باسمه تعالی

جلسه اول

Formal Languages and Automata Theory

نظریه زبانها و ماشین ها

Computer Science

- What kind of things can we compute?
- How fast can we compute?
- How much memory do we need?

Model of computation : theoretical abstract machine

Theory of "computation"

Simple questions , difficult answers – still far from closed

Various models of computation (in order of increasing power) :

① Finite memory : finite automata ; regular expressions

② Finite memory with stack : pushdown automata

③ unrestricted :

Turing machine

λ-calculus

Java , Pascal

(you can add any sufficiently powerful PL to the list)

In parallel : Noam Chomsky formalized notion of language & grammar

① Right-linear

② Context-free

③ Unrestricted

* ⟶ context-sensitive grammars

linear bounded automata

(not in this course)

Practical Applications :

* Regular expressions are used in many systems

grep global regular expression print
findstr (windows)
} Search for a text string in a file

:\> more *.txt

Finite Automata : model protocols & electronic circuits

DTD (Document Type Definition)

DTD defines the structure and legal elements & attributes of an XML document.

DTD is based on context-free grammars. (CFG)

person (name, addr?, child*)

CFGs are used to describe the syntax of every PL.

When developing software : limitations of what software can do

* undecidable : no program whatever can do it

Example : write a compiler that refuses the programs that do not print anything on screen

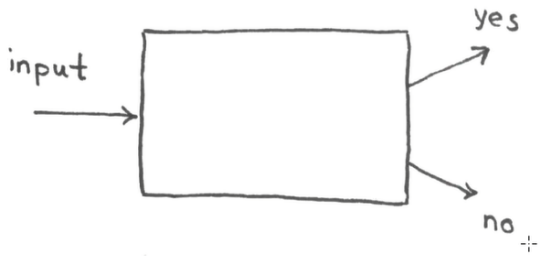
* Intractable : there exists programs, but not fast

NP-completeness

First we simplify the discussion on algorithm problems.

Decision Problem : function with one-bit output : "yes" or "no"

* set A of possible inputs
* subset $B \subseteq A$: "yes"



Example : PRIME , CONNECTED

Abstraction : possible input to decision problem:

finite length strings over some fixed alphabet

Graph, natural numbers, ... can be encoded

alphabet : Σ any finite set $\{0, \cdots, 9\}$ $\{a, b, c, \cdots\}$

String over Σ : any finite-length sequence of elements of Σ

x, y, z refer to strings

$\Sigma = \{a, b\}$ aabb : string

$|x|$: number of symbols in x

empty string : ε (special symbol) null or empty string

$$|\varepsilon| = 0$$

a^n : string of a's of length n.

$a^0 \overset{\text{def}}{=\!=} \varepsilon$

$a^{n+1} \overset{\text{def}}{=\!=} a^n a$

Example. $a^4 = aaaa$ $a^0 = \varepsilon$

Set of all strings over Σ : Σ^*

Example: $\Sigma = \{a, b\}$

$\Sigma^* = \{\varepsilon, a, b, ab, ba, \cdots\}$

$\{a\}^* = \{\varepsilon, a, aa, aaa, \cdots\} = \{a^n \mid n \geq 0\}$

$\emptyset^* = \{\varepsilon\}$

Note : $\emptyset$, $\{\varepsilon\}$, ε are different things

Note. Do not confuse sets and strings

$\{a, b\} = \{b, a\}$ but not $ab \neq ba$

$\{a, a, b\} = \{a, b\}$ but not $aab \neq ab$

Concatenation. takes two strings x and y and makes a new string xy by putting them together

- associative $(xy)z = x(yz)$
- xy and yx not equal in general
- null string ε is identity $\varepsilon x = x\varepsilon = x$

$|xy| = |x| + |y|$

x^n : concatenating n copies of x

$x^0 \overset{\text{def}}{=\!=} \varepsilon$

$x^{n+1} \overset{\text{def}}{=\!=} x^n x$

$(aab)^5 = aabaabaabaabaab$

Language: set of strings (over some alphabet)

Language describes problems with "yes" / "no" answers

L = All strings containing the substring "eh" ($\Sigma = \{a, b, \cdots, z\}$)

tehran , ehsan , mehr are in L

ε , sharif , aban are <u>not</u> in L

$L = \{x \in \Sigma^* \mid x \text{ contains the substring "eh"}\}$

Example. $L = \{s \# s \mid s \in \{a, \cdots, z\}^*\}$

Which of the following are in L?

$ab \# ab$	$ab \# ba$	$a \# \# a \#$
yes	no	no

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Σ^* : set of all strings on Σ

Language : subset of Σ^* (finite or infinite)

Example:

$$\Sigma = \{a, b, c\}$$

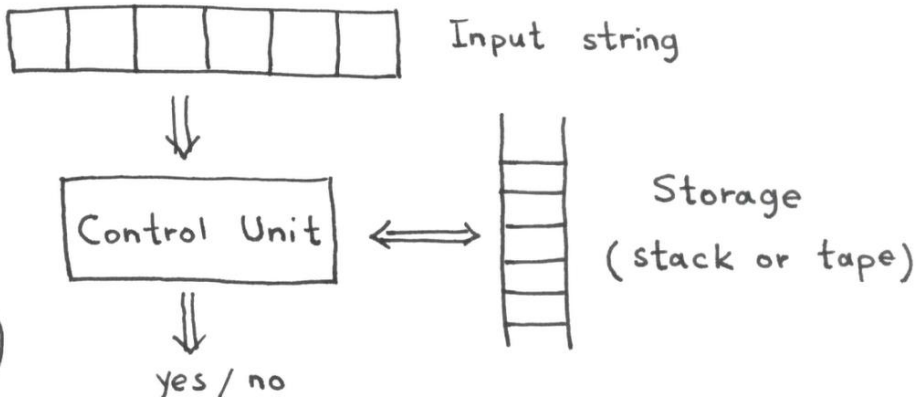
$$L_1 = \{a, ba, abc\}$$

$$L_2 = \{\varepsilon, a, aa, aaa, \cdots\}$$

$$L_3 = \emptyset$$

Automaton (plural : automata)

Abstract model of a computer



(if output is string: transducer)

Control Unit:
- has some finite memory
- keeps track of what step to execute next

Additional Storage is infinite. We never run out of memory

Finite Automata

Automaton without additional storage

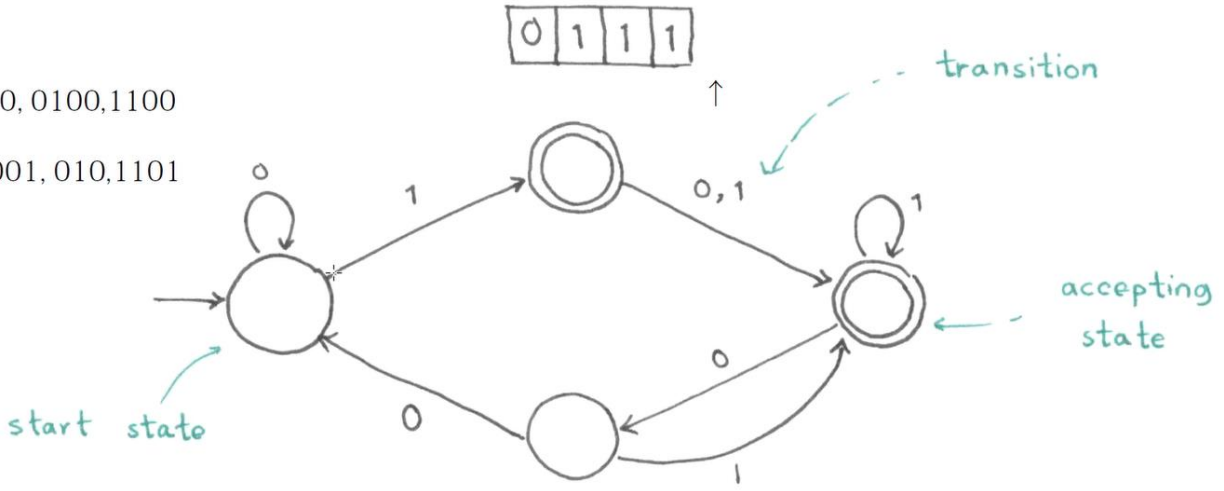
Only the finite memory in the control unit is available

It can remember only a finite number of properties of the past input

②

No: 00, 0100, 1100

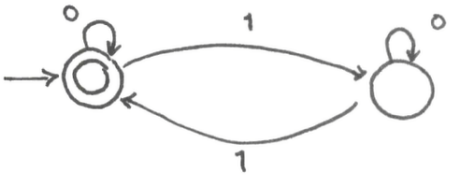
Yes: 001, 010, 1101



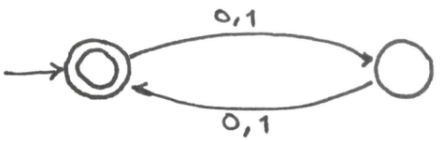
Let M be a finite automaton over Σ

$L(M) = \{ x \in \Sigma^* \mid M \text{ accepts } x \}$

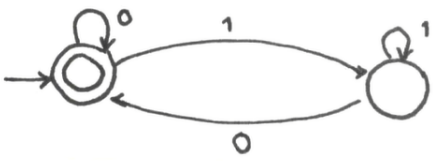
What is the language of the following finite automata?



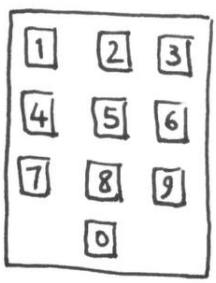
even numbers (1's)

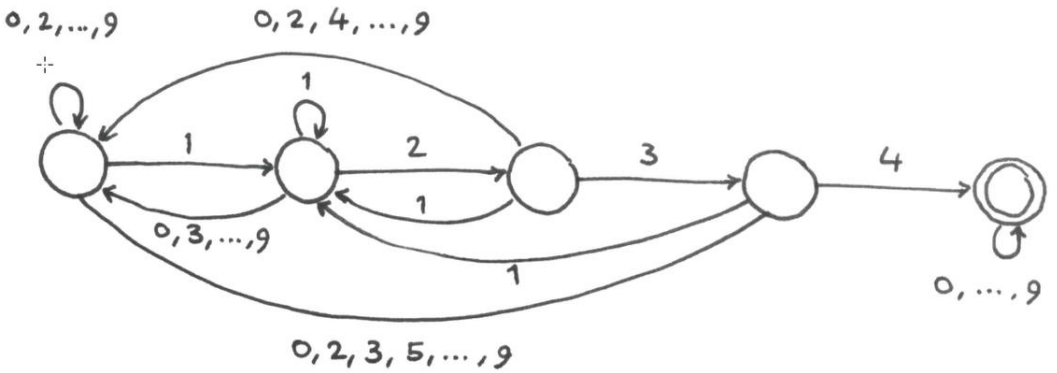


binary string of even length



$\{ x \mid x \text{ ends in } 0 \} \cup \{ \varepsilon \}$





tow can we write a program and code this automaton?

Formal Definition :

Deterministic Finite Automaton (DFA) $M = (Q, \Sigma, \delta, q_0, F)$

Q : a non-empty finite set of states

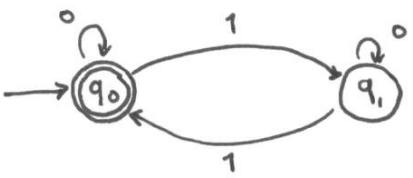
Σ : alphabet

δ : transition function $\quad Q \times \Sigma \longrightarrow Q$

$q_0 \in Q$ start state

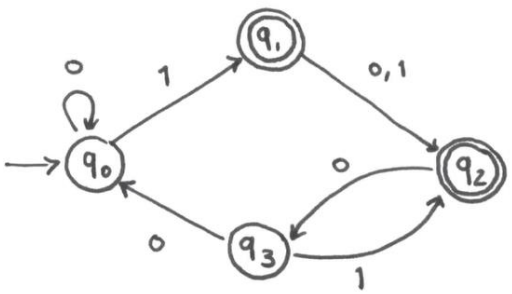
$F \subseteq Q$ set of final states

$\delta(state_0, symbol) = state_1$



$\delta(q_0, 0) = q_0$ $\quad \delta(q_1, 0) = q_1$

$\delta(q_0, 1) = q_1$ $\quad \delta(q_1, 1) = q_0$



$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{0, 1\}$

δ :

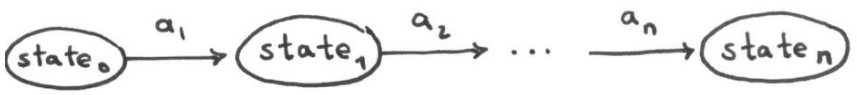
q_0 : start

$F = \{q_1, q_2\}$

δ	0	1
q_0	q_0	q_1
q_1	q_2	q_2
q_2	q_3	q_2
q_3	q_0	q_2

Extended transition function

$\hat{\delta}$: mapping from (state, string) to the corresponding state



$\hat{\delta}(state_0, \underbrace{a_1 \ldots a_n}_{string}) = state_n$

$L(M) = \{x \in \Sigma^* \mid \hat{\delta}(q_0, x) \in F\}$

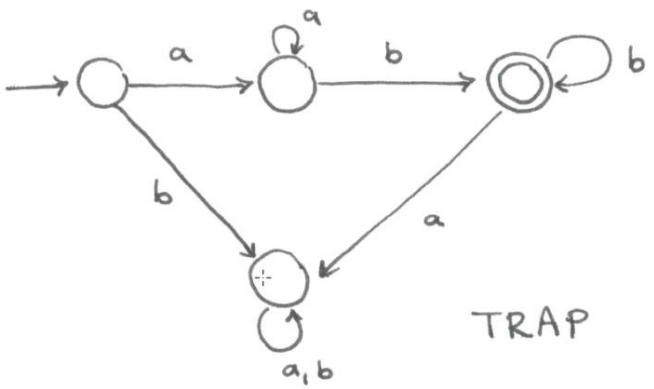
Definition : A language $L \subseteq \Sigma^*$ is regular if a finite automaton accepts it.

Question : Are all languages regular ?

example. $L_1 = \{ 0^n 1^n \mid n \in \mathbb{N} \}$

Remember how many 0's are, check if they are equal to 1's

example. $L_2 = \{ a^n b^m \mid n, m \geq 1 \}$



example. $L_3 = \{ w \mid w$ contains equal number of substrings 01 and 10$\}$

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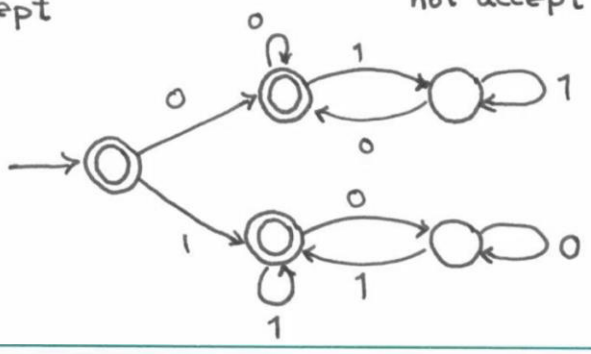
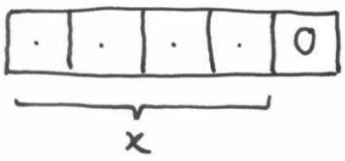
$L_1 = \{ w \mid w \text{ contains equal number of substrings } 01 \text{ and } 10 \}$

$0\,1\ldots1\,0$ — not accept / accept

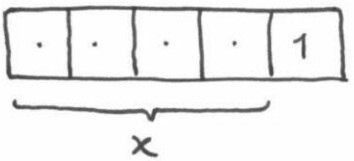
$1\,0\ldots0\,1$ — not accept / accept



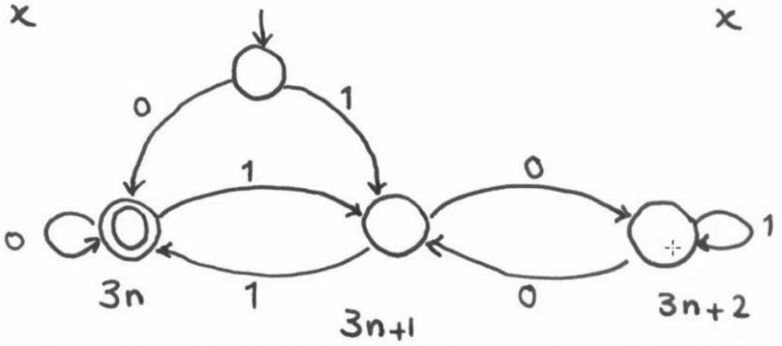
$L_2 = \{ w \text{ in } \{0,1\}^* \mid w \text{ is divisible by } 3 \}$



$= 2x$

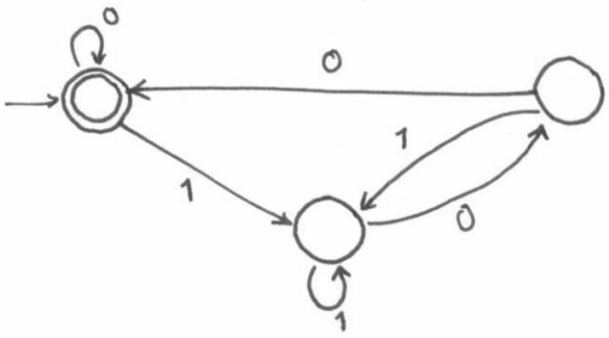


$= 2x+1$

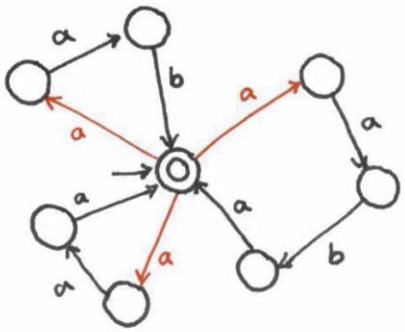


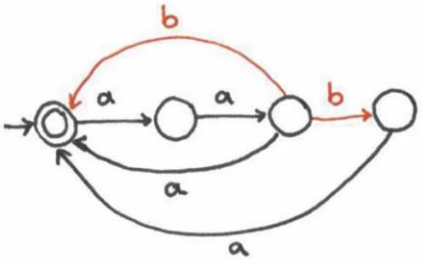
$L_3 = \{ w \text{ in } \{0,1\}^* \mid w \text{ is divisible by } 4 \}$

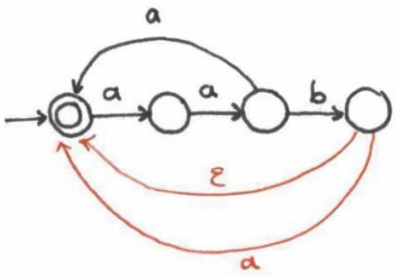
w is of the form ε , 0 , $v\,00$ $(v \in \Sigma^*)$



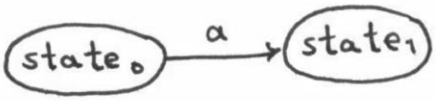
$L = \{aab, aaba, aaa\}^* = \{ x_1 x_2 \dots x_k \mid k \geq 0 \text{ and } x_i \in \{aab, aaba, aaa\} \}$







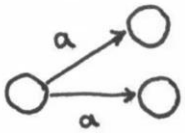
Deterministic finite automaton :



$\delta(state_0, a) = state_1$

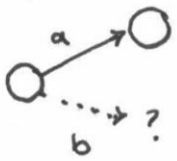
Non-deterministic finite automaton may have:

multiple outgoing edges with same symbol

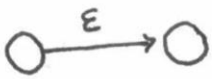


no outgoing edges for some symbols

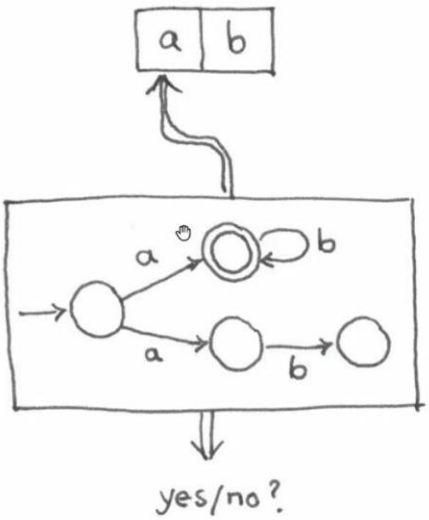
$\Sigma = \{a, b\}$



ε-transitions

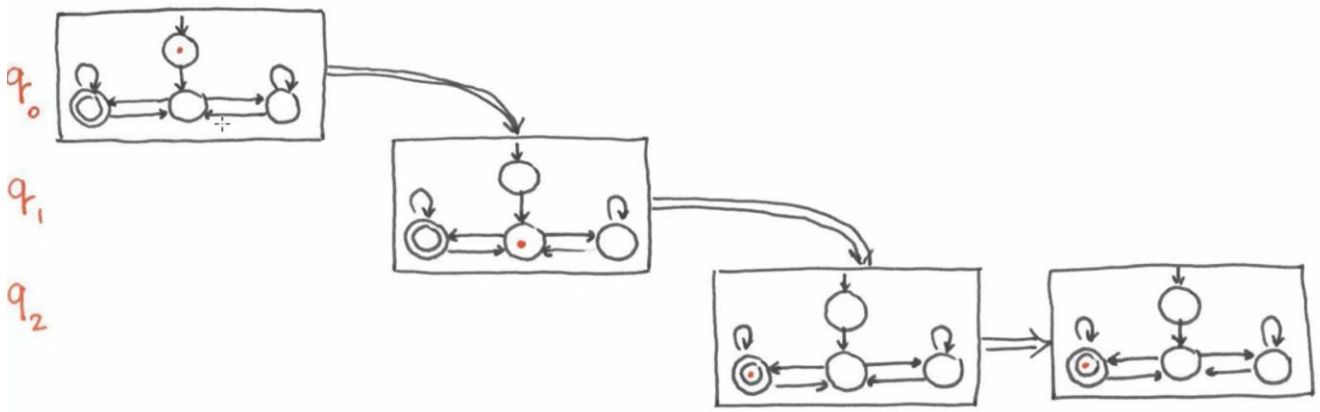


On reading an input symbol, NFA can "choose" to make a transition to one of selected states.



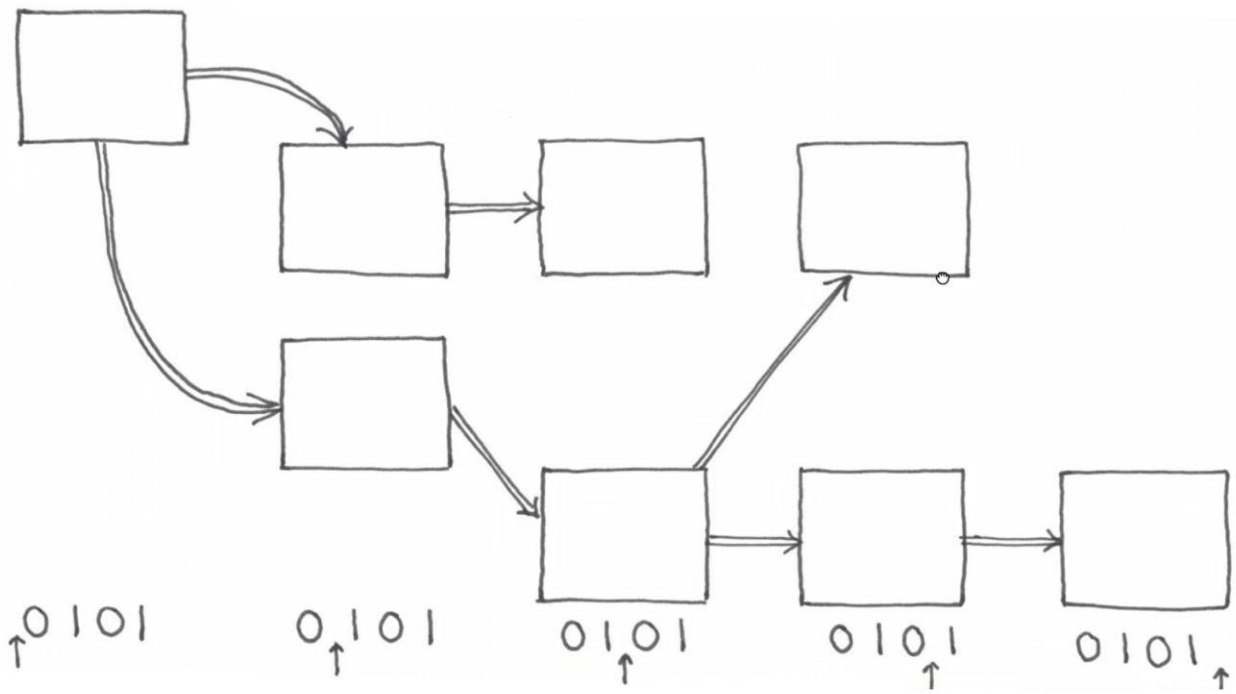
③ Tracking Computation

Configuration: current state and remaining input $(q, w) \in Q \times \Sigma^*$



Deterministic: each step is fully determined by the configuration of the previous step

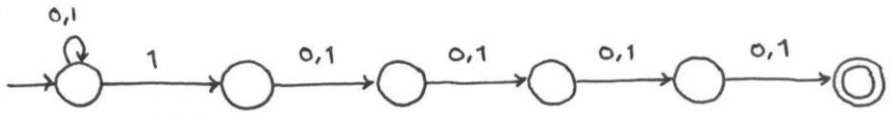
Non-deterministic: at each step the computation is allowed to proceed in multiple ways (zero, one or more)



accept iff at least one configuration accepts

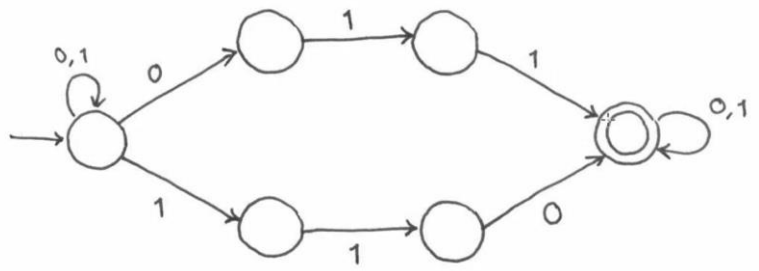
reject iff no configuration accepts

$L = \{ x \in \{0,1\}^* \mid 5^{th} \text{ symbol of } x \text{ from right is } 1 \}$

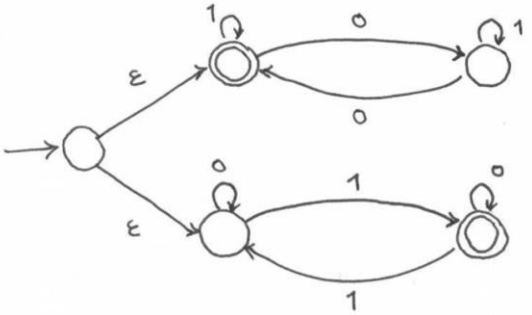


For any input x, if its 5th symbol from right is 1, then there exists some computation path that ends in accepting state

$L = \{ w \mid w \text{ contains } 011 \text{ or } 110 \}$

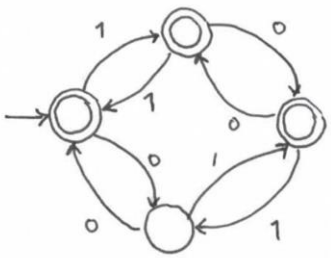


Construct a finite automaton that accepts all strings with an even number of 0's or an odd number of 1's



even number of 0's

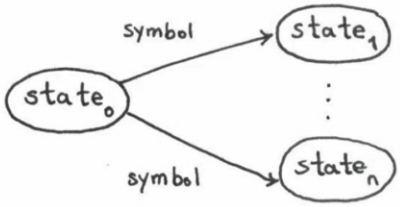
odd number of 1's



Formal Definition

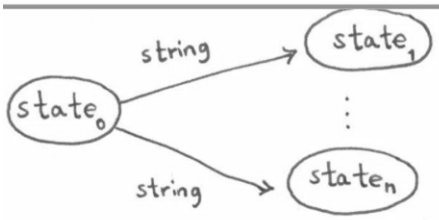
Similar to DFA $N = (Q, \Sigma, \delta, q_0, F)$

$\delta(q, a)$: set of states that N could move from q on input a



$\delta(\text{state}_0, \text{symbol}) = \{\text{state}_1, \ldots, \text{state}_n\}$

$\delta : Q \times \Sigma \longrightarrow 2^Q$



$\hat{\delta}(\text{state}_0, \text{string}) = \{\text{state}_1, \ldots, \text{state}_n\}$

$L(N) = \{ x \in \Sigma^* \mid \hat{\delta}(q_0, x) \in F \}$

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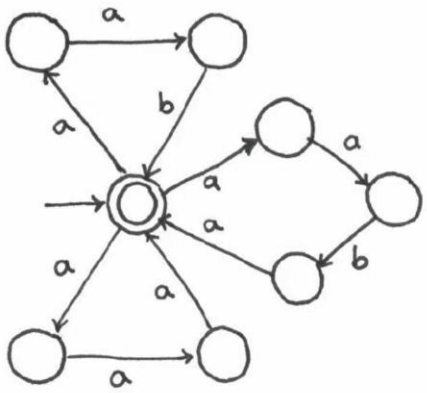
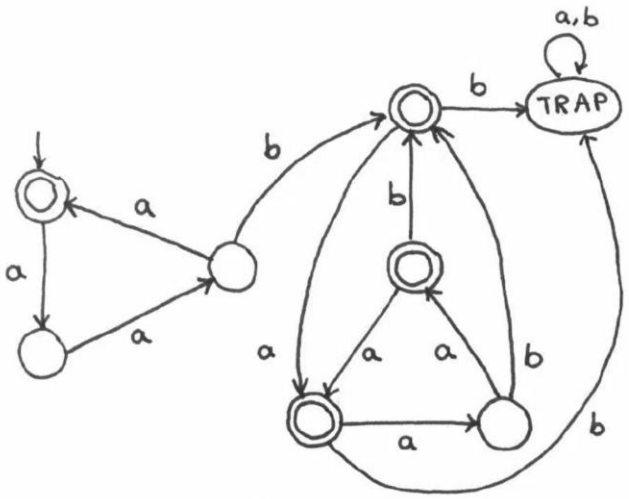
$L = \{aab, aaba, aaa\}^*$



NFA



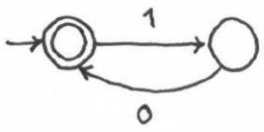
DFA

Equivalence. Two automata are equivalent if their languages are the same. For M_1, M_2: $L(M_1) = L(M_2)$

Example of equivalent automata.

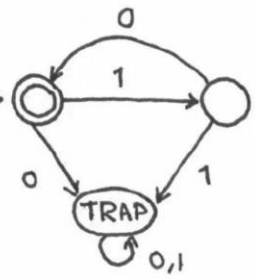
$L(M_1) = \{10\}^*$

NFA



$L(M_2) = \{10\}^*$

DFA



Theorem. Every NFA has an equivalent DFA.

Corollary. A language is regular iff it is recognized by an NFA

From NFA to DFA

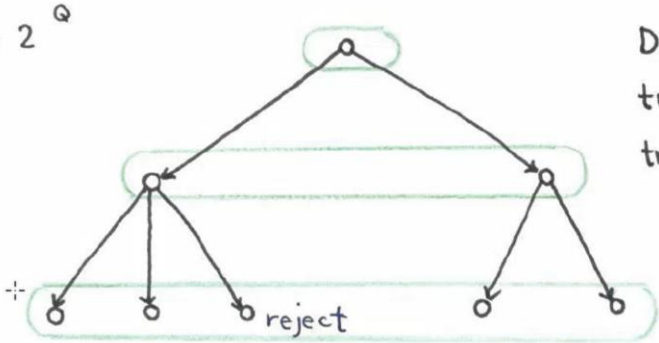
Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

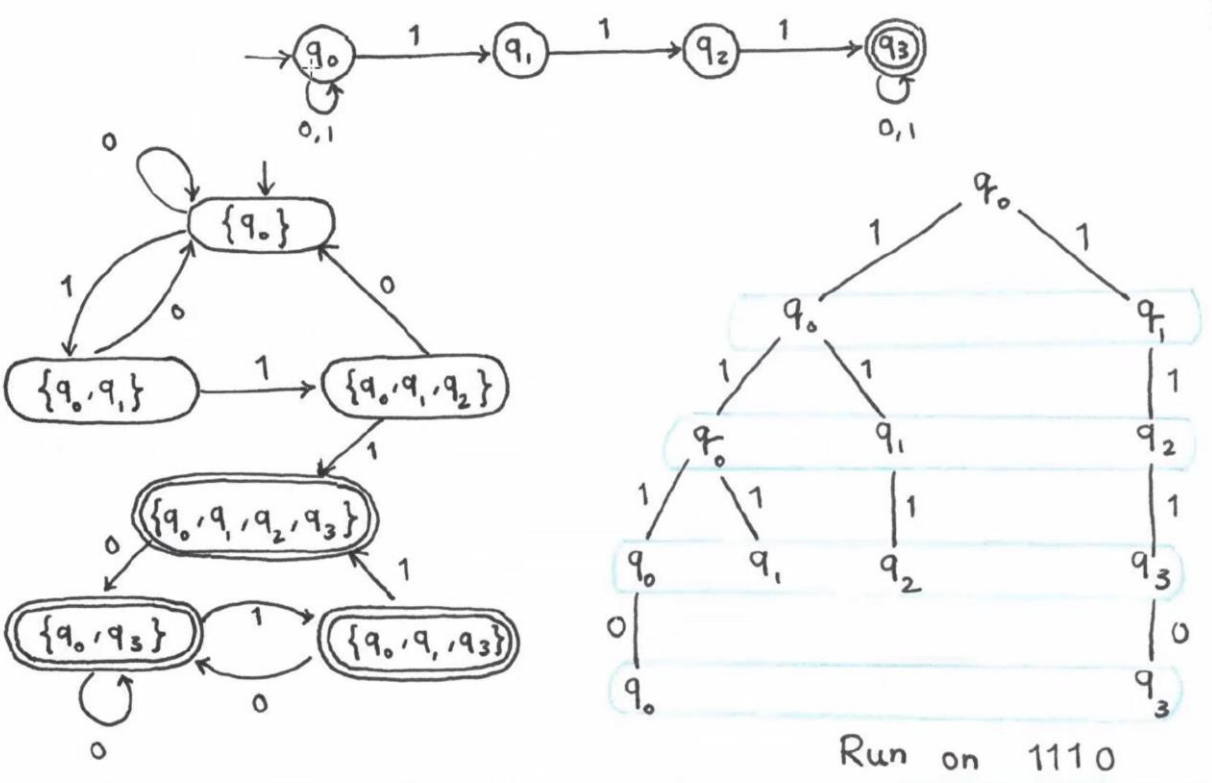
$Q' = ?$

Assume (for now) that there are no ε-transitions

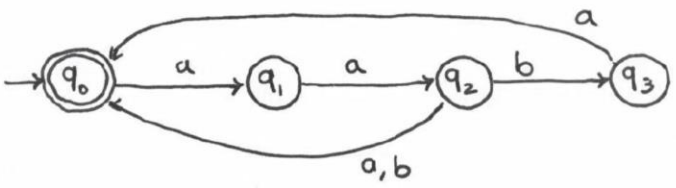
Idea: $Q' = 2^Q$



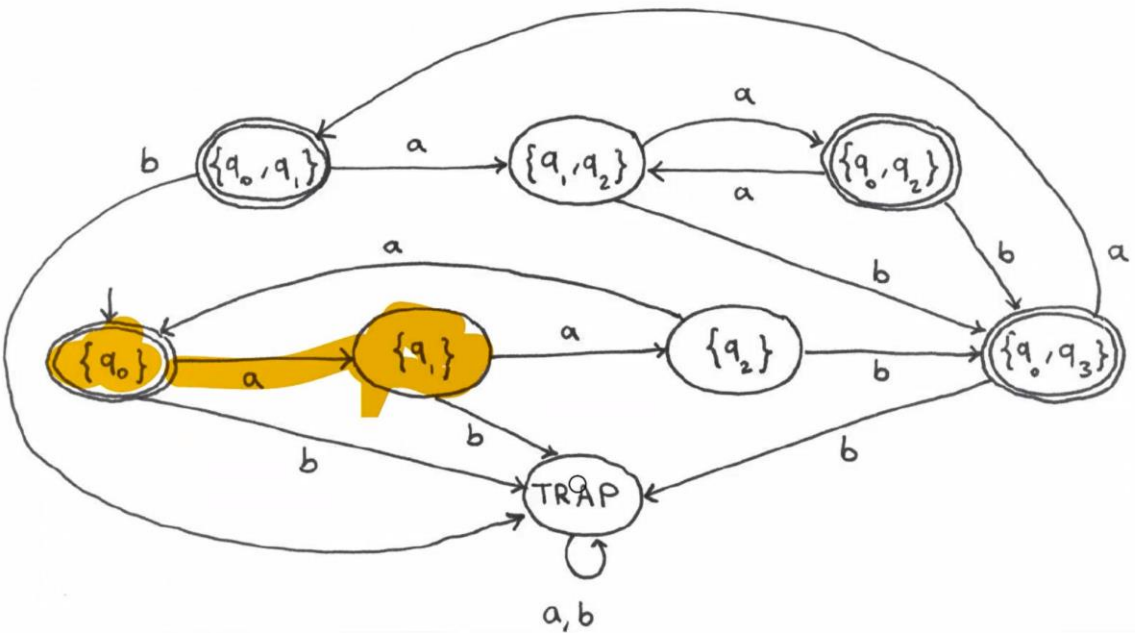
Do a BFS on tree tracking set of states transitioned to



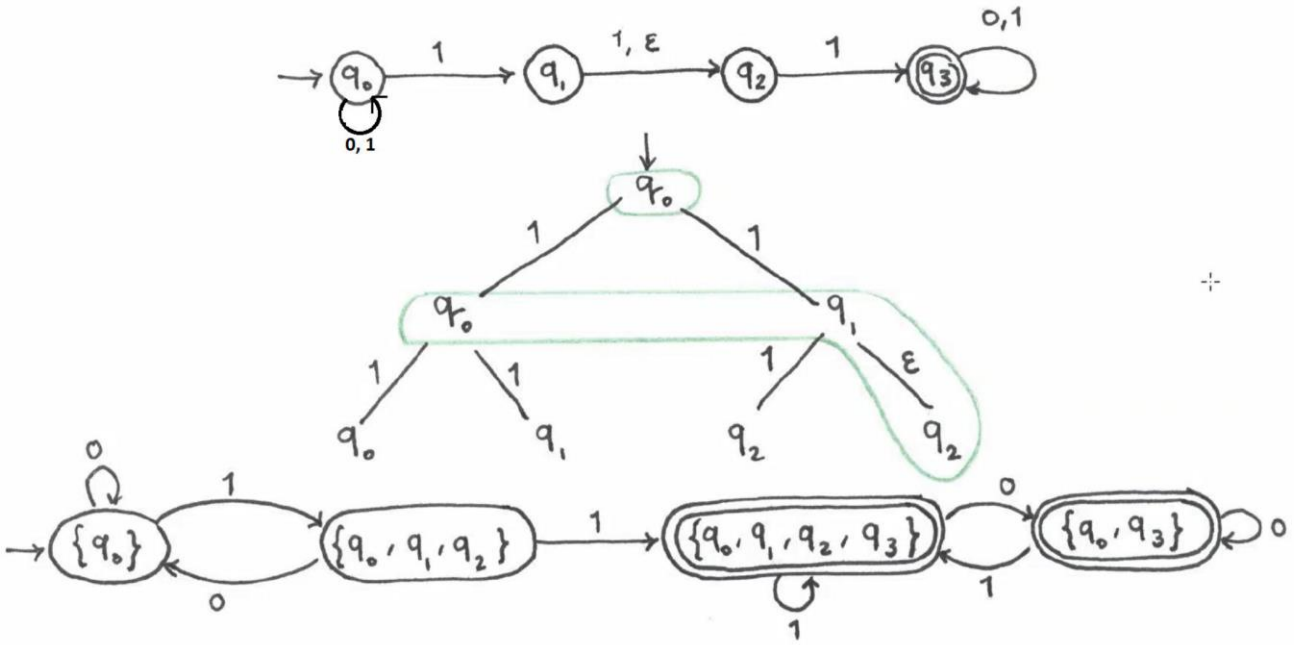
Example. Non-deterministic Finite Automaton:



Equivalent deterministic finite automaton:



What if we have ε-transitions?



$$E(R) = \{ q \mid q \text{ can be reached from } R \text{ by travelling along 0 or more } \varepsilon\text{-transitions} \}$$

$(R \subseteq Q)$

$E(\{q_1\}) = \{q_1, q_2\}$

After reading an input, follow ε-transitions until you cannot anymore

Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

$Q' = 2^Q$

$\delta' : Q' \times \Sigma \to Q'$ $\qquad \delta'(R, a) = \bigcup_{r \in R} E(\delta(r, a))$

$q_0' = E(\{q_0'\})$

$F' = \{ R \in Q' \mid f \in R \text{ for some } f \in F \}$

"Subset Construction"

Worst-case exponentially larger

"Finite Automata and their Decision Problems"

Michael O. Robin & Dana Scott 1959

Subset Construction

Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

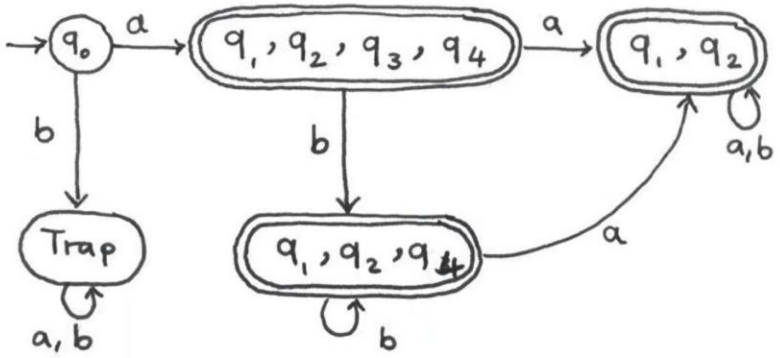
$Q' = 2^Q$

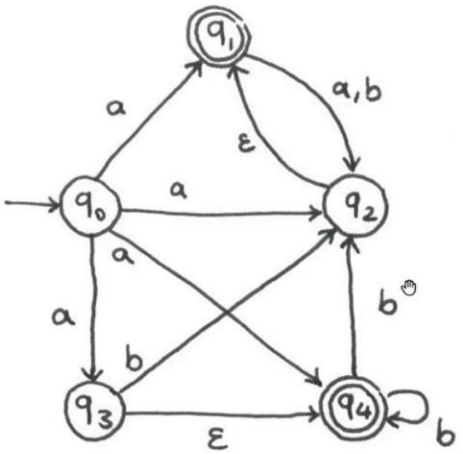
$\delta' : Q' \times \Sigma \longrightarrow Q'$

$\delta'(R, a) = \bigcup_{r \in R} E(\delta(r, a))$

$q_0' = E(\{q_0\})$

$F' = \{R \in Q' \mid f \in R \text{ for some } f \in F\}$





Recall. A language is a set of strings.

We can define the usual set operations on languages.

Given two languages L_1 and L_2

* Union $L_1 \cup L_2 = \{w \mid w \in L_1 \vee w \in L_2\}$
* Intersection $L_1 \cap L_2 = \{w \mid w \in L_1 \wedge w \in L_2\}$
* Complementation $\overline{L_1} = \{w \mid w \notin L_1\}$ $\quad (\overline{L_1} = \Sigma^* - L_1)$

We define three new operations on languages.

* Reverse $L_1^R = \{a_1 a_2 \cdots a_k \mid a_k a_{k-1} \cdots a_1 \in L_1\}$
* Concatenation $L_1 \cdot L_2 = \{vw \mid v \in L_1 \wedge w \in L_2\}$
* Kleene star $L_1^* = \{w_1 w_2 \cdots w_k \mid k \geq 0 \text{ and each } w_i \in L_1\}$

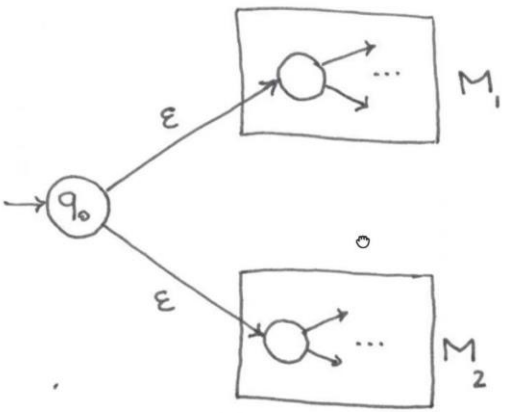
Recall. A language is called regular if it is accepted by a finite state automaton

Definition. A set is closed under an operation if and only if the operation on any two elements of the set produces another element of the set.

example. Set $\{-1, 1\}$ is closed under multiplication but not addition.

Theorem. Regular languages are closed under union, intersection, complementation, reversal, concatenation and star.

$L_1 \cup L_2$:

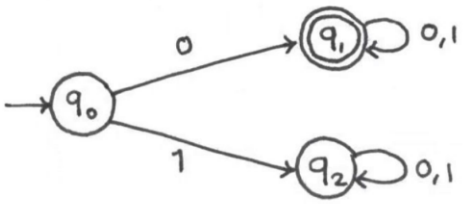


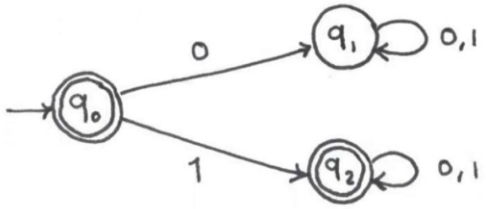
$\overline{L}$ (complementation)

Example: $L_1 = \{0x \mid x \in \{0,1\}^*\}$ strings start with 0

$\overline{L_1} = \{1x \mid x \in \{0,1\}^*\} \cup \{\varepsilon\}$ strings that do not start with 0

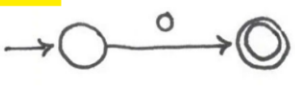
Complement of DFA : Swap accepting states and non-accepting states (and vice versa)



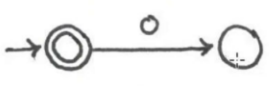


Note. Does not work for NFA

$\Sigma = \{0\}$ $L = \{0\}$



L



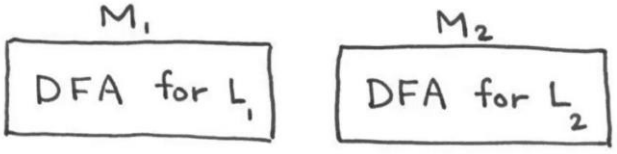
$\overline{L}$ ✗

$L_1 \cap L_2$:

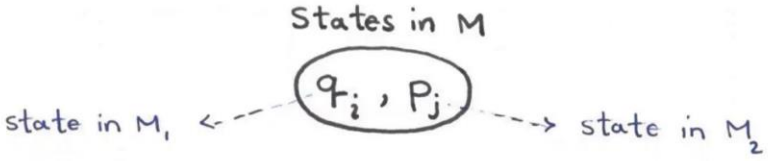
Due to DeMorgan's law we already know regular languages are closed under intersection

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

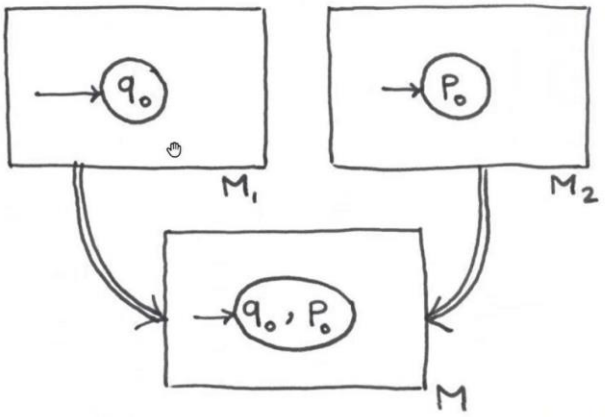
How to construct a new DFA M that recognizes the intersection of the languages of two DFAs?



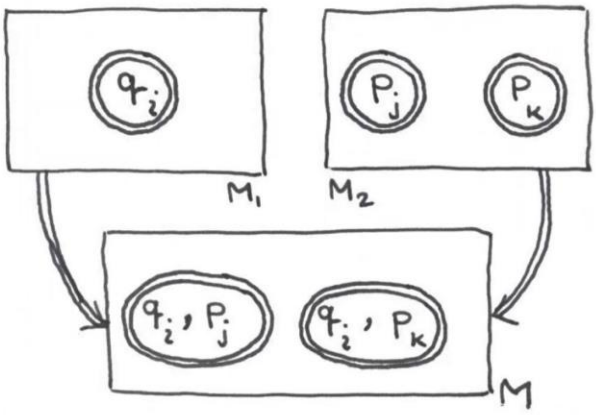
M simulates in parallel M_1 and M_2



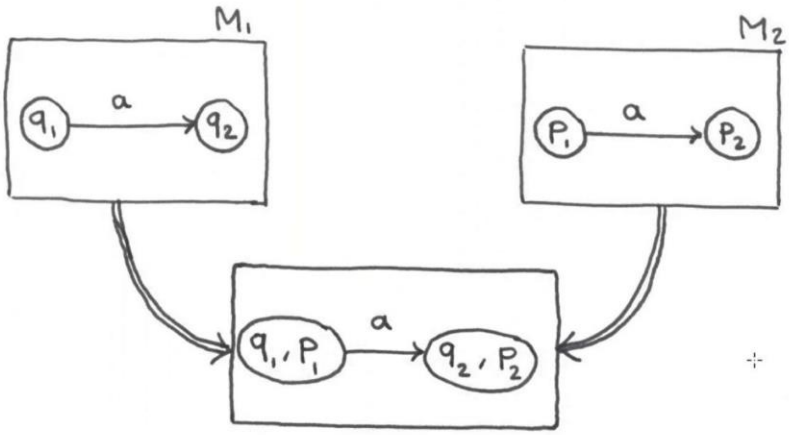
Initial State:



Final State:

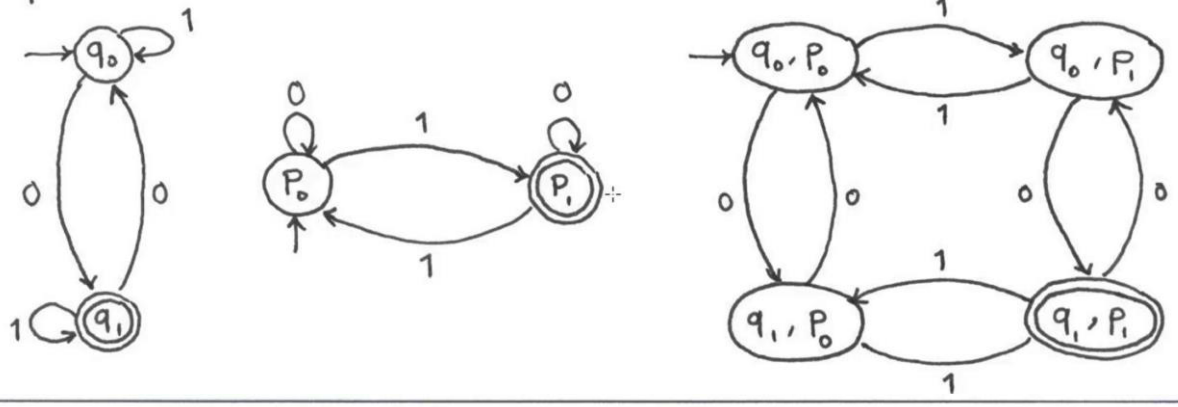


Transitions:

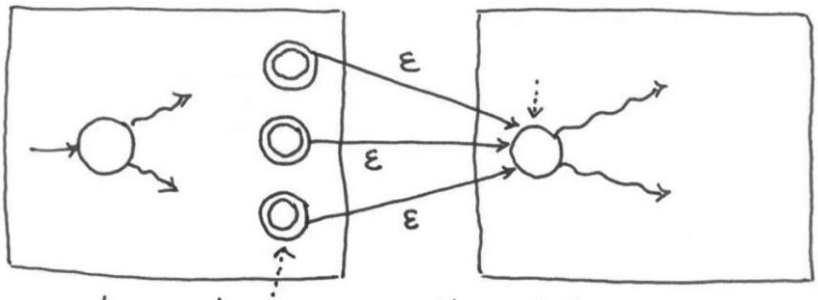


Cross-product DFA : a single DFA which simulates operation of two DFAs in parallel

Example:



$L_1 \cdot L_2$



change to non-accepting state

L^* All possible ways of concatenating any number of copies of strings in L together

$$L^* = \bigcup_{i=0}^{\infty} L^i = L^0 \cup L^1 \cup L^2 \cup L^3 \cup \cdots = \{\varepsilon\} \cup L \cup LL \cup LLL \cdots$$

Question. Can we say L^* is regular because it is the "union" of several regular languages?

$L^0 = \{\varepsilon\}$ regular

$L^1 = L$ regular

$L^2 = L \cdot L$ regular

⋮ ⋮

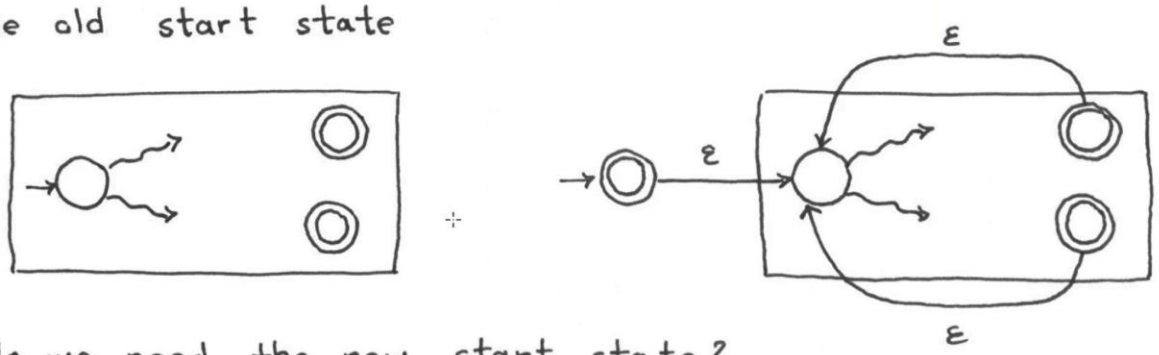
Answer. No. Because it is "infinite" union!

$\{0^n 1^n \mid n \in \mathbb{N}\}$ is not regular

$\varepsilon \cup \{01\} \cup \{0011\} \cup \cdots$ Infinite union

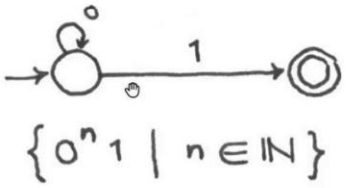
Suppose L is regular. Then there exists a DFA that recognizes L. We construct an NFA that accepts L.

Make a new accepting start state, add ε-transition to the old start state

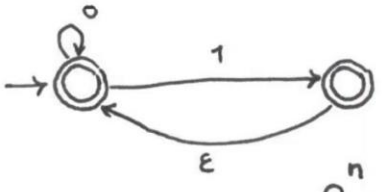


Why do we need the new start state?

Why can't we just change the start state to accepting?



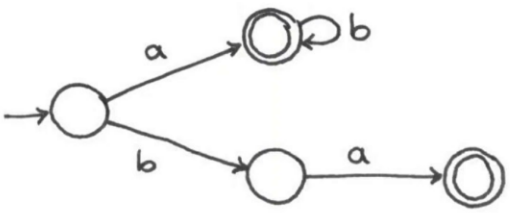
$\{0^n 1 \mid n \in \mathbb{N}\}$

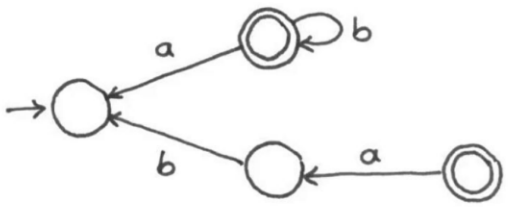


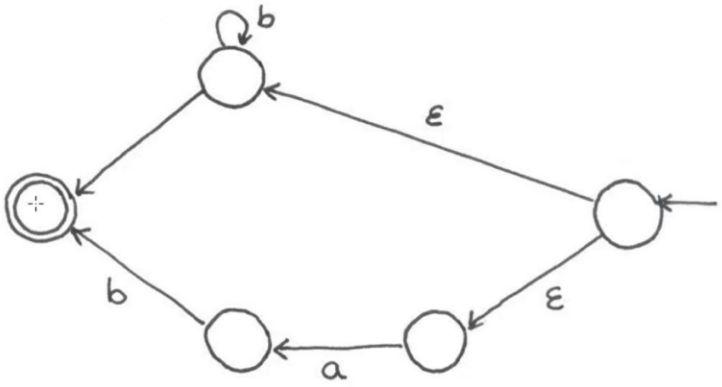
0^n X

L^R

* Reverse all the transitions
* Add a new start state, make ε-transitions to every accepting states
* Turn the "old" initial state to accepting
* Turn the "old" accepting states to accepting







باسمه تعالی

جلسه ششم درس نظریه

DFA : Pictorial representation of a regular language

Regular Expression : Algebraic representation of a regular language

Regular expressions are used by many Unix utilities

rm *.exe (del)
*.bak

Pattern Matching

folder: (a.exe , b.bak , c.exe , abc.bak)

If R_1 and R_2 are regular expressions representing L_1 and L_2

Regular Expression	Regular Language
$\emptyset$	$\{\}$
$a \quad (a \in \Sigma)$	$\{a\}$
ε	$\{\varepsilon\}$
$R_1 + R_2$	$L_1 \cup L_2$
$R_1 R_2$	$L_1 \cdot L_2$
$R_1 *$	$L_1 *$

Example . What does the following represent? $\Sigma = \{0, 1\}$

$(0+1)(0+1)$	$\{\omega \mid \lvert\omega\rvert = 2\}$
$0*10*$	$\{\omega \mid \omega$ contains a single 1$\}$
$((0+1)(0+1))*$	$\{\omega \mid \lvert\omega\rvert$ is even$\}$
$(0+1)*1(0+1)*$	$\{\omega \mid \omega$ has at least one 1$\}$
$(0*10*1)*0*$	$\{\omega \mid \omega$ has even number of 1s$\}$

Operator Precedence

1. *
2. .
3. +

$R_1 * R_2 + R_3 = \left((R_1 *) \cdot R_2\right) + R_3$

$\{w \in \{0,1\}^* \mid w \text{ does not contain the substring } 10\}$

$$0^*1^*$$

$\{w \in \{0,1\}^* \mid w \text{ contains } 00 \text{ as a substring}\}$

$$(0+1)^*00(0+1)^*$$

$\{w \in \{0,1\}^* \mid w \text{ does not contain the substring } 00\}$

$$\underbrace{(01+1)^*}\ \underbrace{(\varepsilon+0)}$$

at most a single 0 between two consecutive 1s ← + → ends in at most a single 0

$\{w \in \{0,1\}^* \mid w \text{ has length } \geq 3 \text{ and its 3rd symbol is } 0\}$

$$(0+1)(0+1)0(0+1)^*$$

$\{w \in \{0,1\}^* \mid w \text{ contains at least three 0s}\}$

$$(0+1)^*0(0+1)^*0(0+1)^*0(0+1)^*$$

$\{w \in \{0,1\}^* \mid w \text{ contains at most one 0}\}$

$$1^*(0+\varepsilon)1^* \qquad\qquad 1^*0?1^*$$

$R?$ shorthand for $(R+\varepsilon)$: zero or one copies of R

$R+$ shorthand for RR^* : one or more copies of R

$\{w \in \{0,1\}^* \mid \text{every run of 0s in } w \text{ has length at least } 3\}$

$$(1+000^*)^*$$

$\{w \in \{0,1\}^* \mid w \text{ does not end in } 00\}$

$$\varepsilon + 0 + 1 + (0+1)^*(01+10+11)$$

$\{w \in \{0,1\}^* \mid |w| \text{ is odd}\}$

$$(0+1)\left((0+1)(0+1)\right)^*$$

Regular Expressions in Java

```
String re = "1*0?1*";
String input = "11101111";
System.out.println(input.matches(re));
```

Is input in the set described by re?

Algebraic Laws for Regular Expressions

Union (U) and concatenation (·) behave sort of like addition(+) and multiplication (*)

- + is commutative and associative
- · is associative (not commutative)

Concatenation distributes over +

Simplification of Expressions

Two regular expressions α and β are equivalent $\alpha \equiv \beta$ if $L(\alpha) = L(\beta)$

If $\alpha \equiv \beta$ we can substitute α for β (or vice versa) in any regular expressions and the resulting expression will be equivalent to the original expression.

Note. Relation $\equiv$ is an equivalence relation.

* Reflexive: for all α $\alpha \equiv \alpha$
* Symmetric: for all α, β if $\alpha \equiv \beta$ then $\beta \equiv \alpha$
* Transitive: for all α, β, γ if $\alpha \equiv \beta$ and $\beta \equiv \gamma$ then $\alpha \equiv \gamma$

Some laws that can be used to simplify regular expressions :

① $\alpha + (\beta + \gamma) \equiv (\alpha + \beta) + \gamma$ — Associativity

② $\alpha + \beta \equiv \beta + \gamma$ — Commutativity

③ $\alpha + \emptyset \equiv \alpha$ — Identity

④ $\alpha + \alpha \equiv \alpha$ — Idempotent

⑤ $\alpha(\beta\gamma) \equiv (\alpha\beta)\gamma$

⑥ $\varepsilon\alpha \equiv \alpha\varepsilon \equiv \alpha$

⑦ $\alpha(\beta + \gamma) \equiv \alpha\beta + \alpha\gamma$ — Left Distributivity

⑧ $(\alpha + \beta)\gamma \equiv \alpha\gamma + \beta\gamma$

⑨ $\emptyset\alpha \equiv \alpha\emptyset \equiv \emptyset$

⑩ $\varepsilon + \alpha\alpha^* \equiv \alpha^*$

⑪ $\varepsilon + \alpha^*\alpha \equiv \alpha^*$

Let's prove ⑦ law :

$\omega \in \alpha(\beta + \gamma)$

$\Leftrightarrow \exists u.\ \exists v.\ (\omega = uv) \wedge u \in L(\alpha) \wedge v \in L(\beta + \gamma)$

$\Leftrightarrow \exists u.\ \exists v.\ (\omega = uv) \wedge u \in L(\alpha) \wedge \left(v \in L(\beta) \vee v \in L(\gamma)\right)$

$\Leftrightarrow \exists u.\ \exists v.\ (\omega = uv) \wedge \left((u \in L(\alpha) \wedge v \in L(\beta)) \vee (u \in L(\beta) \wedge v \in L(\gamma))\right)$

$\Leftrightarrow \exists u.\exists v.\ \left(\omega = uv \wedge u \in L(\alpha) \wedge v \in L(\beta)\right) \vee \left(\omega = uv \wedge u \in L(\beta) \wedge v \in L(\gamma)\right)$

$\Leftrightarrow \quad \omega \in L(\alpha\beta) \quad \vee \quad \omega \in L(\beta\gamma)$

$\Leftrightarrow \quad \omega \in L(\alpha\beta + \beta\gamma)$

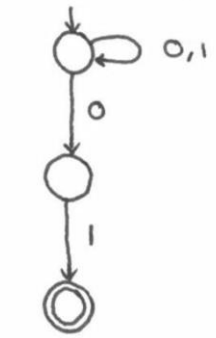
Prove : $(0110+01)(10)^* \equiv 01(10)^*$

$(0110+01)(10)^* \equiv (0110+01\varepsilon)(10)^*$	identity
$\equiv (01)(10+\varepsilon)(10)^*$	left distributivity
$\equiv (01)((10+\varepsilon)(10)^*)$	associativity
$\equiv (01)(10(10)^* + \varepsilon(10)^*)$	right distributivity
$\equiv (01)(10(10)^* + (10)^*)$	identity
$\equiv (01)(10(10)^* + \varepsilon + 10(10)^*)$	rule (10)
$\equiv (01)(10(10)^* + 10(10)^* + \varepsilon)$	commutativity
$\equiv (01)(10(10)^* + \varepsilon)$	idempotent
$\equiv (01)(10)^*$	rule (10)

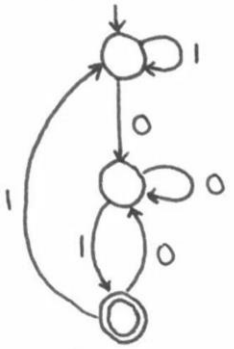
باسمه تعالی

جلسه هفتم درس نظریه

$L = \{ x \in \{0,1\}^* \mid x \text{ ends in } 01 \}$



NFA



DFA

$(0+1)^* 01$

Regular Expression

Theorem. A language is regular if and only if some regular expression describes it.

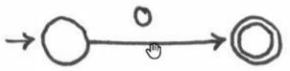
Proof has two parts:

① If a language is described by some regular expression, then it is a regular language.

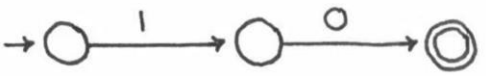
② If a language is regular, then it is described by some regular exp.

Example. regular expression $\Longrightarrow$ NFA

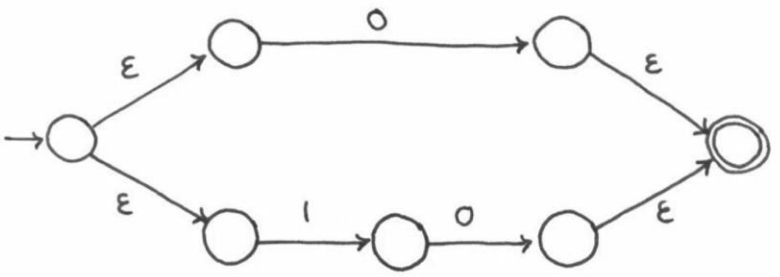
$R_1 = 0$



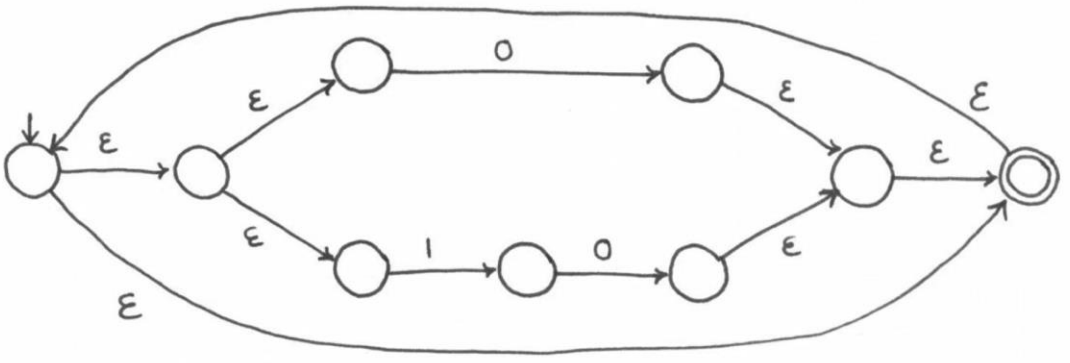
$R_2 = 10$



$R_3 = 0 + 10$



$R_4 = (0+10)^*$

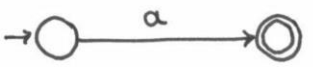


Thompson's Algorithm: an algorithm for converting any regular expression into an NFA.

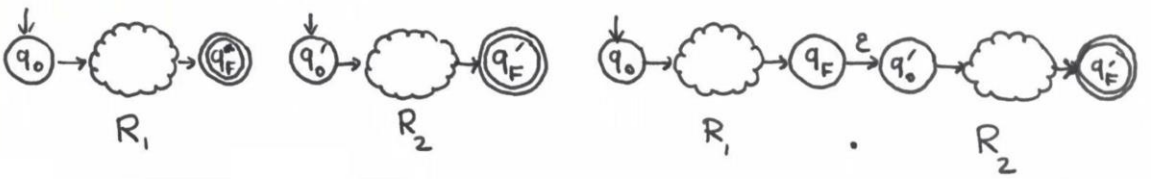
Base Cases.

$\emptyset$ 

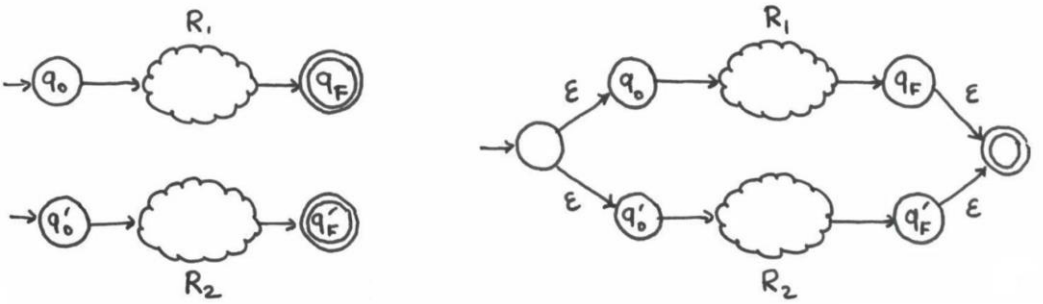
ε 

$a \ (a \in \Sigma)$ 

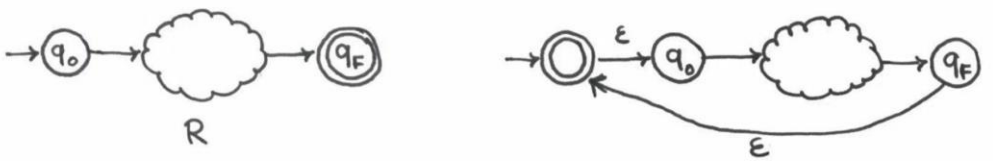
Concatenation $R_1 \ R_2$



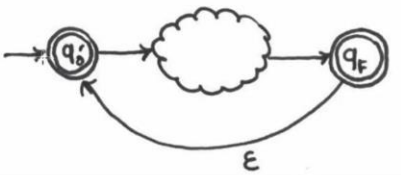
Union $R_1 + R_2$



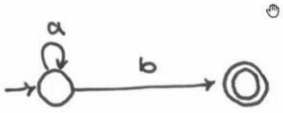
Kleene Star $R*$



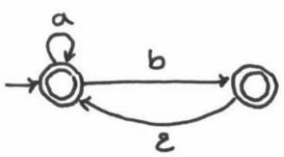
Why doesn't this construction work?



Consider:



$a * b$

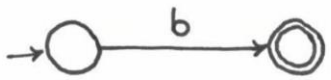


This is not $(a*b)*$ because it accepts ba which is not in $(a*b)*$

Example. $L = ab + a*$

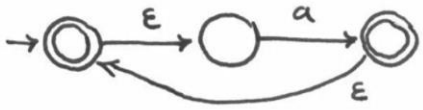
a:

b:

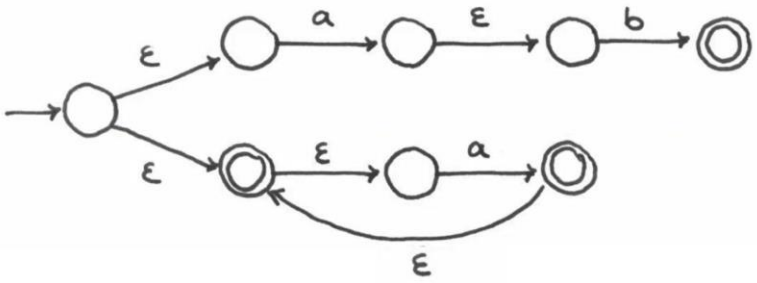


ab:

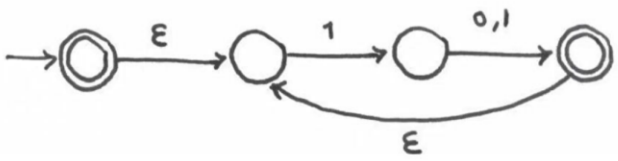
a*:



ab + a*



Example. $L = (1(0+1))*$



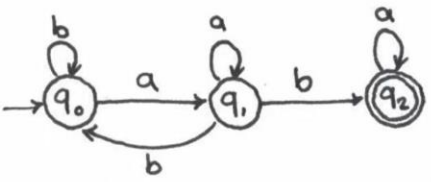
Why does this matter?

Many software tools work by matching regular expression against text

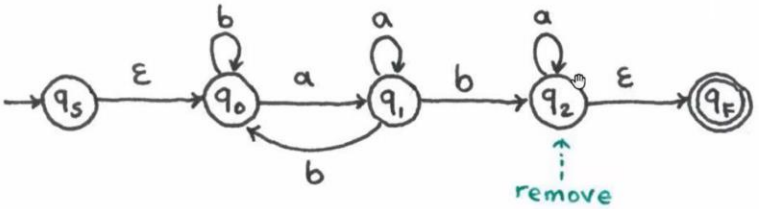
- Convert regular expression to NFA (Tompson's Algorithm)
- Convert NFA to DFA
- Run text through DFA

Theorem. If L is a regular language, then there is a regular expression R with $L = L(R)$.

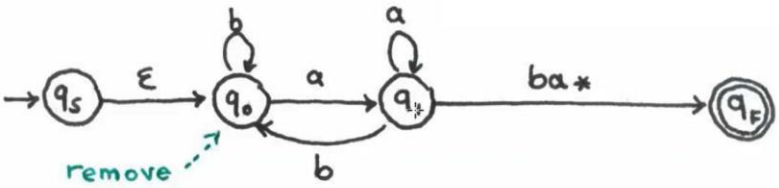
Example.



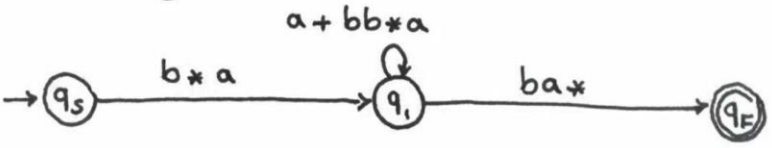
Convert NFA to a special form with: – only one final state
– no incoming edges to start
– no outcoming edges from final



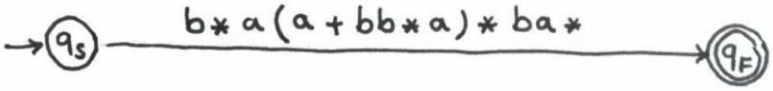
Now we remove states one at a time, replace labels with more complicated regular expression



$ba*$: all strings that can move automaton from q_1 to q_F visiting just q_2 any number of times



$a + bb*a$: all strings that can move automaton from q_1 to q_1 visiting just q_0 any number of times



Final label is the regular expression we are looking for it.

Generalized NFA (gNFA)

Same as NFA, but:

- only one accept state ≠ start state
- start state has no incoming edges, accept state no outgoing edges
- edges are labeled with regular expressions

First convert the original NFA to gNFA

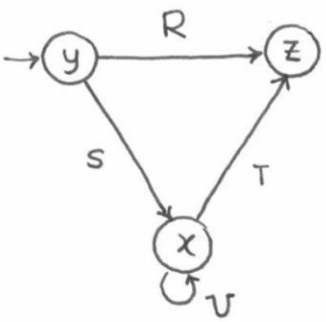
Successively transform gNFA to equivalent gNFAs (recognizing same language) each time removing one state

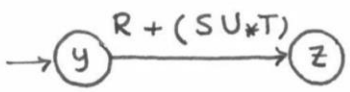
To remove a state x, consider every pair of states y and z (including y = z)

New label for edge (y, z) is the union of two expressions:

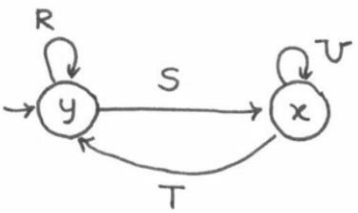
- what was there before
- one for paths through just x

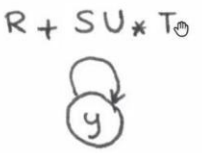
if $y \neq z$



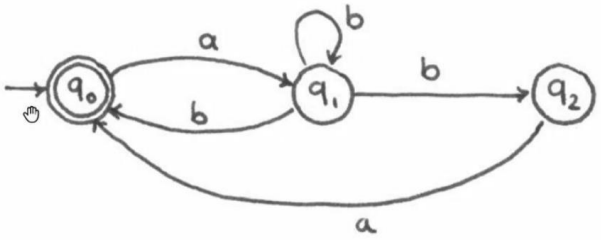


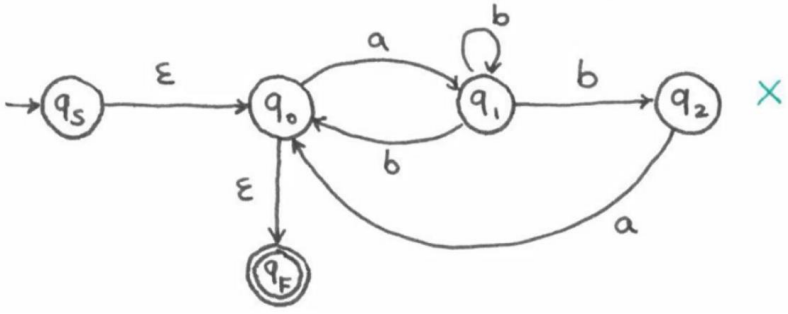
if $y = z$





Example. Convert NFA to regular expression





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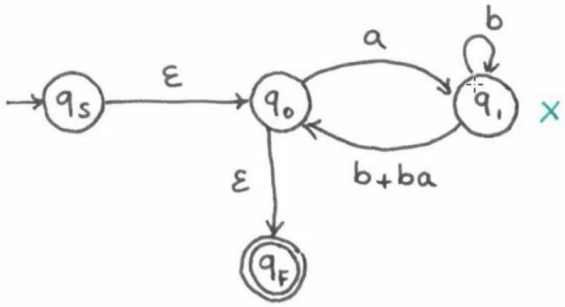

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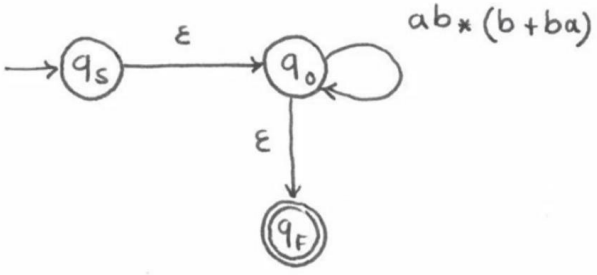


$ab*(b+ba)$

$(ab*(b+ba))*$

باسمه تعالی

جلسهٔ هشتم درس نظریه

Are these languages regular?

$\{0^n 1^n \mid n \geq 0\}$

$\{ww \mid w \in \{0,1\}^*\}$

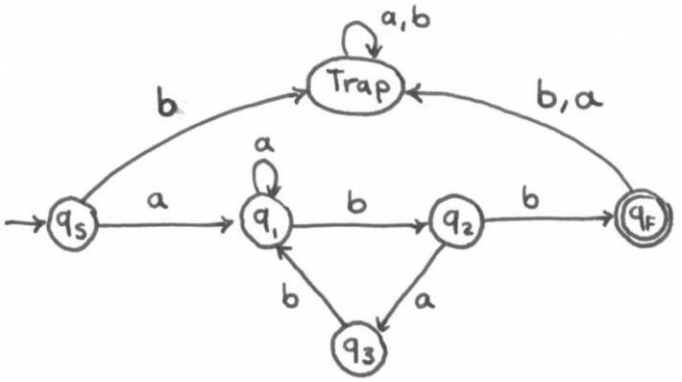
$\{ww^R \mid w \in \{0,1\}^*\}$

$\{0^n 1^m \mid n > m\}$

$\{0^n 1^m \mid n \leq m\}$

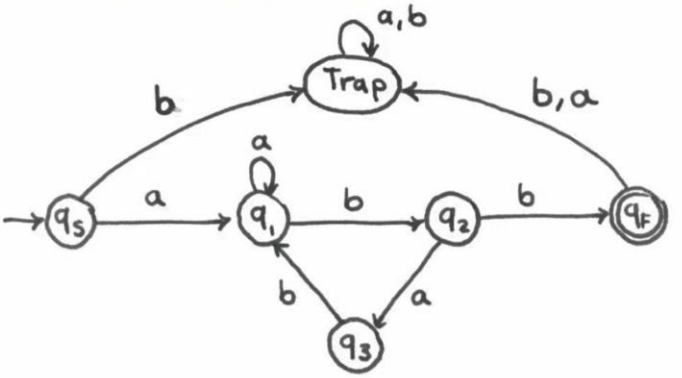
$\{w \mid w$ contains equal number of 0's and 1's$\}$

$\{w \mid w$ contains equal number of substrings 01 and 10$\}$



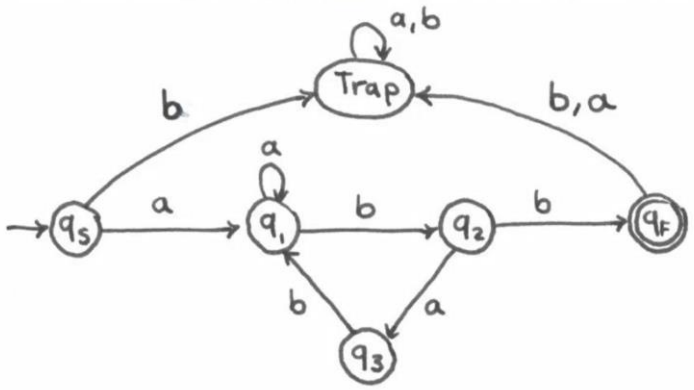
a b a b b a b b b

q_S q_1 q_2 q_3 q_1 q_2 q_3 q_1 q_2 q_F



a b a b b b

q_S q_1 q_2 q_3 q_1 q_2 q_F



a b b

q_S q_1 q_2 q_F

Let L be a regular language. (2)

Consider a string $w \in L$ that is "sufficiently long"

We can split w into three pieces and "pump" middle

- Write $w = xyz$
- Then xy^0z, xy^1z, xy^2z, xy^3z, ... are all in L

If L is regular then

∃ integer $p \geq 1$

∀ string $w \in L$ with $|w| \geq p$

∃ strings x, y, z satisfying $w = xyz$ with

- $|xy| \leq p$
- $|y| > 0$

∀ integer $i \geq 0$, $xy^iz \in L$

$$w = \overbrace{a_1\ a_2\ a_3\ \cdots}^{x}\ \overbrace{\cdots}^{y}\ \cdots\ a_p$$

DFA: p states

states $\longrightarrow$ $q_0 \mid q_1 \mid q_2 \mid \cdots q_i \mid \cdots q_i \cdots \mid q_p$

$A \Rightarrow B \equiv \neg B \Rightarrow \neg A$

(اگر A نتیجه بدهد B)

$\neg\ \forall x \in \mathbb{N}.\ A(x) \equiv \exists x \in \mathbb{N}.\ \neg A(x)$

$\neg\ \exists x \in \mathbb{N}.\ A(x) \equiv \forall x \in \mathbb{N}.\ \neg A(x)$

If

∀ integer $p \geq 1$

∃ string $w \in L$, with $|w| \geq p$

∀ strings x, y, z s.t. $w = xyz$

- $|xy| \leq p$
- $|y| > 0 \quad (y \neq \varepsilon)$

∃ integer $i \geq 0$, $xy^i z \notin L$

then L is not regular

Demon, forall ∀ player

Demon picks p

you pick $w \in L$

Demon picks x, y, z

you pick i

you win if $xy^i z \notin L$

PUMPING LEMMA.

(3)

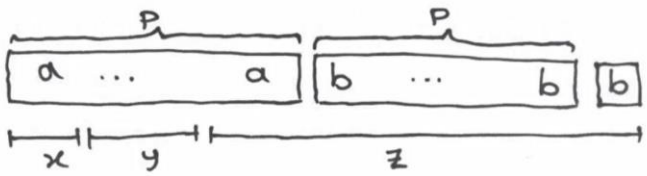
L is not regular if you have a winning strategy.

$L = \{a^n b^n \mid n \geq 0\}$ is not regular

① Demon picks $p \geq 1$

② You pick $w = a^p b^p$ $\quad w \in L$ and $|w| \geq p$

③ $w = xyz$ $\quad |xy| = p' \leq p$ and $y \neq \varepsilon$ +



$x = a^l \quad y = a^j \quad z = a^{p-p'} b^{p+1}$

$(l + j) = p' \quad , \quad j \geq 1$

④ Take $i = 3$ then $xy^i z = a^{p+j\times 2} b^{p+1}$

$L_1 = \{ a^n b^m \mid n \leq m \}$

Proof by contradiction:

$L_1^R = \{ b^m a^n \mid n \leq m \}$ ✓

$L_2 = \{ a^n b^m \mid n \geq m \}$ $\Sigma = \{a, b\}$ ✓

$L_1 \cap L_2 = \{ a^n b^n \mid n \in \mathbb{N} \}$ ×

$L = \{ w \in \{a,b\}^* \mid n_a(w) = n_b(w) \}$

$L \cap a^* b^* = \{ a^n b^n \mid n \geq 0 \}$ (Proof by contradiction)

$L = \{ a^n \mid n$ is a power of $2 \}$

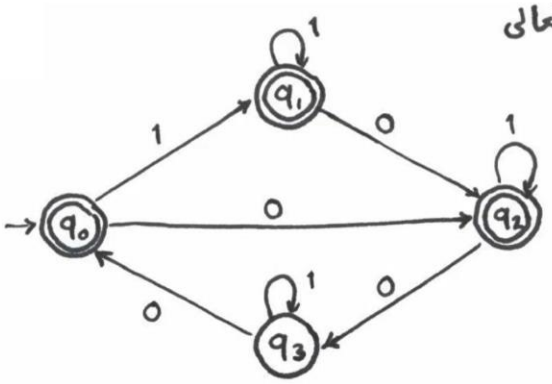
① Demon picks $p \geq 1$

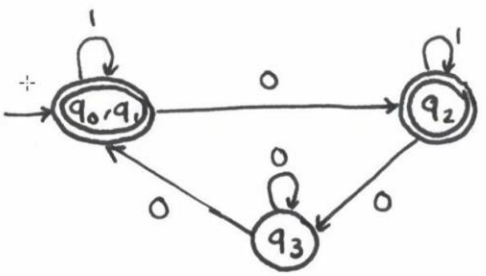
② You pick $w = a^{2^p}$ $|w| = 2^p \geq p$

③ $w = xyz$ $x = a^\ell$, $y = a^j$, $z = a^k$

④ Take $i = 2$ then $xy^2z = a^{2^p + j}$

$$2^p + j \leq 2^p + p < 2^p + 2^p = 2^{p+1}$$



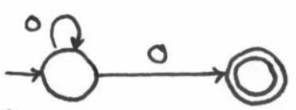


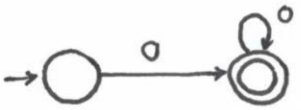
For every regular language L, there exists a unique, minimal DFA that recognizes L (uniqueness up to relabeling of states)

What about NFA minimization?

No unique!

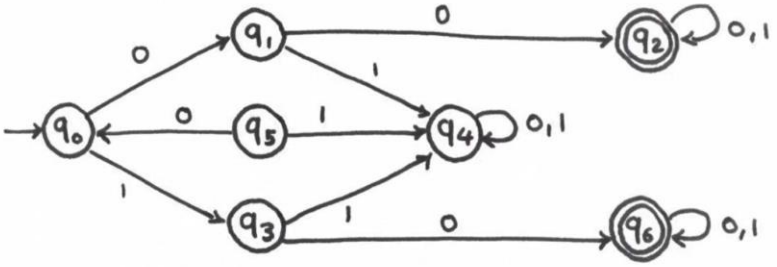
(Kameda-Weiner Algorithm)



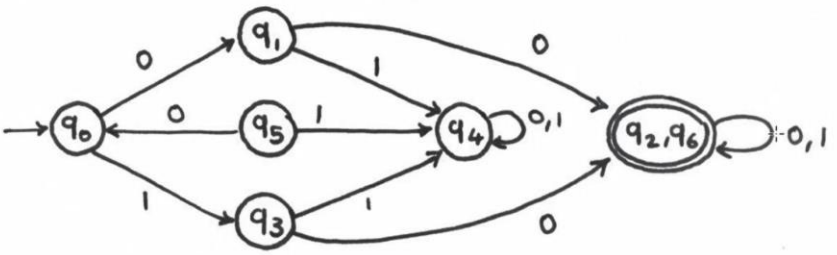


Why does it matter? Use less memory (hardware)

Idea : merge equivalent states



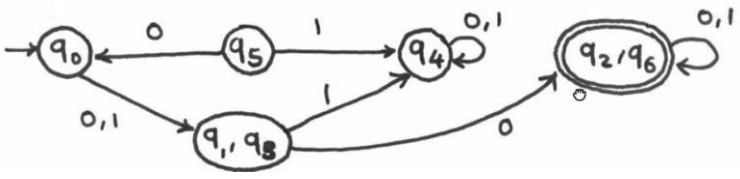
q_2 and q_6 : any strings which reaches them is guaranteed to be accepted later. Any subsequent suffixes produce identical results



if you are in q_1 or q_3 : ① if string ends, reject in both cases

② if next symbol is 0, forever accept in both cases.

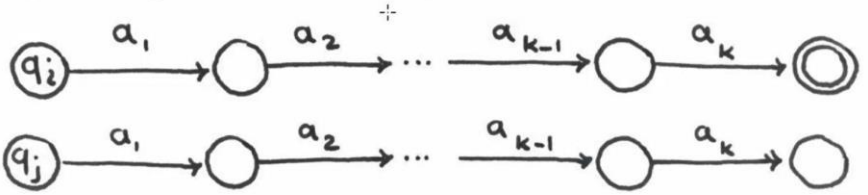
③ if " " " 1, forever reject " " " .



Unreachable states do not affect the accepted language

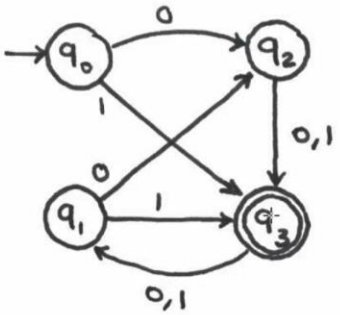
Intuition: two states are equivalent (indistinguishable) if all subsequent behavior from those states is the same.

Two states q_i and q_j are "distinguishable" if



$$\omega = a_1 a_2 \cdots a_{k-1} a_k$$

Example. Which of the following pairs are distinguishable? by which string?



(q_0, q_3)	ε	(q_1, q_2)	0
(q_1, q_3)	ε	(q_0, q_2)	0
(q_2, q_3)	ε	(q_0, q_1)	indistinguishable

Two states q and q' in a DFA $M = (Q, \Sigma, \delta, q_0, F)$ are equivalent if for all strings $\omega \in \Sigma^*$, the states on which ω ends on when reading from q and q' are both accept or reject

* Equivalent states can be merged together

DFA: an automaton is irreducible if:

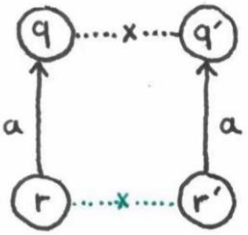
a) it contains no unreachable states
b) no two distinct states are equivalent

Finding indistinguishabe states

Phase 1: 

For each accepting q and rejecting q'

Mark (q, q') as distinguishable x

Phase 2: 

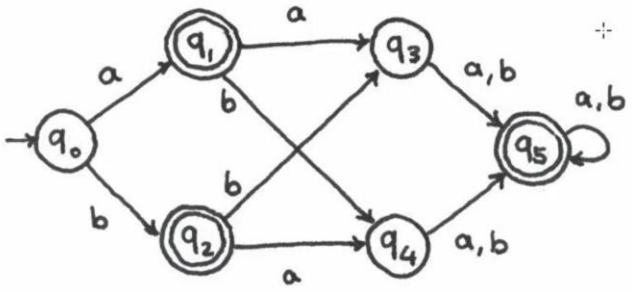
If (q, q') are marked and

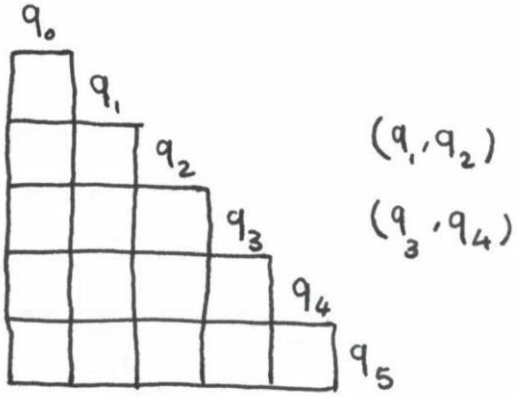
$r \xrightarrow{a} q$ $\quad$ $r' \xrightarrow{a} q'$

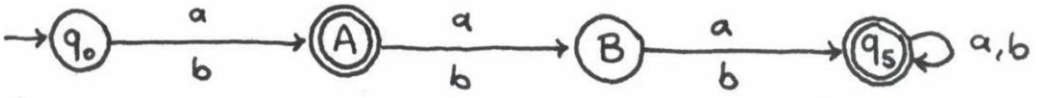
Mark (r, r') as distinguishable

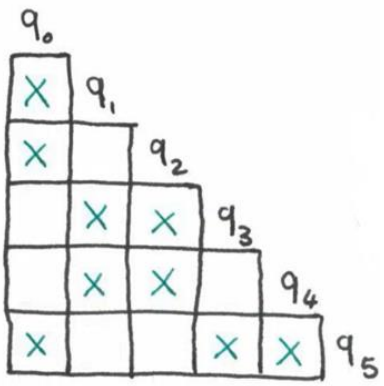
Phase 3: Unmarked pairs are indistinguishable. Merge them!

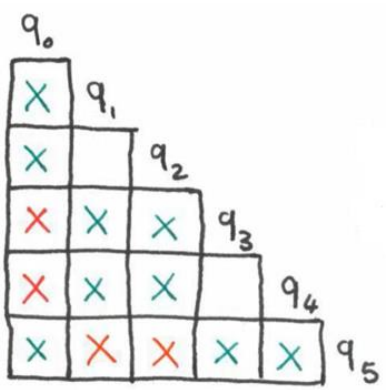
We set up a table in which we can keep track of all pairs of states

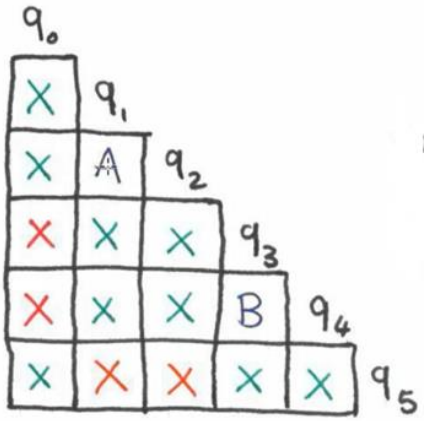


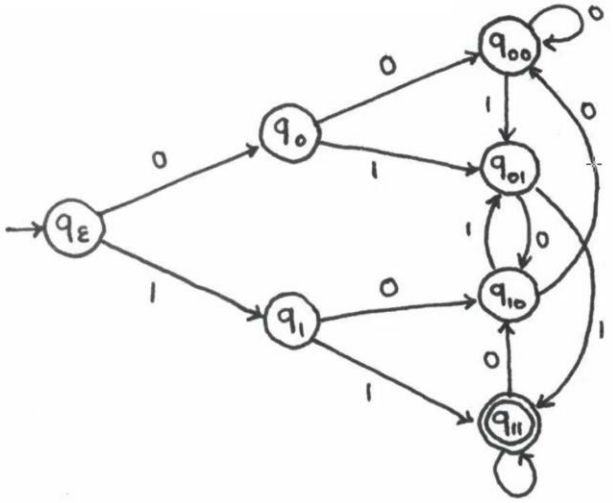




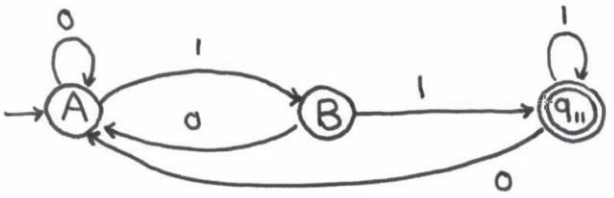








q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}	q_{11}
A						
X	X					
A	A	X				
X	X	B	X			
A	A	X	A	X		
X	X	X	X	X	X	



Decision Problems about Regular Languages

Remember: A decision problem is a problem with yes/no answer

① Given a string w and a language L, determine whether $w \in L$

Assume L is represented by a DFA M. Simulate the action of M on the sequence of symbols of w.

② Given a language L, determine whether it is empty.

Compute the set of reachable states from the start state of the DFA.
Check if any final state is reachable.

③ Given a language L, determine whether it is finite.

Compute all the states that belong to some cycle in the DFA.
Determine if any of them is on a path from an inital to final state.

④ Give L_1 and L_2 determine if they are identical.

Minimize the DFAs representing the language L_1 and L_2.
Check if they are isomorph.

We used pumping lemma to prove L is not regular:

$L = \{a^n b^n \mid n \geq 0\}$

$\omega = a^n b^n = a \underbrace{\boxed{a^{n-1} b^{n-1}}}_{\omega'} b$ recursive structure

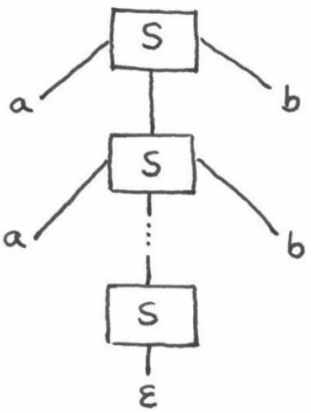
Basic Idea: use "variables" to stand for sets of strings

Variables are defined recursively in terms of one another

Recursive rules ("productions") involve only concatenation

Alternative rules for a variable allow union

$$\begin{cases} S \to a S b \\ S \to \varepsilon \end{cases}$$



$S \Rightarrow a S b \Rightarrow aa S bb \Rightarrow aa \varepsilon bb$

$= aabb$

Terminals = symbols of the alphabet of the language

Variables = non-terminals = a finite set of other symbols, each of which represents a language

Start symbol = the variable whose language is the one being defined

Grammar:

Set of rules capable of generating all the words of a language

(Chomsky ~ 1950)

Left ⟶ Right

Anything matches on left-side can be replaced by what is on the right. Left and Right can be any sequence of variables (non-terminals) and symbols (terminals)

$Exp \rightarrow Var \mid Num \mid (Exp)$

$Exp \rightarrow Exp + Exp$

$Exp \rightarrow Exp * Exp$

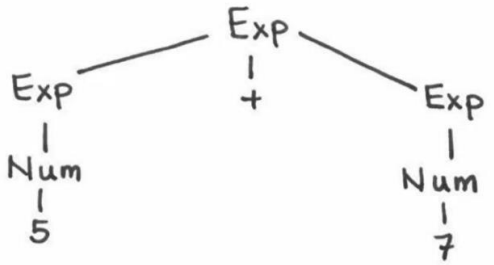
$Var \rightarrow x \mid y \mid z$

$Num \rightarrow 0 \mid \cdots \mid 9$

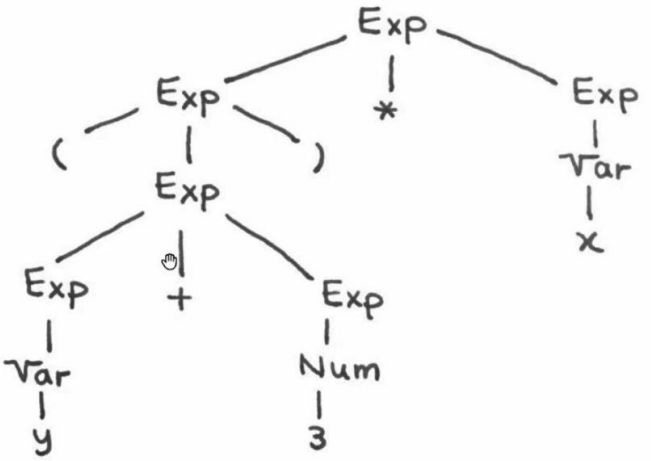
- Start symbol: Exp
- Finite set of variables:

  $\{Exp, Var, Num\}$
- Finite set of terminals:

  $\{x, y, z, 0, \cdots, 9, (,)\}$
- Finite set of rules



5 + 7



(y + 3) * x

- Interior nodes: variable
- Leaves: terminals
- Yield: strings of terminals at the bottom

When designing grammars:

- Think recursively: Build up bigger structures from smaller ones
- Have construction plan:

  Know in what order you build up string
- Store information in non-terminals

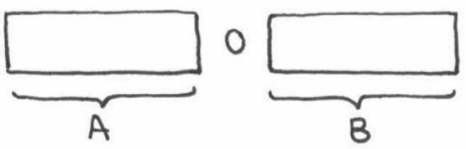
  Have each non-terminal correspond to some useful information

$L = \{ w \in \{0,1\}^* \mid w \text{ contains at least three } 1 \}$

$S \to X\,1\,X\,1\,X\,1\,X$

$X \to 0X \mid 1X \mid \varepsilon$

$L = \{ w \in \{0,1\}^* \mid \text{length of } w \text{ is odd and middle symbol is } 0 \}$



$|A| = |B|$

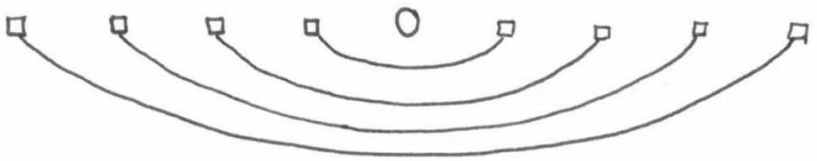
First Attempt:

$$\begin{cases} S \to X\,0\,X \\ X \to 0X \mid 1X \mid \varepsilon \end{cases}$$

wrong!

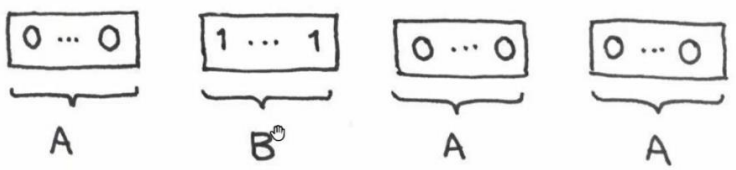
$S \Rightarrow X\,0\,X \Rightarrow 0X0X \Rightarrow 0\,\varepsilon\,0X \Rightarrow 00X \Rightarrow 00\varepsilon \Rightarrow 00$

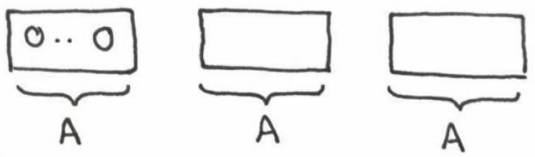
Second Attempt:



$S \to 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid 0$

$L = \{ 0^n 1^m 0^{2n} \mid n, m \geq 0 \}$





$$\begin{cases} S \to 0S00 \\ \quad \mid \varepsilon \end{cases}$$

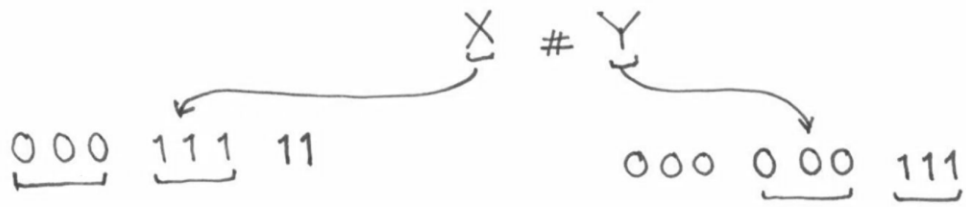
$$\begin{cases} S \to 0S00 \mid X \\ X \to 1X \mid \varepsilon \end{cases}$$

$L = \{ \omega \in \{0,1\}^* \mid \omega \text{ starts and ends with same symbol} \}$

$S \to 0X0 \mid 1X1 \mid \varepsilon$

$X \to 0X \mid 1X \mid \varepsilon$

$L = \{ 0^i 1^j \# 0^k 1^\ell \mid i < j \text{ and } k > \ell \}$



$S \to X \# Y$

$E \to aEb \mid \varepsilon$

$X \to X1 \mid E1$

$Y \to 0Y \mid 0E$

$L = \{ \omega \in \{a,b\}^* \mid \omega = \omega^R \}$

$S \to aSa \mid bSb \mid \varepsilon \mid a \mid b$

$L = \{ \omega \in \{a,b\}^* \mid \omega \text{ contains more a's than b's} \}$

$S \to TaT$

$T \to aT \mid aTbT \mid bTaT \mid \varepsilon$

Types of Grammars:

0) Unrestricted Grammars. no restrictions on rewriting rules

1) Context sensitive Grammars. $\alpha A \beta \longrightarrow \alpha \gamma \beta$

A non-terminal, α, β, γ strings of terminals and non-terminals

γ is non-empty

2) Context free Grammars $A \longrightarrow \gamma$

γ potentially empty string of terminals and non-terminals

3) Regular Grammars

$A \rightarrow aB \mid a$

A regular grammar is either right- or left- linear.

Note. If we mix left-linear and right-linear rules the result is not necessarily regular.

$S \rightarrow aA \mid \varepsilon$ (Right Linear)

$A \rightarrow Sb$ (Left Linear)

$L = \{a^n b^n \mid n \geq 0\}$

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Context_free Grammar $G = (V, \Sigma, R, S)$

V: finite set (variables)

Σ: finite set disjoint from V (terminals)

R: finite non-empty set of rules

$S \in V$ start variable

Example $G = (V, \Sigma, R, S)$

$V = \{S\}$

$\Sigma = \{a, b\}$

$R = \{ S \to aSb, S \to \varepsilon \}$

$L(G) = \{ a^n b^n \mid n \geq 0 \}$

Start Symbol : S

Show that a string w belongs to the language of grammar G

- Start with start symbol S
- Repeatedly replace one of the non-terminals by RHS of a rule
- Stop when sentence contains only terminals

Derivation Step $A \in V$, $\alpha, \beta, \gamma \in (V \cup \Sigma)^*$ $A \to \gamma \in R$

$\alpha A \beta \Rightarrow \alpha \gamma \beta$ derivation step

Derivation Sequence of derivation steps

$\gamma_0 \Rightarrow \gamma_1 \Rightarrow \cdots \Rightarrow \gamma_n \quad (n \geq 0) \qquad \gamma_i \in (V \cup \Sigma)^*$

If there is a derivation from γ_0 to γ_n we write is as $\gamma_0 \overset{*}{\Rightarrow} \gamma_n$

Example. $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aa\varepsilon bb \Rightarrow aabb$

$S \overset{*}{\Rightarrow} aabb$

Language of grammar $L(G) = \{ w \in \Sigma^* \mid S \overset{*}{\Rightarrow} w \}$

Set of strings derivable from start (Context-Free Languages)

How do we know which rule to apply on every step?

- Leftmost derivation : always expand leftmost variable
- Rightmost derivation : always expand rightmost variable

Example. $G = (\{S\}, \{(,)\}, R, S)$ $\quad R = \{ S \to SS \,(①) \mid (S) \,(②) \mid () \,(③) \}$

Derivation : $S \overset{1}{\Rightarrow} SS \overset{2}{\Rightarrow} (S)S \overset{3}{\Rightarrow} (S)() \overset{3}{\Rightarrow} (())()$

Leftmost Derivation : $S \overset{1}{\Rightarrow} SS \overset{2}{\Rightarrow} (S)S \overset{3}{\Rightarrow} (())S \overset{3}{\Rightarrow} (())()$

Rightmost Derivation : $S \overset{1}{\Rightarrow} SS \overset{3}{\Rightarrow} S() \overset{2}{\Rightarrow} (S)() \overset{3}{\Rightarrow} (())()$

Leftmost & Rightmost : allow us to describe derivation by listing sequence of rules

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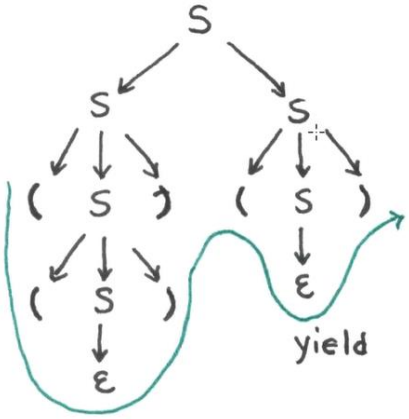


Parse (Derivation) tree shows derivation of a string as tree

a) root is labelled S

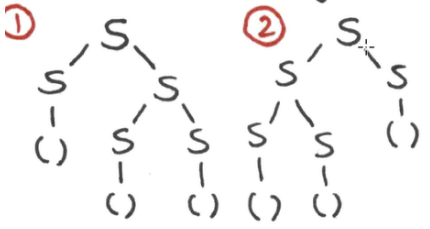
b) every leaf has a label from $\Sigma \cup \{\varepsilon\}$

c) every interior has a label from V

d) if an interior $A \in V$ has children $B_1 \cdots B_n$ then R must have $A \to B_1 \cdots B_n$ $\quad (B_i \in V \cup \Sigma)$

A given CFG G is ambiguous if a string $w \in L(G)$ is the yield of two different parse trees.

Equivalently: G is ambiguous if a string $w \in L(G)$ has two different leftmost or rightmost derivations.



① $S \Rightarrow SS \Rightarrow ()S \Rightarrow ()SS \Rightarrow ()()S \Rightarrow ()()()$

② $S \Rightarrow SS \Rightarrow (S)S \Rightarrow (())S \Rightarrow (())() \Rightarrow ()()()$

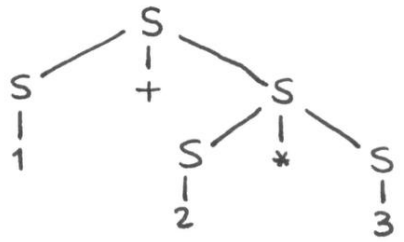
Ambiguity is a property of a grammar, not the language it generates

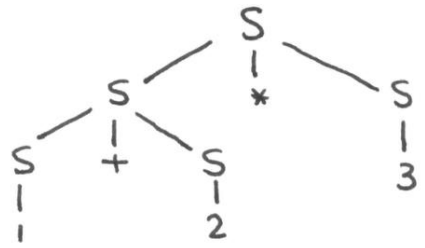
$S \to \varepsilon \mid (S)\, S$ (unambiguous grammar) ③

There is no algorithm to detect if an arbitrary grammar is ambiguous

$S \to S+S \mid S*S \mid 0 \mid \cdots \mid 9$

$1 + 2 * 3$





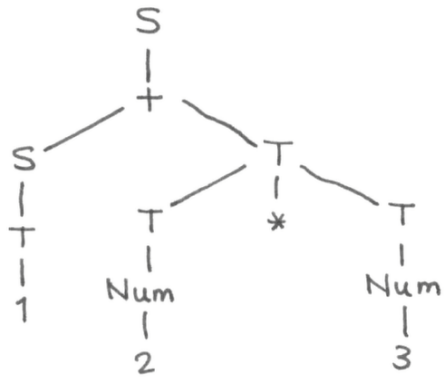
Importance: Compiler does not only say yes/no – has to give semantics

Does this grammar solve the problem?

$S \to (S+S) \mid (S*S) \mid 0 \mid \cdots \mid 9$

$$\begin{cases} S \to S+T \mid T \\ T \to T * \text{Num} \mid \text{Num} \\ \text{Num} \to 0 \mid \cdots \mid 9 \end{cases}$$

$\boxed{x*y*z} + \boxed{a*b} + \boxed{d}$



Is every CFL recognizable by a grammar that has only one variable?

No. Consider the language $L = \{0\} \cup \{\varepsilon, 1, 11, 111, \ldots\}$

$(0 + 1*)$

Assume G has only one non-terminal S

Since the grammar has only finite number of rules, there exists some n such that $1^n \in L$ and it cannot be generated in one step $S \Rightarrow^* 1^n$

$S \Rightarrow^* \alpha S \beta \Rightarrow^* 1^n$ α and β are not both empty

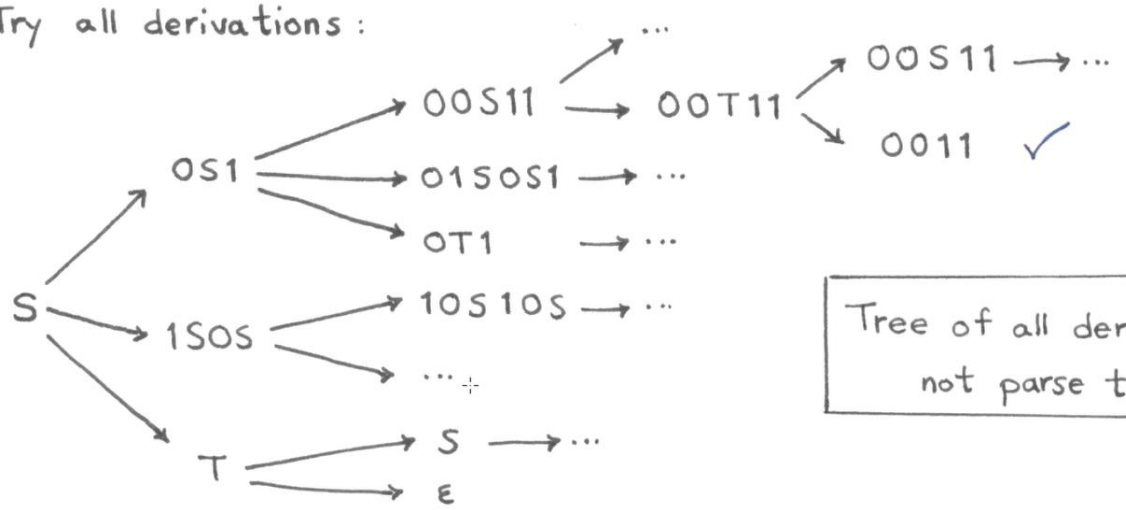
$S \Rightarrow^* 1^a S 1^b$ replace all S with 1 (a,b are not both 0)

$S \Rightarrow^* 1^a 0 1^b$ ×

$$\begin{cases} S \to 0S1 \mid 1S0S \mid T \\ T \to S \mid \varepsilon \end{cases}$$

Is $0011 \in L$? If so, how to build a parse tree?

Try all derivations:



Tree of all derivations, not parse tree

Problems.

1) Trying all derivations may take too long

2) If input is not in the language, parsing will never stop

Let's tackle the second problem

Idea: Stop when |derived string| > |input|

Problems:

* Derived string may shrink because of ε-productions

$$S \Rightarrow 0S1 \Rightarrow 0T1 \Rightarrow 01$$

* Derivation may loop because of unit-productions

$$S \Rightarrow T \Rightarrow S \Rightarrow T \Rightarrow S \Rightarrow \cdots$$

→ We need to remove ε and unit productions.

* Eliminating ε-productions

Given a grammar G, produce an equivalent grammar G′ ($L(G) = L(G')$) such that G′ does not have any rule of the form $A \to \varepsilon$ except possibly $S \to \varepsilon$, and S does not appear on the right hand side of any rule. (Note. If S can appear on the RHS of a rule, say $S \to SS$ then when there is a rule $S \to \varepsilon$ we can again have long intermediate strings yielding short final strings

1) Add a new start symbol S' and the rule $S' \to S$ where S was the original start symbol - this guarantees the new start symbol is not on the RHS of any rule

2) For every rule $A \to \varepsilon$ where A is not the (new) start var

a) Remove the rule $A \to \varepsilon$

b) If you see $C \to \alpha A \beta$ add a new rule $C \to \alpha\beta$

This must be done for each occurance of A

$C \to \alpha A \beta A \gamma$ becomes $C \to \alpha\beta A\gamma \mid \alpha A \beta\gamma \mid \alpha\beta\gamma$

$C \to A$ becomes $C \to \varepsilon$

If $C \to \varepsilon$ was removed earlier, don't add it back

---

$S' \to S$

$S \to ABCD \mid \varepsilon$

$A \to SaS$

$B \to \varepsilon$

$C \to CSE \mid \varepsilon$

$D \to CB \mid b$

$E \to aB$

$S \to \varepsilon$

$S' \to \varepsilon$

$C \to CE$

$A \to Sa$

$A \to aS$

$A \to a$

$B \to \varepsilon$

$S \to ACD$

$D \to C$

$E \to a$

$C \to \varepsilon$

$S \to ABD$

$C \to SE$

$D \to B$

$C \to E$

$S \to AD$

$D \to \varepsilon$

$D \to \varepsilon$

$S \to ABC$

$S \to AC$

$S \to AB$

$S \to A$

A unit production is a production of the form $A \to B$

Grammar:

$S \to 0S1 \mid 1S0S \mid T$

$T \to S \mid R \mid \varepsilon$

$R \to 0SR$

Unit production Graph

① If there is a cycle of unit productions $A \to B \to \cdots \to C \to A$ delete it and replace every thing with A (any var in cycle)

$S \to 0S1 \mid 1S0S$

$S \to R \mid \varepsilon$

$R \to 0SR$

② replace any chain $A \to B \to \dots \to C \to \alpha$

by $A \to \alpha$, $B \to \alpha$, $\dots$, $C \to \alpha$

$S \to 0S1 \mid 1S0S \mid R \mid \varepsilon$

$R \to 0SR$

S
↓
R

$S \to 0S1 \mid 1S0S \mid 0SR \mid \varepsilon$

$R \to 0SR$

Problems. 1) Trying all derivations may take too long

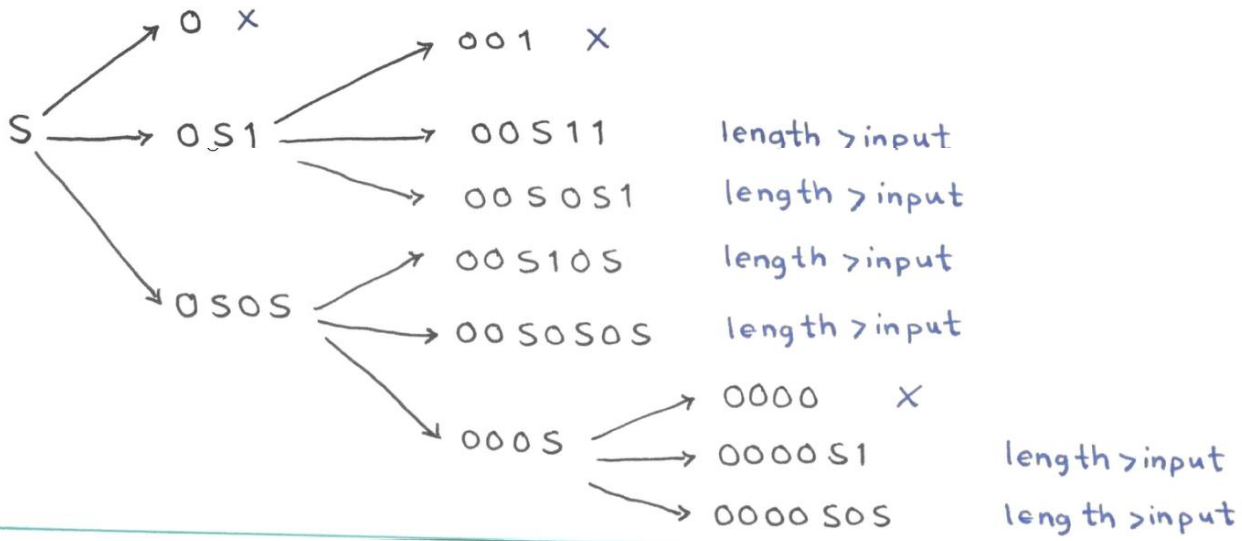
2) If input is not in the language, parsing will never stop ✓

→ Eliminate ε- and unit- productions

Try all possible derivations but stop searching when |derived str| > |in|

$$\begin{cases} S \to 0S1 \mid 0S0S \mid T \\ T \to S \mid 0 \end{cases} \Rightarrow \quad S \to 0S1 \mid 0S0S \mid 0$$

input : 0011



Cocke - Younger - Kasami (CYK Algorithm)

Convert grammar to Chomsky Normal Form

A CFG is in CNF if every production is one of the following:

$A \to BC$	Exactly two non-start variables on right
$A \to a$	Exactly one terminal on right
$S \to \varepsilon$	ε-production only allowed for start

For any context-free grammar there exists an equivalent grammar is CNF.

Step 1. Eliminate ε and unit productions

- Every production RHS is either a single terminal or of length at least 2

Step 2. For each RHS (not single terminal) make RHS all vars

- For each terminal a create a new variable A_a and $A_a \to a$
- Replace a by A_a in RHS of length >2

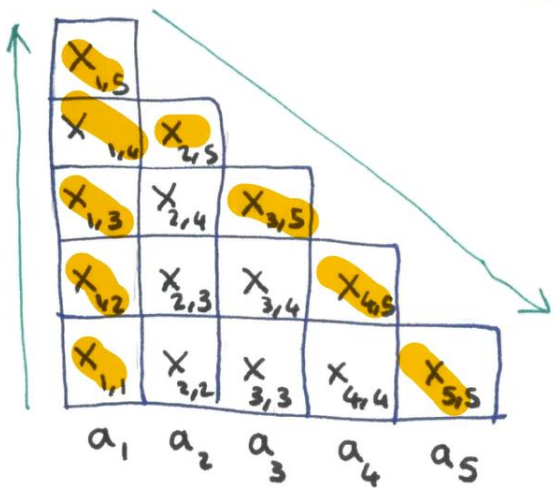
Example. $A \to BcDe$

$A_c \to c$, $A_e \to e$ $\Longrightarrow$ $A \to BA_cDA_e$

Step 3. Break RHS longer than 2 into a chain of productions with RHS of two variables

Example. $A \to BCDE$ $\quad A \to BF$, $F \to CG$ $\quad G \to DE$

(F: fresh)

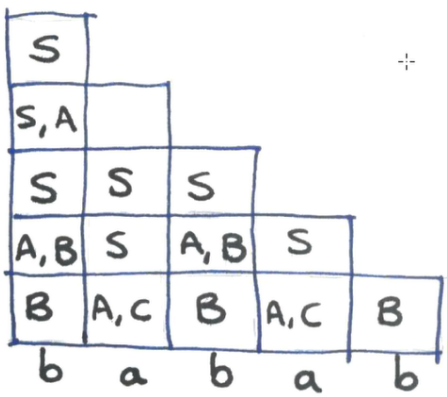


$X_{i,j}$: set of non-terminals deriving $a_i \dots a_j$

Start symbol derives $w = a_1 \dots a_n$ if and only if S is a member of $X_{1,n}$

---

if B belongs to $X_{1,1}$ and C to $X_{2,5}$ and if $A \to BC$ is a grammar rule add A to $X_{1,5}$



$S \to AB \mid CB$

$A \to BA \mid a$

$B \to BC \mid b$

$C \to AC \mid a$

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Two grammars G_1 and G_2 are equivalent if they generate the same language $L(G_1) = L(G_2)$

Theorem. Given a context-free grammar G there exists an equivalent Chomsky Normal Form (CNF) G′ such that $L(G) = L(G')$

CNF. A CFG is in CNF if every production is of the form

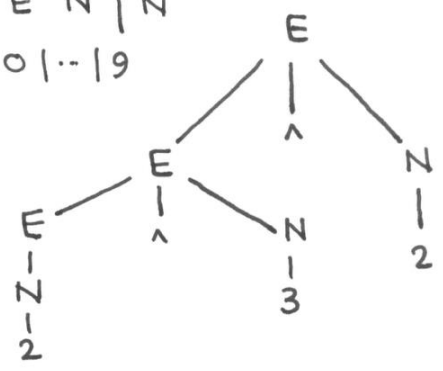
$A \to BC$

$A \to a$

$S \to \varepsilon$ (if $\varepsilon \in L(G)$ and S does not appear on the RHS of any production)

$E \to E \hat{} N \mid N$

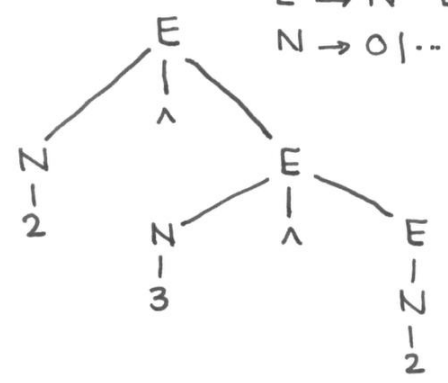
$N \to 0 \mid \cdots \mid 9$



$2 \hat{} 3 \hat{} 2 = 8^2 = 64$

$E \to N \hat{} E \mid N$

$N \to 0 \mid \cdots \mid 9$



$2 \hat{} 3 \hat{} 2 = 2 \hat{} 9 = 512$

Left-recursive Grammar: It has a non-terminal A such that there is a derivation $A \overset{+}{\Longrightarrow} A\alpha$ for some string α

Right-recursive Grammar: It has a non-terminal A such that there is a derivation $A \overset{+}{\Longrightarrow} \alpha A$ for some string α

Transformation of immediate left recursion (direct)

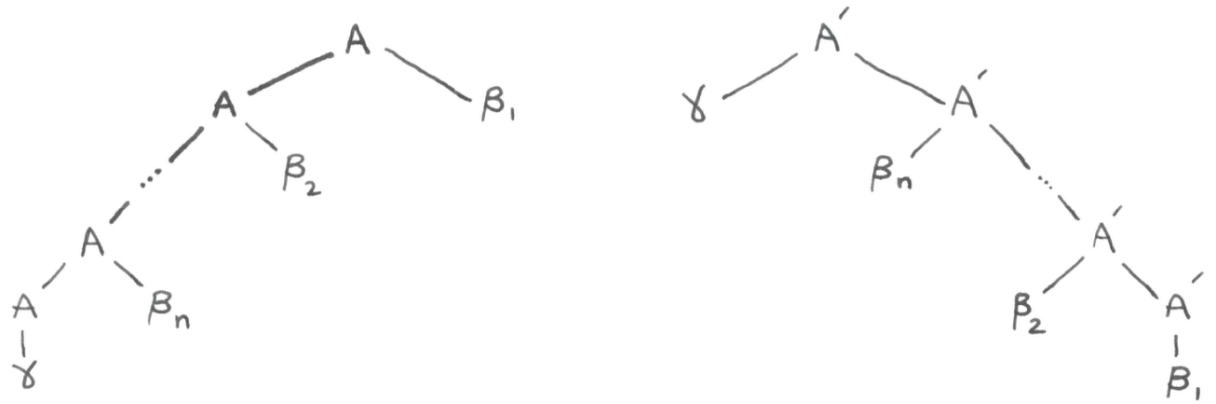
$$\begin{cases} A \to A\beta_1 \mid A\beta_2 \mid \cdots \mid A\beta_h & (h \geq 1) \quad \text{no } \beta_i \text{ string is empty} \\ A \to \gamma_1 \mid \gamma_2 \mid \cdots \mid \gamma_k & (K \geq 1) \end{cases}$$

Create a new fresh non-terminal A′

$$\begin{cases} A \to \gamma_1 A' \mid \gamma_2 A' \mid \cdots \mid \gamma_k A' \\ A' \to \beta_1 A' \mid \beta_2 A' \mid \cdots \mid \beta_h A' \mid \varepsilon \end{cases}$$

Or, after removing the ε-production rule:

$$\begin{cases} A \to \gamma_1 A' \mid \gamma_2 A' \mid \cdots \mid \gamma_k A' \mid \gamma_1 \mid \gamma_2 \mid \cdots \mid \gamma_k \\ A' \to \beta_1 A' \mid \beta_2 A' \mid \cdots \mid \beta_h A' \mid \beta_1 \mid \beta_2 \mid \cdots \mid \beta_h \end{cases}$$



Every leftmost derivation is replaced by rightmost

$A \Rightarrow A\beta_2 \Rightarrow A\beta_3\beta_2 \Rightarrow \gamma_1\beta_3\beta_2$

$A \Rightarrow \gamma_1 A' \Rightarrow \gamma_1\beta_3 A' \Rightarrow \gamma_1\beta_3\beta_2$

Example. Conversion of immediate left recursion

$$\begin{cases} E \to E+T \mid T \\ T \to T \times F \mid F \\ F \to (E) \mid n \end{cases} \qquad \begin{cases} E \to TE' \mid T \\ E' \to +TE' \mid +T \end{cases} \qquad \begin{cases} T \to FT' \mid F \\ T' \to \times FT' \mid \times F \end{cases}$$

Simpler solution in this case:

$$\begin{cases} E \to T+E \mid T \\ T \to F \times T \mid F \\ F \to (E) \mid n \end{cases}$$

Indirect Left Recursion

Example.

$$\begin{cases} A \to Ba \mid \varepsilon \\ B \to Cb \\ C \to Ac \end{cases}$$

$A \Rightarrow Ba \Rightarrow Cba \Rightarrow Acba \Rightarrow \cdots$

Idea. First convert indirect left recursion to direct:

$A \to Acba \mid \varepsilon$

Then remove direct left recursion:

$A \to cbaA \mid \varepsilon$

$$\begin{cases} B \to A_1 \alpha_1 \\ A_1 \to A_2 \alpha_2 \\ \quad \vdots \\ A_{n-1} \to B \alpha_n \end{cases} \qquad B \Rightarrow A_1 \alpha_1 \Rightarrow A_2 \alpha_2 \alpha_1 \Rightarrow \cdots \Rightarrow B \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1$$

Algorithm for grammar with no
- ε-production
- cycle $A \overset{*}{\Rightarrow} \alpha \overset{*}{\Rightarrow} A$

Pick an arbitrary order for all non-terminals $\{A_1, A_2, \cdots A_m\}$

Eliminate all left recursions that do not increase the index of A_i

Eliminate all rules of the form $A_i \to A_j \alpha$ with $j \leq i$

If we do this for each i from 1 to m, no loop can happen

Why? Every step in such a loop would only increase the index thus the original index cannot be reached again.

```
for i:=1 to m do
    for j:=1 to i-1 do
        replace each rule of type A_i → A_j α (i>j) by the rules:
            A_i → γ_1 α | γ_2 α | ··· | γ_k α
        where A_j → γ_1 | γ_2 | ··· | γ_k are alternatives of A_j
        // note this may create immediate left recursion
eliminate all the immediate 1-recursions that may have appeared as
alternatives of A_i
```


$$\begin{cases} A_1 \to A_2 a \mid b \\ A_2 \to A_2 c \mid A_1 d \mid e \end{cases}$$

i	j	Algorithm Action	Transformation
1	skipped	Eliminate immediate 1-recursion of A_1	Unchanged
2	1	Replace rule $A_2 \to A_1 d$ Expand non-terminal A_1	$A_1 \to A_2 a \mid b$ $A_2 \to A_2 c \mid A_2 ad \mid bd \mid e$
2	terminated	Eliminate two immediate left recursions of A_2	$A_1 \to A_2 a \mid b$ $A_2 \to bd A_2' \mid e A_2' \mid bd \mid e$ $A_2 \to c A_2' \mid ad A_2' \mid c \mid ad$

Greibach Normal Form

$A \to a\alpha$ $\quad a \in \Sigma$, $\alpha \in V^*$ $\quad S \to \varepsilon$ (if $\varepsilon \in L(G)$)

Every rule starts with a terminal, followed by zero or more vars

kind of rules that violate Greibach Normal Form

- Epsilon Transitions $\quad A \to \varepsilon$
- Unit productions $\quad A \to B$
- Productions such as $X \to AaB$ (terminal other than leftmost)
- Productions such as $X \to AB$ whose RHS starts with var

Transformation:

- Remove all left recursions
- Expand any variable that occurs in the first position of RHS of a rule

$$\begin{cases} X \to Y\alpha \\ Y \to \gamma_1 \mid \gamma_2 \mid \cdots \mid \gamma_n \end{cases} \qquad X \to \gamma_1\alpha \mid \gamma_2\alpha \mid \cdots \mid \gamma_n\alpha$$

---

⑤

$$\begin{cases} A_1 \to A_2\, a \\ A_2 \to A_1 c \mid b A_1 \mid d \end{cases}$$

① Eliminate left recursion

$$\begin{cases} A_1 \to A_2\, a \\ A_2 \to A_2\, ac \mid bA_1 \mid d \end{cases} \qquad \begin{cases} A_1 \to A_2 a \\ A_2 \to bA_1A_2' \mid dA_2' \mid d \mid bA_1 \\ A_2' \to acA_2' \mid ac \end{cases}$$

② Expand non-terminals in the first position until a terminal prefix appears

$$\begin{cases} A_1 \to bA_1A_2'a \mid dA_2'a \mid da \mid bA_1a \\ A_2 \to bA_1A_2' \mid dA_2' \mid d \mid bA_1 \\ A_2' \to acA_2' \mid ac \end{cases}$$

③ Substitute non-terminals for any terminal not in first place

$$A_1 \to b A_1 A_2' X_a \mid d A_2' X_a \mid da \mid b A_1 X_a$$

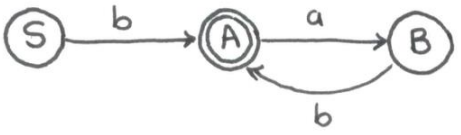
$$A_2 \to b A_1 A_2' \mid d A_2' \mid d \mid b A_1$$

$$A_2' \to a X_c A_2' \mid a X_c$$

$$X_a \to a \qquad X_c \to c$$

Some Interesting Features of GNF:

Every derivation of string w contains $|w|$ rule applications



$S \to bA$

$S \to aS$

$A \to \varepsilon$

$B \to bA$

$A \to aB$

$L = \{a^n b^n \mid n \geq 0\}$

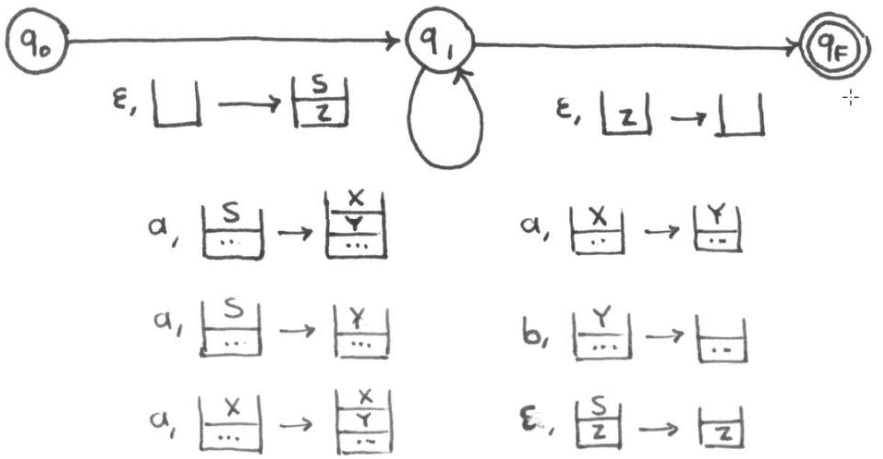
Grammar: $S \to aSb \mid \varepsilon$

Equivalent GNF :

$S \to \varepsilon \mid aXY \mid aY$

$X \to aXY \mid aY$

$Y \to b$



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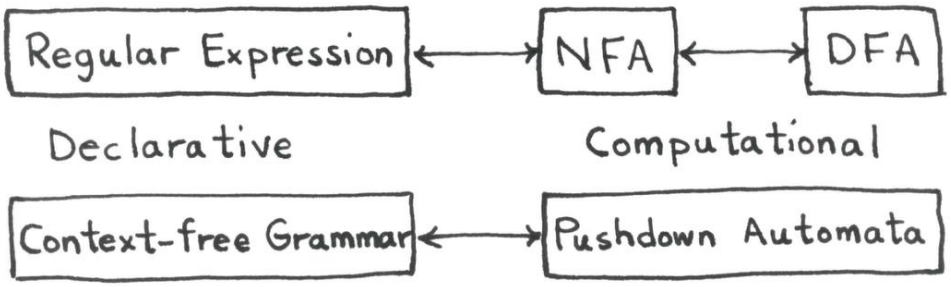
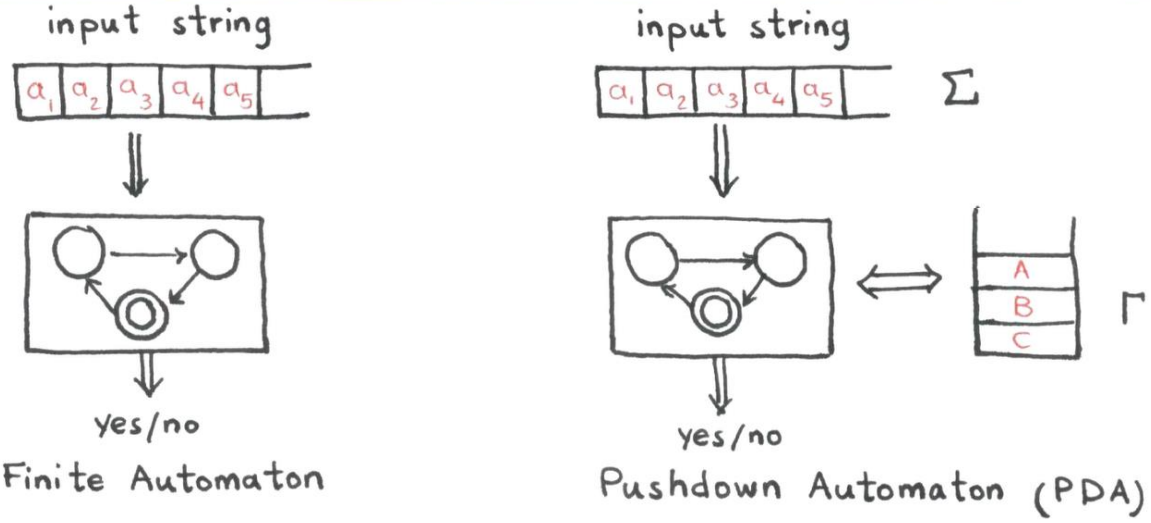


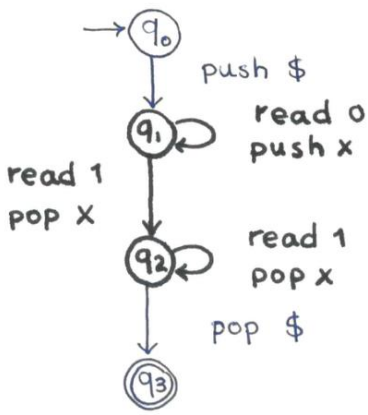


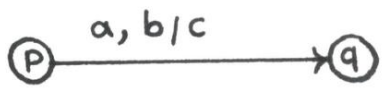
$L = \{ 0^n 1^n \mid n \geq 1 \}$

Remember each 0 by pushing X onto stack

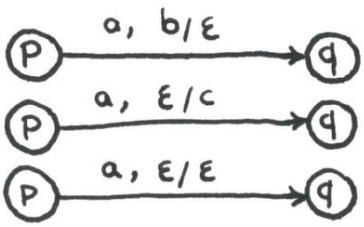
Upon reading a 1, pop X from stack

Accept? Hit stack bottom



Read a, Pop b / Push c 

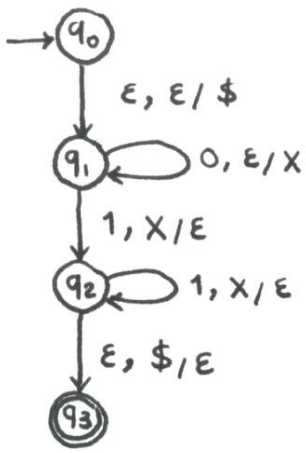
- if next symbol is 'a' and top of stack is 'b' then read 'a', pop 'b' and push 'c'
- if $a = \varepsilon$ don't read next symbol
- if $b = \varepsilon$ don't pop next symbol



only pop

only push

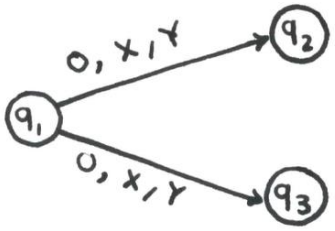
ignore stack



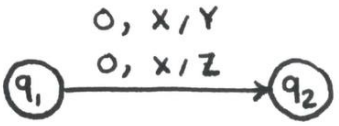
By default, PDA means non-deterministic pushdown automata.

Non-determinism is essential: there are languages that cannot be recognized by DPDAs but can be recognized by NPDAs (lecture 17)

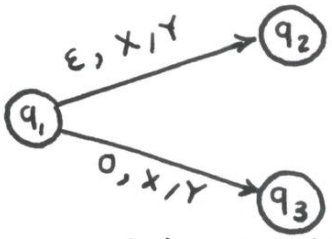
Deterministic pushdown automaton: in each state, there is at most one transition for every combination $(a, b) \in \Sigma \times \Gamma$ +



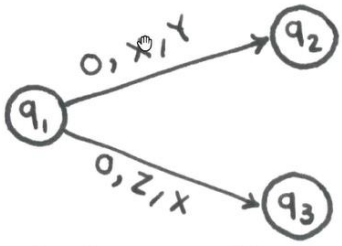
Non-deterministic



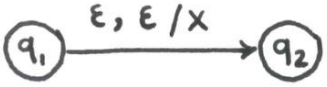
Non-deterministic



Non-deterministic
next state not clear when input is 0 and top stack: X



Deterministic



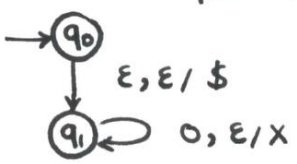
No other transition
Deterministic

A pushdown automaton is $(Q, \Sigma, \Gamma, \delta, q_0, F)$

Q	finite set of states
Σ	finite set of input alphabet
Γ	finite set of stack alphabet
δ	transition function
$q_0 \in Q$	initial state
$F \subseteq Q$	set of accepting states

$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\}) \longrightarrow$ subsets of $\{Q \times (\Gamma \cup \{\varepsilon\})\}$

Q: state; $\Sigma \cup \{\varepsilon\}$: input symbol; $\Gamma \cup \{\varepsilon\}$: pop symbol; Q: state; $\Gamma \cup \{\varepsilon\}$: push symbol



$\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$

$\delta(q_0, \varepsilon, X) = \emptyset$

$\delta(q_0, \varepsilon, \$) = \emptyset$

Note. Finite automaton has a standard definition.
There are different and equivalent definitions of PDA. We use the one described in Sipser.

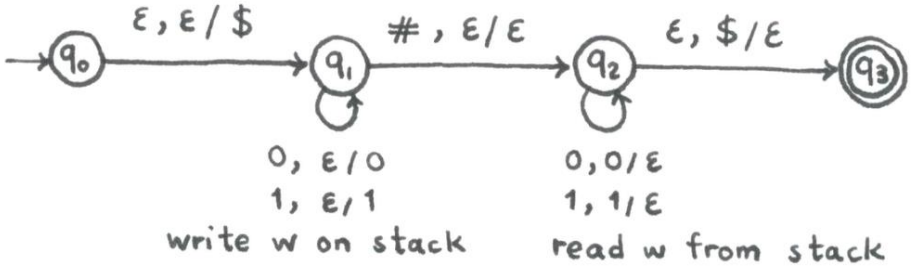
A PDA accepts input x if, for some computational path, PDA finishes reading all input symbols and stop at an accepting state.
When accepting an input string, stack needs not be empty.
Language of a PDA is the set of all strings in Σ^* it accepts

---

$L = \{ w \# w^R \mid w \in \{0,1\}^* \}$ $\quad \Sigma = \{0, 1, \#\}$ $\quad \Gamma = \{0, 1, \$\}$

$\#$, $0\#0$, $01\#10$ in L

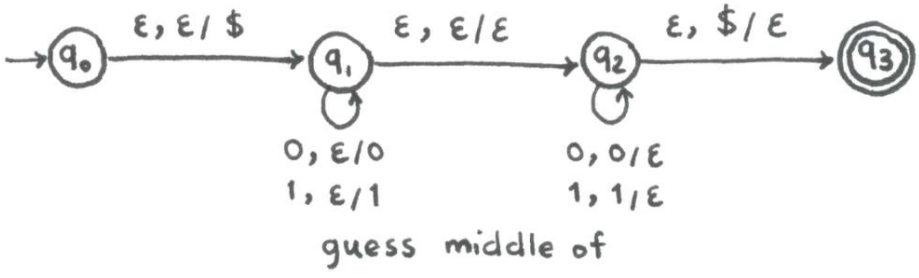
ε , $01\#1$, $0\#\#0$ not in L



$L = \{ w w^R \mid w \in \Sigma^* \}$ $\quad \Sigma = \{0, 1\}$

ε , 00 , 0110 in L

011 , 010 not in L

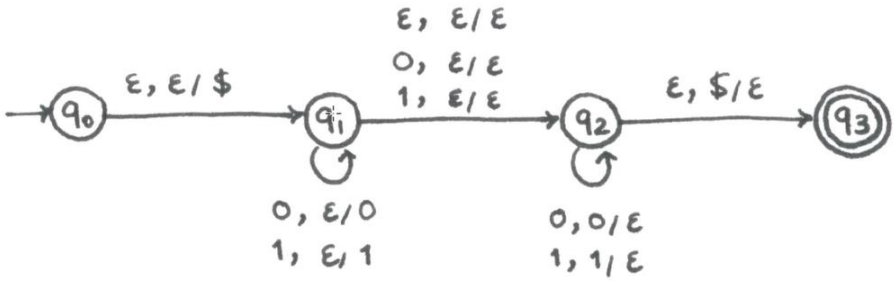


$L = \{ w \in \Sigma^* \mid w = w^R \}$ $\quad \Sigma = \{0, 1\}$

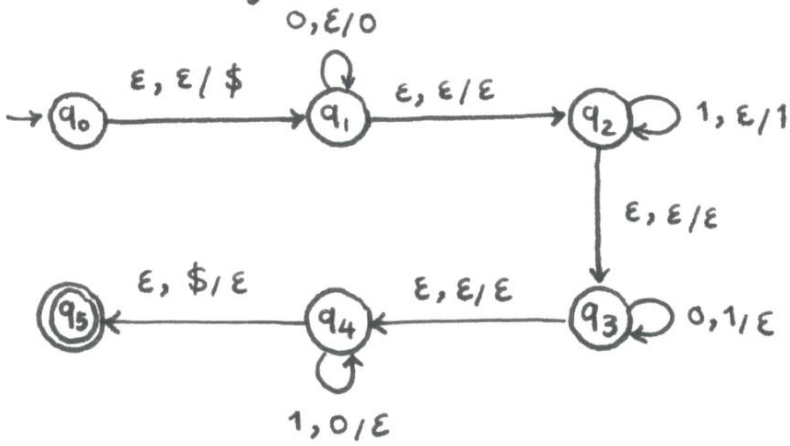
ε, 00, 010, 0110 in L

011 not in L

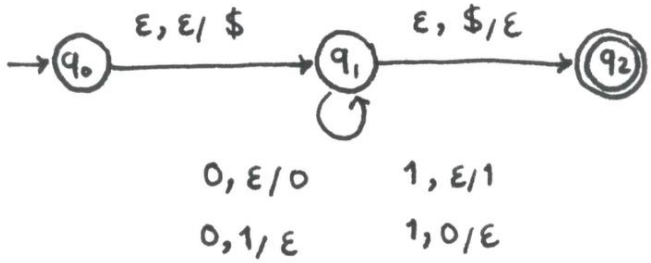
$\underbrace{0010}_{x}\ \underbrace{0100}_{x^R}$ $\qquad$ $\underbrace{0010}_{x}\ \underbrace{10100}_{x^R}$



$L = \{0^n 1^m 0^m 1^n \mid n \geq 0,\ m \geq 0\}$ $\qquad$ $\Sigma = \{0, 1\}$



L = same number of 0's and 1's $\qquad$ $\Sigma = \{0, 1\}$



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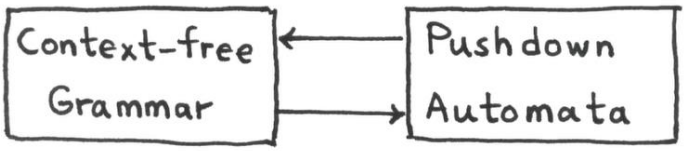
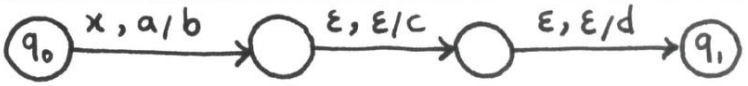
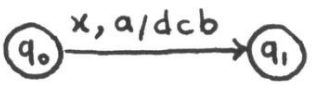
L has a context-free grammar if and only if it is accepted by some PDA



Sequence of transitions like



will be abbreviated as



Given a CFG G, convert it into PDA M with $L(G) = L(M)$

Idea. Build PDA that simulates a leftmost derivation

$$\begin{cases} S \to 0TS1 \mid 1T0 \\ T \to 1 \end{cases}$$

$S \Rightarrow 0TS1 \Rightarrow 01S1 \Rightarrow 011T01 \Rightarrow 011101$

PDA works as follows :

① Write start symbol onto stack

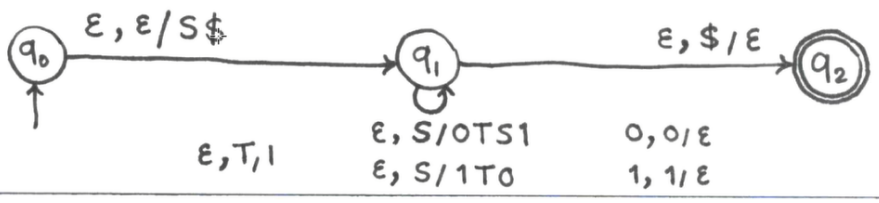
② Rewrite variable on top of stack according to production rule

③ Pop terminal if it matches input

PDA Control		Stack	Input
write start symbol	$\varepsilon, \varepsilon/S\$$	$\$ \underline{S}$	011101
replace by production rule	$\varepsilon, S/0TS1$	$\$1S\underline{T}0$	011101
pop terminal and match	$0, 0/\varepsilon$	$\$1S\underline{T}$	11101
replace by production rule	$\varepsilon, T/1$	$\$1S\underline{1}$	11101
⋮	⋮	⋮	⋮

$S \to 0TS1 \mid 1T0$

$T \to 1$



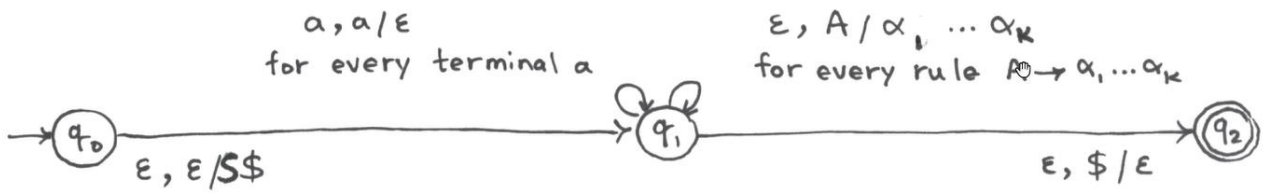
input: ↑011101 ⇒ ↑011101 ⇒ 0↑11101 ⇒ 0↑11101 ⇒ 01↑1101

stack: $S ⇒ $1ST0 ⇒ $1ST ⇒ $1S1 ⇒ $1S

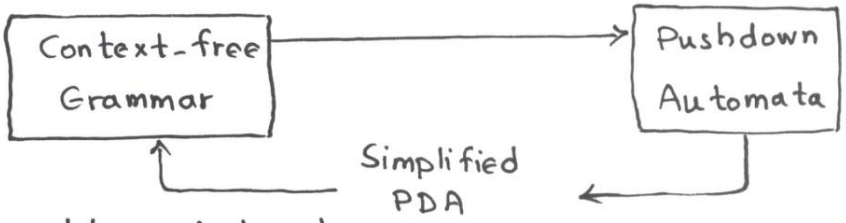
01↑1101 ⇒ 011↑101 ⇒ 011↑101 ⇒ 0111↑01 ⇒ 01110↑1 ⇒ 011101↑

$1OT1 ⇒ $10T ⇒ $101 ⇒ $10 ⇒ $1 ⇒ $

General CFG to PDA Conversion



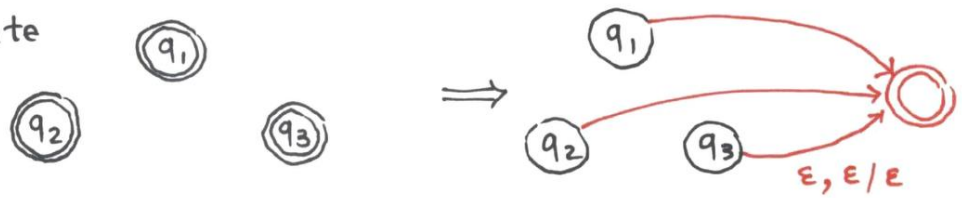
From PDAs to CFGs



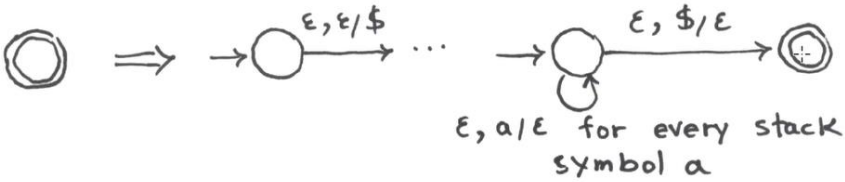
Simplified pushdown Automata:

- Has single accepting state
- Empties its stack before accepting
- Each transition is either a push, or a pop, but not both

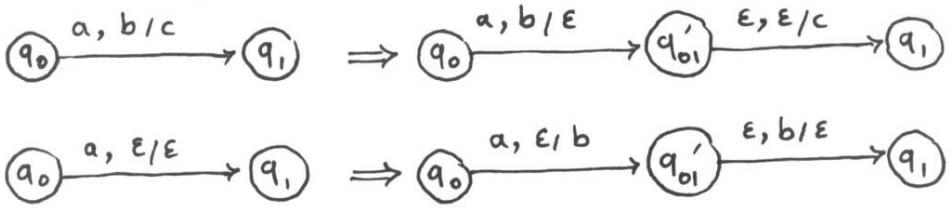
- Single accepting state



- Empties its stack before accepting



- Each transition either push, or pop, but not both



Simplified PDA to CFG

For every pair (q, r) of states in PDA, introduce variable A_{qr} in the context-free grammar

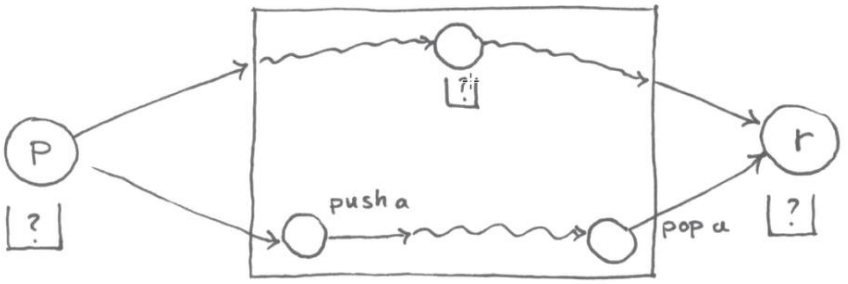
Intention: A_{qr} generates all strings that allow PDA go from q to r with empty stack both at q and r

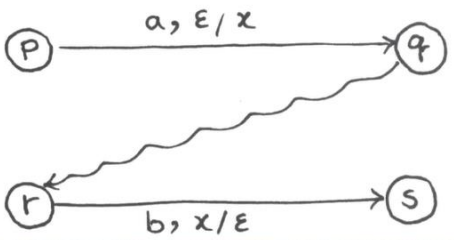
If q_{in} is start and q_{acc} is accepting state, $A_{in\ acc}$ generates all strings

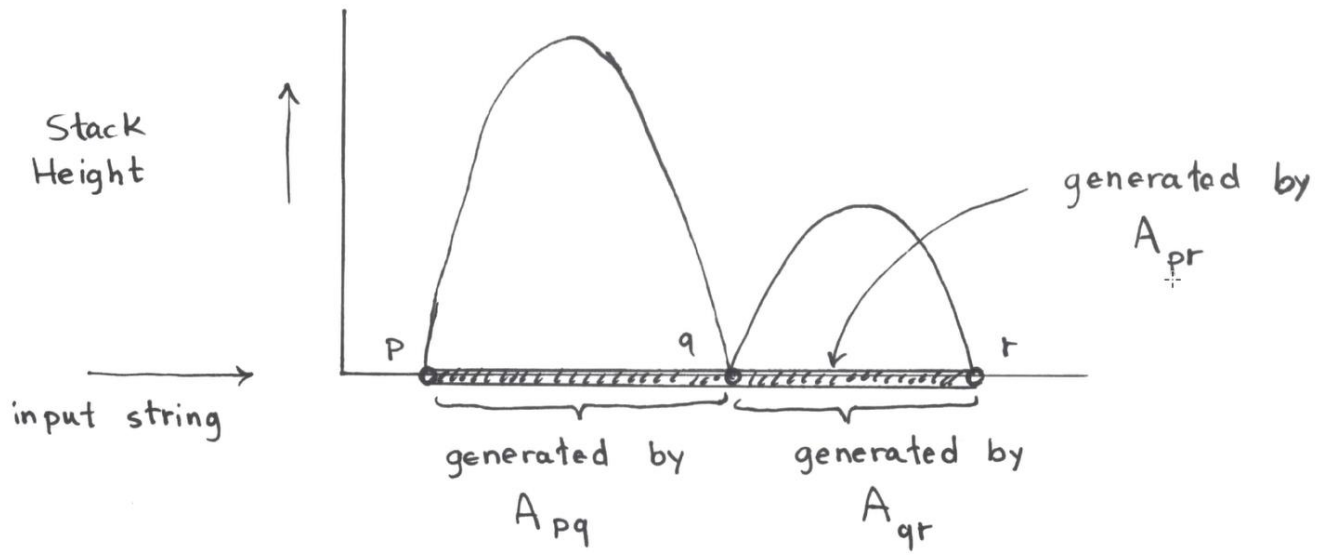
Simplified PDA reading word from p to r

Two cases :

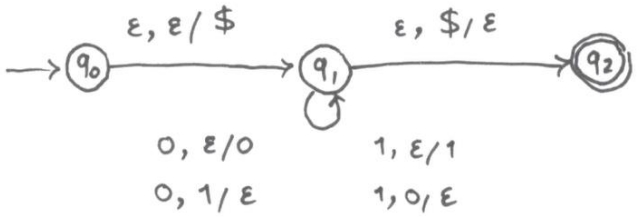
- Stack is empty in the middle
- Stack never empty in the middle



PDA	CFG
$\forall p \in Q$ 	$A_{pp} \to \varepsilon$
$\forall p,q,r \in Q$ 	$A_{pr} \to A_{pq}\ A_{qr}$
$\forall p,q,r,s \in Q$ 	$A_{ps} \to a\ A_{qr}\ b$ $a = \varepsilon$ or $b = \varepsilon$ allowed
Start Variable: A_{pq}	p : initial , q : accepting



Example: Simplified PDA to CFG



variables: A_{00}, A_{11}, A_{22}
A_{01}, A_{02}, A_{12}

start var: A_{02}

$A_{02} \to A_{01} A_{12}$

$A_{01} \to A_{01} A_{11}$

$A_{12} \to A_{11} A_{12}$

$A_{11} \to A_{11} A_{11}$

$A_{11} \to 0 A_{11} 1$

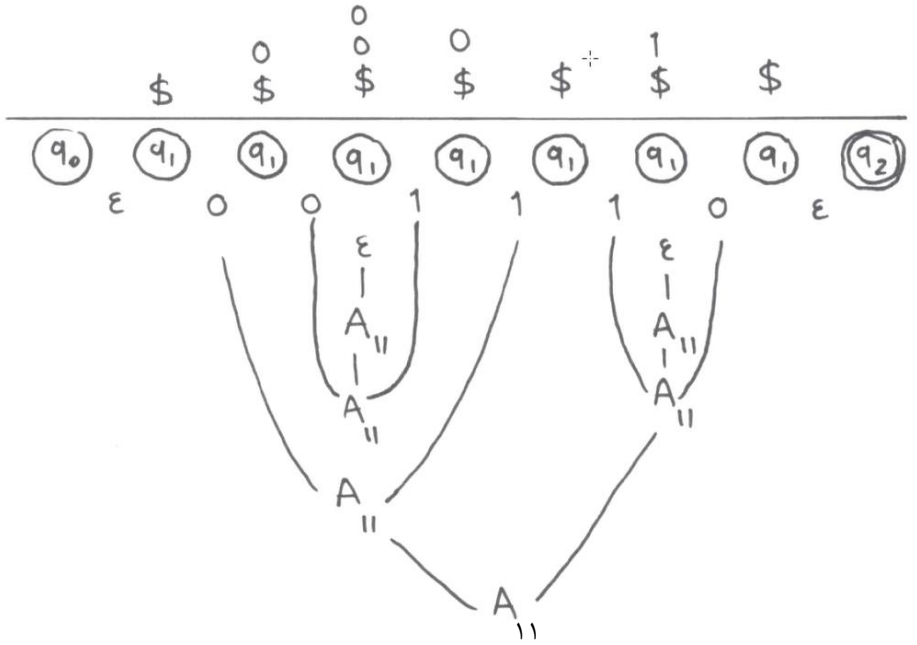
$A_{11} \to 1 A_{11} 0$

$A_{02} \to A_{11}$

$A_{00} \to \varepsilon$

$A_{11} \to \varepsilon$

$A_{22} \to \varepsilon$



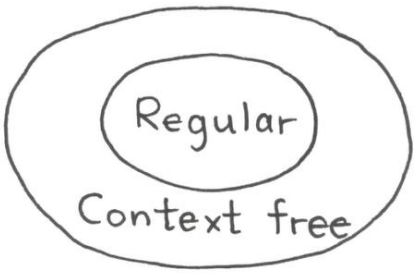
These languages are not regular, are they context-free?

$L_1 = \{a^n b^n \mid n \geq 0\}$

$L_2 = \{a^n b^n c^n \mid n \geq 0\}$

$L_3 = \{ww^R \mid w \in \{a,b\}^*\}$

$L_4 = \{ww \mid w \in \{a,b\}^*\}$



Let's try to design a CFG or PDA for $L_3 = \{a^n b^n c^n \mid n \geq 0\}$

$S \rightarrow aBc \mid \varepsilon$

$B \rightarrow ??$

read a / push x

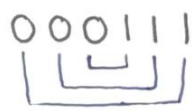
read b / pop x

??

Intuition: If L has nested matchings then "probably" context-free

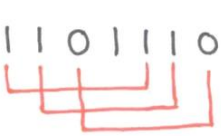
If not, "probably" not context-free

$\{0^n 1^n \mid n \geq 0\}$ context-free



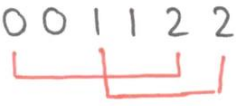
nested

$\{ww \mid w \in \{0,1\}^*\}$ not context-free



not nested

$\{0^n 1^n 2^n \mid n \geq 0\}$ not context-free



not nested

We use a similar idea to the pumping lemma for regular languages to prove a language is not context-free

- Regular Languages : if a string is long enough, there must be some state that is repeated in the computation
- Context-free Languages: if a string is long enough, there must be a non-terminal that is repeated in a derivation

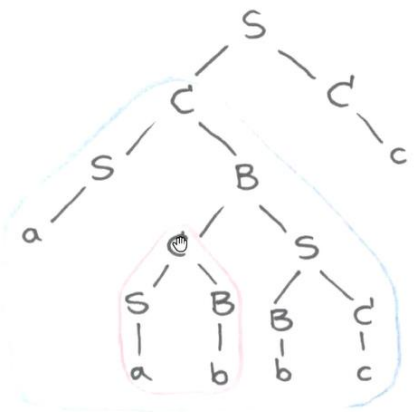
$$\begin{cases} S \to CC \mid BC \mid a \\ B \to CS \mid b \\ C \to SB \mid c \end{cases}$$

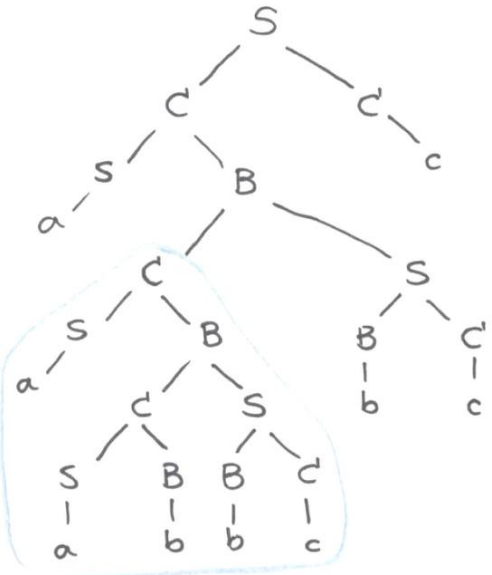
If a derivation is long enough, some variable must have appear twice on some root-to-leaf path in a derivation tree

$S \Rightarrow CC \Rightarrow SBC \Rightarrow SCSC \Rightarrow SSBSC \Rightarrow SSBBCC$

$\Rightarrow aSBBCC \Rightarrow aaBBCC \Rightarrow aabBCC \Rightarrow aabbCC$

$\Rightarrow aabbcC \Rightarrow aabbcc$





We can "cut" and "paste" part of derivation tree

$aabbcc$

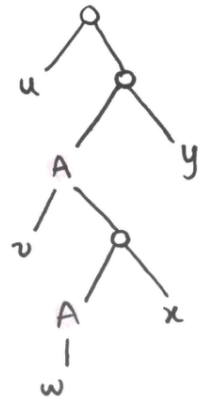
$aaabbcbcc$

We can repeat this many times

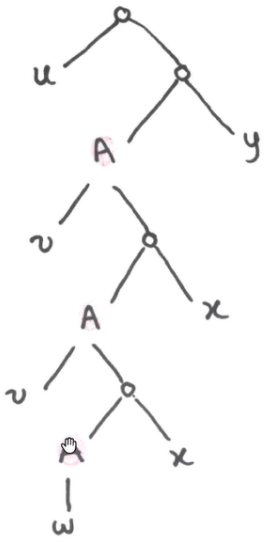
$aabbcc \Rightarrow aaabbcbcc \Rightarrow aaaabbcbcbcc \Rightarrow \cdots$

$\Rightarrow (a)^i ab(bc)^i c$

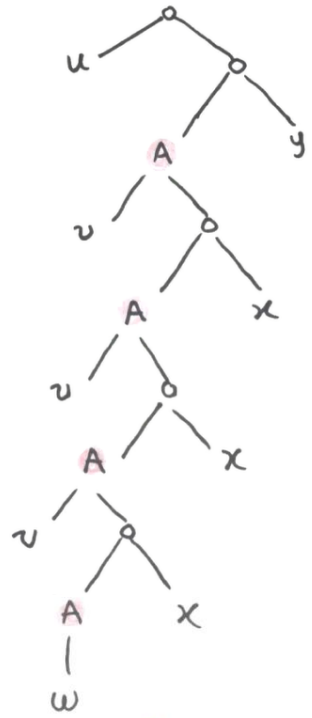
Each sufficiently large derivation will have a middle part that can be repeated indefinitely.



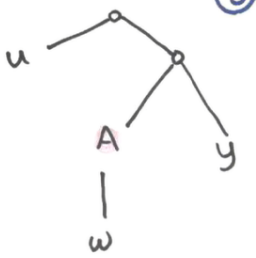
$uvwxy$



uv^2wx^2y



uv^3wx^3y



uwy

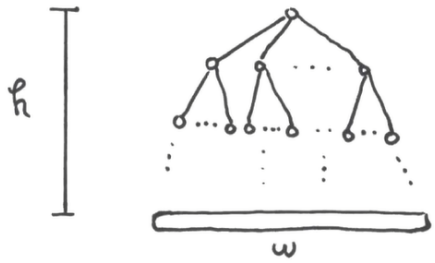
If string $s \in L$ is "long enough" we can pump it

$G = (V, \Sigma, R, S)$ context-free grammar for L

Assume longest rule of G has right hand side length $n \geq 2$

$$A \rightarrow B_1 \cdots B_n$$

Each node in derivation tree has $\leq n$ children



at most n nodes

at most n^2 nodes

$\vdots$

$|w| \leq n^h$ (if $|w| \geq n^h + 1$ then tree height $> h$)

If $|w| \geq p = n^{|V|} + 1$ then tree height $> |V|$

Some variable on longest path in tree are repeated

We can pump the derivation tree

④

L is a context-free Language $\Rightarrow$

$\exists\ p \geqslant 1$

$\forall\ s \in L \cdot |s| \geqslant p$

$\exists\ u, v, w, x, y \cdot s = uvwxy$

① $|vx| > 0$ second and forth are not both null

② $|vwx| \leqslant p$ middle three segments are not too long

$\forall\ i \geqslant 0 \cdot uv^i wx^i y \in L$

If

$\forall\ p \geqslant 1$ — Demon picks p

$\exists\ s \in L \cdot |s| \geqslant p$ — You pick $s \in L$

$\forall\ u, v, w, x, y \cdot s = uvwxy$ — Demon picks u, v, w, x, y

- $|vx| > 0$
- $|vwx| \leqslant p$

$\exists\ i \geqslant 0 \cdot uv^i wx^i y \notin L$ — You pick i

then L is not context-free

Demon: forall ∀ player

You win if $uv^i wx^i y \notin L$

If you want to show that L is not context-free, it suffices to show you have a winning strategy:

no matter what demon does in 1 and 3, you have moves in 2 & 4

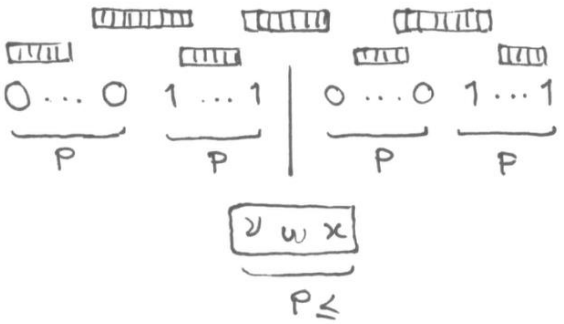
$L = \{ a^n b^n c^n \mid n \geq 0 \}$

- Demon picks $p \geq 1$
- You pick $s = a^p b^p c^p$ $\quad s \in L \quad |s| = 3p \geq p$
- Demon picks u, v, w, x, y $\quad s = uvwxy \quad vx \neq \varepsilon \quad |vwx| \leq p$
- You pick $i = 0$ $\quad uv^0wx^0y = uwy$

Since $|vwx| \leq p$, vwx cannot contain all three symbols.
So vwx either does not have a's, b's or c's.
By $i = 0$ we are changing the balance of a's, b's and c's.

$L = \{ ww \mid w \in \{0,1\}^* \}$

- Demon picks $p \geq 1$
- You pick $s = 0^p 1^p 0^p 1^p$ $\quad s \in L \quad |s| = 4p \geq p$
- Demon picks u, v, w, x, y $\quad s = uvwxy \quad vx \neq \varepsilon \quad |vwx| \leq p$



Suppose vwx is entirely in the first or second half.
$\rightarrow uv^2wx^2y$ is not of the form ww

Suppose vwx is in the middle
$\rightarrow uv^0wx^0y$ is of the form $0^p 1^i 0^j 1^p$ where either i or j is not p.

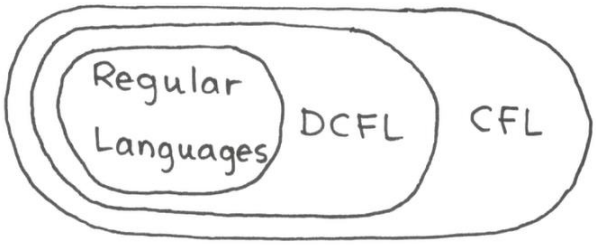
باسمه تعالی

جلسهٔ هفدهم درس نظریه

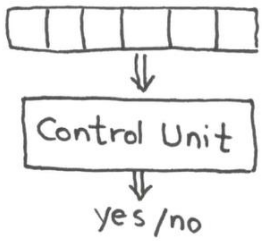
Deterministic PDA

Parsers are DPDA : class of languages in DPDA gives insight what constructs are usable in programming languages

A context-free language L is called a Deterministic Context-Free Language (DCFL) if there is some DPDA that recognizes L

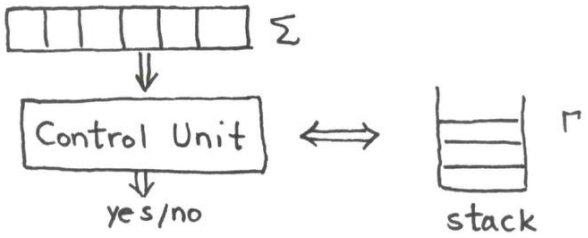


Finite Automata



Deterministic: in each state there is exactly one successor state for $\forall a \in \Sigma$

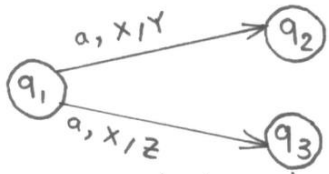
Pushdown Automata



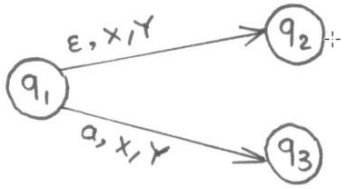
Deterministic: in each state there is at most one successor state for every combination $(a,b) \in \Sigma \times \Gamma$

In a DPDA at any state q :

- At most one choice of move for any input symbol 'a' and stack symbol 'x'



- No choice between using ε or real input



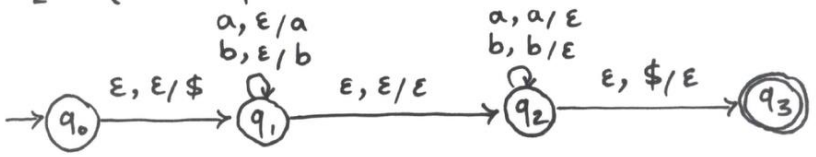
$M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ is deterministic if it satisfies :

- For any $q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $x \in \Gamma$ the set $\delta(q, a, x)$ has at most one element
- For any $q \in Q$, $x \in \Gamma$, if $\delta(q, \varepsilon, x) \neq \{\}$ then $\delta(q, a, x) = \{\}$ (for any $a \in \Sigma$)

Example : $L_1 = \{a^n b^n \mid n \geq 0\}$

DPDA

Example : $L_2 = \{ w w^R \mid w \in \{a, b\}^* \}$



NPDA

{ Deterministic Context_free Language (DCFL) } $\subset$ { Context_free Languages }

- Every DPDA is also NPDA
- There exists a context-free language L which is not accepted by any DPDA

$L = \{a^n b^n \mid n \geq 0\} \cup \{a^n b^{2n}\}$ $\quad n \geq 0$

Language L is context - free

$S \rightarrow S_1 \mid S_2$ $\quad \{a^n b^n\} \cup \{a^n b^{2n}\}$

$S_1 \rightarrow a S_1 b \mid \varepsilon$ $\quad \{a^n b^n\}$

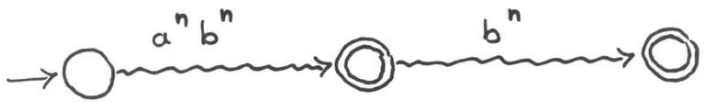
$S_2 \rightarrow a S_2 bb \mid \varepsilon$ $\quad \{a^n b^{2n}\}$

There is no DPDA that accepts L

Assume for contradiction that $L = \{a^n b^n\} \cup \{a^n b^{2n}\}$ is deterministic context-free.

There is a DPDA M with $L(M) = \{a^n b^n\} \cup \{a^n b^{2n}\}$

Since M is deterministic the following path should exist in M

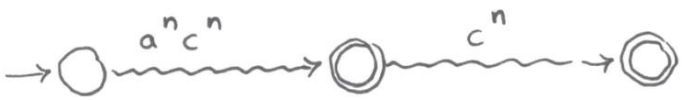


$\{a^n b^n c^n \mid n \geq 0\}$ is not context-free (pumping lemma)

Using M we construct NPDA that accepts $a^n b^n c^n$

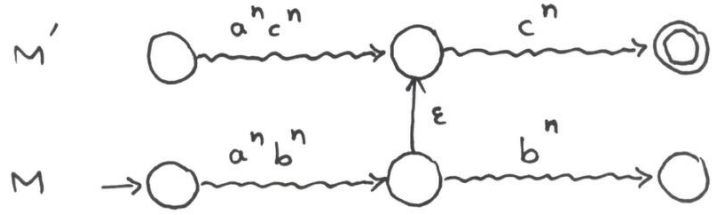
$L(M) = \{a^n b^n\} \cup \{a^n b^{2n}\}$

Modify M ⇓ replace b with c



$L(M') = \{a^n c^n\} \cup \{a^n c^{2n}\}$

NPDA that accepts $\{a^n b^n c^n \mid n \geq 0\}$



NPDA for $\{a^n b^n c^n\}$

Closure Properties:

Union, Concatenation, Kleene closure, Reversal

Non-closure Properties:

Difference, Complement, Intersection

Union

$L_1 = L(G_1)$ $L_2 = L(G_2)$

Rename variables in G_1 so they don't conflict with those of G_2

S_1: start symbol of G_1

S_2: start symbol of G_2

Add production rule $S \to S_1 \mid S_2$

Concatenation

Let S_1 and S_2 be the start symbols of G_1 and G_2

Add a new start symbol S and production $S \rightarrow S_1 S_2$

Star

Let L have grammar G with start symbol S_1

Add production $S \rightarrow S_1 S \mid \varepsilon$

Rightmost derivation of S generates a sequence of zero or more S_1's

Reversal

If L is a CFL with grammar G, form a grammar for L^R by reversing the right side of every production

Example: Production rules of G: $S \rightarrow 0S1 \mid 01$

Reversal of G: $S \rightarrow 1S0 \mid 10$

Non-closure under intersection

(Does not mean it is never)

$L_1 = \{a^n b^n c^n \mid n \geq 1\}$ not CFL

$L_2 = \{a^n b^n c^m \mid n \geq 1, m \geq 1\}$ is CFL

$S \rightarrow AB$

$A \rightarrow 0A1 \mid 01$

$B \rightarrow 2B \mid 2$

$L_3 = \{a^m b^n c^n \mid n \geq 1, m \geq 1\}$ is CFL

$L_1 = L_2 \cap L_3$

Note. Intersection of a context-free language and a regular language is context-free.

Non-closure under difference

Any class of languages that is closed under difference is closed under intersection

$$L \cap M = L \setminus (L \setminus M)$$

Non-closure under complement

1 - Assume complement of every CFL is a CFL

2 - Let L_1 and L_2 be two CFLs

3 - CFL are closed under union $\overline{\overline{L_1} \cup \overline{L_2}}$ X

باسمه تعالی

جلسه هجدهم

Alan Turing (1912 - 1954) Film: "The Imitation Game" بازی تقلید

Turing Test: Classic test of Artificial Intelligence that asks a computer to trick a human into thinking they are talking to a person

Cryptography: Alan Turing broke German encryption machines during WWII

Turing Award: The "Nobel Prize" of computer science

Hilbert's Entscheidungsproblem (Decision Problem)

"Write a program" to solve the following task:

Input: mathematical statement (in first-order logic)

Output: whether the statement is true

(In fact, he didn't ask to "write a program", but to design a decision procedure)

Examples of statements expressible in first-order logic

Fermat's last theorem:

$x^n + y^n = z^n$

has no integer solution for integen $n \geq 3$

Twin prime conjecture:

There are infinitely many pairs of primes of the form p and p+2

Church (1935 - 1936) and Turing (1936-1937) independently showed the procedure that Entscheidungsproblem asks for cannot exist

Church: λ-calculus Turing: Turing machine

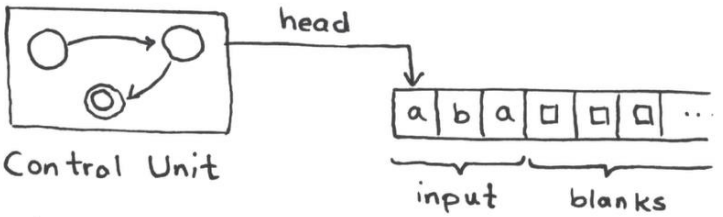
Goal: Model that can compute anything that we can compute

Church-Turing Thesis: Any "algorithm" can be carried out by Turing machine

Intuitive notion of algorithms coincides with those implementable on Turing machines

Turing machine : Finite-state control unit equipped with an infinite tape as its memory

Tape begins with the input to the machine written on it, surrounded with infinitely many blank cells



Machine has a tape head that can read and write on a single memory cell at a time

At each step:

- Write a symbol to the tape cell under the tape head
- Change state
- Move the tape head to the left or to the right

A Turing machine has two alphabets :

① Input alphabet Σ

② Tape alphabet Γ $(\Sigma \subset \Gamma)$

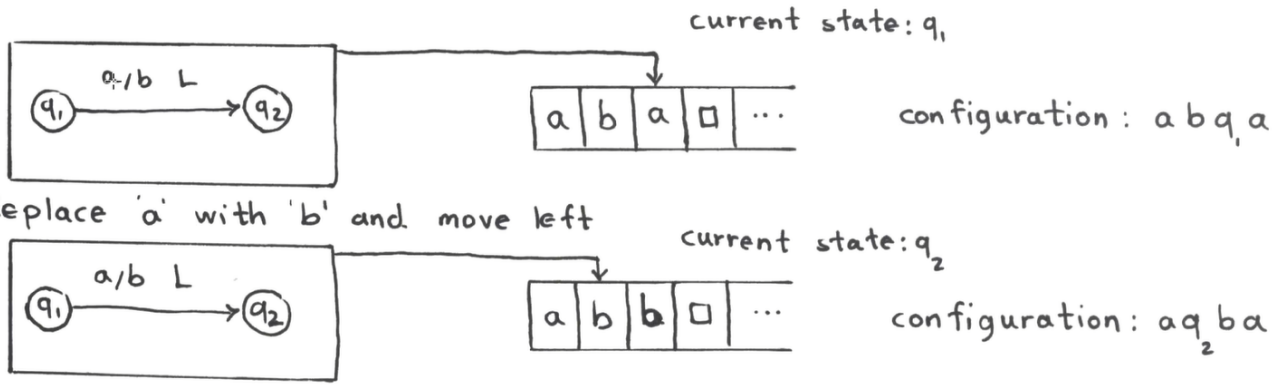
Γ can contain any number of symbols, but always contain at least blank $\square$

Guarantee : $\square \notin \Sigma$

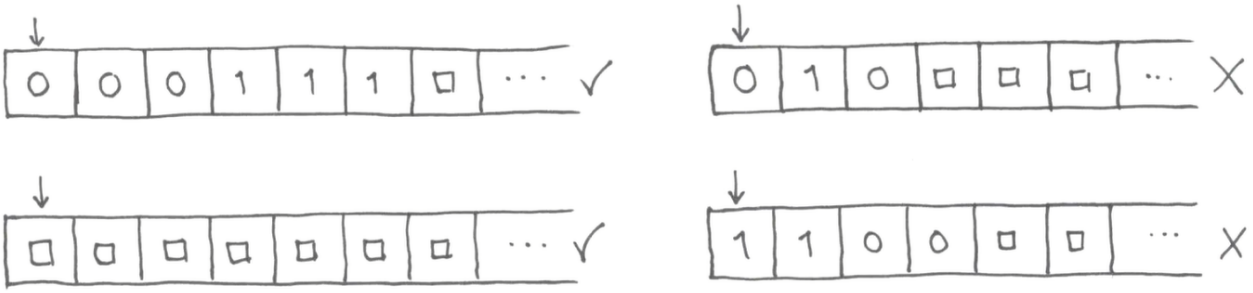
At startup :

Tape contains a finite input string followed by infinite sequence of blanks, and the head is positioned over the first cell on the tape.

How Turing Machine operate

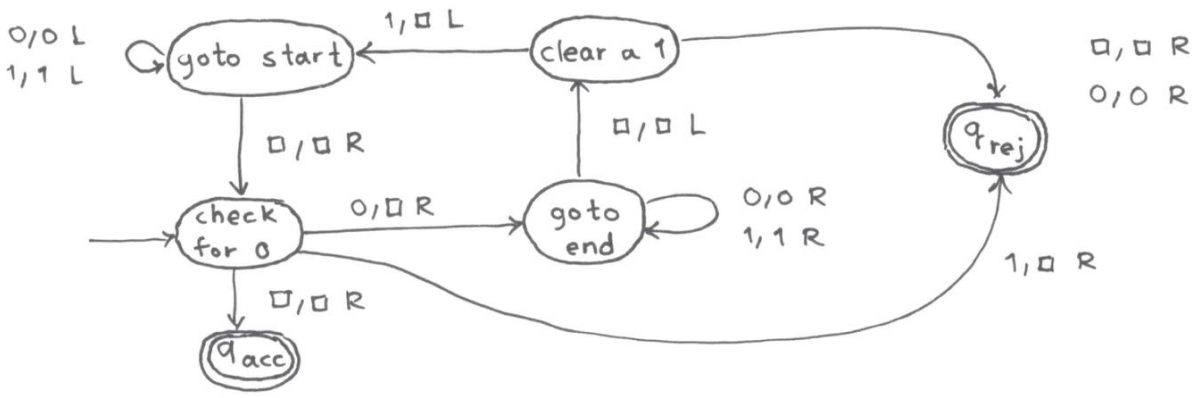


$L = \{ 0^n 1^n \mid n \in \mathbb{N} \}$



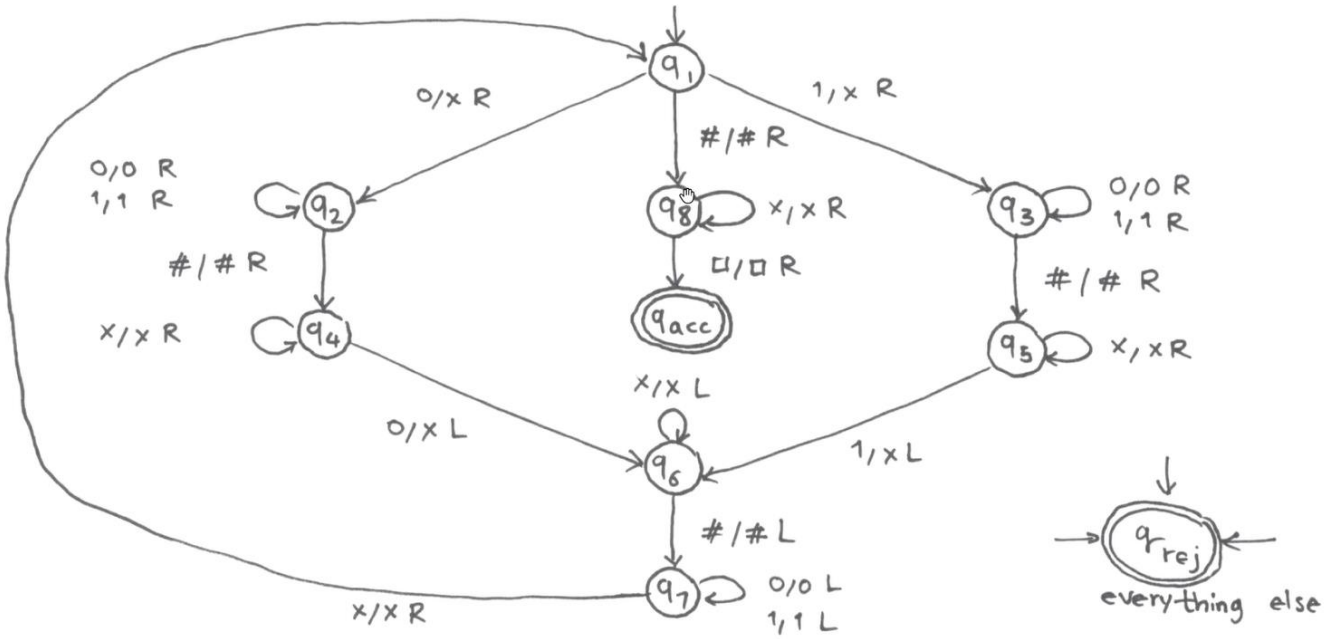
A Recursive Approach:

- The string ε is in L
- The string $0w1$ is in L iff w is in L
- Any string starting with 1 is not in L
- Any string ending with 0 is not in L



$L = \{ w \# w \mid w \in \{0,1\}^* \}$

① Until you reach #
② Read and remember entry — $\times \underline{1}100 \# \times 1100$
③ Write x — $\times \underline{\times}00 \# \times 1100$
④ Move right past # and past all x's — $\times \times 00 \# \times \underline{1}100$
⑤ If this entry is different, reject
⑥ Write x — $\times \times 00 \# \times \underline{\times}00$
⑦ Move left past # and to right of first x — $\times \times \underline{0}0 \# \times \times 00$
⑧ If you see only x's followed by □, accept



Turing Machine $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$

Q finite set of states

Σ finite set of input alphabets not containing □

Γ finite set of tape alphabet including □ $\quad \Sigma \subseteq \Gamma$

$q_0 \in Q$ initial state

q_{acc}, q_{rej} accepting and rejecting states $\quad (q_{acc} \neq q_{rej})$

δ transition function

$$\delta : (Q \setminus \{q_{acc}, q_{rej}\}) \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$$

Turing machines are deterministic. They do not get stuck.

Starting configuration $q_0 w$

Accepting configuration contains q_{acc}

Rejecting configuration contains q_{rej}

Turing machine accepts x if there is a sequence of configrations $C_0, C_1, \ldots, C_k$ where

C_0 is starting $\quad C_i$ yields $C_{i+1} \quad C_k$ accepting

Language recognized by M: set of all strings that M accepts

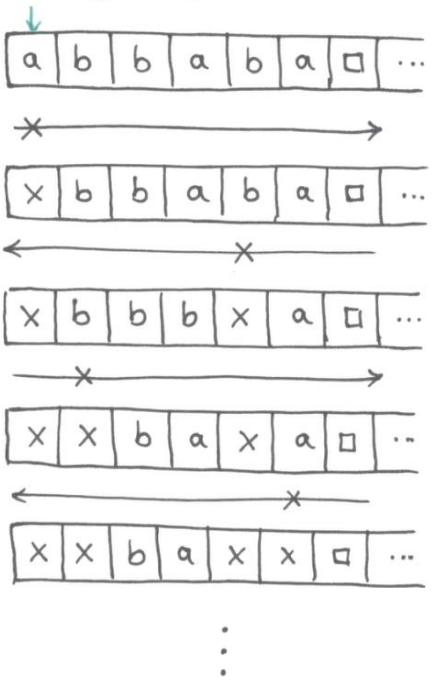
جلسهٔ نوزدهم درس نظریه

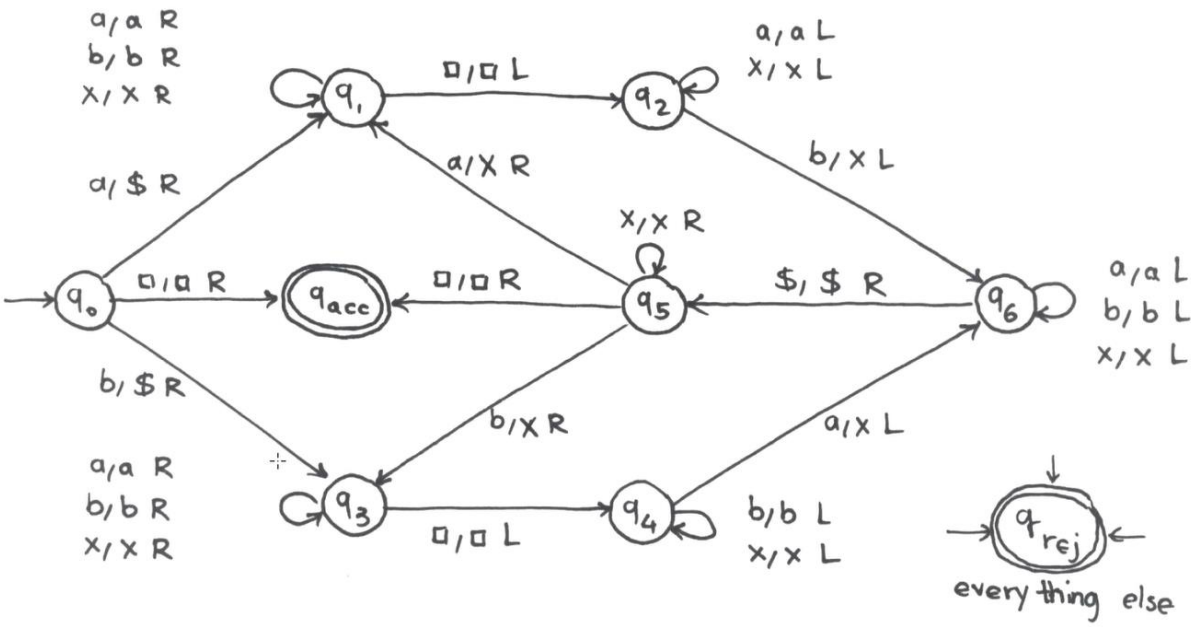
باسمه تعالی

Church - Turing Thesis : Intuitive notion of Algorithm equals Turing machine algorithms

(Sipser, p. 183)

$L = \{ w \in \{a,b\}^* \mid n_a(w) = n_b(w) \}$





$L = \{ w \in a^* \mid |w| = 3^n, n \geq 0 \}$

High-level idea: repeatedly divide by 3

Separate a's into groups of three a's

Three cases:

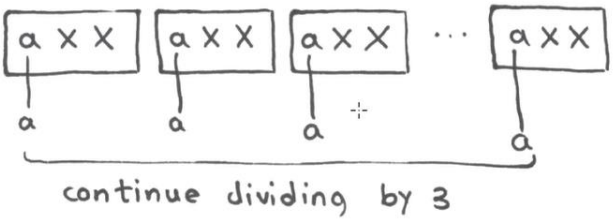
[aaa] [aaa] ··· [aaa] [aaa] ← continue with this case

[aaa] [aaa] ··· [aaa] a a ✗ reject

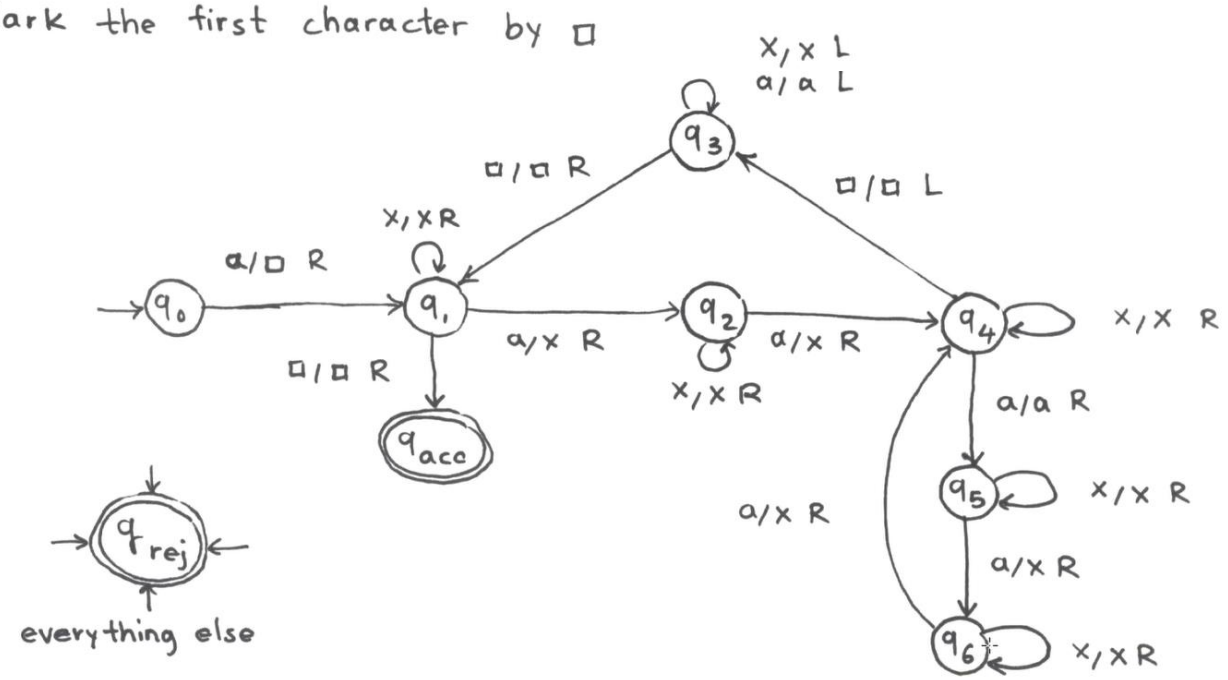
[aaa] [aaa] ··· [aaa] a ✗ reject

Cross off the last two a's in each block

Continue only if the last block ends as [aXX]



Mark the first character by □



$L = \{ a^i b^j c^k \mid ij = k \text{ and } i, j, k > 0 \}$

High-level description of TM:

① For every a:

② | Cross off a

③ | Cross off same number of b's and c's

④ | Uncross the crossed b's (but not c's)

⑤ If all a's and c's are crossed off, accept

Example:

① a a b b c c c c

② ∅ a b b c c c c

③ ∅ a ∅ ∅ ¢ ¢ c c

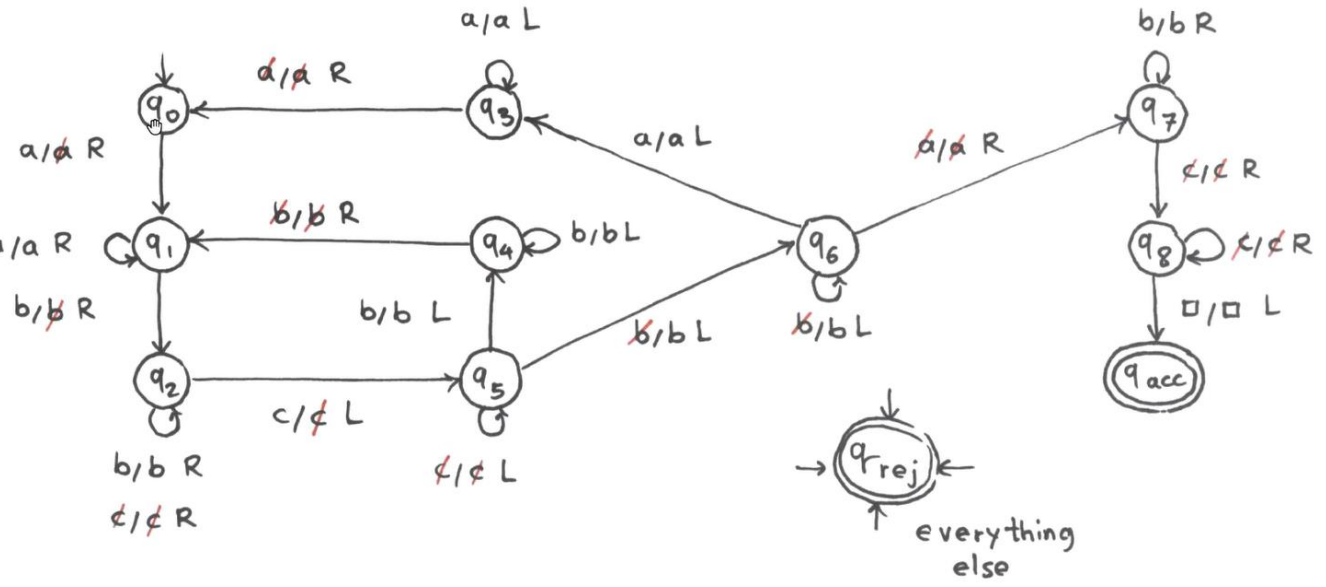
④ ∅ a b b ¢ ¢ c c

② ∅ ∅ b b ¢ ¢ c c

③ ∅ ∅ ∅ ∅ ¢ ¢ ¢ ¢

④ ∅ ∅ b b ¢ ¢ ¢ ¢

$\Sigma = \{a, b, c\}$ $\Gamma = \{a, b, c, \emptyset, \$, ¢, \square\}$



Unlike for DFAs, NFAs, PDAs we rarely give complete state diagrams of Turing machines

We usually give a high-level description

We are interested in algorithms behind Turing machines

Element Distinctness Problem

$L = \{ \# x_1 \# x_2 \cdots \# x_m \mid x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j \}$

Set a left marker (red) at the first #

$$\#\, x_1\, \#\, x_2\, \#\, x_3 \cdots \#\, x_m$$

Scan to right until you reach a # and set a right marker (blue)

$$\#\, x_1\, \#\, x_2\, \#\, x_3 \cdots \#\, x_m$$

Zig zag compare x_1 and x_2:

If every character of x_1 matches x_2 with no extra character → reject

If not, move right marker to next # and compare x_1 and x_3

$$\#\, x_1\, \#\, x_2\, \#\, x_3 \cdots \#\, x_m$$

If x_1 is different from $x_2, \ldots, x_m$ go back and advance red

Reset the blue marker and repeat

$$\#\, x_1\, \#\, x_2\, \#\, x_3 \cdots \#\, x_m$$

$L = \{ a^p \mid p \text{ is a prime number} \}$

Sieve of Eratosthenes غربال اراتوستن

Suppose we want to check if p is prime

We write all numbers from 2 to p in order

Repeat:

- Find the smallest number in the list: declare it prime
- Cross all multiples of that number
- Until all numbers are declared prime or crossed

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~

13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ 23

...

Tape: a a a a □

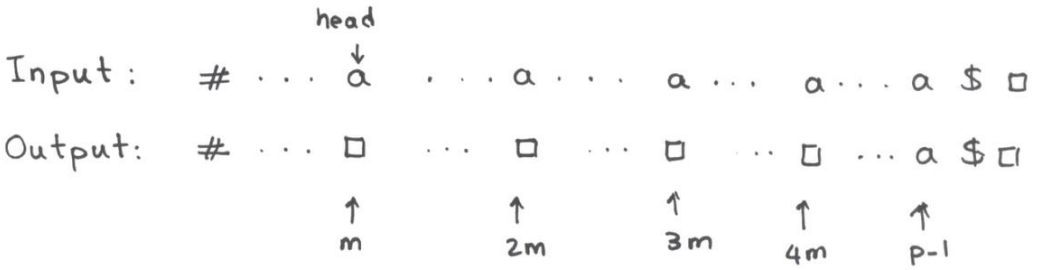
Put # at 1 and $ at p to know the start and end of the tape

a a ... a a $ □

↑ 1 ↑ p

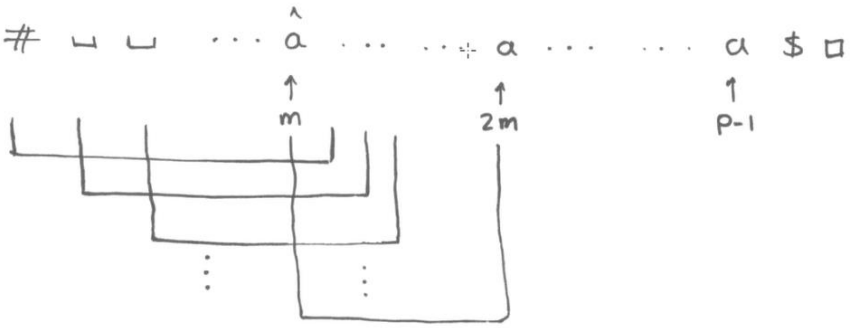
We first solve the following subproblem:

SUBPROBLEM: Assume the head at position m of tape. Convert all the multiples of m to ⊔. If $ is a multiple of m convert it to $*.



First put a special symbol â at position m.

Zig-zag and match the characters between # and â.



This will get us to position 2m. Mark it with a*.

Zig-zag and match characters between â and a* to reach position 3m

Now replace â with ⊔, a* with â and position 3m with a*.

Continue zig-zag matching using â and a*. This way we can find all the multiples of m and remove them.

During the zig-zag matching:

- If the head passes over $, stop. Convert â and a* to ⊔.
- If the head wants to replace $ ($ is on a position which is multiple of m) replace it with $*.

PROBLEM: Start at position 2.

Do the following until the head reaches $* or $:

- Convert all the multiples of the position of head to blank (subproblem above)
- Continue to next position which is not ⊔

If head reaches $ accept. If head reaches $* reject.

باسمه تعالی

جلسهٔ بیستم درس نظریه

$L = \{ \langle G \rangle \mid G \text{ is a connected undirected graph} \}$

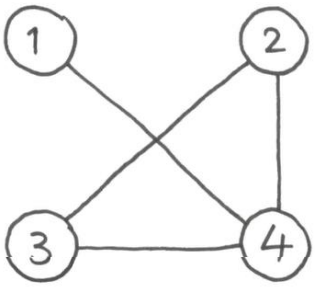
- How do we give a graph to a Turing machine as input?
- How to encode a graph G as a string $\langle G \rangle$?

One way to encode a graph:

(nodes) (edges)

$(1, 2, 3, 4)((1, 4), (2, 3), (3, 4), (4, 2))$

- no node appears twice
- edges are pairs (first node, second node)



On input $\langle G \rangle$:

0. Verify that $\langle G \rangle$ is a description of a graph

   No node/edge repeats: similar to element distinctness

   Edge endpoints are nodes: also similar to element distinctness

1. Mark the first node of G

   Mark the leftmost digit with a dot, e.g. 12 becomes $\dot{1}2$

2. Repeat until no new nodes are marked

   2.1 For each node, mark it if it is attached to an already marked node

   For every dotted u and every undotted node v:

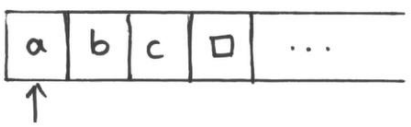
   Underline both u and v from the list

   Try to match them with an edge from the edge list

   If not found, remove underline from u and/or v and try another

3. If all nodes are marked, accept, otherwise reject

Standard Turing machine:



Extends infinitely to right

(semi-infinite tape)

Turing Machine Variants

We prove each new class has the same power with standard TM

For every machine M_1 of first class

there is a machine M_2 of second class

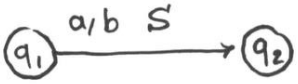
such that $L(M_1) = L(M_2)$ and vice versa

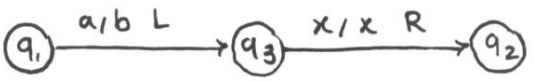
Turing Machine with Stay Option

$\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R, S\}$

* This TM can easily simulate an ordinary TM (just do not use S)
* Ordinary TM can simulate a TM with stay option

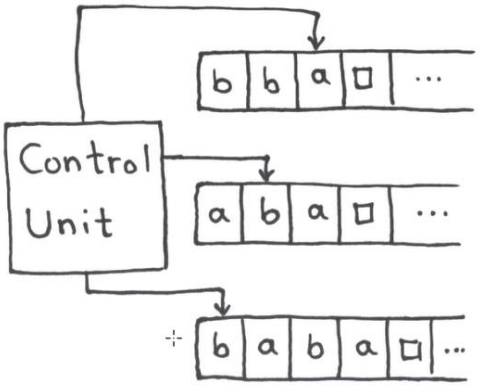
For each S transition introduce a new state and two transitions



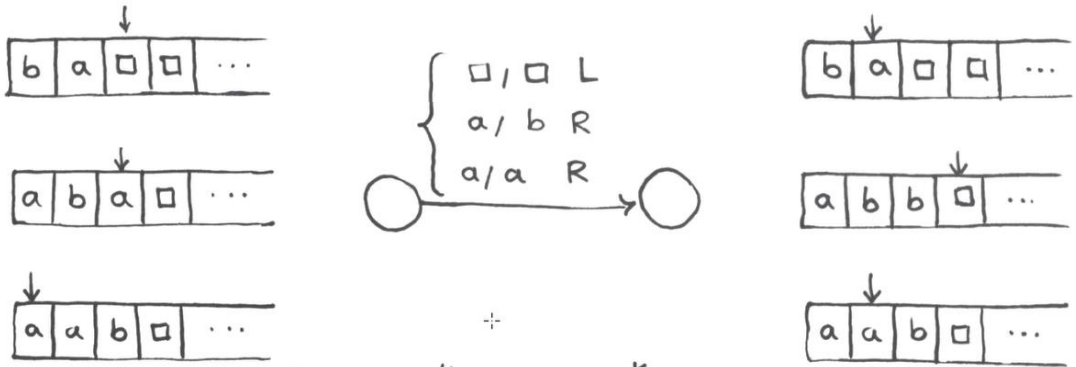


for every symbol x

Multi-tape Turing Machine



* Transitions depend on the contents of all cells under heads
* Different heads can move independent
* Initially: Input is on the first tape and all other tapes are blank



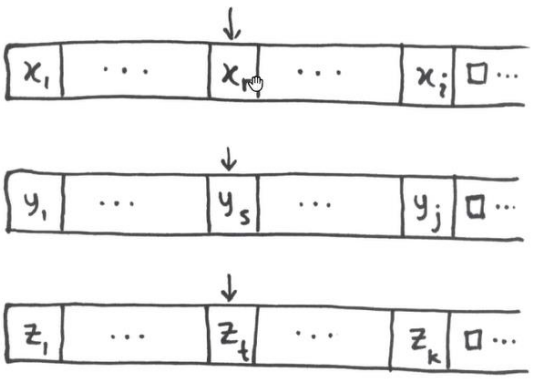
$\delta : Q \times \Gamma^k \longrightarrow Q \times \Gamma^k \times \{L, R\}^k$

Multiple tapes are convenient

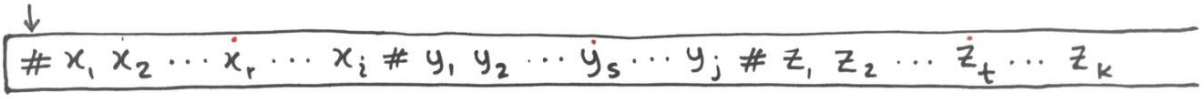
One tape can serve as temporary storage

How to simulate a multi-tape TM on a single-tape TM?

Multi-tape TM (M)

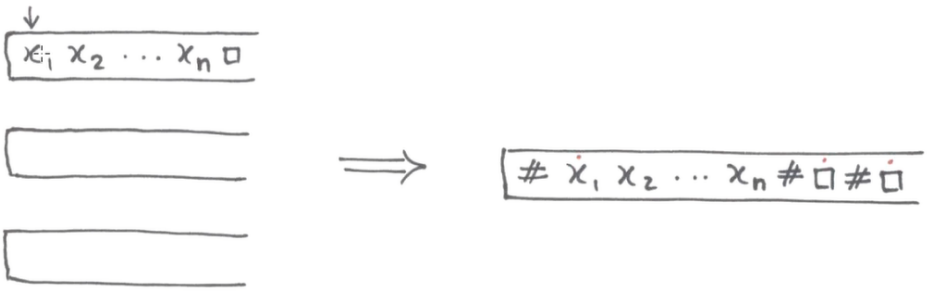


Single-tape TM (S)

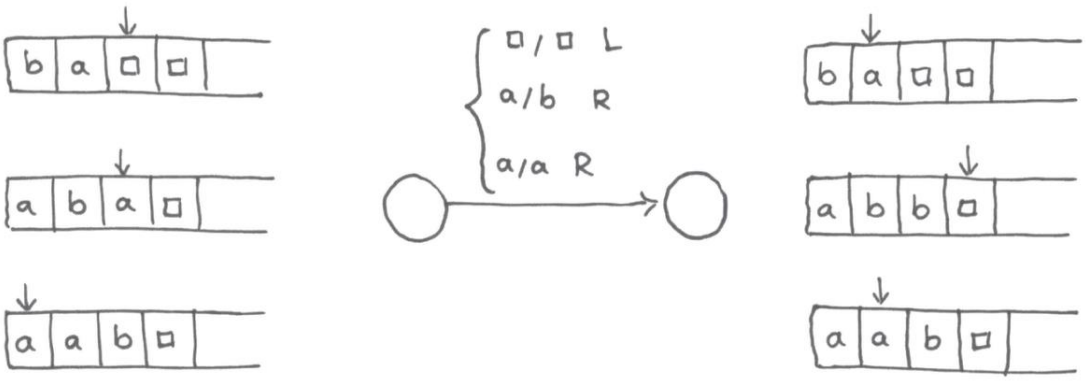


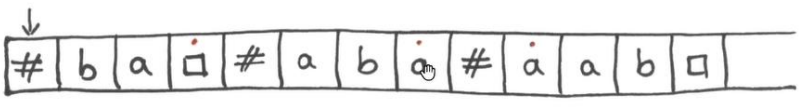
$\Gamma' = \Gamma \cup \{\dot{t} \mid t \in \Gamma\} \cup \{\#\}$

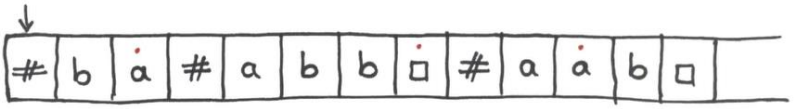
Initialization:



We simulate a move of M on single-tape TM S like this:



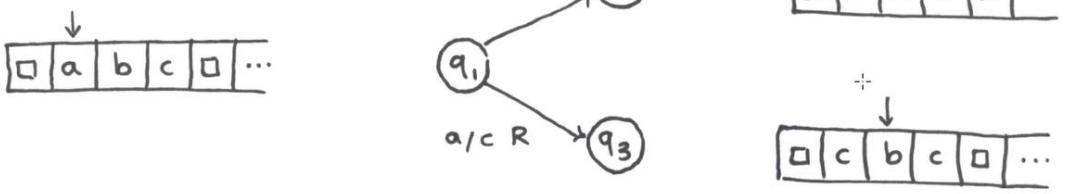




Make a pass over tape to find dotted elements. Update state/tape accordingly. If M reaches accept (reject) state, S accepts (rejects)

Non-deterministic Turing Machine

At each step of execution the machine can choose from a set of possible transitions.



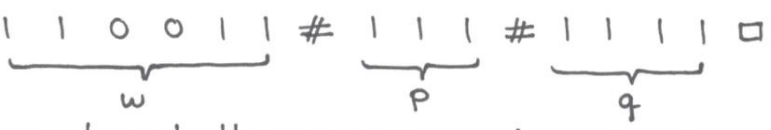
M accepts w iff at least one of its computations accepts

M rejects w iff all of its computations reject

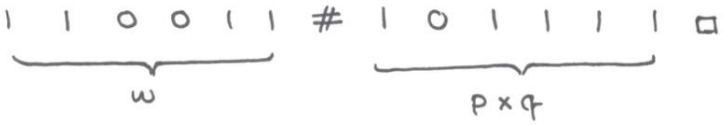
$L = \{ w \in \{0,1\}^* \mid w \text{ is a binary encoding of a composite number} \}$

① Non-deterministically choose two binary numbers $p, q > 1$ such that $2 \leq |p|$ and $|q| \leq |w|$

Write them on tape after w

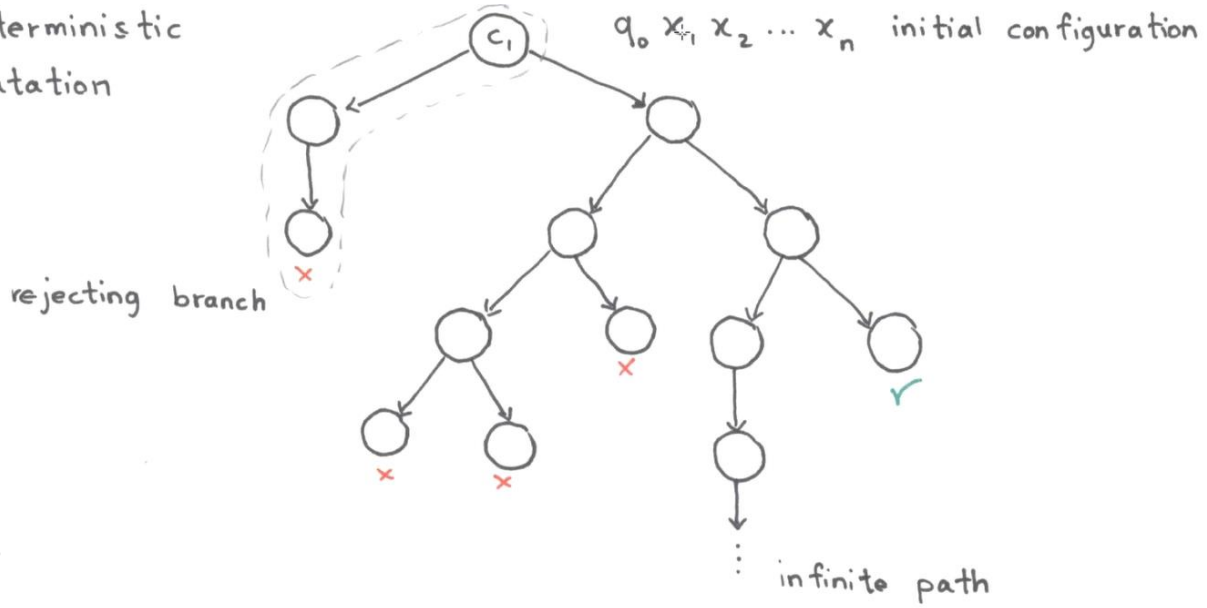


② Multiple p and q and put the answer on tape in place of p and q

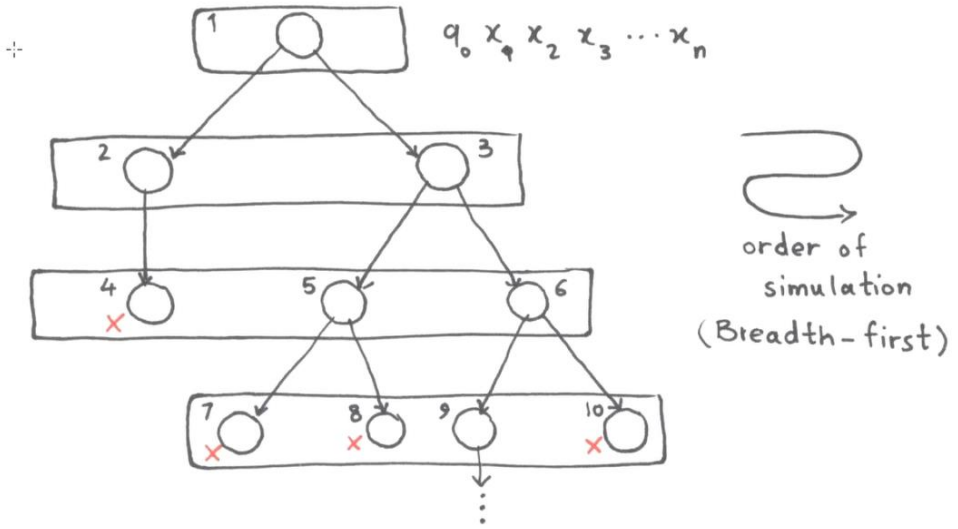


③ Compare w and p×q. If equal, accept. Else, reject.

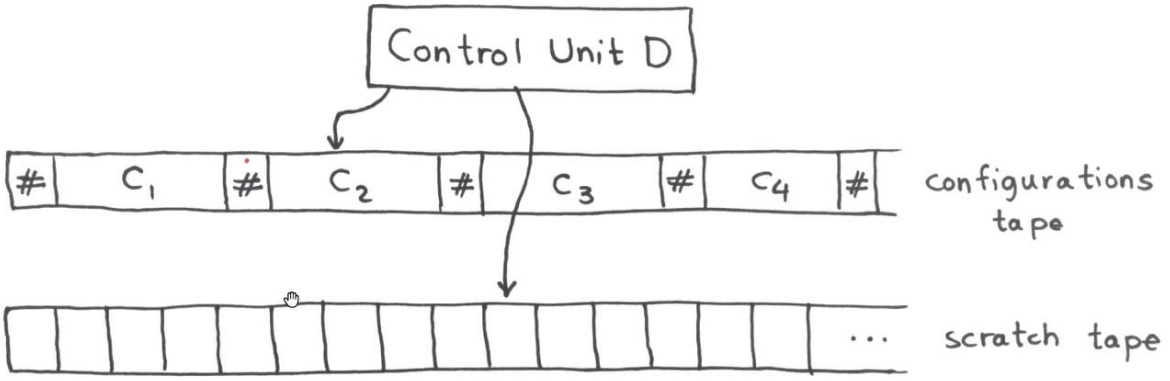
Non-deterministic Computation



Deterministic TM D simulates the non-deterministic TM N



2-tape DTM D simulates N



Build into finite control of D is the knowledge of what moves N has for each state and input

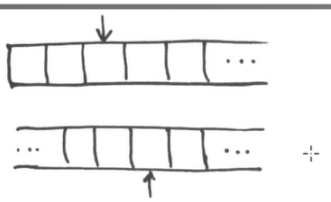
- D examines state and input symbol of current configuration (right after #)
- If the state of the current config is the accept state of N, D accepts and stops
- D copies K copies of the current configuration to the scratch tape
- D applies one non-deterministic move of N to each copy
- D copies the new configurations from scratch tape back to end of tape 1 and clears the scratch type
- D returns to the marked current configuration, erases the mark, and marks the next configuration
- D returns to first step. If there is no next configuration, reject.

Simulation can take exponentially more time than non-deterministic TM.

It is not known whether or not this exponential slowdown is necessary.

1-way tape Turing machine (standard)

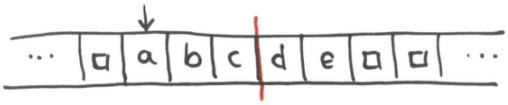
2-way tape Turing machine



2-way tape Turing machine simulates 1-way tape Turing machine (trivial)

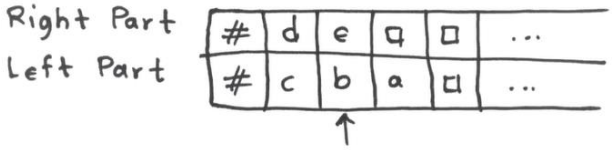
1-way tape Turing machine simulates 2-way tape Turing machine

2-way tape Turing machine

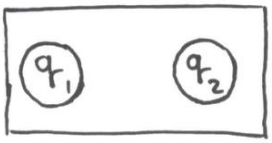


reference point

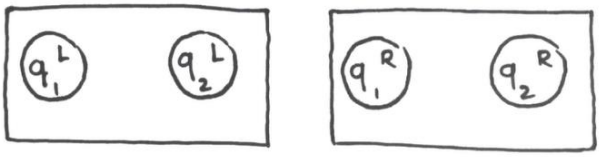
1-way tape Turing machine with two tracks



2-way tape Turing Machine

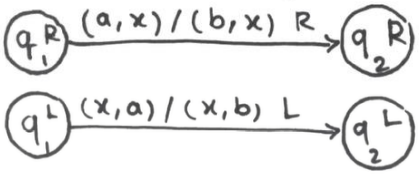


1-way tape Turing Machine



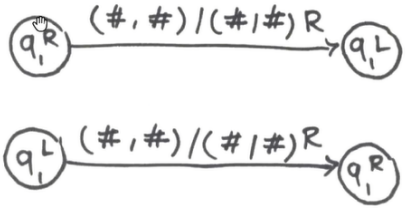
2-way tape Turing Machine

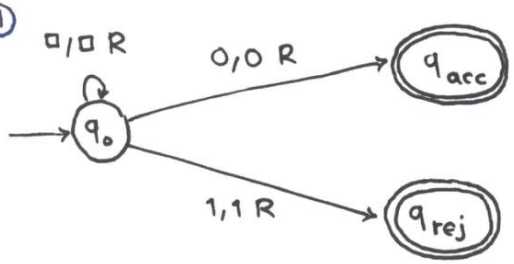
1-way tape Turing Machine



for every symbol x

At the border:



① 

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$\Sigma = \{0,1\}$

input: ε

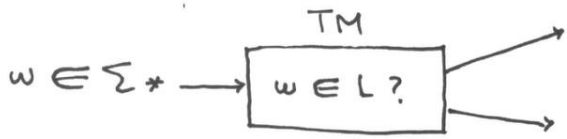
Turing machine may not halt

Inputs to Turing machine can be divided into 3 types:

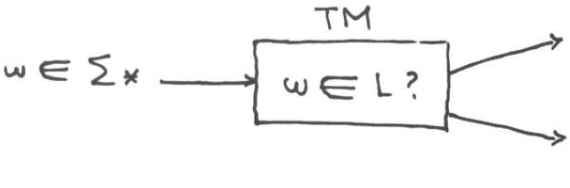
Accept

Reject

Infinite Loop

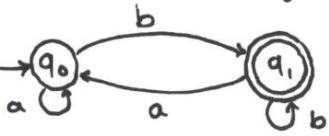


L is decidable (recursive)



L is recognizable (recursively enumerable)

Some examples of languages that are decidable by algorithms

Does 

accept input abb?

We can formulate this question as a language

$$A_{DFA} = \{ \langle D, w \rangle \mid D \text{ is a DFA that accepts input } w \}$$

Is A_{DFA} decidable?

One possible way to encode a DFA $D = (Q, \Sigma, \delta, q_0, F)$ and input w

$$\underbrace{((q_0, q_1)}_{Q}\ \underbrace{(a,b)}_{\Sigma}\ \underbrace{((q_0,a,q_0)(q_0,b,q_1)(q_1,a,q_0)(q_1,b,q_1))}_{\delta}\ \underbrace{(q_0)}_{q_0}\ \underbrace{(q_1)}_{F})\ \underbrace{(abb)}_{w}$$

Pseudo code

On input $\langle D, w \rangle$ where $D = (Q, \Sigma, \delta, q_0, F)$

Set $q \leftarrow q_0$

For $i \leftarrow 1$ to length(w)

$q \leftarrow \delta(q, w_i)$

If $q \in F$ accept, else reject

TM Description

On input $\langle D, w \rangle$ where D is a DFA, w is a string

Simulate D on input w

If simulation ends in an accept state, accept; else reject

Turing Machine Details:

Check input is in correct format (transition function complete, no duplicate)

Perform simulation (very high-level)

Put markers on start state of D and first symbol of w

Until marker for w reaches last symbol:

Update both markers

If state marker is on accepting state, accept; else reject

Conclusion: A_{DFA} is decidable

$((\dot{q}_0, q_1)(a,b)((q_0,a,q_0),(q_0,b,q_1),(q_1,a,q_0),(q_1,b,q_1))(q_0)(q_1))(\dot{a}\,b\,b)$

$((\dot{q}_0, q_1)(a,b)((q_0,a,q_0),(q_0,b,q_1),(q_1,a,q_0),(q_1,b,q_1))(q_0)(q_1))(a\,\dot{b}\,b)$

$((q_0, \dot{q}_1)(a,b)((q_0,a,q_0),(q_0,b,q_1),(q_1,a,q_0),(q_1,b,q_1))(q_0)(q_1))(a\,b\,\dot{b})$

$((q_0, \dot{q}_1)(a,b)((q_0,a,q_0),(q_0,b,q_1),(q_1,a,q_0),(q_1,b,q_1))(q_0)(q_1))(a\,b\,\dot{b})$

$A_{NFA} = \{ \langle N, w \rangle \mid N \text{ is an NFA that accepts input } w \}$

Convert N to a DFA D using the conversion procedure

Run TM M for A_{DFA} on input $\langle D, w \rangle$

If M accepts, accept; else reject

A_{NFA} is decidable

$A_{REX} = \{ \langle R, w \rangle \mid R \text{ is a regular expression that generates } w \}$

Convert R to an NFA N using the conversion procedure

Run TM for A_{NFA} on input $\langle N, w \rangle$

A_{REX} is decidable

$MIN_{DFA} = \{ \langle D \rangle \mid D \text{ is a minimal DFA} \}$

Run DFA minimization algorithm from Lecture 9 on D

If every pair of states is distinguishable, accept; else reject

MIN_{DFA} is decidable

$EQ_{DFA} = \{ \langle D_1, D_2 \rangle \mid D_1 \text{ and } D_2 \text{ are DFAs and } L(D_1) = L(D_2) \}$

Run DFA minimization algorithm on D_1 to obtain a minimal DFA D_1'

Run DFA minimization algorithm on D_2 to obtain a minimal DFA D_2'

If $D_1' = D_2'$ accept; else reject

EQ_{DFA} is decidable

$E_{DFA} = \{ \langle D \rangle \mid D \text{ is a DFA and } L(D) \text{ is empty} \}$

Run the TM S for EQ_{DFA} on input $\langle D, D' \rangle$ where D' is any DFA that accepts no input

If S accepts, accept, else reject

E_{DFA} is decidable

$A_{CFG} = \{ \langle G, w \rangle \mid G \text{ is a CFG that generates } w \}$

Convert G into Chomsky Normal Form G'

Run Cocke-Younger-Kasami algorithm on $\langle G', w \rangle$

If CYK algorithm finds a parse tree, accept; else reject

A_{CFG} is decidable

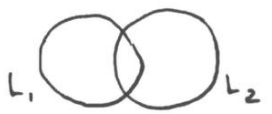
$EQ_{CFG} = \{ \langle G_1, G_2 \rangle \mid G_1, G_2 \text{ are CFGs and } L(G_1) = L(G_2) \}$

What is the difference between EQ_{DFA} and EQ_{CFG}?

To decide EQ_{DFA} we minimize both DFAs

But there is no method that, given a CFG or PDA, produce a unique equivalent minimal CFG or PDA

Note. Since CFGs are not closed under intersection or complement, we cannot use symmetric difference



if $L_1 = L_2 \Rightarrow$ symmetric difference is empty

EQ_{CFG} is undecidable.

$E_{CFG} = \{ \langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset \}$

- Convert G into Chomsky Normal Form
- Mark all non-terminals A which have some rule $A \rightarrow a$
- Repeat until no new non-terminal are marked :
  - Mark the non-terminal A if there is a rule $A \rightarrow BC$ such that B and C are already marked
- If S is marked ($L(G) \neq \emptyset$) then reject , else accept

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Encodings:

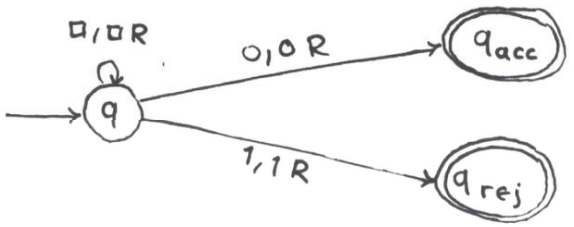
We represent the string encoding of O with angle brackets: $\langle O \rangle$

Example. $MIN_{DFA} = \{\langle D \rangle \mid D \text{ is a minimal DFA}\}$

We represent the string encoding of $O_1, \dots, O_n$ with $\langle O_1, \dots, O_n \rangle$

Example. $A_{CFG} = \{\langle G, w \rangle \mid G \text{ is a context-free grammar that generates } w\}$

How can we encode a Turing machine?



Turing Machine

$(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$

$$\langle M \rangle = \left((q, q_{acc}, q_{rej}), (0,1), (0,1,\square), \left((q, q, \square/\square R), (q, q_{acc}, 0/0R), (q, q_{rej}, 1/1R) \right), (q), (q_{acc}), (q_{rej}) \right)$$

We can encode any Turing machine M in binary alphabet $\{0,1\}$

Analogy: Program source code

- Data given to a program is encoded using $\{0,1\}$
- Program itself is also represented in $\{0,1\}$ on disk

One possible encoding:

Alphabet encoding:

Symbols: a b c d ...

Encodings: 1 11 111 1111 ...

State encoding:

States: q_1 q_2 q_3 ...

Encodings: 1 11 111 ...

Transition Encoding:

$\delta(q_1, a) = (q_2, b, L)$

1 0 1 0 1 1 0 1 1 0 1

Head move: L: 1 R: 11

separator

Machine Encoding:

Transitions:

Encoding:

$\delta(q_1, a) = (q_2, b, L)$ $\quad$ $\delta(q_2, b) = (q_3, c, R)$

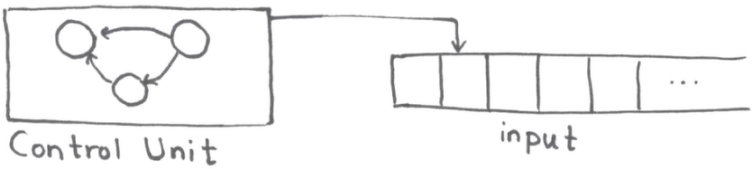
1 0 1 0 1 1 0 1 1 0 1 **0 0** 1 1 0 1 1 0 1 1 1 0 1 1 1 0 1 1

Separator

The set of Turing machines forms a language, each string of the language is the binary encoding of a Turing machine

$L = \{$ 010100101, 001001001011, 1110100111100101, ... $\}$

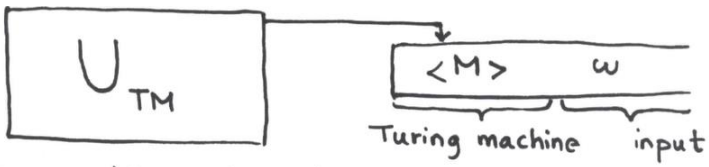
↑ Turing machine 1 ↑ Turing machine 2 ↑ Turing machine 3



So far we only built "dedicated machine"

- Only run one program
- Specified by transitions on states

Can TM be a general-purpose computer? Can we have a TM that simulates an arbitrary TM on arbitrary input?



Universal Turing machine U_{TM} takes $\langle M, w \rangle$ as input (where M is a Turing machine and w is a string) and simulates running M on w

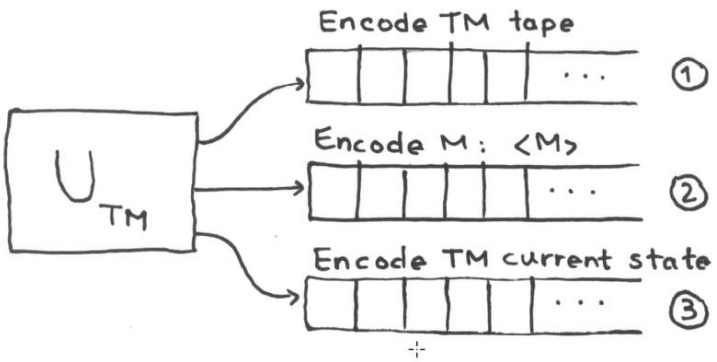
U_{TM} = "On input $\langle M, w \rangle$, where M is a TM and $w \in \Sigma^*$

Run M on w

If M accepts w, U_{TM} accepts $\langle M, w \rangle$.

If M rejects w, U_{TM} rejects $\langle M, w \rangle$."

Note. If M loops on w, U_{TM} loops as well.



U_{TM} simulates the TM

- Read tape 1
- Read tape 3
- Consult tape 2 what to do
- Write tape 1
- Move head tape 1
- Move head tape 3

Since U_{TM} is a Turing machine, it has a language.

What is the language of the universal Turing machine?

The language of the universal Turing machine : A_{TM}

$A_{TM} = L(U_{TM})$ Acceptance language for Turing machine

$= \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

Useful facts: $\langle M, w\rangle \in A_{TM} \iff M$ accepts w

---

Recall : Definition

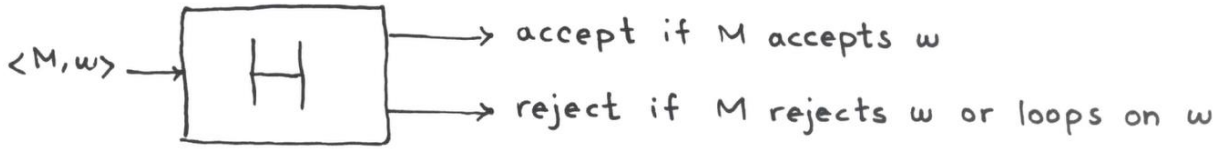
A Turing machine M is said to recognize a language L if $L = L(M)$

A Turing machine M is said to decide a language L if $L = L(M)$ and M halts on every input.

Because $A_{TM} = L(U_{TM})$ we know A_{TM} is Turing-recognizable.

Turing's Theorem: The language A_{TM} is undecidable.

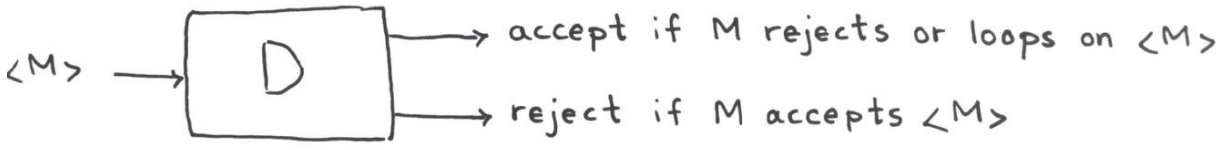
Proof by contradiction: Suppose A_{TM} is decidable, then some TM H decides it



Construct a new TM D (that uses H as a subroutine)

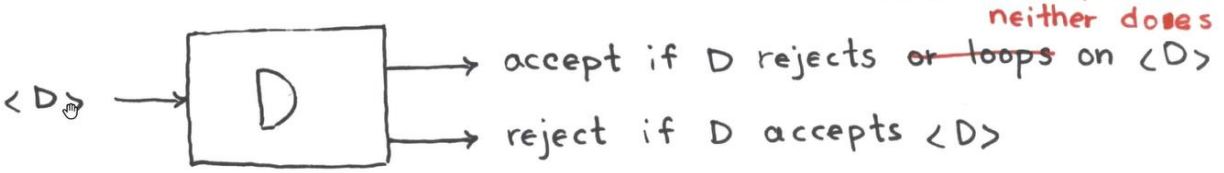
On input $\langle M\rangle$ (i.e. description of a Turing machine M)

1. Run H on input $\langle M, \langle M\rangle\rangle$
2. Output the opposite of H : if H accepts, D rejects, if H reject, D accepts



What happens if $M = D$?

H never loops indefinitely, neither does D



If D rejects $\langle D \rangle$, then D accepts $\langle D \rangle$

If D accepts $\langle D \rangle$, then D rejects $\langle D \rangle$

Contradiction! D cannot exist, H cannot exist!

Assume A_{TM} is decidable $\Rightarrow$ Then there are TM H and D

$\Rightarrow$ But D cannot exist

Proof using Diagnolization

Write an infinite table for pairs (M, w)

	ε	0	1	00	...
M_1	acc	rej	loop	rej	...
M_2	loop	acc	rej	loop	...
M_3	⋮	⋮	⋮	⋮	
M_4					
⋮					

all possible inputs

all possible Turing machines

Only look at those w that describe Turing machine:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	...
M_1	acc	loop	rej	...
M_2	rej	acc	loop	...
M_3	acc	rej	rej	...
M_4	rej	acc	loop	...
⋮	⋮	⋮	⋮	⋮

If H exists, table of outputs of H:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	...	$\langle D \rangle$	...
M_1	acc	rej	rej	...	acc	...
M_2	rej	rej	acc	...	rej	...
M_3	acc	rej	acc	...	rej	...
M_4	rej	acc	rej	...	acc	...
⋮	⋮	⋮	⋮		⋮	
D	rej	acc	rej		?	

D on $\langle M_i \rangle$ accepts iff M_i on $\langle M_i \rangle$ rejects

D on $\langle M_i \rangle$: opposite of M_i on M_i

So, D on $\langle D \rangle$ will accept iff D on $\langle D \rangle$ rejects $\Rightarrow$ contradiction!

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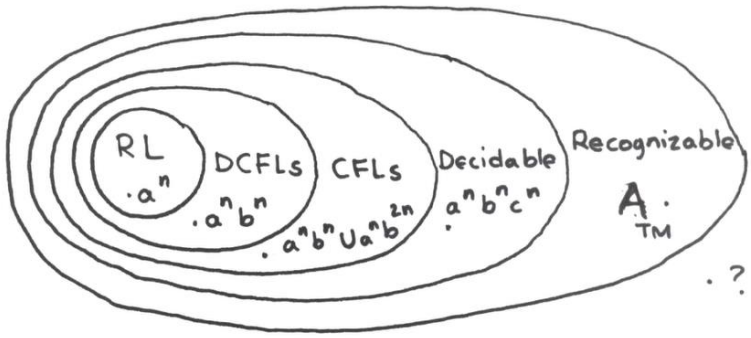
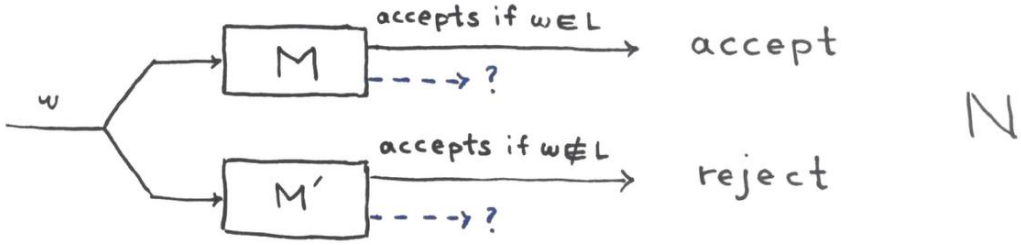
Language hierarchy:



Language A_{TM} is recognizable but not decidable

How about languages that are not recognizable?

Theorem. If L and $\bar{L}$ are both recognizable, then L is decidable



Suppose M accepts L and M′ accepts $\bar{L}$

Turing machine N decides L:

1) Simulate M on input w. If M accepts, N accepts

2) Simulate M′ on input w. If M′ accepts, N rejects

Important detail: M and M′ must run in parallel, not sequentially

Run in parallel: Perform one step of M, followed by one step of M′ alternating between two machines

---

Claim. Language $\overline{A_{TM}}$ is not recognizable.

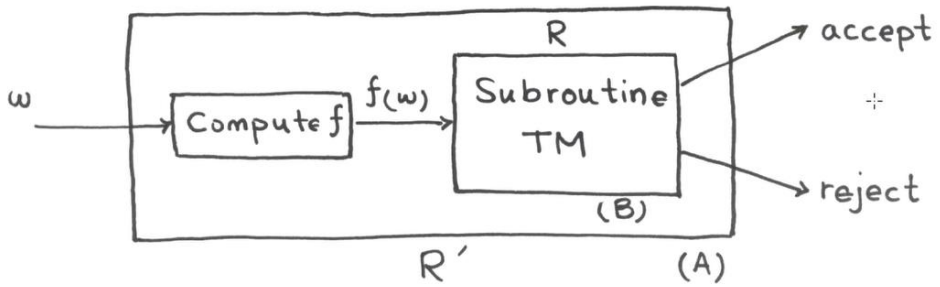
Proof. We know A_{TM} is recognizable.

If $\overline{A_{TM}}$ was also, then A_{TM} would be decidable

But Turing's theorem says A_{TM} is not decidable.

Suppose you have a program R that solves problem B

Now we want to solve a problem A. If you can reduce A to B then you can solve problem A, using R as a subroutine

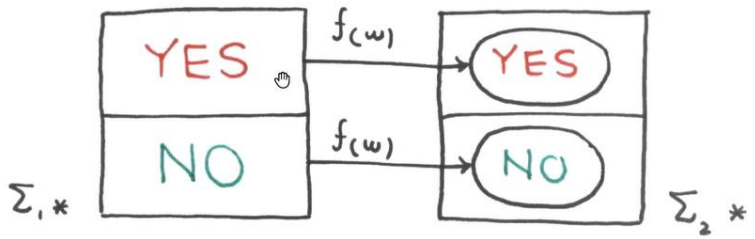


R': on input w:

- Transform the input w to f(w)
- Run machine R on f(w)
- If R accepts f(w), then R' accepts w
- If R rejects f(w), then R' rejects w

Defining Reductions

A reduction from A to B is a function $f: \Sigma_1^* \longrightarrow \Sigma_2^*$ such that for any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$



Every $w \in A$ maps to some $f(w) \in B$

Every $w \notin A$ maps to some $f(w) \notin B$

f does not need to be injective or surjective

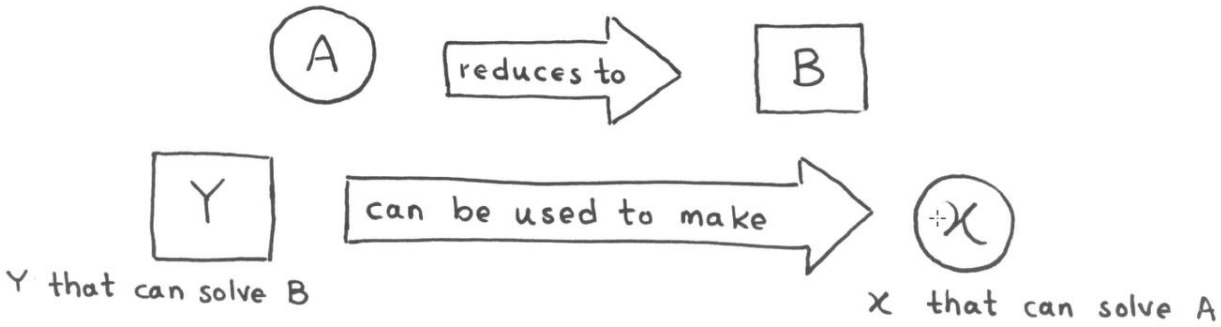
Example.

$A_{DFA} = \{ \langle D, w \rangle \mid D \text{ is a DFA that accepts input } w \}$

$A_{NFA} = \{ \langle N, w \rangle \mid N \text{ is an NFA that accepts input } w \}$

A_{NFA} reduces to A_{DFA} (by converting NFA to DFA)

Problem A reduces to problem B iff a solver for B can be used to solve problem A



$$A \leqslant B$$

then this one is "easy" to
If this one is "hard"

If this one is "easy"
then this one is "hard" too

$A \leqslant B$ and B decidable $\Longrightarrow$ A decidable

$A \leqslant B$ and A undecidable $\Longrightarrow$ B undecidable

---

Halting Problem

Input: Program P in some programming language

Output: true if P terminates, false if P runs forever

```
halts("2+2")
```
True

```
halts("def f(n):
          if n==0 : return 1
          else: return n * f(n-1)
      f(10)")
```
True

```
halts("def f(n):
          if n==0 : return 1
          else: return n * f(n-1)
      f(10.5)")
```
False

```
halts("x = randint()
      while (x>1):
          if (x % 2 ==0) : x = x/2
          else x = 3*x+1")
```
Unknown

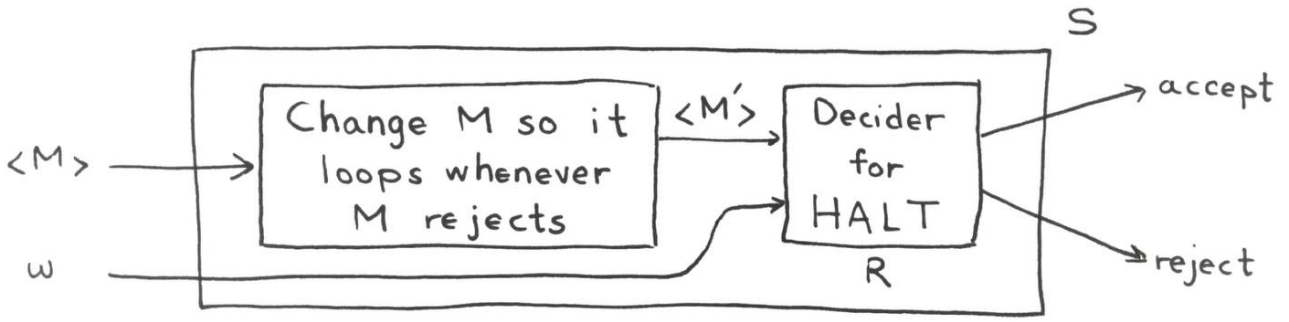
(Collatz Conjecture)

$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$

We reduce A_{TM} to the language $HALT_{TM}$: $A_{TM} \leq HALT_{TM}$

Suppose for a proof by contradiction that $HALT_{TM}$ is decidable.

Then it has a TM R that decides it.



If M never rejects, then M accepts iff M halts on w

$\langle M, w\rangle \in A_{TM}$ iff $\langle M, w\rangle \in HALT$

How can we transform M?

Assume $\Sigma = \{a, b, c\}$. We convert 

to 

Machine S is a decider for HALT (contradiction)!

S = " On input $\langle M, w\rangle$

Transform M into M' by making M loop instead of rejecting

Run R on $\langle M', w\rangle$

If R accepts $\langle M', w\rangle$, then S accepts $\langle M, w\rangle$

If R rejects $\langle M', w\rangle$, then S rejects $\langle M, w\rangle$. "

Both A_{TM} and $HALT_{TM}$ are undecidable

There is no way to decide whether a TM will accept or eventually terminate.

Both A_{TM} and $HALT_{TM}$ are recognizable

Reduce A to B:

$$A \leq B$$

then A is "easy" if B is "easy"

If A is "hard" then B is "hard"

"easy" : decidable, recoanizable

"hard" : undecidable, unrecognizable

$$A_{TM}^{\varepsilon} = \{ \langle M \rangle \mid M \text{ is a TM that accepts input } \varepsilon \}$$

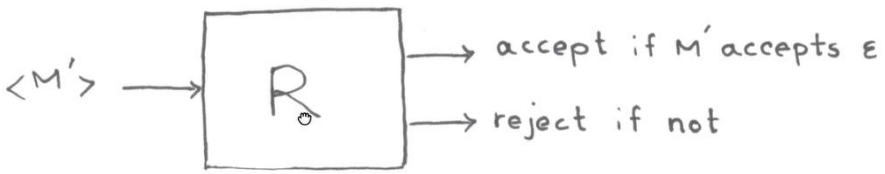
Undecidable.

Intuitive reason: to know whether M accepts ε require simulating M

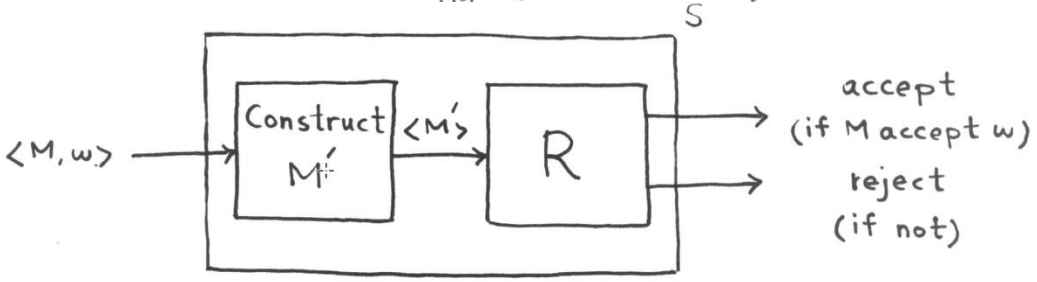
But we need to know whether M halts

$$A_{TM} \leq A_{TM}^{\varepsilon}$$

Suppose R decides A_{TM}^{ε} (proof by contradiction)

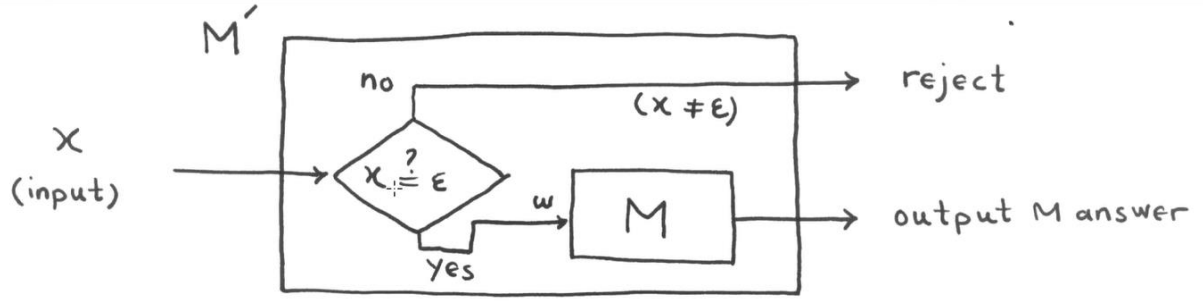


We show we can use R to decide A_{TM} (contradiction)



M' accepts ε if and only if M accept w

②



- If M accepts w

  input is ε and M' accepts it

- If M rejects w

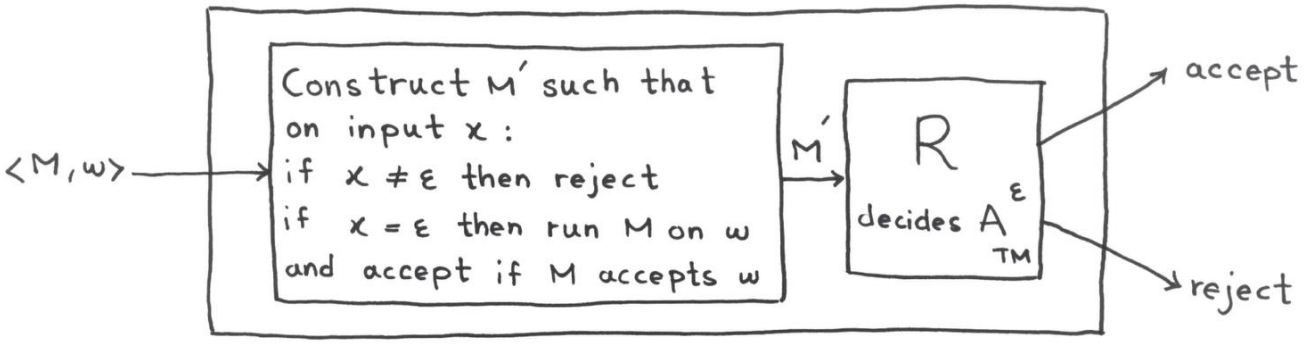
  case $x = \varepsilon$: M' rejects

  case $x \neq \varepsilon$: M' rejects

- If M loops on w

  case $x = \varepsilon$: M' loops

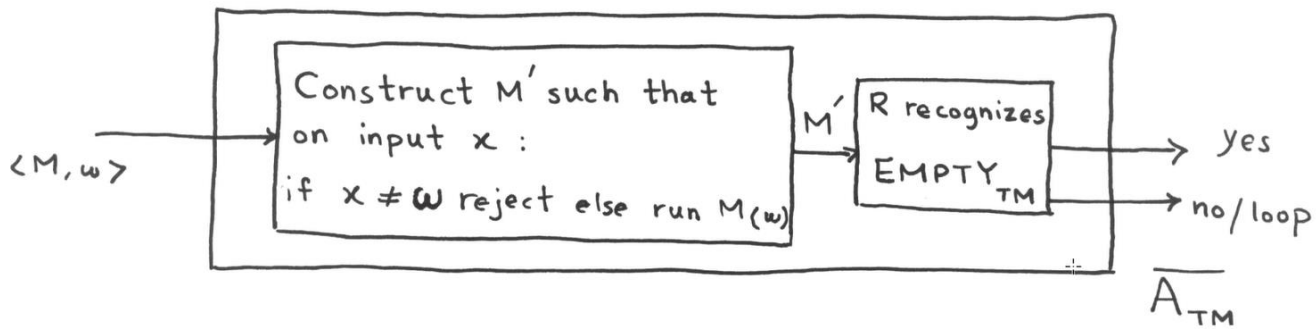
  case $x \neq \varepsilon$: M' rejects



$EMPTY_{TM} = \{ \langle M \rangle \mid M \text{ is a TM such that } L(M) = \emptyset \}$

Unrecognizable $\qquad \overline{A_{TM}} \leq EMPTY_{TM}$

Suppose R recognizes $EMPTY_{TM}$ (Proof by contradiction)



Either $L(M') = \emptyset$ or $L(M') = \{w\}$ (when M accepts w).

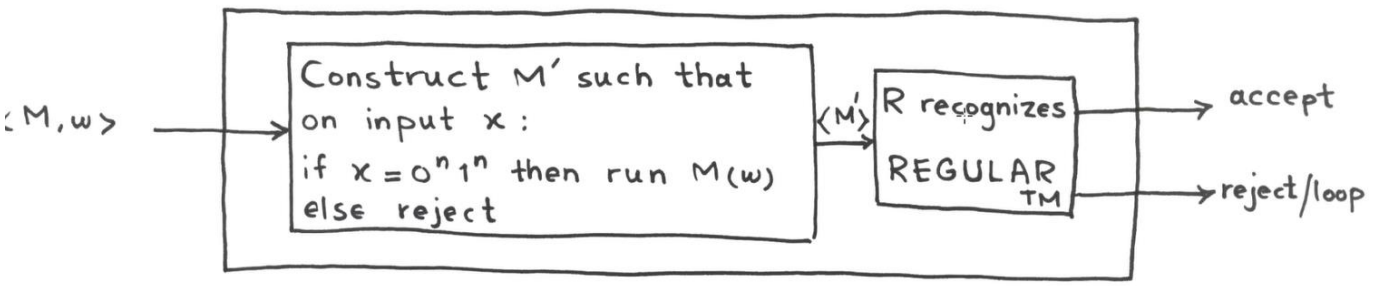
$$\langle M, w\rangle \in \overline{A_{TM}} \iff \langle M, w\rangle \notin A_{TM}$$
$$\iff M \text{ does not accept } w$$
$$\iff L(M') = \emptyset$$
$$\iff M' \in EMPTY_{TM}$$

$$REGULAR_{TM} = \{\langle M\rangle \mid M \text{ is a TM such that } L(M) \text{ is regular}\}$$

Given a program, is it equivalent to some DFA?

Unrecognizable

$$\overline{A_{TM}} \leq REGULAR_{TM}$$



$\langle M, w\rangle \in A_{TM} \iff M'$ accepts $\{0^n 1^n\}$

$\langle M, w\rangle \notin A_{TM} \iff M'$ accepts nothing

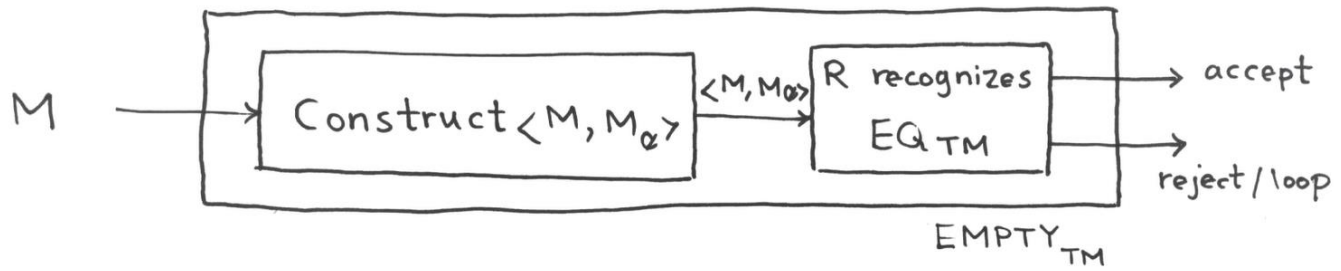
So we conclude: $\langle M, w\rangle \notin A_{TM} \iff M' \in REGULAR_{TM}$

$$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs such that } L(M_1) = L(M_2)\}$$

Unrecognizable

$$EMPTY_{TM} \leq EQ_{TM}$$

Let $M_\emptyset$ be some TM with no path from start to accept state



Software is the most complex artifact that human beings build routinely. Analyzing programs is extremely hard !

$L_1 = \{\langle M, w\rangle \mid M \text{ is a TM that on input } w \text{, tries to move its head past the left end of the tape at some point}\}$ — Undecidable

$L_2 = \{\langle M, w\rangle \mid M \text{ is a TM that on input } w \text{, moves its head left at some point}\}$ — Decidable

L_1 is undecidable. $\quad A_{TM} \leq L_1$

Idea. Construct a TM M' which simulates M but never goes to the left of the starting symbol.

If M halts, then proceed to go all the way left, until it goes beyond the starting symbol.

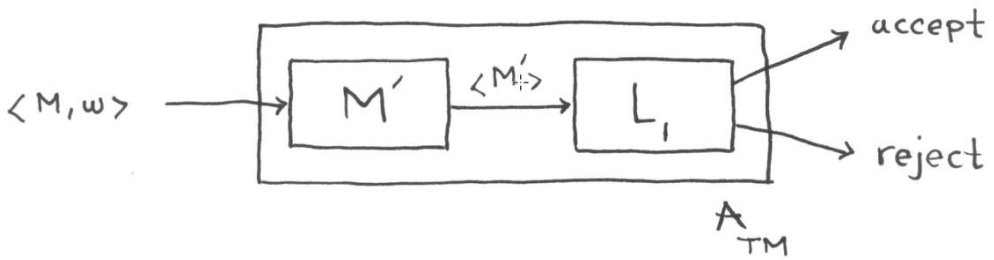
Construct M':

Copy w one position right and put special symbol # before it

Simulate M on w with two changes:

① If head moves to # but has not yet accepted move head back to right and stay in the same state

② If M accepts, enter a special state where the head just keeps moving left



L_2 is decidable

Does a TM M on input w ever try to move its head left?

On input $\langle M, w \rangle$:

Simulate M on w for $|w|$ steps.

If M hasn't made any left moves, then head must be pointing to first ␣ after w.

Search in the state diagram of M by following all transitions that have ␣ as input. Search is similar to the one we did for ε-closure.

We are only interested in these transitions, because if the machine never moves left, the only available input symbol will be ␣.

If none of the transitions move left, accept, otherwise reject.

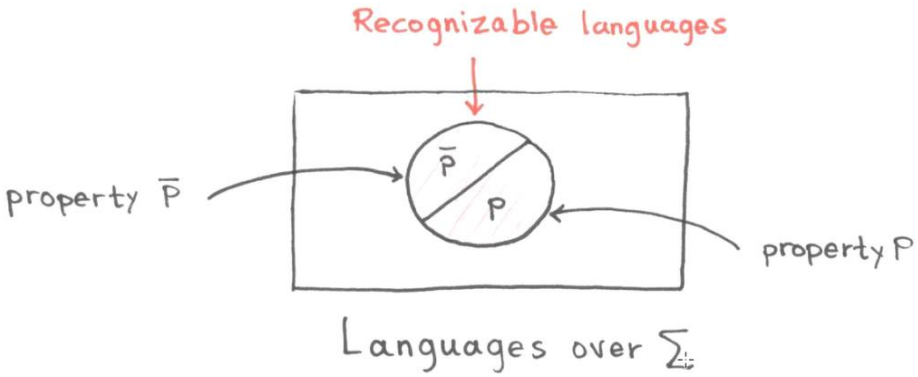
Every time we wish to prove L is undecidable

Do we need to reduce A_{TM} to it? Isn't there any shortcut?

Rice theorem states that Turing machines cannot test whether another Turing machine satisfies a non-trivial property.

Rice Theorem: Let P any subset of the Turing recognizable class such that P and its complement are both non-empty.

Then the language $L_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable



P_1 : L is empty

✓ $L = \emptyset$

✗ $L = \{\text{University}\}$

✗ $L = \{\text{Alan}, \text{Turing}\}$

P_2 : L is regular

✓ $L = \emptyset$

✓ $L = \{a^n \mid n \geq 0\}$

✗ $L = \{a^n b^n \mid n \geq 0\}$

P_3 : L has size 3

✗ $L = \emptyset$

✗ $L = \{\text{University}\}$

✓ $L = \{\text{Alan}, \text{Turing}, \text{Machine}\}$

P_4 : L has size at least 0

True for all languages

P_5 : L is accepted by some Turing machine

True for all Turing-recognizable languages

باسه تعالی

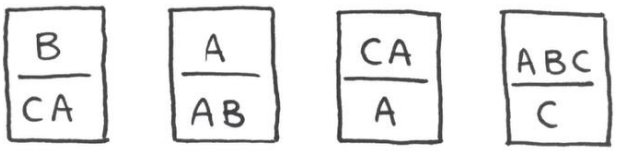
جلسهٔ بیست و پنجم

PCP : Post Correspondence Problem (مسألهٔ تناظر پُست)

Classic undecidable problem that was introduced by Emil Post in 1946

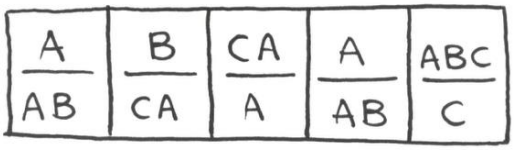
An undecidable problem that does not directly involve Turing machines

Dominos :



We get as many of each type as we need.

Goal: Find a finite sequence of dominos such that the top and bottom strings are the same.



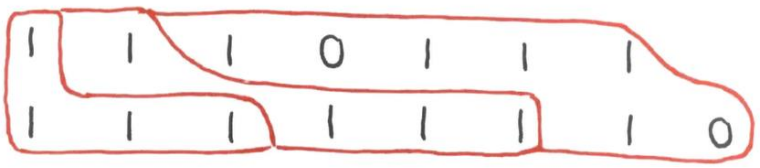
A B C A A A B C
| | | | | | | |
A B C A A A B C

Does a solution exist? The problem is undecidable.

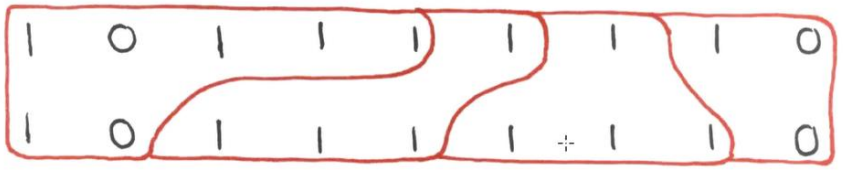
Another instance of PCP:

	A	B
①	1	111
②	101111	10
③	10	0

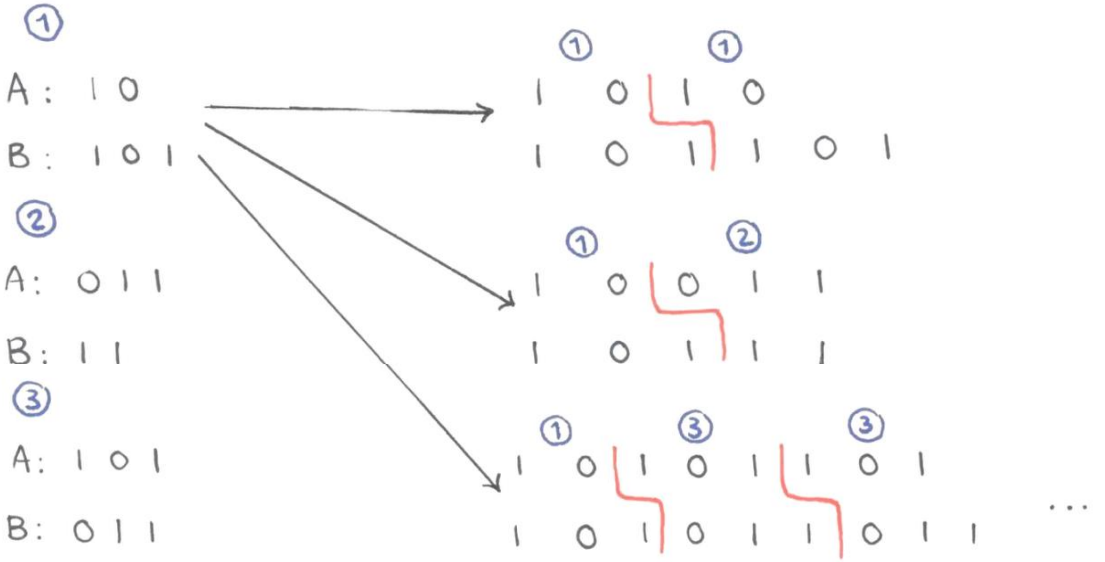
A non-solution: ① ① ②



A solution: ② ① ① ③

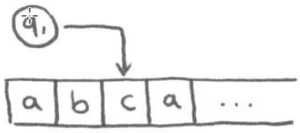


	A	B
(1)	1 0	1 0 1
(2)	0 1 1	1 1
(3)	1 0 1	0 1 1



How can we prove that PCP is undecidable?

Configuration :



$C : a\,b\,q_i\,c\,a \ldots$

A sequence of configurations:

$$\ldots \Rightarrow C_7 \Rightarrow C_8 \Rightarrow C_9 \Rightarrow \ldots$$

step in the computation

An "accepting computation history"

$$C_1 \Rightarrow C_2 \Rightarrow C_3 \Rightarrow C_4 \Rightarrow \cdots \Rightarrow C_{n-1} \Rightarrow C_n$$

C_1: initial starting configuration

$C_2 \Rightarrow C_3 \Rightarrow C_4$: legal steps for machine

C_n: accepting configuration

A "rejecting computation history"

$$C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_n$$

Same, but the last configuration has a rejecting state

A computation history is a finite sequence.

If the machine does not halt ⇒ NO HISTORY

On a given input :

- Deterministic machine : at most one history
- Non-deterministic machine : may have many histories

In this lecture we only consider deterministic TMs.

There is either { an accepting history / a rejecting history / no history / loop } on a particular input.

We can represent a computation history with a string.

$$\# q_0 1 0 1 \# 1 q_4 0 1 \# \cdots \# 0 1 1 q_{acc} 1 \#$$

↑ initial state at the left end of tape

↑ this is an accepting history

Theorem. The Post Correspondence Problem is undecidable

Proof Sketch :

Reduce A_{TM} to PCP $\qquad A_{TM} \leq PCP$

If A_{TM} accepts then

- M computes on w and accepts
- there is a finite computation history which describes the computation of M on w

We encode $\langle M, w \rangle$ into a PCP instance.

There is a solution to PCP iff there is an accepting computation history

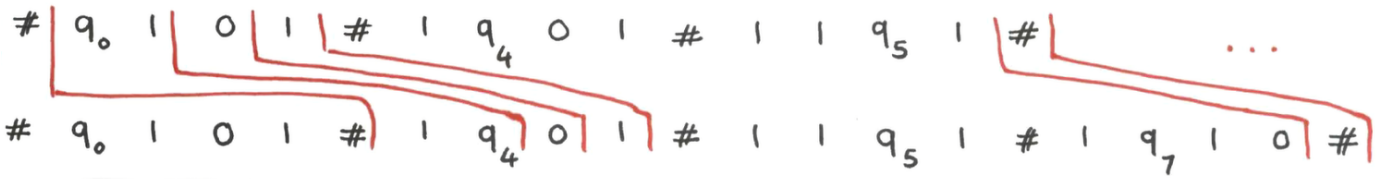
If we could decide this instance of PCP,

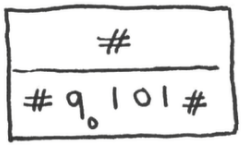
then we could decide A_{TM} .

But we cannot decide A_{TM} !

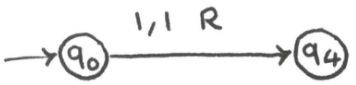
Solution to this instance of PCP ⟹ you have an accepting history

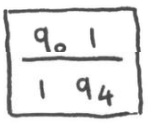
A solution will look like this :





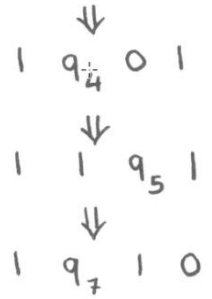
Special Starting Domino





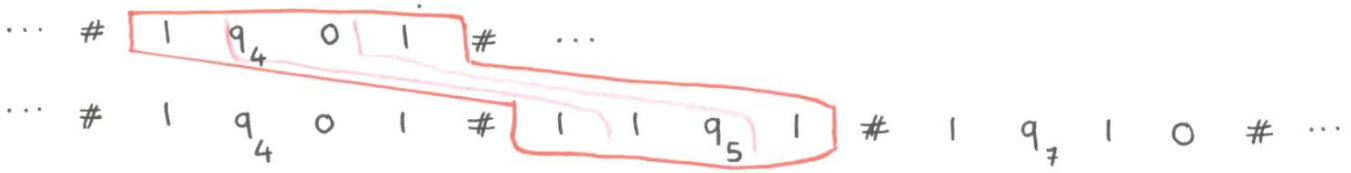
One domino for each transition

Computation History

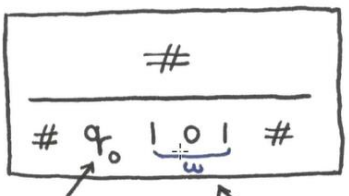


0,1 R

1,0 L

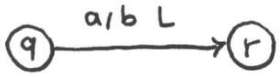


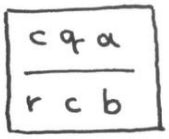
Special Starting Domino :



RIGHT MOVE

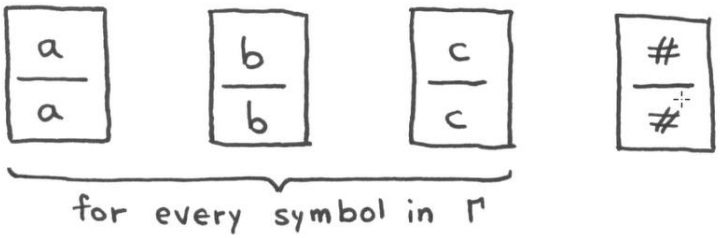
LEFT MOVE





For every $c \in \Gamma$, Add one domino like this

Dominos to "copy" the tape:



Used to copy the other parts of the tape that are not next to head

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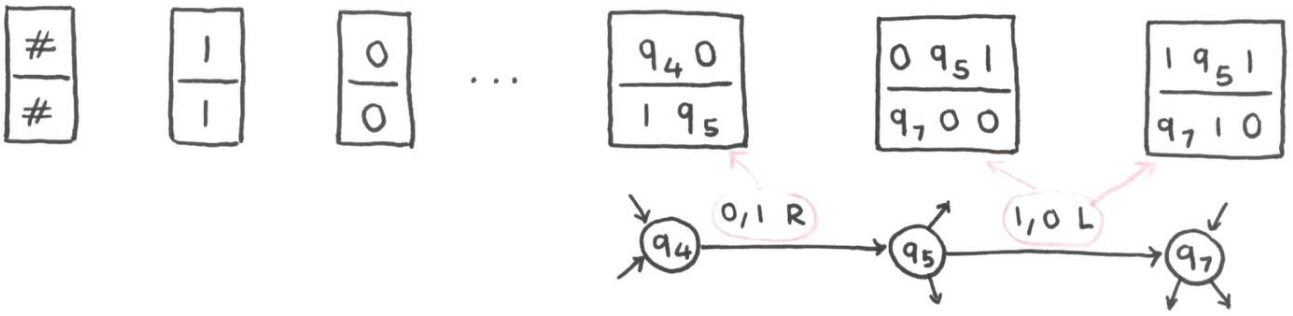
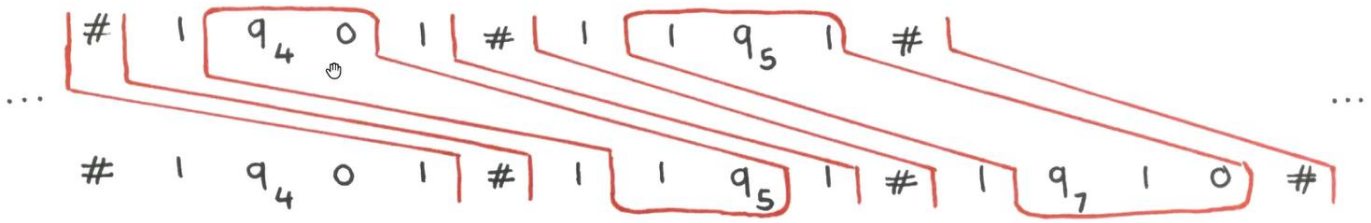
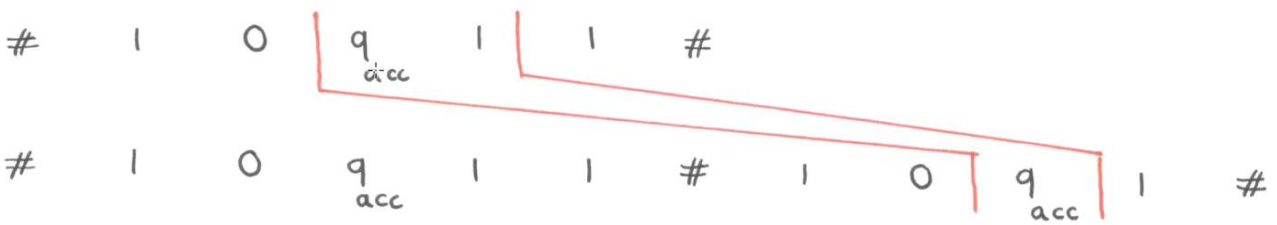




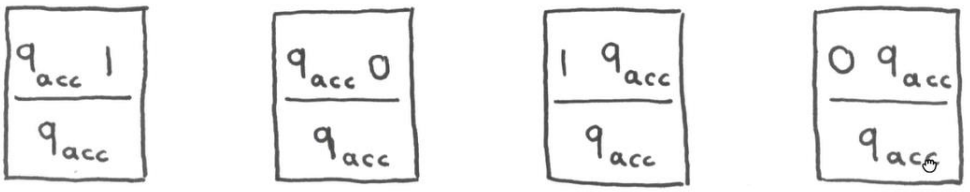
Question: How do we accept?

Answer: Complete the match!

Add special dominos to allow q_{acc} to "eat" the symbols on tape

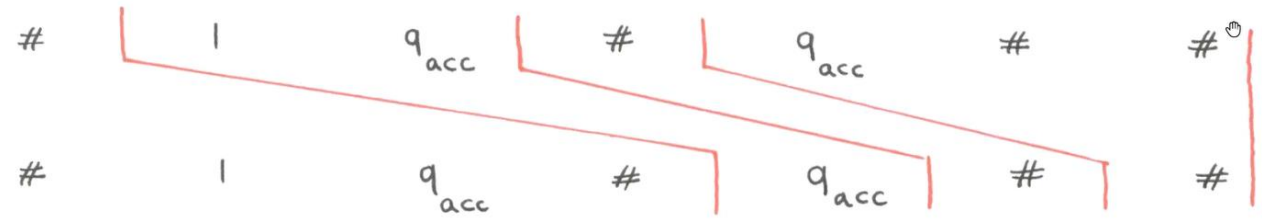


For every symbol in Γ:



Eventually, nothing remains except q_{acc}.

Add a domino to finish the match.



q_{acc} # #
#

Review of the proof:

If you can find a solution to this instance of PCP, then you have found a legal accepting computation history, in which machine M accepts string w.

* Does a solution exist?
* If you can decide the answer, then you can decide whether M accepts w

We know A_{TM} is undecidable!

Undecidability of ambiguity problem for CFG

We illustrate the reduction using the following example:

	A	B
①	a	ab
②	b	ca
③	ca	a
④	abc	c

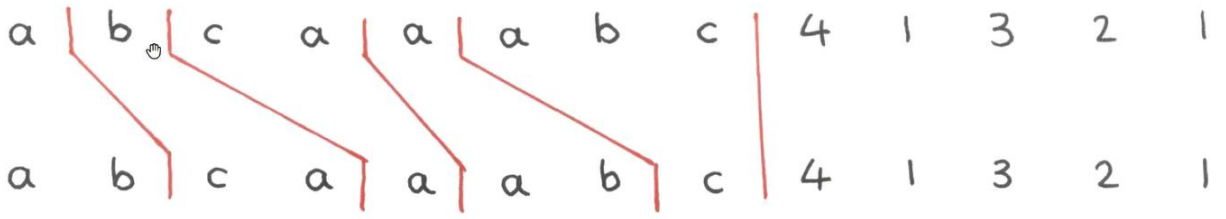
$S \rightarrow A \mid B$

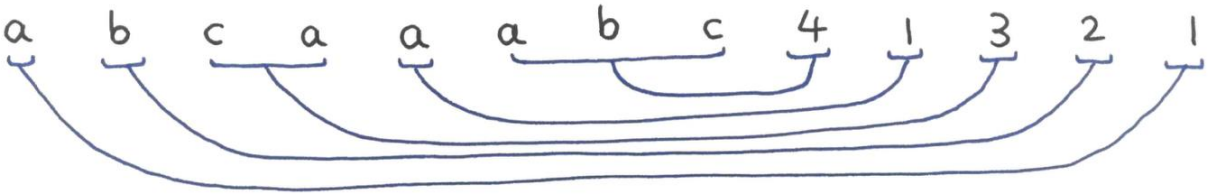
$A \rightarrow aA1 \mid bA2 \mid caA3 \mid abcA4$
$\mid a1 \mid b2 \mid ca3 \mid abc4$

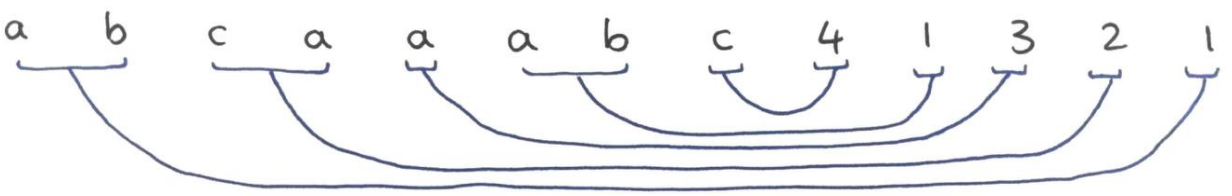
$B \rightarrow abB1 \mid caB2 \mid aB3 \mid cB4$
$\mid ab1 \mid ca2 \mid a3 \mid c4$

1, 2, 3, 4 : record which domino was used

There are two distinct leftmost derivations for the string abcaaabc41321 because this instance of PCP has solutions. (7)







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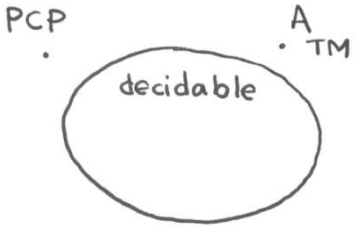
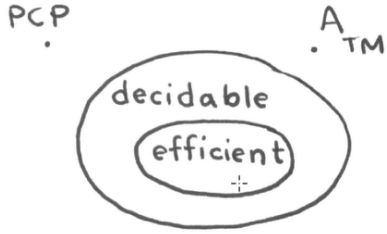


Undecidable Problems :

We cannot find solutions in any finite amount of time.

Decidable Problems :

We can solve them, but it may take a very long time.



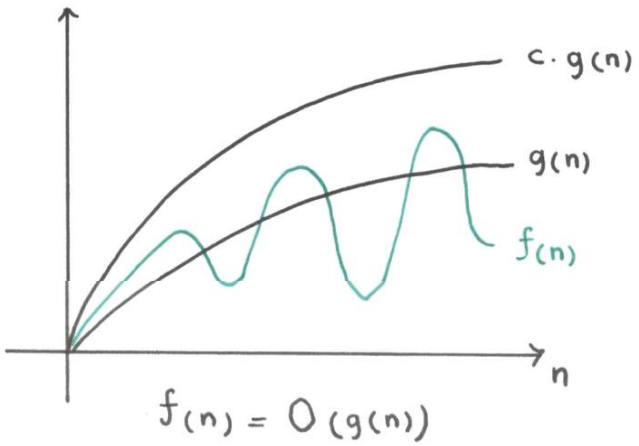
We typically look for efficient algorithms.
Running time depends on the input.
For longer inputs, we should allow more time.
Efficiency is measured as a function of input size.

Asymptotic Analysis

تجزیه و تحلیل مجانبی

The asymptotic behavior of a function $f(n)$ refers to the growth of $f(n)$ as n gets large. We typically ignore small values of n, since we are usually interested in estimating how slow the program is on large input.

Eventually $|f(n)| \leq c \cdot |g(n)|$



We measure run-time in terms of input size "n"

E.g. Run-time grows on the order of input size $O(n)$

$$5n^3 + 23n^2 + 6n \lg n + 186 = O(n^3)$$

Running time of Turing machine M is the function $t_M(n)$:

$t_M(n)$: maximum number of steps that M takes on any input of length n

---

$L = \{ w \# w \mid w \in \{a,b\}^* \}$

a b a a b # a b a a b
~~a~~ b a a b # a b a a b
~~a~~ b a a b # ~~a~~ b a a b
~~a~~ ~~b~~ a a b # ~~a~~ b a a b

M : On input x, until you reach # — $O(n)$ times

- Read and cross off first a or b before #
- // // // after #
- If mismatch, reject

} $O(n)$ steps

If all symbols except # are crossed off, accept — $O(n)$ steps

running time : $O(n^2)$

$L = \{ 0^n 1^n \mid n \geq 0 \}$

M : On input x,

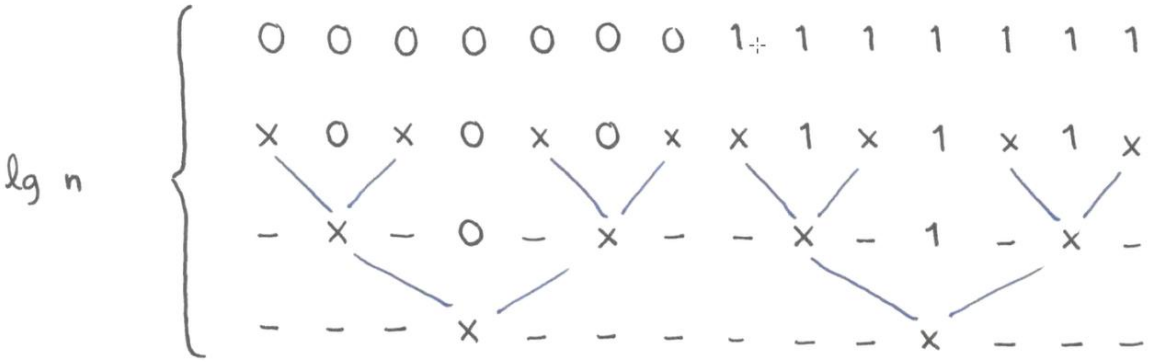
- Check input is of the form 0^*1^* — $O(n)$ steps
- Until everything is crossed off : — $O(n)$ times
  - Cross off the leftmost 0
  - Cross off the following 1

  } $O(n)$ steps
- If everything is crossed off, accept — $O(n)$ steps

running time : $O(n^2)$

Can we find a faster algorithm to decide L?

Cross off every other 0, cross off every other 1



If the parities of 0's and 1's don't match, reject

(odd) 0 0 0 0 0 0 0 1 1 1 1 1 (odd)

(odd) x 0 x 0 x 0 x x 1 x 1 x (even)

M: On input x,

Check input is of the form 0*1* — O(n) steps

Until everything is crossed off: — O(lg n) times

- Find parity of the number of 0s
- Find parity of the number of 1s
- If the parities don't match, reject
- Cross off every other 0 and every other 1

} O(n) steps

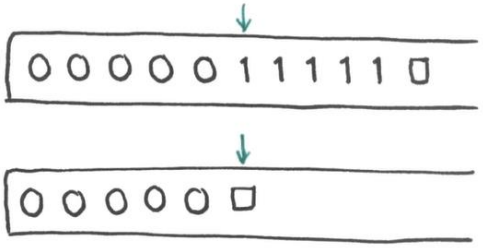
If everything is crossed off, accept — O(n) steps

---

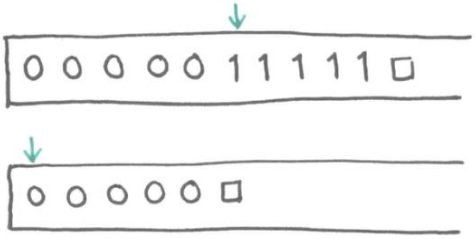
running time: O(n lgn)

What if we have a two-tape TM?

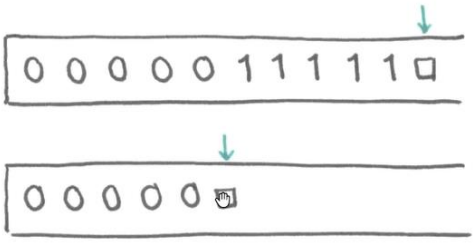
- Copy all 0's to tape 2.



- Reposition tape 2 to beginning



- Scan both tape simultaneously, make sure both heads hit □ at the same time



M: On input x,

Check input is of the form 0^*1^*	$O(n)$ steps
Copy 0^* part of the input to second tape	$O(n)$ steps
Until □ is reached:	$O(n)$ steps
Cross off next 1 from first tape	
Cross off next 0 from second tape	
If both tapes reach □ simultaneously, accept	$O(n)$ steps

running time: $O(n)$

Running time can change depending on the computation model

1-Tape Turing machine: $O(n \lg n)$

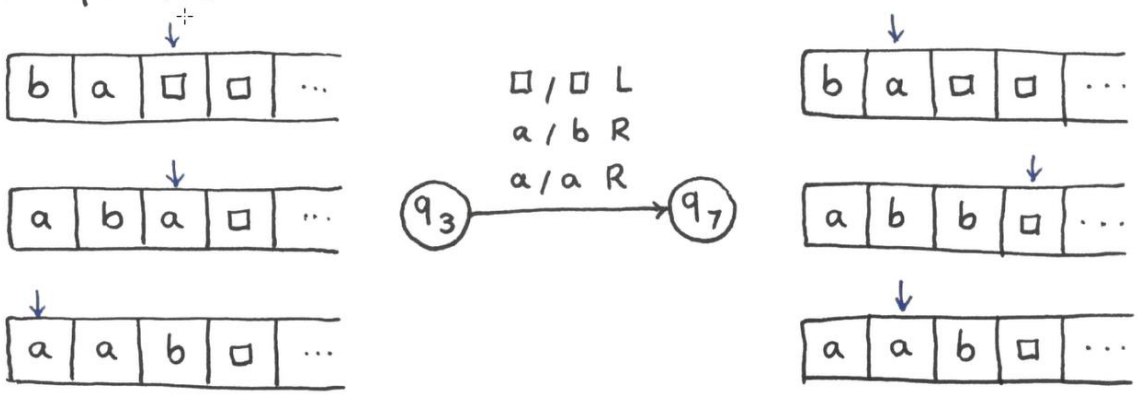
2-Tape Turing machine: $O(n)$

Church-Turing thesis states that any feasible model of computation is no more powerful than a TM.

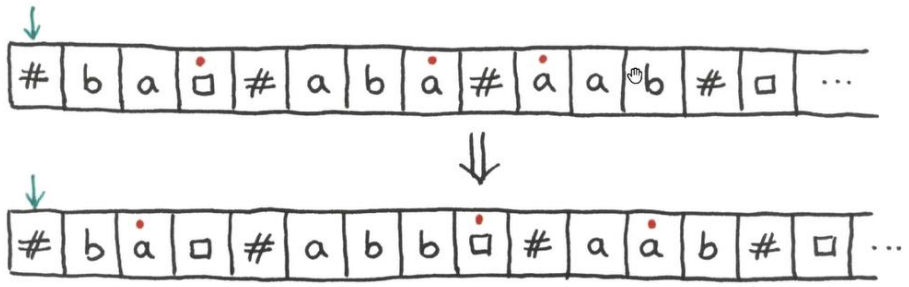
However, some models of computation are more efficient in terms of computation time.

Model of computation matters for complexity!

Theorem: Let $t(n)$ be a function such that $t(n) \geq n$.
Then every $t(n)$-time multi-tape TM has an equivalent $O(t(n)^2)$ single tape TM



1-tape TM simulates multi-tape TM: simulate each step using 2 scans over non-blank portion of tapes



How big can the non-blank portion of the multi-tape TM's tapes become?

- Initially, n for the input
- In $t(n)$ steps, no bigger than $t(n)$, because that's how far the heads can travel (starts at left)

So the number of steps by the 1-tape TM is at most

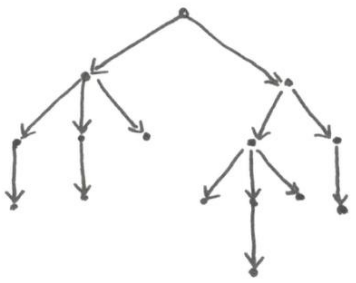
$$t(n) \times ct(n) \quad \text{hence} \quad O(t^2(n))$$

Number of steps of multi-tape machine

steps taken by the scans, to emulate one step of multi-tape TM

Theorem: Let $t(n)$ be a function where $t(n) \geq n$.
Then every $t(n)$-time non-deterministic single-tape TM has an equivalent $2^{O(t(n))}$-time deterministic single-tape TM.

Deterministic Computation

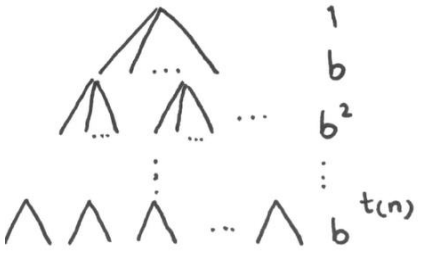


Non-deterministic Computation

We construct a deterministic TM by simulating an NTM using breadth first search. On an input of length n, every branch of the NTM's non-deterministic computation tree has a length of at most $t(n)$.

Every node in the tree can have at most b children, where b is the max number of legal choices given by NTM's transition function. Total number of leaves in the tree is at most $b^{t(n)}$

Lemma: Total number of nodes in the tree is less than twice the maximum number of leaves.



$$\frac{\text{Total number of nodes}}{\text{Number of leaves}} = \frac{1 + b + b^2 + \cdots + b^{t(n)}}{b^{t(n)}}$$

$$= \frac{\frac{b^{t(n)+1} - 1}{b-1}}{b^{t(n)}} < \frac{\frac{b^{t(n)+1}}{b-1}}{b^{t(n)}} = \frac{b}{b-1} \leq 2$$

From Lemma we conclude the total number of nodes is bounded by the number of leaves $O(b^{t(n)})$. The time it takes to start from the root and travel down to a node is $O(t(n))$.
The total running time is $O(t(n)\, b^{t(n)})$

$$t(n)\, b^{t(n)} = 2^{\lg(t(n)) + \lg(b^{t(n)})} = 2^{\lg(t(n)) + t(n)\lg b} \doteq 2^{O(t(n))}$$

Deterministic k-tape TM $\xleftrightarrow{\text{polynomial difference}}$ Single tape TM

Non-deterministic TM $\xleftrightarrow{\text{exponential difference}}$ Deterministic TM

Polynomial time algorithms are considered tractable

Exponential time algorithms are rarely usable for large input size

$$\lim_{n \to \infty} \frac{2^n}{p(n)} \qquad p(n) \text{ polynomial}$$

All reasonable deterministic computational models are polynomially equivalent.

Any of them can simulate another with only a polynomial increase in running time.

- Polynomial functions "scale well".
  - Small changes to the size of the input do not typically induce enormous changes to the overall time.
- Exponential functions scale terribly
  - Small changes to the size of the input induce huge changes in the overall runtime.

A language L can be **decided efficiently** iff there is a TM that decides it in polynomial time.

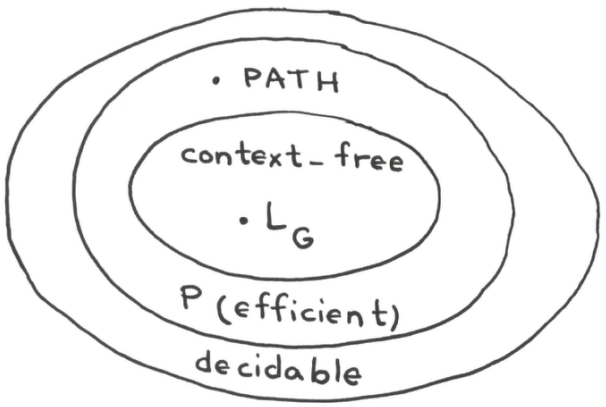
The **time complexity class** $TIME(f(n))$ is the set of languages decidable by a single-tape TM with run-time $O(f(n))$

P is the class of languages that are decidable in polynomial time on a deterministic single-tape TM

$$P = \bigcup_k TIME(n^k)$$

① P is invariant for all models of computation that are polynomially equivalent to the deterministic single-tap TM

② P roughly corresponds to the class of problems that are realistically solvable on a computer.



$PATH = \{\langle G, s, t\rangle \mid$ Graph G has a path from node s to node t$\}$

$L_G = \{w \mid$ CFG G generates $w\}$

$PATH = \{\langle G, s, t\rangle \mid$ Graph G has a path from node s to node t$\}$

M = On input $\langle G, s, t \rangle$

where G is a graph with nodes s and t

Place a mark on node s

Repeat until no additional nodes are marked : $O(n)$

Scan the edges of G $O(m)$

If some edges has both marked and unmarked endpoints

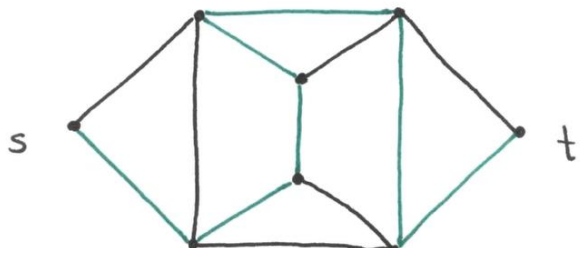
Mark the unmarked endpoint

If t is marked, accept

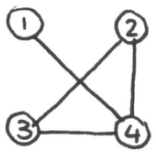
(G has n vertices, m edges) running time: $O(mn)$

A Hamiltonian path in G is a path that visits every node exactly once

$$HAMPATH = \left\{ \langle G, s, t \rangle \mid \begin{array}{c} \text{Graph G has a Hamiltonian path from} \\ \text{node s to node t} \end{array} \right\}$$



We do not know if HAMPATH is in P, we believe it is not !



* Clique : subset of vertices that are pairwise adjacent

(induced subgraph is complete)

$\{1,4\}$ $\{2,3,4\}$ $\{1\}$ are cliques

* Independent set : subset of vertices that are pairwise non-adjacent

$\{1,2\}$ $\{1,3\}$ $\{4\}$ are independent sets

* Vertex cover : subset of vertices that touches (covers) all edges

$\{2,4\}$ $\{3,4\}$ $\{1,2,3\}$ are vertex covers

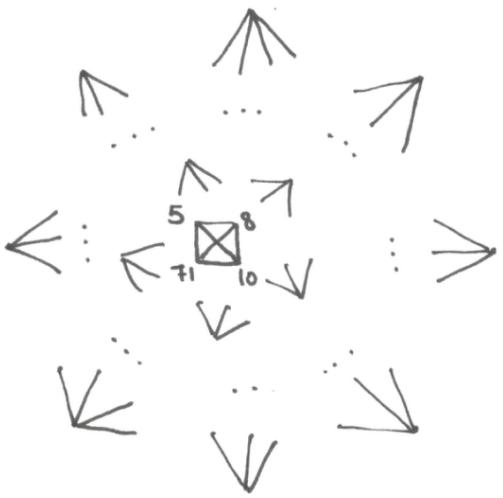
$$\text{CLIQUE} = \{\langle G, k\rangle \mid G \text{ is a graph having a clique of } k \text{ vertices}\}$$

$$\text{INDEPENDENT-SET} = \{\langle G, k\rangle \mid G \text{ is a graph having an independent set of } k \text{ vertices}\}$$

$$\text{VERTEX-COVER} = \{\langle G, k\rangle \mid G \text{ is a graph having a vertex cover of } k \text{ verties}\}$$

What do these problems have in common?

1. Given a candidate solution, we can quickly check if it is valid
2. We don't know how to solve these problems quickly



Is $\langle G, 4\rangle \in$ CLIQUE?

Candidate solution : $\{5, 8, 10, 71\}$

Question: Does graph G have a clique of size k?

M = "On input $\langle G, k\rangle$:

- Non-deterministically guess a k-subset of nodes of G
- Deterministically check whether it is complete
- If so, accept; if not, reject."

Recall. NTM has the following pattern:

M = "On input w :

- Nondeterministically guess some string x,
- Deterministically check whether x solves w.
- If so, accept; otherwise, reject."

The complexity class NP (non-deterministic polynomial time) contains all problems that can be solved in polynomial time by a single tape NTM

Alternative way to characterize the NP class:

A polynomial-time verifier is a deterministic TM of the form:

V = "On input $\langle w, x \rangle$:

- Deterministically check whether x solves w.
- If so, accept; otherwise, reject."

such that V runs in polynomial time in the length of w (not x)

(The string x is called a certificate or a witness for w)

NP: class of languages that have polynomial-time verifiers

CLIQUE is in NP:

V = "On input $\langle G, k \rangle$, a set of nodes C,

If C has size k and all edges between nodes C are present in G

accept

Otherwise reject"

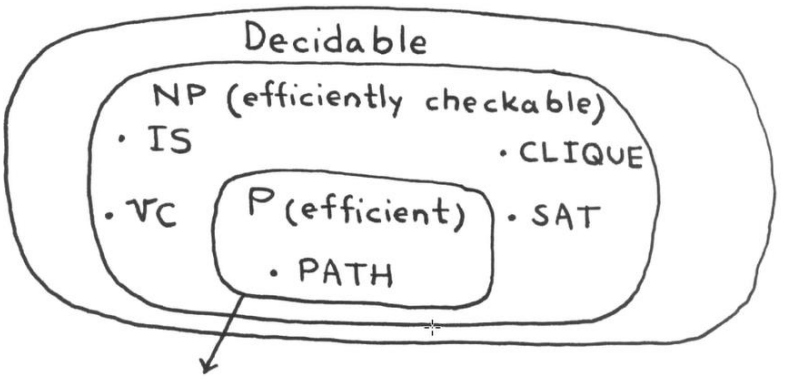
Running time: $O(k^2)$

P is contained in NP: because verifier can ignore candidate solution

IS: INDEPENDENT SET

VC: VERTEX COVER

(will see SAT later)



can we remove this line? Is P = NP?

We do not know (yet!). One reason to believe P ≠ NP is that intuitively searching for a solution is harder than verifying its correctness.

For example, solving homework problems (searching for solutions) is harder than grading (verifying the candidate solution is correct) معمّا چو حل گشت آسان شود

CLIQUE $= \{\langle G, k\rangle \mid G \text{ is a graph having a clique of } k \text{ vertices}\}$

M = On input $\langle G, k\rangle$:

For all subsets S of vertices of size k

If every pair $u, v \in S$ are adjacent

accept

else reject

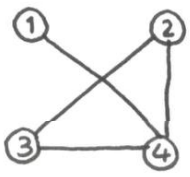
Example.

input : $\langle G, 3\rangle$

subsets : $\{1,2,3\}$ $\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$

× × × ✓

Graph G :



Running time analysis

M = On input $\langle G, k\rangle$:

For all subsets S of vertices of size k — $\binom{n}{k}$ subsets

If every pair $u, v \in S$ are adjacent — k^2 pairs

accept

else reject

---

running time : $k^2 \binom{n}{k}$

According to Stirling's formula $k^2 \binom{n}{k} \geq 2^n$ when $k = \frac{n}{2}$

We strongly suspect that problems like CLIQUE, IS, VC require roughly 2^n time to solve

We do not know how to prove this, but we can prove :

If any one of them can be solved efficiently,

then all of them can be solved efficiently

Next lecture: All problems such as CLIQUE, IS, VC are as hard as one another.

Moreover, they are at least as hard as any problem in NP !

$P = \{ L \mid$ There is a polynomial time decider for $L \}$

$NP = \{ L \mid$ There is a non-deterministic polynomial-time decider for $L \}$

$P \stackrel{?}{=} NP$ Most important question in theoretical computer science

If a solution to a problem can be verified efficiently,
can that problem be solved efficiently?

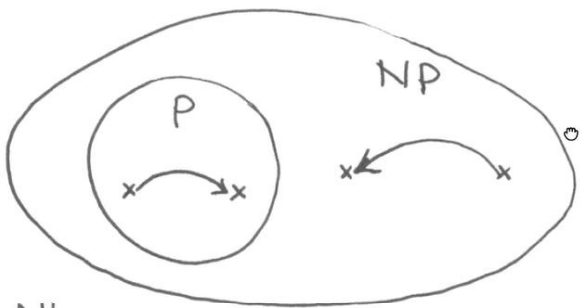
A polynomial-time reduction from L_1 to L_2 is a function $f: \Sigma^* \to \Sigma^*$ such that

- for any $w \in \Sigma^*$, $w \in L_1$ iff $f(w) \in L_2$
- the function f can be computed in polynomial time.

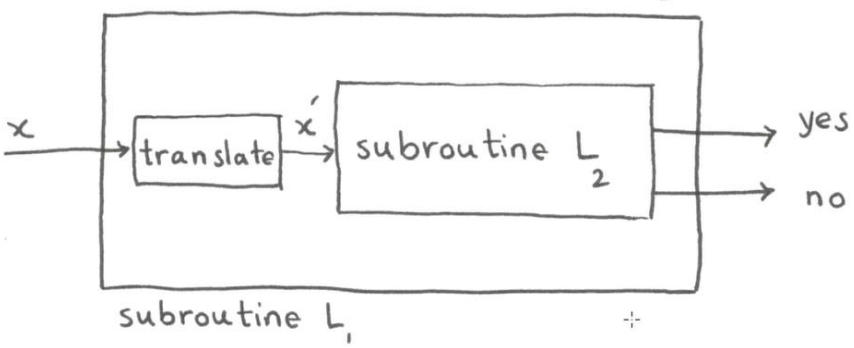
> If $L_1 \leq_P L_2$ and $L_2 \in P$, then $L_1 \in P$

Contrapositive: If $L_1 \leq_P L_2$ and $L_1 \notin P$, then $L_2 \notin P$

> If $L_1 \leq_P L_2$ and $L_2 \in NP$, then $L_1 \in NP$



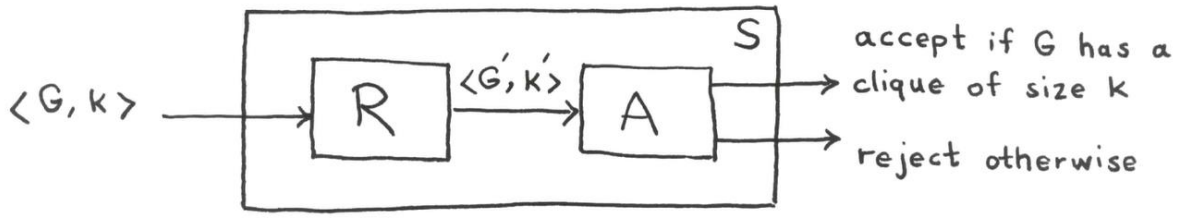
Reduce L_1 to L_2: Allows us to solve L_1 in poly-time if we have a poly-time solver for L_2



Theorem. CLIQUE $\leq_P$ INDEPENDENT-SET

Suppose IS is decided by a poly-time TM A.

We want to build a TM S that uses A to solve CLIQUE.

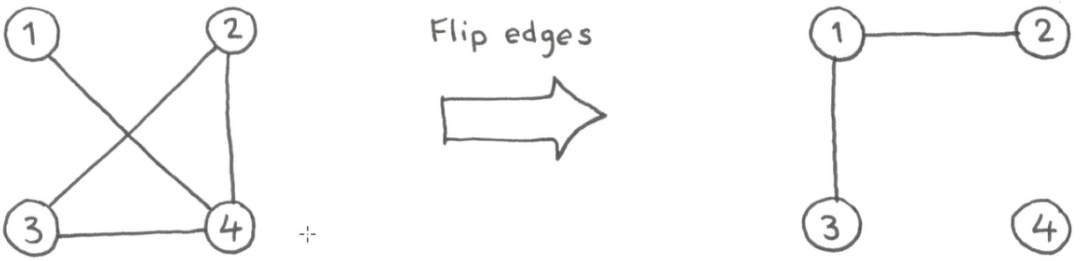


We look for a polynomial-time Turing machine R that turns question

"Does G have a cliaue of size k?"

into

"Does G′ have an independent set (IS) of size k′?"



Graph G Graph G′

clique of size k $k = k'$ IS of size k′

Cliques in G ⟷ Independent sets in G′

* If G has a clique of size k,
  then G′ has an independent set of size k
* If G′ does not have a clique of size k,
  then G′ does not have an independent set of size k

On input ⟨G, k⟩:

Construct G′ by flipping all edges of G

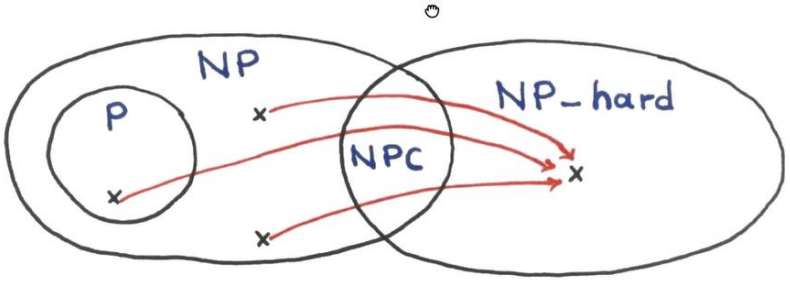
Set $k' = k$

Output ⟨G′, k′⟩

A language L is called NP-hard iff for every $L' \in NP$: $L' \leq_p L$

A language L is called NP-complete iff L is NP-hard and $L \in NP$

The class NPC is the set of NP-complete problems



NPC: subclass of NP which contains the hardest problems of NP
If you can efficiently solve one NP-complete problem, then you can efficiently solve all NP problems

If you find a polynomial time solution to an NP-complete problem:

- You have a bug
- Get ready to receive 1000,000 $!

The first NP-complete problem found was the satisfiability problem.

A Boolean formula is an expression made of variables, ANDs, ORs, and negation, like:

$$\varphi = (x_1 \vee \bar{x}_3) \wedge (x_2 \vee x_1 \vee x_3) \wedge (\bar{x}_1)$$

Task: Assign True / False values to variables so that the formula evaluates to True

$$x_1 = \text{False} \quad x_2 = \text{True} \quad x_3 = \text{False}$$

$$SAT = \{ \langle \varphi \rangle \mid \varphi \text{ is a satisfiable Boolean formula} \}$$

Theorem (Cook-Levin) : SAT is NP-complete.

Finding a solution :

Try all possible assignments

FFF FFT FTF FTT
TFF TFT TTF TTT

For n variables, there are 2^n possible assignment (Takes exp time)

Verifying a solution :

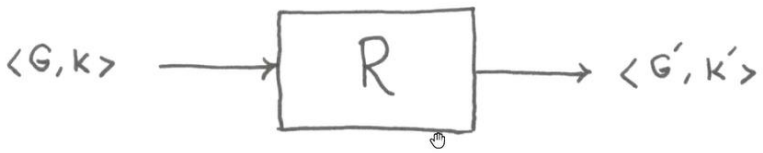
substitute $x_1 = F$, $x_2 = T$, $x_3 = F$

evaluate the formula:

$\varphi = (x_1 \vee \bar{x}_3) \wedge (x_2 \vee x_1 \vee x_3) \wedge (\bar{x}_1)$

This can be done in linear time.

Theorem : $\text{INDEPENDENT\_SET} \leq_P \text{VERTEX\_COVER}$

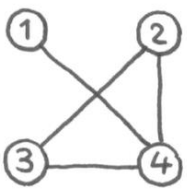


G has an IS of size k $\longleftrightarrow$ G' has a VC of size k'

Example :

Independent sets:

$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$,
$\{1,2\}$, $\{1,3\}$



vertex covers:

$\{2,4\}$, $\{3,4\}$,
$\{1,2,3\}$, $\{1,2,4\}$,
$\{1,3,4\}$, $\{2,3,4\}$,
$\{1,2,3,4\}$

Claim: S is an independent set if and only if $\bar{S}$ is a vertex cover

Proof :

S is an independent set

$\Updownarrow$

no edge has both endpoints in S

$\Updownarrow$

every edge has an endpoint in $\bar{S}$

$\Updownarrow$

$\bar{S}$ is a vertex cover

IS	VC
$\emptyset$	$\{1,2,3,4\}$
$\{1\}$	$\{2,3,4\}$
$\{2\}$	$\{1,3,4\}$
$\{3\}$	$\{1,2,4\}$
$\{4\}$	$\{1,2,3\}$
$\{1,2\}$	$\{3,4\}$
$\{1,3\}$	$\{2,4\}$

Reducing IS to VC

R : On input ⟨G, k⟩

Output ⟨G, n-k⟩

G has an IS of size k ⟷ G has a VC of size n-k

Overall sequence of reductions :

$$SAT \rightarrow 3SAT \rightarrow CLIQUE \rightarrow IS \rightarrow VC$$

What is 3SAT ?

First some definitions :

- A **literal** in a Boolean formula is a variable or its negation

  x , $\bar{y}$ are literals

  $x \wedge y$ is not

- A **clause** is disjunction (OR) of literals.

  $(\bar{x} \vee y \vee z)$, $\bar{x}$ are clauses

  $x \vee \overline{(y \vee z)}$ is not

- A Boolean formula is in **conjuctive normal form (CNF)** if it is the AND of clauses.

  $(x \vee y \vee z) \wedge (\bar{x} \vee y) \wedge (x)$ is in CNF

  $(x \vee (y \wedge z)) \wedge (\bar{x} \vee \bar{y})$ is not in CNF

- A Boolean formula is in 3-CNF if

  * It is in CNF
  * Every clause has exactly three literals

Example: $(x \vee x \vee y) \wedge (y \vee \bar{y} \vee z) \wedge (x \vee w \vee z)$

But not: $(x \vee y \vee z \vee w) \wedge (x \vee y \vee z)$

The language 3SAT is defined as:

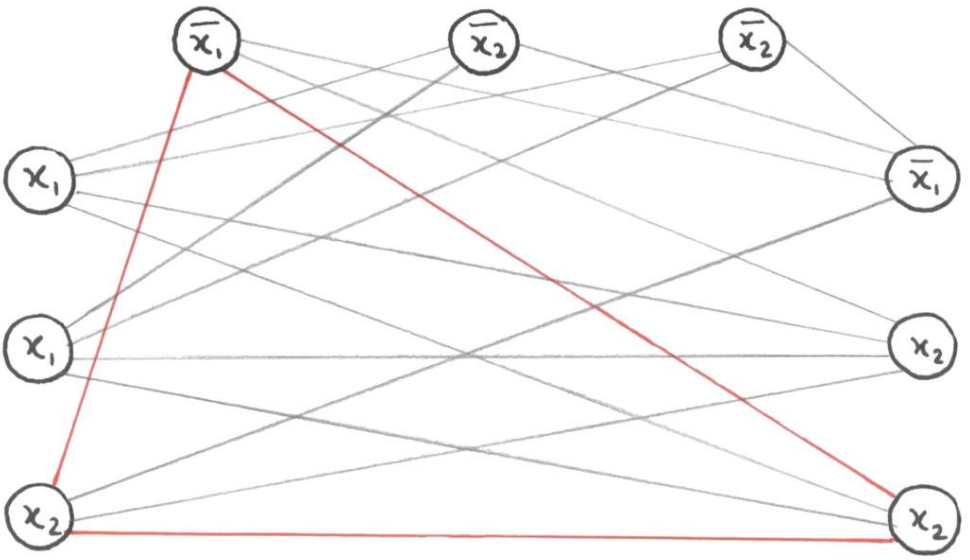
$$3SAT = \{ \langle \varphi \rangle \mid \varphi \text{ is a satisfiable 3-CNF formula} \}$$

Theorem: $3SAT \leq_p CLIQUE$

Transform a 3-CNF formula φ into $\langle G, k \rangle$ such that

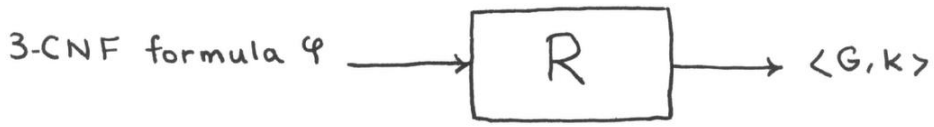
$$\varphi \in 3SAT \Leftrightarrow \langle G, k \rangle \in CLIQUE$$

Example: $(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$



Put edges between all pairs of nodes in different clusters, except pairs of the form $\{x, \overline{x}\}$ (inconsistent pairs)

Do not connect any nodes in the same cluster



R: On input φ, where φ is a 3CNF formula with k clauses

Construct the following graph G:

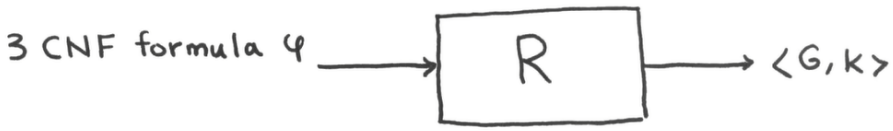
G has 3k vertices, divided into k groups

One for each literal occurrence in φ

If vertices u and v are in different groups and consistent

Add an edge $\langle u, v \rangle$

Output $\langle G, k \rangle$



Every satisfying assignment of φ gives a clique of size k in G

Conversely, every clique of size k in G gives a satisfying assignment of φ

Overall sequence of reductions:

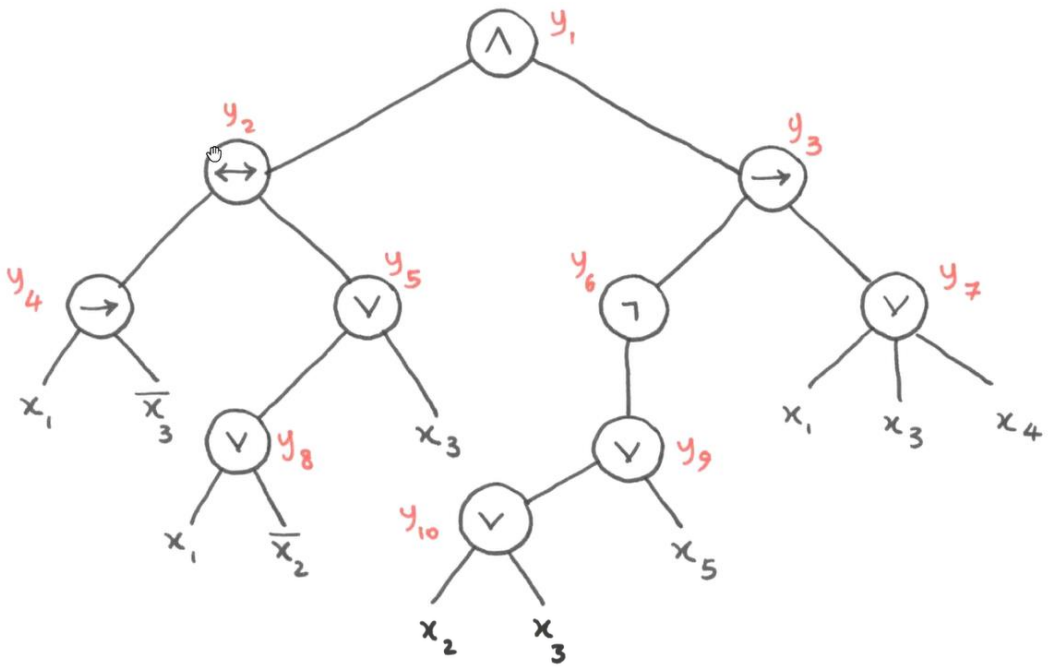
$$SAT \rightarrow 3SAT \rightarrow CLIQUE \rightarrow IS \rightarrow VC$$

Convert arbitrary Boolean formula φ into formula φ^* in 3-CNF

* Straightforward approach may give exponential size φ^*

First. Convert formula to tree representation

$$((x_1 \rightarrow \overline{x_3}) \leftrightarrow (x_1 \vee \overline{x_2} \vee x_3)) \wedge ((\overline{x_2 \vee x_3 \vee x_5}) \rightarrow (x_1 \vee x_3 \vee x_4))$$



Second.

* Introduce extra variables y_i for output of every internal node
* Write formulas for relations between variables

$$y_1 \leftrightarrow (y_2 \wedge y_3)$$

$$y_2 \leftrightarrow (y_4 \leftrightarrow y_5)$$

$$y_3 \leftrightarrow (y_6 \rightarrow y_7)$$

$$\vdots$$

Third. Write formula to express satisfiability of the whole thing

y_1 ↑ final output

$$\wedge \underbrace{(y_1 \leftrightarrow (y_2 \wedge y_3))}_{\text{output of nodes consistent with children}} \wedge \underbrace{(y_2 \leftrightarrow (y_4 \leftrightarrow y_5))}_{\text{output of nodes consistent with children}} \wedge \underbrace{\cdots}_{\text{output of nodes consistent with children}}$$

Fourth. Rewrite each clause into CNF-form

y_1	y_2	y_3	$y_1 \leftrightarrow (y_2 \wedge y_3)$
1	1	1	1
1	1	0	0
1	0	1	0
	...		

clause is equivalent to

$$\overline{\left((y_1 \wedge y_2 \wedge \bar{y}_3) \vee (y_1 \wedge \bar{y}_2 \wedge y_3) \vee \cdots\right)}$$

use De Morgan $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

$$(\bar{y}_1 \vee \bar{y}_2 \vee y_3) \wedge (\bar{y}_1 \vee y_2 \vee \bar{y}_3) \wedge \cdots$$

After the fourth step we have formula in CNF-form

$$y_1 \wedge (\bar{y}_1 \vee \bar{y}_2 \vee y_3) \wedge (\bar{y}_1 \vee y_2 \vee \bar{y}_3) \wedge \cdots$$

But some clauses have only one or two literals

Fifth. Add extra variables and use them to fill up these clauses

Example. Use extra variables p, q to replace y_1 by

$$(y_1 \vee p \vee q) \wedge (y_1 \vee \bar{p} \vee q) \wedge (y_1 \vee p \vee \bar{q}) \wedge (y_1 \vee \bar{p} \vee \bar{q})$$

After fifth step we have

- formula φ^* in 3-CNF form
- that is satisfiable if and only if the original formula φ is satisfiable
- and conversion can be done in polynomial-time

Overall Sequence of reductions.

$$SAT \xrightarrow{\checkmark} 3SAT \xrightarrow{\checkmark} CLIQUE \xrightarrow{\checkmark} IS \xrightarrow{\checkmark} VC$$