlinear if we assume that the coefficient E is no longer a constant but a function of the axial displacement, u i.e. $E(u)$. Let's simply assume that,

$$E(u) = \alpha\,u \tag{4.2}$$

where α is constant. Inserting Eq. (5.44) into Eq. (4.1) makes the latter a nonlinear equation, given as

$$\alpha\, u\, A\, \frac{d^2\, u}{dx^2} = -q \tag{4.3}$$

4.3 Discretization by Galerkin Method

Discretizing Eq. (4.3) using the same shape functions as given by Eq. (1.39), we get

$$\alpha A\, N_k u_k\, \frac{d^2\left(N_j u_j\right)}{dx^2} \neq -q \tag{4.4}$$

Weighting Eq. (4.4) by shape functions, N_i we then obtain

$$N_i\, \left(\alpha A\, N_k u_k\, \frac{d^2\left(N_j u_j\right)}{dx^2} + q\right) \neq 0 \tag{4.5}$$

As usual, the equivalent algebraic forms for Eq. (4.5) can be established by integrating the equation over the length of the bar, thus

$$\int_0^L N_i\, \left(\alpha A\, N_k u_k\, \frac{d^2\left(N_j u_j\right)}{dx^2} + q\right) dx = 0 \tag{4.6}$$

Employing IBP to Eq. (4.6) gives:

$$\int_0^L \left(\alpha A\, N_k u_k\, \frac{dN_i}{dx}\, \frac{dN_j}{dx}\right) dx\, u_j = \int_0^L q\, N_i\, dx + b_i \tag{4.7}$$

where b_i refers to boundary terms or nodal loads.

4.4 Stiffness matrix, $[k]$ and the Nonlinearity

Eq. (4.7) can be given in matrix forms as,

$$k_{ij}\, u_j = f_i \tag{4.8}$$

or

$$[\,k\,]\{u\} = \{f\} \tag{4.9}$$

where k_{ij} or $[k]$ is termed as the local stiffness matrix of the bar element. Thus,

$$k_{ij} = [\,k\,] = \int_0^L \left(\alpha A\, N_k u_k\, \frac{dN_i}{dx}\, \frac{dN_j}{dx}\right) dx \tag{4.10}$$

In an expanded matrix forms, it can be given that,

For linear bar element

$$[\,k\,] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{4.11}$$

where

$$k_{11} = \int_0^L \alpha A\,(N_1 u_1 + N_2 u_2) \left(\frac{dN_1}{dx}\, \frac{dN_1}{dx}\right) dx$$
$$k_{12} = \int_0^L \alpha A\,(N_1 u_1 + N_2 u_2) \left(\frac{dN_1}{dx}\, \frac{dN_2}{dx}\right) dx$$
$$k_{21} = \int_0^L \alpha A\,(N_1 u_1 + N_2 u_2) \left(\frac{dN_2}{dx}\, \frac{dN_1}{dx}\right) dx$$
$$k_{22} = \int_0^L \alpha A\,(N_1 u_1 + N_2 u_2) \left(\frac{dN_2}{dx}\, \frac{dN_2}{dx}\right) dx$$

For quadratic bar element

$$[\,k\,] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{4.12}$$

where

$$k_{11} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_1}{dx} \frac{dN_1}{dx}\right) dx$$
$$k_{12} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_1}{dx} \frac{dN_2}{dx}\right) dx$$
$$k_{13} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_1}{dx} \frac{dN_3}{dx}\right) dx$$
$$k_{21} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_2}{dx} \frac{dN_1}{dx}\right) dx$$
$$k_{22} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_2}{dx} \frac{dN_2}{dx}\right) dx$$
$$k_{23} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_2}{dx} \frac{dN_3}{dx}\right) dx$$
$$k_{31} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_3}{dx} \frac{dN_1}{dx}\right) dx$$
$$k_{32} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_3}{dx} \frac{dN_2}{dx}\right) dx$$
$$k_{33} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_3}{dx} \frac{dN_3}{dx}\right) dx$$

For demonstration purposes, the integration of k_{12} is shown below for both linear and quadratic elements. Their fully integrated stiffness matrix is then given.

For linear bar element

$$\begin{aligned} k_{12} &= \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2\right) \left(\frac{dN_1}{dx} \frac{dN_2}{dx}\right) dx \\ &= \int_0^L \alpha A \left(\left(\frac{L-x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2 \right) \left(\left(-\frac{1}{L}\right) \left(\frac{1}{L}\right) \right) dx \qquad (4.13) \\ &= -\frac{\alpha A \left(u_1 + u_2\right)}{2L} \end{aligned}$$

For quadratic bar element

$$
\begin{aligned}
k_{12} &= \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right)\left(\frac{dN_1}{dx}\frac{dN_2}{dx}\right) dx \\
&= \int_0^L \alpha A \Bigg(\left(\frac{L-2x}{L}\right)\left(\frac{L-x}{L}\right) u_1 + \frac{4x}{L}\left(\frac{L-x}{L}\right) u_2 \\
&\qquad - \frac{x}{L}\left(\frac{L-2x}{L}\right) u_3 \Bigg)\Bigg(\left(-\frac{3}{L} + \frac{4x}{L^2}\right)\left(\frac{4}{L} - \frac{8x}{L^2}\right)\Bigg) dx \\
&= -\frac{\alpha A}{30L}\left(44u_1 + 32u_2 + 4u_3\right)
\end{aligned}
\tag{4.14}
$$

The fully (or analytically) integrated stiffness matrix for both elements can be given as

For linear bar element

$$
[\,k\,] = \frac{\alpha A}{2L}\begin{bmatrix} s & -s \\ -s & s \end{bmatrix} \tag{4.15}
$$

where

$$
s = u_1 + u_2
$$

For quadratic bar element

$$
[\,k\,] = \frac{\alpha A}{30L}\begin{bmatrix} s_1 & s_2 & s_3 \\ s_2 & s_4 & s_5 \\ s_3 & s_5 & s_6 \end{bmatrix} \tag{4.16}
$$

where

$$
\begin{aligned}
s_1 &= 37u_1 + 36u_2 - 3u_3 & \qquad s_4 &= 48u_1 + 64u_2 + 48u_3 \\
s_2 &= -44u_1 - 32u_2 - 4u_3 & \qquad s_5 &= -4u_1 - 32u_2 - 44u_3 \\
s_3 &= 7u_1 - 4u_2 + 7u_3 & \qquad s_6 &= -3u_1 + 36u_2 + 37u_3
\end{aligned}
$$

Matrices given by Eqs. (4.15) and (4.16) are the sources of nonlinearity. This can be made obvious if we compare the ODEs and the resulting matrices between the two problems (i.e. linear vs nonlinear).

For a linear problem, the ODE of the bar has been given as Eq. (4.1)

$$E(\text{constant})\, A\, \frac{d^2u}{dx^2} = -q \tag{4.17}$$

whilst for the nonlinear problem, the ODE is as previously given by Eq. (4.3), which is rewritten herein as

$$E(u)\, A\, \frac{d^2u}{dx^2} = -q \tag{4.18}$$

or

$$\alpha A\, u\, \frac{d^2u}{dx^2} = -q \tag{4.19}$$

As can be seen, the difference between Eqs. (4.17) and (4.18) lies in the constitutive relationship of the material property, E where it is a constant in the former and a function of u (i.e. $E(u)$) in the latter.

And as far as the corresponding stiffness matrices are concerned, for linear problem (as given by Eq. (1.68a)),

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{4.20}$$

For nonlinear problem (as given by Eq. (4.15))

$$\frac{\alpha A}{2L}\begin{bmatrix} u_1 + u_2 & -u_1 - u_2 \\ -u_1 - u_2 & u_1 + u_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{4.21}$$

By comparing Eqs. (4.20) and (4.21), it can be seen that, whilst the stiffness matrix for the linear problem consists of constant coefficients, the stiffness matrix for the nonlinear problem contains the unknown coefficients (or the dof), i.e. u_1, u_2. The nonlinearity can further be made obvious if we expand the two matrices as follows.

For linear problem (expansion of Eq. (4.20))

$$\begin{aligned} \frac{EA}{L}u_1 - \frac{EA}{L}u_2 &= f_1 \\ -\frac{EA}{L}u_1 + \frac{EA}{L}u_2 &= f_2 \end{aligned} \tag{4.22}$$

For nonlinear problem (expansion of Eq. (4.21))

$$
\begin{aligned}
\frac{\alpha A\left(u_1{}^2+u_1u_2\right)}{2L}-\frac{\alpha A\left(u_1u_2+u_2{}^2\right)}{2L}&=f_1\\
-\frac{\alpha A\left(u_1{}^2+u_1u_2\right)}{2L}+\frac{\alpha A\left(u_1u_2+u_2{}^2\right)}{2L}&=f_2
\end{aligned}
\tag{4.23}
$$

Simultaneous equation of Eq. (4.22) is linear thus it can be solved explicitly for u_1 and u_2. In contrast, Eq. (4.23) is quadratic in u_1 and u_2 thus cannot be solved directly. Based on this, it can be said that, solving a nonlinear problem is also an act of solving for the roots of a set of polynomial equations, thus requires iterative schemes.

Finally, due to the difference between linear and nonlinear problems, there is a need to express the dependency on u in the expression of the stiffness matrix of the latter, resulting in the following customary expression of the matrix system,

For local equilibrium

$$[\,k(u)\,]\,\{u\}=\{f\} \tag{4.24}$$

For global equilibrium

$$[\,K(U)\,]\,\{U\}=\{F\} \tag{4.25}$$

In the next discussion, we are going to consider two iterative schemes; Picard's and Newton-Raphson's.

4.5 Iterative Schemes

Consider the nonlinear equilibrium equation as given by Eq. (4.25). Basically, what we have is a dilemma where the vector $\{U\}$ which we want to solve (i.e. by inversing the stiffness matrix, $[K(U)]$) in turn, is what is needed for the calculation of the stiffness matrix. So, the best option for us to proceed, (in fact the only option that we have), is to use a known (or assumed) set of $\{U\}$ in calculating $[K(U)]$ and once calculated, the latter is inversed to solve for $\{U\}$. One can expect that this is going to be a sequential process; a process which sequence can be mathematically expressed with the assistance of a new superscript system, detailed as

$$\left[\,K(U)^{r-1}\,\right]\{U\}^{r} \neq \{F\} \tag{4.26}$$

where r refers to the present stage or present iteration. This then makes $(r-1)$ to represent the immediate previous stage or iteration. For simplification, no superscript is assigned to the load vector, meaning, we assume that the magnitude and the distribution of $\{F\}$ are independent of $\{U\}$ although in various situations this might not be the case.

In general $\{U\}^{r-1}$ and $\{U\}^{r}$ are not similar (except when the solution has converged), thus the left hand side of Eq. (4.26) cannot be equal to the load vector, $\{F\}$ hence the inequality sign. In fact, the difference between these terms would yield a residual, thus

$$\{R\} = \left[\,K(U)^{r-1}\,\right]\{U\}^{r} - \{F\} \tag{4.27}$$

The aim of the iteration process is to reduce the residual, $\{R\}$ to a point of convergence, that is, $\{R\}$ is small enough for us to consider $\{U\}^{r}$ as the solution.

4.5.1 Picard Scheme

Picard scheme is the simplest iterative scheme. It is also known as the direct iteration method. But, its simplicity is not without a price. Picard scheme usually works for mild nonlinear problems but diverges for severe nonlinearity.

The step by step procedure of the scheme is as follows.

i. Take or assume a known value of $\{U\}^{r-1}$
ii. Use the known $\{U\}^{r-1}$ to calculate $[K(U)^{r-1}]$ (i.e. Eq. (4.15) or Eq. (4.16))
iii. Solve $\{U\}^{r}$
iv. Calculate the residual, $\{R\}$ using Eq. (4.27) and check for convergence

v. If converge, stop iteration else solve $\{U\}^r$ (i.e. Eq. (4.25))
vi. Repeat step ii. by using the just solved $\{U\}^r$ as $\{U\}^{r-1}$

The flowchart of the above procedure is shown in Fig. 4.1.

4.5.2 Newton-Raphson Scheme

An alternative iterative scheme which performs better when the nonlinearity is more severe is the Newton-Raphson scheme. The basic concept of the scheme is based on the Taylor's series expansion of the equilibrium equation. The residual of the equilibrium equation can be given as

$$\{R(U)\} \equiv [\,K(U)\,]\,\{U\} - \{F\} \tag{4.28}$$

Expanding Eq. (4.28) by Taylor's series about the known $(r-1)^{\text{th}}$ solution and insisting the residual to vanish give

$$\{R(U)\} = 0 = \{R(U)^{r-1}\} + \frac{\partial\{R(U)^{r-1}\}}{\partial\{U\}^{r-1}}\{\Delta U\} \tag{4.29}$$

where the series has been truncated up to linear terms only. Rearranging gives

$$\frac{\partial\{R(U)^{r-1}\}}{\partial\{U\}^{r-1}}\{\Delta U\} = -\{R(U)^{r-1}\} \tag{4.30}$$

Let's define a new matrix and call it as tangent stiffness matrix, $[T(U)^{r-1}]$, thus,

$$\left[\,T(U)^{r-1}\,\right] = \frac{\partial\{R(U)^{r-1}\}}{\partial\{U\}^{r-1}} \tag{4.31}$$

By inserting Eq. (4.31) into Eq. (4.30), the following is obtained.

$$\left[\,T(U)^{r-1}\,\right]\{\Delta U\} = -\{R(U)^{r-1}\} \tag{4.32}$$

Examining Eq. (4.32), one can observe that $\{\Delta U\}$ is now solvable since the other terms i.e. $[T(U)^{r-1}]$ and $\{R(U)^{r-1}\}$ are known already since they use the results from the previous iteration i.e. $(r-1)^{\text{th}}$. By inversing $[T(U)^{r-1}]$, $\{\Delta U\}$ is thus obtained as

$$\{\Delta U\} = \left[\,T(U)^{r-1}\,\right]^{-1}\left(-\{R(U)^{r-1}\}\right) \tag{4.33}$$

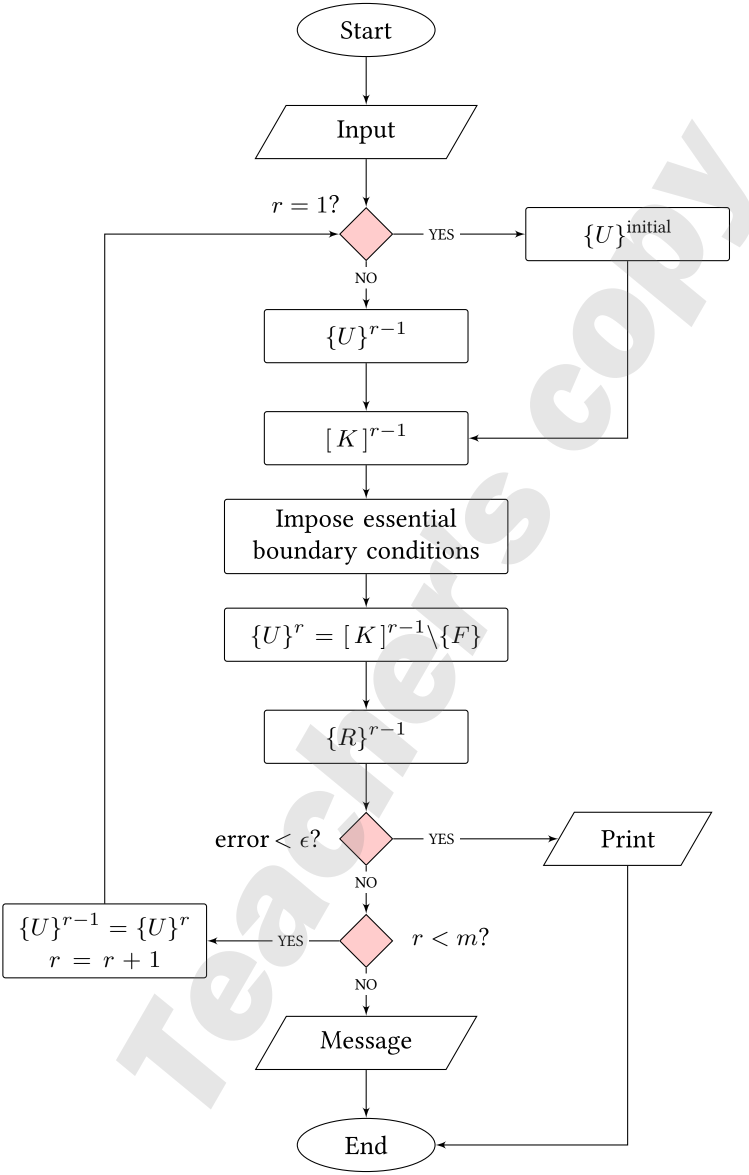

Figure 4.1: Picard flowchart

Once $\{\Delta U\}$ is solved, the solution at the r^{th} iteration is thus

$$\{U\}^r = \{U\}^{r-1} + \{\Delta U\} \tag{4.34}$$

In the next iteration, $\{U\}^r$ of Eq. (4.34) will be inserted back into Eq. (4.32) and take the role as $\{U\}^{r-1}$ in the determination of $[T(U)^{r-1}]$ and $\{R(U)^{r-1}\}$. This process is repeated until the solution converges.

Tangent Stifness Matrix, $[T(U)^{r-1}]$

In the previous discussion, definition of tangent stiffness was given. Herein, the detailed formulation of the matrix is detailed out. For this purpose, Eq. (4.31) is rewritten below.

$$\left[\,T(U)^{r-1}\,\right] = \frac{\partial\{R(U)^{r-1}\}}{\partial\{U\}^{r-1}} \tag{4.35}$$

The numerator on the right hand side terms of Eq. (4.35) is actually the residual of the equilibrium equation at the $(r-1)^{\text{th}}$ iteration which can be given as

$$\{R(U)^{r-1}\} \equiv \left[\,K(U)^{r-1}\,\right]\{U\}^{r-1} - \{F\} \tag{4.36}$$

By inserting Eq. (4.36) into Eq. (4.35) gives

$$\left[\,T(U)^{r-1}\right] = \frac{\partial\left\{\left[\,K(U)^{r-1}\,\right]\{U\}^{r-1} - \{F\}\right\}}{\partial\{U\}^{r-1}} \tag{4.37}$$

By employing chain-rule and since $\{F\}$ does not depend on $\{U\}$ (we are not considering geometric nonlinearity), Eq. (4.37) can be expanded to

$$\begin{aligned}
\left[\,T(U)^{r-1}\,\right] &= \frac{\partial\left[\,K(U)^{r-1}\,\right]}{\partial\{U\}^{r-1}}\{U\}^{r-1} + \frac{\partial\{U\}^{r-1}}{\partial\{U\}^{r-1}}\left[\,K(U)^{r-1}\,\right] \\
&= \frac{\partial\left[\,K(U)^{r-1}\,\right]}{\partial\{U\}^{r-1}}\{U\}^{r-1} + \left[\,K(U)^{r-1}\,\right] \\
&= \left[\hat{K}(U)^{r-1}\right] + \left[K(U)^{r-1}\right]
\end{aligned} \tag{4.38}$$

Eq. (4.38) is derived at global level. It is more convenient to derive the tangent stiffness matrix at local. Therefore, for local and in terms of components, the tangent stiffness matrix of an element can be given as

$$\begin{aligned}\left[\, t_{ij}(u)^{r-1} \,\right] &= \frac{\partial \left[\, k_{ik}(u)^{r-1} \,\right]}{\partial \{u_j\}^{r-1}} \{u_k\}^{r-1} + \left[\, k_{ij}(u)^{r-1} \,\right] \\ &= \left[\hat{k}_{ij}(u)^{r-1}\right] + \left[k_{ij}(u)^{r-1}\right]\end{aligned} \tag{4.39}$$

whilst $k_{ij}(u)^{r-1}$ has been derived as the stiffness matrix of the element as given by Eq. (4.15) for the hypothetical bar element, $\hat{k}_{ij}(u)^{r-1}$ deserves further and more detailed formulation. This is given next.

Based on Eq. (4.39), it can be given that,

$$\hat{k}_{ij}(u)^{r-1} = \frac{\partial \left[\, k_{ik}(u)^{r-1} \,\right]}{\partial \{u_j\}^{r-1}} \{u_k\}^{r-1} \tag{4.40}$$

By inserting Eq. (4.10) into Eq. (4.40) gives

$$\hat{k}_{ij}(u)^{r-1} = \frac{\partial \left[\displaystyle\int_0^L \left(\alpha A \, N_l u_l^{r-1} \frac{dN_i}{dx} \frac{dN_k}{dx} \right) dx \right]}{\partial \{u_j\}^{r-1}} \{u_k\}^{r-1} \tag{4.41}$$

but it can be shown that,

$$\frac{\partial \left(N_l u_l^{r-1} \right)}{\partial \{u_j\}^{r-1}} = N_j \tag{4.42}$$

For example, consider linear bar element and take $j = 2$, thus

$$\frac{\partial \left(N_1 u_1^{r-1} + N_2 u_2^{r-1} \right)}{\partial \{u_2\}^{r-1}} = N_2 \tag{4.43}$$

A more mathematically elegant way is to use of Kronecker delta, δ_{ij}. It can be shown that

$$\frac{\partial \{u_l\}^{r-1}}{\partial \{u_j\}^{r-1}} = \begin{cases} 1, & l = j \\ 0, & l \neq j \end{cases} = \delta_{lj} \tag{4.44}$$

By inserting Eq. (4.44) into Eq. (4.42) and since Kronecker delta, δ_{ij} acts as substitution operator, it can thus be obtained that,

$$\frac{\partial\left(N_l u_l^{r-1}\right)}{\partial\{u_j\}^{r-1}} = N_l\,\delta_{lj} = N_j \tag{4.45}$$

hence, Eq. (4.42) is shown. By inserting either Eq. (4.42) or Eq. (4.45) into Eq. (4.41), the following is obtained:

$$\hat{k}_{ij}(u)^{r-1} = \int_0^L \left(\alpha A\, N_j \frac{dN_i}{dx}\frac{dN_k}{dx}\right) dx\ \{u_k\}^{r-1} \tag{4.46}$$

Since we are free to place $\{u_k\}^{r-1}$ either inside or the outside the integral (since it is a vector of constant), it is preferable to place it inside; a preference which is to be obvious later. Thus,

$$\hat{k}_{ij}(u)^{r-1} = \int_0^L \left(\alpha A\, N_j \frac{dN_i}{dx}\frac{d\left(N_k u_k^{r-1}\right)}{dx}\right) dx \tag{4.47}$$

The association of u_k^{r-1} with N_k in the Eq. (4.47) above is appropriate because together, they would represent the interpolation function, $u(x) = N_k u_k^{r-1}$. Having established all these, we are now in the position to formulate the fully integrated tangent stiffness matrix, $t_{ij}(u)^{r-1}$ for both linear and quadratic bar element. Using Eq. (4.47) and the appropriate shape functions as given in Chapter 1, it can be established that,

For linear bar element

$$\left[\,t_{ij}\,\right]^{r-1} = \begin{bmatrix}\hat{k}_{11} & \hat{k}_{12}\\ \hat{k}_{21} & \hat{k}_{22}\end{bmatrix}^{r-1} + \begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}^{r-1} \tag{4.48a}$$

where

$$\hat{k}_{11} = \int_0^L \alpha A\, N_1 \frac{dN_1}{dx}\frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1}\right)}{dx}\, dx$$

$$\hat{k}_{12} = \int_0^L \alpha A\, N_1 \frac{dN_2}{dx}\frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1}\right)}{dx}\, dx$$

$$\hat{k}_{21} = \int_0^L \alpha A\, N_2 \frac{dN_1}{dx}\frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1}\right)}{dx}\, dx$$

$$\hat{k}_{22} = \int_0^L \alpha A\, N_2 \frac{dN_2}{dx}\frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1}\right)}{dx}\, dx$$

For quadratic bar element

$$\left[\, t_{ij} \,\right]^{r-1} = \begin{bmatrix} \hat{k}_{11} & \hat{k}_{12} & \hat{k}_{13} \\ \hat{k}_{21} & \hat{k}_{22} & \hat{k}_{23} \\ \hat{k}_{31} & \hat{k}_{32} & \hat{k}_{33} \end{bmatrix}^{r-1} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}^{r-1} \tag{4.48b}$$

where

$$\hat{k}_{11} = \int_0^L \alpha A\, N_1 \frac{dN_1}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$
$$\hat{k}_{12} = \int_0^L \alpha A\, N_1 \frac{dN_2}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$
$$\hat{k}_{13} = \int_0^L \alpha A\, N_1 \frac{dN_3}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$
$$\hat{k}_{21} = \int_0^L \alpha A\, N_2 \frac{dN_1}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$
$$\hat{k}_{22} = \int_0^L \alpha A\, N_2 \frac{dN_2}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$

$$\hat{k}_{23} = \int_0^L \alpha A\, N_2 \frac{dN_3}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$
$$\hat{k}_{31} = \int_0^L \alpha A\, N_3 \frac{dN_1}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$
$$\hat{k}_{32} = \int_0^L \alpha A\, N_3 \frac{dN_2}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$
$$\hat{k}_{33} = \int_0^L \alpha A\, N_3 \frac{dN_3}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx$$

Stiffness matrices which are given by Eq. (4.48) are still in integral forms. For demonstration purposes, the integration of k_{12}^{r-1} are shown below for both linear and quadratic elements. The fully integrated tangent stiffness matrix for both elements is then given.

For linear bar element

$$
\begin{aligned}
\hat{k}_{12}^{r-1} &= \int_0^L \alpha A\, N_2 \frac{dN_1}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1}\right)}{dx}\, dx \\
&= \int_0^L \alpha A \left(\frac{x}{L}\right)\left(-\frac{1}{L}\right)\left(\left(-\frac{1}{L}\right) u_1^{r-1} + \left(\frac{1}{L}\right) u_2^{r-1}\right) dx \qquad (4.49a) \\
&= -\frac{\alpha A\left(u_2^{r-1} - u_1^{r-1}\right)}{2L}
\end{aligned}
$$

For quadratic bar element

$$
\begin{aligned}
\hat{k}_{12}^{r-1} &= \int_0^L \alpha A\, N_2 \frac{dN_1}{dx} \frac{d\left(N_1 u_1^{r-1} + N_2 u_2^{r-1} + N_3 u_3^{r-1}\right)}{dx}\, dx \\
&= \int_0^L \alpha A \left(\frac{4x}{L}\left(\frac{L-x}{L}\right)\right)\left(-\frac{3}{L} + \frac{4x}{L^2}\right)\left(\left(-\frac{3}{L} + \frac{4x}{L^2}\right) u_1^{r-1} + \right. \\
&\qquad \left. \left(\frac{4}{L} - \frac{8x}{L^2}\right) u_2^{r-1} + \left(-\frac{1}{L} + \frac{4x}{L^2}\right) u_3^{r-1}\right) dx \\
&= \alpha A \left(\frac{18u_1^{r-1} - 16u_2^{r-1} - 2u_3^{r-1}}{15L}\right)
\end{aligned}
\qquad (4.49b)
$$

By repeating the process for other elements of the matrix, the fully integrated tangent stiffness matrix for both elements can be given as

For linear bar element

$$
\begin{aligned}
\left[\, t_{ij} \,\right]^{r-1} &= \begin{bmatrix} \hat{k}_{11} & \hat{k}_{12} \\ \hat{k}_{21} & \hat{k}_{22} \end{bmatrix}^{r-1} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{r-1} \\
&= \frac{\alpha A}{L} \begin{bmatrix} u_1 & -u_2 \\ -u_1 & u_2 \end{bmatrix}^{r-1}
\end{aligned}
\qquad (4.50a)
$$

For quadratic bar element

$$
\begin{aligned}
\left[\, t_{ij} \,\right]^{r-1} &= \begin{bmatrix} \hat{k}_{11} & \hat{k}_{12} & \hat{k}_{13} \\ \hat{k}_{21} & \hat{k}_{22} & \hat{k}_{23} \\ \hat{k}_{31} & \hat{k}_{32} & \hat{k}_{33} \end{bmatrix}^{r-1} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}^{r-1} \\
&= \frac{\alpha A}{15L} \left[\begin{array}{ccc} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ s_7 & s_2 & s_8 \end{array} \right]^{r-1}
\end{aligned}
\tag{4.50b}
$$

where

$$
\begin{aligned}
s_1 &= 37\,u_1 - 4\,u_2 + 2\,u_3 & s_5 &= 8\,u_1 + 64\,u_2 + 8\,u_3 \\
s_2 &= -4\,u_1 - 32\,u_2 - 4\,u_3 & s_6 &= -4\,u_1 + 8\,u_2 - 44\,u_3 \\
s_3 &= 2\,u_1 - 4\,u_2 + 7\,u_3 & s_7 &= 7\,u_1 - 4\,u_2 + 2\,u_3 \\
s_4 &= -44\,u_1 + 8\,u_2 - 4\,u_3 & s_8 &= 2\,u_1 - 4\,u_2 + 37\,u_3
\end{aligned}
$$

The step by step procedure of the scheme is as follows.

i. Take or assume a known value of $\{U\}^{r-1}$
ii. Use the known $\{U\}^{r-1}$ to calculate the global tangent stiffness matrix, $[T]^{r-1}$ using Eq. (4.50)
iii. Calculate the residual, $\{R\}$ using Eq. (4.28)
iv. Solve the vector of incremental dof, ΔU (i.e. Eq. (4.33))
v. Update $\{U\}^r$ using Eq. (4.34) and check for convergence
vi. If converge, stop iteration else repeat Step ii. by using the just updated$\{U\}^r$ as $\{U\}^{r-1}$

The flowchart of the above procedure is shown in Fig. 4.2.

4.6 Worked Example

Herein, we are going to provide the numerical example to demonstrate the use of the procedures which we have discussed previously. The example is given for Newton-Raphson scheme with linear bar element.

The example solves the problem of a bar structure as shown in Fig. 4.3

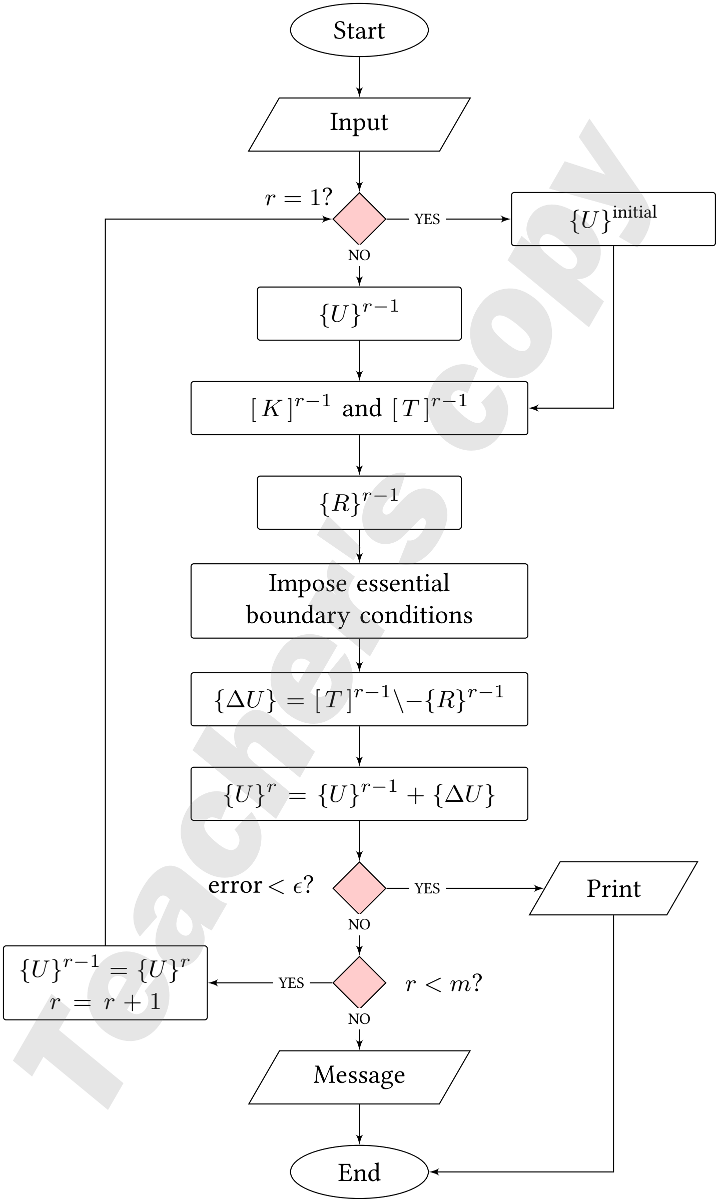

Figure 4.2: Newton-Raphson flowchart

by discretizing it (or meshing it) with two assembled bar elements. The material and the geometrical data for the analysis are given as follows

- Cross-sectional area, $A = 1\,\mathrm{m}^2$
- Material coefficient, $\alpha = 1$
- Total length of structure (bar) $L = 1\,\mathrm{m}$

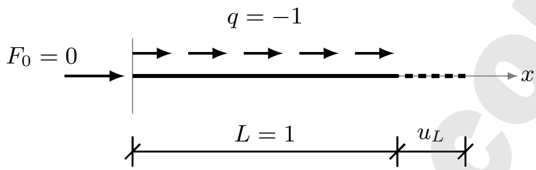

Figure 4.3: Nonlinear bar problem

Boundary conditions:

- Natural (or force): a distributed loading with an intensity of -1 is acting throughout the length of structures the structures and no nodal loads are considered, thus $q = -1$ force/m
- Essential (or displacement): a specified displacement of 1.4142 is applied at the far-right end/edge of the structure, thus u_2^2 (Linear element) = u_3^2 (Quadratic element) = 1.4142

Initial data (i.e. $r = 0$):

- Linear element: $U_1^0 = U_2^0 = 1$ whilst $U_3^0 = 1.4142$
- Quadratic element: $U_1^0 = U_2^0 = U_3^0 = U_4^0 = 1$ whilst $U_5^0 = 1.4142$

Convergence criterion: $\epsilon = 10^{-3}$

Exact solutions (or closed–form solution) of the ODE (for comparison purpose):

- at $x = 0$, $u = 1$
- at $x = L$, $u = 1.4142$

4.6.1 Newton-Raphson Scheme with Linear Bar Element

First iteration, $(r = 1)$

By using the given initial data and Eq. (4.50a) the tangent stiffness matrices at the first iteration (i.e $(r = 1)$), $[t_{ij}]^{1,0}$ and $[t_{ij}]^{2,0}$ for element 1 and 2, respectively are thus given as

$$\left[\, t_{ij} \,\right]^{1,0} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}^{1,0} \tag{4.51}$$

$$\left[\, t_{ij} \,\right]^{2,0} = \begin{bmatrix} 2 & -2.829 \\ -2 & 2.829 \end{bmatrix}^{2,0} \tag{4.52}$$

And the load vector as well as the vector of dof can be given as

Element 1

$$\begin{aligned} \{f_i\}^1 &= \begin{Bmatrix} \dfrac{qL}{2} \\ \dfrac{qL}{2} \end{Bmatrix}^1 = \begin{Bmatrix} -0.25 \\ -0.25 \end{Bmatrix}^1 \\ \{u_i\}^{1,1} &= \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^{1,1} \end{aligned} \tag{4.53}$$

Element 2

$$\begin{aligned} \{f_i\}^2 &= \begin{Bmatrix} \dfrac{qL}{2} \\ \dfrac{qL}{2} \end{Bmatrix}^2 = \begin{Bmatrix} -0.25 \\ -0.25 \end{Bmatrix}^2 \\ \{u_i\}^{2,1} &= \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^{2,1} = \begin{Bmatrix} u_1 \\ 1.4142 \end{Bmatrix}^{2,1} \end{aligned} \tag{4.54}$$

By assembling the two elements, we thus obtain the following. This is done by inserting the values derived in Eqs. (4.53) and (4.54) into Eq. (4.32).

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2+2 & -2.829 \\ 0 & -2 & -2.829 \end{bmatrix}^0 \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \\ 0 \end{Bmatrix}^1 =$$
$$- \left[\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2+2.4142 & -2.4142 \\ 0 & -2.4142 & 2.4142 \end{bmatrix}^0 \begin{Bmatrix} 1 \\ 1 \\ 1.4142 \end{Bmatrix}^0 - \begin{Bmatrix} -0.25 \\ -0.25-0.25 \\ -0.25 \end{Bmatrix}^0 \right] \tag{4.55}$$

Simplifying the right hand side of Eq. (4.55) gives

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2.8284 \\ 0 & -2 & -2.8284 \end{bmatrix}^0 \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \\ 0 \end{Bmatrix}^1 = - \begin{Bmatrix} 0.25 \\ -0.50 \\ 1.25 \end{Bmatrix}^0 \tag{4.56}$$

Take note that U_3 is a specified boundary condition thus does not change. This unchanged value means there is no variation in U_3 thus ΔU_3 is zero throughout the analysis, as shown in Eqs. (4.55) and (4.56) above. A condensed system will then give us

$$\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}^0 \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \end{Bmatrix}^1 = - \begin{Bmatrix} 0.25 \\ -0.50 \end{Bmatrix}^0 \tag{4.57}$$

Solving Eq. (4.57), we thus obtain the solution as

$$\begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \end{Bmatrix}^1 = - \begin{Bmatrix} 0.000 \\ 0.125 \end{Bmatrix}^1 \tag{4.58}$$

and the updated solution according to Eq. (4.34) can be obtained as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}^1 = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}^0 + \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \\ \Delta U_3 \end{Bmatrix}^1 = \begin{Bmatrix} 1.0000 \\ 1.0000 \\ 1.4142 \end{Bmatrix}^0 + \begin{Bmatrix} 0.0000 \\ 0.1250 \\ 0.0000 \end{Bmatrix}^1 = \begin{Bmatrix} 1.0000 \\ 1.1250 \\ 1.4142 \end{Bmatrix}^1 \tag{4.59}$$

Second iteration, $(r = 2)$

In this second iteration, values updated in the first iteration (i.e. Eq. (4.59)) are used for the determination of the relevant matrices thus the tangent stiffness matrices for element 1 and element 2 are given as follows

$$\left[\, t_{ij} \,\right]^{1,1} = \begin{bmatrix} 2.000 & -2.250 \\ -2.000 & 2.250 \end{bmatrix}^{1,1} \tag{4.60}$$

$$\left[\, t_{ij} \,\right]^{2,1} = \begin{bmatrix} 2.250 & -2.828 \\ -2.250 & 2.828 \end{bmatrix}^{2,1} \tag{4.61}$$

By assembling Eqs. (4.60) and (4.61) we obtain the global system as

$$\begin{aligned} &\begin{bmatrix} 2.000 & -2.250 & 0.000 \\ -2.000 & 4.500 & -2.828 \\ 0.000 & -2.250 & -2.828 \end{bmatrix}^{1} \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \\ 0 \end{Bmatrix}^{2} \\ &= - \left[\begin{bmatrix} 2.1250 & -2.1250 & 0.000 \\ -2.1250 & 4.6642 & -2.5392 \\ 0.000 & -2.5392 & 2.5392 \end{bmatrix}^{1} \begin{Bmatrix} 1.0000 \\ 1.1250 \\ 1.4142 \end{Bmatrix}^{1} - \begin{Bmatrix} -0.25 \\ -0.25 - 0.25 \\ -0.25 \end{Bmatrix}^{1} \right] \\ &= - \begin{Bmatrix} -0.016 \\ 0.031 \\ 0.984 \end{Bmatrix}^{1} \end{aligned} \tag{4.62}$$

A condensed system will then give us

$$\begin{bmatrix} 2.000 & -2.250 \\ -2.000 & 4.500 \end{bmatrix}^{1} \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \end{Bmatrix}^{2} = - \begin{Bmatrix} -0.016 \\ 0.031 \end{Bmatrix}^{1} \tag{4.63}$$

Solving Eq. (4.63) we thus obtain the solution as

$$\begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \end{Bmatrix}^{2} = - \begin{Bmatrix} 0.0000 \\ -0.0007 \end{Bmatrix}^{2} \tag{4.64}$$

and the updated solution according to Eq. (4.34) can be obtained as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}^{2} = - \begin{Bmatrix} 1.0000 \\ 1.1180 \\ 1.4142 \end{Bmatrix}^{2} \tag{4.65}$$

Next, we checked for convergence and this is done by calculating the root-mean square error taken about the solution obtained from the previous iteration (1st iteration), thus,

$$\sqrt{\sum_{I=1}^{N} {R_I}^2} = 0.0034 > 10^{-3} \tag{4.66}$$

Based on the above, it can be seen that the solutions still have not converged so the iteration must be continued. Updated values as given in Eq. (4.64) is then used in the next iteration i.e. ($r = 3$). If we repeat the whole procedure hence proceed with the 3rd iteration, the following results can be obtained

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}^3 = -\begin{Bmatrix} 1.0000 \\ 1.1180 \\ 1.4142 \end{Bmatrix}^3 \tag{4.67}$$

The iteration process is stopped at the 3rd iteration because it can be shown that the convergence criterion has been met. In fact, these values coincide perfectly with the exact solutions (i.e. those obtained from the closed-form solution of the ODE).

4.6.2 Source Code for Picard: Linear Element

```
% Clear data
clc; clear; close all

% --------------------
% Input parameters
% --------------------

% Domain and material properties
Lb          = 1;                % Bar length [m]
alpha       = 1;                % Material coefficient
A           = 1;                % Cross sectional area [m^2]
q           = -1;               % Distributed load [N/m]

% Meshing and boundary condition
node_xy     = [0 Lb/2 Lb];      % Coordinate
elm_node    = [1 2; 2 3];       % Connectivity matrix
node_cond   = [0 0 1];          % Node condition
Uo          = [1;1;1.4142];     % Guessed and boundary values for dof

num_node    = length(node_xy);  % Total number of node/dof
num_elm     = size(elm_node,1); % Total number of element

% Convergence tolerance
tol         = 1e-3;

% --------------------
% Carry out nonlinear iteration
% --------------------

for iter = 1:10  % Maximum 10 iteration

%   --------------------
%   Calculate the local matrix and force vector
%   --------------------

%   Initialize global stiffness matrix and load vector
    K   = zeros(num_node);
    F   = zeros(num_node,1);

%   Assemble local to global
    for i=1:num_elm

%       Variables index
        iu = elm_node(i,:);

%       Elemental length
        L  = node_xy(elm_node(i,2))-node_xy(elm_node(i,1));

%       Elemental dof from previous iteration
        u  = Uo(iu);

%       Local stiffness matrix and force vector
        k  =  alpha*A/2/L*[u(1)+u(2) -u(1)-u(2);-u(1)-u(2) u(1)+u(2)];
        f  = q*L/2*[1; 1];

%       Assemble local to global matrix and vector
        K(iu,iu) = K(iu,iu) + k;
        F(iu)    = F(iu) + f;
```

```
    end

%   --------------------
%   Impose boundary conditions
%   --------------------

%   Known dof index
    dof_k = find(node_cond);

%   Backup force vector for residual calculation
    Fo = F;

%   Loop for all nodes
    for i = 1:num_node

%       Check if node is boundary node
        if node_cond(i)

%           Modify force vector to take into account boundary condition
            F = F - K(:,i)*Uo(i);

        end

    end

%   --------------------
%   Solve the matrix system
%   --------------------

%   Initialize dof
    U = Uo;

%   Unknown degree of freedom
    dof_u = setdiff(1:num_node,dof_k);

%   Solve for the unknown
    U(dof_u) = K(dof_u,dof_u)\F(dof_u);

%   --------------------
%   Convergence check
%   --------------------

%   Error between current and previous iteration
    err = max(abs(U-Uo));

%   Check for convergence
    if err < tol
        fprintf('Solution converged!\n')
        fprintf('Iteration = %i, error = %.1e\n', i, err)
        break
    end

%   Use the new dof value for next iteration
    Uo = U;

end
```

4.6.3 Source Code for Newton Raphson: Linear Element

```
% Clear data
clc; clear; close all

% --------------------
% Input parameters
% --------------------

% Domain and material properties
Lb           = 1;                  % Bar length [m]
alpha        = 1;                  % Material coefficient
A            = 1;                  % Cross sectional area [m^2]
q            = -1;                 % Distributed load [N/m]

% Meshing and boundary condition
node_xy      = [0 Lb/2 Lb];        % Coordinate
elm_node     = [1 2; 2 3];         % Connectivity matrix
node_cond    = [0 0 1];            % Node condition
Uo           = [1;1;1.4142];       % Guessed and boundary values for dof

num_node     = length(node_xy);    % Total number of node/dof
num_elm      = size(elm_node,1);   % Total number of element

% Convergence tolerance
tol          = 1e-3;

% --------------------
% Carry out nonlinear iteration
% --------------------

for iter = 1:10  % Maximum 10 iteration

%   --------------------
%   Calculate the local matrix and force vector
%   --------------------

%   Initialize global tangent stiffness matrix and residual vector
    T   = zeros(num_node);
    R   = zeros(num_node,1);

%   Assemble local to global
    for i=1:num_elm

%       Variables index
        iu = elm_node(i,:);

%       Elemental length
        L  = node_xy(elm_node(i,2))-node_xy(elm_node(i,1));

%       Elemental dof from previous iteration
        u  = Uo(iu);

%       Local stiffness  and tangent matrix
        k  =  alpha*A/2/L*[u(1)+u(2) -u(1)-u(2);-u(1)-u(2) u(1)+u(2)];
        t  =  alpha*A/L*[u(1) -u(2);-u(1) u(2)];

%       Local force and residual vector
        f  = q*L/2*[1; 1];
```

```
        r  = k*u-f;

%       Assemble local to global matrix and vector
        T(iu,iu) = T(iu,iu) + t;
        R(iu)    = R(iu) + r;

    end

%   --------------------
%   Solve the matrix system
%   --------------------

%   Known dof index
    dof_k = find(node_cond);

%   Initialize incremental solution
    dU = zeros(num_node,1);

%   Unknown degree of freedom
    dof_u = setdiff(1:num_node,dof_k);

%   Solve for the incremental solution
    dU(dof_u) = T(dof_u,dof_u)\-R(dof_u);

%   Update solution
    U = Uo + dU;

%   --------------------
%   Convergence check
%   --------------------

%   Error between current and previous iteration
    err = max(abs(U-Uo));

%   Check for convergence
    if err < tol
        fprintf('Solution converged!\n')
        fprintf('Iteration = %i, error = %.1e\n', i, err)
        break
    end

%   Use the new dof value for next iteration
    Uo = U;

end
```

5 Fluid Dynamics: Navier-Stokes Equations

5.1 Introduction

The behaviour of flow of fluids and gases obey the three conservation laws; mass, momentum and energy, all must conserve. The complete modelling of these laws hence the actual flow behaviour is extremely difficult. It is the practice, therefore, to reduce the complexity of the problem by introducing appropriate assumptions. Herein, it is assumed that;

i. flow is incompressible
ii. flow is isothermal
iii. flow behaviour in width direction is constant
iv. fluid is materially linear (Newtonian) and homogenous isotropic.

These assumptions allow us to deal with a simpler set of equations but sufficient enough for its expansion to the more general cases becomes obvious.

5.2 Derivation of Navier-Stokes Partial Differential Equations

By employing assumption ii. which means that there is no variation in temperature within the flow, a flow problem can be completely described by the mass (continuity) and momentum equations alone as the energy equation is now uncoupled from the latter. The derivation of mass (continuity) equation is given next, followed by the derivation of the momentum equation. To note, from now on, mass equation will be referred as continuity equation.

5.2.1 Continuity Equation

The continuity equation can be derived by considering a differential element of the flow as shown in Fig. 5.1. As shown in the figure, the variables to be considered are the density, ρ, velocity components in x-direction, u, in y-direction, v and in z-direction, w. Similar to previous chapters, the variables are expanded by Taylor series at all sides (surfaces).

The conservation of mass principle requires that the time rate of decrease of mass inside the differential element must be equal to the net mass flowing out of the differential element through its surface. The former can be given as

$$\text{The time rate of decrease of mass } = \frac{\partial \rho}{\partial t}\,(dx\,dy\,dz) \tag{5.1}$$

In x-direction, by balancing terms on the right and left sides of the differential element, we obtain

$$\left(\rho u + \frac{\partial(\rho u)}{\partial x}\,dx\right)dy\,dz - (\rho u)\,dy\,dz = \frac{\partial(\rho u)}{\partial x}\,dx\,dy\,dz \tag{5.2}$$

By similar argument, the net outflow in y-direction is given as

$$\left(\rho v + \frac{\partial(\rho v)}{\partial y}\,dy\right)dx\,dz - (\rho v)\,dx\,dz = \frac{\partial(\rho v)}{\partial y}\,dx\,dy\,dz \tag{5.3}$$

and in z-direction as

$$\left(\rho w + \frac{\partial(\rho w)}{\partial z}\,dz\right)dx\,dy - (\rho w)\,dx\,dy = \frac{\partial(\rho w)}{\partial z}\,dx\,dy\,dz \tag{5.4}$$

Having established the net flow in all direction, the net mass flow of the fluid out of the differential element through its surfaces can thus be given as

$$\text{Net mass flow } = \left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)dx\,dy\,dz \tag{5.5}$$

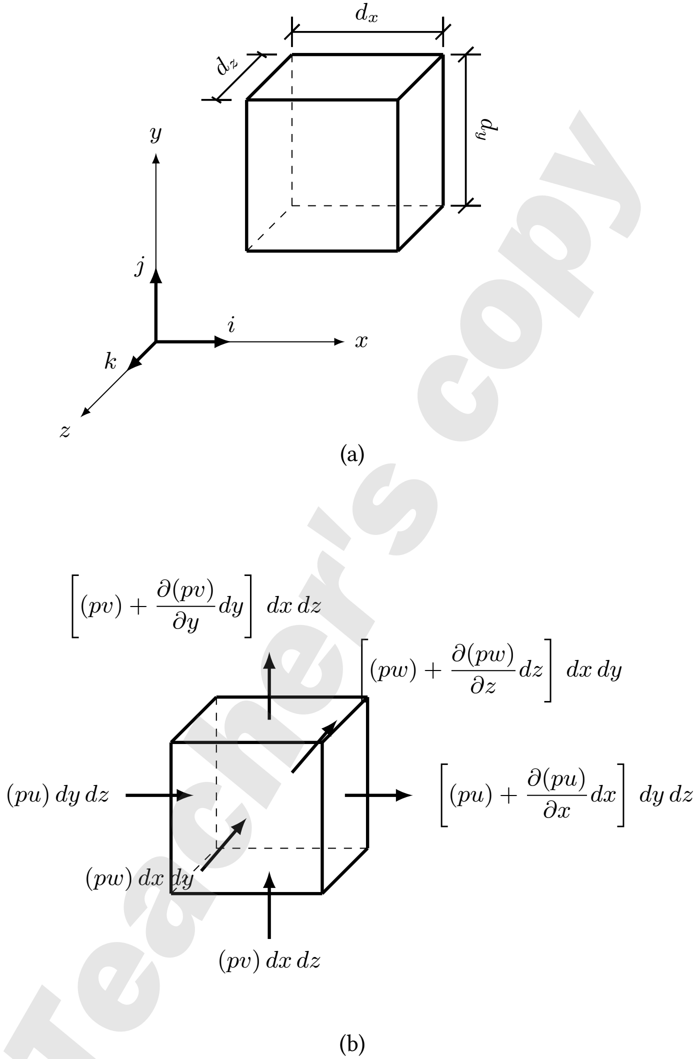

Figure 5.1: Flow mass differential element

The principle of mass conservation requires Eq. (5.5) to be equal to Eq. (5.1), thus,

$$-\frac{\partial \rho}{\partial t}\,dx\,dy\,dz = \left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right) dx\,dy\,dz \tag{5.6}$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{5.7}$$

or in vector forms as

$$\frac{\partial \rho}{\partial t} + \{\partial\}\{\rho u\}^T = 0 \tag{5.8}$$

where

$$\{\partial\} = \left\{ \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right\} \tag{5.9}$$

$$\{\rho u\} = \left\{ \rho u \quad \rho v \quad \rho w \right\} \tag{5.10}$$

Eqs. (5.7) and (5.8) are the continuity equation for the flow given in the form termed as conservation forms. In this form, the product of ρu, ρv and ρw are themselves treated as the dependent variables. However, sometime, it would be more convenient to deal directly with the velocity components, u, v and w, termed as primitive variables as this would be the more familiar forms we have encountered so far.

To express Eqs. (5.7) and (5.8) explicitly in terms of the primitive variables, we employ the following derivative called material derivative. We omit the theoretical derivation and discussion of this derivative but suffice to say, it is a natural product of Eulerian formulation; a formulation usually employed in fluid mechanics whilst its counterpart, the Lagrangian being usually employed in solid mechanics. The material derivative is given as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \{u\}\{\partial\}^T \tag{5.11}$$

Now, since Eqs. (5.7) and (5.8) involves the derivative of the product of two functions, we can expand this derivative by chain-rule as

$$\frac{\partial \rho}{\partial t} + \rho\{\partial\}\{u\}^T + \{u\}\{\partial\}^T \rho = 0 \tag{5.12}$$

If we observe carefully, we will notice that the differential operators in Eq. (5.12) which operates on the density, ρ, can be replaced by the material derivative. By inserting Eq. (5.11) into Eq. (5.12) we obtain

$$\frac{D\rho}{Dt} + \rho\{\partial\}\{u\}^T = 0 \tag{5.13}$$

Having established Eq. (5.13), the continuity equation of the flow is now in the non-conservation form. It can be expanded to give

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{5.14}$$

Now, we employ assumption i. that is, the flow is incompressible. Incompressible fluid is a fluid having constant density in time. Accordingly, incompressible flow refers to the state where the rate of change of density is assumed zero. It is worth to note that, while incompressible flow does not really refer to incompressible fluid as the latter is an ideal state of fluid, the terms are usually interchangeable. Due to assumption i., Eq. (5.14) is reduced to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.15}$$

since the assumption would mean $\dfrac{D\rho}{Dt} = 0$.

We have completed the derivation of the continuity equation (mass conservation equation) for incompressible flow. Next, we are going to derive the momentum equations.

5.2.2 Momentum Equations

The momentum equation is the principle based on Newton's 2nd Law which basic form is

$$\sum F = m\,a \tag{5.16}$$

where F is the total force acting on a particle whilst m and a are the mass and the acceleration of the particle, respectively. For a continuum of higher dimension, Eq. (5.16) is expanded into

$$\sum F_x = m\,a_x \tag{5.17a}$$

$$\sum F_y = m\,a_y \tag{5.17b}$$

$$\sum F_z = m\,a_z \tag{5.17c}$$

Eq. (5.17) represent the equilibrium of forces in all direction. Since we base our discussion on Eulerian formulation, the accelerative terms on the right and side of the equations should be expressed in terms of the material derivatives, thus,

$$\sum F_x = \rho \, \frac{Du}{Dt} \, dx \, dy \, dz \tag{5.18a}$$

$$\sum F_y = \rho \, \frac{Dv}{Dt} \, dx \, dy \, dz \tag{5.18b}$$

$$\sum F_z = \rho \, \frac{Dw}{Dt} \, dx \, dy \, dz \tag{5.18c}$$

since $m = \rho \, dx \, dy \, dz$. Terms on the left hand side of Eq. (5.18) can be determined by considering the differential element of the flow as shown in Fig. 5.2.

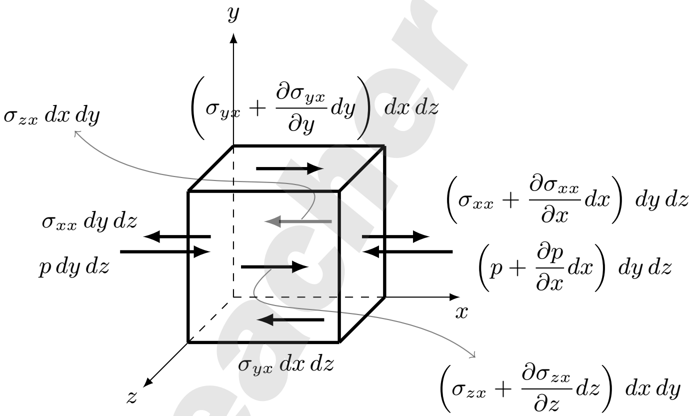

Figure 5.2: Flow forces in differential element (shown for plane x-y only)

As shown in the figure, the variables to be considered are pressure, p and viscous stresses; normal and shear stresses in x-direction, y-direction and z-direction (i.e. σ_{xx}, σ_{yy}, σ_{zz}, σ_{yx}, σ_{zx}, σ_{yz}). Similar to previous chapters, the variables are expanded by Taylor series at all sides (surfaces).

By considering the equilibrium of forces in x-direction, we obtain

$$
\begin{aligned}
\sum F_x = &\left(p - \left(p + \frac{\partial p}{\partial x} dx\right)\right) dy\, dz + \\
&\left(\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right) - \sigma_{xx}\right) dy\, dz + \\
&\left(\left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy\right) - \sigma_{yx}\right) dx\, dz + \\
&\left(\left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z} dz\right) - \sigma_{zx}\right) dx\, dy + \rho\, f_x\, dx\, dy\, dz
\end{aligned}
\tag{5.19}
$$

where $\rho\, f_x\, dx\, dy\, dz$ is the body force in the x-direction. By inserting Eq. (5.19) into Eq. (5.18a), expanding and cancelling, the following is obtained

$$
-\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho\, f_x = \rho \frac{Du}{Dt} \tag{5.20}
$$

Eq. (5.20) is the momentum equation for the flow in the x-direction. Applying the same procedure to the other directions, momentum equations in y-direction and z-direction can be given, respectively as

$$
-\frac{\partial p}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho\, f_y = \rho \frac{Dv}{Dt} \tag{5.21}
$$

$$
-\frac{\partial p}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho\, f_z = \rho \frac{Dw}{Dt} \tag{5.22}
$$

Eqs. (5.3), (5.4) and (5.20) are the complete momentum equations for 3 dimensional flow. By inserting the material derivative as given by Eq. (5.11) into the equations, we can expand the equations into

$$
-\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho\, f_x = \rho \frac{\partial u}{\partial t} + \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}\right) \tag{5.23}
$$

$$
-\frac{\partial p}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho\, f_y = \rho \frac{\partial v}{\partial t} + \rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}\right) \tag{5.24}
$$

$$
-\frac{\partial p}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho\, f_z = \rho \frac{\partial w}{\partial t} + \rho \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}\right) \tag{5.25}
$$

Eq. (5.15) and Eqs. (5.23) to (5.25) are the 3 dimensional continuity and the momentum equations of the flow which are famously known as Navier-Stokes equations. These equations are supplemented by boundary condition equations which discussion is going to be given next. Also, since the equations can be time-dependent or unsteady (if time derivative terms are not omitted), the solution of the PDE would require initial conditions to be prescribed.

5.2.3 Boundary Condition Equations

Like other PDEs, the continuity and the momentum equations must be supplemented by boundary equations so as to have a well-posed problem. There are two types of boundary conditions; Neumann (natural) and Dirichlet (essential) boundary conditions.

Neumann (natural) boundary conditions

i. in x-direction

$$b_x = -p\, n_x + \sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z \tag{5.26}$$

ii. in y-direction

$$b_y = -p\, n_y + \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z \tag{5.27}$$

iii. in z-direction

$$b_z = -p\, n_z + \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z \tag{5.28}$$

where, n_x, n_y and n_z are the components of the unit normal vector of the surface under consideration.

It can be shown that the secondary terms produced by the integration by parts conducted on the viscous and the pressure terms of the momentum equations will be in similar forms as to Eqs. (5.26) to (5.28). Due to this, equations above are also known as natural boundary conditions.

Dirichlet (essential) boundary conditions

Dirichlet boundary condition requires that the velocity components at the boundary where natural conditions are unknown (to be solved) must be known thus specified.

$$u = \bar{u}(t) \tag{5.29}$$

$$v = \bar{v}(t) \tag{5.30}$$

$$w = \bar{w}(t) \tag{5.31}$$

For no slip condition, normally the specified value of the velocities (i.e. $\bar{u}(t)$, $\bar{v}(t)$, $\bar{w}(t)$) can either be zero or equal to the velocity of the boundary.

There are other possible boundary conditions for a flow such as slip condition and free surface condition. However, these are not covered herein.

5.3 Two-dimensional Navier Stokes Equations

Eq. (5.15), Eqs. (5.23) to (5.25) are the three-dimensional Navier-Stokes equations together with their boundary equations given by Eqs. (5.26) to (5.28) and Eqs. (5.29) to (5.31). If we employ assumption iii) that the flow behaviour is assumed constant in the width direction, the equations can be reduced to two-dimensional which can be given as follows.

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.32}$$

Momentum equation

$$-\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \rho\, f_x = \rho\, \frac{\partial u}{\partial t} + \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \tag{5.33}$$

$$-\frac{\partial p}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho\, f_y = \rho\, \frac{\partial v}{\partial t} + \rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) \tag{5.34}$$

Neumann (natural) boundary conditions

$$b_x = -p\, n_x + \sigma_{xx} n_x + \sigma_{yx} n_y \tag{5.35}$$

$$b_y = -p\, n_y + \sigma_{xy} n_x + \sigma_{yy} n_y \tag{5.36}$$

Dirichlet (essential) boundary conditions

$$u = \bar{u}(t) \tag{5.37}$$

$$v = \bar{v}(t) \tag{5.38}$$

It is convenient to express the continuity equation (Eq. (5.32) and the momentum equations (Eqs. (5.33) and (5.34)) in vector and matrix forms, as we did for heat and plane stress elements so as to allow for easy tracing of the extension of the procedure. This way, the evolution of the procedure can easier be followed.

In matrix and vector forms, the continuity equation (Eq. (5.32)) can be given as

$$\left\{ \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right\} \left\{ \begin{matrix} u \\ v \end{matrix} \right\} = 0 \tag{5.39}$$

or

$$\{\partial\}\{u\}^T = 0 \tag{5.40}$$

where $\{u\}^T = \left\{ \begin{matrix} u \\ v \end{matrix} \right\}$. When expressed in vector forms, $\{u\}$ is taken as representing both velocities, u and v. Transpose of the vector is employed because, all vectors and matrices are treated as if they are originally arranged in a row-wise manner; a normal practice in computer programming works.

To express the momentum equations in matrix and vector forms, it would be an advantage if we highlight the similarity of the viscous terms to previously discussed plane stress formulation as this, not only allow for easy tracing of the evolution of the procedure, emphasizes the applicability of the algorithm developed for the former to be directly used in the latter, maybe with some slight modification.

To highlight the similarity, the viscous terms are boxed in the following equation.

$$\boxed{A}$$

$$\begin{aligned}
-\frac{\partial p}{\partial x} &+ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \rho\, f_x = \rho\,\frac{\partial u}{\partial t} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \\
-\frac{\partial p}{\partial y} &+ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho\, f_y = \rho\,\frac{\partial v}{\partial t} + \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)
\end{aligned} \tag{5.41}$$

By observing box A, we can notice how similar the terms inside the box to the ones we encountered in the plane stress problem (refer **Eqn. (3.71)**). Due to this, similar matrix and vector forms can be employed, thus

$$\begin{Bmatrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \tag{5.42}$$

or

$$\begin{Bmatrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} \end{Bmatrix} = \left[\partial\right]\left\{\sigma\right\}^T \tag{5.43}$$

since it can be shown that $\sigma_{xy} = \sigma_{yx}$. $[\partial]$ is termed as differential operator matrix.

Now, by employing assumption iv. that is, the fluid is assumed as materially linear (Newtonian) and isotropic, as well as assumption i. that is, the flow is incompressible, the terms $\lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$, which initially exists in the normal stress relationship where λ is the second viscosity coefficient, is excluded.

The constitutive equations of the viscous stresses can thus be given as

$$\begin{aligned}
\sigma_{xx} &= 2\mu\frac{\partial u}{\partial x} \\
\sigma_{yy} &= 2\mu\frac{\partial v}{\partial y} \\
\sigma_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned} \tag{5.44}$$

which can be arranged in matrix forms as

$$\{\sigma\}^T = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{5.45}$$

or

$$\{\sigma\}^T = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{5.46}$$

Observe that the second matrix on the right hand side of the equation above is also the transpose of the differential operator matrix, $[\partial]$ previously defined in Eqs. (5.43) and (5.44). Based on this and by introducing the constitutive matrix, $[E]$, Eq. (5.46) can be given as

$$\{\sigma\}^T = [\,E\,]\,[\,\partial\,]^T\,\{u\}^T \tag{5.47}$$

where

$$[\,E\,] = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{5.48}$$

By inserting Eq. (5.47) into Eq. (5.43), the viscous terms can be completely expressed in terms of the primitive variables, u and v as

$$\begin{Bmatrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} \end{Bmatrix} = [\,\partial\,]\,[\,E\,]\,[\,\partial\,]^T\,\{u\}^T \tag{5.49}$$

Now let's deal with the rest of Eq. (5.41). To maintain the same line of argument, boxes are also provided for other terms as shown below.

$$\overset{\boxed{B}}{\boxed{\begin{matrix} -\dfrac{\partial p}{\partial x} \\[3ex] -\dfrac{\partial p}{\partial y} \end{matrix}}} \begin{matrix} +\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} \\[3ex] +\dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} \end{matrix} \overset{\boxed{C}}{\boxed{\begin{matrix} +\rho\, f_x \\[3ex] +\rho\, f_y \end{matrix}}} \begin{matrix} = \\[3ex] = \end{matrix} \overset{\boxed{D}}{\boxed{\begin{matrix} \rho\,\dfrac{\partial u}{\partial t} \\[3ex] \rho\,\dfrac{\partial v}{\partial t} \end{matrix}}} \overset{\boxed{E}}{\boxed{\begin{matrix} +\rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) \\[3ex] +\rho\left(u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right) \end{matrix}}} \tag{5.50}$$

Terms in box B can be arranged in vector form as

$$\begin{Bmatrix} -\dfrac{\partial p}{\partial x} \\ -\dfrac{\partial p}{\partial y} \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{Bmatrix}^T p = -\{\partial\}^T p \tag{5.51}$$

Terms in box C can be given in vector form as

$$\begin{Bmatrix} \rho\, f_x \\ \rho\, f_y \end{Bmatrix} = \rho \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} = \rho\,\{f\}^T \tag{5.52}$$

Terms in box D can be given in vector form as

$$\begin{Bmatrix} \rho\,\dfrac{\partial u}{\partial t} \\ \rho\,\dfrac{\partial v}{\partial t} \end{Bmatrix} = \rho \begin{Bmatrix} \dfrac{\partial u}{\partial t} \\ \dfrac{\partial v}{\partial t} \end{Bmatrix} = \rho \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \rho\,\{\dot{u}\}^T \tag{5.53}$$

Finally, terms in box E can be given in matrix form as

$$\begin{aligned} \begin{Bmatrix} \rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) \\ \rho\left(u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right) \end{Bmatrix} &= \rho \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \\ &= \rho \begin{Bmatrix} u \\ v \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{Bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \\ &= \rho\,\{u\}^T\{\partial\}\{u\}^T \end{aligned} \tag{5.54}$$

Now by using Eq. (5.49), and Eqs. (5.51) to (5.54), the momentum equations of Eq. (5.33) and Eq. (5.34) can be given in matrix and vector forms as

$$-\{\partial\}^T p + [\,\partial\,]\,[\,E\,]\,[\,\partial\,]^T \{u\}^T + \rho\,\{f\}^T = \rho\,\{\dot{u}\}^T + \rho\,\{u\}^T\{\partial\}\{u\}^T \tag{5.55}$$

Reminder: please note the difference between $\{\partial\}$ and $[\,\partial\,]$.

Eqs. (5.40) and (5.55) are the continuity and momentum equation for 2 dimensional viscous incompressible fluid problems.

5.4 Discretization by Galerkin WRM

Having established the PDE of the problem, we are all set to descritize the equations so as to obtain the FEM algebraic formulation. There are various ways to this for flow problems but two favoured formulations are the mixed and the penalty formulations. However, herein we will concern only with the former.

The term 'mixed' refers to the fact that we have variables from different families; velocities and pressure. Whilst the former is displacement related quantities, the latter is force-related quantity. Previous studies have shown that, for the formulation to work well, the order of interpolation of velocities must be higher than the pressure. Due to this, to interpolate the velocities, quadratic shape functions are employed while for pressure, bilinear shape functions are used. For rectangular elements these mean

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & \dots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & \dots & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix} \tag{5.56}$$

$$= [\,N\,]\,\{\hat{u}\}^T$$

$$p = \{L_1 \quad L_2 \quad L_3 \quad L_4\} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \{L\}\{\hat{p}\}^T \tag{5.57}$$

$[N]$ and $[L]$ are similar to those previous derived for scalar (heat transfer) and plane stress elements. Due to this, the arrangement of nodes for a rectangular element is shown in Fig. 5.3.

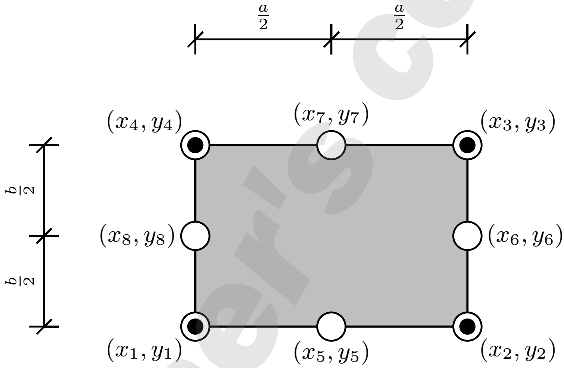

Figure 5.3: Rectangular element for fluid

For a rectangular element, it is still convenient to provide the shape functions in the physical coordinates (x, y) and conduct analytical integration. For demonstration purposes and to ease the tracing of the formulation, herein these are pursued. But it must be noted that, in practice, the shape function functions are expressed in natural coordinates, (ξ, η) and numerical integration is employed as these would allow for easier treatment of elements with irregular shape.

Inserting Eqs. (5.56) and (5.57) into Eqs. (5.40) and (5.55),we obtain

Continuity

$$\{\partial\}\,[\,N\,]\,\{\hat{u}\}^T \neq 0 \tag{5.58}$$

Momentum

$$
\begin{aligned}
-\{\partial\}^T\{L\}\{\hat{p}\}^T + [\,\partial\,]\,[\,E\,]\,[\,\partial\,]^T\,[\,N\,]\,\{\hat{u}\}^T + \rho\,\{f\}^T \\
\neq \rho\,[\,N\,]\,\{\dot{\hat{u}}\}^T + \rho\,[\,N\,]\,\{\hat{u}\}^T\{\partial\}\,[\,N\,]\,\{\hat{u}\}^T
\end{aligned}
\tag{5.59}
$$

Note that, for the time derivative terms, since the shape function $[N]$ are not function of time, the dofs are thus differentiated by time, i.e $\{\dot{\hat{u}}\}^T$.

By observing Fig. 5.3 and Eq. (5.56) as well as Eq. (5.57), we can identify that, for an element, we have 20 dofs, i.e. $\{u_1 \ldots u_8, v_1 \ldots v_8, p_1 \ldots p_8\}$ with only 3 equations. To obtain sufficient number of equations, we will multiply and then integrate mass equation (Eq. (5.58)) with pressure shape function $\{L\}$ to produce 4 independent equations, and the momentum equations with velocities shape function $[\,N\,]$ to produce 16 independent equations. Together, they will make up 20 simultaneous algebraic equations as required. Multiplying Eq. (5.58) with pressure shape functions $\{L\}^T$ and integrating give

$$
\int_x \int_y \{L\}^T\{\partial\}\,[\,N\,]\,\{\hat{u}\}^T\,dy\,dx = 0 \tag{5.60}
$$

By multiplying the velocity shape functions $[N]^T$ to the momentum equations (Eq. (5.59)) and integrating give

$$
\begin{aligned}
&-\int_x \int_y [\,N\,]^T\,\{\partial\}^T\{L\}\{\hat{p}\}^T\,dy\,dx \\
&+\int_x \int_y [\,N\,]^T\,[\,\partial\,]\,[\,E\,]\,[\,\partial\,]^T\,[\,N\,]\,\{\hat{u}\}^T\,dy\,dx \\
&+\rho\int_x \int_y [\,N\,]^T\,\{f\}^T\,dy\,dx = \rho\int_x \int_y [\,N\,]^T\,[\,N\,]\,\{\dot{\hat{u}}\}^T\,dy\,dx \\
&+\rho\int_x \int_y [\,N\,]^T\,[\,N\,]\,\{\hat{u}\}^T\{\partial\}\,[\,N\,]\,\{\hat{u}\}^T\,dy\,dx
\end{aligned}
\tag{5.61}
$$

Integration by Parts (IBP)

As mentioned, the intentions of IBP are:

i. To relax the continuity requirement

ii. To induce explicit the natural boundary conditions

Such intentions will not be fulfilled if IBP is conducted to the continuity equation. In the context of point ii. even if we conduct IBP to the continuity equation, the resulting boundary terms, would not have physical meaning.

Also, based on point i. and ii., IBP should be conducted onto pressure and viscous terms only as these will induce the natural boundary conditions as previously given by Eqs. (5.35) and (5.36). Therefore, by conducting IBP to Eq. (5.61) but to the pressure and viscous terms only, results in

$$
\begin{aligned}
&\int_x \int_y [N]^T \{\partial\}^T \{L\}\{\hat{p}\}^T dy\, dx \\
&- \int_x \int_y [N]^T [\partial][E][\partial]^T [N]\{\hat{u}\}^T dy\, dx \\
&+\rho \int_x \int_y [N]^T \{f\}^T dy\, dx = \rho \int_x \int_y [N]^T [N]\{\dot{\hat{u}}\}^T dy\, dx \\
&+\rho \int_x \int_y [N]^T [N]\{\hat{u}\}^T \{\partial\}[N]\{\hat{u}\}^T dy\, dx - \int_s [N]^T \{b_s\}^T ds
\end{aligned}
\tag{5.62}
$$

Now, let's introduce the matrices.

5.4.1 Matrices and the Discretized Equation of Motion of Fluid

Since the viscous terms are similar in form to those derived for plane stress problem (refer Chapter 3); let's call the corresponding matrix as stiffness matrix $[k]$ where

$$
[k] = \int_x \int_y [N]^T [\partial][E][\partial]^T [N]\, dy\, dx \tag{5.63}
$$

Due to the similarity, stiffness matrix algorithms previously established for plane stress problem can thus be applied. Next, for the terms associated with time derivative of dofs, let's call the corresponding matrix as mass matrix $[m]$ given as

$$[\,m\,] = \rho \int_x \int_y [\,N\,]^T\,[\,N\,]\,dy\,dx \tag{5.64}$$

Next, by considering the pressure terms, let's introduce a coefficient matrix, $[c]$ where

$$[\,c\,] = \int_x \int_y [\,N\,]^T\,\{\partial\}^T \{L\}\,dy\,dx \tag{5.65}$$

The second last terms on the right hand side of Eq. (5.62) exhibits nonlinearity. Accordingly, we introduce a matrix termed as nonlinear matrix $[k_N]$ where

$$[\,k_N\,] = \rho \int_x \int_y [\,N\,]^T\,[\,N\,]\,\{\hat{u}\}^T \{\partial\}\,[\,N\,]\,dy\,dx \tag{5.66}$$

Finally, we introduce a body force vector $\{f_B\}$ and natural force vector $\{b\}$ as

$$\{f_B\} = \rho \int_x \int_y [\,N\,]^T\,\{f\}^T dy\,dx \tag{5.67}$$

$$\{b\} = \int_s [\,N\,]^T\,\{b_s\}^T ds \tag{5.68}$$

Having established these corresponding matrices and vectors, the continuity and the momentum equations can be compactly expressed. For the continuity equation, it can be shown that,

$$\int_x \int_y \{L\}^T \{\partial\}\,[\,N\,]\,dy\,dx = -\left[\int_x \int_y \{N\}^T \{\partial\}^T\,[\,L\,]\,dy\,dx\right]^T = -\,[\,c\,]^T \tag{5.69}$$

By inserting Eq. (5.63) to Eq. (5.63) into Eqs. (5.62) and (5.63) and by extracting the dofs from the integral sign, we obtain the following

Continuity

$$-\left[\,c\,\right]^{T}\left\{\hat{u}\right\}^{T}=0 \tag{5.70}$$

Momentum

$$\left[\,c\,\right]\left\{\hat{p}\right\}^{T}-\left[\,k\,\right]\left\{\hat{u}\right\}^{T}+\left\{f_{B}\right\}=\left[\,m\,\right]\left\{\dot{\hat{u}}\right\}^{T}+\left[\,k_{N}\,\right]\left\{\hat{u}\right\}^{T}-\left\{b\right\} \tag{5.71}$$

Since IBP is conducted only on some terms whilst there are others which are not, it is important to note that, those terms which are subjected to IBP, the multiplication between terms inside the integral must be carried out from left to right while for those which are not, the multiplication must be carried out from right to left. To demonstrate this and to provide example on how to build the above matrices, the determination of element (3,4) for each matrix and element (3,1) for each vector above are shown next. Although in practice, numerical integration is employed which detailed procedure is given in the next chapter, herein analytical integration is employed so as to allow for easy tracing of the procedure and to allow for its immediate application. Such analytical integrations are possible herein because we limit our discussion to rectangular domain thus rectangular elements only.

Mass matrix, $[m]$

We begin our demonstration by showing the derivation for element m_{34} of the mass matrix, $[m]$. Since this term has no derivative, no dilemma on whether to conduct IBP or not. Thus,

$$\begin{aligned} m_{34} &= \int_x \int_y \begin{bmatrix} \vdots & \vdots \\ N_2 & 0 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & N_2 & \cdots \end{bmatrix} dy\,dx \\ &= \int_x \int_y (N_2)(0) + (0)(N_2)\,dy\,dx \;= 0 \end{aligned} \tag{5.72}$$

Repeating the procedure above (Step 1 to Step 5) for all components, the complete integrated values of $[\,m\,]$ can be given as

$$[\,m\,] = \left[\begin{array}{ccc} S_a & | & S_b \\ -- & -- & -- \\ S_b^T & | & S_c \end{array}\right] \tag{5.73}$$

where

$$[\,S_a\,] = \begin{bmatrix} s_1 & 0 & s_2 & 0 & s_3 & 0 & s_2 & 0 \\ 0 & s_1 & 0 & s_2 & 0 & s_3 & 0 & s_2 \\ s_2 & 0 & s_1 & 0 & s_2 & 0 & s_3 & 0 \\ 0 & s_2 & 0 & s_1 & 0 & s_2 & 0 & s_3 \\ s_3 & 0 & s_2 & 0 & s_1 & 0 & s_2 & 0 \\ 0 & s_3 & 0 & s_2 & 0 & s_1 & 0 & s_2 \\ s_2 & 0 & s_3 & 0 & s_2 & 0 & s_1 & 0 \\ 0 & s_2 & 0 & s_3 & 0 & s_2 & 0 & s_1 \end{bmatrix}$$

$$[\,S_b\,] = \begin{bmatrix} s_4 & 0 & s_5 & 0 & s_5 & 0 & s_4 & 0 \\ 0 & s_4 & 0 & s_5 & 0 & s_5 & 0 & s_4 \\ s_4 & 0 & s_4 & 0 & s_5 & 0 & s_5 & 0 \\ 0 & s_4 & 0 & s_4 & 0 & s_5 & 0 & s_5 \\ s_5 & 0 & s_4 & 0 & s_4 & 0 & s_5 & 0 \\ 0 & s_5 & 0 & s_4 & 0 & s_4 & 0 & s_5 \\ s_5 & 0 & s_5 & 0 & s_4 & 0 & s_4 & 0 \\ 0 & s_5 & 0 & s_5 & 0 & s_4 & 0 & s_4 \end{bmatrix}$$

$$[\,S_c\,] = \begin{bmatrix} s_6 & 0 & s_7 & 0 & s_8 & 0 & s_7 & 0 \\ 0 & s_6 & 0 & s_7 & 0 & s_8 & 0 & s_7 \\ s_7 & 0 & s_6 & 0 & s_7 & 0 & s_8 & 0 \\ 0 & s_7 & 0 & s_6 & 0 & s_7 & 0 & s_8 \\ s_8 & 0 & s_7 & 0 & s_6 & 0 & s_7 & 0 \\ 0 & s_8 & 0 & s_7 & 0 & s_6 & 0 & s_7 \\ s_7 & 0 & s_8 & 0 & s_7 & 0 & s_6 & 0 \\ 0 & s_7 & 0 & s_8 & 0 & s_7 & 0 & s_6 \end{bmatrix}$$

$$s_1 = \frac{a\,b\,\rho}{30} \qquad s_2 = \frac{a\,b\,\rho}{90}$$

$$s_3 = \frac{a\,b\,\rho}{60} \qquad s_4 = -\frac{a\,b\,\rho}{30}$$

$$s_5 = -\frac{2\,a\,b\,\rho}{45} \qquad s_6 = \frac{8\,a\,b\,\rho}{45}$$

$$s_7 = \frac{a\,b\,\rho}{9} \qquad s_8 = \frac{4\,a\,b\,\rho}{45}$$

Stiffness matrix, $[k]$

stiffness matrix, $[k]$ involves IBP thus the matrix multiplication must be carried out from left to right. We will do this in step-by-step manner.

Step 1

$$\ldots \begin{bmatrix} \vdots & \vdots \\ N_2 & 0 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \ldots \Rightarrow \ldots \begin{bmatrix} \vdots & \vdots & \vdots \\ \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_2}{\partial y} \\ \vdots & \vdots & \vdots \end{bmatrix} \ldots \tag{5.74}$$

Step 2

$$\begin{aligned} &\ldots \begin{bmatrix} \vdots & \vdots & \vdots \\ \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_2}{\partial y} \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \ldots \\ &\Rightarrow \ldots \begin{bmatrix} \vdots & \vdots & \vdots \\ 2\mu \frac{\partial N_2}{\partial x} & 0 & \mu \frac{\partial N_2}{\partial y} \\ \vdots & \vdots & \vdots \end{bmatrix} \ldots \end{aligned} \tag{5.75}$$

Step 3

$$
\begin{aligned}
&\cdots \begin{bmatrix} \vdots & \vdots & \vdots \\ 2\mu \dfrac{\partial N_2}{\partial x} & 0 & \mu \dfrac{\partial N_2}{\partial y} \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \cdots \\
&\qquad \Rightarrow \ \cdots \begin{bmatrix} \vdots & \vdots \\ 2\mu \dfrac{\partial N_2}{\partial x} \dfrac{\partial}{\partial x} + \mu \dfrac{\partial N_2}{\partial y} \dfrac{\partial}{\partial y} & \mu \dfrac{\partial N_2}{\partial y} \dfrac{\partial}{\partial x} \\ \vdots & \vdots \end{bmatrix} \cdots
\end{aligned}
\tag{5.76}
$$

Step 4

$$
\begin{aligned}
&\cdots \begin{bmatrix} \vdots & \vdots \\ 2\mu \dfrac{\partial N_2}{\partial x} \dfrac{\partial}{\partial x} + \mu \dfrac{\partial N_2}{\partial y} \dfrac{\partial}{\partial y} & \mu \dfrac{\partial N_2}{\partial y} \dfrac{\partial}{\partial x} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & N_2 & \cdots \end{bmatrix} \cdots \\
&\qquad \Rightarrow \cdots \left(2\mu \frac{\partial N_2}{\partial x} \frac{\partial}{\partial x} + \mu \frac{\partial N_2}{\partial y} \frac{\partial}{\partial y} \right) (0) + \mu \frac{\partial N_2}{\partial y} \frac{\partial N_2}{\partial x} \cdots \\
&\qquad \Rightarrow \cdots \mu \frac{\partial N_2}{\partial y} \frac{\partial N_2}{\partial x} \cdots
\end{aligned}
\tag{5.77}
$$

Step 5 (analytical integration)

Step 1 until Step 4 would give k_{34} as

$$
k_{34} = \int_x \int_y \mu \frac{\partial N_2}{\partial y} \frac{\partial N_2}{\partial x} \, dy \, dx \tag{5.78}
$$

For a rectangular element with a width of a and a height of b, as shown in Fig. 5.3, k_{34} can be re-given as

$$
\begin{aligned}
k_{34} &= \int_0^a \int_0^b \mu \left(\frac{4\,x\,y}{a\,b^2} - \frac{2\,x^2}{a^2\,b} - \frac{x}{a\,b} \right) \\
&\qquad \left(\frac{4\,x}{a^2} - \frac{1}{a} - \frac{y}{a\,b} + \frac{2\,y^2}{a\,b^2} - \frac{4\,x\,y}{a^2\,b} \right) dy\,dx \\
&= -\frac{17\mu}{36}
\end{aligned}
\tag{5.79}
$$

Repeating the procedure above (Step 1 to Step 5) for all components, the complete integrated values of $[\,k\,]$ can be given as

$$
[\,k\,] = \left[\begin{array}{ccc} S_a & | & S_b \\ -- & -- & -- \\ S_b^T & | & S_c \end{array} \right] \tag{5.80}
$$

where

$$
[\,S_a\,] = \begin{bmatrix}
s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\
s_2 & s_{16} & s_8 & s_{17} & s_6 & s_{18} & s_4 & s_{19} \\
s_3 & s_8 & s_1 & s_{24} & s_7 & s_4 & s_5 & s_{25} \\
s_4 & s_{17} & s_{24} & s_{16} & s_8 & s_{19} & s_{25} & s_{18} \\
s_5 & s_6 & s_7 & s_8 & s_1 & s_2 & s_3 & s_4 \\
s_6 & s_{18} & s_4 & s_{19} & s_2 & s_{16} & s_8 & s_{17} \\
s_7 & s_4 & s_5 & s_{25} & s_3 & s_8 & s_1 & s_{24} \\
s_8 & s_{19} & s_{25} & s_{18} & s_4 & s_{17} & s_{24} & s_{16}
\end{bmatrix}
$$

$$
[\,S_b\,] = \begin{bmatrix}
s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{12} & s_{14} & s_{15} \\
s_{15} & s_{20} & s_{12} & s_{21} & s_{12} & s_{22} & s_{10} & s_{23} \\
s_9 & s_{26} & s_{14} & s_{12} & s_{13} & s_{15} & s_{11} & s_{15} \\
s_{12} & s_{20} & s_{26} & s_{23} & s_{15} & s_{22} & s_{15} & s_{21} \\
s_{13} & s_{12} & s_{14} & s_{15} & s_9 & s_{10} & s_{11} & s_{12} \\
s_{12} & s_{22} & s_{10} & s_{23} & s_{15} & s_{20} & s_{12} & s_{21} \\
s_{13} & s_{15} & s_{11} & s_{15} & s_9 & s_{26} & s_{14} & s_{12} \\
s_{15} & s_{22} & s_{15} & s_{21} & s_{12} & s_{20} & s_{26} & s_{23}
\end{bmatrix}
$$

$$[\,S_c\,] = \begin{bmatrix} s_{27} & 0 & 0 & s_{28} & s_{29} & 0 & 0 & s_{30} \\ 0 & s_{31} & s_{28} & 0 & 0 & s_{32} & s_{30} & 0 \\ 0 & s_{28} & s_{33} & 0 & 0 & s_{30} & s_{34} & 0 \\ s_{28} & 0 & 0 & s_{35} & s_{30} & 0 & 0 & s_{36} \\ s_{29} & 0 & 0 & s_{30} & s_{27} & 0 & 0 & s_{28} \\ 0 & s_{32} & s_{30} & 0 & 0 & s_{31} & s_{28} & 0 \\ 0 & s_{30} & s_{34} & 0 & 0 & s_{28} & s_{33} & 0 \\ s_{30} & 0 & 0 & s_{36} & s_{28} & 0 & 0 & s_{35} \end{bmatrix}$$

$$s_1 = \frac{26\,\mu}{45\,a\,b}\,\left(a^2 + 2\,b^2\right) \qquad s_2 = \frac{17\,\mu}{36}$$

$$s_3 = \frac{\mu}{90\,a\,b}\,\left(17\,a^2 + 56\,b^2\right) \qquad s_4 = \frac{\mu}{12}$$

$$s_5 = \frac{23\,\mu}{90\,a\,b}\,\left(a^2 + 2\,b^2\right) \qquad s_6 = \frac{7\,\mu}{36}$$

$$s_7 = \frac{\mu}{45\,a\,b}\,\left(14\,a^2 + 17\,b^2\right) \qquad s_8 = -\frac{\mu}{12}$$

$$s_9 = \frac{\mu}{45\,a\,b}\,\left(3\,a^2 - 80\,b^2\right) \qquad s_{10} = -\frac{5\,\mu}{9}$$

$$s_{11} = -\frac{2\,\mu}{45\,a\,b}\,\left(10\,a^2 + 3\,b^2\right) \qquad s_{12} = -\frac{\mu}{9}$$

$$s_{13} = -\frac{\mu}{45\,a\,b}\,\left(3\,a^2 + 40\,b^2\right) \qquad s_{14} = -\frac{2\,\mu}{45\,a\,b}\,\left(20\,a^2 - 3\,b^2\right)$$

$$s_{15} = \frac{\mu}{9} \qquad s_{16} = \frac{26\,\mu}{45\,a\,b}\,\left(2\,a^2 + b^2\right)$$

$$s_{17} = \frac{\mu}{45\,a\,b}\,\left(17\,a^2 + 14\,b^2\right) \qquad s_{18} = \frac{23\,\mu}{90\,a\,b}\,\left(2\,a^2 + b^2\right)$$

$$s_{19} = \frac{\mu}{90\,a\,b}\,\left(56\,a^2 + 17\,b^2\right) \qquad s_{20} = \frac{2\,\mu}{45\,a\,b}\,\left(3\,a^2 - 20\,b^2\right)$$

$$s_{21} = -\frac{\mu}{45\,a\,b}\,\left(40\,a^2 + 3\,b^2\right) \qquad s_{22} = -\frac{2\,\mu}{45\,a\,b}\,\left(3\,a^2 + 10\,b^2\right)$$

$$s_{23} = -\frac{\mu}{45\,a\,b}\left(80\,a^2 - 3\,b^2\right) \qquad s_{24} = -\frac{17\,\mu}{36}$$

$$s_{25} = -\frac{7\,\mu}{36} \qquad s_{26} = \frac{5\,\mu}{9}$$

$$s_{27} = \frac{8\,\mu}{45\,a\,b}\left(3\,a^2 + 20\,b^2\right) \qquad s_{28} = -\frac{4\,\mu}{9}$$

$$s_{29} = -\frac{8\,\mu}{45\,a\,b}\left(3\,a^2 - 10\,b^2\right) \qquad s_{30} = \frac{4\,\mu}{9}$$

$$s_{31} = \frac{16\,\mu}{45\,a\,b}\left(3\,a^2 + 5\,b^2\right) \qquad s_{32} = -\frac{8\,\mu}{45\,a\,b}\left(6\,a^2 - 5\,b^2\right)$$

$$s_{33} = \frac{16\,\mu}{45\,a\,b}\left(5\,a^2 + 3\,b^2\right) \qquad s_{34} = \frac{8\,\mu}{45\,a\,b}\left(5\,a^2 - 6\,b^2\right)$$

$$s_{35} = \frac{8\,\mu}{45\,a\,b}\left(20\,a^2 + 3\,b^2\right) \qquad s_{36} = \frac{8\,\mu}{45\,a\,b}\left(10\,a^2 - 3\,b^2\right)$$

Coefficient matrix, $[\,c\,]$

Coefficient matrix, $[\,c\,]$ involves IBP thus the matrix multiplication must be carried out from left to right.

Step 1

$$\ldots\begin{bmatrix} \vdots & \vdots \\ N_2 & 0 \\ \vdots & \vdots \end{bmatrix}\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix}\ldots \Rightarrow \ldots\begin{Bmatrix} \vdots \\ \dfrac{\partial N_2}{\partial x} \\ \vdots \end{Bmatrix}\ldots \tag{5.81}$$

Step 2

$$\ldots\begin{Bmatrix} \vdots \\ \dfrac{\partial N_2}{\partial x} \\ \vdots \end{Bmatrix}\{\cdots \quad L_4 \quad \cdots\}\ldots \Rightarrow \ldots\frac{\partial N_2}{\partial x}L_4\ldots \tag{5.82}$$

Step 3 (analytical integration)

$$\begin{aligned} c_{34} &= \int_x \int_y \frac{\partial N_2}{\partial x} L_4 \, dy \, dx \\ &= \int_0^a \int_0^b \left(\frac{4\,x}{a^2} - \frac{1}{a} - \frac{y}{a\,b} + \frac{2\,y^2}{a\,b^2} - \frac{4\,x\,y}{a^2\,b} \right) \left(\frac{y\,(a-x)}{a\,b} \right) dy\,dx \\ &= -\frac{b}{18} \end{aligned} \tag{5.83}$$

Repeating the procedure above (Step 1 to Step 5) for all components, the complete integrated values of $[\,c\,]$ can be given as

$$[\,c\,] = \begin{bmatrix} s_1 & s_{15} & s_{17} & s_5 \\ s_2 & s_4 & s_{20} & s_{21} \\ s_3 & s_{16} & s_{17} & s_5 \\ s_4 & s_2 & s_{21} & s_{20} \\ s_5 & s_{17} & s_{16} & s_3 \\ s_4 & s_6 & s_{22} & s_{20} \\ s_5 & s_{17} & s_{15} & s_1 \\ s_6 & s_4 & s_{20} & s_{22} \\ s_7 & s_{18} & s_{19} & s_{11} \\ s_8 & s_8 & s_8 & s_8 \\ s_9 & s_9 & s_9 & s_9 \\ s_{10} & s_{14} & s_{23} & s_{24} \\ s_{11} & s_{19} & s_{18} & s_7 \\ s_{12} & s_{12} & s_{12} & s_{12} \\ s_{13} & s_{13} & s_{13} & s_{13} \\ s_{14} & s_{10} & s_{24} & s_{23} \end{bmatrix} \tag{5.84}$$

where

$$s_1 = -\frac{7\,b}{36} \qquad s_2 = -\frac{7\,a}{36} \qquad s_3 = -\frac{b}{36} \qquad s_4 = -\frac{a}{18}$$

$$s_5 = -\frac{b}{18} \qquad s_6 = -\frac{a}{36} \qquad s_7 = \frac{2\,b}{9} \qquad s_8 = -\frac{a}{6}$$

$$s_9 = \frac{b}{6} \qquad s_{10} = \frac{a}{9} \qquad s_{11} = \frac{b}{9} \qquad s_{12} = \frac{a}{6}$$

$$s_{13} = -\frac{b}{6} \qquad s_{14} = \frac{2\,a}{9} \qquad s_{15} = \frac{b}{36} \qquad s_{16} = \frac{7\,b}{36}$$

$$s_{17} = \frac{b}{18} \qquad s_{18} = -\frac{2\,b}{9} \qquad s_{19} = -\frac{b}{9} \qquad s_{20} = \frac{a}{18}$$

$$s_{21} = \frac{a}{36} \qquad s_{22} = \frac{7\,a}{36} \qquad s_{23} = -\frac{2\,a}{9} \qquad s_{24} = -\frac{a}{9}$$

Nonlinear matrix, $[k_N]$

Nonlinear matrix, $[k_N]$ does not involves IBP thus the matrix multiplication must be carried out from right to left.

Step 1

$$\dots\left\{\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right\}\begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & N_2 & \cdots \end{bmatrix}\dots \Rightarrow \dots\left\{\cdots \quad \frac{\partial N_2}{\partial y} \quad \cdots\right\}\dots \tag{5.85}$$

Step 2

$$\cdots \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix} \left\{ \cdots \quad \frac{\partial N_2}{\partial y} \quad \cdots \right\} \cdots \Rightarrow \cdots \left[\cdots \quad \frac{\partial N_2}{\partial y} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix} \quad \cdots \right] \cdots \tag{5.86}$$

Step 3

$$\cdots \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_8 \end{bmatrix} \left[\cdots \quad \frac{\partial N_2}{\partial y} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix} \quad \cdots \right] \cdots$$

$$\Rightarrow \cdots \begin{bmatrix} N_1 \dfrac{\partial N_2}{\partial y} u_1 + N_2 \dfrac{\partial N_2}{\partial y} u_2 + N_3 \dfrac{\partial N_2}{\partial y} u_3 + \dots + N_8 \dfrac{\partial N_2}{\partial y} u_8 \\ N_1 \dfrac{\partial N_2}{\partial y} v_1 + N_2 \dfrac{\partial N_2}{\partial y} v_2 + N_3 \dfrac{\partial N_2}{\partial y} v_3 + \dots + N_8 \dfrac{\partial N_2}{\partial y} v_8 \end{bmatrix} \cdots \tag{5.87}$$

Step 4

$$\cdots\begin{bmatrix} \vdots & \vdots \\ N_2 & 0 \\ \vdots & \vdots \end{bmatrix}\begin{bmatrix} N_1\dfrac{\partial N_2}{\partial y}u_1 + N_2\dfrac{\partial N_2}{\partial y}u_2 + \ \ldots \ + N_8\dfrac{\partial N_2}{\partial y}u_8 \\ N_1\dfrac{\partial N_2}{\partial y}v_1 + N_2\dfrac{\partial N_2}{\partial y}v_2 + \ \ldots \ + N_8\dfrac{\partial N_2}{\partial y}v_8 \end{bmatrix}\cdots$$

$$\Rightarrow \ldots\left(N_2N_1\frac{\partial N_2}{\partial y}u_1 + N_2N_2\frac{\partial N_2}{\partial y}u_2 + \ \ldots \ + N_2N_8\frac{\partial N_2}{\partial y}u_8\right)\ldots \tag{5.88}$$

Step 5 (analytical integration)

$$k_{N,34} = \rho\int_0^a\int_0^b\left(N_2N_1\frac{\partial N_2}{\partial y}u_1 + \ \ldots \ + N_2N_8\frac{\partial N_2}{\partial y}u_8\right)dy\,dx$$

$$= \frac{a\rho}{12600}\left(72u_1 - 390u_2 + 43u_3 + 15u_4 - 172u_5 - 238u_6 - 128u_7 - 42u_8\right) \tag{5.89}$$

Repeating the procedure above (Step 1 to Step 5) for all components, the complete integrated values of $[\,k_N\,]$ can be given as

$$[\,k_N\,] = \begin{bmatrix} S_a & | & S_b \\ -- & -- & -- \\ S_c & | & S_d \end{bmatrix} \tag{5.90}$$

where

$$[\,S_a\,] = \begin{bmatrix} s_1 & s_{17} & s_{33} & s_{49} & s_{65} & s_{79} & s_{93} & s_{107} \\ s_2 & s_{18} & s_{34} & s_{50} & s_{66} & s_{80} & s_{94} & s_{108} \\ s_3 & s_{19} & s_{35} & s_{51} & s_{67} & s_{81} & s_{95} & s_{109} \\ s_4 & s_{20} & s_{36} & s_{52} & s_{68} & s_{82} & s_{96} & s_{110} \\ s_5 & s_{21} & s_{37} & s_{53} & s_{69} & s_{83} & s_{97} & s_{111} \\ s_6 & s_{22} & s_{38} & s_{54} & s_{70} & s_{84} & s_{98} & s_{112} \\ s_7 & s_{23} & s_{39} & s_{55} & s_{71} & s_{85} & s_{99} & s_{113} \\ s_8 & s_{24} & s_{40} & s_{56} & s_{72} & s_{86} & s_{100} & s_{114} \end{bmatrix}$$

$$[\,S_b\,] = \begin{bmatrix} s_{121} & s_{137} & s_{153} & s_{169} & s_{185} & s_{199} & s_{215} & s_{231} \\ s_{122} & s_{138} & s_{154} & s_{170} & s_{186} & s_{200} & s_{216} & s_{232} \\ s_{123} & s_{139} & s_{155} & s_{171} & s_{187} & s_{201} & s_{217} & s_{233} \\ s_{124} & s_{140} & s_{156} & s_{172} & s_{188} & s_{202} & s_{218} & s_{234} \\ s_{125} & s_{141} & s_{157} & s_{173} & s_{189} & s_{203} & s_{219} & s_{235} \\ s_{126} & s_{142} & s_{158} & s_{174} & s_{190} & s_{204} & s_{220} & s_{236} \\ s_{127} & s_{143} & s_{159} & s_{175} & s_{191} & s_{205} & s_{221} & s_{237} \\ s_{128} & s_{144} & s_{160} & s_{176} & s_{192} & s_{206} & s_{222} & s_{238} \end{bmatrix}$$

$$[\,S_c\,] = \begin{bmatrix} s_9 & s_{25} & s_{41} & s_{57} & s_{13} & s_{87} & s_{45} & s_{115} \\ s_{10} & s_{26} & s_{42} & s_{58} & s_{14} & s_{88} & s_{46} & s_{116} \\ s_{11} & s_{27} & s_{43} & s_{59} & s_{73} & s_{89} & s_{101} & s_{63} \\ s_{12} & s_{28} & s_{44} & s_{60} & s_{74} & s_{90} & s_{102} & s_{64} \\ s_{13} & s_{29} & s_{45} & s_{61} & s_{75} & s_{91} & s_{103} & s_{117} \\ s_{14} & s_{30} & s_{46} & s_{62} & s_{76} & s_{92} & s_{104} & s_{118} \\ s_{15} & s_{31} & s_{47} & s_{63} & s_{77} & s_{27} & s_{105} & s_{119} \\ s_{16} & s_{32} & s_{48} & s_{64} & s_{78} & s_{28} & s_{106} & s_{120} \end{bmatrix}$$

$$[\,S_d\,] = \begin{bmatrix} s_{129} & s_{145} & s_{161} & s_{177} & s_{133} & s_{207} & s_{223} & s_{239} \\ s_{130} & s_{146} & s_{162} & s_{178} & s_{134} & s_{208} & s_{224} & s_{240} \\ s_{131} & s_{147} & s_{163} & s_{179} & s_{193} & s_{209} & s_{225} & s_{183} \\ s_{132} & s_{148} & s_{164} & s_{180} & s_{194} & s_{210} & s_{226} & s_{184} \\ s_{133} & s_{149} & s_{165} & s_{181} & s_{195} & s_{211} & s_{227} & s_{241} \\ s_{134} & s_{150} & s_{166} & s_{182} & s_{196} & s_{212} & s_{228} & s_{242} \\ s_{135} & s_{151} & s_{167} & s_{183} & s_{197} & s_{213} & s_{229} & s_{243} \\ s_{136} & s_{152} & s_{168} & s_{184} & s_{198} & s_{214} & s_{230} & s_{244} \end{bmatrix}$$

(details for s_1 to s_{244} are given in the appendix).

5.4.2 All the Matrices are Combined Together

The continuity equation and the momentum equations, as given by Eqs. (5.70) and (5.71), respectively can be combined together in the following way. If the dof are grouped together in such a way that

$$\{u\} = \begin{Bmatrix} \{\hat{u}\}^T \\ \{\hat{p}\}^T \end{Bmatrix} \tag{5.91}$$

then Eqs. (5.70) and (5.71) can be arranged as

$$\begin{bmatrix} [\,m\,] & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \{\dot{\hat{u}}\}^T \\ \{\dot{\hat{p}}\}^T \end{Bmatrix} + \begin{bmatrix} [\,k\,] + [\,k_N\,] & -[\,c\,] \\ -[\,c\,]^T & 0 \end{bmatrix} \begin{Bmatrix} \{\hat{u}\}^T \\ \{\hat{p}\}^T \end{Bmatrix} = \begin{Bmatrix} \{f_B\} + \{b\} \\ 0 \end{Bmatrix} \tag{5.92}$$

Finally Eq. (5.92) can be given as

$$[\,M\,]\{\dot{U}\} + [\,K(U)\,]\{U\} = \{R\} \tag{5.93}$$

where the use of capital letters are not only to highlight the representation of the various sub-matrices by a combined matrix where

$$[\,K(U)\,] = \begin{bmatrix} [\,k\,] + [\,k_N\,] & -[\,c\,] \\ -[\,c\,]^T & 0 \end{bmatrix} \tag{5.94}$$

and

$$[\,M\,] = \begin{bmatrix} [\,m\,] & 0 \\ 0 & 0 \end{bmatrix} \tag{5.95}$$

but also to highlight that, our discussion from hereafter assumes that the elements have been assembled.

To note, $[\,K(U)\,]$ is expressed in such a way so as to remind that the matrix contains nonlinear matrix $[\,k_N\,]$ which in turn contains the dofs, $\{U\}$ hence the nonlinear nature.

Next, we are going to discuss about time integration and nonlinear scheme required in the solution of Eq. (5.93).

5.5 Time Integration

Eq. (5.93) involves the first order derivative of the dof's in time, expressed by $\{\dot{U}\}$. As a time derivative, it can be discretized, which the common practice would be, by finite difference method. Basically there are two ways to do this;

i. Forward Euler scheme
ii. Backward Euler scheme

Despite the two available techniques, herein, we will concern only with Backward Euler scheme. Readers who are interested to know about the other technique can refer to books on finite difference method and finite element method.

Based on Backward Euler, Eq. (5.93) can be discretized in time as

$$\left[\frac{1}{\Delta_t}\,[\,M\,] + \left[\,K(U^{t+1})\,\right]\right]\{U^{t+1}\} = \frac{1}{\Delta_t}\,[\,M\,]\,\{U^t\} + \{R^{t+1}\} \tag{5.96}$$

where Δ_t is a time step interval, $t+1$ refers to the present time step and t represents the previous time step.

Eq. (5.96) is the discretized equation of Navier-Stokes in time by Backward Euler (finite difference). However, the equation is yet to be solved as it is nonlinear. We must resort to a nonlinear solution scheme which is to be discussed next.

5.6 Picard Nonlinear Scheme (Direct Substitution)

In previous chapter, the concept of nonlinearity and iterative schemes were detailed. Although two schemes were discussed, herein, in avoiding the lengthy discussion, especially on the derivation of the tangent stiffness matrix, only Picard scheme is demonstrated.

Eq. (5.96) can be given conveniently for nonlinear discussion as

$$\left[\,\bar{K}(U^{\,t+1})\,\right]\{U^{\,t+1}\} = \{\bar{R}^{\,t+1}(U^t)\} \tag{5.97}$$

where

$$\left[\,\bar{K}(U^{\,t+1})\,\right] = \left[\frac{1}{\Delta_t}\,[\,M\,] + \left[\,K(U^{\,t+1})\,\right]\right] \tag{5.98}$$

$$\{\bar{R}^{\,t+1}(U^{\,t})\} = \frac{1}{\Delta_t}\,[\,M\,]\,\{U^{\,t}\} + \{R^{\,t+1}\} \tag{5.99}$$

Picard scheme is the most straightforward scheme which is also known as direct substitution method. To solve Eq. (5.97) by Picard scheme, we use previous iteration values of U i.e. $U^{\,t+1,n}$ where n represents the last iteration, in the construction of the matrix, $\left[\,\bar{K}(U^{\,t+1})\,\right]$. With this, Eq. (5.97) can be re-represented as

$$\left[\,\bar{K}(U^{\,t+1,\,n})\,\right]\{U^{\,t+1,\,n+1}\} = \{\bar{R}^{\,t+1,\,n}\} \tag{5.100}$$

where $n+1$ represent current stage of iteration. Having established Eq. (5.100), we are now all set to solve the problem. We need to iterate until the solution $\{U^{\,t+1}\}$ is converged. Once converged, we then go to the next time step. A step-by-step procedure is given as follows.

i. take or assume known values of $\{U^{t+1,n}\}$
ii. use these known values of $\{U^{t+1,n}\}$ to calculate $[K(U^{t+1,n})]$
iii. solve $\{U^{t+1,n+1}\}$
iv. calculate residual and check for convergence
v. if converged, stop iteration and go the next time step, else repeat step ii. by using the just solved $\{U^{t+1,n+1}\}$ as $\{U^{t+1,n}\}$

5.7 Example

Herein, we are going to demonstrate the FEM formulation of Navier-Stokes equations by solving a benchmark problem known as lid-driven cavity. For a step-by-step demonstration (hand-calculation), only four assembled elements are used, so as to allow for easy tracing of the procedure. However, it must be noted that such usage is for demonstration purpose only as the result will be very poor. After the step-by-step calculation, the procedure is run using a sufficient number of elements (denser mesh). The results are then validated against an established data for proof of correctness.

Consider the lid-driven cavity as in Fig. 5.4 below.

The parameters used in this problem are:

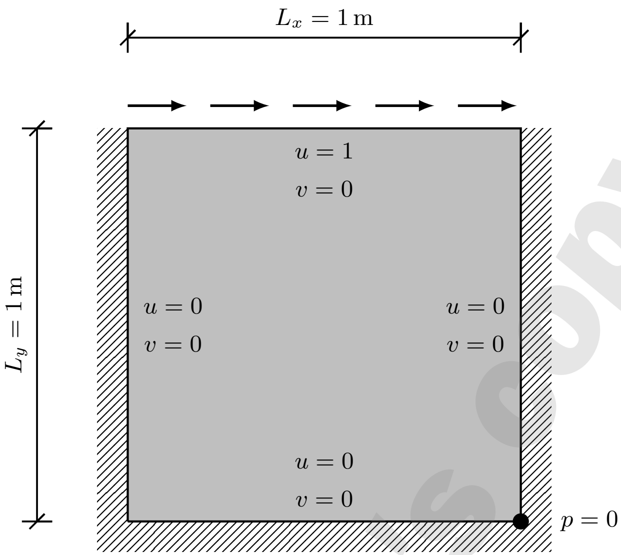

Figure 5.4: Lid-driven cavity problem

- Density, $\rho = 1\,\mathrm{kg\,m^{-3}}$
- Viscosity, $\mu = 1.1\,\mathrm{Pa \cdot s}$
- Lid velocity, $u_{lid} = 1\,\mathrm{m\,s^{-1}}$
- Time step, $\Delta t = 0.1\,\mathrm{s}$

Fig. 5.5 shows the node and element numbering whilst Fig. 5.6 shows the numbering of the global dofs.

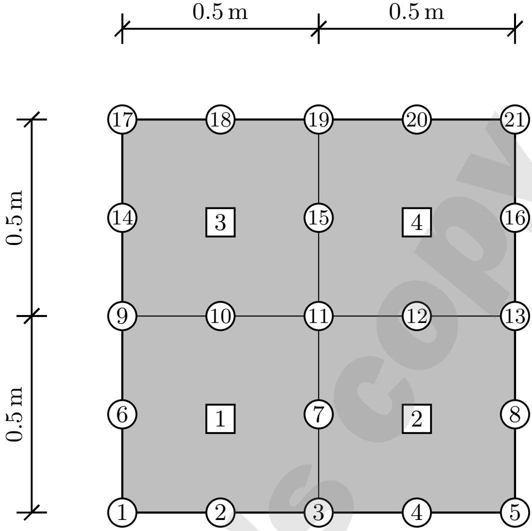

Figure 5.5: The arrangement of node and element numbering

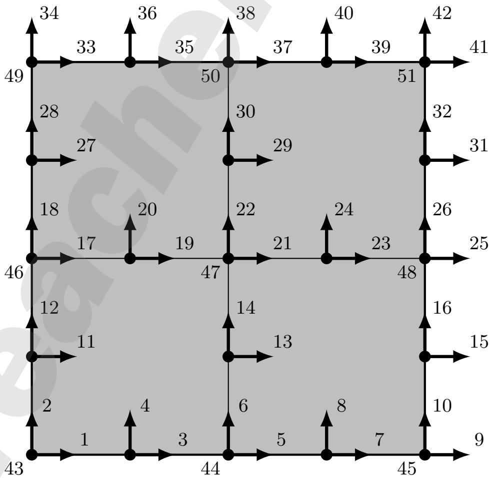

Figure 5.6: The arrangement of global dofs

Time step 1, Iteration 1:

Due to the symmetricality, each element would have similar local stiffness matrix $[\,k\,]$, mass matrix $[\,m\,]$, coefficient matrix, $[\,c\,]$, and nonlinear matrix $[k_N]$.

Based on Eq. (5.80), local stiffness matrix $[\,k\,]$ is given as

$$
[\,k\,] = \begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 15 & 16 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \\ 15 \\ 16 \end{array} &
\begin{bmatrix}
1.907 & 0.519 & 0.892 & \ldots & -0.831 & 0.122 \\
0.519 & 1.907 & -0.092 & \ldots & -0.611 & -1.882 \\
0.892 & -0.092 & 1.907 & \ldots & -0.636 & 0.122 \\
0.092 & 0.758 & -0.519 & \ldots & 0.122 & -1.051 \\
0.843 & 0.214 & 0.758 & \ldots & -0.636 & -0.122 \\
0.214 & 0.843 & 0.092 & \ldots & -0.122 & -1.051 \\
0.758 & 0.092 & 0.843 & \ldots & -0.831 & -0.122 \\
-0.092 & 0.892 & -0.214 & \ldots & 0.611 & -1.882 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-0.831 & -0.611 & -0.636 & \ldots & 3.129 & 0.000 \\
0.122 & -1.882 & 0.122 & \ldots & 0.000 & 4.498
\end{bmatrix}
\end{array} \tag{5.101}
$$

Using Eq. (5.73), local mass matrix $[\,m\,]$ is given as

$$
[\,m\,] = \begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 15 & 16 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \\ 15 \\ 16 \end{array} &
\begin{bmatrix}
0.833 & 0.000 & 0.278 & \ldots & -0.833 & 0.000 \\
0.000 & 0.833 & 0.000 & \ldots & 0.000 & -0.833 \\
0.278 & 0.000 & 0.833 & \ldots & -1.111 & 0.000 \\
0.000 & 0.278 & 0.000 & \ldots & 0.000 & -1.111 \\
0.417 & 0.000 & 0.278 & \ldots & -1.111 & 0.000 \\
0.000 & 0.417 & 0.000 & \ldots & 0.000 & -1.111 \\
0.278 & 0.000 & 0.417 & \ldots & -0.833 & 0.000 \\
0.000 & 0.278 & 0.000 & \ldots & 0.000 & -0.833 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-0.833 & 0.000 & -1.111 & \ldots & 4.444 & 0.000 \\
0.000 & -0.833 & 0.000 & \ldots & 0.000 & 4.444
\end{bmatrix}
\end{array} \times 10^{-2} \tag{5.102}
$$

Using on Eq. (5.84), coefficient matrix $[\,c\,]$ is given as

$$[\,c\,] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \\ 15 \\ 16 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\ \left[\begin{array}{rrrr} -0.097 & 0.014 & 0.028 & -0.028 \\ -0.097 & -0.028 & 0.028 & 0.014 \\ -0.014 & 0.097 & 0.028 & -0.028 \\ -0.028 & -0.097 & 0.014 & 0.028 \\ -0.028 & 0.028 & 0.097 & -0.014 \\ -0.028 & -0.014 & 0.097 & 0.028 \\ -0.028 & 0.028 & 0.014 & -0.097 \\ -0.014 & -0.028 & 0.028 & 0.097 \\ \vdots & \vdots & \vdots & \vdots \\ -0.083 & -0.083 & -0.083 & -0.083 \\ 0.111 & 0.056 & -0.056 & -0.111 \end{array} \right] \end{array} \tag{5.103}$$

By assuming an initial condition of $\{U\}^{1,0} = 0$, based on Eq. (5.66), non-linear matrix $[k_N]$ is simply zeros.

$$[k_N] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \\ 15 \\ 16 \end{array} \begin{array}{c} \begin{array}{cccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \cdots & \quad 15 \quad & \quad 16 \quad \end{array} \\ \left[\begin{array}{cccccc} 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & \ldots & 0.000 & 0.000 \end{array} \right] \end{array} \tag{5.104}$$

Since the size of an assembled stiffness matrix would be very large, herein, reduced size of assembled stiffness matrix with boundary conditions have been imposed, are presented (i.e. $u_1 = v_1 = u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = u_5 = v_5 = u_6 = v_6 = u_8 = v_8 = u_9 = v_9 = u_{13} = v_{13} = u_{14} = v_{14} = u_{16} = v_{16} = u_{17} = v_{17} = u_{21} = v_{21} = p_1 = p_2 = p_3 = p_4 = p_6 = p_7 = p_9 = 0$). We obtain the matrix system as given by

$$
\begin{array}{c} \\ 13 \\ 14 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 29 \\ 30 \\ 35 \\ 36 \\ 37 \\ 38 \\ 39 \\ 40 \\ 47 \\ 50 \end{array}
\begin{array}{c}
\begin{array}{ccccc} \quad 13 & \quad 14 & \cdots & \quad 47 & \quad 50 \end{array} \\
\begin{bmatrix}
7.147 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & 9.884 & \dots & 0.222 & 0.000 \\
0.278 & 0.489 & \dots & 0.222 & 0.056 \\
0.489 & 0.278 & \dots & 0.000 & 0.083 \\
-1.829 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & -3.931 & \dots & 0.000 & -0.028 \\
0.278 & -0.489 & \dots & -0.222 & -0.056 \\
-0.489 & 0.278 & \dots & 0.000 & 0.083 \\
0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & -0.222 & 0.222 \\
0.000 & 0.000 & \dots & 0.056 & 0.111 \\
0.000 & 0.000 & \dots & -0.083 & -0.083 \\
0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & 0.028 & -0.194 \\
0.000 & 0.000 & \dots & -0.056 & -0.111 \\
0.000 & 0.000 & \dots & -0.083 & -0.083 \\
0.000 & 0.222 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & 0.000 & 0.000
\end{bmatrix}
\end{array}
\begin{Bmatrix} u_7^{1,1} \\ v_7^{1,1} \\ u_{10}^{1,1} \\ v_{10}^{1,1} \\ u_{11}^{1,1} \\ v_{11}^{1,1} \\ u_{12}^{1,1} \\ v_{12}^{1,1} \\ u_{15}^{1,1} \\ v_{15}^{1,1} \\ u_{18}^{1,1} \\ v_{18}^{1,1} \\ u_{19}^{1,1} \\ v_{19}^{1,1} \\ u_{20}^{1,1} \\ v_{20}^{1,1} \\ p_5^{1,1} \\ p_8^{1,1} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
\tag{5.105}
$$

By solving Eq. (5.105) using Matlab command "\", $\{U^{1,1}\}$ are thus obtained as

$$\{U^{1,1}\} = \begin{Bmatrix} u_7^{1,1} \\ v_7^{1,1} \\ u_{10}^{1,1} \\ v_{10}^{1,1} \\ u_{11}^{1,1} \\ v_{11}^{1,1} \\ u_{12}^{1,1} \\ v_{12}^{1,1} \\ u_{15}^{1,1} \\ v_{15}^{1,1} \\ u_{18}^{1,1} \\ v_{18}^{1,1} \\ u_{19}^{1,1} \\ v_{19}^{1,1} \\ u_{20}^{1,1} \\ v_{20}^{1,1} \\ p_5^{1,1} \\ p_8^{1,1} \end{Bmatrix} = \begin{Bmatrix} 0.017 \\ 0.000 \\ 0.075 \\ 0.057 \\ 0.119 \\ 0.000 \\ 0.075 \\ -0.057 \\ 0.391 \\ 0.000 \\ 1.137 \\ 0.265 \\ 1.886 \\ 0.000 \\ 1.137 \\ -0.265 \\ 0.000 \\ 0.000 \end{Bmatrix} \tag{5.106}$$

The maximum residual between current and previous result are 2.5431.

Time step 1, Iteration 2:

Using the result from previous iteration, the nonlinear matrix $[k_N]$ is given as

$$
[k_N] = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \cdots & \quad 15 \quad & \quad 16 \quad \end{array} \\
\begin{bmatrix}
-0.028 & -0.085 & -0.009 & \dots & 0.299 & 0.224 \\
0.000 & 0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.009 & -0.048 & 0.028 & \dots & 0.299 & 0.150 \\
0.000 & 0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.028 & -0.021 & 0.009 & \dots & 0.262 & 0.150 \\
0.000 & 0.000 & 0.000 & \dots & 0.000 & 0.000 \\
-0.009 & 0.035 & -0.028 & \dots & 0.262 & -0.150 \\
0.000 & 0.000 & 0.000 & \dots & 0.000 & 0.000 \\
-0.075 & -0.053 & 0.075 & \dots & -0.598 & 0.000 \\
0.000 & 0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.150 & 0.150 & 0.150 & \dots & -0.748 & -0.299 \\
0.000 & 0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.075 & 0.641 & -0.075 & \dots & -0.898 & -1.496 \\
0.000 & 0.000 & 0.000 & \dots & 0.000 & 0.000 \\
-0.150 & 0.037 & -0.150 & \dots & -0.748 & -0.449 \\
0.000 & 0.000 & 0.000 & \dots & 0.000 & 0.000
\end{bmatrix}
\end{array}
\times 10^{-3}
\tag{5.107}
$$

Thus, the assembled stiffness matrix can be given as

$$
\begin{array}{c}
 \\ 13 \\ 14 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 29 \\ 30 \\ 35 \\ 36 \\ 37 \\ 38 \\ 39 \\ 40 \\ 47 \\ 50
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 13 & 14 & \cdots & 47 & 50 \end{array} \\
\begin{bmatrix}
7.147 & -0.001 & \dots & 0.000 & 0.000 \\
0.000 & 9.884 & \dots & 0.222 & 0.000 \\
0.279 & 0.487 & \dots & 0.222 & 0.056 \\
0.489 & 0.278 & \dots & 0.000 & 0.083 \\
-1.829 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & -3.931 & \dots & 0.000 & -0.028 \\
0.277 & -0.490 & \dots & -0.222 & -0.056 \\
-0.489 & 0.278 & \dots & 0.000 & 0.083 \\
0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & -0.222 & 0.222 \\
0.000 & 0.000 & \dots & 0.056 & 0.111 \\
0.000 & 0.000 & \dots & -0.083 & -0.083 \\
0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & 0.028 & -0.194 \\
0.000 & 0.000 & \dots & -0.056 & -0.111 \\
0.000 & 0.000 & \dots & -0.083 & -0.083 \\
0.000 & 0.222 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & 0.000 & 0.000
\end{bmatrix}
\end{array}
\begin{Bmatrix}
u_7^{1,2} \\ v_7^{1,2} \\ u_{10}^{1,2} \\ v_{10}^{1,2} \\ u_{11}^{1,2} \\ v_{11}^{1,2} \\ u_{12}^{1,2} \\ v_{12}^{1,2} \\ u_{15}^{1,2} \\ v_{15}^{1,2} \\ u_{18}^{1,2} \\ v_{18}^{1,2} \\ u_{19}^{1,2} \\ v_{19}^{1,2} \\ u_{20}^{1,2} \\ v_{20}^{1,2} \\ p_5^{1,2} \\ p_8^{1,2}
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0
\end{Bmatrix}
\tag{5.108}
$$

By solving Eq. (5.108) using Matlab command "\", $\{U^{1,2}\}$ are thus obtained as

$$\{U^{1,2}\} = \begin{Bmatrix} u_7^{1,2} \\ v_7^{1,2} \\ u_{10}^{1,2} \\ v_{10}^{1,2} \\ u_{11}^{1,2} \\ v_{11}^{1,2} \\ u_{12}^{1,2} \\ v_{12}^{1,2} \\ u_{15}^{1,2} \\ v_{15}^{1,2} \\ u_{18}^{1,2} \\ v_{18}^{1,2} \\ u_{19}^{1,2} \\ v_{19}^{1,2} \\ u_{20}^{1,2} \\ v_{20}^{1,2} \\ p_5^{1,2} \\ p_8^{1,2} \end{Bmatrix} = \begin{Bmatrix} 0.017 \\ 0.000 \\ 0.075 \\ 0.057 \\ 0.119 \\ 0.000 \\ 0.075 \\ -0.057 \\ 0.391 \\ 0.000 \\ 1.137 \\ 0.265 \\ 1.886 \\ 0.000 \\ 1.137 \\ -0.265 \\ -0.001 \\ -0.003 \end{Bmatrix} \tag{5.109}$$

The maximum residual between current and previous result are 0.0034.

For the purpose of demonstration, let assume that the result has converged. We can now break the nonlinear iteration loop and calculate the value at the second time step using the current results as the starting values value.

Time step 2, Iteration 1:

Using the result from previous iteration, the nonlinear matrix $[k_N]$ is given as

$$
[k_N] = \begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 15 & 16 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{array} &
\begin{bmatrix}
-0.028 & -0.085 & -0.009 & \dots & 0.299 & 0.224 \\
0.000 & 0.000 & 0.000 & \dots & 0.001 & 0.001 \\
0.009 & -0.048 & 0.028 & \dots & 0.299 & 0.150 \\
0.000 & 0.000 & 0.000 & \dots & 0.001 & 0.001 \\
0.028 & -0.021 & 0.009 & \dots & 0.262 & 0.150 \\
0.000 & 0.000 & 0.000 & \dots & 0.001 & 0.001 \\
-0.009 & 0.035 & -0.028 & \dots & 0.262 & -0.150 \\
0.000 & 0.000 & 0.000 & \dots & 0.001 & -0.001 \\
-0.075 & -0.053 & 0.075 & \dots & -0.598 & 0.000 \\
0.000 & 0.000 & 0.000 & \dots & -0.002 & 0.000 \\
0.150 & 0.150 & 0.150 & \dots & -0.748 & -0.299 \\
0.001 & 0.001 & 0.001 & \dots & -0.003 & -0.001 \\
0.075 & 0.641 & -0.075 & \dots & -0.898 & -1.496 \\
0.000 & 0.002 & 0.000 & \dots & -0.003 & -0.005 \\
-0.150 & 0.037 & -0.150 & \dots & -0.748 & -0.449 \\
-0.001 & 0.000 & -0.001 & \dots & -0.003 & -0.002
\end{bmatrix}
\end{array} \times 10^{-3} \tag{5.110}
$$

Thus, the assembled stiffness matrix can be given as

$$
\begin{array}{c}
\\ 13 \\ 14 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 29 \\ 30 \\ 35 \\ 36 \\ 37 \\ 38 \\ 39 \\ 40 \\ 47 \\ 50
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 13 & 14 & \cdots & 47 & 50 \end{array} \\
\left[\begin{array}{rrcrr}
7.147 & -0.001 & \dots & 0.000 & 0.000 \\
0.000 & 9.884 & \dots & 0.222 & 0.000 \\
0.279 & 0.487 & \dots & 0.222 & 0.056 \\
0.489 & 0.278 & \dots & 0.000 & 0.083 \\
-1.829 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & -3.931 & \dots & 0.000 & -0.028 \\
0.277 & -0.490 & \dots & -0.222 & -0.056 \\
-0.489 & 0.278 & \dots & 0.000 & 0.083 \\
0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & -0.222 & 0.222 \\
0.000 & 0.000 & \dots & 0.056 & 0.111 \\
0.000 & 0.000 & \dots & -0.083 & -0.083 \\
0.000 & 0.000 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & 0.028 & -0.194 \\
0.000 & 0.000 & \dots & -0.056 & -0.111 \\
0.000 & 0.000 & \dots & -0.083 & -0.083 \\
0.000 & 0.222 & \dots & 0.000 & 0.000 \\
0.000 & 0.000 & \dots & 0.000 & 0.000
\end{array}\right]
\end{array}
\begin{Bmatrix}
u_7^{2,1} \\ v_7^{2,1} \\ u_{10}^{2,1} \\ v_{10}^{2,1} \\ u_{11}^{2,1} \\ v_{11}^{2,1} \\ u_{12}^{2,1} \\ v_{12}^{2,1} \\ u_{15}^{2,1} \\ v_{15}^{2,1} \\ u_{18}^{2,1} \\ v_{18}^{2,1} \\ u_{19}^{2,1} \\ v_{19}^{2,1} \\ u_{20}^{2,1} \\ v_{20}^{2,1} \\ p_5^{2,1} \\ p_8^{2,1}
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0
\end{Bmatrix}
\tag{5.111}
$$

By solving Eq. (5.111) using Matlab command "\", $\{U^{2,1}\}$ are thus obtained as:

$$\{U^{2,1}\} = \begin{Bmatrix} u_7^{2,1} \\ v_7^{2,1} \\ u_{10}^{2,1} \\ v_{10}^{2,1} \\ u_{11}^{2,1} \\ v_{11}^{2,1} \\ u_{12}^{2,1} \\ v_{12}^{2,1} \\ u_{15}^{2,1} \\ v_{15}^{2,1} \\ u_{18}^{2,1} \\ v_{18}^{2,1} \\ u_{19}^{2,1} \\ v_{19}^{2,1} \\ u_{20}^{2,1} \\ v_{20}^{2,1} \\ p_5^{2,1} \\ p_8^{2,1} \end{Bmatrix} = \begin{Bmatrix} 0.022 \\ 0.000 \\ 0.109 \\ 0.082 \\ 0.144 \\ 0.000 \\ 0.109 \\ -0.082 \\ 0.515 \\ 0.000 \\ 1.169 \\ 0.302 \\ 1.999 \\ 0.000 \\ 1.170 \\ -0.303 \\ -0.002 \\ -0.003 \end{Bmatrix} \tag{5.112}$$

The maximum residual between current and previous result are 0.1938.

Time step 2, Iteration 2:

Using the result from previous iteration, the nonlinear matrix $[k_N]$ is given as

$$[k_N] = \begin{array}{c} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16 \end{array} \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 15 & 16 \end{array} \\ \begin{bmatrix} -0.037 & -0.113 & -0.012 & \dots & 0.395 & 0.296 \\ 0.000 & 0.000 & 0.000 & \dots & 0.001 & 0.001 \\ 0.012 & -0.063 & 0.037 & \dots & 0.395 & 0.197 \\ 0.000 & 0.000 & 0.000 & \dots & 0.001 & 0.001 \\ 0.037 & -0.028 & 0.012 & \dots & 0.345 & 0.197 \\ 0.000 & 0.000 & 0.000 & \dots & 0.001 & 0.001 \\ -0.012 & 0.046 & -0.037 & \dots & 0.345 & -0.197 \\ 0.000 & 0.000 & 0.000 & \dots & 0.001 & -0.001 \\ -0.099 & -0.070 & 0.099 & \dots & -0.789 & 0.000 \\ 0.000 & 0.000 & 0.000 & \dots & -0.003 & 0.000 \\ 0.197 & 0.197 & 0.197 & \dots & -0.987 & -0.395 \\ 0.001 & 0.001 & 0.001 & \dots & -0.003 & -0.001 \\ 0.099 & 0.846 & -0.099 & \dots & -1.184 & -1.973 \\ 0.000 & 0.003 & 0.000 & \dots & -0.004 & -0.007 \\ -0.197 & 0.049 & -0.197 & \dots & -0.987 & -0.592 \\ -0.001 & 0.000 & -0.001 & \dots & -0.003 & -0.002 \end{bmatrix} \end{array} \times 10^{-3} \tag{5.113}$$

Thus, the assembled stiffness matrix can be given as

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 13 & 14 & \cdots & 47 & 50 \end{array} \\
\begin{array}{c} 13\\14\\19\\20\\21\\22\\23\\24\\29\\30\\35\\36\\37\\38\\39\\40\\47\\50 \end{array} &
\begin{bmatrix}
7.147 & -0.001 & \ldots & 0.000 & 0.000 \\
0.000 & 9.884 & \ldots & 0.222 & 0.000 \\
0.279 & 0.487 & \ldots & 0.222 & 0.056 \\
0.489 & 0.278 & \ldots & 0.000 & 0.083 \\
-1.829 & 0.000 & \ldots & 0.000 & 0.000 \\
0.000 & -3.931 & \ldots & 0.000 & -0.028 \\
0.277 & -0.491 & \ldots & -0.222 & -0.056 \\
-0.489 & 0.278 & \ldots & 0.000 & 0.083 \\
0.000 & 0.000 & \ldots & 0.000 & 0.000 \\
0.000 & 0.000 & \ldots & -0.222 & 0.222 \\
0.000 & 0.000 & \ldots & 0.056 & 0.111 \\
0.000 & 0.000 & \ldots & -0.083 & -0.083 \\
0.000 & 0.000 & \ldots & 0.000 & 0.000 \\
0.000 & 0.000 & \ldots & 0.028 & -0.194 \\
0.000 & 0.000 & \ldots & -0.056 & -0.111 \\
0.000 & 0.000 & \ldots & -0.083 & -0.083 \\
0.000 & 0.222 & \ldots & 0.000 & 0.000 \\
0.000 & 0.000 & \ldots & 0.000 & 0.000
\end{bmatrix}
\end{array}
\begin{Bmatrix}
u_7^{2,2} \\ v_7^{2,2} \\ u_{10}^{2,2} \\ v_{10}^{2,2} \\ u_{11}^{2,2} \\ v_{11}^{2,2} \\ u_{12}^{2,2} \\ v_{12}^{2,2} \\ u_{15}^{2,2} \\ v_{15}^{2,2} \\ u_{18}^{2,2} \\ v_{18}^{2,2} \\ u_{19}^{2,2} \\ v_{19}^{2,2} \\ u_{20}^{2,2} \\ v_{20}^{2,2} \\ p_5^{2,2} \\ p_8^{2,2}
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0
\end{Bmatrix}
\tag{5.114}
$$

By solving Eq. (5.114) using Matlab command "\", $\{U^{2,2}\}$ are thus obtained as

$$\{U^{2,2}\} = \begin{Bmatrix} u_7^{2,2} \\ v_7^{2,2} \\ u_{10}^{2,2} \\ v_{10}^{2,2} \\ u_{11}^{2,2} \\ v_{11}^{2,2} \\ u_{12}^{2,2} \\ v_{12}^{2,2} \\ u_{15}^{2,2} \\ v_{15}^{2,2} \\ u_{18}^{2,2} \\ v_{18}^{2,2} \\ u_{19}^{2,2} \\ v_{19}^{2,2} \\ u_{20}^{2,2} \\ v_{20}^{2,2} \\ p_5^{2,2} \\ p_8^{2,2} \end{Bmatrix} = \begin{Bmatrix} 0.022 \\ 0.000 \\ 0.109 \\ 0.082 \\ 0.144 \\ 0.000 \\ 0.109 \\ -0.082 \\ 0.515 \\ 0.000 \\ 1.169 \\ 0.302 \\ 1.999 \\ 0.000 \\ 1.170 \\ -0.303 \\ -0.002 \\ -0.004 \end{Bmatrix} \tag{5.115}$$

The maximum residual between current and previous result are 0.0012. The obtained results after total time $t = 10$ are validated herein for various mesh number against those obtained from commercial software. Table 5.1 gives the numerical values of the results for various mesh densities and those from commercial software.

Table 5.1: Validation of global displacement

	u_{10}	v_{10}	u_{11}	v_{11}	p_5
Mesh (4x4)	-0.6649	0.5292	-0.2030	0.0000	-45.629
Mesh (10x10)	-0.2589	0.3577	-0.4104	0.0000	-0.7403
Software	-0.2480	0.3481	-0.3964	0.0023	-0.8659

Fig. 5.7 and Fig. 5.8 show the comparison of plot of velocity and pressure contours, respectively between Matlab code and a commercial software.

Fig. 5.9 and Fig. 5.10 show the plot of velocity and pressure profiles at $x = 0.5$ unit and $y = 0.5$ unit, respectively.

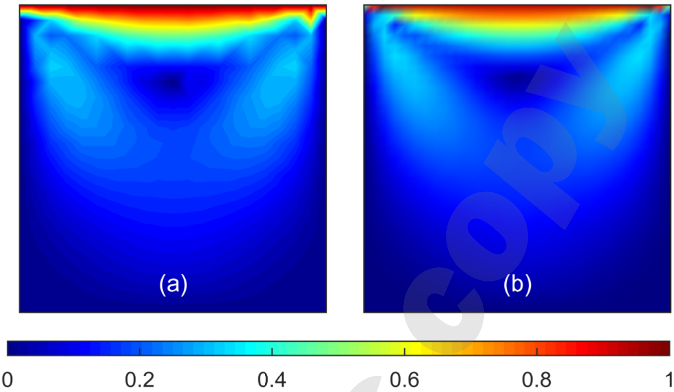

Figure 5.7: Velocity contour of (a) 8-nodes element (10×10) and (b)software

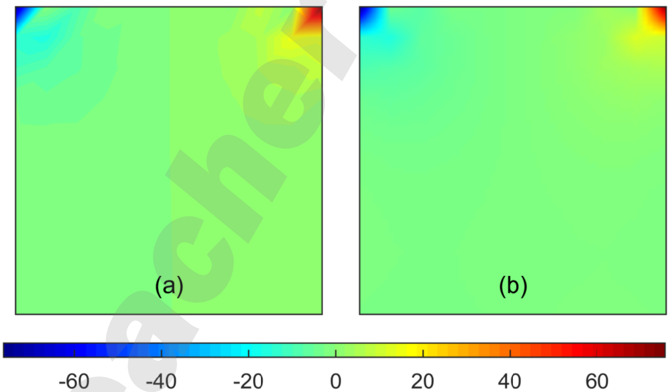

Figure 5.8: Pressure contour of (a) 8-nodes element (10×10) and (b)software

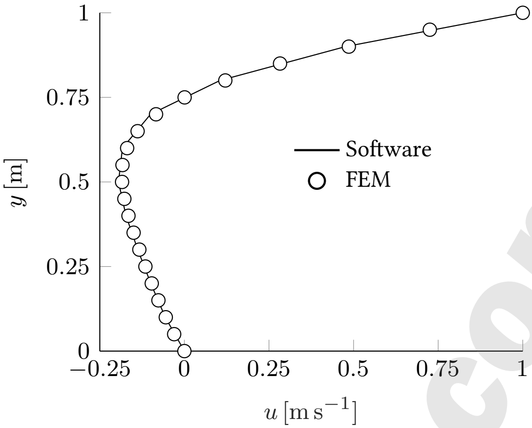

Figure 5.9: Velocity profile at $x = 0.5$

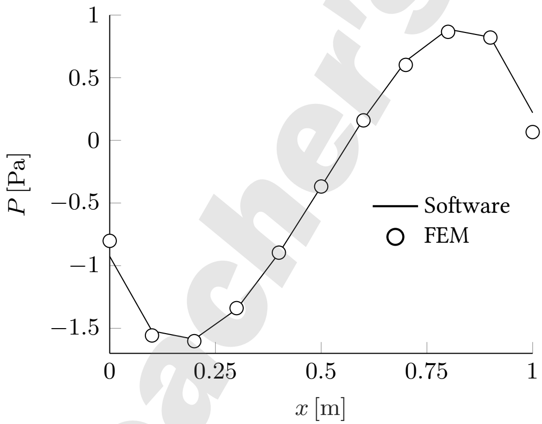

Figure 5.10: Pressure profile at $y = 0.5$

5.7.1 Source Code for Navier Stokes Equations

```
% Clear data
clc; clear; close all

% --------------------
% Input (fluid properties)
% --------------------

rho  = 1;       % Density [kg/m^3]
```

```
mu   = 1e-3;    % Fluid viscosity [Pa.s]

% --------------------
% Input (mesh information)
% --------------------

% Node coordinate
node_xy = [  0     0
          0.25     0
          0.50     0
          0.75     0
          1.00     0
             0  0.25
          0.50  0.25
          1.00  0.25
             0  0.50
          0.25  0.50
          0.50  0.50
          0.75  0.50
          1.00  0.50
             0  0.75
          0.50  0.75
          1.00  0.75
             0  1.00
          0.25  1.00
          0.50  1.00
          0.75  1.00
          1.00  1.00];

% Connectivity matrix
elem_node = [ 1   3  11   9   2   7  10   6
              3   5  13  11   4   8  12   7
              9  11  19  17  10  15  18  14
             11  13  21  19  12  16  20  15];

% Get additional mesh info
node_num = size(node_xy,1);
elem_num = size(elem_node,1);

% --------------------
% Input (solver setting)
% --------------------

dt  = 1;                % Time step
t   = 0:dt:100;         % Simulation time [s]
tol = 1e-3;             % Error tolerance

% --------------------
% Dof numbering
% --------------------
node_u_var = [ 1   3   5   7   9  11  13  15  17  19  ...
              21  23  25  27  29  31  33  35  37  39  41];
node_v_var = [ 2   4   6   8  10  12  14  16  18  20  ...
              22  24  26  28  30  32  34  36  38  40  42];
node_p_var = [43   0  44   0  45   0   0   0  46   0  ...
              47   0  48   0   0   0  49   0  50   0  51];

% Total dof
ndof = 51;

% --------------------
```

```
% Boundary conditions
% --------------------

% Node condition [1-boundary node, 0-internal node]
node_u_cond = [  1   1   1   1   1   1   0   1   1   0 ...
                 0   0   1   1   0   1   1   1   1   1   1];
node_v_cond = [  1   1   1   1   1   1   0   1   1   0 ...
                 0   0   1   1   0   1   1   1   1   1   1];
node_p_cond = [  0   0   0   0   1   0   0   0   0   0 ...
                 0   0   0   0   0   0   0   0   0   0   0];

% Boundary conditions
node_u_bc   = [  0   0   0   0   0   0   0   0   0   0 ...
                 0   0   0   0   0   0   1   1   1   1   1];
node_v_bc   = [  0   0   0   0   0   0   0   0   0   0 ...
                 0   0   0   0   0   0   0   0   0   0   0];
node_p_bc   = [  0   0   0   0   0   0   0   0   0   0 ...
                 0   0   0   0   0   0   0   0   0   0   0];

% --------------------
% Initial condition
% --------------------

% Initialize value
UVP = zeros(ndof,1);

% Dof with boundary condition
UVP(node_u_var(node_u_cond==1)) = node_u_bc(node_u_cond==1);

% Dof previous time step
UVP0 = UVP;

% --------------------
% Time marching
% --------------------

for it=1:length(t)

% --------------------
% Nonlinear iteration
% --------------------

for iter=1:10

%   --------------------
%   Assemble the global matrix & vector
%   --------------------

%   Initialize global matrix and vector
    K = zeros(ndof,ndof);
    M = zeros(ndof,ndof);
    F = zeros(ndof,1);

%   Assembly of global matrix and vector
    for ie = 1:elem_num

%       Local dof numbering for velocity and pressure
        iu = node_u_var(elem_node(ie,:));
        iv = node_v_var(elem_node(ie,:));
        ip = node_p_var(elem_node(ie,1:4));
```

```
%       Local combined dof numbering
        iuvp = zeros(1,20);
        iuvp(1:2:16) = iu;
        iuvp(2:2:16) = iv;
        iuvp(17:20)  = ip;

%       Coordinates for element
        xy = node_xy(elem_node(ie,:),:);

%       Side length
        a = xy(2,1) - xy(1,1);
        b = xy(3,2) - xy(2,2);

%       Local velocity
        u = UVP(iu);
        v = UVP(iv);

%       Local k, m, c, kn matrices (refer appendix)
        [k, m, c, kn] = localMatrices(mu, rho, a, b, u, v);

%       Local to global assembly
        K(iuvp,iuvp) = K(iuvp,iuvp) + [k+kn -c; -c' zeros(4)];
        M(iuvp,iuvp) = M(iuvp,iuvp) + [m zeros(16,4);zeros(4,16) zeros
             (4)];
    end

%   Effective stiffness & force vector
    Keff = 1/dt*M + K;
    Feff = 1/dt*(M*UVP0) + F;

%   --------------------
%   Impose boundary conditions
%   --------------------

    for i = 1:node_num

%       Local dof for velocity
        iu = node_u_var(i);
        iv = node_v_var(i);
        ip = node_p_var(i);

%       Velocity, u
        if node_u_cond(i)
            Feff          = Feff - Keff(:,iu) * node_u_bc(i);
            Keff(iu,:)    = 0;
            Keff(:,iu)    = 0;
            Keff(iu,iu)   = 1;
            Feff(iu)      = node_u_bc(i);
        end

%       Velocity,v
        if node_v_cond(i)
            Feff          = Feff - Keff(:,iv) * node_v_bc(i);
            Keff(iv,:)    = 0;
            Keff(:,iv)    = 0;
            Keff(iv,iv)   = 1;
            Feff(iv)      = node_v_bc(i);
        end

%       Pressure, p
        if node_p_cond(i)
```

```
            Feff            = Feff - Keff(:,ip) * node_p_bc(i);
            Keff(ip,:)      = 0;
            Keff(:,ip)      = 0;
            Keff(ip,ip)     = 1;
            Feff(ip)        = node_p_bc(i);
        end

    end

%   --------------------
%   Solve the matrix system
%   --------------------

%   Solve for the unknown
    uvp = Keff\Feff;

%   Check convergence
    maxErr = max(abs(UVP-uvp));
    if maxErr<tol
%       Final value
        UVP  = uvp;
        UVP0 = uvp;
        break
    else
%       Next iteration
        UVP = uvp;
    end

end

end
```

6 Numerical Integration

6.1 Introduction

All the integrations in previous chapters are done analytically. As mentioned, this is to allow for easy tracing of the procedure and immediate determination of the matrices and vectors. However, in practice, numerical integration is employed, for the following reasons (among others),

i. to cater for the irregular shape of elements hence domains
ii. to employ reduced integration in reducing the effect of "overstiffness" and speeding up computing time

The basic concept of numerical integration is to map a "distorted" element in the physical domain into a regular shape element in the natural domain. If Gauss-Legendre quadrature is employed, this natural element is usually in the particular form of a rectangular with a side length of 2 units for two-dimensional problem and a line of 2 units of length for one-dimensional problem. This mapping is shown in Fig. 6.1.

As can be seen, the mapping process requires the coordinates of the physical element to be expressed as an interpolated functions in terms of shape functions and nodal coordinates, which, for one-dimensional problem, can be given as,

$$x = \{N(\xi)\}\{\hat{x}\}^T \tag{6.1}$$

and for two-dimensional as,

$$x = \{N(\xi, \eta)\}\{\hat{x}\}^T \tag{6.2a}$$

$$y = \{N(\xi, \eta)\}\{\hat{y}\}^T \tag{6.2b}$$

where $N(\xi)$ and $N(\xi, \eta)$ are the shape functions derived in the natural coordinates for 1D and 2D formulations, respectively whilst $\{\hat{x}\}^T$ and $\{\hat{y}\}^T$

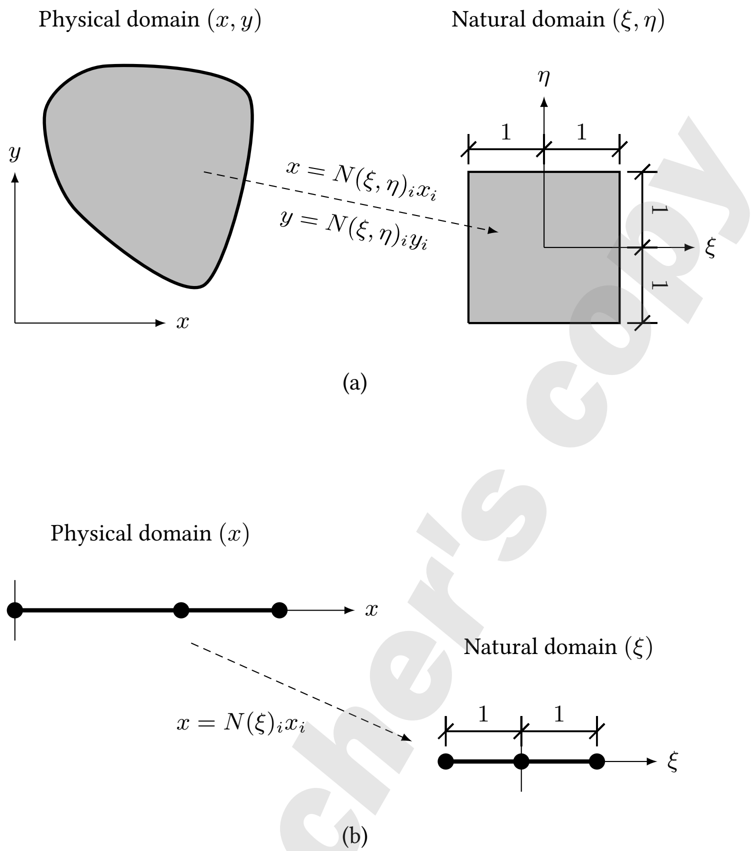

Figure 6.1: (a) Two-dimensional, and (b) one-dimensional mapping of element

are the nodal coordinates of the element in the physical domain. The procedures for the derivation of the natural shape functions are similar to those outlined previously except that, the variables are evaluated at a specific set of values of, say, at $\xi = -1, 1$ for a linear bar element and at $\xi = -1, 0, 1$ for a quadratic bar element. For a bilinear 2D element, the variables are evaluated at $(\xi, \eta) = (-1, -1), (1, -1), (1, 1), (-1, 1)$ while for quadratic 2D element, evaluation is carried out at $(\xi, \eta) = (-1, -1), (1, -1), (1, 1), (-1, 1), (0, -1), (1, 0), (0, 1), (-1, 0)$.

Omitting the detailed derivation, the shape functions for each type of elements above is given below.

Shape functions for linear bar element

$$N_1 = \frac{1}{2}(1-\xi) \tag{6.3a}$$

$$N_2 = \frac{1}{2}(1+\xi) \tag{6.3b}$$

Shape functions for quadratic bar element

$$N_1 = \frac{1}{2}(-\xi+\xi^2) \tag{6.4a}$$

$$N_2 = (1-\xi^2) \tag{6.4b}$$

$$N_3 = \frac{1}{2}(\xi+\xi^2) \tag{6.4c}$$

Shape functions for bilinear quadrilateral

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \tag{6.5a}$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) \tag{6.5b}$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \tag{6.5c}$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) \tag{6.5d}$$

Shape functions for quadratic quadrilateral

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1) \tag{6.6a}$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \tag{6.6b}$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \tag{6.6c}$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1) \tag{6.6d}$$

$$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta) \tag{6.6e}$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2) \tag{6.6f}$$

$$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta) \tag{6.6g}$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2) \tag{6.6h}$$

6.2 1D Numerical Integration Demonstration

For demonstration purposes, let's determine the value of one of the stiffness matrix terms of the quadratic nonlinear bar element, say k_{12} (which was previously given by Eq. (4.14),

$$k_{12} = \int_0^L \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right) \left(\frac{dN_1}{dx}\frac{dN_2}{dx}\right) dx \tag{6.7}$$

It must be noted that, in k_{12}, all terms including the shape functions, N_i are expressed in terms of the physical coordinates, x. However, to carry out Gauss-Legendre numerical integration, we must express everything in terms of the natural coordinates, ξ. To do this, we first obtain the equivalent differential through the following change of variable:

$$dx = \frac{\partial x}{\partial \xi} d\xi = J d\xi \tag{6.8}$$

where J is termed as the Jacobian. Based on Eq. (6.8) the Jacobian, J can be determined as follows. Note that, since quadratic bar element is considered, the use of Eq. (6.4) would give,

$$J = \frac{\partial x}{\partial \xi} = \frac{\partial (N_i x_i)}{\partial \xi} = \frac{\partial \left(\frac{1}{2}(-\xi + \xi^2)x_1 + (1 - \xi^2)x_2 + \frac{1}{2}(\xi + \xi^2)x_3 \right)}{\partial \xi} \tag{6.9}$$

For the case where node 2 is at mid-point of the physical domain (actual element) i.e. $x_2 = \frac{x_3 - x_1}{2}$, the Jacobian, J can be shown as, simply,

$$J = \frac{L}{2} \tag{6.10}$$

Inserting Eq. (6.10) into Eq. (6.8), the differential can be given as,

$$dx = \frac{L}{2} d\xi \tag{6.11}$$

Next, let's determine the equivalent derivatives terms. Again, based on change of variable, the differential operator, $\frac{d}{dx}$ can be expressed as:

$$\frac{d}{dx} = \frac{d\xi}{dx} \frac{d}{d\xi} \tag{6.12}$$

Based on Eq. (6.9) and Eq. (6.10), Eq. (6.12) can then be given as,

$$\frac{d}{dx} = \frac{1}{J} \frac{d}{d\xi} = J^{-1} \frac{d}{d\xi} = \frac{2}{L} \frac{d}{d\xi} \tag{6.13}$$

where, we have a specific name for J^{-1} that is inverse of Jacobian. Having established the differential and the differential operator in the natural domain, the equivalent k_{12} can thus finally be given as (by inserting Eqs. (6.11) and (6.13) into Eq. (6.7)),

$$\begin{aligned} k_{12} &= \int_{-1}^{1} \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3 \right) \left(\frac{1}{J} \frac{dN_1}{d\xi} \right) \left(\frac{1}{J} \frac{dN_2}{d\xi} \right) J d\xi \\ &= \int_{-1}^{1} \alpha A \left(N_1 u_1 + N_2 u_2 + N_3 u_3 \right) \left(\frac{2}{L} \frac{dN_1}{d\xi} \right) \left(\frac{2}{L} \frac{dN_2}{d\xi} \right) \frac{L}{2} d\xi \end{aligned} \tag{6.14}$$

Observe that, once converted into the natural domain, the integration limit now ranges from -1 to 1. And note again that, all the shape functions, N_i in Eq. (6.14) are those given by Eq. (6.4). Now, the numerical integration of Eq. (6.14) can be given as,

$$k_{12} = \sum_{i=1}^{m} \phi_i \, \alpha A \ \left(N_1 u_1 + N_2 u_2 + N_3 u_3\right)\big|_i \left(\frac{2}{L}\frac{dN_1}{d\xi}\right)\bigg|_i \left(\frac{2}{L}\frac{dN_2}{d\xi}\right)\bigg|_i \frac{L}{2} \tag{6.15}$$

where, m and i are the order and the location of Gauss points, respectively and ϕ_i is the Gauss weight. Their values are given in Table 6.1. Fig. 6.2 shows the arrangement of Gauss point for $m = 2$ and $m = 3$.

Table 6.1: Gauss points and their weight for 1D integration

Gauss Point, m	Location of Gauss Point, g_i	Gauss Weight, ϕ_i
2	(-0.577),(+0.577)	(1.000),(1.000)
3	(-0.775),(0.0),(+0.775)	(0.555),(0.888),(0.555)
4	($\pm$ 0.861),($\pm$ 0.339)	(0.348),(0.652)
5	($\pm$ 0.538),(0.0),($\pm$ 0.906)	(0.478),(0.568),(0.236)

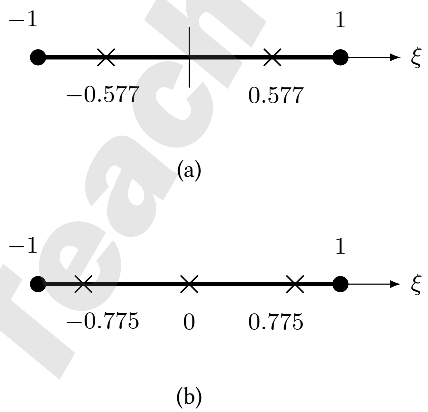

Figure 6.2: Locations of Gauss point for 1D integration, (a) $m = 2$ and (b) $m = 3$

Now, let's demonstrate the numerical integration of k_{12} for $m = 3$. Inserting the shape functions of Eq. (6.4), into Eq. (6.15) yields,

$$
\begin{aligned}
k_{12} = \sum_{i=1}^{m} & \phi_i \, \alpha A \times \\
& \left(\left(\frac{1}{2}(-\xi + \xi^2) \right) u_1 + \left(1 - \xi^2 \right) u_2 + \left(\frac{1}{2}(\xi + \xi^2) \right) u_3 \right) \Bigg|_i \\
& \left(\frac{2}{L} \, \frac{d}{d\xi} \left(\frac{1}{2}(-\xi + \xi^2) \right) \right) \Bigg|_i \left(\frac{2}{L} \, \frac{d}{d\xi} \left(1 - \xi^2 \right) \right) \Bigg|_i \frac{L}{2}
\end{aligned}
\tag{6.16}
$$

Based on the values from Table 6.1 for the case $m = 3$, Eq. (6.16) can be expanded as.

$$
\begin{aligned}
k_{12} = \phi_1 \, \alpha A & \left(\left(\frac{1}{2}(-\xi + \xi^2) \right) u_1 + \left(1 - \xi^2 \right) u_2 + \left(\frac{1}{2}(\xi + \xi^2) \right) u_3 \right) \Bigg|_{g_1} \\
& \left(\frac{1}{L}(-1 + 2\xi) \right) \Bigg|_{g_1} \left(-\frac{4\xi}{L} \right) \Bigg|_{g_1} \frac{L}{2} + \\
\phi_2 \, \alpha A & \left(\left(\frac{1}{2}(-\xi + \xi^2) \right) u_1 + \left(1 - \xi^2 \right) u_2 + \left(\frac{1}{2}(\xi + \xi^2) \right) u_3 \right) \Bigg|_{g_2} \\
& \left(\frac{1}{L}(-1 + 2\xi) \right) \Bigg|_{g_2} \left(-\frac{4\xi}{L} \right) \Bigg|_{g_2} \frac{L}{2} + \\
\phi_3 \, \alpha A & \left(\left(\frac{1}{2}(-\xi + \xi^2) \right) u_1 + \left(1 - \xi^2 \right) u_2 + \left(\frac{1}{2}(\xi + \xi^2) \right) u_3 \right) \Bigg|_{g_3} \\
& \left(\frac{1}{L}(-1 + 2\xi) \right) \Bigg|_{g_3} \left(-\frac{4\xi}{L} \right) \Bigg|_{g_3} \frac{L}{2}
\end{aligned}
\tag{6.17}
$$

or

$$
\begin{aligned}
k_{12} = \phi_1\,\alpha A \Bigg(& \left(\frac{1}{2}(-g_1 + {g_1}^2)\right) u_1 + \left(1 - {g_1}^2\right) u_2 + \\
& \left(\frac{1}{2}(g_1 + {g_1}^2)\right) u_3 \Bigg) \left(\frac{1}{L}(-1 + 2g_1)\right) \left(-\frac{4g_1}{L}\right) \frac{L}{2} + \\
& \phi_2\,\alpha A \Bigg(\left(\frac{1}{2}(-g_2 + {g_2}^2)\right) u_1 + \left(1 - {g_2}^2\right) u_2 + \\
& \left(\frac{1}{2}(g_2 + {g_2}^2)\right) u_3 \Bigg) \left(\frac{1}{L}(-1 + 2g_2)\right) \left(-\frac{4g_2}{L}\right) \frac{L}{2} + \\
& \phi_3\,\alpha A \Bigg(\left(\frac{1}{2}(-g_3 + {g_3}^2)\right) u_1 + \left(1 - {g_3}^2\right) u_2 + \\
& \left(\frac{1}{2}(g_3 + {g_3}^2)\right) u_3 \Bigg) \left(\frac{1}{L}(-1 + 2g_3)\right) \left(-\frac{4g_3}{L}\right) \frac{L}{2}
\end{aligned}
\tag{6.18}
$$

thus

$$k_{12} = -\alpha\frac{A}{L}(-1.4667u_1 - 1.0667u_2 - 0.133u_3) \tag{6.19}$$

Eq. (6.19) is similar to the exact integration as given by Eq. (4.14). This is due to the fact that $m = 3$ was chosen as the order of quadrature. A full integration will be obtained when the function to be integrated is of degree $2n - 1$ and the number of quadrature chosen is at least, n.

If quadrature higher than n is chosen, it will not increase the accuracy of the integration whilst quadrature lesser than n will result in the so called reduced integration. In the present case, due to the nonlinearity, the degree k_{12} is 4 (i.e. ξ^4) thus n must be at least 3 if exact integration is to be obtained.

6.3 2D Numerical Integration Demonstration

To demonstrate the application of 2D Gauss-Legendre numerical integration scheme, let's determine the stiffness matrix of plane stress bilinear element as given by Eq. (3.35) in Chapter 3. It is re-given here in as,

$$[k] = \int_y \int_x [N]^T [\partial][E][\partial]^T [N]\, dx\, dy \tag{6.20}$$

where

$$[E] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix}$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$[\partial]^T = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

It is a normal practise to introduce a matrix $[B]$ as follows,

$$[B] = [\partial]^T [N] \tag{6.21}$$

Thus, Eq. (6.20) becomes

$$[k] = \int_y \int_x [B]^T [E][B]\, dx\, dy \tag{6.22}$$

Omitting the derivation, it can be shown that the differential area of the physical domain, $dx\, dy$ relates to the differential area of the natural domain, $d\xi\, d\eta$ through

$$dx\, dy = |J|\, d\xi\, d\eta \tag{6.23}$$

where $|J|$ is termed as determinant of the Jacobian matrix, $[J]$. In turn, the Jacobian matrix, $[J]$ can be obtained through the following change of variables

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} \tag{6.24a}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} \tag{6.24b}$$

In matrix form, Eq. (6.24) can be given as

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \tag{6.25}$$

or

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = [\,J\,] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \tag{6.26}$$

hence the Jacobian matrix, $[\,J\,]$ is defined as

$$[\,J\,] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{6.27}$$

Inserting the interpolation of the physical coordinates as given by Eq. (6.2), $[\,J\,]$ is given as

$$\begin{aligned} [\,J\,] &= \begin{bmatrix} \dfrac{\partial \{N\}\{\hat{x}\}^T}{\partial \xi} & \dfrac{\partial \{N\}\{\hat{y}\}^T}{\partial \xi} \\ \dfrac{\partial \{N\}\{\hat{x}\}^T}{\partial \eta} & \dfrac{\partial \{N\}\{\hat{y}\}^T}{\partial \eta} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\partial \{N\}}{\partial \xi} \\ \dfrac{\partial \{N\}}{\partial \eta} \end{bmatrix} \begin{bmatrix} \{\hat{x}\}^T & \{\hat{y}\}^T \end{bmatrix} \end{aligned} \tag{6.28}$$

Expanding Eq. (6.28) gives

$$[\,J\,] = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & \dfrac{\partial N_2}{\partial \xi} & \dfrac{\partial N_3}{\partial \xi} & \dfrac{\partial N_4}{\partial \xi} \\ \dfrac{\partial N_1}{\partial \eta} & \dfrac{\partial N_2}{\partial \eta} & \dfrac{\partial N_3}{\partial \eta} & \dfrac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{6.29}$$

Inserting Eq. (6.5) into Eq. (6.29)

$$[\,J\,] = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{6.30}$$

By carrying out the matrix multiplication of Eq. (6.30), $[\,J\,]$ can be compactly given as

$$[\,J\,] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{6.31}$$

where

$$J_{11} = -\frac{1}{4}(1-\eta)\,x_1 + \frac{1}{4}(1-\eta)\,x_2 + \frac{1}{4}(1+\eta)\,x_3 - \frac{1}{4}(1+\eta)\,x_4$$

$$J_{12} = -\frac{1}{4}(1-\eta)\,y_1 + \frac{1}{4}(1-\eta)\,y_2 + \frac{1}{4}(1+\eta)\,y_3 - \frac{1}{4}(1+\eta)\,y_4$$

$$J_{21} = -\frac{1}{4}(1-\xi)\,x_1 - \frac{1}{4}(1+\xi)\,x_2 + \frac{1}{4}(1+\xi)\,x_3 + \frac{1}{4}(1-\xi)\,x_4$$

$$J_{22} = -\frac{1}{4}(1-\xi)\,y_1 - \frac{1}{4}(1+\xi)\,y_2 + \frac{1}{4}(1+\xi)\,y_3 + \frac{1}{4}(1-\xi)\,y_4$$

Having determined the Jacobian matrix $[\,J\,]$, its determinant, $|J|$ as required in Eq. (6.23), can be given as

$$|\,J\,| = J_{22}\,J_{11} - J_{12}\,J_{21} \tag{6.32}$$

Next, let's determine the equivalent derivative terms in Eq. (6.20), through change of variables,

$$[\,\partial\,]^T[\,N(\xi,\eta)\,] = [\,\partial\,]^T[\,J\,][\,N(x,y)\,] \tag{6.33}$$

where the dependency of the shape function on their coordinate system is highlighted (i.e. $N(\xi,\eta), N(x,y)$). Since we need to express everything in natural coordinates, thus

$$[\,\partial\,]^T[\,N(x,y)\,] = [\,\partial\,]^T[\,J\,]^{-1}[\,N(\xi,\eta)\,] \tag{6.34}$$

where the inverse of the Jacobian matrix, $[\,J\,]^{-1}$ is determined as

$$[\,J\,]^{-1} = \frac{\begin{bmatrix} J_{22} & -J_{21} \\ -J_{12} & J_{11} \end{bmatrix}}{|\,J\,|} \tag{6.35}$$

By inserting Eq. (6.35) into Eq. (6.20), we then have completely mapped the integral into the natural domain given as

$$\begin{aligned}[\,k\,] = \int_{-1}^{1}\int_{-1}^{1} \Big[[\,\partial\,]^T[\,J\,]^{-1}[\,N(\xi,\eta)\,]\Big]^T[\,E\,] \\ \Big[[\,\partial\,]^T[\,J\,]^{-1}[\,N(\xi,\eta)\,]\Big]|\,J\,|\,d\xi\,d\eta\end{aligned} \tag{6.36}$$

where the dependency of the shape function on the natural coordinates is understood hence its notation is omitted. Now, the numerical integration of Eq. (6.36) can be given as

$$\begin{aligned}[\,k\,] = \sum_{i=1}^{m}\sum_{j=1}^{m} \phi_i\,\phi_j \Big[[\,\partial\,]^T[\,J\,]^{-1}[\,N(\xi,\eta)\,]\Big]\Big|_{ij}^{T}[\,E\,] \\ \Big[[\,\partial\,]^T[\,J\,]^{-1}[\,N(\xi,\eta)\,]\Big]\Big|_{ij}|\,J\,|\Big|_{ij}\end{aligned} \tag{6.37}$$

where $|_{ij}$ refers to the evaluation at the location of Gauss point, ij. Fig. 6.3 shows the geometrical arrangement of Gauss point for $m = 2$ and $m = 3$.

Let's nail our understanding with numerical example.

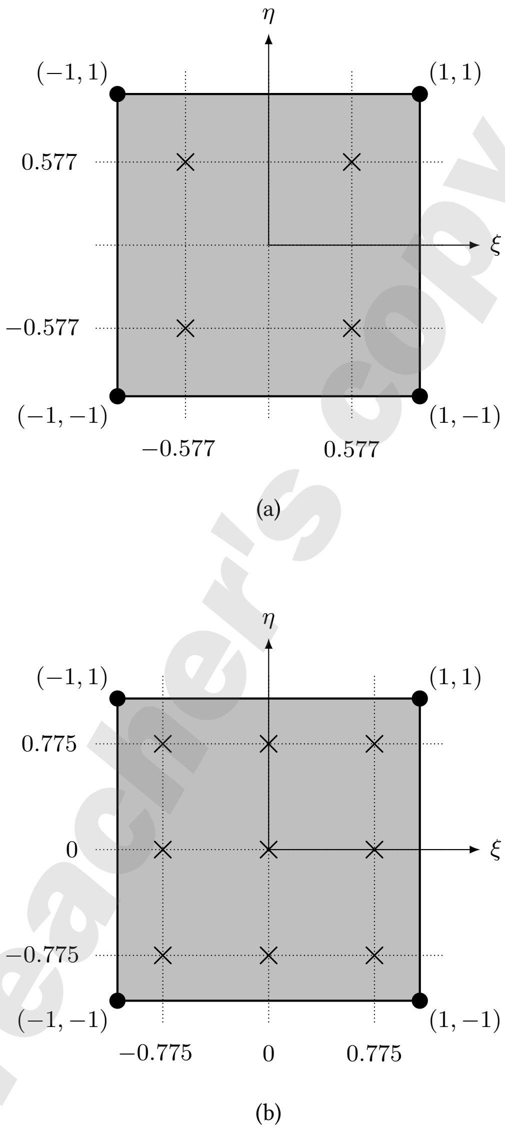

Figure 6.3: Location of Gauss points for 2D integration, (a) $m = 2$ and (b) $m = 3$

6.4 Worked Example: 2D Bilinear Plane Stress

Fig. 6.4 shows a plane stress bilinear quadrilateral element. Determine the stiffness matrix, $[\,k\,]$ of the element having the following material properties

Young modulus, $E = 1\,\mathrm{N\,mm^{-2}}$

Poisson's ratio, $\nu = 0.25$

The nodal physical coordinates are as shown in the figure.

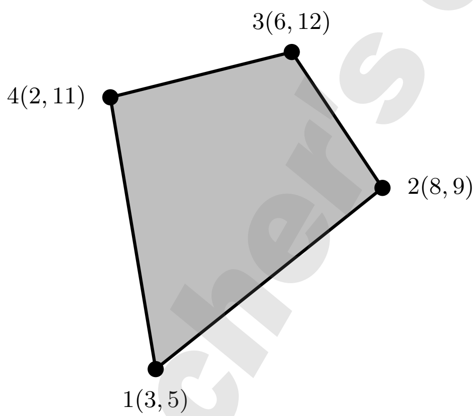

Figure 6.4: Plane stress bilinear quadrilateral element

From Fig. 6.4, the nodal physical coordinates of the element are given as

$$\left[\{\hat{x}\}^T \ \ \{\hat{y}\}^T\right] = \begin{bmatrix} 3 & 5 \\ 8 & 9 \\ 6 & 12 \\ 2 & 11 \end{bmatrix} \tag{6.38}$$

Using Eq. (6.31) the Jacobian matrix, $[\,J\,]$ is determined as

$$[\,J\,] = \begin{bmatrix} \frac{9}{4} - \frac{1}{4}\eta & \frac{5}{4} - \frac{3}{4}\eta \\ -\frac{3}{4} - \frac{1}{4}\xi & \frac{9}{4} - \frac{3}{4}\xi \end{bmatrix} \tag{6.39}$$

The determinant of Jacobian, $|J|$ is determined from Eq. (6.32) as

$$|J| = \left(\frac{9}{4} - \frac{1}{4}\eta\right)\left(\frac{9}{4} - \frac{3}{4}\xi\right) - \left(\frac{5}{4} - \frac{3}{4}\eta\right)\left(-\frac{3}{4} - \frac{1}{4}\xi\right) \tag{6.40}$$

And the inverse of Jacobian matrix, $[\,J\,]^{-1}$ is determined using Eq. (6.35)

$$[\,J\,]^{-1} = \frac{\begin{bmatrix} \frac{9}{4} - \frac{3}{4}\xi & -\frac{5}{4} + \frac{3}{4}\eta \\ \frac{3}{4} + \frac{1}{4}\xi & \frac{9}{4} - \frac{1}{4}\eta \end{bmatrix}}{|J|} \tag{6.41}$$

The derivative term in Eq. (6.36) can thus be calculated as

$$[\,\partial\,]^T[\,J\,]^{-1}[\,N\,] = \begin{bmatrix} s_1 & 0 & s_3 & 0 & s_5 & 0 & s_7 & 0 \\ 0 & s_2 & 0 & s_4 & 0 & s_6 & 0 & s_8 \\ s_2 & s_1 & s_4 & s_3 & s_6 & s_5 & s_8 & s_7 \end{bmatrix} \tag{6.42}$$

where:

$$s_1 = \frac{\xi - 3\eta + 2}{9\eta + 11\xi - 48}$$

$$s_2 = -\frac{2\eta + 4\xi - 6}{9\eta + 11\xi - 48}$$

$$s_3 = -\frac{\xi - 6\eta + 7}{9\eta + 11\xi - 48}$$

$$s_4 = \frac{\eta + 4\xi + 3}{9\eta + 11\xi - 48}$$

$$s_5 = -\frac{6\eta - 4\xi + 2}{9\eta + 11\xi - 48}$$

$$s_6 = -\frac{\eta + 5\xi + 6}{9\eta + 11\xi - 48}$$

$$s_7 = \frac{3\eta - 4\xi + 7}{9\eta + 11\xi - 48}$$

$$s_8 = \frac{2\eta + 5\xi - 3}{9\eta + 11\xi - 48}$$

Numerical values of $[\,k\,]$ at the 1st gauss point $(i = 1, j = 1)$

Gauss point location, $\xi = -0.557, \eta = -0.557$

Gauss weight, $\phi_1 = 1.000, \phi_1 = 1.000$

$$\begin{aligned} [B]\,|_{1,1} &= [\partial]^T [J]^{-1} [N]\,|_{1,1} \\ &= \begin{bmatrix} -0.053 & 0 & 0.166 & \cdots & -0.127 & 0 \\ 0 & -0.159 & 0 & \cdots & 0 & 0.118 \\ -0.159 & -0.053 & -0.002 & \cdots & 0.118 & -0.127 \end{bmatrix} \end{aligned} \tag{6.43}$$

$$
\begin{aligned}
[\,k\,]\,|_{1,1} &= \phi_1\phi_1\,[B]\,|_{1,1}^{T}\,[E]\,[B]\,|_{1,1}\,(|J|)\,|_{1,1} \\
&= (1)(1)\begin{bmatrix} -0.053 & 0 & -0.159 \\ 0 & -0.159 & -0.053 \\ 0.166 & 0 & -0.002 \\ 0 & -0.002 & 0.166 \\ 0.014 & 0 & 0.043 \\ 0 & 0.043 & 0.014 \\ -0.127 & 0 & 0.118 \\ 0 & 0.118 & -0.127 \end{bmatrix} \times \\
&\quad \begin{bmatrix} 1.067 & 0.267 & 0 \\ 0.267 & 1.067 & 0 \\ 0 & 0 & 0.400 \end{bmatrix} \times \\
&\quad \begin{bmatrix} -0.053 & 0 & 0.166 & \cdots & -0.127 & 0 \\ 0 & -0.159 & 0 & \cdots & 0 & 0.118 \\ -0.159 & -0.053 & -0.002 & \cdots & 0.118 & -0.127 \end{bmatrix} \\
&= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & \cdots & 7 & 8 \end{array} \\
\begin{bmatrix} 0.097 & 0.042 & -0.069 & \ldots & -0.002 & 0.048 \\ 0.042 & 0.209 & -0.052 & \ldots & 0.021 & -0.129 \\ -0.069 & -0.052 & 0.219 & \ldots & -0.168 & 0.040 \\ -0.078 & -0.024 & -0.002 & \ldots & 0.059 & -0.065 \\ -0.026 & -0.011 & 0.018 & \ldots & 0.001 & -0.013 \\ -0.011 & -0.056 & 0.014 & \ldots & -0.006 & 0.035 \\ -0.002 & 0.021 & -0.168 & \ldots & 0.170 & -0.075 \\ 0.048 & -0.129 & 0.040 & \ldots & -0.075 & 0.159 \end{bmatrix}
\end{array}
\end{aligned}
\tag{6.44}
$$

Numerical values of $[\,k\,]$ at the 2nd gauss point $(i = 1, j = 2)$

Gauss point location, $\xi = -0.557, \eta = 0.557$

Gauss weight, $\phi_1 = 1.000, \phi_2 = 1.000$

$$
\begin{aligned}
[B]\,|_{1,2} &= [\partial]^T [J]^{-1} [N]\,|_{1,2} \\
&= \begin{bmatrix} 0.006 & 0 & 0.060 & \cdots & -0.225 & 0 \\ 0 & -0.146 & 0 & \cdots & 0 & 0.096 \\ -0.146 & 0.006 & -0.026 & \cdots & 0.096 & -0.225 \end{bmatrix}
\end{aligned} \tag{6.45}
$$

$$
\begin{aligned}
[\,k\,]\,|_{1,2} &= \phi_1 \phi_2\, [B]\,|_{1,2}^T\, [E]\, [B]\,|_{1,2}\, (|J|)\,|_{1,2} \\
&= \begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
0.052 & -0.004 & 0.012 & \ldots & -0.044 & 0.081 \\
-0.004 & 0.139 & -0.015 & \ldots & 0.055 & -0.095 \\
0.012 & -0.015 & 0.025 & \ldots & -0.095 & 0.024 \\
-0.022 & 0.026 & -0.006 & \ldots & 0.024 & -0.050 \\
-0.020 & -0.037 & 0.058 & \ldots & -0.215 & -0.017 \\
-0.056 & -0.069 & -0.003 & \ldots & 0.010 & -0.040 \\
-0.044 & 0.055 & -0.095 & \ldots & 0.353 & -0.089 \\
0.081 & -0.095 & 0.024 & \ldots & -0.089 & 0.185
\end{bmatrix}
\end{array}
\end{aligned} \tag{6.46}
$$

Numerical values of $[\,k\,]$ at the 3rd gauss point $(i = 2, j = 1)$

Gauss point location, $\xi = 0.557, \eta = -0.557$

Gauss weight, $\phi_2 = 1.000, \phi_1 = 1.000$

$$
\begin{aligned}
[B]\,|_{2,1} &= [\partial]^T [J]^{-1} [N]\,|_{2,1} \\
&= \begin{bmatrix} -0.092 & 0 & 0.236 & \cdots & -0.063 & 0 \\ 0 & -0.103 & 0 & \cdots & 0 & 0.027 \\ -0.103 & -0.092 & -0.101 & \cdots & 0.027 & -0.063 \end{bmatrix}
\end{aligned} \tag{6.47}
$$

$$[\,k\,]\,|_{2,1} = \phi_2\phi_1\,[B]\,|_{2,1}^{T}\,[E]\,[B]\,|_{2,1}\,(|J|)\,|_{2,1}$$

$$= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 7 & 8 \end{array} \\ \begin{bmatrix} 0.078 & 0.037 & -0.111 & \dots & 0.030 & 0.011 \\ 0.037 & 0.087 & -0.016 & \dots & 0.004 & -0.004 \\ -0.111 & -0.016 & 0.371 & \dots & -0.099 & 0.025 \\ -0.043 & 0.014 & -0.093 & \dots & 0.025 & -0.052 \\ 0.003 & -0.025 & -0.160 & \dots & 0.043 & -0.030 \\ -0.006 & -0.097 & 0.084 & \dots & -0.023 & 0.042 \\ 0.030 & 0.004 & -0.099 & \dots & 0.027 & -0.007 \\ 0.011 & -0.004 & 0.025 & \dots & -0.007 & 0.014 \end{bmatrix} \end{array} \tag{6.48}$$

Numerical values of $[\,k\,]$ at the 4th gauss point $(i = 2, j = 2)$

Gauss point location, $\xi = 0.557, \eta = 0.557$

Gauss weight, $\phi_2 = 1.000, \phi_2 = 1.000$

$$[B]\,|_{2,2} = [\partial]^T [J]^{-1}\,[N]\,|_{2,2}$$
$$= \begin{bmatrix} -0.023 & 0 & 0.113 & \cdots & -0.176 & 0 \\ 0 & -0.070 & 0 & \cdots & 0 & -0.029 \\ -0.070 & -0.023 & -0.161 & \cdots & -0.029 & -0.176 \end{bmatrix} \tag{6.49}$$

$$[\,k\,]\,|_{2,2} = \phi_2\phi_2\,[B]\,|_{2,2}^{T}\,[E]\,[B]\,|_{2,2}\,(|J|)\,|_{2,2}$$

$$= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 7 & 8 \end{array} \\ \begin{bmatrix} 0.011 & 0.005 & 0.008 & \dots & 0.023 & 0.023 \\ 0.005 & 0.025 & -0.003 & \dots & 0.016 & 0.017 \\ 0.008 & -0.003 & 0.109 & \dots & -0.088 & 0.048 \\ -0.010 & 0.050 & -0.055 & \dots & 0.029 & -0.014 \\ -0.043 & -0.018 & -0.029 & \dots & -0.088 & -0.086 \\ -0.018 & -0.091 & 0.010 & \dots & -0.060 & -0.064 \\ 0.023 & 0.016 & -0.088 & \dots & 0.152 & 0.015 \\ 0.023 & 0.017 & 0.048 & \dots & 0.015 & 0.061 \end{bmatrix} \end{array} \tag{6.50}$$

The final numerical values of the stiffness matrix $[\,k\,]$ is thus

$$[\,k\,] = [\,k\,]|_{1,1} + [\,k\,]|_{1,2} + [\,k\,]|_{2,1} + [\,k\,]|_{2,2}$$

$$= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & 7 & 8 \end{array} \\ \begin{bmatrix} 0.239 & 0.080 & -0.160 & \ldots & 0.007 & 0.164 \\ 0.080 & 0.459 & -0.086 & \ldots & 0.097 & -0.211 \\ -0.160 & -0.086 & 0.724 & \ldots & -0.451 & 0.136 \\ -0.153 & 0.066 & -0.156 & \ldots & 0.136 & -0.180 \\ -0.086 & -0.091 & -0.113 & \ldots & -0.259 & -0.145 \\ -0.091 & -0.314 & 0.106 & \ldots & -0.079 & -0.027 \\ 0.007 & 0.097 & -0.451 & \ldots & 0.703 & -0.155 \\ 0.164 & -0.211 & 0.136 & \ldots & -0.155 & 0.418 \end{bmatrix} \end{array} \tag{6.51}$$

6.4.1 Source Code for Numerical Integration with 2-by-2 Gauss points

```
%Clear data
clc; clear; close all

% --------------------
% Input parameters
% --------------------

% Domain and material properties
E       = 1;          % Young's Modulus [Pa]
nu      = 0.25;       % Poison ratio

% Gauss points & their weight
GP_xi   = [-0.577 -0.577  0.577  0.577];
GP_eta  = [-0.577  0.577 -0.577  0.577];
W_xi    = [1 1 1 1];
W_eta   = [1 1 1 1];

% Physical coordinates
coor_xy = [3   5;
           8   9;
           6  12;
           2  11];

% --------------------
% Calculate the local k matrix using numerical integration
% --------------------

% Initialize k matrix
k = zeros(8,8);

% Loop over Gauss points
for i = 1:4

  % Gauss point
  xi       = GP_xi(i);
```

```
  eta      = GP_eta(i);

  % Weight
  phi_xi   = W_xi(i);
  phi_eta  = W_eta(i);

  % Jacobian matrix
  dN = [-(1-eta)  (1-eta)  (1+eta) -(1+eta);
        -(1-xi)  -(1+xi)   (1+xi)   (1-xi)];
  J  = (1/4) * dN * coor_xy;

  % B matrix
  s1 =  (xi - 3*eta + 2)   / (9*eta + 11*xi - 48);
  s2 = -(2*eta + 4*xi - 6) / (9*eta + 11*xi - 48);
  s3 = -(xi - 6*eta + 7)   / (9*eta + 11*xi - 48);
  s4 =  (eta + 4*xi + 3)   / (9*eta + 11*xi - 48);
  s5 = -(6*eta - 4*xi + 2) / (9*eta + 11*xi - 48);
  s6 = -(eta + 5*xi + 6)   / (9*eta + 11*xi - 48);
  s7 =  (3*eta - 4*xi + 7) / (9*eta + 11*xi - 48);
  s8 =  (2*eta + 5*xi - 3) / (9*eta + 11*xi - 48);

  B = [s1   0  s3   0  s5   0  s7   0;
        0  s2   0  s4   0  s6   0  s8;
       s2  s1  s4  s3  s6  s5  s8  s7];

  % Constitutive matrix
  D = E/(1-nu^2) * [ 1  nu   0;
                    nu   1   0;
                     0   0   (1-nu)/2];

  % Stiffness matrix
  k = k + phi_xi*phi_eta*B'*D*B*det(J);

end
```

6.4.2 Source Code for Numerical Integration with 3-by-3 Gauss points

```
%Clear data
clc; clear; close all

% --------------------
% Input parameters
% --------------------

% Domain and material properties
E       = 1;         % Young's Modulus [Pa]
nu      = 0.25;      % Poison ratio

% Gauss points & their weight
GP_xi   = [-0.775 -0.775 -0.775      0      0      0  0.775 0.775 0.775];
GP_eta  = [-0.775      0  0.775 -0.775      0 0.775 -0.775      0 0.775];
W_xi    = [ 0.555  0.555  0.555  0.889 0.889 0.889  0.555 0.555 0.555];
W_eta   = [ 0.555  0.889  0.555  0.555 0.889 0.555  0.555 0.889 0.555];

% Physical coordinates
coor_xy = [3   5;
           8   9;
```

```
          6  12;
          2  11];

% --------------------
% Calculate the local k matrix using numerical integration
% --------------------

% Initialize k matrix
k = zeros(8,8);

% Loop over Gauss points
for i = 1:9

  % Gauss point
  xi       = GP_xi(i);
  eta      = GP_eta(i);

  % Weight
  phi_xi   = W_xi(i);
  phi_eta  = W_eta(i);

  % Jacobian matrix
  dN = [-(1-eta)  (1-eta)  (1+eta) -(1+eta);
        -(1-xi)  -(1+xi)   (1+xi)   (1-xi)];
  J = (1/4) * dN * coor_xy;

  % B matrix
  s1 =  (xi - 3*eta + 2)   / (9*eta + 11*xi - 48);
  s2 = -(2*eta + 4*xi - 6) / (9*eta + 11*xi - 48);
  s3 = -(xi - 6*eta + 7)   / (9*eta + 11*xi - 48);
  s4 =  (eta + 4*xi + 3)   / (9*eta + 11*xi - 48);
  s5 = -(6*eta - 4*xi + 2) / (9*eta + 11*xi - 48);
  s6 = -(eta + 5*xi + 6)   / (9*eta + 11*xi - 48);
  s7 =  (3*eta - 4*xi + 7) / (9*eta + 11*xi - 48);
  s8 =  (2*eta + 5*xi - 3) / (9*eta + 11*xi - 48);

  B = [s1   0  s3   0  s5   0  s7   0;
        0  s2   0  s4   0  s6   0  s8;
       s2  s1  s4  s3  s6  s5  s8  s7];

  % Constitutive matrix
  D = E/(1-nu^2) * [ 1  nu   0;
                    nu   1   0;
                     0   0   (1-nu)/2];

  % Stiffness matrix
  k = k + phi_xi*phi_eta*B'*D*B*det(J);

end
```

6.5 Exercises

1. Consider a bar element with shown in Fig. 6.5. Compute the stiffness matrix using 3-node shape functions derived in natural coordinates. The modulus elasticity of the bar is $2 \times 10^8\,\mathrm{kN\,m^{-2}}$ and the cross-sectional area is $0.5\,\mathrm{m}^2$. The formulation of the stiffness matrix for bar was previously given by Eqn. (1.59).

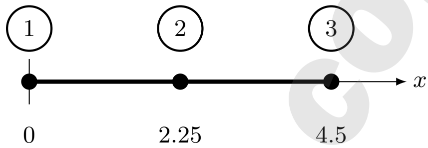

Figure 6.5: 3-node bar element

2. Consider a $0.015\,\mathrm{m}$ thickness rhombus shown in Fig. 6.6. Determine the stiffness matrix using plane stress bilinear element derived in natural coordinates. The modulus elasticity is $2.05 \times 10^8\,\mathrm{kN\,m^{-2}}$ and the Poisson's ratio is 0.3.

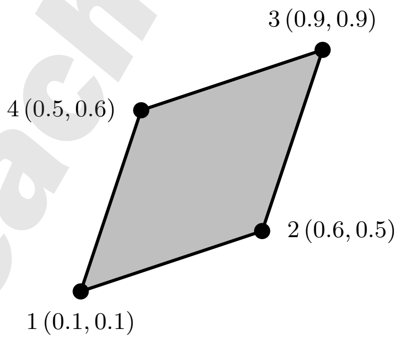

Figure 6.6: A rhombus plane stress element

3. Determine the conductance matrix for the 8-node element having $0.005\,\mathrm{m}$ thickness as shown in Fig. 6.7. The thermal conductivity of the material is $450\,\mathrm{W\,m^{-1}\,K^{-1}}$.

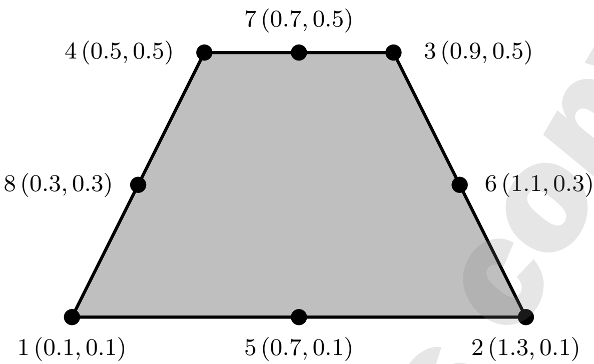

Figure 6.7: 8-node element

Appendix A

Components of matrix $[k_N]$

$$s_1 = \frac{b\rho}{12600}\left(43u_2 - 390u_1 + 15u_3 + 72u_4 - 238u_5 - 128u_6 - 42u_7 - 172u_8\right)$$
$$s_2 = \frac{b\rho}{12600}\left(43v_2 - 390v_1 + 15v_3 + 72v_4 - 238v_5 - 128v_6 - 42v_7 - 172v_8\right)$$
$$s_3 = \frac{b\rho}{12600}\left(43u_1 + 86u_2 - 68u_3 - 41u_4 + 266u_5 + 52u_6 + 14u_7 + 208u_8\right)$$
$$s_4 = \frac{b\rho}{12600}\left(43v_1 + 86v_2 - 68v_3 - 41v_4 + 266v_5 + 52v_6 + 14v_7 + 208v_8\right)$$
$$s_5 = \frac{b\rho}{12600}\left(15u_1 - 68u_2 + 30u_3 - 27u_4 + 98u_5 - 32u_6 + 42u_7 + 152u_8\right)$$
$$s_6 = \frac{b\rho}{12600}\left(15v_1 - 68v_2 + 30v_3 - 27v_4 + 98v_5 - 32v_6 + 42v_7 + 152v_8\right)$$
$$s_7 = \frac{b\rho}{12600}\left(72u_1 - 41u_2 - 27u_3 - 54u_4 + 154u_5 - 72u_6 - 14u_7 + 192u_8\right)$$
$$s_8 = \frac{b\rho}{12600}\left(72v_1 - 41v_2 - 27v_3 - 54v_4 + 154v_5 - 72v_6 - 14v_7 + 192v_8\right)$$
$$s_9 = \frac{b\rho}{900}\left(19u_2 - 17u_1 + 7u_3 + 11u_4 - 72u_5 + 4u_6 - 8u_7 - 44u_8\right)$$
$$s_{10} = \frac{b\rho}{900}\left(19v_2 - 17v_1 + 7v_3 + 11v_4 - 72v_5 + 4v_6 - 8v_7 - 44v_8\right)$$
$$s_{11} = \frac{b\rho}{3150}\left(13u_2 - 32u_1 - 8u_3 - 18u_4 + 14u_5 + 240u_6 + 56u_7 - 20u_8\right)$$
$$s_{12} = \frac{b\rho}{3150}\left(13v_2 - 32v_1 - 8v_3 - 18v_4 + 14v_5 + 240v_6 + 56v_7 - 20v_8\right)$$
$$s_{13} = \frac{b\rho}{900}\left(u_2 - 3u_1 + 3u_3 - u_4 - 8u_5 + 16u_6 + 8u_7 - 16u_8\right)$$
$$s_{14} = \frac{b\rho}{900}\left(v_2 - 3v_1 + 3v_3 - v_4 - 8v_5 + 16v_6 + 8v_7 - 16v_8\right)$$
$$s_{15} = \frac{b\rho}{3150}\left(52u_2 - 43u_1 + 38u_3 + 48u_4 - 154u_5 - 20u_6 - 56u_7 - 320u_8\right)$$
$$s_{16} = \frac{b\rho}{3150}\left(52v_2 - 43v_1 + 38v_3 + 48v_4 - 154v_5 - 20v_6 - 56v_7 - 320v_8\right)$$
$$s_{17} = \frac{a\rho}{12600}\left(72u_2 - 390u_1 + 15u_3 + 43u_4 - 172u_5 - 42u_6 - 128u_7 - 238u_8\right)$$
$$s_{18} = \frac{a\rho}{12600}\left(72v_2 - 390v_1 + 15v_3 + 43v_4 - 172v_5 - 42v_6 - 128v_7 - 238v_8\right)$$
$$s_{19} = \frac{a\rho}{12600}\left(72u_1 - 54u_2 - 27u_3 - 41u_4 + 192u_5 - 14u_6 - 72u_7 + 154u_8\right)$$

$$s_{20} = \frac{a\rho}{12600}\,(72v_1 - 54v_2 - 27v_3 - 41v_4 + 192v_5 - 14v_6 - 72v_7 + 154v_8)$$
$$s_{21} = \frac{a\rho}{12600}\,(15u_1 - 27u_2 + 30u_3 - 68u_4 + 152u_5 + 42u_6 - 32u_7 + 98u_8)$$
$$s_{22} = \frac{a\rho}{12600}\,(15v_1 - 27v_2 + 30v_3 - 68v_4 + 152v_5 + 42v_6 - 32v_7 + 98v_8)$$
$$s_{23} = \frac{a\rho}{12600}\,(43u_1 - 41u_2 - 68u_3 + 86u_4 + 208u_5 + 14u_6 + 52u_7 + 266u_8)$$
$$s_{24} = \frac{a\rho}{12600}\,(43v_1 - 41v_2 - 68v_3 + 86v_4 + 208v_5 + 14v_6 + 52v_7 + 266v_8)$$
$$s_{25} = \frac{a\rho}{3150}\,(48u_2 - 43u_1 + 38u_3 + 52u_4 - 320u_5 - 56u_6 - 20u_7 - 154u_8)$$
$$s_{26} = \frac{a\rho}{3150}\,(48v_2 - 43v_1 + 38v_3 + 52v_4 - 320v_5 - 56v_6 - 20v_7 - 154v_8)$$
$$s_{27} = \frac{a\rho}{900}\,(3u_3 - u_2 - 3u_1 + u_4 - 16u_5 + 8u_6 + 16u_7 - 8u_8)$$
$$s_{28} = \frac{a\rho}{900}\,(3v_3 - v_2 - 3v_1 + v_4 - 16v_5 + 8v_6 + 16v_7 - 8v_8)$$
$$s_{29} = \frac{a\rho}{3150}\,(13u_4 - 18u_2 - 8u_3 - 32u_1 - 20u_5 + 56u_6 + 240u_7 + 14u_8)$$
$$s_{30} = \frac{a\rho}{3150}\,(13v_4 - 18v_2 - 8v_3 - 32v_1 - 20v_5 + 56v_6 + 240v_7 + 14v_8)$$
$$s_{31} = \frac{a\rho}{900}\,(11u_2 - 17u_1 + 7u_3 + 19u_4 - 44u_5 - 8u_6 + 4u_7 - 72u_8)$$
$$s_{32} = \frac{a\rho}{900}\,(11v_2 - 17v_1 + 7v_3 + 19v_4 - 44v_5 - 8v_6 + 4v_7 - 72v_8)$$
$$s_{33} = \frac{b\rho}{12600}\,(41u_3 - 43u_2 - 86u_1 + 68u_4 - 266u_5 - 208u_6 - 14u_7 - 52u_8)$$
$$s_{34} = \frac{b\rho}{12600}\,(41v_3 - 43v_2 - 86v_1 + 68v_4 - 266v_5 - 208v_6 - 14v_7 - 52v_8)$$
$$s_{35} = \frac{b\rho}{12600}\,(390u_2 - 43u_1 - 72u_3 - 15u_4 + 238u_5 + 172u_6 + 42u_7 + 128u_8)$$
$$s_{36} = \frac{b\rho}{12600}\,(390v_2 - 43v_1 - 72v_3 - 15v_4 + 238v_5 + 172v_6 + 42v_7 + 128v_8)$$
$$s_{37} = \frac{b\rho}{12600}\,(41u_1 - 72u_2 + 54u_3 + 27u_4 - 154u_5 - 192u_6 + 14u_7 + 72u_8)$$
$$s_{38} = \frac{b\rho}{12600}\,(41v_1 - 72v_2 + 54v_3 + 27v_4 - 154v_5 - 192v_6 + 14v_7 + 72v_8)$$
$$s_{39} = \frac{b\rho}{12600}\,(68u_1 - 15u_2 + 27u_3 - 30u_4 - 98u_5 - 152u_6 - 42u_7 + 32u_8)$$
$$s_{40} = \frac{b\rho}{12600}\,(68v_1 - 15v_2 + 27v_3 - 30v_4 - 98v_5 - 152v_6 - 42v_7 + 32v_8)$$
$$s_{41} = \frac{b\rho}{900}\,(17u_2 - 19u_1 - 11u_3 - 7u_4 + 72u_5 + 44u_6 + 8u_7 - 4u_8)$$
$$s_{42} = \frac{b\rho}{900}\,(17v_2 - 19v_1 - 11v_3 - 7v_4 + 72v_5 + 44v_6 + 8v_7 - 4v_8)$$
$$s_{43} = \frac{b\rho}{3150}\,(43u_2 - 52u_1 - 48u_3 - 38u_4 + 154u_5 + 320u_6 + 56u_7 + 20u_8)$$
$$s_{44} = \frac{b\rho}{3150}\,(43v_2 - 52v_1 - 48v_3 - 38v_4 + 154v_5 + 320v_6 + 56v_7 + 20v_8)$$
$$s_{45} = \frac{b\rho}{900}\,(3u_2 - u_1 + u_3 - 3u_4 + 8u_5 + 16u_6 - 8u_7 - 16u_8)$$
$$s_{46} = \frac{b\rho}{900}\,(3v_2 - v_1 + v_3 - 3v_4 + 8v_5 + 16v_6 - 8v_7 - 16v_8)$$

$$s_{47} = \frac{b\rho}{3150}\,(32u_2 - 13u_1 + 18u_3 + 8u_4 - 14u_5 + 20u_6 - 56u_7 - 240u_8)$$
$$s_{48} = \frac{b\rho}{3150}\,(32v_2 - 13v_1 + 18v_3 + 8v_4 - 14v_5 + 20v_6 - 56v_7 - 240v_8)$$
$$s_{49} = \frac{a\rho}{12600}\,(72u_2 - 54u_1 - 41u_3 - 27u_4 + 192u_5 + 154u_6 - 72u_7 - 14u_8)$$
$$s_{50} = \frac{a\rho}{12600}\,(72v_2 - 54v_1 - 41v_3 - 27v_4 + 192v_5 + 154v_6 - 72v_7 - 14v_8)$$
$$s_{51} = \frac{a\rho}{12600}\,(72u_1 - 390u_2 + 43u_3 + 15u_4 - 172u_5 - 238u_6 - 128u_7 - 42u_8)$$
$$s_{52} = \frac{a\rho}{12600}\,(72v_1 - 390v_2 + 43v_3 + 15v_4 - 172v_5 - 238v_6 - 128v_7 - 42v_8)$$
$$s_{53} = \frac{a\rho}{12600}\,(43u_2 - 41u_1 + 86u_3 - 68u_4 + 208u_5 + 266u_6 + 52u_7 + 14u_8)$$
$$s_{54} = \frac{a\rho}{12600}\,(43v_2 - 41v_1 + 86v_3 - 68v_4 + 208v_5 + 266v_6 + 52v_7 + 14v_8)$$
$$s_{55} = \frac{a\rho}{12600}\,(15u_2 - 27u_1 - 68u_3 + 30u_4 + 152u_5 + 98u_6 - 32u_7 + 42u_8)$$
$$s_{56} = \frac{a\rho}{12600}\,(15v_2 - 27v_1 - 68v_3 + 30v_4 + 152v_5 + 98v_6 - 32v_7 + 42v_8)$$
$$s_{57} = \frac{a\rho}{3150}\,(48u_1 - 43u_2 + 52u_3 + 38u_4 - 320u_5 - 154u_6 - 20u_7 - 56u_8)$$
$$s_{58} = \frac{a\rho}{3150}\,(48v_1 - 43v_2 + 52v_3 + 38v_4 - 320v_5 - 154v_6 - 20v_7 - 56v_8)$$
$$s_{59} = \frac{a\rho}{900}\,(11u_1 - 17u_2 + 19u_3 + 7u_4 - 44u_5 - 72u_6 + 4u_7 - 8u_8)$$
$$s_{60} = \frac{a\rho}{900}\,(11v_1 - 17v_2 + 19v_3 + 7v_4 - 44v_5 - 72v_6 + 4v_7 - 8v_8)$$
$$s_{61} = \frac{a\rho}{3150}\,(13u_3 - 32u_2 - 18u_1 - 8u_4 - 20u_5 + 14u_6 + 240u_7 + 56u_8)$$
$$s_{62} = \frac{a\rho}{3150}\,(13v_3 - 32v_2 - 18v_1 - 8v_4 - 20v_5 + 14v_6 + 240v_7 + 56v_8)$$
$$s_{63} = \frac{a\rho}{900}\,(u_3 - 3u_2 - u_1 + 3u_4 - 16u_5 - 8u_6 + 16u_7 + 8u_8)$$
$$s_{64} = \frac{a\rho}{900}\,(v_3 - 3v_2 - v_1 + 3v_4 - 16v_5 - 8v_6 + 16v_7 + 8v_8)$$
$$s_{65} = \frac{b\rho}{12600}\,(27u_2 - 30u_1 - 15u_3 + 68u_4 - 42u_5 - 152u_6 - 98u_7 + 32u_8)$$
$$s_{66} = \frac{b\rho}{12600}\,(27v_2 - 30v_1 - 15v_3 + 68v_4 - 42v_5 - 152v_6 - 98v_7 + 32v_8)$$
$$s_{67} = \frac{b\rho}{12600}\,(27u_1 + 54u_2 - 72u_3 + 41u_4 + 14u_5 - 192u_6 - 154u_7 + 72u_8)$$
$$s_{68} = \frac{b\rho}{12600}\,(27v_1 + 54v_2 - 72v_3 + 41v_4 + 14v_5 - 192v_6 - 154v_7 + 72v_8)$$
$$s_{69} = \frac{b\rho}{12600}\,(390u_3 - 72u_2 - 15u_1 - 43u_4 + 42u_5 + 172u_6 + 238u_7 + 128u_8)$$
$$s_{70} = \frac{b\rho}{12600}\,(390v_3 - 72v_2 - 15v_1 - 43v_4 + 42v_5 + 172v_6 + 238v_7 + 128v_8)$$
$$s_{71} = \frac{b\rho}{12600}\,(68u_1 + 41u_2 - 43u_3 - 86u_4 - 14u_5 - 208u_6 - 266u_7 - 52u_8)$$
$$s_{72} = \frac{b\rho}{12600}\,(68v_1 + 41v_2 - 43v_3 - 86v_4 - 14v_5 - 208v_6 - 266v_7 - 52v_8)$$
$$s_{73} = \frac{b\rho}{3150}\,(43u_3 - 48u_2 - 38u_1 - 52u_4 + 56u_5 + 320u_6 + 154u_7 + 20u_8)$$

$$s_{74} = \frac{b\rho}{3150}\left(43v_3 - 48v_2 - 38v_1 - 52v_4 + 56v_5 + 320v_6 + 154v_7 + 20v_8\right)$$
$$s_{75} = \frac{b\rho}{900}\left(17u_3 - 11u_2 - 7u_1 - 19u_4 + 8u_5 + 44u_6 + 72u_7 - 4u_8\right)$$
$$s_{76} = \frac{b\rho}{900}\left(17v_3 - 11v_2 - 7v_1 - 19v_4 + 8v_5 + 44v_6 + 72v_7 - 4v_8\right)$$
$$s_{77} = \frac{b\rho}{3150}\left(8u_1 + 18u_2 + 32u_3 - 13u_4 - 56u_5 + 20u_6 - 14u_7 - 240u_8\right)$$
$$s_{78} = \frac{b\rho}{3150}\left(8v_1 + 18v_2 + 32v_3 - 13v_4 - 56v_5 + 20v_6 - 14v_7 - 240v_8\right)$$
$$s_{79} = \frac{a\rho}{12600}\left(68u_2 - 30u_1 - 15u_3 + 27u_4 + 32u_5 - 98u_6 - 152u_7 - 42u_8\right)$$
$$s_{80} = \frac{a\rho}{12600}\left(68v_2 - 30v_1 - 15v_3 + 27v_4 + 32v_5 - 98v_6 - 152v_7 - 42v_8\right)$$
$$s_{81} = \frac{a\rho}{12600}\left(68u_1 - 86u_2 - 43u_3 + 41u_4 - 52u_5 - 266u_6 - 208u_7 - 14u_8\right)$$
$$s_{82} = \frac{a\rho}{12600}\left(68v_1 - 86v_2 - 43v_3 + 41v_4 - 52v_5 - 266v_6 - 208v_7 - 14v_8\right)$$
$$s_{83} = \frac{a\rho}{12600}\left(390u_3 - 43u_2 - 15u_1 - 72u_4 + 128u_5 + 238u_6 + 172u_7 + 42u_8\right)$$
$$s_{84} = \frac{a\rho}{12600}\left(390v_3 - 43v_2 - 15v_1 - 72v_4 + 128v_5 + 238v_6 + 172v_7 + 42v_8\right)$$
$$s_{85} = \frac{a\rho}{12600}\left(27u_1 + 41u_2 - 72u_3 + 54u_4 + 72u_5 - 154u_6 - 192u_7 + 14u_8\right)$$
$$s_{86} = \frac{a\rho}{12600}\left(27v_1 + 41v_2 - 72v_3 + 54v_4 + 72v_5 - 154v_6 - 192v_7 + 14v_8\right)$$
$$s_{87} = \frac{a\rho}{3150}\left(8u_1 - 13u_2 + 32u_3 + 18u_4 - 240u_5 - 14u_6 + 20u_7 - 56u_8\right)$$
$$s_{88} = \frac{a\rho}{3150}\left(8v_1 - 13v_2 + 32v_3 + 18v_4 - 240v_5 - 14v_6 + 20v_7 - 56v_8\right)$$
$$s_{89} = \frac{a\rho}{900}\left(17u_3 - 19u_2 - 7u_1 - 11u_4 - 4u_5 + 72u_6 + 44u_7 + 8u_8\right)$$
$$s_{90} = \frac{a\rho}{900}\left(17v_3 - 19v_2 - 7v_1 - 11v_4 - 4v_5 + 72v_6 + 44v_7 + 8v_8\right)$$
$$s_{91} = \frac{a\rho}{3150}\left(43u_3 - 52u_2 - 38u_1 - 48u_4 + 20u_5 + 154u_6 + 320u_7 + 56u_8\right)$$
$$s_{92} = \frac{a\rho}{3150}\left(43v_3 - 52v_2 - 38v_1 - 48v_4 + 20v_5 + 154v_6 + 320v_7 + 56v_8\right)$$
$$s_{93} = \frac{b\rho}{12600}\left(72u_4 - 27u_2 - 41u_3 - 54u_1 - 14u_5 - 72u_6 + 154u_7 + 192u_8\right)$$
$$s_{94} = \frac{b\rho}{12600}\left(72v_4 - 27v_2 - 41v_3 - 54v_1 - 14v_5 - 72v_6 + 154v_7 + 192v_8\right)$$
$$s_{95} = \frac{b\rho}{12600}\left(30u_2 - 27u_1 - 68u_3 + 15u_4 + 42u_5 - 32u_6 + 98u_7 + 152u_8\right)$$
$$s_{96} = \frac{b\rho}{12600}\left(30v_2 - 27v_1 - 68v_3 + 15v_4 + 42v_5 - 32v_6 + 98v_7 + 152v_8\right)$$
$$s_{97} = \frac{b\rho}{12600}\left(86u_3 - 68u_2 - 41u_1 + 43u_4 + 14u_5 + 52u_6 + 266u_7 + 208u_8\right)$$
$$s_{98} = \frac{b\rho}{12600}\left(86v_3 - 68v_2 - 41v_1 + 43v_4 + 14v_5 + 52v_6 + 266v_7 + 208v_8\right)$$
$$s_{99} = \frac{b\rho}{12600}\left(72u_1 + 15u_2 + 43u_3 - 390u_4 - 42u_5 - 128u_6 - 238u_7 - 172u_8\right)$$
$$s_{100} = \frac{b\rho}{12600}\left(72v_1 + 15v_2 + 43v_3 - 390v_4 - 42v_5 - 128v_6 - 238v_7 - 172v_8\right)$$

$$s_{101} = \frac{b\rho}{3150}\left(13u_3 - 8u_2 - 18u_1 - 32u_4 + 56u_5 + 240u_6 + 14u_7 - 20u_8\right)$$
$$s_{102} = \frac{b\rho}{3150}\left(13v_3 - 8v_2 - 18v_1 - 32v_4 + 56v_5 + 240v_6 + 14v_7 - 20v_8\right)$$
$$s_{103} = \frac{b\rho}{900}\left(11u_1 + 7u_2 + 19u_3 - 17u_4 - 8u_5 + 4u_6 - 72u_7 - 44u_8\right)$$
$$s_{104} = \frac{b\rho}{900}\left(11v_1 + 7v_2 + 19v_3 - 17v_4 - 8v_5 + 4v_6 - 72v_7 - 44v_8\right)$$
$$s_{105} = \frac{b\rho}{3150}\left(48u_1 + 38u_2 + 52u_3 - 43u_4 - 56u_5 - 20u_6 - 154u_7 - 320u_8\right)$$
$$s_{106} = \frac{b\rho}{3150}\left(48v_1 + 38v_2 + 52v_3 - 43v_4 - 56v_5 - 20v_6 - 154v_7 - 320v_8\right)$$
$$s_{107} = \frac{a\rho}{12600}\left(68u_2 - 86u_1 + 41u_3 - 43u_4 - 52u_5 - 14u_6 - 208u_7 - 266u_8\right)$$
$$s_{108} = \frac{a\rho}{12600}\left(68v_2 - 86v_1 + 41v_3 - 43v_4 - 52v_5 - 14v_6 - 208v_7 - 266v_8\right)$$
$$s_{109} = \frac{a\rho}{12600}\left(68u_1 - 30u_2 + 27u_3 - 15u_4 + 32u_5 - 42u_6 - 152u_7 - 98u_8\right)$$
$$s_{110} = \frac{a\rho}{12600}\left(68v_1 - 30v_2 + 27v_3 - 15v_4 + 32v_5 - 42v_6 - 152v_7 - 98v_8\right)$$
$$s_{111} = \frac{a\rho}{12600}\left(41u_1 + 27u_2 + 54u_3 - 72u_4 + 72u_5 + 14u_6 - 192u_7 - 154u_8\right)$$
$$s_{112} = \frac{a\rho}{12600}\left(41v_1 + 27v_2 + 54v_3 - 72v_4 + 72v_5 + 14v_6 - 192v_7 - 154v_8\right)$$
$$s_{113} = \frac{a\rho}{12600}\left(390u_4 - 15u_2 - 72u_3 - 43u_1 + 128u_5 + 42u_6 + 172u_7 + 238u_8\right)$$
$$s_{114} = \frac{a\rho}{12600}\left(390v_4 - 15v_2 - 72v_3 - 43v_1 + 128v_5 + 42v_6 + 172v_7 + 238v_8\right)$$
$$s_{115} = \frac{a\rho}{3150}\left(8u_2 - 13u_1 + 18u_3 + 32u_4 - 240u_5 - 56u_6 + 20u_7 - 14u_8\right)$$
$$s_{116} = \frac{a\rho}{3150}\left(8v_2 - 13v_1 + 18v_3 + 32v_4 - 240v_5 - 56v_6 + 20v_7 - 14v_8\right)$$
$$s_{117} = \frac{a\rho}{3150}\left(43u_4 - 38u_2 - 48u_3 - 52u_1 + 20u_5 + 56u_6 + 320u_7 + 154u_8\right)$$
$$s_{118} = \frac{a\rho}{3150}\left(43v_4 - 38v_2 - 48v_3 - 52v_1 + 20v_5 + 56v_6 + 320v_7 + 154v_8\right)$$
$$s_{119} = \frac{a\rho}{900}\left(17u_4 - 7u_2 - 11u_3 - 19u_1 - 4u_5 + 8u_6 + 44u_7 + 72u_8\right)$$
$$s_{120} = \frac{a\rho}{900}\left(17v_4 - 7v_2 - 11v_3 - 19v_1 - 4v_5 + 8v_6 + 44v_7 + 72v_8\right)$$
$$s_{121} = \frac{b\rho}{450}\left(17u_1 - 2u_3 - 5u_4 + 18u_5 + 12u_6 + 2u_7 + 8u_8\right)$$
$$s_{122} = \frac{b\rho}{450}\left(17v_1 - 2v_3 - 5v_4 + 18v_5 + 12v_6 + 2v_7 + 8v_8\right)$$
$$s_{123} = \frac{b\rho}{450}\left(5u_3 - 17u_2 + 2u_4 - 18u_5 - 8u_6 - 2u_7 - 12u_8\right)$$
$$s_{124} = \frac{b\rho}{450}\left(5v_3 - 17v_2 + 2v_4 - 18v_5 - 8v_6 - 2v_7 - 12v_8\right)$$
$$s_{125} = \frac{b\rho}{450}\left(5u_2 - 2u_1 - 3u_3 + 2u_5 + 8u_6 - 2u_7 - 8u_8\right)$$
$$s_{126} = \frac{b\rho}{450}\left(5v_2 - 2v_1 - 3v_3 + 2v_5 + 8v_6 - 2v_7 - 8v_8\right)$$
$$s_{127} = \frac{b\rho}{450}\left(2u_2 - 5u_1 + 3u_4 - 2u_5 + 8u_6 + 2u_7 - 8u_8\right)$$

$$s_{128} = \frac{b\rho}{450}\left(2v_2 - 5v_1 + 3v_4 - 2v_5 + 8v_6 + 2v_7 - 8v_8\right)$$
$$s_{129} = \frac{b\rho}{225}\left(9u_1 - 9u_2 + u_3 - u_4 - 12u_6 + 12u_8\right)$$
$$s_{130} = \frac{b\rho}{225}\left(9v_1 - 9v_2 + v_3 - v_4 - 12v_6 + 12v_8\right)$$
$$s_{131} = \frac{b\rho}{225}\left(6u_1 - 4u_2 + 4u_3 + 4u_4 - 12u_5 - 40u_6 - 8u_7\right)$$
$$s_{132} = \frac{b\rho}{225}\left(6v_1 - 4v_2 + 4v_3 + 4v_4 - 12v_5 - 40v_6 - 8v_7\right)$$
$$s_{133} = \frac{b\rho}{225}\left(u_1 - u_2 - u_3 + u_4 - 8u_6 + 8u_8\right)$$
$$s_{134} = \frac{b\rho}{225}\left(v_1 - v_2 - v_3 + v_4 - 8v_6 + 8v_8\right)$$
$$s_{135} = \frac{b\rho}{225}\left(4u_1 - 6u_2 - 4u_3 - 4u_4 + 12u_5 + 8u_7 + 40u_8\right)$$
$$s_{136} = \frac{b\rho}{225}\left(4v_1 - 6v_2 - 4v_3 - 4v_4 + 12v_5 + 8v_7 + 40v_8\right)$$
$$s_{137} = \frac{a\rho}{3150}\left(100u_5 - 36u_2 - 46u_3 - 39u_4 - 64u_1 + 112u_6 + 120u_7 + 98u_8\right)$$
$$s_{138} = \frac{a\rho}{3150}\left(100v_5 - 36v_2 - 46v_3 - 39v_4 - 64v_1 + 112v_6 + 120v_7 + 98v_8\right)$$
$$s_{139} = \frac{a\rho}{3150}\left(100u_5 - 64u_2 - 39u_3 - 46u_4 - 36u_1 + 98u_6 + 120u_7 + 112u_8\right)$$
$$s_{140} = \frac{a\rho}{3150}\left(100v_5 - 64v_2 - 39v_3 - 46v_4 - 36v_1 + 98v_6 + 120v_7 + 112v_8\right)$$
$$s_{141} = \frac{a\rho}{3150}\left(120u_5 - 39u_2 - 64u_3 - 36u_4 - 46u_1 + 98u_6 + 100u_7 + 112u_8\right)$$
$$s_{142} = \frac{a\rho}{3150}\left(120v_5 - 39v_2 - 64v_3 - 36v_4 - 46v_1 + 98v_6 + 100v_7 + 112v_8\right)$$
$$s_{143} = \frac{a\rho}{3150}\left(120u_5 - 46u_2 - 36u_3 - 64u_4 - 39u_1 + 112u_6 + 100u_7 + 98u_8\right)$$
$$s_{144} = \frac{a\rho}{3150}\left(120v_5 - 46v_2 - 36v_3 - 64v_4 - 39v_1 + 112v_6 + 100v_7 + 98v_8\right)$$
$$s_{145} = \frac{a\rho}{315}\left(10u_1 + 10u_2 + 12u_3 + 12u_4 - 48u_5 - 28u_6 - 24u_7 - 28u_8\right)$$
$$s_{146} = \frac{a\rho}{315}\left(10v_1 + 10v_2 + 12v_3 + 12v_4 - 48v_5 - 28v_6 - 24v_7 - 28v_8\right)$$
$$s_{147} = \frac{a\rho}{225}\left(8u_1 + 7u_2 + 7u_3 + 8u_4 - 20u_5 - 24u_6 - 20u_7 - 16u_8\right)$$
$$s_{148} = \frac{a\rho}{225}\left(8v_1 + 7v_2 + 7v_3 + 8v_4 - 20v_5 - 24v_6 - 20v_7 - 16v_8\right)$$
$$s_{149} = \frac{a\rho}{315}\left(12u_1 + 12u_2 + 10u_3 + 10u_4 - 24u_5 - 28u_6 - 48u_7 - 28u_8\right)$$
$$s_{150} = \frac{a\rho}{315}\left(12v_1 + 12v_2 + 10v_3 + 10v_4 - 24v_5 - 28v_6 - 48v_7 - 28v_8\right)$$
$$s_{151} = \frac{a\rho}{225}\left(7u_1 + 8u_2 + 8u_3 + 7u_4 - 20u_5 - 16u_6 - 20u_7 - 24u_8\right)$$
$$s_{152} = \frac{a\rho}{225}\left(7v_1 + 8v_2 + 8v_3 + 7v_4 - 20v_5 - 16v_6 - 20v_7 - 24v_8\right)$$
$$s_{153} = \frac{b\rho}{3150}\left(64u_1 + 39u_2 + 46u_3 + 36u_4 - 98u_5 - 120u_6 - 112u_7 - 100u_8\right)$$
$$s_{154} = \frac{b\rho}{3150}\left(64v_1 + 39v_2 + 46v_3 + 36v_4 - 98v_5 - 120v_6 - 112v_7 - 100v_8\right)$$

$$s_{155} = \frac{b\rho}{3150}\left(39u_1 + 64u_2 + 36u_3 + 46u_4 - 98u_5 - 100u_6 - 112u_7 - 120u_8\right)$$
$$s_{156} = \frac{b\rho}{3150}\left(39v_1 + 64v_2 + 36v_3 + 46v_4 - 98v_5 - 100v_6 - 112v_7 - 120v_8\right)$$
$$s_{157} = \frac{b\rho}{3150}\left(46u_1 + 36u_2 + 64u_3 + 39u_4 - 112u_5 - 100u_6 - 98u_7 - 120u_8\right)$$
$$s_{158} = \frac{b\rho}{3150}\left(46v_1 + 36v_2 + 64v_3 + 39v_4 - 112v_5 - 100v_6 - 98v_7 - 120v_8\right)$$
$$s_{159} = \frac{b\rho}{3150}\left(36u_1 + 46u_2 + 39u_3 + 64u_4 - 112u_5 - 120u_6 - 98u_7 - 100u_8\right)$$
$$s_{160} = \frac{b\rho}{3150}\left(36v_1 + 46v_2 + 39v_3 + 64v_4 - 112v_5 - 120v_6 - 98v_7 - 100v_8\right)$$
$$s_{161} = \frac{b\rho}{225}\left(24u_5 - 7u_2 - 8u_3 - 8u_4 - 7u_1 + 20u_6 + 16u_7 + 20u_8\right)$$
$$s_{162} = \frac{b\rho}{225}\left(24v_5 - 7v_2 - 8v_3 - 8v_4 - 7v_1 + 20v_6 + 16v_7 + 20v_8\right)$$
$$s_{163} = \frac{b\rho}{315}\left(28u_5 - 10u_2 - 10u_3 - 12u_4 - 12u_1 + 48u_6 + 28u_7 + 24u_8\right)$$
$$s_{164} = \frac{b\rho}{315}\left(28v_5 - 10v_2 - 10v_3 - 12v_4 - 12v_1 + 48v_6 + 28v_7 + 24v_8\right)$$
$$s_{165} = \frac{b\rho}{225}\left(16u_5 - 8u_2 - 7u_3 - 7u_4 - 8u_1 + 20u_6 + 24u_7 + 20u_8\right)$$
$$s_{166} = \frac{b\rho}{225}\left(16v_5 - 8v_2 - 7v_3 - 7v_4 - 8v_1 + 20v_6 + 24v_7 + 20v_8\right)$$
$$s_{167} = \frac{b\rho}{315}\left(28u_5 - 12u_2 - 12u_3 - 10u_4 - 10u_1 + 24u_6 + 28u_7 + 48u_8\right)$$
$$s_{168} = \frac{b\rho}{315}\left(28v_5 - 12v_2 - 12v_3 - 10v_4 - 10v_1 + 24v_6 + 28v_7 + 48v_8\right)$$
$$s_{169} = \frac{a\rho}{450}\left(3u_1 - 5u_2 + 2u_3 - 8u_5 - 2u_6 + 8u_7 + 2u_8\right)$$
$$s_{170} = \frac{a\rho}{450}\left(3v_1 - 5v_2 + 2v_3 - 8v_5 - 2v_6 + 8v_7 + 2v_8\right)$$
$$s_{171} = \frac{a\rho}{450}\left(17u_2 - 5u_1 - 2u_4 + 8u_5 + 18u_6 + 12u_7 + 2u_8\right)$$
$$s_{172} = \frac{a\rho}{450}\left(17v_2 - 5v_1 - 2v_4 + 8v_5 + 18v_6 + 12v_7 + 2v_8\right)$$
$$s_{173} = \frac{a\rho}{450}\left(2u_1 - 17u_3 + 5u_4 - 12u_5 - 18u_6 - 8u_7 - 2u_8\right)$$
$$s_{174} = \frac{a\rho}{450}\left(2v_1 - 17v_3 + 5v_4 - 12v_5 - 18v_6 - 8v_7 - 2v_8\right)$$
$$s_{175} = \frac{a\rho}{450}\left(5u_3 - 2u_2 - 3u_4 - 8u_5 + 2u_6 + 8u_7 - 2u_8\right)$$
$$s_{176} = \frac{a\rho}{450}\left(5v_3 - 2v_2 - 3v_4 - 8v_5 + 2v_6 + 8v_7 - 2v_8\right)$$
$$s_{177} = \frac{a\rho}{225}\left(4u_2 - 4u_1 - 6u_3 - 4u_4 + 40u_5 + 12u_6 + 8u_8\right)$$
$$s_{178} = \frac{a\rho}{225}\left(4v_2 - 4v_1 - 6v_3 - 4v_4 + 40v_5 + 12v_6 + 8v_8\right)$$
$$s_{179} = \frac{a\rho}{225}\left(9u_2 - u_1 - 9u_3 + u_4 + 12u_5 - 12u_7\right)$$
$$s_{180} = \frac{a\rho}{225}\left(9v_2 - v_1 - 9v_3 + v_4 + 12v_5 - 12v_7\right)$$
$$s_{181} = \frac{a\rho}{225}\left(4u_1 + 6u_2 - 4u_3 + 4u_4 - 12u_6 - 40u_7 - 8u_8\right)$$

$$s_{182} = \frac{a\rho}{225}\,(4v_1 + 6v_2 - 4v_3 + 4v_4 - 12v_6 - 40v_7 - 8v_8)$$
$$s_{183} = \frac{a\rho}{225}\,(u_1 + u_2 - u_3 - u_4 + 8u_5 - 8u_7)$$
$$s_{184} = \frac{a\rho}{225}\,(v_1 + v_2 - v_3 - v_4 + 8v_5 - 8v_7)$$
$$s_{185} = \frac{b\rho}{450}\,(3u_1 + 2u_3 - 5u_4 + 2u_5 + 8u_6 - 2u_7 - 8u_8)$$
$$s_{186} = \frac{b\rho}{450}\,(3v_1 + 2v_3 - 5v_4 + 2v_5 + 8v_6 - 2v_7 - 8v_8)$$
$$s_{187} = \frac{b\rho}{450}\,(5u_3 - 3u_2 - 2u_4 - 2u_5 + 8u_6 + 2u_7 - 8u_8)$$
$$s_{188} = \frac{b\rho}{450}\,(5v_3 - 3v_2 - 2v_4 - 2v_5 + 8v_6 + 2v_7 - 8v_8)$$
$$s_{189} = \frac{b\rho}{450}\,(2u_1 + 5u_2 - 17u_3 - 2u_5 - 8u_6 - 18u_7 - 12u_8)$$
$$s_{190} = \frac{b\rho}{450}\,(2v_1 + 5v_2 - 17v_3 - 2v_5 - 8v_6 - 18v_7 - 12v_8)$$
$$s_{191} = \frac{b\rho}{450}\,(17u_4 - 2u_2 - 5u_1 + 2u_5 + 12u_6 + 18u_7 + 8u_8)$$
$$s_{192} = \frac{b\rho}{450}\,(17v_4 - 2v_2 - 5v_1 + 2v_5 + 12v_6 + 18v_7 + 8v_8)$$
$$s_{193} = \frac{b\rho}{225}\,(4u_1 + 4u_2 - 4u_3 + 6u_4 - 8u_5 - 40u_6 - 12u_7)$$
$$s_{194} = \frac{b\rho}{225}\,(4v_1 + 4v_2 - 4v_3 + 6v_4 - 8v_5 - 40v_6 - 12v_7)$$
$$s_{195} = \frac{b\rho}{225}\,(u_2 - u_1 - 9u_3 + 9u_4 - 12u_6 + 12u_8)$$
$$s_{196} = \frac{b\rho}{225}\,(v_2 - v_1 - 9v_3 + 9v_4 - 12v_6 + 12v_8)$$
$$s_{197} = \frac{b\rho}{225}\,(4u_4 - 4u_2 - 6u_3 - 4u_1 + 8u_5 + 12u_7 + 40u_8)$$
$$s_{198} = \frac{b\rho}{225}\,(4v_4 - 4v_2 - 6v_3 - 4v_1 + 8v_5 + 12v_7 + 40v_8)$$
$$s_{199} = \frac{a\rho}{3150}\,(64u_1 + 36u_2 + 46u_3 + 39u_4 - 100u_5 - 112u_6 - 120u_7 - 98u_8)$$
$$s_{200} = \frac{a\rho}{3150}\,(64v_1 + 36v_2 + 46v_3 + 39v_4 - 100v_5 - 112v_6 - 120v_7 - 98v_8)$$
$$s_{201} = \frac{a\rho}{3150}\,(36u_1 + 64u_2 + 39u_3 + 46u_4 - 100u_5 - 98u_6 - 120u_7 - 112u_8)$$
$$s_{202} = \frac{a\rho}{3150}\,(36v_1 + 64v_2 + 39v_3 + 46v_4 - 100v_5 - 98v_6 - 120v_7 - 112v_8)$$
$$s_{203} = \frac{a\rho}{3150}\,(46u_1 + 39u_2 + 64u_3 + 36u_4 - 120u_5 - 98u_6 - 100u_7 - 112u_8)$$
$$s_{204} = \frac{a\rho}{3150}\,(46v_1 + 39v_2 + 64v_3 + 36v_4 - 120v_5 - 98v_6 - 100v_7 - 112v_8)$$
$$s_{205} = \frac{a\rho}{3150}\,(39u_1 + 46u_2 + 36u_3 + 64u_4 - 120u_5 - 112u_6 - 100u_7 - 98u_8)$$
$$s_{206} = \frac{a\rho}{3150}\,(39v_1 + 46v_2 + 36v_3 + 64v_4 - 120v_5 - 112v_6 - 100v_7 - 98v_8)$$
$$s_{207} = \frac{a\rho}{315}\,(48u_5 - 10u_2 - 12u_3 - 12u_4 - 10u_1 + 28u_6 + 24u_7 + 28u_8)$$
$$s_{208} = \frac{a\rho}{315}\,(48v_5 - 10v_2 - 12v_3 - 12v_4 - 10v_1 + 28v_6 + 24v_7 + 28v_8)$$

$$s_{209} = \frac{a\rho}{225}\left(20u_5 - 7u_2 - 7u_3 - 8u_4 - 8u_1 + 24u_6 + 20u_7 + 16u_8\right)$$
$$s_{210} = \frac{a\rho}{225}\left(20v_5 - 7v_2 - 7v_3 - 8v_4 - 8v_1 + 24v_6 + 20v_7 + 16v_8\right)$$
$$s_{211} = \frac{a\rho}{315}\left(24u_5 - 12u_2 - 10u_3 - 10u_4 - 12u_1 + 28u_6 + 48u_7 + 28u_8\right)$$
$$s_{212} = \frac{a\rho}{315}\left(24v_5 - 12v_2 - 10v_3 - 10v_4 - 12v_1 + 28v_6 + 48v_7 + 28v_8\right)$$
$$s_{213} = \frac{a\rho}{225}\left(20u_5 - 8u_2 - 8u_3 - 7u_4 - 7u_1 + 16u_6 + 20u_7 + 24u_8\right)$$
$$s_{214} = \frac{a\rho}{225}\left(20v_5 - 8v_2 - 8v_3 - 7v_4 - 7v_1 + 16v_6 + 20v_7 + 24v_8\right)$$
$$s_{215} = \frac{b\rho}{3150}\left(98u_5 - 39u_2 - 46u_3 - 36u_4 - 64u_1 + 120u_6 + 112u_7 + 100u_8\right)$$
$$s_{216} = \frac{b\rho}{3150}\left(98v_5 - 39v_2 - 46v_3 - 36v_4 - 64v_1 + 120v_6 + 112v_7 + 100v_8\right)$$
$$s_{217} = \frac{b\rho}{3150}\left(98u_5 - 64u_2 - 36u_3 - 46u_4 - 39u_1 + 100u_6 + 112u_7 + 120u_8\right)$$
$$s_{218} = \frac{b\rho}{3150}\left(98v_5 - 64v_2 - 36v_3 - 46v_4 - 39v_1 + 100v_6 + 112v_7 + 120v_8\right)$$
$$s_{219} = \frac{b\rho}{3150}\left(112u_5 - 36u_2 - 64u_3 - 39u_4 - 46u_1 + 100u_6 + 98u_7 + 120u_8\right)$$
$$s_{220} = \frac{b\rho}{3150}\left(112v_5 - 36v_2 - 64v_3 - 39v_4 - 46v_1 + 100v_6 + 98v_7 + 120v_8\right)$$
$$s_{221} = \frac{b\rho}{3150}\left(112u_5 - 46u_2 - 39u_3 - 64u_4 - 36u_1 + 120u_6 + 98u_7 + 100u_8\right)$$
$$s_{222} = \frac{b\rho}{3150}\left(112v_5 - 46v_2 - 39v_3 - 64v_4 - 36v_1 + 120v_6 + 98v_7 + 100v_8\right)$$
$$s_{223} = \frac{b\rho}{225}\left(7u_1 + 7u_2 + 8u_3 + 8u_4 - 24u_5 - 20u_6 - 16u_7 - 20u_8\right)$$
$$s_{224} = \frac{b\rho}{225}\left(7v_1 + 7v_2 + 8v_3 + 8v_4 - 24v_5 - 20v_6 - 16v_7 - 20v_8\right)$$
$$s_{225} = \frac{b\rho}{315}\left(12u_1 + 10u_2 + 10u_3 + 12u_4 - 28u_5 - 48u_6 - 28u_7 - 24u_8\right)$$
$$s_{226} = \frac{b\rho}{315}\left(12v_1 + 10v_2 + 10v_3 + 12v_4 - 28v_5 - 48v_6 - 28v_7 - 24v_8\right)$$
$$s_{227} = \frac{b\rho}{225}\left(8u_1 + 8u_2 + 7u_3 + 7u_4 - 16u_5 - 20u_6 - 24u_7 - 20u_8\right)$$
$$s_{228} = \frac{b\rho}{225}\left(8v_1 + 8v_2 + 7v_3 + 7v_4 - 16v_5 - 20v_6 - 24v_7 - 20v_8\right)$$
$$s_{229} = \frac{b\rho}{315}\left(10u_1 + 12u_2 + 12u_3 + 10u_4 - 28u_5 - 24u_6 - 28u_7 - 48u_8\right)$$
$$s_{230} = \frac{b\rho}{315}\left(10v_1 + 12v_2 + 12v_3 + 10v_4 - 28v_5 - 24v_6 - 28v_7 - 48v_8\right)$$
$$s_{231} = \frac{a\rho}{450}\left(17u_1 - 5u_2 - 2u_3 + 8u_5 + 2u_6 + 12u_7 + 18u_8\right)$$
$$s_{232} = \frac{a\rho}{450}\left(17v_1 - 5v_2 - 2v_3 + 8v_5 + 2v_6 + 12v_7 + 18v_8\right)$$
$$s_{233} = \frac{a\rho}{450}\left(3u_2 - 5u_1 + 2u_4 - 8u_5 + 2u_6 + 8u_7 - 2u_8\right)$$
$$s_{234} = \frac{a\rho}{450}\left(3v_2 - 5v_1 + 2v_4 - 8v_5 + 2v_6 + 8v_7 - 2v_8\right)$$

$$s_{235} = \frac{a\rho}{450}\,(5u_4 - 3u_3 - 2u_1 - 8u_5 - 2u_6 + 8u_7 + 2u_8)$$
$$s_{236} = \frac{a\rho}{450}\,(5v_4 - 3v_3 - 2v_1 - 8v_5 - 2v_6 + 8v_7 + 2v_8)$$
$$s_{237} = \frac{a\rho}{450}\,(2u_2 + 5u_3 - 17u_4 - 12u_5 - 2u_6 - 8u_7 - 18u_8)$$
$$s_{238} = \frac{a\rho}{450}\,(2v_2 + 5v_3 - 17v_4 - 12v_5 - 2v_6 - 8v_7 - 18v_8)$$
$$s_{239} = \frac{a\rho}{225}\,(4u_1 - 4u_2 - 4u_3 - 6u_4 + 40u_5 + 8u_6 + 12u_8)$$
$$s_{240} = \frac{a\rho}{225}\,(4v_1 - 4v_2 - 4v_3 - 6v_4 + 40v_5 + 8v_6 + 12v_8)$$
$$s_{241} = \frac{a\rho}{225}\,(6u_1 + 4u_2 + 4u_3 - 4u_4 - 8u_6 - 40u_7 - 12u_8)$$
$$s_{242} = \frac{a\rho}{225}\,(6v_1 + 4v_2 + 4v_3 - 4v_4 - 8v_6 - 40v_7 - 12v_8)$$
$$s_{243} = \frac{a\rho}{225}\,(9u_1 - u_2 + u_3 - 9u_4 + 12u_5 - 12u_7)$$
$$s_{244} = \frac{a\rho}{225}\,(9v_1 - v_2 + v_3 - 9v_4 + 12v_5 - 12v_7)$$

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