MSCE MATHEMATICS: MODEL ANSWERS BY TOPIC

FORM 3 & 4

MICHAEL MASIYA
KELVIN MALUNGA

SECOND EDITION

Masiya Books

Blantyre

Email: mscemodelmath@gmail.com

Phone/WhatsApp: 0995822298 or 0887616933.

Availability in Bookshops: Bookmate-BT, Bookworm-LL, Mzuzu Book Centre-MZ, Fegs-Zomba, God of Miracles-BT, Claim-BT.

Typesetting: Michael Masiya

Layout & Chapter Highlights: Kelvin Malunga

Cover Design: Patrick Kupatsa (patrickkupatsa@gmail.com)

Circle Geometry Construction: Josiah Mwachande Banda (josiah.b62@gmail.com)

Printed in Malawi by Kris Offset & Screen Printers Ltd

First published 2021

ISBN 978 999 080 8926

FOREWORD

MSCE Mathematics Model Answers By Topic (1990-2021) is a timely response to Malawi's Vision 2063 which was launched on Tuesday, January 19, 2021 by The State President of the Republic of Malawi, His Excellency Dr. Lazarus McCarthy Chakwera. Part of the Vision is to enhance Science, Technology, Engineering, Arts and Mathematics (STEAM) courses as key to strengthening innovation and growth creation. This is ultimately expected to transition Malawi into an upper-middle income economy- an inclusively wealthy and self-reliant nation.

Our book fits into the Agenda 2063 by ensuring that students improve scores in Mathematics, one of the STEAM courses. In 2019, it was worrisome to observe a 13 percent drop in the pass-rate of MSCE against the previous years. Subsequently, the 2019 MSCE Chief Examiner's report for Mathematics also bemoaned the poor performance of students in the subject and cited several areas for improvement including: Mathematics clubs in schools, solving in teams, and practicing solving and following instructions for Mathematics questions. Hence, this model answers book can help attain these suggested recommendations by being one of the core books in Mathematics clubs and by demonstrating how to approach questions at MSCE-level. We, therefore, do hope that this book is one of the ingredients that will help improve the pass-rate of the MSCE in the country, and going forward, be a catalyst for uptake of Science courses of which Mathematics is a pre-requisite.

AUTHORS' NOTE

What makes Mathematics interesting is the fact that *"...there are many roads that lead to Rome..."* While we provide what we term as "suggested solutions" in this book, these solutions are not an end in themselves as there can always be an alternative and/or better solution to any Mathematical problem. For instance, if you are asked to rationalize the denominator in $\frac{2}{\sqrt{2}}$.

This is how you could go about it:

I. $$\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{\left(\sqrt{2}\right)^2} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$$

II. $$\frac{2}{\sqrt{2}} = \frac{\left(\sqrt{2}\right)^2}{\sqrt{2}} [\text{ since } \left(\sqrt{a}\right)^2 = a] \Rightarrow \frac{\cancel{\sqrt{2}} \times \sqrt{2}}{\cancel{\sqrt{2}}} = \sqrt{2}$$

III. $$\frac{2}{\sqrt{2}} = \frac{2^1}{2^{1/2}} = 2^{1-\frac{1}{2}} = 2^{1/2} = \sqrt{2}$$

See? Thus, we cannot in this book, due to time, space limitations, or otherwise, provide all possible solutions to a single problem. Hence, with your confidence, we believe you can always identify better solutions. In our experience as teachers, we have at times been wowed by some exceptional and "out of this world" solutions of some students such that we also end up learning! If in the course of solving, you are able to identify a better solution or propose a correction, please do not hesitate to whatsapp/email us and we shall incorporate your solution and where possible, make due acknowledgement: For the First edition, we thank Pastor Lunduka of Nsanje District for his input. We can be reached through Whatsapp numbers 0995822298 or 0887616933. Our email is: mscemodelmath@gmail.com. Enjoy!

KEY FEATURES:

- *Topic by topic arrangement of problems and solutions.*
- *Detailed explanations to the procedures taken in solving.*
- *'Chapter highlights' that outline important concepts in each chapter.*

HOW TO USE THIS BOOK:

- The problems are to be attempted as follow up exercises applying what has already been learnt. We therefore recommend that the student put diligent effort in understanding the concepts before attempting these questions.
- The problems should be attempted before checking the solutions sections.
- The book should be used as an aid for developing mathematical proficiency and problem–solving skills in readiness for MSCE rather than a core text for learning Mathematics at MSCE.
- To get the full benefit of the book, it is essential to regularly and repeatedly engage with the questions. We encourage students to practice Math using the book throughout the school term, rather than cramming when exams are approaching.

WHY USE THIS BOOK?

- The approach taken in this book not only helps students easily recognize appropriate procedures in tackling problems but also provide comments on why such methods and corresponding steps are taken.
- We kindly appreciate suggestions and comments for improvements on solutions in future editions.

ACKNOWLEDGEMENTS

Acknowledgements from Us: We would like to recognize the following individuals who contributed directly to the book project. First, the following distinguished individuals who typed our manual solutions into the computer and drew all the figures: Immaculata Triza Kamalizeni, Sileni Thando Sabvala, Jailosi Phiri, Horristian Karieka, Precious Zimbiri, and Francis Laurent. The following individuals provided other forms of indirect support to the project including transportation, workspace & other forms of facilitation: Mr. Kenneth Malunga Snr, Ulemu Masiya, Mwai Jepter, Grant Mark & Tadala Simbeye. For the design of the book cover, we would like to acknowledge the efforts and talent of Mr. Patrick Kupatsa. We also benefitted from advisory services from Mr. Sungeni Segula and Mr. Martin Masiya on Social Media Marketing which will form a crucial stage of the project in the near future.

Acknowledgements from Michael: To God Almighty for the gift of life and wisdom. Over the years, I have grown to appreciate the presence of God and inspiration and how it manifests in the strong passion and ability for Math. May His Name forever be glorified!

I would like to thank Mr. Kingsley Jika of Chanco Publications who first believed in the author in me. He encouraged me to write and constantly reminded me of what authorship entails. I was able to undertake this project because of his initial trust and encouragement.

I am very grateful to have partnered with Kelvin on this project. It has not been easy to find such a synergy and joint passion for Math! I would also like to recognize initial input of Dr John Mbotwa and Paul Mlaviwa into the early stages of the project.

To my family…I have taken immense time away to put this together. I continually thank Ulemu for the endless support and everytime I felt warm and relaxed being welcomed home into the joyful arms and excitement of Ephraim and Elena. These dear three always kept me eager to return home. They say, "East or West, home is best" indeed.

I would not have been able to solve the simplest of an equation if not for my late grandmother, Mrs. Patricia E. Kasongo (Nee Mbano), who taught me Mathematics from a tender age. May her soul continue to rest in peace. To my mothers: Joanita, Gloria, Sandra, Mariette & Hannah- I remain grateful for your care. I would also do injustice not to mention two great teachers I met in my academic journey who intensified my Mathematical acumen: the late Mr. Kamwendo who taught me in Form 4 at Pius XII Minor Seminary, and Prof. JJ. Namangale at Chancellor College. These two have been icons who have sharpened me along the way.

Acknowledgements from Kelvin: I greatly express my gratitude to my mother, Nora Chimera Malunga, for her visible and tangible love – thank you for overflowing with love and care – it makes me endure and triumph. And to my father, Kenneth Bauleni Malunga, all appreciation is due for being the biggest advocate of mathematics I know – indeed, as you've always put it, "mathematics is the simplest subject."

Special appreciation goes to Alice Malunga for always being an unshakable believer of possibilities and for never running short of motivation - keep aiming for the highest Allie Girl. Elizabeth Kapusa, Elias Seda, Steve Nyirenda, Patrick Munyongah and Faith Nkhoma, I individually thank you for your brilliant inputs in this work.

Kenneth Malunga Jr and Karen Malunga - you guys are great brains. You guys are a force to be reckoned with.

Most of all, my involvement in this project would not have been possible without great and unforgettable mathematics teachers who helped me see my potential: Mrs Mpirangwe and Mrs Kaliya.

TABLE OF CONTENTS

CH 1 FACTORIZATION

Chapter Highlights

In this chapter, we solve problems on factorization of quadratic expressions. The concept of factorization rewrites a quadratic expression into a product of its factors.

A student must be able to:

- Factorize using common factor
- Factorize by inspection
- Identify multiples and factors in a quadratic expression.
- Factorize the difference of two squares.

Binomials

- Factor out the greatest common factor
- Check if it is the difference of squares

i.e. $x^2 - y^2 = (x-y)(x+y)$

Trinomials

- $ax^2 + bx + c$

Multiply $a \times c$ then list all integers that multiply to give the number $a \times c$. Then choose a pair that adds or subtracts to give the middle number b. Rewrite the middle term bx as the sum of these two terms. Divide the new expression into two pairs and factor out the common term.

- Some problems require us to group expressions i.e. $ax + by + mx + my$

1. Factorize completely $ac - 2ab - 3bc + 6b^2$. **[1990 Algebra #2a]**
2. Factorize completely $3y^2 + 17y + 10$. **[1990 Algebra #2b]**
3. Factorize completely $3 - 3k + 7k - 7k^2$. **[1991 Algebra #2a]**
4. Factorize completely $2m^3 + 3m^2 - 2m$. **[1991 Algebra #2a]**
5. Factorize $x^2(x-1) - x + 1$. **[1992 Algebra #3a]**
6. Factorize completely $rd^2 - r$. **[1993 Algebra #2a]**
7. Factorize completely $5h^2 + 4h - 28$. **[1993 Algebra #2b]**
8. Factorize completely $3(x-y) - n(y-x)$. **[1993 Algebra #2c]**
9. Factorize completely $ax - 2a - bx + 2b$. **[1994 Algebra #2a]**
10. Factorize completely $3 + 5x - 2x^2$. **[1994 Algebra #2b]**
11. Factorize completely $(t^2-1)^2 - 9$. **[1994 Algebra #2c]**
12. Factorize completely $2x^2 - 9x + 10$. **[1995 Algebra II #2a]**
13. Factorize completely $(2a-b)^2 - 2a + b$. **[1995 Algebra #2b]**
14. Factorize completely $81x^4 - y^4$. **[1995 Algebra #2c]**
15. Factorize completely $9Kt^2 - K$. **[1996 P1 #4a]**
16. Factorize completely $3y^2 - y - 2$. **[1996 P1 #4a]**
17. Factorize completely $x^2 - 2xy - px + 2py$. **[1997 P1 #4a]**
18. Factorize completely $\frac{1}{4} - a^2$. **[1997 P1 #4b]**
19. Factorize: $x^2t - t^3$. **[1998 P1 #7 mod]**
20. Factorize $2x^2 - 98$. **[1999 P1 #3a]**
21. Factorize completely $2x^2 + x - 6$. **[1999 PII #2b(i)]**
22. Factorize completely $rx - ex + ex^2 - rx^2$. **[1999 PII #2b(ii)]**
23. Factorize completely $(p-q)^2 - 4$. **[2000 P1 #14]**
24. Factorize completely $18m^4 - 2n^2$. **[2001 P1 #2]**
25. Factorize completely $1 - 16(1-y)^2$ in its simplest form. **[2002 P1 #6]**
26. Factorize completely $x^2 + 3x + 4(x+3)$. **[2003 PI #1]**
27. Factorize completely $a^2 + 2ab + b^2 - 4$. **[2003 PII #4b]**
28. Factorize completely $6 + x - 2x^2$. **[2004 P1 #1]**
29. Factorize completely $3t^2 - 4t + 1$. **[2005 P1 #1]**
30. Factorize completely $2x^2 - 4x - 126$. **[2007 PI #3]**
31. Factorize completely $3x^2y + 5xy + 2y$ **[2008 P1 #1]**
32. Factorize completely $2x^2 + 3xy - 35y^2$ **[2010 P1 #1]**

33. Factorize completely $3-5x-2x^2$.
[2011 PI #1]
34. Factorize $2y^2-18$. [2011 PII #1a]
35. Factorize $2m^2n-3mn-5n$ completely.
[2012 P1 #2]
36. Factorise completely $35-30a-5a^2$.
[2013 P1 #1]
37. Factorise completely $2x^2-7x-4$.
[2014 P1 #1]
38. Factorise $12x^2+11x-5$. [2015 PI #1]
39. Factorise completely $4d^2f-22df+10f$.
[2016 PI #1]
40. Factorize completely $1-p-2p^2$. [2017 PI #1]
41. Factorize completely $5x^2-13x-6$.
[2019 PI #1]
42. Factorise completely $a^3b(1-x)+ab(x-1)$.
[2020 Mock PI #1]
43. Factorize completely $5y^2-1\frac{1}{4}$.
[2021 Mock PI #1]

1. **[1990 Algebra #2a]**
$= a(c-2b)-3b(c-2b)$
$= (c-2b)(a-3b)$

2. **[1990 Algebra #2b]**
Factors: $3\times 10 = +30 \Rightarrow +2+15=+17$
$= 3y^2+15y+2y+10$
$= 3y(y+5)+2(y+5)$
$= (3y+2)(y+5)$

3. **[1991 Algebra #2a]**
$= 3(1-k)+7k(1-k)$
$= \cancel{(3+7k)}(3+7k)(1-k)$

4. **[1991 Algebra #2a]**
$= m(2m^2+3m-2)$ since m is common
Factors: $2\times -2 = -4 \Rightarrow +4-1=+3$
$= m(2m^2+4m-m-2)$
$= m\left[2m(m+2)-1(m+2)\right]$
$= m\left[(2m-1)(m+2)\right]$

5. **[1992 Algebra #3a]**
$= x^2(x-1)-1(x-1)$
$= (x-1)(x^2-1)$
$= (x-1)(x-1)(x+1)$ diff.of 2sq.

6. **[1993 Algebra #2a]**
$= r(d^2-1)$
$= r(d-1)(d+1)$ diff.of 2sq.

7. **[1993 Algebra #2b]**
Factors: $5\times -28 = -140 \Rightarrow +14, -10 = +4$
$= 5h^2+14h-10h-28$
$= h(5h+14)-2(5h+14)$
$= (5h+14)(h-2)$

8. **[1993 Algebra #2c]**
$= 3(x-y)-n(-1)(x-y)$
$= 3(x-y)+n(x-y)$
$= (x-y)(3+n)$

9. **[1994 Algebra #2a]**
$= a(x-2)-b(x-2)$
$= (x-2)(a-b)$

10. **[1994 Algebra #2b]**
Factors: $3\times -2 = -6 \Rightarrow +6-1=+5$
$= 3+6x-x-2x^2$
$= 3(1+2x)-x(1+2x)$
$= (1+2x)(3-x)$

11. **[1994 Algebra #2c]**
$= (t^2-1)^2-3^2$
$= \left[(t^2-1)-3\right]\left[(t^2-1)+3\right]$ diff.of 2sq
$= (t^2-4)(t^2+2)$ simplifying
$= (t-2)(t+2)(t^2+2)$ diff.of 2sq

12. **[1995 Algebra II #2a]**
Factors: $2\times 10 = +20 \Rightarrow -5-4=-9$
$= 2x^2-5x-4x+10$
$= x(2x-5)-2(2x-5)$
$= (2x-5)(x-2)$

13. **[1995 Algebra #2b]**
$= (2a-b)^2-2a+b$
$= (2a-b)^2-1(2a-b)$
$= (2a-b)[(2a-b)-1]$
$= (2a-b)(2a-b-1)$

14. **[1995 Algebra #2c]**
$= 81x^4-y^4$
$= \left(9x^2-y^2\right)\left(9x^2+y^2\right)$ diff.of 2sq
$= \left(3x-y\right)\left(3x+y\right)\left(9x^2+y^2\right)$ diff.of 2sq

15. **[1996 P1 #4a]**
$= K\left(9t^2-1\right)$ K is common
$= K\left(3t-1\right)\left(3t+1\right)$ diff.of 2sq

16. **[1996 P1 #4a]**
Factors: $3\times-2=-6\Rightarrow-3+2=-1$
$=3y^2-3y+2y-2$
$=3y(y-1)+2(y-1)$
$=(y-1)(3y+2)$

17. **[1997 P1 #4a]**
$=x^2-2xy-px+2py$
$=x(x-2y)-p(x-2y)$
$=(x-p)(x-2y)$

18. **[1997 P1 #4b]**
$=\frac{1}{4}-a^2$
$=\left(\frac{1}{2}\right)^2-a^2$ diff.of 2sq
$=\left(\frac{1}{2}-a\right)\left(\frac{1}{2}+a\right)$

19. **[1998 P1 #7 mod]**
$=x^2t-t^3$
$=t\left(x^2-t^2\right)$ t is common
$=t(x+t)(x-t)$ diff. of 2sq

20. **[1999 P1 #3a]**
$=2\left(x^2-49\right)$
$=2(x-7)(x+7)$ diff.of 2sq

21. **[1999 PII #2b(i)]**
Factors: $2\times-6=-12\Rightarrow+4-3=+1$
$=2x^2+x-6$
$=2x^2+4x-3x-6$
$=2x(x+2)-3(x+2)$
$=(x+2)(2x-3)$

22. **[1999 PII #2b(ii)]**
$=rx-ex+ex^2-rx^2$
$=x(r-e)-x^2(r-e)$
$=\left(x-x^2\right)(r-e)$
$=x(1-x)(r-e)$ taking x out of 1st bracket

23. **[2000 P1 #14]**
$=(p-q)^2-4$
$=(p-q-2)(p-q+2)$

24. **[2001 P1 #2]**
$=18m^4-2n^2$
$=2\left(9m^4-n^2\right)$ common factor
$=2\left(3m^2-n\right)\left(3m^2+n\right)$

25. **[2002 P1 #6]**
$=1-4^2(1-y)^2$
$=\left[1-4(1-y)\right]\left[1+4(1-y)\right]$ diff. of 2sq
$=(1-4+4y)(1+4-4y)$
$=(4y-3)(5-4y)$

26. **[2003 PI #1]**
$=x^2+3x+4(x+3)$
One bracket is already factorized
$=x(x+3)+4(x+3)$
$=(x+4)(x+3)$

27. **[2003 PII #4b]**
$=\left(a^2+2ab+b^2\right)-4$
$=(a^2+ab+ab+b^2)-4$
$=a(a+b)+b(a+b)-4$
$=(a+b)(a+b)-4$
$=(a+b)^2-4$
$=(a+b)^2-2^2$ diff.of 2sq
$=(a+b+2)(a+b-2)$

28. **[2004 P1 #1]**
Factors: $6\times-2=-12\Rightarrow+4-3=+1$
$=6+4x-3x-2x^2$
$=2(3+2x)-x(3+2x)$
$=(3+2x)(2-x)$

29. **[2005 P1 #1]**
Factors: $1\times 3 = +4 \Rightarrow -3-1=-4$
$3t^2-4t+1.$
$3t^2-3t-t+1.$
$3t(t-1)-1(t-1)$
$(t-1)(3t-1)$

30. **[2007 PI #3]**
$2x^2-4x-126$
$=2(x^2-2x-63)$
Factors: $1\times -63=-63 \Rightarrow -9,+7$
$=2(x-9)(x+7)$. By inspection

31. **[2008 P1 #1]**
$3x^2y+5xy+2y$
$=y(3x^2+5x+2)$ common factor
Factors: $2\times 3=+6 \Rightarrow +3,+2=+5$
$=y(3x^2+3x+2x+2)$
$=y[3x(x+1)+2(x+1)]$
$=y(x+1)(3x+2)$

32. **[2010 P1 #1]**
Factors: $2\times -35=-70 \Rightarrow +10-7=+3$
$=2x^2+10xy-7xy-35y^2$
$=2x(x+5y)-7y(x+5y)$
$=(2x-7y)(x+5y)$

33. **[2011 PI #1]**
Factors: $3\times -2=-6 \Rightarrow -6,+1=-5$
$=3-6x+x-2x^2$
$=3(1-2x)+x(1-2x)$
$=(1-2x)(3+x)$

34. **[2011 PII #1a]**
$=2y^2-18$
$=2(y^2-9)$
$=2(y+3)(y-3)$ diff.of 2sq

35. **[2012 P1 #2]**
$=n(2m^2-3m-5)$ common factor
Factors: $2\times -5=-10 \Rightarrow -5,+2=-3$
$=n(2m^2-5m+2m-5)$
$=n[m(2m-5)+1(2m-5)]$
$=n(2m-5)(m+1)$

36. **[2013 P1 #1]**
$35-30a-5a^2$
$=5(7-6a-a^2)$ common factor
$=5(7-7a+a-a^2)$
$=5[7(1-a)+a(1-a)]$
$=5[(7+a)(1-a)]$
$=5(7+a)(1-a)$

37. **[2014 P1 #1]**
$=2x^2-7x-4$
Factors: $2\times -4=-8 \Rightarrow -8,+1=-7$
$=2x^2-8x+x-4$
$=2x(x-4)+1(x-4)$
$=(2x+1)(x-4)$

38. **[2015 PI #1]**
$12x^2+11x-5$
Factors: $12\times -5=-60 \Rightarrow +15-4=+11$
$=(12x^2-4x)+(15x-5)$
$=4x(3x-1)+5(3x-1)$
$=(4x+5)(3x-1)$

39. **[2016 PI #1]**
$=4d^2f-22df+10f.$
$=2f(2d^2-11d+5)$ common factor
Factors: $2\times 5=+10 \Rightarrow -1,-10=-11$
$=2f(2d^2-10d-d+5)$
$=2f[2d(d-5)-1(d-5)]$
$\therefore 2f(2d-1)(d-5)$

40. **[2017 PI #1]**

Factors: $-2\times1=-2\Rightarrow -2+1=-1$

$=1+p-2p-2p^2$

$=1(1+p)-2p(1+p)$

$=(1-2p)(1+p)$

41. **[2019 PI #1]**

$5\times-6=-30\Rightarrow -15+2=-13$

$=5x^2-15x+2x-6$

$=5x(x-3)+2(x-3)$

$=(5x+2)(x-3)$

42. **[2020 Mock PI #1]**

$a^3b(1-x)+ab(x-1)$

$=a^3b(1-x)-ab(1-x)$

$=ab(1-x)\left[(a^2-1)\right]$

$=ab(1-x)(a+1)(a-1)$

43. **[2021 Mock PI #1]**

$5y^2-1\frac{1}{4}$

$=5y^2-\frac{5}{4}$

$=5\left(y^2-\frac{1}{4}\right)$

$=5\left(y+\frac{1}{2}\right)\left(y-\frac{1}{2}\right)$

CH 2
QUADRATIC EQUATIONS

Chapter Highlights

In this chapter we will focus on solving problems on quadratic equations. Quadratic equations are of the form $ax^2+bx+c=0$. When solving these problems, we will employ factor method, completing the square and quadratic formula. We will also formulate quadratic equations from practical problems.

The quadratic formula

For $ax^2+bx+c=0$ where $a\neq 0$:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Completing the square method

Again, for $ax^2+bx+c=0$,

1. If $a\neq 1$, divide the quadratic eqtn by a.

2. Write the quadratic in the form

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

3. Add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation.

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2+\frac{c}{a}$$

4. Factor the left side of the equation into perfect squares

$$\left(x+\frac{b}{2a}\right)^2=\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

5. Then express the equation in the form:

$$(x+p)^2-q=0$$

Where $p=\frac{b}{2a}$, and $q=\frac{c}{a}+\left(\frac{b}{2a}\right)^2$

In this new form, the point $(-p,q)$ is the turning point and the equation $x=-p$ is the equation of the line of symmetry.

Usually, when given a quadratic equation and asked to solve the answer to 2 decimal places/estimate the answer to some prescribed significant figures, just know that it cannot be solved by factorization. In that case, consider either completing the square method or using the quadratic formula.

1. The sum of n terms of squares of two consecutive odd numbers is 74. If n is the smaller of the two numbers, find the values of n. **[2004 PII #2]**

2. John is twice as old as Mary. If the sum of the squares of their ages is 125, how old is Mary? **[2006 PI #8]**

3. Solve the equation $2=x^2+10x+21$ giving your answer to correct 2 decimal places. **[2006 PII #12b]**

4. Given that the roots of a quadratic equation $y=x^2+ax+b$ are -3 and 2. Find the values of a and b. **[2007 PII #5b]**

5. Solve the equation $x^2-3x-5=0$, giving your answer correct to two decimal places. **[2007 PII #9a]**

6. Solve the equation $3(a+1)^2-3=0$ **[2008 P1 #10]**

7. Solve the equation $5x^2+8x-2=0$, giving your answer correct 2 decimal places. **[2012 P2 #7a]**

8. Solve the equation $2x^2-5x+1=0$, giving your answer correct to two decimal places. **[2014 P1 #1b]**

9. Solve the equation $x^2-6x+1=0$, giving your answer correct to 2 decimal places. **[2017 PII #8a]**

10. Solve the equation $10-5x-x^2=0$, giving your answer correct to one decimal place. **[2019 PI #8]**

1. **[2004 PII #2]**

Since the odd number is n.
$\therefore$ the larger odd number is $n+2$ *(since odd numbers differ by two)*
Since the sum of square of two consecutive odd numbers is 74 then,

$n^2+(n+2)^2=74$

$n^2+n^2+4n+4=74$

$2n^2+4n+4-74=0$

$2n^2+4n-70=0$

$n^2+2n-35=0$ *(divide each term by 2)*

$(n+7)(n-5)=0$ *(by inspection)*

Either $n+7=0$ or $n-5=0$

$n=-7$ or $n=5$

2. **[2006 PI #8]**

Let Mary's age be x
Then John's age $=2x$
Then,

$(2x)^2+x^2=125$

Solving for x

$4x^2+x^2=125$

$\frac{5x^2}{5}=\frac{125}{5}$

$x^2=25$

$x=\pm\sqrt{25}=\pm5$

since age is always positive $x=5$

$\therefore$ Mary is 5 years old

3. **[2006 PII #12b]**

$x^2+10x+21-2=0$

$x^2+10x+19=0$ which is the form:

ax^2+bx+c where,

$a=1, b=10, c=19$

using the quadratic formula

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-10\pm\sqrt{(10)^2-4(1)(19)}}{2(1)}$

$x=\frac{-10\pm\sqrt{100-76}}{2}$

$x=\frac{-10\pm\sqrt{24}}{2}$

$x=\frac{-10\pm4.8990}{2}$

$x=\frac{-10-4.8990}{2}$ or $\frac{-10+4.8990}{2}$

$x=-2.5505$ or $x=-7.4495$

$x=-2.55$ or $x=-7.45$ (to 2.dec places)

4. **[2007 PII #5b]**

Given the quadratic equation

$y=x^2+ax+b$

since the roots of the equation are -3 and 2

$x=-3$ or $x=2$

So $x+3=0$ or $x-2=0$

Thus, $(x+3)(x-2)=0$ since either equals *zero*

$x(x-2)+3(x-2)=0$

$x^2+x-6=0$

$x^2+x-6\equiv x^2+ax+b$

Thus, $a=1; b=-6$

5. **[2007 PII #9a]**

$x^2-3x-5=0 \equiv ax^2+bx+c=0$

from the equation, $a=1, b=-3, c=-5$.

using the quadratic formula,

$x=\frac{-b\pm\sqrt{b^2-4ac}}{`2a}$

$\therefore x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-5)}}{2(1)}$

$x=\frac{3\pm\sqrt{9+20}}{2}$

$x=\frac{3\pm\sqrt{29}}{2}$

$x=\frac{3+\sqrt{29}}{2}$ or $x=\frac{3-\sqrt{29}}{2}$

$x=\frac{3+5.3852}{2}$ or $x=\frac{3-5.3852}{2}$

$\therefore x=4.1926$ or $x=-1.1926$

$\therefore x=4.19$ or $x=-1.19$ (to 2 dp)

6. **[2008 P1 #10]**

$3(a+1)^2-3=0$

$3(a+1)^2=3$

$\frac{3(a+1)^2}{3}=\frac{3}{3}$

$(a+1)^2=1$

$a+1=\pm\sqrt{1}$

$a=\pm1-1$

$\therefore a=1-1=0$ or $a=-1-1=-2$

$\therefore a=0$ or $a=-2$

7. **[2012 P2 #7a]**

Given $5x^2+8x-2=0$

using formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

where $a=5,\ b=8,\ c=-2$

$\Rightarrow x=\frac{-(8)\pm\sqrt{8^2-4(5)(-2)}}{2(5)}$

$=\frac{-8\pm\sqrt{64+40}}{10}$

$=\frac{-8\pm\sqrt{104}}{10}\quad=\frac{-8\pm10.2}{10}$

$=\frac{-8+10.2}{10}$ or $\frac{-8-10.2}{10}$

$=\frac{2.2}{10}$ or $\frac{-18.2}{10}$

$x=0.22$ or -1.82

8. **[2014 P1 #1b]**

$2x^2-5x+1=0$

Use quadratic formula

$a=2,\quad b=-5,\quad c=1$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{(-5)\pm\sqrt{(-5)^2-4(2\times1)}}{2(2)}$

$=\frac{5\pm\sqrt{25-8}}{4}\quad=\frac{5\pm\sqrt{17}}{4}$

$=\frac{5+4.1231}{4}$ or $\frac{5-4.1231}{4}$

$=\frac{9.1231}{4}$ or $\frac{0.8769}{4}$

$=2.8078$ or 0.2192

$=2.80$ or 0.22 (*to 2 dec. places*)

9. **[2017 PII #8a]**

$x^2-6x+1=0\equiv ax^2+bx+c$

$a=1, b=-6, c=1$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-1)}}{2(1)}$

$x=\frac{6\pm\sqrt{36+4}}{2}$

$x=\frac{6\pm\sqrt{40}}{2}$

Either $x=\frac{6+\sqrt{40}}{2}$ or $x=\frac{6-\sqrt{40}}{2}$

$x=\frac{6+5.6569}{2}$ or $x=\frac{6-5.6569}{2}$

$x=\frac{11.6569}{2}$ or $x=\frac{0.3431}{2}$

$x=5.8285$ or 0.1716

$\therefore x=5.83$ or 0.17 (*2decimal places*)

10. **[2019 PI #8]**

$10-5x-x^2=0$

Re-write as $ax^2+bx+c=0$

$-x^2-5x+10=0$

Using the quadratic formula;

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$a=-1,\ b=-5$ and $c=10$

$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(-1)(10)}}{2(-1)}$

$x=\frac{5\pm\sqrt{25+40}}{-2}$

$x=\frac{5\pm\sqrt{65}}{-2}$

Either $x=\frac{5+\sqrt{65}}{-2}$ or $\frac{5-\sqrt{65}}{-2}$

$x=-6.53112$ or 1.5311

$x=-6.5$ or 1.5

CH 3
IRRATIONAL NUMBERS

Chapter Highlights

Irrational numbers are numbers that cannot be expressed as rational numbers. In this chapter, the irrational numbers we will mostly deal with are surds. Surds are, simply put, expressions that include the square roots $\left(\sqrt{\ }\right)$.

In this chapter, we will solve problems that involve simplifying surds using certain rules.

Rules of surds

We can multiply and divide surds using the following rules:

1. $\sqrt{MN} = \sqrt{M} \times \sqrt{N}$
2. $\sqrt{\dfrac{M}{N}} = \dfrac{\sqrt{M}}{\sqrt{N}}$
3. $\sqrt{M} \times \sqrt{M} = \sqrt{M^2} = M$

Surds can also be a square root of a single number:

4. $M\sqrt{N} = \sqrt{M^2 N}$

Note: Like Surds are surds that have been reduced to have the same radicand (in $\sqrt{a}$, a is the radicand). Like surds can be added and subtracted using the following rules:

5. $M\sqrt{N} + P\sqrt{N} = (M + P)\sqrt{N}$
6. $M\sqrt{N} - P\sqrt{N} = (M - P)\sqrt{N}$

Note that individual surds (such as $\sqrt{75}$) can sometimes be simplified by writing the radicand as a product with a possible perfect square. $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$.

We will also use conjugate surds to simplify i.e. the conjugate of $\left(a + \sqrt{b}\right)$ is $\left(a - \sqrt{b}\right)$.

Rationalizing surds

- $\dfrac{1}{\sqrt{b}}$ multiply both numerator and denominator by $\sqrt{b}$
- $\dfrac{1}{a + \sqrt{b}}$ multiply both numerator and denominator by the conjugate of the denominator i.e. $a - \sqrt{b}$

1. Express $\dfrac{3}{\sqrt{2}}$ as a fraction with a rational denominator. **[2003 P1#3]**
2. Express $\dfrac{1}{3\sqrt{2} - 3}$ with a rational denominator in its simplest form. **[2003 PII #5b]**
3. Simplify $\dfrac{\sqrt{2}}{\sqrt{2} - 1}$. Express your answer with a rational denominator. **[2004 PI #14]**
4. Without using a calculator or a four-figure table, simplify $\sqrt{125} + \sqrt{5} - \sqrt{45}$, leaving your answer in surd form. **[2005 PI #2]**
5. Express $\dfrac{\sqrt{3} + 1}{\sqrt{3} - 1}$ with a rational denominator in its simplest form. **[2005 PII #1b]**
6. Simplify $\dfrac{x^2 - 2}{x - \sqrt{2}}$. **[2006 P1 #23]**
7. Simplify $(\sqrt{2} + \sqrt{3})(\sqrt{8} - \sqrt{12})$ without using a calculator and four figure tables. **[2006 PII #1a]**
8. Express $\dfrac{\sqrt{2}}{\sqrt{3}}$ with a rational denominator. **[2007 PI #1]**
9. Simplify $(2 - \sqrt{7})(2 + \sqrt{7})$ without using a calculator or a four-figure table. **[2007 PII #1a]**
10. Without using a calculator or a four-figure table simplify $\dfrac{15\sqrt{6}}{\sqrt{98} - \sqrt{2}}$ to its simplest form. **[2008 P2 #1b]**
11. Without using a calculator or a four figured table, simplify $\sqrt{27} \times \sqrt{32}$ leaving your answer in surd form. **[2008 P1 #2]**

12. Without using a calculator or four figure tables, simplify $\frac{\sqrt{54}+3\sqrt{3}}{\sqrt{3}}$ in its simplest form.

[2010 P1 #5]

13. Without using a calculator or a four figure table, simplify $\frac{5+\sqrt{3}}{\sqrt{7}+\sqrt{5}}$, leaving your answer with a rational denominator. **[2010 PII #2b]**

14. Without using a calculator or four figured tables, simplify $\frac{1}{\sqrt{2}}-\frac{\sqrt{2}}{3}$ leaving your answer with a rational denominator. **[2011 PI #3]**

15. Simplify $\frac{3\sqrt{2}+1}{\sqrt{3}+\sqrt{6}}$ without using four figure table or a calculator. **[2011 PII #4a]**

16. Without using a calculator or four figured tables, simplify $\frac{8\sqrt{2}}{\sqrt{98}-3\sqrt{2}}$, giving your answer in its simplest form. **[2012 P1 #5]**

17. Without using a calculator or four-figured tables, simplify $\frac{3\sqrt{5}-\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$, leaving your answer with a rational denominator. **[2012 P2 #1a]**

18. Without using calculator or a four figure table, simplify $\frac{\sqrt{12}\times\sqrt{75}\times\sqrt{108}}{9\sqrt{6}}$ leaving your answer with a rational denominator. **[2013 P1 #4]**

19. Without using a four figure table or a calculator, simplify $\sqrt{72}-\sqrt{8}+\sqrt{288}$ leaving your answer in surd form. **[2013 P2 #9a]**

20. Without using a calculator or four figure tables, Simplify $\frac{1}{\sqrt{5}}\left(\sqrt{80}-\sqrt{5}\right)$. **[2014 P1 #4]**

21. Without using a calculator or four figure table, Simplify $\left(2\sqrt{5}\right)^2-2\sqrt{45}-\sqrt{5}$. **[2014 PII #8a]**

22. Without using a calculator, simplify $\frac{3}{\sqrt{7}-\sqrt{5}}$, giving the answer with a rational denominator. **[2017 PI #4]**

23. Without using a calculator, simplify $\frac{2\sqrt{3}}{\sqrt{2}\left(\sqrt{5}+1\right)}$, giving the answer with a rational denominator. **[2017 PII #1b]**

24. Without using calculator or four figure tables, simplify $\frac{\sqrt{10}}{\sqrt{60}-\sqrt{15}}$, leaving your answer with a rational denominator. **[2015 PI #4]**

25. Without using a calculator or four figure tables, simplify $\frac{4-\sqrt{5}}{\sqrt{5}-\sqrt{3}}$. **[2015 PII #1b]**

26. Simplify $\sqrt{\frac{162}{5w^2}}$, leaving your answer with a rational denominator. **[2016 PI #4]**

27. Without using a calculator or four figure tables, simplify $\frac{\sqrt{2}+2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$, leaving the answer with a rational denominator. **[2016 PII #8a]**

28. Without using a calculator, simplify $\frac{3}{\sqrt{7}-\sqrt{5}}$,

giving the answer with a rational denominator.
[2017 PI #4]

29. Simplify $\frac{2\sqrt{5}}{\sqrt{35}}$, giving your answer with a rational denominator. [2018 P1 #4]

30. Without using a calculator simplify $\frac{2\sqrt{2}}{5-\sqrt{2}}$,

giving your answer with a rational denominator.
[2018 PII #1]

31. Without using a calculator, simplify $\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}$,

giving your answer with a rational denominator.
[2019 PI #4]

32. Simplify $\frac{a}{\sqrt{a}-b}+\frac{b}{\sqrt{a}+b}$. [2020 Mock PI #2]

1. **[2003 P1#3]**

$\frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$ (Rationalize denominator)

$=\frac{3\times\sqrt{2}}{\sqrt{2\times2}}$

$=\frac{3\sqrt{2}}{2}$

2. **[2003 PII #5b]**

$=\frac{1}{3\sqrt{2}-3}$

$=\frac{1}{3\sqrt{2}-3}\times\frac{(3\sqrt{2}+3)}{(3\sqrt{2}+3)}$ (Multiplying by conjugate of denominator)

$=\frac{3\sqrt{2}+3}{\left(3\sqrt{2}-3\right)\left(3\sqrt{2}+3\right)}$

$=\frac{3\sqrt{2}+3}{\left(3\sqrt{2}\right)^2-3^2}\rightarrow$ (Difference of 2 squares)

$=\frac{3\sqrt{2}+3}{9\times2-9}\Rightarrow\frac{3\sqrt{2}+3}{18-9}$

$=\frac{3\sqrt{2}+3}{9}$

$=\frac{\cancel{3}\left(\sqrt{2}+1\right)}{\cancel{9}}$

$=\frac{\sqrt{2}+1}{3}$

3. **[2004 PI #14]**

$\frac{\sqrt{2}}{\sqrt{2}-1}=\frac{\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$

(Conjugate of $\sqrt{2}-1$ is $\sqrt{2}+1$)

$=\frac{\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$

$=\frac{\sqrt{2}\times\sqrt{2}+\sqrt{2}\times1}{(\sqrt{2})^2-(1)^2}$ (Difference of two squares)

$=\frac{2+\sqrt{2}}{2-1}$

$=\frac{2+\sqrt{2}}{1}$

$=2+\sqrt{2}$

4. **[2005 PI #2]**

$=\sqrt{25\times5}+\sqrt{5}-\sqrt{9\times5}$ (simplify each term)

$=5\sqrt{5}+\sqrt{5}-3\sqrt{5}$ (simplify each term)

$=6\sqrt{5}-3\sqrt{5}$ (addition of surds)

$=3\sqrt{5}$

5. **[2005 PII #1b]**

$\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$ (conjugate of $\sqrt{3}-1$ is $\sqrt{3}+1$)

$=\frac{\sqrt{3}\left(\sqrt{3}+1\right)+1\left(\sqrt{3}+1\right)}{(\sqrt{3})^2-(1)^2}$ (expanding terms)

$=\frac{\left(\sqrt{3}\right)^2+\sqrt{3}+\sqrt{3}+1}{3-1}=\frac{3+2\sqrt{3}+1}{2}$

$=\frac{4+2\sqrt{3}}{2}$ (factor out 2 from numerator)

$=\frac{2(2+\sqrt{3})}{2}$

$=2+\sqrt{3}$

6. **[2006 P1 #23]**

$\frac{x^2-2}{x-\sqrt{2}}\times\frac{x+\sqrt{2}}{x+\sqrt{2}}$ $\left(\begin{array}{c}\text{multiply by conjuagate}\\ \text{of denominator}\end{array}\right)$

$=\frac{(x^2-2)\left(x+\sqrt{2}\right)}{(x)^2-(\sqrt{2})^2}$

$=\frac{(x^2-2)\left(x+\sqrt{2}\right)}{x^2-2}$

$=x+\sqrt{2}$ (cancelling out x^2-2)

7. **[2006 PII #1a]**

$(\sqrt{2}+\sqrt{3})(\sqrt{8}-\sqrt{12})$

$=(\sqrt{2}+\sqrt{3})(\sqrt{4}\times\sqrt{2}-\sqrt{4}\times\sqrt{3})$

$=(\sqrt{2}+\sqrt{3})(2\sqrt{2}-2\sqrt{3})$

$=2\left[(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})\right]$ *(factoring out 2)*

$=2\left[\left(\sqrt{2}\right)^2-\left(\sqrt{3}\right)^2\right]$ *(difference of two squares)*

$=2\,(2-3)$

$=2(-1)$

$=-2$

8. **[2007 PI #1]**

$\frac{\sqrt{2}}{\sqrt{3}}$

$=\frac{\sqrt{2}\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$

$=\frac{\sqrt{2\times3}}{(\sqrt{3})^2}$ since $(\sqrt{a})^2=a$

$=\frac{\sqrt{6}}{3}$

9. **[2007 PII #1a]**

$\left(2-\sqrt{7}\right)\left(2+\sqrt{7}\right)=2^2-(\sqrt{7})^2$

(difference of two squares)

$=(2\times2)-(\sqrt{7}\times\sqrt{7})$

$=4-7$

$=-3$

10. **[2008 P2 #1b]**

$\frac{15\sqrt{6}}{\sqrt{98}-\sqrt{2}}$

$=\frac{15\sqrt{6}}{\sqrt{49\times2}-\sqrt{2}}$

$=\frac{15\sqrt{6}}{7\sqrt{2}-\sqrt{2}}$

$=\frac{15\sqrt{6}}{6\sqrt{2}}$

$=\frac{15}{6}\sqrt{\frac{6}{2}}$ (since $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$)

$=\frac{5}{2}\sqrt{3}$ (simplifying)

$=\frac{5\sqrt{3}}{2}$

11. **[2008 P1 #2]**

$\sqrt{27}\times\sqrt{32}=\sqrt{9\times3}\times\sqrt{16\times2}$

$=\sqrt{9}\times\sqrt{3}\times\sqrt{16}\times\sqrt{2}$ (simplifying)

$=3\times\sqrt{3}\times4\times\sqrt{2}$

$=3\times4\times\sqrt{3\times2}$ (putting like terms together)

$=12\times\sqrt{6}$

$=12\sqrt{6}$

12. **[2010 P1 #5]**

$\frac{\sqrt{54}+3\sqrt{3}}{\sqrt{3}}$

$=\frac{\sqrt{54}+3\sqrt{3}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$ (to rationalize denom.)

$=\frac{\sqrt{3}\left(\sqrt{54}+3\sqrt{3}\right)}{\sqrt{3\times3}}$

$=\frac{\sqrt{54}\times\sqrt{3}+3\sqrt{3}\times\sqrt{3}}{3}$

$=\frac{\sqrt{54\times3}+3\times\sqrt{3\times3}}{3}$

$=\frac{\sqrt{162}+3\times3}{3}$

$=\frac{\sqrt{81\times2}+9}{3}$

$=\frac{9\sqrt{2}+9}{3}$

$=\frac{9\left(\sqrt{2}+1\right)}{3}$

$=3\left(\sqrt{2}+1\right)$

13. **[2010 PII #2b]**

$$\frac{5+\sqrt{3}}{\sqrt{7}+\sqrt{5}}$$

$$\frac{5+\sqrt{3}}{\sqrt{7}+\sqrt{5}}\times\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}-\sqrt{5}} \text{ using conjugates}$$

$$=\frac{\left(5+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{5}\right)}{\left(\sqrt{7}\right)^2-\left(\sqrt{5}\right)^2}$$

$$=\frac{\left(5+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{5}\right)}{7-5}$$

$$=\frac{\left(5+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{5}\right)}{2}$$

14. **[2011 PI #3]**

$$\frac{1}{\sqrt{2}}-\frac{\sqrt{2}}{3}$$

$$=\frac{3-\sqrt{2}\times\sqrt{2}}{3\sqrt{2}} \text{ (common denom. \& simplifying)}$$

$$=\frac{3-\sqrt{2\times2}}{3\sqrt{2}}$$

$$=\frac{3-2}{3\sqrt{2}}$$

$$=\frac{1}{3\sqrt{2}}$$

$$=\frac{1}{3\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} \text{ (rationalizing)}$$

$$=\frac{\sqrt{2}}{3\sqrt{2\times2}}$$

$$=\frac{\sqrt{2}}{3\times2}$$

$$=\frac{\sqrt{2}}{6}$$

15. **[2011 PII #4a]**

multipliying with is conjugate$\sqrt{3}-\sqrt{6}$

$$\frac{3\sqrt{2}+1}{\sqrt{3}+\sqrt{6}}\times\frac{\sqrt{3}-\sqrt{6}}{\sqrt{3}-\sqrt{6}}$$

$$=\frac{\sqrt{3}\left(3\sqrt{2}+1\right)-\sqrt{6}\left(3\sqrt{2}+1\right)}{3-6}$$

$$=\frac{3\sqrt{6}+\sqrt{3}-3\sqrt{12}-\sqrt{6}}{-3}$$

$$=\frac{3\sqrt{6}+\sqrt{3}-3\sqrt{4\times3}-\sqrt{6}}{-3}$$

$$=\frac{3\sqrt{6}+\sqrt{3}-6\sqrt{3}-\sqrt{6}}{-3}$$

$$=\frac{3\sqrt{6}-\sqrt{6}+\sqrt{3}-6\sqrt{3}}{-3}$$

$$=\frac{2\sqrt{6}-5\sqrt{3}}{-3}=\frac{5\sqrt{3}-2\sqrt{6}}{3}$$

16. **[2012 P1 #5]**

$$\frac{8\sqrt{2}}{\sqrt{98}-3\sqrt{2}}$$

$$=\frac{8\sqrt{2}}{\sqrt{49\times2}-3\sqrt{2}}$$

$$=\frac{8\sqrt{2}}{7\sqrt{2}-3\sqrt{2}}$$

$$\frac{8\sqrt{2}}{7\sqrt{2}-3\sqrt{2}}$$

$$=\frac{8\sqrt{2}}{4\sqrt{2}}$$

$$=2$$

17. **[2012 P2 #1a]**

$$\textit{Given}\ \frac{3\sqrt{5}-\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$$

multiplying with the conjugate $2\sqrt{5}-3\sqrt{2}$

$$=\frac{3\sqrt{5}-\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}\times\frac{2\sqrt{5}-3\sqrt{2}}{2\sqrt{5}-3\sqrt{2}}$$

$$=\frac{3\sqrt{5}\left(2\sqrt{5}-3\sqrt{2}\right)-\sqrt{2}\left(2\sqrt{5}-3\sqrt{2}\right)}{(2\sqrt{5})^2-(3\sqrt{2})^2}$$

$$=\frac{6\sqrt{5\times5}-9\sqrt{5\times2}-2\sqrt{2\times5}+3\sqrt{2\times2}}{4\times5-9\times2}$$

$$=\frac{6\times5-9\sqrt{10}-2\sqrt{10}+3\times2}{20-18}$$

$$=\frac{30+6-11\sqrt{10}}{2}$$

$$=\frac{36-11\sqrt{10}}{2}$$

18. **[2013 P1 #4]**

$$=\frac{\sqrt{12}\times\sqrt{75}\times\sqrt{108}}{9\sqrt{6}}$$
$$=\frac{\sqrt{4\times3}\times\sqrt{25\times3}\times\sqrt{36\times3}}{9\sqrt{6}}$$
$$=\frac{2\sqrt{3}\times5\sqrt{3}\times6\sqrt{3}}{9\sqrt{6}}$$
$$=\frac{60\left(\sqrt{3}\times\sqrt{3}\times\sqrt{3}\right)}{9\sqrt{6}}$$
$$=\frac{60\left(\sqrt{27}\right)}{9\sqrt{6}}$$
$$=\frac{\overset{20}{\cancel{60}}\sqrt{3}}{\underset{3}{\cancel{9}}\sqrt{6}}$$
$$\Rightarrow\frac{20}{3}\sqrt{\frac{27}{6}}$$
$$=\frac{20}{3}\sqrt{\frac{9}{2}}=\frac{20}{3}\frac{\sqrt{9}}{\sqrt{2}}=\frac{20}{\cancel{3}}\times\frac{\cancel{3}}{\sqrt{2}}$$
$$=\frac{20}{\sqrt{2}}\text{(rationalize denominator)}$$
$$=\frac{20}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{20\sqrt{2}}{2}$$
$$=10\sqrt{2}$$

19. **[2013 P2 #9a]**

$$\sqrt{72}-\sqrt{8}+\sqrt{288}$$
$$=\sqrt{36\times2}-\sqrt{4\times2}+\sqrt{144\times2}$$
$$=6\sqrt{2}-2\sqrt{2}+12\sqrt{2}$$
$$=4\sqrt{2}+12\sqrt{2}$$
$$=16\sqrt{2}$$

20. **[2014 P1 #4]**

$$\frac{1}{\sqrt{5}}\left(\sqrt{80}-\sqrt{5}\right)$$
$$=\frac{1}{\sqrt{5}}\left(\sqrt{16\times5}-\sqrt{5}\right)$$
$$=\frac{1}{\sqrt{5}}\left(4\sqrt{5}-\sqrt{5}\right)$$
$$=\frac{1}{\sqrt{5}}\left(3\sqrt{5}\right)$$
$$=\frac{3\sqrt{5}}{\sqrt{5}}$$
$$=3$$

21. **[2014 PII #8a]**

$$(2\sqrt{5})^2-2\sqrt{45}-\sqrt{5}$$
$$=(2\times2\times\sqrt{5}\times\sqrt{5})-2(\sqrt{9\times5})-\sqrt{5}$$
$$=(4\times5)-2(3\times\sqrt{5})-\sqrt{5}$$
$$=20-6\sqrt{5}-\sqrt{5}$$
$$=20-7\sqrt{5}$$

22. **[2017 PI #4]**

$$\frac{3}{\sqrt{7}-\sqrt{5}}$$
$$=\frac{3}{\sqrt{7}-\sqrt{5}}\times\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}$$
$$=\frac{3\left(\sqrt{7}+\sqrt{5}\right)}{\left(\sqrt{7}\right)^2-\left(\sqrt{5}\right)^2}$$
$$=\frac{3\left(\sqrt{7}+\sqrt{5}\right)}{7-5}$$
$$=\frac{3\left(\sqrt{7}+\sqrt{5}\right)}{2}$$

23. **[2017 PII #1b]**

$$\frac{2\sqrt{3}}{\sqrt{2}\left(\sqrt{5}+1\right)}$$
$$=\frac{2\sqrt{3}}{\sqrt{10}+\sqrt{2}}$$
$$=\frac{2\sqrt{3}}{\sqrt{10}+\sqrt{2}}\times\frac{\sqrt{10}-\sqrt{2}}{\sqrt{10}-\sqrt{2}}$$

$$=\frac{2\sqrt{3}\left(\sqrt{10}-\sqrt{2}\right)}{\left(\sqrt{10}\right)^2-\left(\sqrt{2}\right)^2}$$

$$=\frac{2\sqrt{30}-2\sqrt{6}}{10-2}$$

$$=\frac{2\left(\sqrt{30}-\sqrt{6}\right)}{8}$$

$$=\frac{\cancel{2}\left(\sqrt{30}-\sqrt{6}\right)}{\cancel{8}_4}$$

$$=\frac{\sqrt{30}-\sqrt{6}}{4}$$

24. **[2015 PI #4]**

$$\frac{\sqrt{10}}{\sqrt{60}-\sqrt{15}}$$

$$=\frac{\sqrt{10}}{\sqrt{4\times15}-\sqrt{15}}$$

$$=\frac{\sqrt{10}}{2\sqrt{15}-\sqrt{15}}$$

$$=\frac{\sqrt{10}}{\sqrt{15}}=\sqrt{\frac{10}{15}}$$

$$=\frac{\sqrt{2}}{\sqrt{3}}$$

$$=\frac{\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\frac{\sqrt{6}}{3}$$

25. **[2015 PII #1b]**

$$\frac{4-\sqrt{5}}{\sqrt{5}-\sqrt{3}}$$

$$=\frac{4-\sqrt{5}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{\left(4-\sqrt{5}\right)\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}$$

$$=\frac{4\left(\sqrt{5}+\sqrt{3}\right)-\sqrt{5}\left(\sqrt{5}+\sqrt{3}\right)}{5-3}$$

$$=\frac{4\sqrt{5}+4\sqrt{3}-5-\sqrt{15}}{2}$$

26. **[2016 PI #4]**

$$\sqrt{\frac{162}{5w^2}}$$

$$=\frac{\sqrt{81}\times\sqrt{2}}{\sqrt{5}\times\sqrt{w^2}}$$

$$=\frac{9\sqrt{2}}{w\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}$$

$$=\frac{9\sqrt{10}}{5w}$$

27. **[2016 PII #8a]**

$$\frac{\sqrt{2}+2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$$

$$=\frac{\left(\sqrt{2}+2\sqrt{5}\right)\left(\sqrt{5}+\sqrt{2}\right)}{\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)}$$

$$=\frac{\sqrt{2}\left(\sqrt{5}+\sqrt{2}\right)+2\sqrt{5}\left(\sqrt{5}+\sqrt{2}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2}$$

$$=\frac{\sqrt{10}+2+10+2\sqrt{10}}{5-2}$$

$$=\frac{3\sqrt{10}+12}{3}$$

$$=\frac{3\left(\sqrt{10}+4\right)}{3}$$

$$=\sqrt{10}+4$$

28. **[2017 PI #4]**

$$=\frac{3}{\sqrt{7}-\sqrt{5}}\text{ (multiplyby denominator conjugate)}$$

$$=\frac{3}{\sqrt{7}-\sqrt{5}}\times\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}$$

$$=\frac{3\left(\sqrt{7}+\sqrt{5}\right)}{\left(\sqrt{7}\right)^2-\left(\sqrt{5}\right)^2}$$

$$=\frac{3\left(\sqrt{7}+\sqrt{5}\right)}{7-5}$$

$$=\frac{3\left(\sqrt{7}+\sqrt{5}\right)}{2}$$

29. **[2018 P1 #4]**

$$=\frac{2\sqrt{5}}{\sqrt{35}}\times\frac{\sqrt{35}}{\sqrt{35}}$$

$$=\frac{2\sqrt{5\times35}}{35}$$

$$=\frac{2\sqrt{175}}{35}$$

$$=\frac{2\sqrt{25\times7}}{35}$$

$$=\frac{2\sqrt{25}\times\sqrt{7}}{35}$$

$$=\frac{2\times5\times\sqrt{7}}{35}$$

$$=\frac{\cancelto{2}{10}\sqrt{7}}{\cancelto{7}{35}}$$

$$=\frac{2\sqrt{7}}{7}$$

30. **[2018 PII #1]**

$$\frac{2\sqrt{2}}{5-\sqrt{2}}\times\frac{5+\sqrt{2}}{5+\sqrt{2}}$$

$$=\frac{2\sqrt{2}\left(5+\sqrt{2}\right)}{(5)^2-\left(\sqrt{2}\right)^2}$$

$$=\frac{10\sqrt{2}+2\times2}{25-2}$$

$$=\frac{10\sqrt{2}+4}{23}$$

$$=\frac{2\left(5\sqrt{2}+2\right)}{23}$$

31. **[2019 PI #4]**

$$\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}\times\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}-\sqrt{2}}\text{ (using conjugates)}$$

$$\frac{\left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}-\sqrt{2}\right)}{\left(\sqrt{x^2}\right)-\left(\sqrt{2}\right)^2}$$

$$=\frac{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)-\sqrt{2}\left(\sqrt{x}-\sqrt{2}\right)}{x-2}$$

$$=\frac{x-\sqrt{2}\sqrt{x}-\sqrt{2}\sqrt{x}+2}{x-2}$$

$$=\frac{x-2\sqrt{2x}+2}{x-2}$$

32. **[2020 Mock PI #2]**

$$\frac{a}{\sqrt{a}-b}+\frac{b}{\sqrt{a}+b}$$

$$=\frac{a\left(\sqrt{a}+b\right)+b\left(\sqrt{a}-b\right)}{\left(\sqrt{a}-b\right)\left(\sqrt{a}+b\right)}\text{ common denominators}$$

$$=\frac{a\sqrt{a}+ab+b\sqrt{a}-b^2}{\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2}\text{ product of conjugates}$$

$$=\frac{a\sqrt{a}+b\sqrt{a}+ab-b^2}{a-b}$$

$$=\frac{\sqrt{a}\left(a+b\right)+b\left(a-b\right)}{a-b}$$

CH 4
CIRCLE GEOMETRY I (CHORD PROPERTIES)

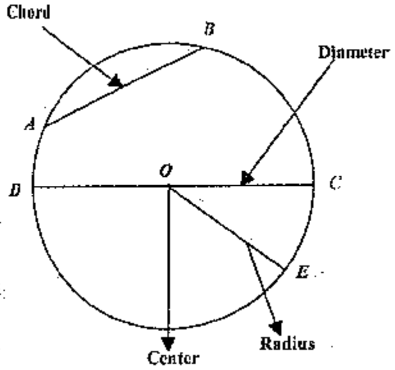
Chapter Highlights

This chapter provides solutions to problems on chord properties of a circle.

Students must be able to:

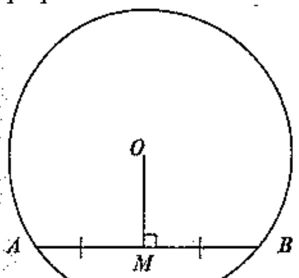
- describe chord properties of a circle.
- apply chord properties to solve problems.

Parts of a circle



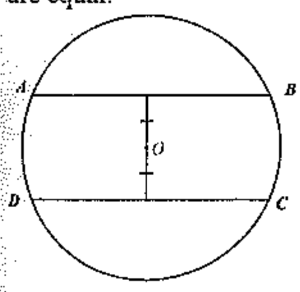
Chord Properties

a. The line from the center of the circle perpendicular to a chord bisects the chord.



Similarly: A line from centre to a chord, which bisects the chord, is perpendicular to the chord.

b. Equal chords are equidistant from the center of the circle or, chords equidistant from the center are equal.



$AB = DC$

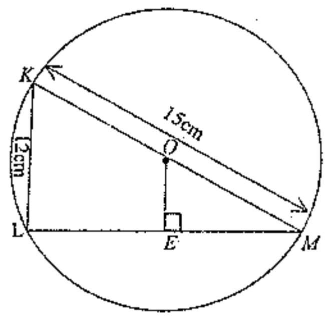
1. A circle has a chord which is 8 cm long. The radius of the circle is 5cm long. Sketch the diagram and calculate the angle subtended by the chord at the center of the circle. **[2003 PI #20]**

2. A chord is 6 cm away from the center of a circle. If the chord is 16cm long, calculate the radius of the circle. **[2004 P1 #13]**

3. A chord of a circle center **O** is 8.4cm long. If the radius of the circle is 7cm, sketch the diagram and calculate the distance of the chord from the center of the circle. **[2005 PI #17]**

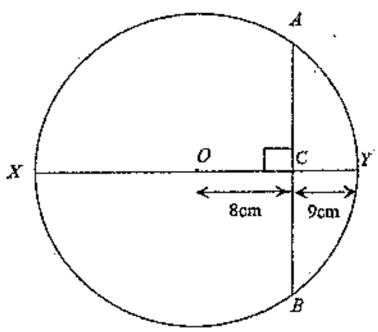
4. An arc of a circle subtends an angle of $54°$ at the center. If the arc is 9cm long, calculate the circumference of the circle. **[2008 P2 #3b]**

5. **Figure 1** shows a circle KLM center O. the diameter KM=15cm and chord KL=12cm. E is a point on LM.



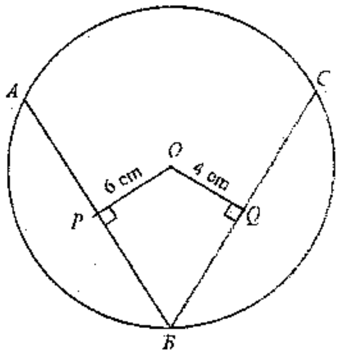
If **OE** is perpendicular to the chord **LM** calculate the length of **EM**. **[2010 P1 #21]**

6. **Figure 2** shows a circle **AYBX** center O. XY cuts AB at C such that angle ACY=$90°$



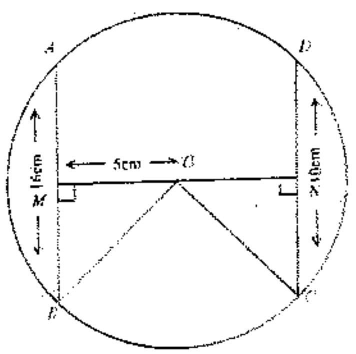
If **OC** = 8cm and **CY** = 9cm, calculate the length of **AB**. **[2012 P1 #12]**

7. **Figure 3** shows of a circle centre O. Lines AB and BC are chords, $AB = 2x$, OQ = $4cm$ and OP= 6cm .



Find BC in terms of x. **[2013 P1 #6]**

8. **Figure 4** shows a circle **ABCD** centre **O. AB** =16cm. **CD**=10cm and **OM**=5cm



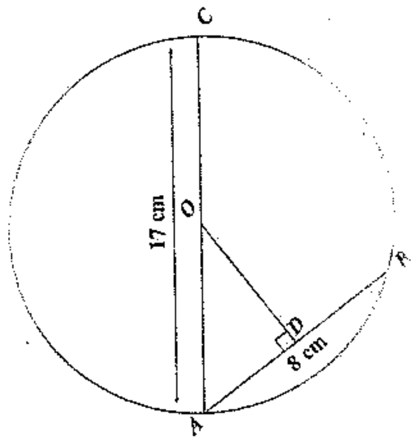
Calculate length of **ON**. **[2014 P1 #9]**

9. A circle has 2 chords **PQ** and **RS.** The chord **PQ** is 12 cm long and 4 cm from the centre of the circle. Given that **RS** is 7 cm from the centre of the circle, calculate the length of **RS,** leaving your answer in surd form. **[2017 PI #7]**

10. A chord of a circle of radius 6cm is 4cm away from the centre. Calculate the length of the chord, giving your answer correct to one decimal place. **[2015 PI #10]**

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

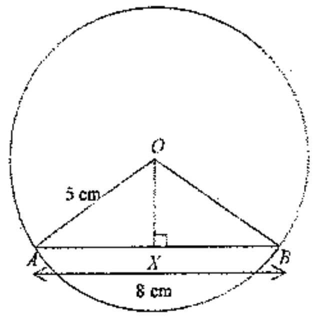
11. **Figure 5** shows a circle **ABC** with centre **O. OD** is perpendicular to chord **AB.** Line **AOC** is a diameter.



If AC =**17 cm** and **AB=8 cm,** calculate the length of **OD.** **[2016 PI #7]**

12. A chord **AB** of a circle is 3.6 cm from the center O of the circle. If the radius of the circle is 6 cm long, calculate the length of the chord **AB.** **[2019 PI #5]**

1. [2003 PI #20]



Need to find $\angle AOB$

AX=XB ($\perp$ from centre bisects chord)

$AX = \frac{8cm}{2} = 4cm$

$\text{Cos}\,\angle OAX = \frac{AX}{AO} = \frac{4}{5}$

$= 0.8$

$\angle OAX = \text{Cos}^{-1}\,0.8$

$\angle OAX = 36.87^{o}$

In ΔAOB

OA=OB (radii)

So, $\angle$OBA=36.87^{o} ($\angle$ opp. equal sides)

Thus. $\angle AOC + 2(36.87^{o}) = 180^{o}$ ($\angle$*sum of a*Δ)

$\angle AOC = 180^{o} - 73.74^{o}$

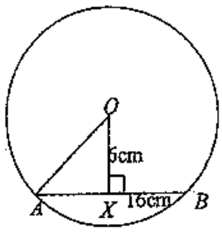
$\angle AOC = 106.26^{o}$

$\therefore$The angle subtended by the chord at the center is 106° (to the nearest degree).

2. [2004 P1 #13]

Let AB be a chord and OX be the distance from the center.

Let OX $\perp$ AB



we need to find a radius OA

now = ½ × 16cm ($\perp$ *line from center bisects chord)*

= 8cm

$OA^2 = OX^2 + AX^2$ (*Pythagoras theorem)*

$= 6^2 + 8^2$

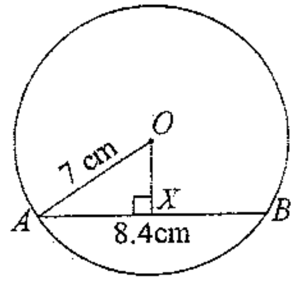
$= 36 + 64$

$= 100$

$\therefore OA = \sqrt{100}$

$\therefore$ radius = 10cm

3. [2005 PI #17]



Construction: join OX: let OX be $\perp$ AB

$AX = \frac{8.2cm}{2} = 4.2cm$ ($\perp$*from center to a chord bisects the chord*)

$OX^2 = OA^2 - AX^2$ (*Pythagoras theorem*)

$= 7^2 - 4.2^2$

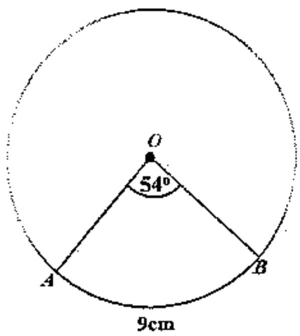
$= 49 - 17.64$

$= 31.36$

$\Rightarrow OX = \sqrt{31.36}\ cm = 5.6\ cm$

$\therefore$ the distance of the chord from the center of the circle is 5.6cm.

4. [2008 P2 #3b]



Arc length is a portion of the circumference (C). The circumference covers 360°.

We use proportions.

Let $x = 360^{o}$

But $Arc\ AB = 54^{o}$

Dividing the two, we get:

$\frac{x}{AB} = \frac{360^{o}}{54^{o}}$

$\frac{x}{9} = \frac{36\cancel{0}^{o}}{\cancel{5}4^{o}}$

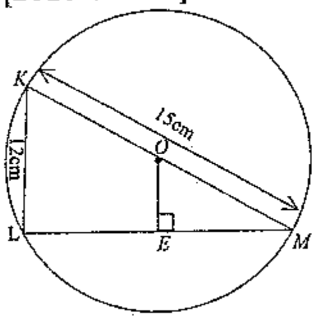
$x = \cancel{9} \times \frac{20}{\cancel{3}}$

$x = 3 \times 20$

$x = 60$

Hence, the circumference of the circle is 60cm.

5. **[2010 P1 #21]**



In ΔKLM,

$\angle KLM = 90^o$ ($\angle$ in a semi-circle)

Using Pythagoras Theorem,

$KL^2 + LM^2 = KM^2$

$LM^2 = KM^2 - KL^2$

$LM^2 = 15^2 - 12^2$

$LM^2 = 225 - 144$

$LM^2 = 81$

$LM = \sqrt{81}$

$LM = 9cm$

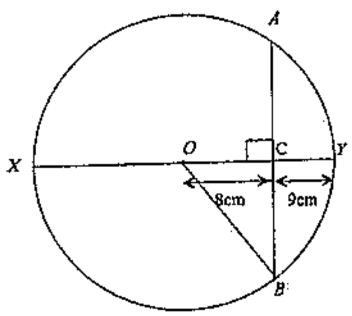
Since OE $\perp$ LM (given)

$EM = \frac{1}{2} LM$ ($\perp$ from centre bisects chord)

$EM = \frac{1}{2} \times 9cm$

$EM = 4.5cm$

6. **[2012 P1 #12]**



Given OC = 8cm, CY =9cm and $\angle$ACY=90°

Construction: Join BO

$OA = OY$ (*radii*)

$\therefore OA = 8 + 9 = 17cm$ (*given*)

$In \Delta OBC$,

$BC^2 = BO^2 - OC^2$ (*Pythagoras*)

$BC^2 = 17^2 - 8^2$

$BC^2 = 225$

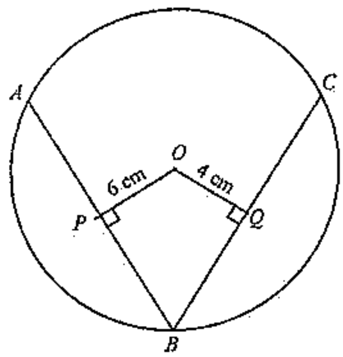
$BC = 15$

$BC = AC$ (line from centre $\perp$ to chord bisects it)

$AB = 15 \times 2$

$= 30cm$

7. **[2013 P1 #6]**



$PB = \frac{1}{2} AB$ ($\perp$ OP bisects chord AB)

$PB = \frac{1}{2}(2x) = x$

In ΔOPB,

$OB^2 = OP^2 + PB^2$ (Pythagoras)

$OB^2 = 6^2 + x^2$

$OB^2 = 36 + x^2$

In ΔOQB,

$OB^2 = OQ^2 + BQ^2$ (Pythagoras)

$OB^2 = 4^2 + BQ^2$

$36 + x^2 = 16 + BQ^2$ (from ΔOPB)

$x^2 + 36 - 16 = BQ^2$

$x^2 + 20 = BQ^2$

$BQ = \sqrt{x^2 + 20}$

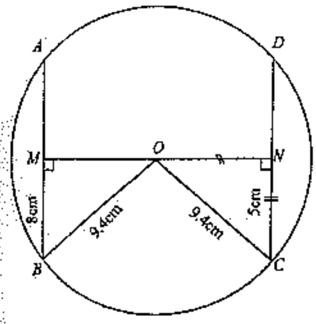
But $BQ = \frac{1}{2} BC$ ($\perp$ OQ bisects chord BC)

Thus, $2BQ = BC$

$2 \times \sqrt{x^2 + 20} = BC$

$\therefore$ in terms of x, $BC = 2\sqrt{x^2 + 20}$

8. **[2014 P1 #9]**



$MB = \frac{1}{2} AB$ ($\perp$ OM bisects chord AB)

$MB = \frac{1}{2} \times 16cm = 8cm$

Similarly,

$CN = \frac{1}{2} CD$ ($\perp$ ON bisects chord CD)

$CN = \frac{1}{2} \times 10cm = 5cm$

In ΔsMBO and NOC

$OM = NC = 5cm$

$OB = OC$ (*radii*)

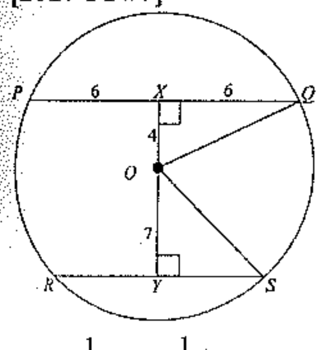
$\angle BMO = \angle ONC$ (*rt.*$\angle s$)

$\therefore \Delta BMO \equiv \Delta ONC$ (*RHS*)

$\therefore ON = 8cm$ (corr.sides of $\equiv \Delta$s)

---

9. **[2017 PI #7]**



$QX = \frac{1}{2} PQ = \frac{1}{2}(12)$ ($\perp$ OX bisects PQ)

$= 6cm$

ΔOXQ is right-angled

$OQ^2 = OX^2 + XQ^2$ (*Pythagoras theorem*)

$OQ^2 = 4^2 + 6^2$

$OQ = \sqrt{6^2 + 4^2}$

$OQ = \sqrt{52}$

$OQ = OS$ (*radii*)

$\therefore OS = \sqrt{52}$

In ΔOYS, $\angle OYS = 90^o$ ($\perp$ *from centre*)

$OY^2 + YS^2 = OS^2$ (*Pythagoras theorem*)

$YS^2 = OS^2 - OY^2$

$YS^2 = \left(\sqrt{52}\right)^2 - (7)^2$

$YS^2 = 52 - 49$

$YS = \sqrt{3}$

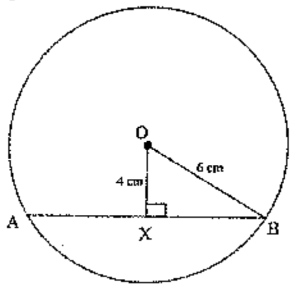
But $YS = \frac{1}{2} RS$ ($\perp OY$ bisects RS)

$RS = 2YS$

$= 2\sqrt{3}cm$

---

10. **[2015 PI #10]**



Let $OX \perp AB$

In ΔOBC

$OB^2 = OX^2 + XB^2$ (*pythagoras theorem*)

$6^2 = 4^2 + XB^2$

$36 = XB^2 + 16$

$XB^2 = 36 - 16$

$XB^2 = 20$

$XB = \pm\sqrt{20}$

$XB = \pm 4.472$

$XB = 4.472cm$

But $XB = \frac{1}{2} AB$ ($\perp OX$ bisects chord AB)

Thus, $AB = 2XB$

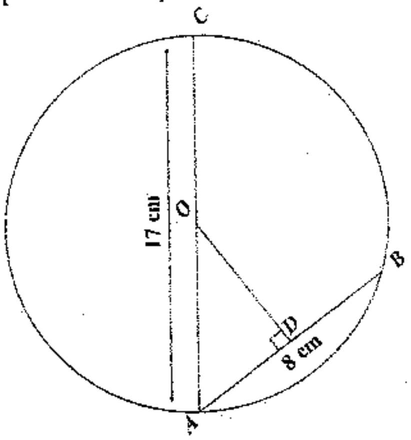
$AB = 2(4.472)$

$AB = 8.944$

$AB = 8.9cm$ (to 1.dec.place)

---

11. **[2016 PI #7]**



$AD = \frac{1}{2}AB$ $(\perp OD$ bisects chord $AB)$

$AD = \frac{1}{2}(8) = 4cm$

$AO = \frac{1}{2} \times AC$ $\left(radius = \frac{1}{2} diameter\right)$

$AO = 8.5cm$

In ΔAOD,

$AO^2 = OD^2 + AD^2$ $(pythagoras)$

$OD^2 = AO^2 - AD^2$

$OD^2 = (8.5)^2 - (4)$

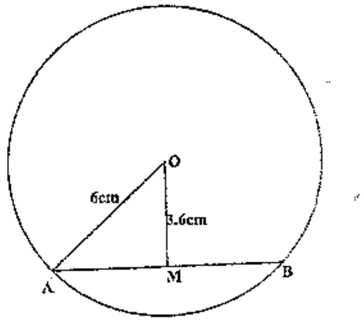
$= 72.25 - 16$

$OD = \sqrt{72.25 - 16}$

$= \sqrt{56.25}$

$= 7.5cm$

12. **[2019 PI #5]**



Let the chord be AB and M be the midpoint of AB

OM $\perp$ AB (OM from centre bisects AB)

$\therefore \Delta AOM$ is right angled

$AM^2 = AO^2 - MO^2$ (pythagoras)

$AM^2 = 6^2 - (3.6)^2$

$AM = \sqrt{36 - 12.96}$

$AM = \sqrt{23.04}$

AM=4.8cm

AB=2(AM) (M is midpoint)

AB=2(4.8)

AB=9.6cm

CH 5
ALGEBRAIC FRACTIONS

Learning Objectives

By the end of the chapter, a student must be able to:

- express fractions to their lowest term.
- simplify algebraic fractions.

Chapter Highlights

Algebraic fractions are simply fractions with a quadratic expression in the numerator or denominator, or both. In dealing with quadratic fractions, we will use knowledge on factorization.

- We should be able to simplify the numerator and denominator by factorization and expressing the fraction in its lowest term by canceling like terms.
- Adding and subtraction of algebraic fractions involve the same procedure of adding or subtraction numerical fractions: where we find the common denominator (usually a product of the denominators).
- We will also encounter problems to solve equations involving quadratic fractions.

1. Express $\frac{1}{x^2-x-2}-\frac{1}{x+1}$ as a single fraction. **[2003 PI #10]**

2. Express $\frac{m-2}{m-3}+\frac{m+3}{m+2}$ as a single fraction at its lowest term. **[2004 PI #16]**

3. Express $1-\frac{1}{x}-\frac{x-2}{x-1}$ as a single fraction in its simplest form. **[2004 P1 #8a]**

4. Simplify $\frac{x^2-y^2}{x^2+xy}$. **[2005 P1 #4]**

5. Simplify to the lowest term, $\frac{x-2}{x^2-3x+2}\div\frac{x}{x-1}$. **[2005 P2 #1a]**

6. Simplify $\frac{3x+6}{x^2+x-2}$. **[2006 P1 #1]**

7. Simplify $\frac{1}{a-b}+\frac{1}{a+b}$. **[2007 PI #10]**

8. Simplify $\frac{4x^2-2x}{x-2}\div\frac{x(2x-1)}{x^2-5x+6}$. **[2007 P2 #10a]**

9. Simplify $\frac{1}{2}-\frac{1}{(x+2y)}-1$. **[2008 P2 #2a]**

10. Simplify $\frac{x^2-1}{x}\times\frac{x^2}{x-1}$. **[2008 P1 #3]**

11. Simplify $\frac{d-1}{3}-\frac{2d+1}{7}$. **[2010 P1 #15]**

12. Simplify $\frac{v-a}{v+a}-\frac{v+a}{v-a}$. **[2010 PII #6a]**

13. Simplify $\frac{m}{m+2}-\frac{6}{m^2+m-2}$. **[2011 PI #8]**

14. Simplify $\frac{3}{y}-\frac{y-1}{y(y+2)}$. **[2012 P1 #3]**

15. Simplify $\frac{5}{t+3}-\frac{3(t-3)}{t^2-9}$. **[2012 P2 #8a]**

16. Simplify $\frac{ay+2a+2b+by}{y+2}\div\frac{y^2-1}{2y+2}$. **[2013 P2 #2b]**

17. Simplify $\frac{3a^2}{a-2}\div\frac{a}{2a^2-5a+2}$. **[2014 P1 #3]**

18. Solve the equation $\frac{5y+2}{y+1}=\frac{y+1}{y}$. **[2020 P1 #10a]**

19. Simplify $\frac{3xy-12xy^3}{1+2y}$. **[2017 PII #2a]**

20. Express $\frac{p+q}{pq}-\frac{q+t}{qt}$ as a single fraction in its simplest form. **[2015 PII #2b]**

21. Simplify $\frac{(y-3)(y-2)}{y} \div \left(\frac{y-3}{y}\right)^2$.

[2016 PII #2a]

22. Simplify $\frac{x}{x-7} - \frac{7x-35}{x^2-12x+35}$. **[2018 P1 #5]**

23. Express $\frac{2x}{x^2-1} + \frac{1}{1-x}$ as a single fraction in the lowest form. **[2018 PII #17]**

24. Simplify $\frac{3}{3x+y} \div \frac{9y}{3xy+y^2}$. **[2019 PII #1a]**

1. **[2003 PI #10]**

$$=\frac{1}{x^2-x-2}-\frac{1}{x+1}$$

$$=\frac{1}{(x+1)(x-2)}-\frac{1}{x+1} \text{ (factorizing denominator)}$$

$$=\frac{1-1(x-2)}{(x+1)(x-2)}$$

$$=\frac{1-x+2}{(x+1)(x-2)}$$

$$=\frac{3-x}{(x+1)(x-2)}$$

2. **[2004 PI #16]**

$$=\frac{m-2}{m-3}+\frac{m+3}{m+2}$$

$$=\frac{(m+2)(m-2)+(m-3)(m+3)}{(m-3)(m+2)}$$

$$=\frac{m^2-2^2+m^2-3^2}{(m-3)(m+2)} \textit{ (difference of two squares)}$$

$$=\frac{m^2+m^2-4-9}{(m-3)(m+2)} \textit{ (simplifying)}$$

$$=\frac{2m^2-13}{(m-3)(m+2)}$$

3. **[2004 P1 #8a]**

$$=\frac{1}{1}-\frac{1}{x}-\frac{x-2}{x-1}$$

$$=\frac{x(x-1)-(x-1)-x(x-2)}{x(x-1)} \textit{ (common denominator)}$$

$$=\frac{x^2-x-x+1-x^2+2x}{x(x-1)}$$

$$=\frac{x^2-2x+1-x^2+2x}{x(x-1)}$$

$$=\frac{x^2-x^2-2x+2x+1}{x(x-1)}$$

$$=\frac{1}{x(x-1)}$$

4. **[2005 P1 #4]**

$$=\frac{x^2-y^2}{x^2+xy}$$

$$=\frac{(x+y)(x-y)}{x(x-y)} \qquad \textit{(difference of two squares)}$$

$$=\frac{x+y}{x}$$

5. **[2005 P2 #1a]**

$$=\frac{x-2}{x^2-3x+2}\div\frac{x}{x-1}$$

$$=\frac{x-2}{x^2-3x+2}\times\frac{x-1}{x}$$ *(flipping fraction when dividing)*

$$=\frac{x-2}{(x-2)(x-1)}\times\frac{x-1}{x}$$ *(factoring out the denominator)*

$$=\frac{1}{x} \textit{ (cancelling out common terms)}$$

6. **[2006 P1 #1]**

$$=\frac{3x+6}{(x-1)(x+2)} \textit{ (factorize the denominator)}$$

$$=\frac{3(x+2)}{(x-1)(x+2)}$$

$$=\frac{3}{x-1}$$

7. **[2007 PI #10]**

$$=\frac{1}{a-b}+\frac{1}{a+b}$$

$$=\frac{1(a+b)+1(a-b)}{(a+b)(a-b)} \text{ common denominator}$$

$$=\frac{a+b+a-b}{(a+b)(a-b)} \text{ simplifying terms}$$

$$=\frac{a+a+b-b}{(a+b)(a-b)}$$

$$=\frac{2a}{(a+b)(a-b)}$$

8. **[2007 P2 #10a]**

$$=\frac{4x^2-2}{x-2}\div\frac{x(2x-1)}{x^2-5x+6}$$

$$=\frac{4x^2-2x}{x-2}\times\frac{x^2-5x+6}{x(2x-1)}\ \text{(flipping fractions)}$$

$$=\frac{2x(2x-1)}{x-2}\times\frac{(x-2)(x-3)}{x(2x-1)}\ \text{(factorising)}$$

$$=2\times(x-3)$$

$$=2(x-3)$$

---

9. **[2008 P2 #2a]**

$$=\frac{1}{x}-\frac{1}{x+2y}-1$$

$$=\frac{(x+2y)-x-x(x+2)}{x(x+2y)}$$

$$=\frac{x+2y-x-x^2-2x}{x(x+2y)}$$

$$=\frac{2y-x^2-2xy}{x(x+2y)}$$

---

10. **[2008 P1 #3]**

$$=\frac{x^2-1}{x}\times\frac{x^2}{x-1}$$

$$=\frac{(x-1)(x+1)}{x}\times\frac{x^2}{x-1}\text{(difference of two squares)}$$

$$=\frac{x(x+1)}{1}$$

$$=x(x+1)$$

---

11. **[2010 P1 #15]**

$$=\frac{d-1}{3}\quad-\quad\frac{2d+1}{7}$$

$$=\frac{7(d-1)\ -\ 3(2d+1)}{21}$$

$$=\frac{7d-7\ -\ 6d-3}{21}\quad\textit{(Arrange like terms together)}$$

$$=\frac{7d-6d-7-3}{21}$$

$$=\frac{d-10}{21}$$

---

12. **[2010 PII #6a]**

$$=\frac{v-a}{v+a}-\frac{v+a}{v-a}$$

$$=\frac{(v-a)(v-a)-\left[(v+a)(v+a)\right]}{(v+a)(v-a)}$$

$$=\frac{v^2-va-va+a^2-\left[v^2+va+va+a^2\right]}{(v+a)(v-a)}$$

$$=\frac{v^2-2va+a^2-v^2-2va-a^2}{(v+a)(v-a)}$$

$$=\frac{-4va}{(v+a)(v-a)}$$

$$\therefore\frac{-4va}{(v+a)(v-a)}$$

---

13. **[2011 PI #8]**

$$=\frac{m}{m+2}-\frac{6}{(m+2)(m-1)}$$

$$=\frac{m(m-1)-6}{(m+2)(m-1)}$$

$$=\frac{m^2-m-6}{(m+2)(m-1)}$$

$$=\frac{(m-3)(m+2)}{(m+2)(m-1)}\text{(factorizing numerator)}$$

$$=\frac{(m-3)}{(m-1)}$$

---

14. **[2012 P1 #3]**

$$=\frac{3}{y}-\frac{y-1}{y(y+2)}$$

$$=\frac{3(y+2)-(y-1)}{y(y+2)}$$

$$=\frac{3y+6-y+1}{y(y+2)}$$

$$=\frac{2y+7}{y(y+2)}$$

---

15. **[2012 P2 #8a]**

factorising t^2-9 as $(t+3)(t-3)$

$\Rightarrow \frac{5}{t+3}-\frac{3(t-3)}{(t+3)(t-3)}$

$=\frac{5}{t+3}-\frac{3}{t+3}$

$=\frac{5-3}{t+3}$

$=\frac{2}{t+3}$

16. **[2013 P2 #2b]**

$=\frac{ay+2a+2b+by}{y+2}\div\frac{y^2-1}{2y+2}$

$=\frac{ay+2a+2b+by}{y+2}\times\frac{2(y+1)}{(y+1)(y-1)}$ (flipping fraction diff of 2 sqrs)

$=\frac{y(a+b)+2(a+b)}{y+2}\times\frac{2(y+1)}{(y+1)(y-1)}$

$=\frac{(y+2)(a+b)}{(y+2)}\times\frac{2}{(y-1)}$

$=\frac{2(a+b)}{(y-1)}$

17. **[2014 P1 #3]**

$\frac{3a^2}{a-2}\div\frac{a}{2a^2-5a+2}$

$\frac{3a^2}{a-2}\div\frac{a}{2a^2-4a-a+2}$ (factoring denominator)

$\frac{3a^2}{a-2}\div\frac{a}{2a(a-2)-1(a-2)}$ (grouping terms)

$\frac{3a^2}{a-2}\div\frac{a}{(2a-1)(a-2)}$

$\frac{3a^2}{a-2}\times\frac{(2a-1)(a-2)}{a}$ (flipping fractions)

$3a(2a-1)$

18. **[2017 PII #2a]**

$=\frac{3xy-12xy^3}{1+2y}$

$=\frac{3xy(1-4y^2)}{1+2y}$ (taking out common factor)

$=\frac{3xy\cancel{(1+2y)}(1-2y)}{\cancel{1+2y}}$ (difference of 2 sqrs)

$=3xy(1-2y)$

19. **[2020 P1 #10a]**

$\frac{5y+2}{y+1}=\frac{y+1}{y}$ (*cross* multiply)

$y(5y+2)=(y+1)(y+1)$ Expand both sides

$5y^2+2y=y(y+1)+1(y+1)$

$5y^2+2y=y^2+2y+1$ (group like terms)

$5y^2+2y-2y-y^2-1=0$

$4y^2-1=0$ (dif of two squares)

$(2y+1)(2y-1)=0$

Either $(2y+1)=0$ or $(2y-1)=0$

$2y=-1$ or $2y=1$

$\frac{2y}{2}=\frac{-1}{2}$ or $\frac{2y}{2}=\frac{1}{2}$

$\therefore y=-\frac{1}{2}$ or $y=\frac{1}{2}$

20. **[2015 PII #2b]**

$\frac{p+q}{pq}-\frac{q+t}{qt}$

$=\frac{t(p+q)-p(q+t)}{pqt}$

$=\frac{tp+tq-pq-pt}{pqt}$

$=\frac{tp-pt+tq-pq}{pqt}$ (since $pt=tp$, they cancel out)

$=\frac{tq-pq}{pqt}$

$=\frac{q(t-p)}{pqt}$

$=\frac{t-p}{pt}$

21. **[2016 PII #2a]**

$$=\frac{(y-3)(y-2)}{y}\div\left(\frac{y-3}{y}\right)^2$$

$$=\frac{(y-3)(y-2)}{y}\div\frac{(y-3)^2}{y^2}$$

$$=\frac{(y-3)(y-2)}{y}\times\frac{y^2}{(y-3)^2}\text{(flipping fractions)}$$

$$=\frac{\cancel{(y-3)}(y-2)}{\cancel{y}}\times\frac{y^{\cancel{2}}}{\cancel{(y-3)}(y-3)}$$

$$=\frac{y(y-2)}{(y-3)}$$

22. **[2018 P1 #5]**

$$=\frac{x}{x-7}-\frac{7x-35}{x^2-12x+35}$$

$$=\frac{x}{x-7}-\frac{7(x-5)}{(x-7)(x-5)}\text{(factorizing denominator)}$$

$$=\frac{x}{x-7}-\frac{7}{x-7}\text{ (canceling } (x-5)\text{)}$$

$$=\frac{x-7}{x-7}$$

$$=1$$

23. **[2018 PII #17]**

$$\frac{2x}{x^2-1}+\frac{1}{1-x}$$

$$=\frac{2x}{(x+1)(x-1)}+\frac{1}{-(x-1)}$$

$$=\frac{2x}{(x+1)(x-1)}-\frac{1}{(x-1)}$$

$$=\frac{2x-(x+1)}{(x+1)(x-1)}$$

$$=\frac{2x-x-1}{(x+1)(x-1)}$$

$$=\frac{x-1}{(x+1)(x-1)}$$

$$=\frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}$$

$$=\frac{1}{x+1}$$

24. **[2019 PII #1a]**

$$=\frac{3x}{3x+y}\div\frac{9y^{\cancel{2}}}{\cancel{y}(3x+y)}$$

$$=\frac{3x}{3x+y}\div\frac{9y}{3x+y}$$

$$=\frac{\cancel{3x}^{1}}{\cancel{3x+y}_{1}}\times\frac{\cancel{3x+y}^{1}}{\cancel{9y}_{3}}\text{(flipping fractions)}$$

$$=\frac{x}{3y}$$

CH 6
SETS

Chapter Highlights

Sets are a collection of district objects. To solve the problems in this chapter, we are required to have knowledge on 'set language'. Set language consists of various words that enable us to identity and represent elements, union, intersection, compliments of sets etc.

Symbols used in set language

- $A \cup B$ (The union of set A and B)
- $A \cap B$ (The intersection of set A and B)
- n(A) (Number of elements in set A)
- A' (the compliment of set A)
- $A \subset B$ (A is a subset of B)
- $A \not\subset B$ (A is not a subset of B)
- $\varnothing$ (empty set)

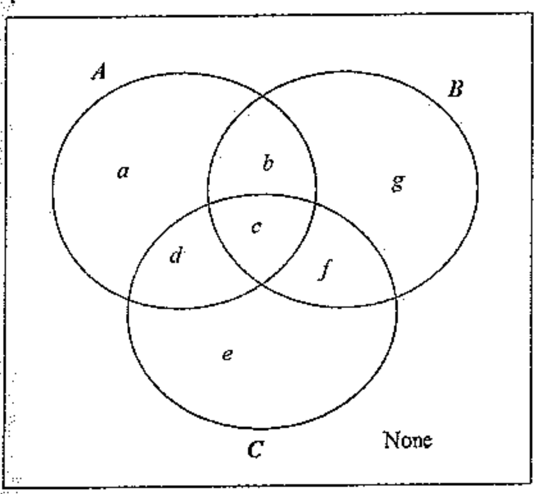
Important formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

With this formula, the needed term can be obtained by changing the subject of the formula.

Venn diagram with three variables

μ



A = a+b+c+d Only **A** = a

B = g+b+c+f Only **B** = g

C =d+c+f+e Only **C** = e

A and **B** only =b+c

Elements in **A** , **B** and **C** = c

B and C only = c+f and A and **C** only = c+d

1. The universal set $(\xi) = \{10, 20, 30, 40, 50, 60, 70\}$, A={10,30, 60} and B={20,40, 50}, evaluate A'$\cap$ B. **[2003 PP1 #5]**

2. A class of 50 students wrote tests in Mathematics, Biology and physical science. The results of the tests were as follows:
   - 12 passed Mathematics and Physical science
   - 19 passed Mathematics and Biology
   - 17 passed Biology and Physical Science
   - 2 passed Physical Science only
   - 5 passed Mathematics only
   - 6 passed Biology only

   If 5 students failed all the **three** subjects and x passed all the subjects, use venn-diagram to calculate the value of x. **[2003 P2#11b]**

3. Given that the universal set ξ ={11,14,15,17,18,20,23,26}, set X={11,14,15,17,18, 20} and set Y={15,17,18,20,23,26}. Find $X' \cup Y'$. **[2004 PI #22]**

4. Given that X = {11, 12, 15, 17}
   Y = {11, 15, 17, 19, 20} and
   Z = {11,17,21}

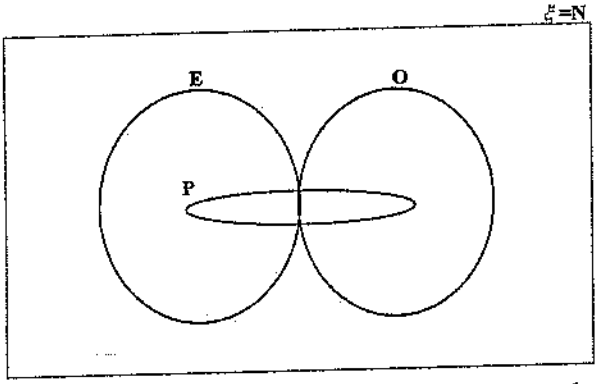
   i. Show the three sets on a venn diagram

   ii. Find $n(X \cap Y \cap Z)$. **[2004 PII #9a]**

5. Given that $\{a,c,e,h,i,l,m,s,t,w\}$ is a universal set and Y=$\{a,c,e,h,l,m,s\}$. Find $n(Y')$. **[2005 P1 #5]**

6. Given that **A, B** and **C** are sets,
   a. Draw a venn diagram and shade the region representing A'∩B'∩C.
   b. Find n(A'∩B'∩C), if n(A∪B)=8 and n(A∪B∪C)=12. **[2005 PII # 11b]**

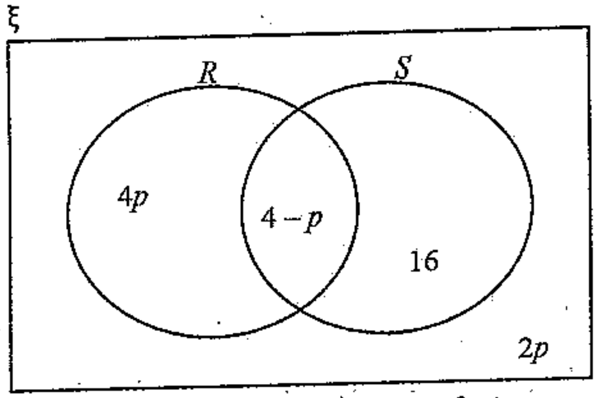
7. **Figure 1,** shows a Venn diagram representing set of all numbers (N), set of even numbers (E), set of odd numbers (O) and set of prime numbers (P).



Copy the Venn diagram and place the numbers $\frac{1}{2}$, 2, 6, 9 and 13 in the right places. **[2006 P1 #15]**

8. A church congregation has a youth group and music group. There are 400 people in the congregation out of which 40 people belong to both the youth group and the music group. There are 60 members who belong to the youth group only while 220 belong to neither the youth group nor music group.
   i. Draw a venn diagram illustrating this information.
   ii. Calculate the number of people who belong to the music group only. **[2006 PII #9a]**

9. **Figure 2** is a Venn diagram showing the number of elements in sets **R, S** and universal set ξ.



If n (R∪S) = 29, calculate the value of p.
[2007 PI #8]

10. Given that the universal set ξ= {11, 12, 13, 14, 15, 16, 17, 18, and 19}. Set B= {numbers greater than 16} and set C= {multiples of 3}. Find the elements of;
   i. Set B
   ii. Set C
   iii. Set (B∪C)'.

[2007 PII #6a]

11. At Mpini secondary school, 72 students like watching football, 64 like watching basketball and 62 like watching netball; 18 like watching football and basketball, 24 like watching netball and football and 20 like watching basketball and netball, 8 students like watching all three games and 56 don't like watching any game.
   i. Draw a Venn diagram representing this information
   ii. From the Venn diagram, calculate number of students at the school. **[2008 P2 #5b]**

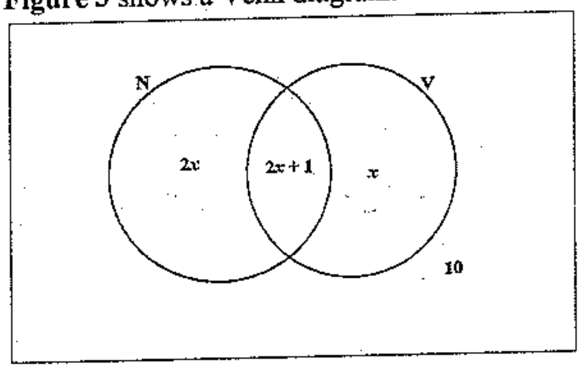
12. Given that $n(x) = 18$, $n(y) = 24$ and $n(x \cup y) = 40$. Find $n(x \cap y)$. **[2008 P1 #18]**

13. Given that X={a, e}; Y={b, c, d, e} and Z={c, d, e, f}, find (X∪Y) ∩ Z. **[2010 P1 #2]**

14. In a form four class, students learn French, Latin and history. 20 students learn French, 55 students learn latin and 37 students learn history. 7 students learn French and latin only, 5 learn Latin and history only, 2 learn French and history only. 10 do not learn any of these subjects while x students learn all the three subjects in the class. If there are 100 students in the class:
   i. Draw a venn diagram to present this information
   ii. Use your venn diagram to calculate the number of students who learn latin only.

[2010 PII #10b]

15. **Figure 3** shows a Venn diagram.



In the Venn diagram,

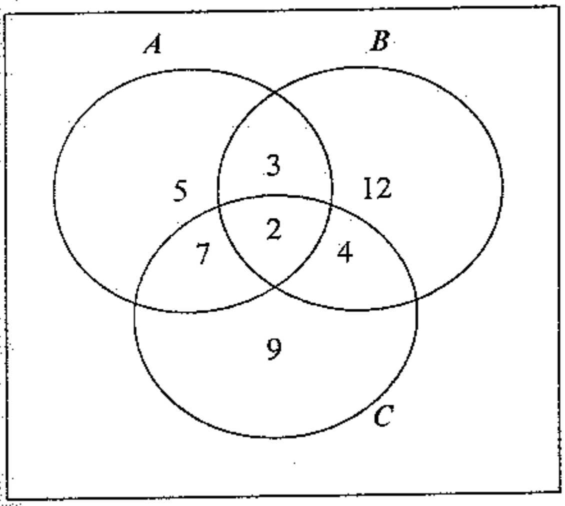
ξ = {girls in form three}

N={girls that play netball}

V={girls that play volleyball}

Given that there are 21 girls in the class, find how many girls play both netball and volleyball.
[2011 PI #11]

16. **Figure 4** Shows a Venn diagram of sets **A, B and C.**



Find $A' \cup (B \cap C)$. **[2011 PII #11b]**

17. Given that P and Q are two sets such that $n(P \cup Q) = 37$, $n(P) = 30$, and $n(P \cap Q) = 5$, find $n(Q)$. **[2012 P1 #13]**

18. In a class of 50 students, each of the student ate at least one of the following types of fruits: Bananas, mango and orange. It was found that:
- $(x+1)$ students ate all the three types of fruits
- 9 students ate mangoes and oranges only
- 8 students ate bananas and mangoes only
- 5 students ate bananas and oranges only
- x students ate bananas only
- $(x-1)$ students ate mangoes only
- $(x+4)$ students ate oranges only.

i. Illustrate the information using a Venn diagram.
ii. Find the number of students who ate mangoes.
[2012 P2 #9b]

19. Given that $\xi = \{1,2,3,4,5,6,7,8\}$, $A = \{1,2,3,7\}$ and $B = \{3,4,5,6,7\}$, draw a venn diagram to illustrate this information.
[2013 P1 #7]

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

20. There are 60 vitamin pills which contains at least one of the following vitamins: A, B and C.
- 12 pills contain vitamin A only.
- 7 pills contain vitamin B only.
- 11 pills contain vitamin C only
- 6 pills contain all the three vitamins.
- $n(B \cap C \cap A') = x$
- $n(A \cap C \cap B') = 2x$
- $n(A \cap B \cap C') = 3x$

Using a Venn diagram, calculate the number of pills which contain Vitamin A.
[2013 P2 #10b]

21. Given that the universal set $\xi = \{1,2,3,4,6,9\}$, set A and B are subsets of ξ such that A = {even numbers} and B = {perfect squares}.
Find $n(A \cap B)'$. **[2014 P1 #8]**

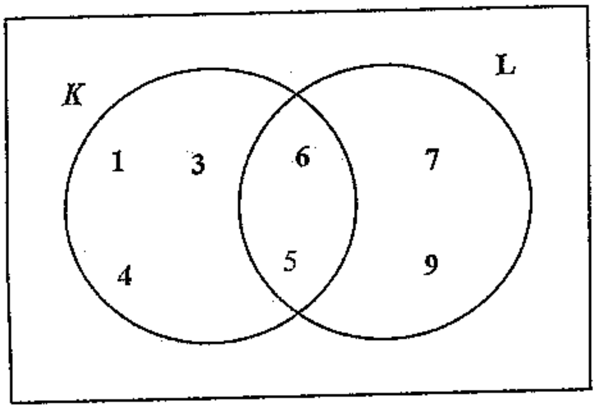
22. Given that A and B are sets such that $n(A) = 12$, $n(B) = 15$ and $n(A \cap B) = x$, find $n(A' \cup B')$ in terms of x. **[2014 PII #4a]**

23. Given that $\mu = \{1,3,6,7,9,15,17,18,19,22,24\}$ $M = \{3,15,18,19\}$, $N = \{1,6,7,18,24\}$ and $P = \{3,6,15,19,24\}$. Find $M \cup N' \cap P$.
[2017 PII #4a]

24. In a class of 55 students, 36 drink tea, 18 drink coffee and 10 neither. If the number of students who drink both tea and coffee is x, draw a Venn diagram and use it to find the value of x.
[2015 PI #7]

25. Given that $\mu = \{1,4,5,9,10,16,17,19,23,25,32,36,45\}$, sets A and G are subset of μ such that:
$A = \{r : 4 < r < 25\}$ and
$G = \{x : x \text{ is an odd number}\}$, find $A \cap G$.
[2015 PII #5a]

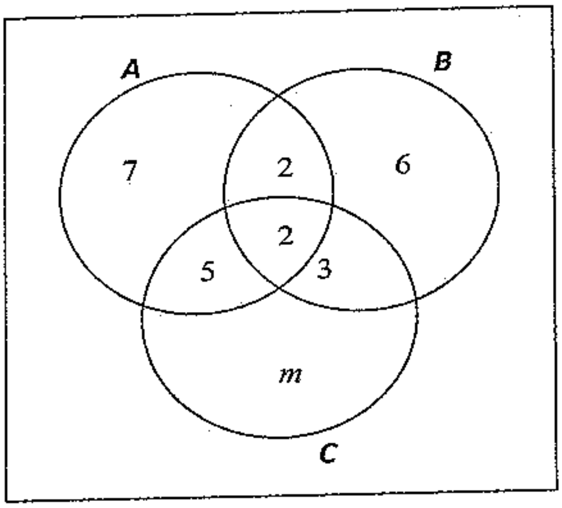
26. **Figure 5** is a Venn diagram showing elements in set **K** and set **L**.



Find $(K \cup L)' \cup (K \cap L)'$. **[2016 PI #15]**

27. Given that $n(\mu) = 60$, $n(\mathbf{L}) = 44$, $n(\mathbf{R}) = 34$, and $n(\mathbf{L} \cup \mathbf{R})' = 6$. Calculate $n(\mathbf{L} \cap \mathbf{R})$.
[2016 PII #10a]

28. **Figure 6** is a Venn diagram showing number of elements of sets **A, B** and **C.**



If $n(A \cup B \cup C) = 30$, calculate the value of m.
[2018 P1 #7]

29. Given that $\mu = \{a, b, c, d, e, f, g\}$, set **P** and **Q** are subsets of μ such that $\mathbf{P} = \{a, b, c, e\}$ and $\mathbf{Q} = \{a, b, d, e\}$, find $\mathbf{P} \cup (\mathbf{P}' \cap \mathbf{Q})$.
[2018 PII #5]

30. In a class, 40 students were asked to choose among agriculture, Home economics and Geography.
   - 8 students chose all 8 subjects
   - 7 students chose Agriculture and Home Economics only.
   - 4 students chose agriculture and Geography only.
   - 2 students chose Home Economics and Geography only
   - 10 students chose geography only.
   - $x+1$ chose Agriculture only.
   - x student chose Home Economics only.

   a) Present this information in a Venn diagram.
   b) Use the Venn diagram to calculate the number of students took Agriculture only.
[2019 PII #12]

1. **[2003 PP1 #5]**

Given $\xi = \{10,20,30,40,50,60,70\}$,

$A = \{10,30,60\}, B = \{20,40,50\}$

$A' = \{20,40,50,70\}$

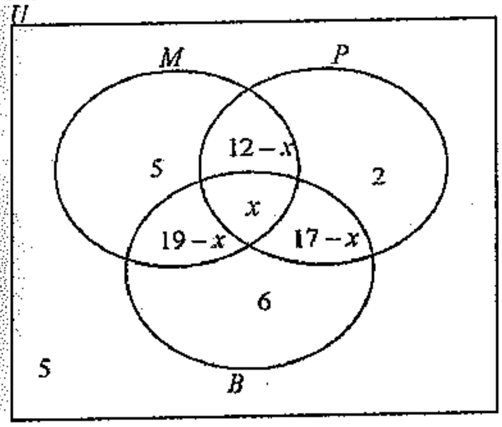
$B = \{20,40,50\}$

$A' \cap B$ consists of all the elements that are found in A' and B.
$\therefore A' \cap B = \{20, 40, 50\}$.

---

2. **[2003 P2#11b]**

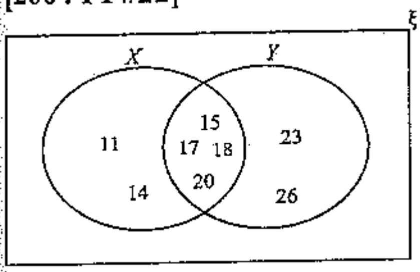
The Venn diagram will look as follows:
Let **M** be the number of students that passed mathematics.
Let **B** be the number of students that passed biology.
Let **P** be the number of students that passed physics.



We are given that the total number of students who wrote the test = 50.
But from the Venn diagram, total number of students who write the test =
$x+(12-x)+(19-x) + (17-x)+5+2+6+5$
$\Rightarrow x+12-x+19-x+17-x+5+2+6+5=50$
Now arranging like terms together:
$x-x-x-x+12+19+17+5+2+6+5=50$
$x-3x+66=50 \Rightarrow -2x=50-66$
$\Rightarrow -2x=-16$ (divide by -2 both sides)
$\Rightarrow x=8$
$\therefore$ The value of $x = 8$

---

3. **[2004 PI #22]**



$\xi = \{11, 14, 15, 17, 18, 20, 23, 26\}$

$X = \{11, 14, 15, 17, 18, 20\}$

$Y = \{15, 17, 18, 20, 23, 26\}$

$X' = \{23,26\}$ (elements not in X')

$Y' = \{11,14\}$ (elements not in Y)

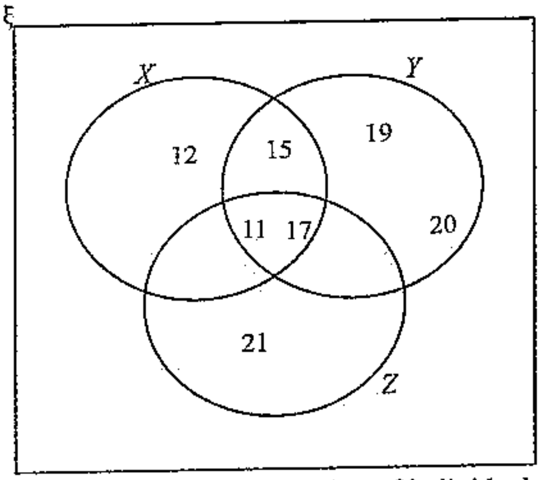
$X' \cup Y' = \{11,14,23,26\}$

---

4. **[2004 PII #9a]**

i. Start with inserting elements found in all three sets: 11 and 17.

Then inset those found in pairs of sets only i.e put 15 in the space where X intersect Y only.
Then in the space where the set doen't intersect with an other set, insert numbers found only in the set: 12 in X only, 19 and 20 in Y only and 21 in Z only



ii. n(X∩Y∩Z) is the counting of individual elements in the intersection of the three sets.
X∩Y∩Z = {11, 17}
$\therefore n$(X∩Y∩Z) = 2 elements.

---

5. **[2005 P1 #5]**

Given that $\xi = \{a, c, e, \text{h}, \text{i}, l, \text{m}, \text{s}, \text{t}, \text{w}\}$

And $Y = \{a,c,e,h,i,m,s\}$

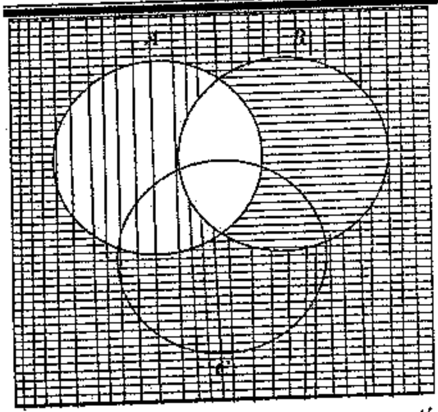
$Y' = \{l, t, \text{m}\}$ (Elements not in y)

So n(Y') = 3 (Count individual members of Y')

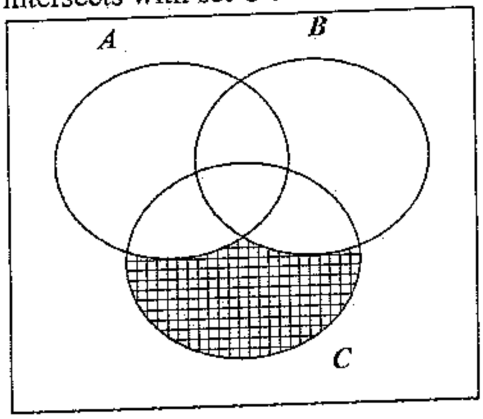
---

6. **[2005 PII # 11b]**

To carefully detect the region, we start with shading A' with horizontal lines, then shade B' with vertical lines. The region $A' \cap B'$ is the region forming boxes.



Then check which of the region $A' \cap B'$ intersects with set C to find $A' \cap B' \cap C$:



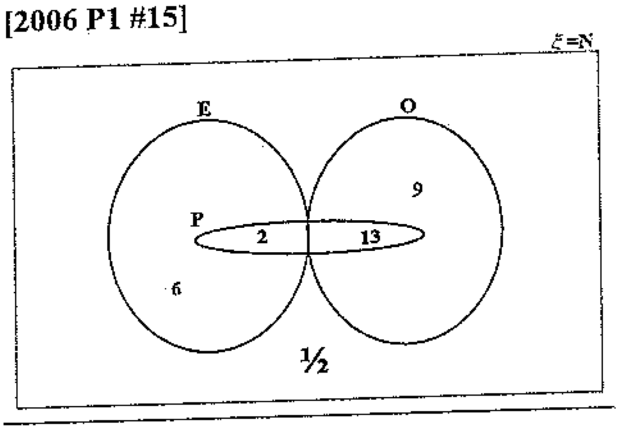
ii. From the shaded region, we can clearly see that $n(A' \cap B' \cap C) = n(A \cup B \cup C) - n(A \cup B)$

$\Rightarrow n(A' \cap B' \cap C) = 12 - 8$

$\therefore n(A' \cap B' \cap C) = 4$

---

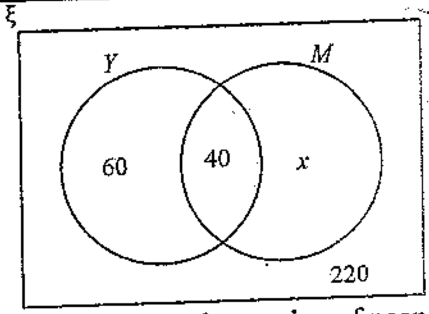
7. **[2006 P1 #15]**



---

8. **[2006 PII #9a]**

Let ξ be the total number of people in the congregation. We are given that ξ = 400.
Let Y = Set of people in the youth group,
M = Set of people in the music group.
Y∩M is the number of people that belong to both the youth group and the music group.
$\Rightarrow$ Y∩M = 40 (given)
The Venn diagram will be as follows:



Let x represent the number of people who belong to music group only.
$\Rightarrow x + 40 + 60 + 220 = 400$
$\Rightarrow x + 320 = 400$
$\Rightarrow x = 80$
$\therefore$ 80 people belong to the music group only.

---

9. **[2007 PI #8]**

$n(R \cup S) = 29$

$n(R \cup S) = 4p + (7 - p) + 16$

$\Rightarrow 4p + (7 - p) + 16 = 29$

$3p + 23 = 29 \Rightarrow 3p = 29 - 23$

$\Rightarrow \frac{3p}{3} = \frac{6}{3}$

$\therefore p = 2$

---

10. **[2007 PII #6a]**

Given $\xi = \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$

Given that $B = \{\text{numbers greater than } 16\}$

i.) $\Rightarrow B = \{17, 18, 19\}$

Given that $C = \{\text{multiple of } 3\}$

ii.) $\Rightarrow C = \{12, 15, 18\}$

iii.) $\Rightarrow (B \cup C) = \{12, 15, 17, 18, 19\}$

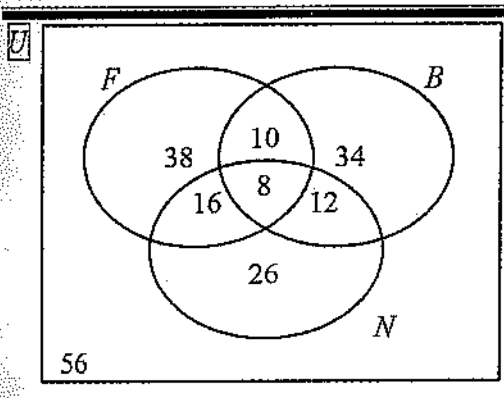
$\therefore (B \cup C)' = \{11, 13, 14, 16\}$ (what is not in ξ)

---

11. **[2008 PII #5b]**

i. Let **F** represent the set of students who like watching football

Let **B** represent the set of students who like watching basketball.
Let **N** be the set of students who like watching netball.
- Start with 8 who watch all
- B and N only = 20 - 8 = 12
- F and N only = 24 - 8 = 16
- B and F only = 18 – 8 = 10
Then
B only = 64 – (12+8+10) = 34
F only = 72 - (16+8+10) = 38
N only = 62 – (16+8+12) = 26



ii. From the Venn diagram, we calculate the number of students at the school:

$38+10+8+16+34+12+26+56=200$

$\therefore$ There are 200 students at the school.

12. **[2008 P1 #18]**

Given $n(x)=18, n(y)=24, n(x\cup y)=40$

need to find $n(x\cap y)$:

now: $n(x\cup y)=n(x)+n(y)-n(x\cap y)$

$40=18+24-n(x\cap y)$

$40-18-24=-n(x\cap y)$

$-2=-n(x\cap y)$

$2=n(x\cap y)$

$\therefore n(x\cap y)=2$

13. **[2010 P1 #2]**

Given $X=\{a,\ e\}$; $Y=\{b,\ c,\ d,\ e\}$

$\Rightarrow (X\cup Y)=\{a,b,c,d,e\}$

And $Z=\{c,\ d,\ e,\ f\}$

$\therefore (X\cup Y)\cap Z=\{c,d,e\}$

14. **[2010 PII #10b]**

i. The venn diagram:

Let **F** represent the set of students who learn French

Let **L** represent the set of students who learn Latin

Let **H** represent the set of students who learn History.

Those who learn French only:

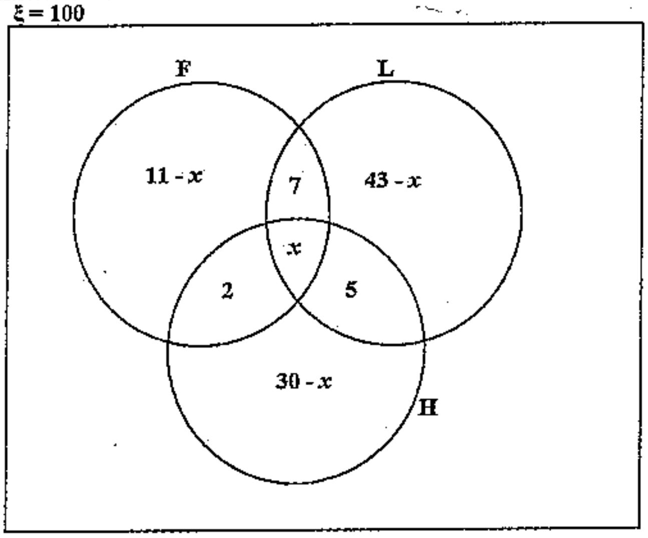
$20-(7+x+2)=11-x$.

Those who learn Latin only:

$55-(7+x+5)=43-x$

Those who learn History only:

$37-(5+x+2)=30-x$



ii. We should first find the value of x:

We are given that there are 100 students in the class.

Using the Venn diagram, number of students

$=(43-x)+(11-x)+(30-x)+7+5+2+10++x$

$=108-2x$

$\Rightarrow 108-2x=100$

$\Rightarrow 108-100=2x$

$\Rightarrow 2x=8$ (divide 2 both sides)

$\Rightarrow x=4$

From the Venn diagram, number of students learning Latin only= 43-x

=43-4

=39.

15. **[2011 PI #11]**

We are given that $\xi=21$

From the Venn diagram,

$\xi=2x+2x+1+x+10$

$\Rightarrow 5x+11=21 \Rightarrow 5x=21-11$

$\Rightarrow 5x=10$

$\therefore x=2$

From Venn diagram girls playing both netball and volleyball $=2x+1$

$=2(2)+1$

$=5$

$\therefore$ 5 girls play both netball and volleyball.

16. **[2011 PII #11b]**

From the Venn diagram $\xi=\{5, 7, 3, 4, 2, 12, 9\}$

$A=\{2,3,5,7\}$,

$A'=\{4,9,12\}$ (elements not in A but in ξ)

$B=\{2,3,4,12\}$, $C=\{2,4,7,9\}$

$\Rightarrow B\cap C=\{2,4\}$

$A'\cup(B\cap C)=\{2,4,9,12\}$

17. **[2012 P1 #13]**

We know that $n(P\cup Q)=n(P)+n(Q)-n(P\cap Q)$

Given that $n(P\cup Q)=37$, $n(P)=30$,

$n(P\cap Q)=5$.

$n(Q)=n(P\cup Q)-n(P)+n(P\cap Q)$

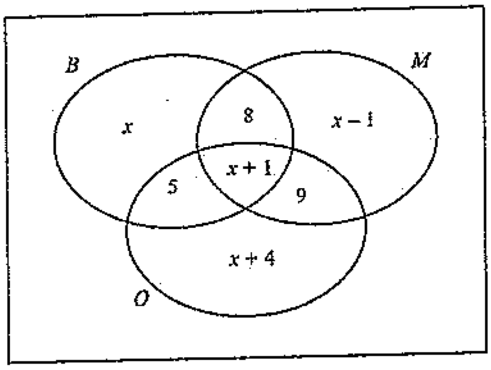
$n(Q)=37-30+5$

$\therefore n(Q)=12$

18. **[2012 P2 #9b]**

Let **B** be the set of students who ate Bananas
Let **M** be the set of students who ate Mangoes
Let **O** be the set of students who ate Oranges

i.



i. We are given that the total number of students $=50$.

ii. Using the Venn diagram, the total number of student $=x+8+x-1+5+x+1+9+x+4$.

$=4x+26$

$\Rightarrow 4x+26=50$

$4x=50-26$

$4x=24$(divide both sides by 4)

$x=6$

ii.

From the Venn diagram, number of students who ate mangoes is $=8+9+x+1+x-1$

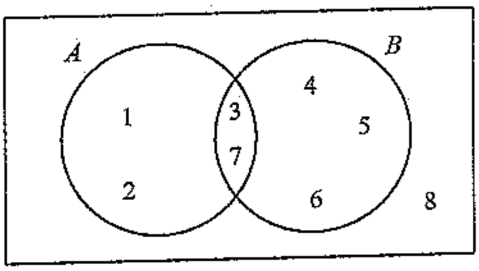
Substituting x,

$=8+9+6+1+6-1=29$

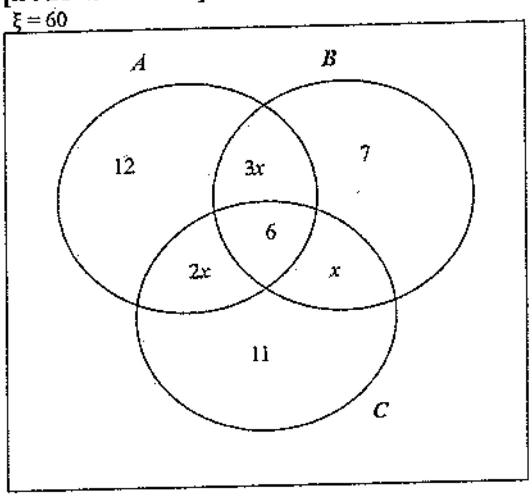
$\therefore$ 29 students ate mangoes in the class.

19. **[2013 P1 #7]**

Below is the Venn diagram:



20. **[2013 P2 #10b]**



We are given that there are 60 pills

From the Venn diagramn total pills $=$

$6+x+2x+3x+12+7+11=6x+36$

$\Rightarrow 6x+36=60$ $\Rightarrow 6x=60-36$

$\Rightarrow 6x=24$

$\Rightarrow \frac{6x}{6}=\frac{24}{6}$

$\Rightarrow x=4$

Venn diagram shows number of pills containing vitamin $A=12+6+3x+2x$

$=18+5x$

$=18+5(4)$

$=38$

$\therefore$ 38 pills contain vitamins A.

21. **[2014 P1 #8]**

Given that $\xi = \{1, 2, 3, 4, 6, 9\}$,

A={Even numbers}

$\Rightarrow A = \{2, 4, 6\}$

$B = \{\text{Perfect squares}\}$

$\Rightarrow$ B={1, 4, 9}

$A \cap B = \{4\}$

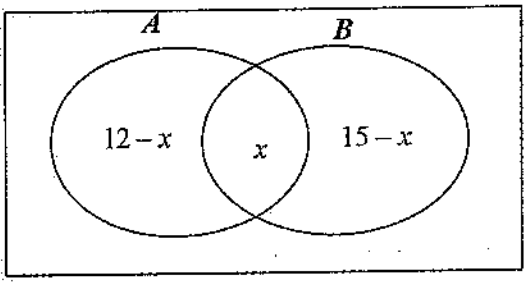
$(A \cap B)' = \{1, 2, 3, 6, 9\}$

$n(A \cap B)' = 5$

---

22. **[2014 PII #4a]**

$n(A) = 12, n(B) = 15, n(A \cap B) = x$



$n(A') = 15 - x$ (number of elements outside A)

$n(B') = 12 - x$ (number of elements outside B)

$\therefore n(A' \cup B') = n(A') + n(B')$

$= (15 - x) + (12 - x)$

$n(A' \cup B') = 15 + 12 - x - x$

$n(A' \cup B') = 27 - 2x$

---

23. **2017 PII #4a]**

Given $\mu = \{1, 3, 6, 7, 9, 15, 17, 18, 19, 22, 24\}$,

M={3,15,18,19},

$N = \{1, 6, 7, 18, 24\} \Rightarrow N' = \{3, 9, 15, 17, 19, 22\}$

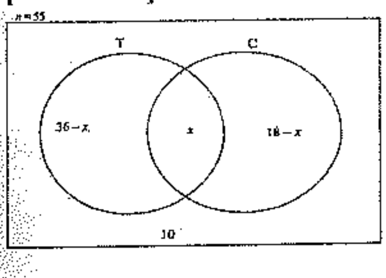
$M \cup N' = \{3, 9, 15, 17, 18, 19, 22\}$

P={3, 6, 15, 19, 24}

$\therefore M \cup N' \cap P = \{3, 15, 19\}$

---

24. **[2015 PI #7]**



$10 + (36 - x) + x + (18 - x) = 55$

$10 + 36 + 18 - x + x - x = 55$

$64 - x = 55$

$x = 64 - 55$

$x = 9$

$\therefore$ 9 students drink both tea and coffee

---

25. **[2015 PII #5a]**

$\mu = \{1, 4, 5, 9, 10, 16, 17, 19, 23, 25, 32, 36, 45\}$

$A = \{r : 4 < r < 25\}$

$\Rightarrow A = \{5, 9, 10, 16, 17, 19, 23\}$

and $G = \{x : x \text{ is an odd number}\}$,

$G = \{1, 5, 9, 17, 19, 23, 25, 45\}$

$A \cap G = \{5, 9, 17, 19, 23\}$

---

26. **[2016 PI #15]**

$K \cup L = \{1, 3, 4, 6, 5, 7, 9\}$

$\therefore (K \cup L)' = \{\ \}$ (empty set)

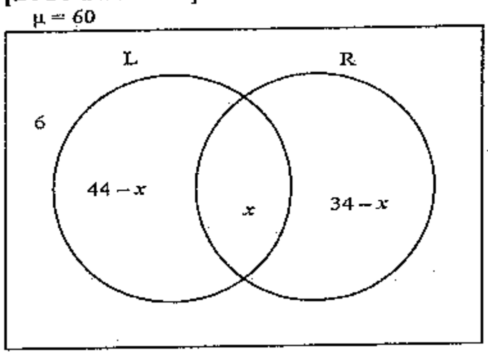
$K \cap L = \{5, 6\}$

$\therefore (K \cap L)' = \{1, 3, 4, 7, 9\}$

$\therefore (K \cup L)' \cup (K \cap L)' = \{1, 3, 4, 7, 9\}$

---

27. **[2016 PII #10a]**



Let $n(L \cap R)$ be x

Given that μ=60.

Also from Venn diagram:

$\mu = 6 + (44 - x) + x + (34 - x)$

$6 + 44 + 34 + x - x - x = 60$

$84 - x = 60$

$x = 84 - 60$

$x = 24$

$\therefore n(L \cap R) = 24$.

---

28. **[2018 P1 #7]**

$(A \cup B \cup C) = 30$................(given)

from the Venn diagram,

$(A \cup B \cup C) = 7 + 6 + m + 5 + 3 + 2 + 2$

$= 25 + m$....(from the venn diagram)

$\Rightarrow 25 + m = 30$

$m = 30 - 25$

$\therefore m = 5$

29. **[2018 PII #5]**

To find $P \cup (P' \cap Q)$

$\mu = \{a, b, c, d, e, f, g\}$

$P = \{a, b, c, e\}$ and $Q = \{a, b, d, e\}$

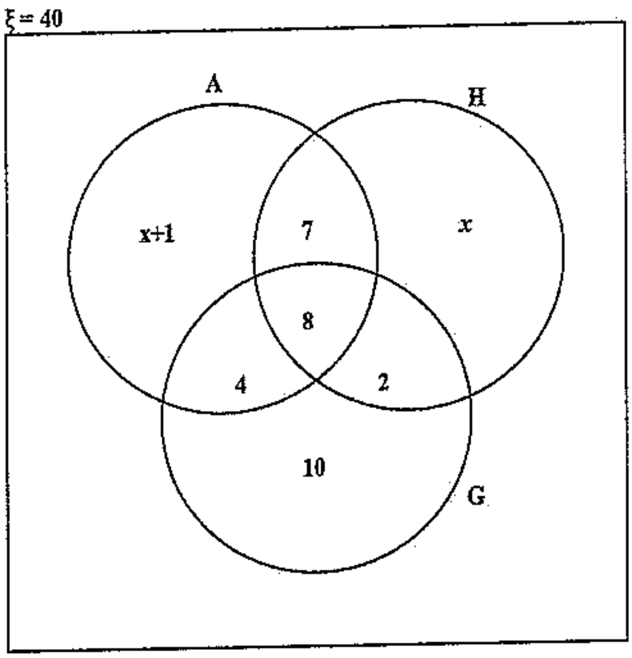
$P' = \{d, f, g\}$

$\therefore P' \cap Q = \{d\}$

$\therefore P \cup (P' \cap Q) = \{a, b, c, d, e\}$

30. **[2019 PII #12]**

a. The Venn diagram



b. Number of students who took Agriculture only

First, solve for x as follows:

$(x+1) + x + 10 + 7 + 2 + 4 + 8 = 40$

$2x + 32 = 40$

$2x = 40 - 32$

$2x = 8$

$\frac{\cancel{2}x}{\cancel{2}} = \frac{\cancel{8}^4}{\cancel{2}}$

$x = 4$

Students who took Agriculture only

$= x + 1$

$= 4 + 1$

$= 5$

$\therefore 5$ students took Agriculture only

CH 7 MAPPING AND FUNCTIONS

Chapter Highlights

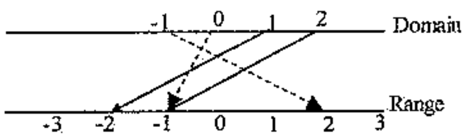
In mapping and functions, we have to be familiar with the relations that exist between the domain and range. The domain and range are sets of input and output values respectively. Mapping and function generally deals with a special relation between two sets that connects each member of the domain to only one element in the range. Functions will be denoted as $f(x)$, $g(x)$, $h(x)$ etc.

To safely attempt the problems in this section, students require knowledge from topics such as Sets and Subject of the formula. The problems below require us to find either the range or domain given the other. We shall also be required to evaluate functions at some given values or solve for missing values in some functions.

1. Given that $f(x)=x^3-x$, calculate $f(-2)$. **[2003 PI #1]**

2. In **Figure 1,** the function $f: x \rightarrow x^2-2x-1$ is defined on the domain {-1, 0, 1, 2}.

   Copy and complete the mapping diagram for the function.



[2003 PP II #3a]

3. Given that $g(x)=\dfrac{3x}{x+1}$ calculate the value of x when $g(x)=2$. **[2004 P1#3]**

4. If $f(x)=\dfrac{3x}{8x-a}$, find $f(-a)$ in its simplest form. **[2004 P2#1a]**

5. The function $y=2+x$ has the range {3,6}. Find its domain. **[2005 PI #6]**

6. The function $f(x)=\dfrac{2x-1}{x}$ is defined on the domain {-1, ½, 2}. Draw the arrow diagram to represent this function. **[2005 PII #4b]**

7. If $f(x)=8^x-6$. Find $f(2/3)$. **[2006 PI #4]**

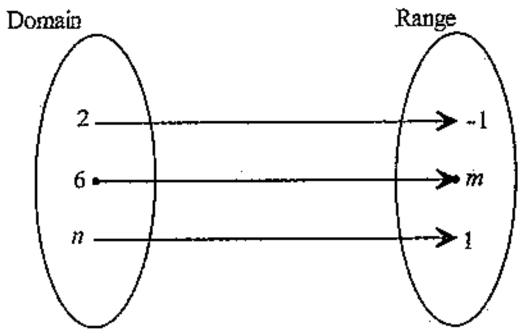
8. Given that $f(x)=x+3$ and $f(2)=2+x+g(x)$, to find $g(x)$. **[2006 PII #1b]**

9. The function $f(x)=\dfrac{1}{3x-1}$. Given that {-1, 0, 2} is the domain. Find the range. **[2007 PI #4]**

10. Given the function $g(x)=dx-5$ and that g (2) =1. Find the value of d. **[2007 PII #2a]**

11. The image of a set $\{-1,0\}$ under function ax^2+b is $\{7,-5\}$ .calculate the value of a. **[2008 P2 #1a]**

12. **Figure 2** shows an arrow diagram for the function $f(x)=2x-5$



Calculate the values of m and n . **[2008 P1 #7]**

13. The function $f(y)=3y+2$. Given that {5} is the range. Find the domain. **[2010 P1 #4]**

14. Given that $f(y)=\dfrac{y^2}{3}+1$, calculate the values of y when $f(y)=4$. **[2010 PII #3a]**

15. Given that $g(x)=\dfrac{2x^3}{3}+1$; find $g(-1)$ in its simplified form. **[2011 PI #2]**

16. Given that $g(x)=ax^2+bx$, find the values of a and b if $g(1)=-1$ and, $g(3)=15$. **[2011 PII #3b]**

17. Given that $f(x) = bx + 4$ and $f(4) = 24$, find b. **[2012 P1 #1]**

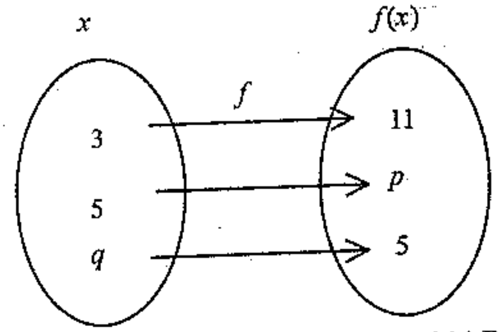
18. Given that $h(x) = \frac{2x+1}{3x}$, find p when $h(p) = \frac{1}{2}$. **[2012 P2 #2b]**

19. The function $f(x)$ is defined as $f(x) = 3x^2 + 3$, calculate $f(\sqrt{3})$. **[2013 P1 #3]**

20. Given that $f(x) = x^3$, find the value of x when $f(x) = f(2) + 19$. **[2013 P2 #3a]**

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

21. **Figure 3** shows an arrow diagram for the function $f(x) = 3x + 2$



Find the values of p and q. **[2014 P1 #2]**

22. Given that $f(x) = 9^x$, calculate $f\left(\frac{3}{2}\right)$ **[2014 PII #1a]**

23. Given that $f(x) = \frac{1-2x}{3-x}$, find $f(-2)$. **[2015 PI #2]**

24. Given that $f(x) = ax^2 + bx - 1$, $f(-1) = 2$, and $f(2) = 5$, find the value of a and b. **[2015 PII #2a]**

25. Given the functions $f(x) = x^3 + 5$. calculate $f(2^{1/3})$. **[2016 PI #3]**

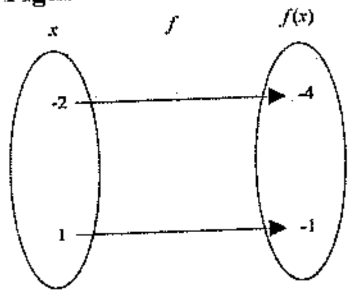
26. Given that $f(x) = \sqrt[3]{x^2 - 1}$ and $f(x) = 2$, find the values of x. **[2016 PII #1a]**

27. Given that $f(x) = \frac{1-x}{3}$. Evaluate $f(-8)$. **[2017 PI #2]**

28. Given that $g : u \to \frac{u-1}{2u+3}$, calculate $g(4)$. **[2017 PII #1a]**

29. Given that $f(x) = \frac{5-7x}{3}$, calculate $f(2)$. **[2018 P1 #2]**

30. **Figure 4** shows a function $f : x \to ax + b$



Calculate the value of a and the value of b. **[2018 PII #15]**

31. Given that $h(x) = 3x^2$ and $h(a+1) = 3$, calculate the value of a. **[2019 PI #2]**

32. Given that $f(1-x) = x^2 + g(x) - 2$ and $g(x) = 1 - x$, find $f(2)$. **[2020 Mock PI #11]**

33. Given that $g(x) = \frac{x}{1-x}$, find the value of a if $g(3-a) = \frac{2}{3}$. **[2021 Mock PII #1a]**

1. **[2003 PI #1]**

$f(x) = x^3 - x$

$f(2) = (-2)^3 - (-2)$

$= -8 + 2$

$= -6$

2. **[2003 PP II #3a]**

Domain= {-1, 0, 1, 2}

$f(x) = x^2 - 2x - 1$

$f(-1) = (-1)^2 - 2(-1) - 1$

$= 1 + 2 - 1$

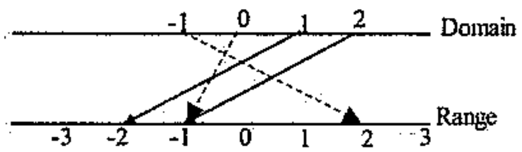
$= 2$

$f(0) = 0 - 2 \times 0 - 1$

$= 0 - 0 - 1$

$= -1$

The completed mapping:



3. **[2004 P1#3]**

Let g(x)= 2, then replacing g(x) with 2:

We have, $2 = \frac{3x}{x+1}$

$2(x+1) = 3x$ (*multiply both sides by* $x+1$)

$2x + 2 = 3x$

$2x - 3x = -2$

$-x = -2$

$x = 2$ (*multiply both sides by -1*)

4. **[2004 P2#1a]**

$f(x) = \frac{3x}{8x - a}$

So $f(-a) = \frac{3(-a)}{8(-a) - a}$ (*substituting* $-a$ *for* x)

$= \frac{-3a}{-8a - a}$

$= \frac{-3a}{-9a}$

$= \frac{1}{3}$

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

5. **[2005 PI #6]**

Range is a set of outputs (y-values)

At $y = 3$,

$3 = 2 + x$

$3 - 2 = x$

$1 = x$

$\therefore x = 1$

At $y = 6$,

$6 = 2 + x$

$6 - 2 = x$

$4 = x$

$\therefore x = 4$

Thus, Domain is {1, 4}

6. **[2005 PII #4b]**

Solving for range:

$f(-1) = \frac{2(-1) - 1}{-1}$

$= \frac{-2 - 1}{-1} = \frac{-3}{-1} = 3$

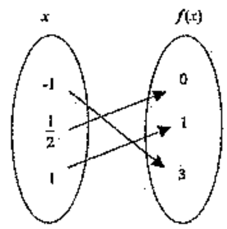
$f\left(\frac{1}{2}\right) = \frac{2\left(\frac{1}{2}\right) - 1}{\frac{1}{2}}$

$= \frac{1 - 1}{\frac{1}{2}} = \frac{0}{\frac{1}{2}} = 0$

$f(1) = \frac{2(1) - 1}{1}$

$= \frac{2 - 1}{1} = \frac{1}{1} = 1$

$\therefore$the range is {3, 0, 1}



7. **[2006 PI #4]**

$f\left(\frac{2}{3}\right) = 8^{2/3} - 6$

$= \sqrt[3]{8^2} - 6$ since $a^{m/n} = \sqrt[n]{a^m}$

$= \sqrt[3]{64} - 6$

$= 4 - 6$

$= -2$

8. **[2006 PII #1b]**

$f(x) = x + 3$
$\Rightarrow f(2) = 2 + 3$
$= 5$

But $f(2) = 2 + x + g(x)$

$\therefore 2 + x + g(x) = 5 \quad [both = f(2)]$

$g(x) = 5 - 2 - x$
$\therefore g(x) = 3 - x$

9. **[2007 PI #4]**

Given $f(x) = \dfrac{1}{3x-1}$; domain $\{-1, 0, 2\}$

Range:

$f(-1) = \dfrac{1}{3(-1)-1} = \dfrac{1}{-3-1} = -\dfrac{1}{4}$

$f(0) = \dfrac{1}{3(0)-1} = \dfrac{1}{0-1} = \dfrac{1}{-1} = -1$

$f(2) = \dfrac{1}{3(2)-1} = \dfrac{1}{6-1} = \dfrac{1}{5}$

$\therefore$ The range is $\left\{-\dfrac{1}{4}, -1, \dfrac{1}{5}\right\}$

10. **[2007 PII #2a]**

$g(x) = dx - 5$

to evaluate $g(2)$:

$g(2) = d(2) - 5$

$1 = 2d - 5 \quad (\text{since } g(2) = 1)$

$1 + 5 = 2d$

$6 = 2d$

$\dfrac{\cancel{6}}{\cancel{2}} = \dfrac{\cancel{2}}{\cancel{2}}d$

$d = 3$

11. **[2008 P2 #1a]**

let $f(x) = ax^2 + b$

$f(-1) = a(-1)^2 + b$

$7 = a + b$ [Since $f(-1) = 7$].....(i)

$f(0) = a(0)^2 + b$

$-5 = 0 + b$

$-5 = b$ [Since $f(0) = -5$].....(ii)

Substituting (ii) in (i):

$7 = a + (-5)$

$7 + 5 = a$

$a = 12$

$\therefore a = 12$ and $b = -5$

12. **[2008 P1 #7]**

From the figure,

$f(6) = m$ and $f(n) = 1$

Taking $f(x) = 2x - 5$

$f(6) = 2(6) - 5$

$m = 12 - 5$

$m = 7$

$f(n) = 2(n) - 5$

$2n - 5 = 1$

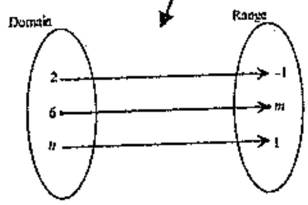
$2n = 1 + 5$

$2n = 6$

$\dfrac{\cancel{2}}{\cancel{2}}n = \dfrac{\cancel{6}}{\cancel{2}}$

$n = 3$

$\therefore m = 7$ and $n = 3$



13. **[2010 P1 #4]**

$f(y) = 3y + 2$

Given range is 5,

$5 = 3y + 2$

$5 - 2 = 3y$

$3y = 3$

$y = 1$

The domain is $\{1\}$

14. **[2010 PII #3a]**

$f(y)=\frac{y^2}{3}+1;$

$f(y)=4$

$4=\frac{y^2}{3}+1$

$4-1=\frac{y^2}{3}$

$3=\frac{y^2}{3}$

$y^2=9$

$y=\pm\sqrt{9}$

$y=3$ or -3

---

15. **[2011 PI #2]**

$g(-1)=\frac{2(-1)^3}{3}+1$

$=\frac{-2}{3}+1$

$=\frac{1}{3}$

---

16. **[2011 PII #3b]**

Given that $g(x)=ax^2+bx$

$g(1)=a(1)^2+b(1)$

$-1=a+b$ [Since $g(1)=-1$]

$a+b=-1$ .....(i)

$g(3)=a(3)^2+b(3)$

$15=9a+3b$ [Since $g(3)=15$]

$3a+b=5$......(ii) (dividing through by 3)

from eqt(i); $a=-1-b$ ....(iii)

substituting in (ii),

$3(-1-b)+b=5$

$-3-3b+b=5$

$-2b=5+3$

$\therefore -2b=8$

$\frac{\cancel{-2}b}{\cancel{-2}}=\frac{\cancel{8}}{\cancel{-2}}$

substituting $b=-4$ in (i)

$a=-b-1$

$a=-(-4)-1$

$a=4-1$

$=3$

$\therefore a=3$ and, $b=-4$

---

17. **[2012 P1 #1]**

$f(x)=bx+4$

$f(4)=b(4)+4$

$24=4b+4$ [Since $f(4)=24$]

$24-4=4b$

$4b=20$

$b=\frac{2\cancel{0}}{\cancel{4}}$

$b=5$

---

18. **[2012 P2 #2b]**

Given that $h(x)=\frac{2(x+1)}{3x}$

$h(p)=\frac{2p+1}{3p}$

$\frac{1}{2}=\frac{2p+1}{3p}$

$3p=2(2p+1)$

$3p=4p+2$

$3p-4p=2$

$-p=2$

$\therefore p=-2$

---

19. **[2013 P1 #3]**

$f(x)=3x^2+3$

$f(\sqrt{3})=3(\sqrt{3})^2+3$

$f(\sqrt{3})=3(3)+3$ Since $(\sqrt{a})^2=a$

$f(\sqrt{3})=9+3$

$f(\sqrt{3})=12$

---

20. **[2013 P2 #3a]**

$f(x)=x^3$

$f(x)=f(2)+19$

$\therefore x^3=f(2)+19$ [both equal to $f(x)$]

$x^3=2^3+19$

$x^3=8+19$

$x^3=27$

$x=\sqrt[3]{27}$

$x=3$

21. **[2014 P1 #2]**

From the fig,

$f(5)=p$ and $f(q)=5$

$f(x)=3x+2$

$f(5)=3(5)+2$

$p=15+2$ [since $f(5)=p$]

$p=17$

$f(x)=3x+2$

$f(q)=3q+2$

$5=3q+2$ [since $f(q)=5$]

$5-2=3q$

$3=3q$

$\frac{\cancel{3}}{\cancel{3}}=\frac{\cancel{3}}{\cancel{3}}q$

$q=1$

$\therefore p=17$ and $q=1$

22. **[2014 PII #1a]**

$f(x)=9^x$

$f\left(\frac{3}{2}\right)=9^{\frac{3}{2}}$

$=\left(\sqrt{9}\right)^3$ since $a^{m/n}=\sqrt[n]{a^m}=\left(\sqrt[n]{a}\right)^m$

$=3^3$

$=27$

Alternatively; $9^{\frac{3}{2}}=\left(3^2\right)^{3/2}=3^{\cancel{2}\times\frac{3}{\cancel{2}}}=3^3=27$

23. **[2015 PI #2]**

$f(x)=\frac{1-2x}{3-x}$

$f(-2)=\frac{1-2(-2)}{3-(-2)}$

$f(-2)=\frac{1+4}{3+2}$

$f(-2)=\frac{5}{5}$

$f(-2)=1$

24. **[2015 PII #2a]**

$f(x)=ax^2+bx-1$

When $f(-1)=2$

$a(-1)^2-b-1=2$

$a-b=2+1$

$a-b=3$..................(i)

When $f(2)=5$

$a(2)^2+b(2)-1=5$

$4a+2b=5+1$

$4a+2b=6$

$2a+b=3$..............(ii)

Make a in equation (i) the subject and substitute in (ii)

$a=3+b$..............(iii)

$2a+b=3$

$2(3+b)+b=3$

$6+2b+b=3$

$3b=3-6$

$3b=-3$

$\frac{\cancel{3}}{\cancel{3}}b=-\frac{\cancel{3}}{\cancel{3}}$

$b=-1$

Substitute $b=-1$ in (iii)

$a=3+b$

$a=3+-1$

$a=2$

$\therefore$ When $a=2, b=-1$

25. **[2016 PI #3]**

$f(x)=x^3+5$

$f\left(2^{1/3}\right)=\left(2^{1/3}\right)^3+5$

$=\left(2^{3\times\frac{1}{3}}\right)+5$

$=2^1+5$

$=2+5$

$=7$

26. **[2016 PII #1a]**

$f(x)=\sqrt[3]{x^2-1}$

$f(x)=2$

$\sqrt[3]{x^2-1}=2$

$\left(\sqrt[3]{x^2-1}\right)^3=2^3$

$x^2-1=8$

$x^2=8+1$

$x^2=9$

$x=\pm\sqrt{9}$

$\therefore x=3$ or -3

27. **[2017 PI #2]**

$f(-8)=\frac{1-(-8)}{3}$

$=\frac{1+8}{3}$

$=\frac{9}{3}$

$=3$

28. **[2017 PII #1a]**

$g(u)\to\frac{u-1}{2u+3}$

$g(4)=\frac{4-1}{2(4)+3}$

$g(4)=\frac{4-1}{8+3}$

$g(4)=\frac{3}{11}$

29. **[2018 P1 #2]**

$f(\text{x})=\frac{5-7x}{3}$

$f(2)=\frac{5-7(2)}{3}$

$f(2)=\frac{5-14}{3}$

$f(2)=\frac{-9}{3}$

$f(2)=-3$

30. **[2018 PII #15]**

$f(x)=ax+b$

$f(-2)=-2a+b=-4$

$-2a+b=-4$..................(i)

$f(1)=a+b=-1$

$a+b=-1$.............(ii)

solving the two equations simultaneously; make a the subject in equation (i) and substitute in equation (ii)

$a=-b-1$ [a subject in (i)]

$-2a+b=-4$ (ii)

$-2(-b-1)+b=-4$ [substitute a in (ii)]

$2b+2+b=-4$

$3b=-4-2$

$\frac{\cancel{3}b}{\cancel{3}}=\frac{-\cancel{6}}{\cancel{3}}$

$b=-2$

$a=-b-1$

$a=-(-2)-1$

$a=2-1$

$a=1$

Thus, when $a=1$, $b=-2$

31. **[2019 PI #2]**

$h(x) = 3x^2$

$h(a+1) = 3(a+1)^2$

But $h(a+1) = 3$ (given)

so, $3(a+1)^2 = 3$

$\frac{\cancel{3}(a+1)^2}{\cancel{3}} = \frac{\cancel{3}}{\cancel{3}}$

$(a+1)^2 = 1$

$a+1 = \pm\sqrt{1}$

Either $a+1=1$ or $a+1=-1$

$a=-1+1$ or $a=-1-1$

$a=0$ or $a=-2$

32. **[2020 Mock PI # 11]**

$f(1-x) = x^2 + g(x) - 2$

$= x^2 + 1 - x - 2$ [Since $g(x) = 1-x$]

$= x^2 - x - 1$

Now to find $f(2)$

Let $2 = 1-x$

$x = 1-2$

$x = -1$

So,

$f(2) = (-1)^2 - (-1) - 1$

$f(2) = 1$

33. **[2021 Mock PII #1a]**

$g(3-a) = \frac{3-a}{1-(3-a)}$

But $g(3-a) = \frac{2}{3}$ (given)

So, $\frac{3-a}{1-(3-a)} = \frac{2}{3}$

$\frac{3(3-a)}{1-3+a} = 2$

$9-3a = 2(a-2)$

$9-3a = 2a-4$

$-3a-2a = -4-9$

$-5a = -13$

$a = \frac{-13}{-5}$

$a = 2.6$

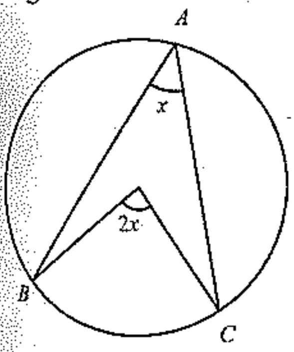
CH 8
CIRCLE GEOMETRY II (ANGLE PROPERTIES)

Chapter Highlights

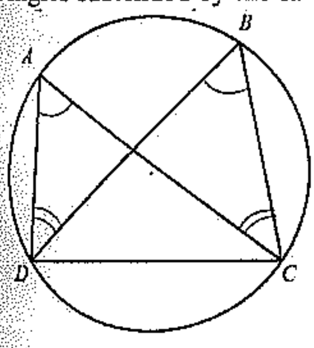
In this chapter, we will solve problems involving angle properties of a circle. We will analyze the relationships that exist between parts of a circle and the angles they subtend. We are not only expected to use the angle properties but also prove them. Students should further be familiar with solving problems requiring such.

Angle properties of a circle

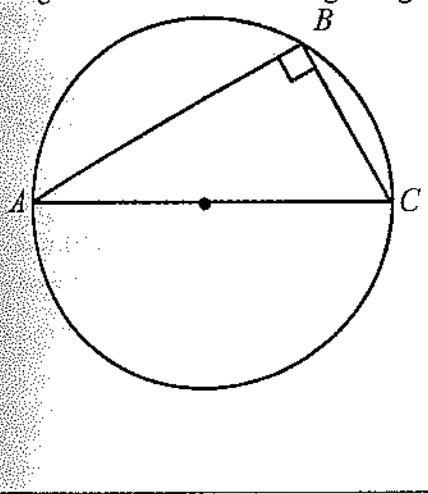
- Angle subtended by an arc at the center is twice the angle subtended at the circumference.



- Angles subtended by the same arc/chord are equal.

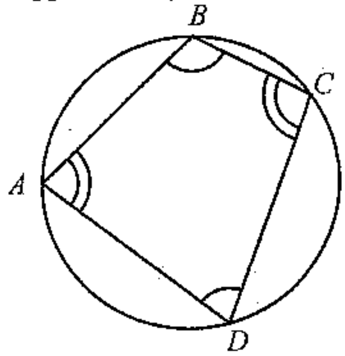


- Angle in a semi-circle is a right angle.



Properties of a cyclic quadrilateral

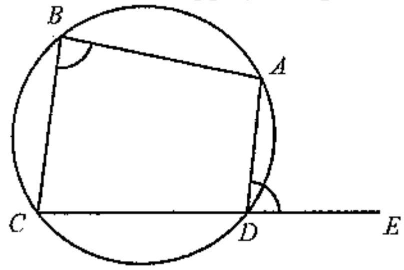
- Opposite angle of a cyclic quadrilateral are supplementary.



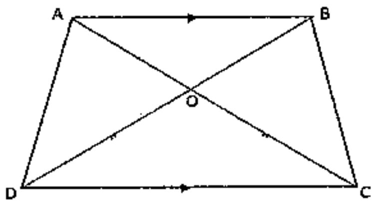
$\hat{B}+\hat{D}=180°$

$\hat{A}+\hat{C}=180°$

- The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

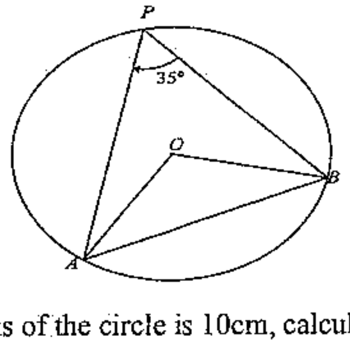


1. In **Figure 1, ABCD** is a trapezium in which **AB** is parallel to **DC**, the diagonals **AC** and **BD** intersect at **O** such that **OD=OC.**



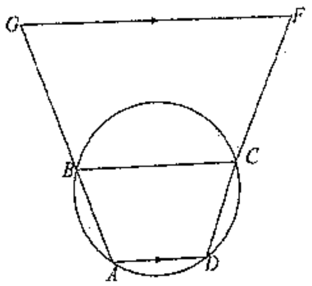
Prove that the points **A, B, C** and **D** are concyclic.
[2003 PI #22]

2. **Figure 2,** shows a chord **AB** subtending an angle of 35° at **P** on the circumference of a circle center **O.**



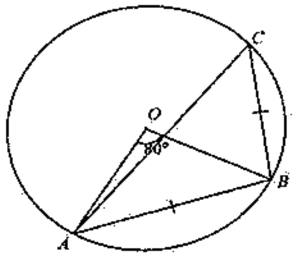
If the radius of the circle is 10cm, calculate the length of the chord **AB**. **[2004 PII #6a]**

3. **Figure 3,** shows a circle **ABCD** in which **AB** and **DC** are produced to **G** and **F** respectively such that **GF** is parallel to **AD.**



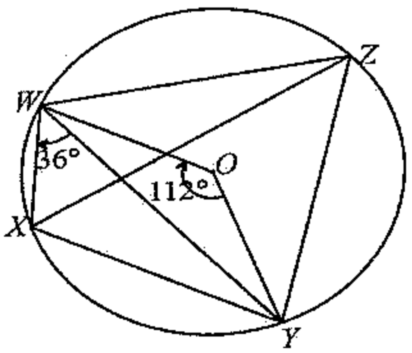
Prove that quadrilateral **GBCF** is cyclic.
[2006 P1 #19]

4. **Figure 4** shows a circle **AOB** Centre **O. ABC and ABO** are triangles in which **AB=BC.**



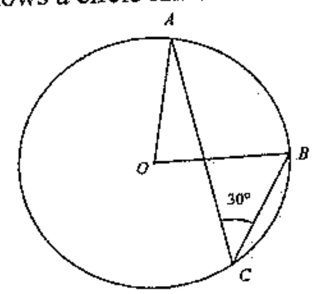
If angle **AOB=**$80°$ Calculate angle **OAC.**
[2008 P2 #6b]

5. **Figure 5** shows a circle **WXYZ** centre **O**. angle **WOY**= 112° and **XWY**=36°.



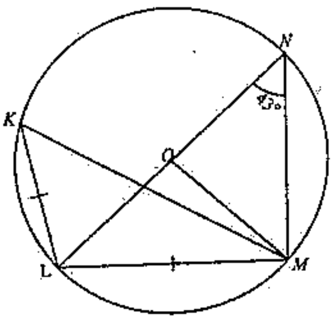
Calculate the size of angle **WZX**. **[2010 P1 #3]**

6. **Figure 6** shows a circle **ABC** with center **O.**



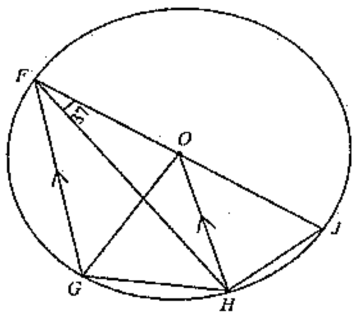
Given that angle **ACB=**$30°$, calculate angle **ABO.** **[2011 PII #4b]**

7. **Figure 7** shows a circle KLMN centre O. Line LON is a diameter, KL=LM and angle LMN = 43°



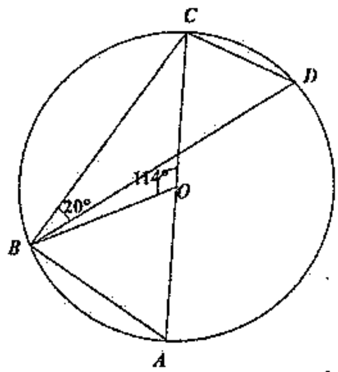
Calculate the value of angle KMO. **[2013 P1 #10]**

8. **Figure 8** shows a circle FGHJ with centre O. FG is parallel to OH and FOJ is a diameter.



If angle HFJ is 37°, calculate angle GHF.
[2013 P2 #8b]

9. **Figure 9** shows a circle **ABCD** with center **O.**



If angle **BOC** = 114° and angle **CBD** = 20°, calculate the value of angle **DBO.** **[2018 PII #10]**

10. Prove that angles subtended by the same chord or arc are equal. **[2019 PII #4a]**

CIRCLE GE
To buy o
or whatsa
in bulk, c

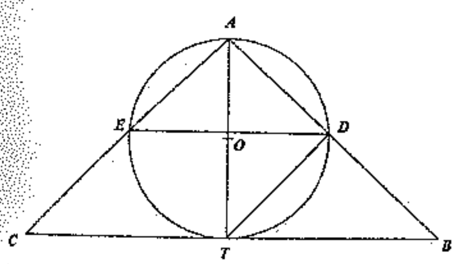
11. **Figure**
which C
AC and

Show th

12. **Figure**
to BC.

Prove
supple

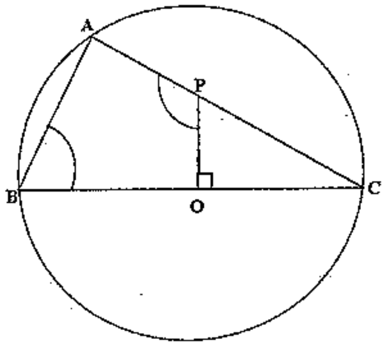
11. **Figure 10** is a circle **ADTE** with center **O** in which **CTB** is a tangent at **T** and **AT** is a diameter. **AC** and **AD** cut the circle at **E** and **D** respectively.



Show that **CEDB** is a cyclic quadrilateral.

[2019 PII #4b]

12. **Figure 11** is a circle centre O. PO is perpendicular to BC.



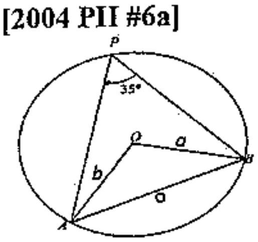
Prove that angles APO and ABO are supplementary. **[2021 PII #6b]**

1. **[2003 PI #22]**
<u>Given</u>; Trapezium ABCD with AB/DC diagonal AC and BD intersecting at O and OD=OC.
<u>To prove:</u> A, B, C and D are concyclic points.

<u>Proof:</u>

$B\hat{A}C = A\hat{C}D$ $(alt.\angle s, AB \parallel DC)$
In ΔDOC,
OD=OC (given)
Hence, $O\hat{D}C = O\hat{C}D$ ($\angle$s opp. equal sides)
But, $A\hat{C}D = O\hat{C}D$ (same $\angle$)
and, $B\hat{D}C = O\hat{D}C$ (same $\angle$)
Hence, $A\hat{C}D = B\hat{D}C$
and so, $B\hat{A}C = B\hat{D}C$ (both equal to $A\hat{C}D$)
$\therefore$ Points A,B,C,D are concyclic & BC is a chord
($\angle$s subtended by the same chord are equal)

2. **[2004 PII #6a]**



<u>Given</u>: circle center O, radius = 10cm
<u>To find</u>: length of chord AB
<u>Solution:</u> using notation in the figure
$\angle AOB = 2 \times 35°$ (*$\angle$ at center is twice the $\angle$ at circumference*)
$\angle AOB = 70°$
In ΔAOB, a=b=10cm (*radii*)
Using notation;
$o^2 = a^2 + b^2 - 2ab\cos O$
$= 10^2 + 10^2 - 2(10 \times 10)\cos 70°$
$= 100 + 100 - 200\cos 70°$
$= 200 - 200(0.3420)$
$= 200 - 68.4040$
$= 131.5960$
$o = \sqrt{131.5960}$
$= 11.47$
$\therefore AB \approx 11.47$ cm (to 2 decimal places)

3. **[2006 P1 #19]**

$F\hat{G}B + B\hat{A}D = 180°$ (Allied $\angle$s GF $\parallel$ AD)
but $B\hat{A}D = B\hat{C}F$ (ext.$\angle$ = int.opp.$\angle$)
so $F\hat{G}B + B\hat{C}F = 180°$
$\therefore$ quad. GBCF is cyclic (opp. $\angle s$ are suppl.)

4. **[2008 P2 #6b]**
Given: Circle ABC centre O, ΔAOB and ΔABC with AB, $\angle AOB = 80°$
required to find: Angle OAC.
$2\angle ACB = \angle AOB$ ($\angle$ at centre $= 2 \times \angle$ at circumf.)
$\Rightarrow \angle ACB = \frac{\angle AOB}{2}$ (Divide both sides by 2)
$= \frac{80°}{2}$
$= 40°$
$AB = BC$ (Given)
$\therefore \Delta ABC$ is isoscles
$\therefore \angle CAB = \angle ACB$ ($\angle$s opp. equal sides)
$= 40°$
Now, $OA = OB$ (radii)
$\therefore \angle OAB = \angle OBA$ ($\angle$s opp. equal sides)
$\angle OAB = \frac{180° - 80°}{2}$ ($\angle$sum of $\triangle$)
$= \frac{100°}{2}$
$= 50°$
but $\angle OAB = \angle OAC + \angle CAB$ (bisected $\angle$)
$\Rightarrow \angle OAC = \angle OAB - \angle CAB$
$\Rightarrow 50° - 40°$
$= 10°$
The value of angle OAC is $10°$

5. **[2010 P1 #3]**
$\angle WOY = 2 \times \angle WZY$
($\angle$ at the $\odot = 2 \times \angle$ on circumf)
$\therefore \angle WZY = \frac{1}{2}(112°)$
$\angle WZY = 56°$
But $\angle XWY = \angle XZY$ ($\angle s$ in same seg)
$\therefore \angle XZY = 36°$
$\angle WZY = \angle WZX + \angle XZY$ (bisected $\angle$)
$\angle WZX = \angle WZY - \angle XZY$
$\angle WZX = 56° - 36°$
$\angle WZX = 20°$

6. **[2011 PII #4b]**

To find $\angle ABO$

In ΔAOB

$\angle AOB = 2\times\angle ACB$ ($\angle$at $\odot = 2\times\angle$at O^{ce})

$= 2\times 30°$

$= 60°$

but OA=OB (radii)

So $\angle OAB=\angle ABO$ ($\angle$s opp equal sides)

but $\angle AOB + \angle OAB + \angle ABO = 180°$ ($\angle$sum of a Δ)

$\angle AOB + 2\angle ABO = 180°$ (since $\angle OAB = \angle ABO$)

$60° + 2\angle ABO = 180°$

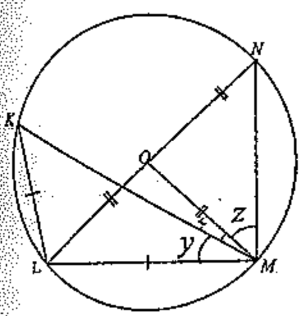
$2\angle ABO = 180° - 60°$

$2\angle ABO = 120°$

$\frac{2\angle ABO}{2} = \frac{120°}{2}$

$\angle ABO = 60°$

7. **[2013 P1 #10]**



Using notation in the figure: x, y, z

$ON = OM$ (radii)

$\angle OMN = z = 43°$ ($\angle s$ opposite equal sides)

$\angle LKM = 43°$ ($\angle s$ in same seg)

But LK=LM (given)

$\angle LMK = y = 43°$ ($\angle s$ opposite equal sides)

But, $y + x + z = 90°$ ($\angle$ in a semi-$\odot$)

$43° + x + 43° = 90°$

$x + 86° = 90°$

$x = 90° - 86°$

$x = 4°$

Thus, $\angle KMO = 4°$

8. **[2013 P2 #8b]**

In ΔFOH,

OF = OH (radii)

$\angle OHF = 37°$ ($\angle$s opp. equal sides)

But $\angle GFH = \angle OHF = 37°$ (alt. $\angle s$ FG || OH)

So in ΔOFG,

$\angle OFG = \angle GFH + \angle OFH$ (bisected $\angle$)

$= 37° + 37°$

$= 74°$

but OF = OG (radii)

so $\angle FGO = 74°$ ($\angle s$ opp equal sides) [*]

Thus, $\angle FOG + 74° + 74° = 180°$ ($\angle$sum of a Δ)

$\angle FOG + 148° = 180°$

$\angle FOG = 180° - 148°$

$\angle FOG = 32°$

But $\angle FOG = 2\angle GHF$ ($\angle$at $\odot = 2\times\angle$at O^{ce})

$32° = 2\angle GHF$

$\frac{32°}{2} = \frac{2\angle GHF}{2}$

$\angle GHF = 16°$

Altenatively, from step [*]

$\angle FGO = 74°$

In quad FGHJ

$\angle FGO + \angle GHF = 180°$ (opp $\angle$s of cyclic quad)

so $74° + \angle GHF = 180°$

$\angle GHF = 180° - 74°$

$\angle GHF = 106°$

but $\angle GHJ = \angle GHF + \angle FHJ$ (bisected)

$\angle FHJ = 90°$ (angles in semi circle)

so $106° = \angle GHF + 90°$

$106° - 90° = \angle GHF$

$16° = \angle GHF$

$\angle GHF = 16°$

9. **[2018 PII #10]**

$\Rightarrow$ In ΔOBC

OC = OB (radii)

$\Rightarrow \angle OBC = \angle OCB$ (base $\angle s$ of isosceles Δ)

$\angle BOC + \angle OCB + \angle OBC = 180°$ (Sum $\angle$ in Δ)

$114° + 2\angle OBC = 180°$

$\Rightarrow \angle OBC = \dfrac{180° - 114°}{2} \Rightarrow \angle OBC = \dfrac{66}{2}$

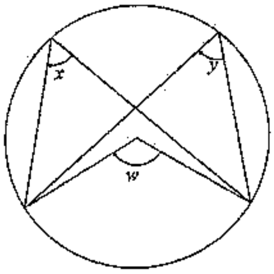
$\therefore \angle OBC = 33$

$\angle OBC = \angle CBD + \angle DBO$ (bisected)

$33° = 20° + \angle DBO \Rightarrow 33 - 20° = \angle DBO$

$\therefore \angle DBO = 13°$

10. **[2019 PII #4a]**



$<w = 2\times < y\left(\angle \text{at centre} = 2 \times \angle \text{at } \odot^{ce}\right)$

$< w = 2\times < x\left(\angle \text{at centre} = 2 \times \angle \text{at } \odot^{ce}\right)$

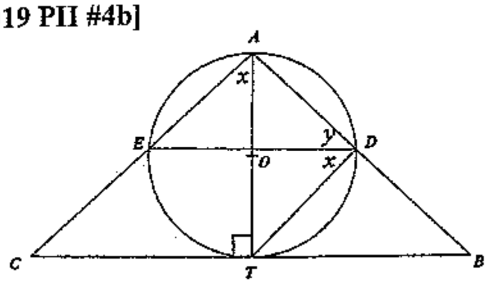
But $<w$ is common

$\Leftrightarrow 2\times < y = 2\times < x$

$\therefore < y = < x$

$\therefore$ Angles subtended by the same chord or arc are equal(QED)

11. **[2019 PII #4b]**



Using notation in the figure

In ΔADT,

$x + y = 90°....(i)$ ($\angle$in a semi-$\odot$)

$y = 90° - x$

In ΔCAT,

$\angle CAT = x$ ($\angle s$ in same seg)

$\angle ATC = 90°$ (*radius* OT $\perp$ tangent CB)

Thus,

$\angle ACT + x + 90° = 180°$ ($\angle sum$ of a Δ)

$\angle ACT + x = 180° - 90°$

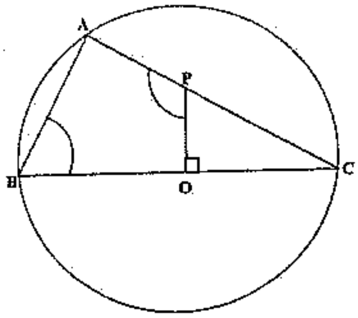
$\angle ACT = 90° - x$

at the same time, $y = 90° - x$

Thus, $\angle ACT = y$ (both equal to $90 - x$)

$\therefore$ *CEDB* is a cyclic quad (ext.$\angle$ = int.*opp.*$\angle$)

12. **[2021 PII #6b]**



$\angle BAC = 90°$ ($\angle$in a semi-$\odot$)

$\angle BAP = 90°$ (*same*$\angle$)

But in quad BAPO,

ext$\angle$ $POC = 90°$ (given)

$\angle BAP = \angle POC = 90°$

$\therefore$ quad BAPO is cyclic (ext.$\angle$ = int.opp$\angle$)

$\therefore \angle APO$ & $\angle ABO$ are supplementary (opp$\angle s$ of cyclic.quad)

Alternative

BC is a straight line (a diameter)

$\angle POC = 90°$ (given)

$\angle BOP = 90°$ (supplimentary angles – *with* $\angle POC$)

$\angle BAP = 90°$ ($\angle s$ angles in semicircle)

In quadrialateral ABOP,

$\angle ABO + \angle BOP + \angle APO + \angle PAO = 360°$

(sum of interior angles in a quadrilateral)

$\angle ABO + 90° + \angle APO + 90° = 360°$

$\angle ABO + \angle APO + 180° = 360°$

$\angle ABO + \angle APO = 360° - 180°$

$\angle ABO + \angle APO = 180°$

$\therefore \angle ABO$ and $\angle APO$ are suppplimentary angles of cyclic quadrilateral.

CH 9 TRANSFORMATIONS

Chapter Highlights
In this chapter, we solve problems on transformation. Transformation is the change in shape or position of an object in the cartesian plane. This chapter solves problems on two types of translation: enlargement and translation.

Translation
This is a shift of an object in the Cartesian plane using a translation vector (T). Given a figure's coordinate A, the image of translation is given as:
$A+T=A'$
A is rewritten in vector form as:
e.g. $A=(3,2) \Rightarrow A=\begin{pmatrix}3\\2\end{pmatrix}$
Given Matrix M, $A'=MA$.
Enlargement
Involves the enlargement of figures using a scale factor.
$A \times$ scale factor $= A'$

$\begin{pmatrix}x\\y\end{pmatrix} \times$ scale factor $= \begin{pmatrix}x'\\y'\end{pmatrix}$

Then $\frac{x'}{x}=\frac{y'}{y}=$ scale factor
Center of enlargement
It is the intersection point of lines through a point and its image.
The formula for calculating new image is:
Image $=\begin{pmatrix}x\\y\end{pmatrix} \times$ scale factor $-$ center of enlargement
One can change subject of formula to find centre of enlargement.

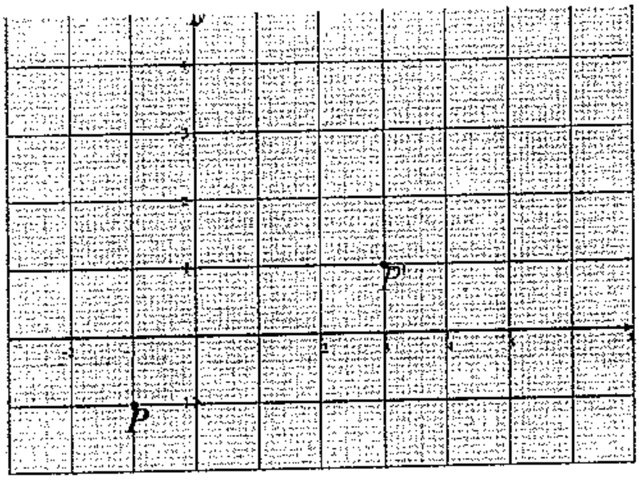
1. A point T has the coordinates $\begin{pmatrix}3\\2\end{pmatrix}$. The matrix which transforms T into T' is $\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$. Calculate the coordinates of T'. **[2003 PI #6]**

2. Given that $C'=(5,-5)$ is the image of $C(1,1)$ after translation. Find the translation vector that maps C to C'. **[2004 PP1 #5]**

3. A triangle **DEF** has vertices D (0, 2), E(4, 6) and F(6, 2).
i. Using scale of 2cm to represent two units on both axes, draw the triangle **DEF** on the graph paper provided.
ii. On the same axes draw the enlargement D E' F' using $D(0,2)$ as center of enlargement and scale factor of 2. **[2004 P2 #7a]**

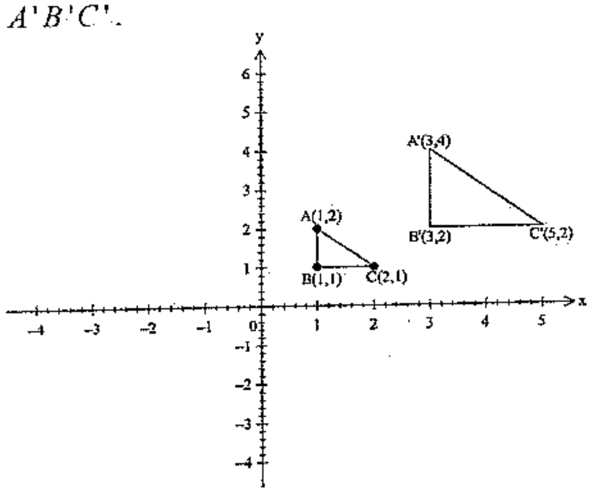
4. A point $P(-2,4)$ is translated to a point P', if P' is 5 units down and 3 units to the right of point P, find the coordinates of P'. **[2005 P1 #7]**

5. **Figure 1** shows the image P' of P after translation.



Find the translation vector that maps P into P'. **[2007 PI #16]**

6. **Figure 2** shows triangle **ABC** and its enlargement $A'B'C'$.



Copy and complete the figure to show in the same diagram the coordinates for the center of enlargement. **[2008 P1 #17]**

7. Given that vector $A = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is translated by the vector $B = \begin{pmatrix} a \\ b \end{pmatrix}$ to vector $A' = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$. Calculate the values of a and b. **[2008 P2 #5a]**

8. On a graph paper, using scale of 2cm to represent 1 unit on both axes, draw a triangle whose vertices are A(2, 2), B(3, 5) and C(5, 3). On the same axes draw the image of triangle **ABC** using $-\frac{1}{2}$ as a scale factor and (0, 0) as the center of enlargement. **[2010 P1 #22]**

9. Triangle RSU has vertices R(0, 6), S(2, 6) and U(2, 8).

   Using a scale of 2cm to represent 2 units on both axes. Draw on a graph paper, triangle RSU and its images R'S'U' under a translation $T = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$. **[2010 PII #9a]**

10. A point P has been translated by the translation vector $T = \begin{pmatrix} -3 \\ -11 \end{pmatrix}$ to $p'\begin{pmatrix} -1 \\ -4 \end{pmatrix}$. Find the coordinates of point P. **[2011 PII #2b]**

11. A point D (-4, 1) is translated to a point D` is 7 units to the right and 3 units down of D, find the coordinates of D'. **[2012 P2 #4b]**

12. A point $A(3,4)$ is translated to a $A'(7,9)$ Calculate the translation in column vector. **[2013 P2 #1]**

13. The coordinates of a triangle ABC are $A(3,1)$, $B(1,2)$ *and* $C(3,5)$. If the Centre of enlargement is $(0,0)$ and the scale factor is 3, calculate the coordinates of the vertices of its $A'B'C'$. **[2014 PI #10]**

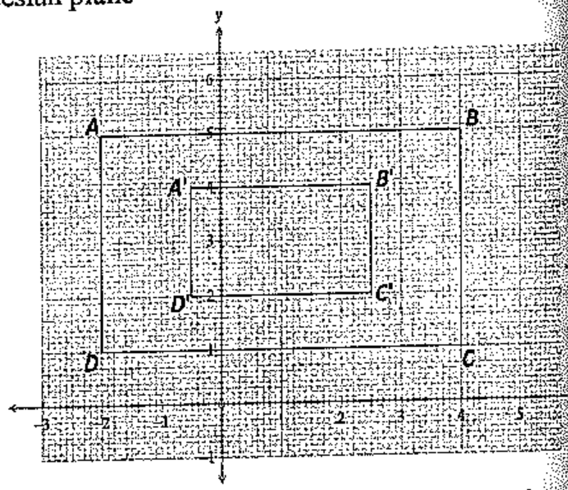
14. A point $R(1,5)$ was transformed to point $R'(-3,4)$ after translation. Calculate translation vector. **[2015 PI #3]**

15. If $M = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$, find the image of the point $(1,-1)$ under translation $2M$. **[2016 PI #6]**

16. A triangle **LMN** has vertices **L**$(2,1)$, **M**$(-2,3)$ and **N**$(1,-2)$. If triangle **LMN** is translated 5 units to the left and 6 units up, find the coordinates of the image of triangle **LMN.** **[2017 PI #9]**

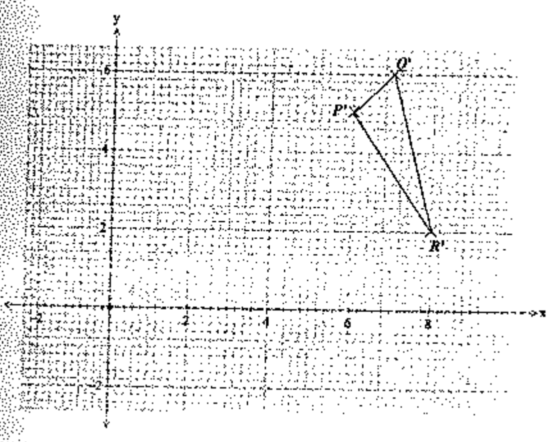
17. Using a scale of 2 cm to represent 1 unit on both axes:
    (i) Draw triangle K(5, 1), L(5, -2) and M(2, -3)
    (ii) On the same scale axes, draw the image of the triangle KLM under a translation $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$.
    [2017 PII #7a]

18. **Figure 3** shows a rectangle **ABCD** and its enlargement and its enlargement **A'B'C'D'** on a cartesian plane



i. By drawing lines in the figure, show the centre of enlargement.
ii. Write down the coordinates of the centre of enlargement. **[2018 PII #7]**

19. **Figure 4** shows the image **P'Q'R'** of triangle **PQR** after translation.



If the translation vector was $\begin{pmatrix}3\\2\end{pmatrix}$, in the same **Figure 4** draw triangle **PQR.** **[2019 PI #7]**

20. Point R($3a$, 4) was translated into image R$'(-2, b-1)$ using a translation vector T(4, 2). Calculate the values of a and b.
[2020 Mock PII #10a]

1. **[2003 PI #6]**

The matrix (M) that transforms T into

$$T' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

So, $T' = MT$

$$T' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times3+0\times2 \\ 1\times3+1\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 6+0 \\ 3+2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

Hence, the coordinates of $T'$are $(6,5)$.

2. **[2004 PP1 #5]**

The translation formula is given as:
Object + translation vector = image
Here, $C + T = C'$ we solve for T

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + T = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

$$T = \begin{pmatrix} 5 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \textit{making T the subject}$$

$$T = \begin{pmatrix} 5 \\ -5 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \textit{removing negative}$$

$$T = \begin{pmatrix} 5-1 \\ -5-1 \end{pmatrix} \quad \textit{adding}$$

$$T = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

$\therefore$ the translation vector is $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$.

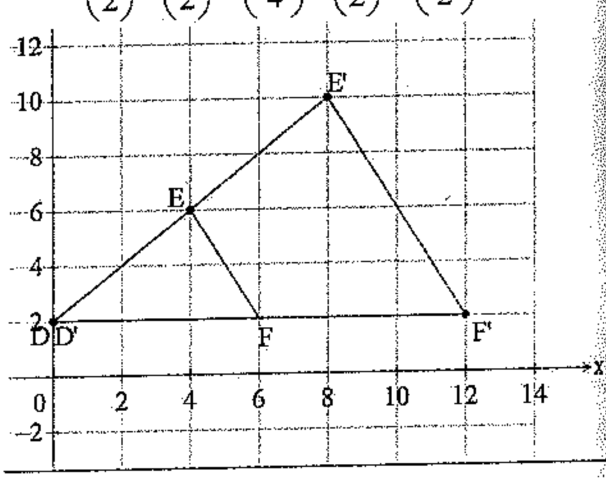
3. **[2004 P2 #7a]**

(ii) Centre of enlargement = D(0, 2)
Scale factor = 2
So image = scale factor × object − center of enlargement

Thus, $D' = 2\begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = D$

$E' = 2\begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$

$F' = 2\begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$



4. **[2005 P1 #7]**

Since P' is 5 units down and 3 units to the right of point P, then the translation vector $T = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$P' = P + T$

$$\therefore P' = \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} -2+3 \\ 4+(-5) \end{pmatrix}$$

$$= \begin{pmatrix} -2+3 \\ 4-5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore$ the coordinates of P' are (1, -1)

5. **[2007 PI #16]**

$P + T = P'$

From the graph, $P = (-1,-1)$ its image P' =(3,1).

So T= $P' - P$

$$\therefore \text{Translation vector} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

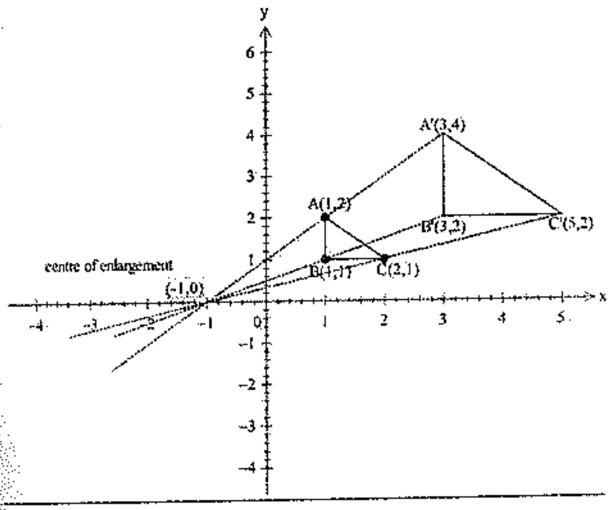
$$= \begin{pmatrix} 3-(-1) \\ 1-(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 3+1 \\ 1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$\therefore$ The translation vectors is $= \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

6. **[2008 P1 #17]**
Need to extend A'A, B'B, an C'C
From the graph the extended lines have converged at (-1,0)
$\therefore$ centre of of enlargement is(-1,0)



7. **[2008 P2 #5a]**

$A+B=A'$

$\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}5\\7\end{pmatrix}$

$\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}5\\7\end{pmatrix}$

$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}5\\7\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix}$

$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}5-2\\7-3\end{pmatrix}$

$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$

$\therefore a=3$ and $b=4$

8. **[2010 P1 #22]**
Vertices of the image are:
So image = scale factor × object – center of enlargement

$A(2,2), A'=-\frac{1}{2}\begin{pmatrix}2\\2\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}$

$A'=\begin{pmatrix}-\frac{1}{2}\times 2\\-\frac{1}{2}\times 2\end{pmatrix}$

$A'=(-1,-1)$

$B(3,5), B'=-\frac{1}{2}\begin{pmatrix}3\\5\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}$

$B'=(-1.5,-2.5)$

$C(5,3), C'=-\frac{1}{2}\begin{pmatrix}5\\3\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}$

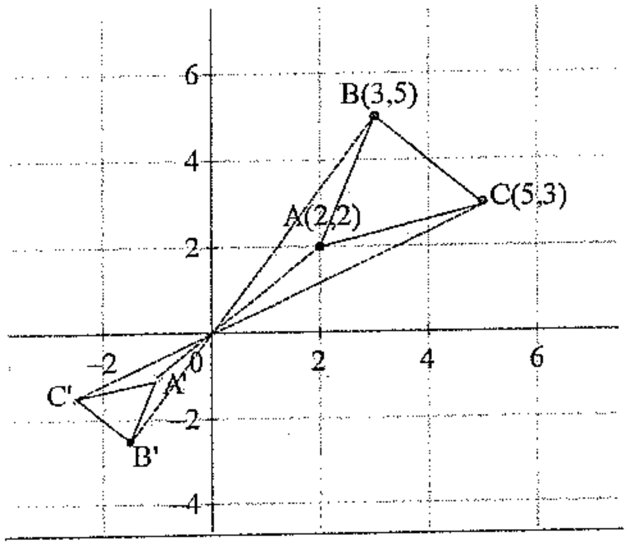
$C'=(-2.5,-1.5)$

The coordinates are as follows

$A(2,2), A'(-1,-1)$

$B(3,5), B'(-1.5,-2.5)$

$C(5,3), C'(-2.5,-1.5)$



9. **[2010 PII #9a]**

$R(0,6)$ $S(2,6)$ $U(2,8)$ and $T=\begin{pmatrix}4\\-4\end{pmatrix}$

Vertices of Image=object +Translation

$\Leftrightarrow R(0,6)$ $R'=\begin{pmatrix}0\\6\end{pmatrix}+\begin{pmatrix}4\\-4\end{pmatrix}$

$R'=\begin{pmatrix}4\\2\end{pmatrix}$

$\therefore R'(4,2)$

$S(2,6)$, $S'=\begin{pmatrix}2\\6\end{pmatrix}+\begin{pmatrix}4\\-4\end{pmatrix}$

$S'=\begin{pmatrix}6\\2\end{pmatrix}$

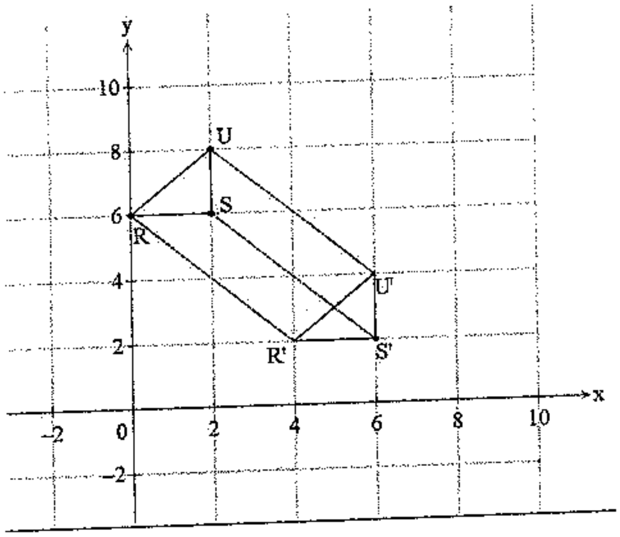
$\therefore S'(6,2)$

$U(2,8)$, $U'=\begin{pmatrix}2\\8\end{pmatrix}+\begin{pmatrix}4\\-4\end{pmatrix}$

$$U' = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$U'(6,4)$$

Plot Δs RSU and R'S'U' on the graph paper



10. **[2011 PII #2b]**

Translation

object + translation = image

$$P + T = P'$$

$$P + \begin{pmatrix} -3 \\ -11 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 \\ -4 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 - -3 \\ -4 - -11 \end{pmatrix}$$

$$= \begin{pmatrix} -1 + 3 \\ -4 + 11 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

∴ the coordinates of point P are (2, 7)

11. **[2012 P2 #4b]**

Point D(-4, 1) and translation vector $T = \begin{pmatrix} +7 \\ -3 \end{pmatrix}$

$$D + T = D'$$

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} -4+7 \\ 1+(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Hence, coordinates of D' are (3,-2)

12. **[2013 P2 #1]**

Let the transformation vector be T

$$A + T = A'$$

$$A + T = A'$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} + T = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$$T + \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$$T = \begin{pmatrix} 7 \\ 9 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$T = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

The translation in column vector is $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

13. **[2014 PI #10]**

So image = scale factor × object – center of enlargement

coordinates image if scale factor is 3

$$A' = 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$$

$$B' = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$C' = 3\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 15 \end{pmatrix}$$

$A'B'C'$ are $A'(9,3)$; $B'(3,6)$, and $C'(9,15)$.

14. **[2015 PI #3]**

Given $R(1,5)$ $R'(-3,4)$

Let the translation vector T

$R+T=R'$

$$\begin{pmatrix}1\\5\end{pmatrix}+T=\begin{pmatrix}-3\\4\end{pmatrix}$$

$$T=\begin{pmatrix}-3\\4\end{pmatrix}-\begin{pmatrix}1\\5\end{pmatrix}$$

$$T=\begin{pmatrix}-3-1\\4-5\end{pmatrix}$$

$$T=\begin{pmatrix}-4\\-1\end{pmatrix}$$

The translation vector is $\begin{pmatrix}-4\\-1\end{pmatrix}$

15. **[2016 PI #6]**

$T=2M$

$$T=2\begin{pmatrix}-5\\3\end{pmatrix}$$

$$T=\begin{pmatrix}-10\\6\end{pmatrix}$$

So taking point P=$\begin{pmatrix}1\\-1\end{pmatrix}$,

$P'=P+T$

$$=\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-10\\6\end{pmatrix}$$

$$=\begin{pmatrix}-9\\5\end{pmatrix}$$

Hence, the image of point (1,-1) under translation 2M is (-9,5).

16. **[2017 PI #9]**

$L(2,1)$ $M(-2,3)$ $N(1,-2)$

Translation vector, $T=\begin{pmatrix}-5\\6\end{pmatrix}$

ΔL'M'N'=ΔLMN+T

$L'=L+T$

$$L'=\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-5\\6\end{pmatrix}=\begin{pmatrix}-3\\7\end{pmatrix}$$

$M'=M+T$

$$M'=\begin{pmatrix}-2\\3\end{pmatrix}+\begin{pmatrix}-5\\6\end{pmatrix}=\begin{pmatrix}-7\\9\end{pmatrix}$$

$N'=N+T$

$$N'=\begin{pmatrix}1\\-2\end{pmatrix}+\begin{pmatrix}-5\\6\end{pmatrix}=\begin{pmatrix}-4\\4\end{pmatrix}$$

$\therefore$ coordinates are $L'(-3,7)$; $M'(-7,9)$; and $N'(-4,4)$

17. **[2017 PII #7a]**

$K(5,1), L(5,-2), M(2,-3)$

Translation vector $(T)=\begin{pmatrix}-5\\+6\end{pmatrix}$

$\therefore$ Coordinates of $K'L'M'$ are;

$K'=K+T$

$$K'=\begin{pmatrix}5\\1\end{pmatrix}+\begin{pmatrix}-4\\2\end{pmatrix}$$

$$K'=\begin{pmatrix}1\\3\end{pmatrix}$$

$\therefore K'=(1,3)$

$L'=L+T$

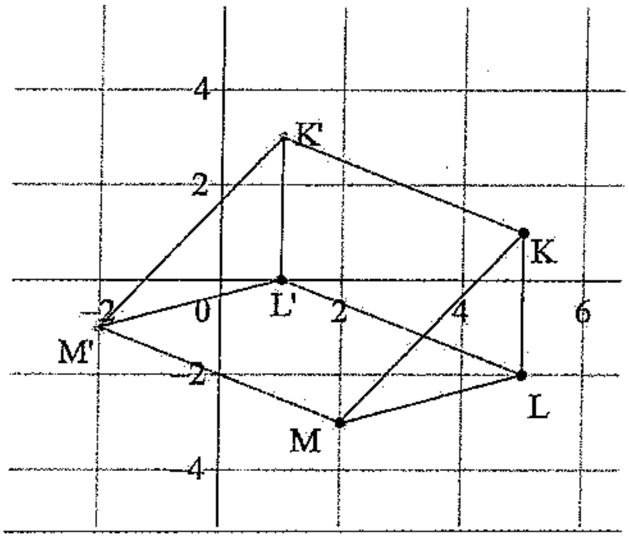
$$L'=\begin{pmatrix}5\\-2\end{pmatrix}+\begin{pmatrix}-4\\2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

$L'=(1,0)$

$M'=M+T$

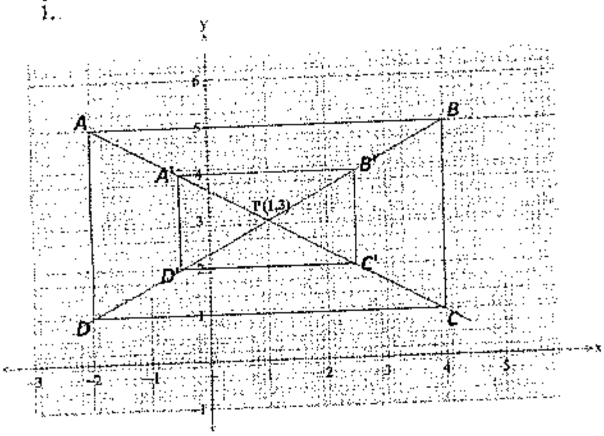
$$M'=\begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}-4\\2\end{pmatrix}=\begin{pmatrix}-2\\-1\end{pmatrix}$$

$M'=(-2,-1)$



18. **[2018 PII #7]**

i.



ii. Centre of enlargement is point, P(1,3)

19. **[2019 PI #7]**

We are given that $T = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$P + T = P'$

$So,\ P = P' - T$

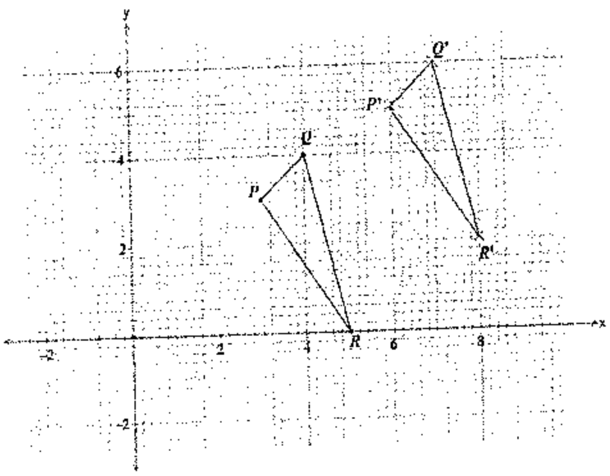
$= \begin{pmatrix} 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6-3 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$So,\ \mathrm{Q} = Q' - T$

$= \begin{pmatrix} 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7-3 \\ 6-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$So,\ \mathrm{R} = R' - T$

$= \begin{pmatrix} 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 8-3 \\ 2-2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$



20. **[2020 Mock PII #10a]**

$\mathrm{R} + \mathrm{T} = \mathrm{R}'$

$\begin{pmatrix} 3a \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ b-1 \end{pmatrix}$

$\begin{pmatrix} 3a+4 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ b-1 \end{pmatrix}$

so, $3a + 4 = -2$ (i)

$3a = -2 - 4$

$3a = -6$

$\frac{3a}{3} = \frac{-6}{3}$

$a = -2$

and $6 = b - 1$ (ii)

$6 + 1 = b$

$7 = b$

$a = -2$ and $b = 7$

CH 10 CHANGE OF SUBJECT OF FORMULA

Chapter Highlights

This involves rearranging variables in an equation to make a different variable the new subject. In order to effectively change the subject of the formula, we need skills in algebra.

Remarks:

- What we do on one side when changing subject of the formula, we must do on the other side.
- When given equations with fractions, it is useful to eliminate the fractions. You can 'cross multiply' or multiply each term by the common denominator to clear an equation of fractions. e.g. problem 1,2, 3, 11, etc.
- When a problem has two or more terms of the same variable, place them on one side and factor out the variable e.g. problem 3, 7, 6 and 21.
- Take logarithms on both sides of the equation and apply laws of logarithms to make a power the subject.
- We will use some of the following opposite operations (and vice versa) i.e.
  - The opposite of addition is subtraction.
  - The opposite of multiplication is division.
  - The opposite of squaring is taking the square root.
  - The opposite of cubing is taking the cube root.

1. Make m the subject of the formula $y=\frac{m}{1+m}$. **[2003 PI #11]**
2. Make x the subject of the formula $y=\frac{p}{2x^3+q}$ **[2003 PII #3b]**
3. Make y the subject of the formula $\frac{y+d}{c}=\frac{3d}{y-d}$. **[2004 PI #3]**
4. Make r the subject of the formula $S=\pi(2r)^2$. **[2005 P1 #9]**
5. Make n the subject of the formula $\log y^n=2$. **[2005 PII #2a]**
6. Make y the subject of the formula $y^{\frac{1}{2}}=\frac{w+w(xy)^{\frac{1}{2}}}{x^{\frac{1}{2}}}$. **[2006 PII #8a]**
7. Express b in terms of a and c in the formula $c=ab-\frac{b}{a}$. **[2007 PI #5]**
8. Make t the subject of the formula $L=W\frac{(k-t)}{M}-t$. **[2007 PII #3a]**
9. Make R the subject of the formula $A=P\left(1+\frac{R}{100}\right)^T$. **[2008 PII #4a]**
10. Make t the subject of the formula $M=K+\frac{3y^2}{t}$. **[2008 P1 #4]**
11. Make a the subject of the formula: $x=\frac{\sqrt{a}}{y}+b$. **[2010 PI #13]**
12. Make n the subject of the formula; $T=ar^n$. **[2010 PII #4a]**
13. Make y the subject of the formula $p=\sqrt{\frac{y-a}{y+1}}$. **[2011 PI #1]**
14. Make k the subject of the formula $r=(\sqrt{k}+1)^2$ **[2011 PII #5a]**
15. Make q the subject of the formula $a-\frac{3}{q^2}=p$. **[2012 P1 #4]**
16. Make m the subject of the formula $r=\frac{n}{\sqrt[3]{k-m}}$ **[2012 PII #1b]**
17. Make p the subject of the formula $a^3\sqrt{p}+1=m$ **[2013 P1 #5]**

18. Make y the subject of the formula $p=\frac{1}{4}xy^3h.$

[2013 PII #2a]

19. Make x the subject of the formula $a=b\frac{\sqrt{x^2-n}}{m}$. **[2014 P1 #5]**

20. Make y subject of the formula $n=\frac{4y-3}{m}+y$.

[2014 PII #2a]

21. Make x the subject of the formula $p-x=\frac{2+x}{p}.$

[2015 PI #5]

22. Make y the subject of the formula $p-\sqrt[3]{\frac{y}{x+1}}=\frac{1}{a}$ **[2015 PII #1a]**

23. Make m the subject of the formula $k=\frac{m^3-n^2}{2p}.$

[2016 PI #2]

24. Make d the subject of the formula $a=b\sqrt[n]{\frac{d}{f}}-g.$

[2016 PII #1b]

25. Make n the subject of the equation $\frac{a}{n}+mn=tn$

[2017 PI #3]

26. Make p the subject of the formula $y=3\log_a(p^2+5).$ **[2017 PII #3a]**

27. Make q the subject of the formula in $p=\left(\frac{r}{q+r}\right)^2.$ **[2018 P1 #3]**

28. Make g the subject of the equation $\frac{f}{a-g}=\frac{1}{2g}.$

[2018 PII #1]

29. Make w the subject of the formula $s=\frac{f}{m-w^2p}.$

[2019 PI #3]

1. **[2003 PP1 #11]**

$y = \frac{m}{1+m}$ (multiply $1+m$ on both sides)
$y(1+m) = m$ (expand the LHS)
$y + my = m$ (put the m's on one side)
$my - m = -y$ (factor out m)
$m(y-1) = -y$ (divide $y-1$ both sides)
$m = -\frac{y}{y-1}$

2. **[2003 PP II #3b]**

$y(2x^3+q) = P$ (Expand LHS)
$2x^3y + qy = P$
$2x^3y = P - qy$ (divide $2y$ both sides)
$x^3 = \frac{P-qy}{2y}$ (take cube root both sides)
$x = \sqrt[3]{\frac{P-qy}{2y}}$

3. **[2004 PP1 #3]**

$\frac{y+d}{c} = \frac{3d}{y-d}$ (cross multiply)
$(y+d)(y-d) = c \times 3d$ (expand LHS)
$y^2 - d^2 = 3cd$
$\Rightarrow y^2 = 3cd + d^2$ (take square root both sides)
$\Rightarrow y = \pm\sqrt{3cd + d^2}$

4. **[2005 P1 #9]**

$S = \pi(2r)^2 \Leftrightarrow \pi(2r)^2 = S$
$(2r)^2 = \frac{S}{\pi}$ (Take square root both sides)
$2r = \pm\sqrt{\frac{S}{\pi}}$ (Multiply $\frac{1}{2}$ both sides)
$\frac{1}{2} \times 2r = \frac{1}{2} \times \pm\sqrt{\frac{S}{\pi}}$
$\therefore r = \frac{1}{2}\pm\sqrt{\frac{S}{\pi}}$

5. **[2005 PII #2a]**

$\log y^n = 2$
$n \log y = 2$ (laws of logarithm)
$\frac{n \log y}{\log y} = \frac{2}{\log y}$ (divide by $\log y$ both sides)
$\Rightarrow n = \frac{2}{\log y}$

6. **[2006 PII #8a]**

$y^{\frac{1}{2}} = \frac{w + w(xy)^{\frac{1}{2}}}{x^{\frac{1}{2}}}$
$= x^{½}y^{½} = w + w(xy)^{½}$
$= (xy)^{½} = w + w(xy)^{½}$
$= (xy)^{½} - w(xy)^{½} = w$
$= (xy)^{½}(1-w) = w$ *(factoring out $(xy)^{½}$)*
$(xy)^{½} = \frac{w}{1-w}$ *(dividing both sides by $(1-w)$)*

Squaring both sides of the equation, we have

$\left[(xy)^{½}\right]^2 = \left(\frac{w}{1-w}\right)^2$

$\Rightarrow xy = \left(\frac{w}{1-w}\right)^2$

$\frac{xy}{x} = \frac{\left(\frac{w}{1-w}\right)^2}{x}$

$\therefore y = \frac{1}{x}\left(\frac{w}{1-w}\right)^2$

7. **[2007 PI #5]**

$c = ab - \frac{b}{a} \Leftrightarrow ab - \frac{b}{a} = c$

$ab - \frac{b}{a} = c$ (Multiply each term by a)
$a^2b - b = ac$ (Factor out b)
$b(a^2-1) = ac$ (divide each term by a^2-1)
$\frac{b(a^2-1)}{a^2-1} = \frac{ac}{a^2-1}$

$$\Rightarrow b = \frac{ac}{a^2-1} = \frac{ac}{(a-1)(a+1)}$$

Alternatively

$c = b\left(\frac{a}{1} - \frac{1}{a}\right)$ factory out of b

$$c = b\left(\frac{a^2-1}{a}\right) \Rightarrow c = \frac{b\left(a^2-1\right)}{a}$$

$c = \frac{b\left(a^2-1\right)}{a}$ multiply both sides by $\frac{a}{a^2-1}$

$$\Rightarrow \frac{a}{a^2-1} \times c = b$$

$$\Rightarrow b = \frac{ac}{a^2-1} = \frac{ac}{(a-1)(a+1)}$$

8. **[2007 PII #3a]**

$$L = W\frac{k-t}{M} - t$$

multiply each term by M

$$\Rightarrow L \times M = W\frac{k-t}{M} \times M - t \times M$$

$$\Rightarrow LM = W(\text{k}-t) - Mt$$

$$\Rightarrow LM = Wk - Wt - tM$$

Put the t's on one side

$$Wt + Mt = Wk - LM$$

$t(W+M) = Wk - LM$ (factor out t)

$\therefore t = \frac{Wk - LM}{W+M}$ (divide both side by $W+M$)

9. **[2008 P2 #4a]**

$A = P(1+\frac{R}{100})^T$ (divide both sides by P)

$$\frac{A}{P} = \frac{P(1+\frac{R}{100})^T}{P}$$

$\Rightarrow \frac{A}{P} = (1+\frac{R}{100})^T$ (Take T– th root both sides)

$\sqrt[T]{(\frac{A}{P})} = 1 + \frac{R}{100}$ (subtract 1 from both sides)

$\sqrt[T]{(\frac{A}{P})} - 1 = \frac{R}{100}$ (multiply both sides by 100)

$$100\left[\sqrt[T]{(\tfrac{A}{P})} - 1\right] = R$$

$$\therefore \text{R} = 100\left[\sqrt[T]{(\tfrac{A}{P})} - 1\right]$$

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

10. **[2008 P1 #4]**

$M = K + \frac{3y^2}{t}$ (Multiply each term by t)

$Mt = Kt + 3y^2$ (bring the t's on one side)

$Mt - Kt = 3y^2$ (factor out t)

$t(M-K) = 3y^2$ (divide both sides by $M-K$)

$$\therefore t = \frac{3y^2}{M-K}$$

11. **[2010 P1 #13]**

$x = \frac{\sqrt{a}}{y} + b$ (multiply each term by y)

$$xy = y \times \frac{\sqrt{a}}{y} + b \times y$$

$$xy = \sqrt{a} + by$$

$xy - by = \sqrt{a}$ (take 'by' to the other side)

$\left(\sqrt{a}\right)^2 = \left(xy - by\right)^2$ (square both sides)

$$a = \left(xy - by\right)^2$$

12. **[2010 PII #4a]**

$T = ar^n$ (divide by a both sides)

$$\frac{T}{a} = \frac{ar^n}{a}$$

$\frac{T}{a} = r^n$ (take log on both sides)

$\log\left(\frac{T}{a}\right) = \log r^n$ (using a laws of log)

$\log\left(\frac{T}{a}\right) = n\log r$ (divide by $\log r$ both sides)

$$\Leftrightarrow \frac{n\log r}{\log r} = \frac{\log\left(\frac{T}{a}\right)}{\log r}$$

$$n = \frac{\log\left(\frac{T}{a}\right)}{\log r}$$

13. **[2011 P1 #1]**

$p=\sqrt{\frac{y-a}{y+1}}$ (squaring both sides)

$(p)^2=\left(\sqrt{\frac{y-a}{y+1}}\right)^2 \Rightarrow p^2=\frac{y-a}{y+1}$

$p^2=\frac{y-a}{y+1}$

$p^2(y+1)=y-a$ (multiply by $y+1$)

$p^2y+p^2=y-a$

$p^2y-y=-a-p^2$ (place terms with y on one side)

$y(p^2-1)=-a-p^2$ (factor out y on the LHS)

$y=\frac{-a-p^2}{p^2-1}$ (divide by p^2-1 both sides)

$\Rightarrow y=\frac{-1(a+p^2)}{-1(-p^2+1)}$ (factor out -1)

$y=\frac{a+p^2}{1-p^2} \Rightarrow y=\frac{a+p^2}{(1-p)(1+p)}$

14. **[2011 PII #5a]**

$r=(\sqrt{k}+1)^2$

$\pm\sqrt{r}=\sqrt{\left(\sqrt{k}+1\right)^2}$ (take square root both sides)

$\pm\sqrt{r}=\sqrt{k}+1$

$\pm\sqrt{r}-1=\sqrt{k}$ (subtract 1 both sides)

$(\pm\sqrt{r}-1)^2=k$ (square both sides)

$\therefore k=\left(\pm\sqrt{r}-1\right)^2$

15. **[2012 P1 #4]**

$a-\frac{3}{q^2}=p \Leftrightarrow a-p=\frac{3}{q^2}$

$a-p=\frac{3}{q^2}$

$q^2(a-p)=3$ (multiply each with q^2)

$q^2=\frac{3}{a-p}$ Dividing both side with $(a-p)$

$\sqrt{q^2}=\pm\sqrt{\frac{3}{a-p}}$ (square root of both sides)

$q=\pm\sqrt{\frac{3}{a-p}}$

16. **[2012 P2 #1b]**

Given $r=\frac{n}{\sqrt[3]{k-m}}$,

multiply both sides with $\sqrt[3]{k-m}$

$r(\sqrt[3]{k-m})=n$

diving both sides with r

$\sqrt[3]{k-m}=\frac{n}{r}$(cubing both sides)

$\left(\sqrt[3]{k-m}\right)^3=\left(\frac{n}{r}\right)^3$

$k-m=\frac{n^3}{r^3}$

$-m=\frac{n^3}{r^3}-k$

$m=-\frac{n^3}{r^3}+k$ (divide each term by -1)

$\therefore m=k-\frac{n^3}{r^3} \Rightarrow m=k-\left(\frac{n}{r}\right)^3$

17. **[2013 P1 #5]**

$a\sqrt[3]{p}+1=m$

$a\sqrt[3]{p}=m-1$ (divide by a both sides)

$\frac{a\sqrt[3]{p}}{a}=\frac{m-1}{a}$

$\sqrt[3]{p}=\frac{m-1}{a}$

$\Rightarrow\left(\sqrt[3]{p}\right)^3=\frac{(m-1)^3}{(a)^3}$

$\therefore p=\frac{(m-1)^3}{a^3}$

18. **[2013 P2 #2a]**

$P=\frac{1}{4}xy^3h$

$4p=\frac{xy^3h}{4}\times4$ (multiply by 4 both sides)

$4p=xy^3h$ (divide by xh both sides)

$\frac{4p}{xh}=y^3$

$\sqrt[3]{\frac{4p}{xh}} = y$ (take cube root both sides)

$\Rightarrow y = \sqrt[3]{\frac{4p}{xh}}$

19. **[2014 P1 #5]**

$a = b\frac{\sqrt{x^2-n}}{m}$ (multiply m both sides)

$ma = b\frac{\sqrt{x^2-n}}{\cancel{m}} \times \cancel{m}$

$am = b\sqrt{x^2-n}$

$\frac{am}{b} = \cancel{b}\frac{\sqrt{x^2-n}}{\cancel{b}}$ (divide by b both sides)

$\frac{am}{b} = \sqrt{x^2-n}$

$\therefore \left(\frac{am}{b}\right)^2 = \left(\sqrt{x^2-n}\right)^2$ (squaring both sides)

$\left(\frac{am}{b}\right)^2 = x^2 - n$ (swap sides)

$x^2 - n = \left(\frac{am}{b}\right)^2$

$x^2 = \frac{a^2m^2 + b^2n}{b^2}$ (write LHS as as single fraction)

$\sqrt{x^2} = \pm\sqrt{\frac{a^2m^2+b^2n}{b^2}}$ (take square root both sides)

separate square root to numerator and denominator

$x = \frac{\pm\sqrt{a^2m^2+b^2n}}{b}$

20. **[2014 PII #2a]**

$n = \frac{4y-3}{m} + y$ (multiply each term by m)

leave only terms with y on left side

$nm = 4y - 3 + ym$

$nm + 3 = 4y + ym$(factor out y)

$nm + 3 = y(4+m)$ (divide $(4+m)$ both sides)

$\frac{nm+3}{4+m} = y$

$\therefore y = \frac{nm+3}{4+m}$.

21. **[2015 PI #5]**

$p - x = \frac{2+x}{p}$ (multiply each term by p)

$p^2 - px = \frac{2+x}{p} \times p$

$p^2 - px = 2 + x$

$p^2 - 2 = px + x$(terms with x put on one sides)

$px + x = p^2 - 2$ (swap sides)

$x(p+1) = p^2 - 2$ (factor out x)

$\frac{x\cancel{(p+1)}}{\cancel{p+1}} = \frac{p^2-2}{p+1}$ (divide by $p+1$ both sides)

$x = \frac{p^2-2}{p+1}$.

22. **[2015 PII #1a]**

$p - \sqrt[3]{\frac{y}{x+1}} = \frac{1}{a}$

move $-\sqrt[3]{\frac{y}{x+1}}$ to right and $\frac{1}{a}$ to the left

$p - \frac{1}{a} = \sqrt[3]{\frac{y}{x+1}}$

$\left(\frac{ap-1}{a}\right)^3 = \left(\sqrt[3]{\frac{y}{x+1}}\right)^3$ (take cube root both sides)

$\frac{(ap-1)^3}{a^3} = \frac{y}{x+1}$

$(x+1) \times \frac{(ap-1)^3}{a^3} = \frac{y}{\cancel{x+1}} \times \cancel{(x+1)}$ (multiply $x+1$ both

$\therefore y = \frac{(x+1)(ap-1)^3}{a^3}$

23. **[2016 PI #2]**

$k = \frac{m^3 - n^2}{2p}$

$2pk = m^3 - n^2$ (multiply $2p$ on both sides)

$2pk + n^2 = m^3$

$\sqrt[3]{2pk + n^2} = \sqrt[3]{m^3}$ (take cube root both sides)

$\therefore m = \sqrt[3]{2pk + n^2}$

24. **[2016 PII #1b]**

$a = b\sqrt[n]{\frac{d}{f}} - g$ (add g both sides)

$a + g = b\sqrt[n]{\frac{d}{f}}$

$\frac{a+g}{b} = \frac{b\sqrt[n]{\frac{d}{f}}}{b}$ (divide by b both sides)

$\frac{a+g}{b} = \sqrt[n]{\frac{d}{f}}$

$\left(\frac{a+g}{b}\right)^n = \left(\sqrt[n]{\frac{d}{f}}\right)^n$ (raise both sides to power n)

$\left(\frac{a+g}{b}\right)^n = \frac{d}{f}$

$\left(\frac{a+g}{b}\right)^n \times f = f \times \frac{d}{f}$ (multiply both sides f)

$f\left(\frac{a+g}{b}\right)^n = d$

$\therefore d = f\left(\frac{a+g}{b}\right)^n$

---

25. **[2017 PI #3]**

$\frac{a}{n} + mn = tn$ (multiply by n on each term)

$n \times \frac{a}{n} + n(mn) = n(tn)$

$a + mn^2 = tn^2$

$tn^2 - mn^2 = a$

$n^2(t-m) = a$ (put terms with n on one side)

$\frac{n^2\cancel{(t-m)}}{\cancel{(t-m)}} = \frac{a}{(t-m)}$ [divide each term by $(t-m)$]

$n^2 = \frac{a}{(t-m)}$

$\sqrt{n^2} = \pm\sqrt{\frac{a}{t-m}}$ (take square root both sides)

$n = \pm\sqrt{\frac{a}{t-m}}$

26. **[2017 PII #3a]**

$y = 3\log_a(p^2+5)$

$\frac{y}{3} = \frac{3\log_a\left(p^2+5\right)}{3}$ (divide both sides by 3)

$\frac{y}{3} = \log_a\left(p^2+5\right)$

$p^2 + 5 = a^{\frac{y}{3}}$ (change log to indices)

$p^2 = a^{\frac{y}{3}} - 5$ (make p^2 subject of the formula)

$\sqrt{p^2} = \pm\sqrt{a^{\frac{y}{3}} - 5}$ (take square root both sides)

$p = \pm\sqrt{a^{\frac{y}{3}} - 5}$

---

27. **[2018 P1 #3]**

$p = \left(\frac{r}{q+r}\right)^2$

$\sqrt{p} = \frac{r}{q+r}$ (take square root both sides)

$\sqrt{p}(q+r) = r$ (multiply by $q+r$ both sides)

$q + r = \frac{r}{\sqrt{p}}$ (divide by $\sqrt{p}$ both sides)

$q = \frac{r}{\sqrt{p}} - r$ (subtract r both sides)

---

28. **[2018 PII #1]**

$\frac{f}{a-g} = \frac{1}{2g}$

$2fg = a - g$ (cross multiply)

$2fg + g = a$ (place terms with g on one side)

$g(2f+1) = a$ (factor out g)

$\frac{g\cancel{(2f+1)}}{\cancel{2f+1}} = \frac{a}{2f+1}$ (divide by $2f+1$ both sides)

$\Rightarrow g = \frac{a}{2f+1}$

---

29. **[2019 PI #3]**

$s = \dfrac{f}{m - w^2 p}$

$s\left(m - w^2 p\right) = f$ (cross-multiplying)

$sm - sw^2 p = f$ (expanding)

$sm - f = w^2 sp$ (making terms with w subject)

$w^2 sp = sm - f$

$\dfrac{w^2 \cancel{sp}}{\cancel{sp}} = \dfrac{sm - f}{sp}$ (dividing both sides by sp)

$w^2 = \dfrac{sm - f}{sp}$

$w = \pm\sqrt{\dfrac{sm - f}{sp}}$ (taking sqrts of both sides)

$w = \pm\sqrt{\dfrac{sm - f}{sp}}$

CH 11
EXPONENTIAL AND LOGARITHMIC EQUATIONS

Chapter Highlights

In this chapter, we will solve problems relating to exponential and logarithmic equations. Exponential equations are equations in which the variable/unknown value is an exponent.

Before we start solving these problems:

- We should be able to convert between logarithmic equations and exponential equations i.e.:
  from $y = a^x$ to $x = \log_a y$ (for $a > 1$)
  from $x = \log_a y$ to $y = a^x$
- We should express numbers as powers of a given base using the following rules of exponents, for $a \neq 0$, $b \neq 0$,
  1) $a^x \times a^y = a^{x+y}$
  2) $\frac{a^x}{a^y} = a^{x-y}$
  3) $\left(a^x\right)^y = a^{xy}$
  4) $a^0 = 1$
  5) $a^{-x} = \frac{1}{a^x}$
  6) $a^{\frac{x}{y}} = \sqrt[y]{a^x} = \left(\sqrt[y]{a}\right)^x$
- Derive and use rules of logarithms:
  1) $\log_a pq = \log_a p + \log_a q$
  2) $\log_a \frac{p}{q} = \log_a p - \log_a q$
  3) $\log_a a = 1$
  4) $\log_a p^n = n\log_a p$

1. Given $\log_a 2 = 0.6110$ and $\log_a 3 = 0.7039$, calculate $\log_a 6$. **[2003 PI #8]**

2. Calculate the value of x if $10^x = 0.01$. **[2003 PP II #1.a]**

3. Solve the equation $\log_{10}(2m+6) = 1 + \log_{10}(m-1)$. **[2004 PP1 #19]**

4. Solve the equation $\log_9 27^k = K+1$. **[2004 PII #10a]**

5. Given that $\log_m 27 = 3$ find m. **[2005 PI #8]**

6. Solve for x if $(2^x)^2 - 9(2^x) + 8 = 0$. **[2005 PII #9a]**

7. Given that $\log_{10} n - \log_{10} m = 2\log_{10} h$, show that $n = mh^2$. **[2006 PI #20]**

8. Evaluate $2\log_a a + 3\log_a 1 + \log_2 \sqrt{2}$. **[2006 PII #2a]**

9. Given that $\log_a 2 = 0.668$ and $\log_a 3 = 0.884$, evaluate $\log_a 12$. **[2007 PI #11]**

10. Simplify $\frac{4^x \times 8^{x-1}}{32^x}$. **[2007 PII #11a]**

11. Solve the equation $\log_7 343 = 2x - 5$. **[2008 PII #4b]**

12. Given that $\log_x\left(\frac{1}{2}\right) + \log_x 16 = 3$, find the value of x. **[2008 P1 #6]**

13. Given that $\log_5 x + \log_5 y = 3\log_5 q$ show that $x = \frac{q^3}{y}$. **[2010 P1 #8]**

14. Given that $\log_y 864 - \log_y 6 = 2$, find the value of y. **[2010 PII #5a]**

15. Solve the equation $2^{2a} - 5(2^a) + 4 = 0$. **[2010 PII #8a]**

16. Given that $\log_x 6\frac{1}{4} = 2$. Find the value of x. **[2011 PI #4]**

17. Without using four-figured tables or a calculator evaluate $\log_a 2 - \log_a 6 + \log_a 3$. **[2011 PII #1b]**

18. Given that $\log p - \log q = 2\log r$, find the p in terms of q and r. **[2012 P1 #11]**

19. Given that $\log_a c = 0.4475$, evaluate $\log_a\left(\frac{a^2}{c}\right)$. **[2012 P2 #10a]**

20. Solve the equation $\frac{1}{3^{(n+1)}} = 81$. **[2013 P1 #14]**

21. Evaluate $\log_4 256 + \log_3 27 - \frac{1}{2}\log_2 16$. **[2013 P2 #10a]**

22. Given that $\log_2\left(\frac{4}{\sqrt{p}}\right) = 3$, calculate the value of p leaving your answer in its simplest form. **[2014 P1 #7]**

23. Solve for b in equation $2^{2b+1} - 17\left(2^b\right) + 8 = 0$. **[2014 PII #7b]**

24. Solve the equation $8^x \times 2^{(3x-1)} = 2^{35}$. **[2015 PI #17]**

25. solve the equation $2\log_5 x = \log_5(5x-6)$. **[2015 PII #10a]**

26. Given that $\log_2\left(\frac{1}{16}\right) = 2x-1$, find the value of x. **[2016 PI #11]**

27. Without using a calculator or four figure tables, evaluate $\frac{\log\sqrt{x}}{\log\sqrt[3]{x^2}}$. **[2016 PII #3a]**

28. Given that $\log 2 = 0.3010$ and $\log 3 = 0.4771$, without using a calculator, evaluate $\log 6 - \log 5$. **[2017 PI #12]**

29. Solve the equation
$\log_2(x^2+3x-2) = 2\log_2 x + 1$. **[2017 PII #10a]**

30. Given that $a\log_2 8 = 6$, calculate the value of a. **[2018 P1 #12]**

31. Given that $\log_2\left(x^2+3x-2\right) = 1 + 2\log_2 x$, calculate the values of x. **[2018 PII #18]**

32. Given that $\log_3\frac{3}{a} = -2$, find the value of a. **[2019 PII #2a]**

33. Given that $\log_{10} 2 = 0.3010$, and $\log_{10} 3 = 0.4771$, without using a calculator or four-figure tables, evaluate $\log_{10} 0.6$. **[2020 Mock PI #3]**

34. Solve for x in $6^{x+1} + 7\times 6^x = 78$. **[2021 Mock PI #6]**

35. Given that $2^k \times 7^d = 392$, find the values of k and d. **[2021 Mock PII #1b]**

1. **[2003 PI #8]**

$\log_a 6 = \log_a(2\times 3)$ Expressing 6 in terms of 2 and 3

$\log_a 2 + \log_a 3$ (addition rule)

$= 0.6110 + 0.7039$ (subtrcting given values)

$= 1.3149$

$\therefore \log_a 6 = 1.3149$

2. **[2003 PP II #1.a]**

$10^x = 0.01$

$10^x = \frac{1}{100}$ (converting decimal to fraction)

$10^x = \frac{1}{10^2}$

$10^x = 10^{-2}$ (rules of exponents)

$x = -2$ (equal bases)

3. **[2004 PP1 #19]**

$\log_{10}(2m+6) = 1 + \log_{10}(m-1)$

$\log_{10}(2m+6) = \log_{10}10 + \log_{10}(m-1)$

(since $\log_n 10 = 1$*)*

$\log_{10}(2m+6) = \log_{10}10(m-1)$ *(addition rule)*

Thus, $2m + 6 = 10(m-1)$ *(equality of logs)*

$2m + 6 = 10m - 10$

$2m - 10m = -10 - 6$

$\frac{-8m}{-8} = \frac{-16}{-8}$

$\therefore m = 2.$

4. **[2004 PII #10a]**

$\log_9 27^k = K + 1.$

$\Rightarrow 9^{(K+1)} = 27^K$

$3^{3(K+1)} = 3^{3K}$ (expressing both sides with base 3)

$3^{2K+2} = 3^{3K}$

$\therefore 2K + 2 = 3K$ *(equal bases)*

$3K - 2K = 2$

$\therefore K = 2$

5. **[2005 PI #8]**

$\log_m 27 = 3 \Rightarrow m^3 = 27$ (log to exponential)

$m^3 = 3^3$

$\therefore m = 3$ *(equal bases)*

6. **[2005 PII #9a]**

Let $2^x = y$

Then $y^2 - 9y + 8 = 0.$

By inspection, $(y-8)(y-1) = 0$

Either $y - 8 = 0$ *or* $y - 1 = 0$

$y = 8$ *or* $y = 1$

re-substituting 2^x *for* y

$2^x = 8$ $\quad 2^x = 1$

$2^x = 2^3$ *or* $2^x = 2^0$

$\therefore x = 3$ *or* $x = 0$

7. **[2006 PI #20]**

$\log_{10} n - \log_{10} m = 2\log_{10} h$

$\log_{10}\left(\frac{n}{m}\right) = 2\log_{10} h$ (difference of logs)

$\log_{10}\left(\frac{n}{m}\right) = \log_{10} h^2$ (power of logs)

$\frac{n}{m} = h^2$ (equality of logs)

$n = h^2 m$ (making n subject)

8. **[2006 PII #2a]**

$= 2\log_a a + 3\log_a 1 + \log_a \sqrt{2}$

$= 2\log_a a + 3\log_a 1 + \log_2 2^{1/2}$ (since $\sqrt{a} = a^{\frac{1}{2}}$)

Since $\log_a a = 1, \log_a 1 = 0,$ *and*

$\log_a 2^{1/2} = \frac{1}{2}\log_2 2$ *then*

$= 2\log_a a + 3\log_a 1 + \frac{1}{2}\log_2 2$

$= 2(1) + 3(0) + \frac{1}{2}(1)$

$= 2 + 0 + \frac{1}{2}$

$= 2\frac{1}{2}$

9. **[2007 PI #11]**

Given $\log_a 2 = 0.668$ and $\log_a 3 = 0884$

$\log_a 12 = \log_a(4\times 3)$ (expressing 12 in terms of 4 and 3)

$= \log_a 4 + \log_a 3$ (laws of logs)

$= \log_a 2^2 + \log_a 3$

$= 2\log_a 2 + \log_a 3$ (laws of logs)

$= 2\times 0.668 + 0.884$ (substituting values)

$= 1.336 + 0.884$

$= 2.22$

10. **[2007 PII #11a]**

$\dfrac{4^x \times 8^{x-1}}{32^x}$

$= \dfrac{2^{2x} \times 2^{3(x-1)}}{2^{5x}}$ (expressing each term to base 2)

$= \dfrac{2^{2x} \times 2^{3x-3}}{2^{5x}}$ (expanding power)

$= \dfrac{2^{2x+3x-3}}{2^{5x}}$ (rules of exponent)

$= \dfrac{2^{5x-3}}{2^{5x}}$ (simplifying)

$= 2^{5x-3-5x}$

$= 2^{-3}$

$= \dfrac{1}{2^3} = \dfrac{1}{8}$

11. **[2008 PII #4b]**

$\log_7 343 = 2x - 5 \Rightarrow$

$7^{2x-5} = 343$ (converting log to exponential)

$7^{2x-5} = 7^3$

$\therefore 2x - 5 = 3$ (equal bases)

$2x = 3 + 5$

$2x = 8$

$\dfrac{2x}{2} = \dfrac{8}{2}$

$x = 4$

$\therefore$ The value of x is 4.

12. **[2008 P1 #6]**

$\log_x(\frac{1}{2}) + \log_x 16 = 3$

$\log_x(\frac{1}{2} \times 16) = 3$ (addition rule)

$\log_x 8 = 3$

$x^3 = 8$ (converting log to exponential)

$x^3 = 2^3$

$x = 2$ (equal powers imply equal bases)

$\therefore$ The value of x is 2

13. **[2010 P1 #8]**

$\log_5 x + \log_5 y = 3\log_5 q$

$\log_5(xy) = \log_5 q^3$ (additon rule)

$xy = q^3$ (taking anti-logs/equality of logs)

$\dfrac{xy}{y} = \dfrac{q^3}{y}$

$x = \dfrac{q^3}{y}$

14. **[2010 PII #5a]**

$\log_y 864 - \log_y 6 = 2$

$\log_y\left(\dfrac{864}{6}\right) = 2$ (division rule)

$\log_y 144 = 2$ (converting log to exponential)

$144 = y^2$

$12^2 = y^2$ (exponentiating LHS)

$\therefore y = 12$ (equal powers imply equal bases)

15. **[2010 PII #8a]**

$2^{2a} - 5(2^a) + 4 = 0$

$(2^a)^2 - 5(2^a) + 4 = 0$

Let 2^a be y

$y^2 - 5y + 4 = 0$

$(y-4)(y-1) = 0$ (by inspection)

Either $(y-4) = 0$ or $(y-1) = 0$

$\therefore y = 4$ or $y = 1$

replacing $2^a = y$

When $y = 4$; $2^a = 4$

$2^a = 2^2$

$\therefore a = 2$

When $y = 1$; $2^a = 1$

$2^a = 2^0$

$\therefore a = 0$

$\therefore a = 2$ or $a = 0$

16. **[2011 PI #4]**

$\log_x 6\frac{1}{4} = 2$

$\log_x \frac{25}{4} = 2$ (creating an improper fraction)

$\dfrac{25}{4} = x^2$ (converting log to exponential)

$\sqrt{\frac{25}{4}} = \sqrt{x^2}$ (taking roots)

$\sqrt{\left(\frac{5}{2}\right)^2} = \sqrt{x^2}$

$\frac{5}{2} = x$

17. **[2011 PII #1b]**

$log_a 2 - \log_a 6 + \log\ 3$

$= \log_a 2 + \log_a 3 - \log_a 6$

$= \log_a \frac{2\times3}{6}$ (addition rule and division rule)

$= \log_a 1$

$= 0$

18. **[2012 P1 #11]**

Given that $\log p - \log q = 2\log r$

$\log p - \log q = \log\frac{p}{q}$ *(division rule)*

$2\log r = \log r^2$ (log of a power)

$\log\frac{p}{q} = \log r^2$

$\therefore \frac{p}{q} = r^2$ (taking anti-logs)

$\therefore p = r^2 q$

19. **[2012 P2 #10a]**

Given $\log_a c = 0.4475$

$\log_a\left(\frac{a^2}{c}\right) = \log_a a^2 - \log_a c$

$= 2\log_a a - \log_a c$

$= 2(1) - 0.4475$ (Since $\log_a a = 1$)

$= 2 - 0.4475$

$= 1.5525$

20. **[2013 P1 #14]**

$\frac{1}{3^{(n+1)}} = 81$

$3^{-1(n+1)} = 3^4$ (expressing to base 3)

$3^{-1-n} = 3^4$

$-1-n = 4$ (equal bases)

$-n = 4+1$

$-n = 5$ (diving by -1 both sides)

$\therefore n = -5$

21. **[2013 P2 #10a]**

$\log_4 256 + \log_3 27 - \frac{1}{2}\log_2 16$

$= \log_4 4^4 + \log_3 3^3 - \log_2 16^{\frac{1}{2}}$ (simplifying)

$= 4\log_4 4 + 3\log_3 3 - \log_2 \sqrt{16}$ (rules of exponents)

$= 4(1) + 3(1) - \log_2 4$ (Since $\log_a a = 1$)

$= 4 + 3 - \log_2 2^2$

$= 7 - 2\log_2 2$

$= 7 - 2(1)$ [since $\log a^a = 1$]

$= 7 - 2$

$= 5$

22. **[2014 P1 #7]**

$\log_2\left(\frac{4}{\sqrt{p}}\right) = 3$

$\frac{4}{\sqrt{p}} = 2^3$ (converting exponential to log)

$\frac{4}{\sqrt{p}} = 8$

$4 = 8\sqrt{p}$ (cross- multiplying)

$\therefore \sqrt{p} = \frac{\cancel{4}}{\cancel{8}}$ (simplifying)

$\left(\sqrt{p}\right)^2 = \left(\frac{1}{2}\right)^2$ (squaring both sides)

$p = \frac{1}{4}$

23. **[2014 PII #7b]**

$2^{2b+1} - 17(2^b) + 8 = 0$

$2^{2b} \times 2^1 - 17(2^b) + 8 = 0$ (rules of exponents)

$2\left(2^b\right)^2 - 17(2^b) + 8 = 0$ (rules of exponents)

Let $2^b = y$

$2y^2 - 17y + 8 = 0,$

Factors: $2\times8 = +16 \Rightarrow -16 - 1 = -17$

$2y^2 - 16 - y + 8 = 0$

$2y(y-8) - 1(y-8) = 0$

$(y-8)(2y-1)=0$ after factorizing

Either $(y-8)=0$ or $(2y-1)=0$,

$y=8$ or $y=\frac{1}{2}$

When $y=8$;

$2^b=8$

$2^b=2^3$

$\therefore b=3$ (equal bases)

When $y=\frac{1}{2}$;

$2^b=\frac{1}{2}$

$2^b=2^{-1}$

$\therefore b=-1$ (equal bases)

24. **[2015 PI #17]**

$8^x \times 2^{(3x-1)}=2^{35}$

$\left(2^3\right)^x \times 2^{(3x-1)}=2^{35}$

$2^{3x}\times 2^{(3x-1)}=2^{35}$

$2^{3x+(3x-1)}=2^{35}$ (when multiplying, we add powers)

$2^{6x-1}=2^{35}$

$\therefore 6x-1=35$ (equal bases)

$6x=35+1$

$6x=36$

$\frac{\cancel{6}x}{\cancel{6}}=\frac{3\cancel{6}}{\cancel{6}}$

$x=6$

25. **[2015 PII #10a]**

$2\log_5 x=\log_5(5x-6)$

$\log_5 x^2=\log_5(5x-6)$

$x^2=5x-6$ (taking anti-log on both sides)

$x^2-5x+6=0$

$(x-3)(x-2)=0$ (by inspection)

either $(x-3)=0$ or $(x-2)=0$

$\therefore$ Either $x=3$ or $x=2$

26. **[2016 PI #11]**

$\log_2\left(\frac{1}{16}\right)=2x-1,$

$\frac{1}{16}=2^{2x-1}$ (log to exponent)

$16^{-1}=2^{2x-1}$ (reciprocal)

$2^{-4}=2^{2x-1}$

$-4=2x-1$ (equal bases)

$-2x=-1+4$

$-2x=3$

$\frac{-2x}{-2}=\frac{3}{-2}$

$\therefore x=-1.5$

27. **[2016 PII #3a]**

$\frac{\log\sqrt{x}}{\log\sqrt[3]{x^2}}$

$=\frac{\log x^{\frac{1}{2}}}{\log x^{\frac{2}{3}}}$ since $\sqrt[n]{a^m}=a^{m/n}$

$=\frac{\frac{1}{2}\cancel{\log x}}{\frac{2}{3}\cancel{\log x}}$ by the power rule

$=\frac{1}{2}\div\frac{2}{3}$

$=\frac{1}{2}\times\frac{3}{2}$ (flipping fraction)

$=\frac{3}{4}$

28. **[2017 PI #12]**

$\log 6-\log 5$

$=\log(2\times 3)-\log(10\div 2)$ (expressing in terms of given values)

$=(\log 2+\log 3)-(\log 10-\log 2)$ (addition+division rule)

$=(0.3010+0.4771)-(1-0.3010)$ (substituting values)

$=(0.7781)-(0.699)$

$=0.0791$

29. **[2017 PII #10a]**

$\log_2(x^2+3x-2)=2\log_2 x+1$

$\log_2(x^2+3x-2)=\log_2(x+1)^2$ (power rule)

Take antilog$_2$ on both sides

$x^2+3x-2=(x+1)^2$ (equal logs)

$x^2+3x-2=(x+1)(x+1)$

$x^2+3x-2=x^2+2x+1$ (expanding)

$x^2-x^2+3x-2x=1+2$ (collecting terms)

$\therefore x=3$

30. **[2018 P1 #12]**

$a\log_2 8=6$

$\log_2 2^{3a}=6$ (power rule)

$3a\log_2 2=6$

$3a(1)=6$ (Since $\log_a a=1$)

$\frac{\cancel{3}}{\cancel{3}}a=\frac{\cancel{6}}{\cancel{3}}$

$a=2$

31. **[2018 PII #18]**

$\log_2(x^2+3x-2)=1+2\log_2 x$

$\log_2(x^2+3x-2)=\log_2 2+\log_2 x^2$ $\left(\begin{array}{c}\text{power rule}\\ \text{and } \log_a a=1\end{array}\right)$

$\log_2(x^2+3x-2)=\log_2(2x^2)$

Take the antilog$_2$ on both sides

$x^2+3x-2=2x^2$

$2x^2-x^2-3x+2=0$

$x^2-3x+2=0$

$(x-1)(x-2)=0$

Either $(x-2)=0$ or $(x-1)=0$

$x=2$ or $x=1$

32. **[2019 PII #2a]**

$\log_3\frac{3}{a}=-2$

$\frac{3}{a}=3^{-2}$ (log to exponent)

$\frac{3}{a}=\frac{1}{3^2}$

$\frac{3}{a}=\frac{1}{9}$

$a=3(9)$

$a=27$

33. **[2020 Mock PI #3]**

$\log_{10}0.6$

$=\log_{10}\left(\frac{6}{10}\right)$ (decimal to fraction)

$=\log_{10}\left(\frac{2\times3}{10}\right)$ (expressing 6 in terms of 2 and 3)

$=\log_{10}2+\log_{10}3-\log_{10}10$ (addition and division rule)

$=0.3010+0.4771-1$

$=-0.2219$

34. **[2021 Mock PI #6]**

$6^{x+1}+7\times6^x=78$

$6^x6^1+7\times6^x=78$ (rules of exponents)

$6^x(6+7)=78$ (taking out 6^x)

$6^x(13)=78$ (simplifying)

$6^x=\frac{78}{13}$

$6^x=6$

$6^x=6^1$

$\therefore x=1$ (equal bases)

35. **[2021 Mock PII #1b]**

$2^k\times7^d=392$

Find all factors of 392:

2	392
2	196
2	98
7	49
7	7
	1

$392=2\times2\times2\times7\times7=2^3\times7^2$

$2^k\times7^d=2^3\times7^2$ (grouping factors)

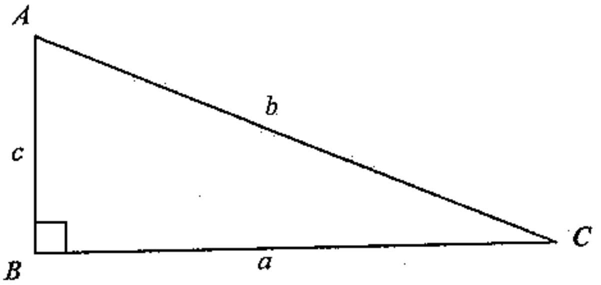
$\therefore k=3; d=2$ (equal bases)

CH 12 TRIGONOMETRY I

Chapter Highlights

In this chapter, our focus will be right angled triangles. We will solve problems on trigonometric triangles. We will solve problems on trigonometric ratios: sine ratio, cosine ratio and tangent ratio. We should be able to work out problems involving trigonometry: finding angles given the ratios, calculating sides of right angled triangle using trigonometric ratios.

Using the right angle below:



Pythagoras' theorem:-

$$c^2 = a^2 + b^2$$

Trig ratios for angle A:-

$\sin A = \frac{a}{b}$, $\cos A = \frac{c}{b}$

$\text{Tan } A = \frac{a}{c}$

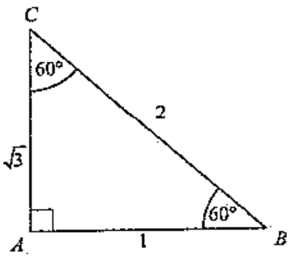
Trig ratios for angle C:-

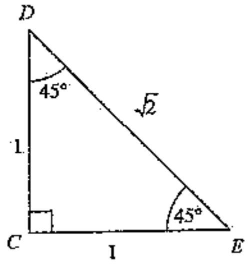
$\sin C = \frac{c}{b}$,

$\cos C = \frac{a}{b}$ $\text{Tan } C = \frac{c}{a}$

Special Angles:

Using the triangles below, we will derive the trigonometric ratios for the following special angles: $30^\circ, 45^\circ$ and 60°.





Note that:

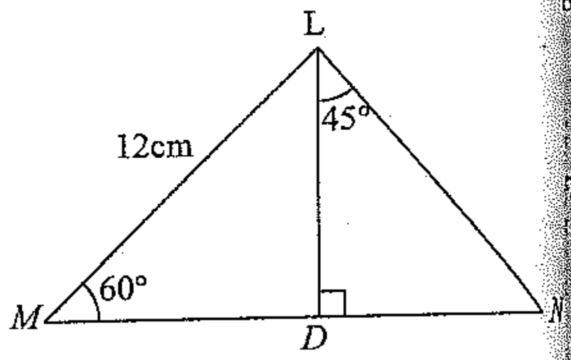
$\sin 0^\circ = 0$ $\cos 0^\circ = 1$ $\tan 0^\circ = 0s$

$\sin 90^\circ = 1$ $\cos 90 = 0$ $\tan 90 = undefined$

Table of special angles

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

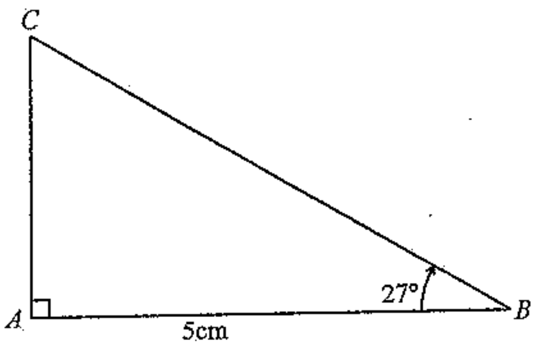
1. In **Figure 1, LMN** is a triangle. **D** is a point on MN such that angle **LDN**= 90°, angle **LMN**= 60° and angle **NLD**= 45°



Given that **LM**=12 cm, calculate the length of DN.

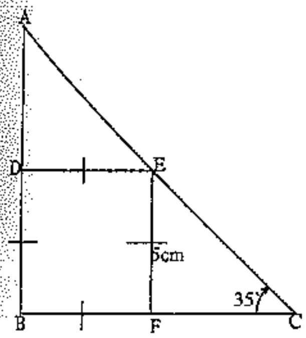
[2003 PP II #1b]

2. In **figure 2, ABC** is a triangle in which angle **BAC**=90°, angle **ABC**=27° and **AB**=5 cm



Calculate the length of **AC.** **[2004 PI #6]**

3. **Figure 3** shows a square **DEFB** of side 5cm inside triangle **ABC**



If angle **ACB**= 35°, calculate the length of **AB.**

[2005 PII #3a]

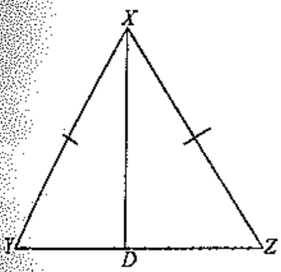
4. Given that $\sin\theta = \frac{1}{3}$. Find $\cos\theta$, leaving your answer in surd form. **[2005 PII #5a]**

5. Without using a calculator or four-figure table, find cos B if sin B=0.8. **[2006 P1 #9]**

6. The angle of depression of a car from the top of a pole is 35°. If the top of the pole is 25m from the ground, calculate the distance of the car from the pole. **[2007 PI #21]**

7. An aeroplane takes off at an angle of $23°$ to the horizontal ground and flies for 13km. calculate its height above the ground, leaving your answer correct to 1 decimal place. **[2008 P1 #12]**

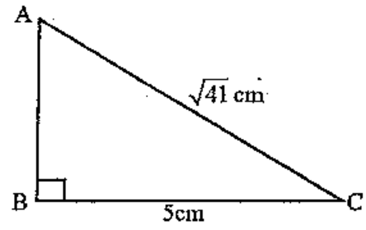
8. **Figure 4** shows an isosceles triangle XYZ with sides XY=XZ=9cm and YZ=12cm. XD is a perpendicular bisector of YZ.



Calculate angle YXZ giving your answer correct to the nearest degree. **[2010 P1 #23]**

9. A boat is 100m away from the bottom of a cliff. If the cliff is 16m high, calculate the angle of depression of the boat from the top of the cliff to the nearest degree. **[2010 PII #7a]**

10. **Figure 5** shows triangle ABC right angled at B. BC= 5 cm and AC $=\sqrt{41}$ cm. Calculate angle ACB, giving your answer correct to the nearest degree.

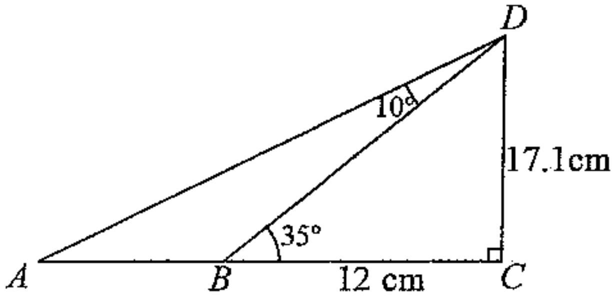


[2012 P1 #15]

11. A plane leaves an airport **A** at an angle of elevation of $40°$. It covers a distance of 6km to point **B** which is x kilometers high above the ground. It then reduces the angle of elevation to $20°$ and flies a distance of 18km to y km higher than before to point **C.** Calculate, to the nearest degree, the height of the plane above the ground at point **C**. **[2016 PI #19]**

12. A point **P** on the level ground is 15 m away from the foot of the wall of a house. The angle of elevation of the bottom **B** of a window from **P** is $12°$ and the angle of elevation to the top **T** of the window from the same point **P** is $16°$. Calculate the height of the window, giving your answer correct to the nearest metre. **[2016 PII #6a]**

13. **Figure 6** shows a right-angled triangle **ABC.** Angle **BDC** $=35°$, angle **ABD** $=10°$, **BC** $=12$ cm and **CD** = 17.1 cm.



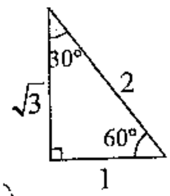
Calculate the length of **AB,** giving your answer correct to one decimal place. **[2017 PI #20]**

14. The angles of elevation of the top of a tower from two girls: Anna (A) and Belita (B) are $22°$ and $18°$ respectively. If the two girls are on opposite sides of the tower and the tower is 100m high, calculate the distance between the girls, giving the answer correct to the nearest whole number. **[2019 PI #14]**

1. **[2003 PP II #1b]**

Given right ΔLDM : $\sin 60° = \dfrac{LD}{LM}$

$$\sin 60 = \frac{LD}{12cm}$$



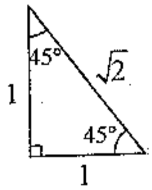
$\Rightarrow LD = 12\sin 60°$

$LD = 12 \times \dfrac{\sqrt{3}}{2}$ $\qquad \left(\sin 60 = \dfrac{\sqrt{3}}{2}\right)$

$LD = 6\sqrt{3}$

In ΔLDN

$\tan 45° = \dfrac{DN}{LD} \Rightarrow \tan 45° = \dfrac{DN}{6\sqrt{3}}$



$\Rightarrow DN = 6\sqrt{3}\tan 45° \quad \tan 45° = 1$

$\Rightarrow DN = 6\sqrt{3} \times 1$

$\therefore DN = 6\sqrt{3}cm$

Altenatively :

upon finding $LD = 6\sqrt{3}cm$

In ΔLDN

$\angle LDN + \angle NLD + \angle LND = 180°$ ($\angle$sum in Δ)

$90° + 45° + \angle LND = 180°$

$\Rightarrow \angle LND = 180° - 135°$

$\Rightarrow \angle LND = 45°$ also $\angle NLD = 45°$

$\Rightarrow \Delta LDN$ is an isosceles triangle (base $\angle s$ equal)

$\therefore$ DN=LD

$\therefore$ DN=$6\sqrt{3}cm$.

---

2. **[2004 PI #6]**

Since ABC is a right angled triangle, we use trig ratios:

$$\tan 27° = \frac{opp}{adj}$$
$$= \frac{AC}{5cm}$$

AC $= 5 \times \tan 27°$

AC $= 5 \times 0.5095$ *(using a calculator)*

AC $= 2.55$ cm (to 2 decimal places)

---

3. **[2005 PII #3a]**

$DEFB$ is a square and ABC is a triangle

BFC is a straight line

$\angle EFB = 90°$

in ΔEFC

$\tan 35 = \dfrac{EF}{FC} \Rightarrow \tan 35 = \dfrac{5}{FC}$

$\Rightarrow FC = \dfrac{5}{\tan 35}$

$\Rightarrow FC = 7.1407cm$

$BC = BF + FC$

$BC = 5cm + 7.1407cm$

$\therefore BC = 12.1407cm$

in ΔABC $\tan 35 = \dfrac{AB}{BC} \Rightarrow \tan 35 = \dfrac{\text{AB}}{12.1407}$

$\Rightarrow AB = 12.1407\tan 35$

$\Rightarrow AB = 8.50100966$

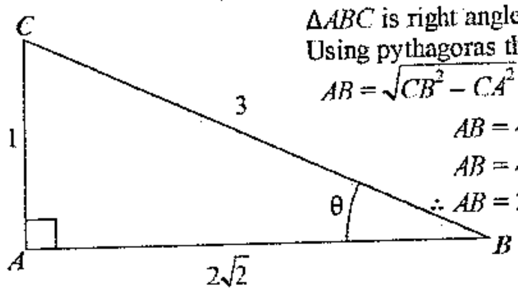
$\therefore AB = 8.50cm$ (to 2 decimal places)

---

4. **[2005 PII 5a]**

Since trigonometric ratios are used on right angled triangles, we construct a geometric representation

of $\sin\theta = \dfrac{1}{3}$ $\begin{matrix}\rightarrow \text{opposite of } \theta \\ \rightarrow \text{hypotenuse of } \theta\end{matrix}$



ΔABC is right angled.

Using pythagoras theorem:

$AB = \sqrt{CB^2 - CA^2}$

$AB = \sqrt{3^2 - 1^2}$

$AB = \sqrt{8}$

$\therefore AB = 2\sqrt{2}$

$\cos\theta = \dfrac{adj}{hyp} \Rightarrow \cos\theta = \dfrac{2\sqrt{2}}{3}$

$\therefore \cos\theta = \dfrac{2\sqrt{2}}{3}$

---

5. **[2006 P1 #9]**

Best way is to reconstruct a triangle of the trigonometric ratios.

We construct a triangle representing $\sin B = 0.8$

$\sin B = 0.8 \Rightarrow \sin B = \dfrac{8}{10} \Rightarrow \sin B = \dfrac{4}{5}$

$\sin B = \frac{4 \to opp}{5 \to hyp}$

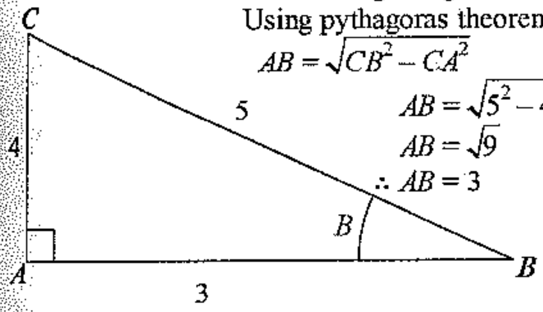
ΔABC is right angled.
Using pythagoras theorem:

$AB = \sqrt{CB^2 - CA^2}$

$AB = \sqrt{5^2 - 4^2}$

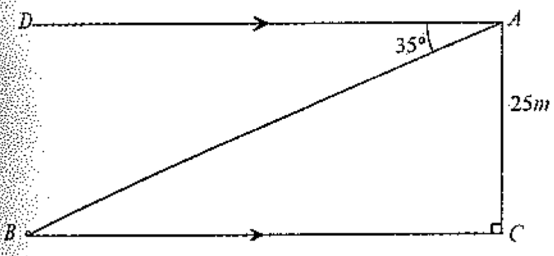
$AB = \sqrt{9}$

$\therefore AB = 3$



$\therefore \cos B = \frac{adj}{hyp} = \frac{3}{5} = 0.6$

6. **[2007 PI #21]**



Need to find BC

$\angle ABC = \angle DAB = 35^\circ$ (alt.$\angle$s;$DA // BC$)

Now: $\tan \angle ABC = \frac{AB}{BC}$

$\tan 35^\circ = \frac{25m}{BC}$

$\Rightarrow BC = \frac{25m}{\tan 35^\circ}$

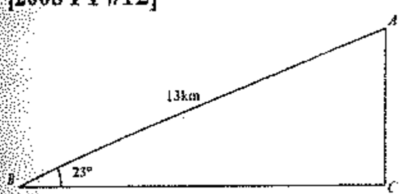
$= \frac{25m}{0.7002}$

$= 35.704084$

$= 35.70$ (to 2 decimal place)

$\therefore$ The distance of the car from the pole is 35.70m

7. **[2008 P1 #12]**



$\sin B = \frac{AC}{AB}$

$\sin 23^\circ = \frac{AC}{13km}$

$AC = 13km \times \sin 23^\circ$

$AC = 3km \times 0.3907 = 5.07950$

$AC = 5.1km$ (to 1 decimal place)

$\therefore$ the height of the airplane above the ground is 5.1km

8. **[2010 P1 #23]**

In ΔXYD, XY=9 cm (given)

$YD = \frac{12}{2} \Rightarrow YD = 6cm$ (XD $\perp$ YZ)

$\cos \angle XYD = \frac{6}{9} \Rightarrow \angle XYD = \cos^{-1}\frac{6}{9}$

$\Rightarrow \angle XYD = 48.1896^\circ$

Since $\angle XYD = \angle XZD$ (isosceles Δ)

$\Rightarrow \angle XYD = 48.1896^\circ$

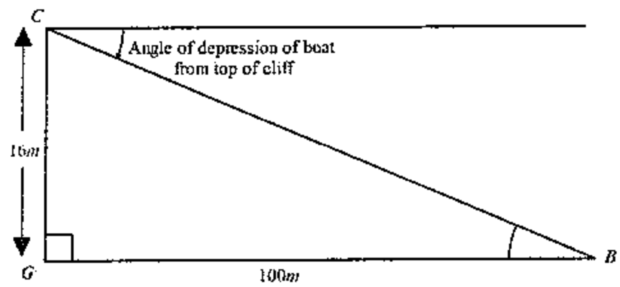
$\angle YXZ = 180^\circ - \angle XYD - \angle XZD$

$\angle YXZ = 180^\circ - 48.1896 - 48.1896$

$= 83.6206$

$\therefore \angle YXZ = 84^\circ$ (*to the nearest* degrees)

9. **[2010 PII #7a]**



Angle of depression $= \angle CBG$ (alternate angles)

In right ΔCBG

$\tan \angle CBG = \frac{CG}{BG} \Rightarrow \tan \angle CBG = \frac{16}{100}$

$\Rightarrow \tan \angle CBG = 0.16$

$\Rightarrow \angle CBG = \tan^{-1} 0.16 \Rightarrow \angle CBG = 9.09^\circ$

$\therefore \angle CBG = 9^\circ$ (to the nearest degree)

10. **[2012 P1 #15]**

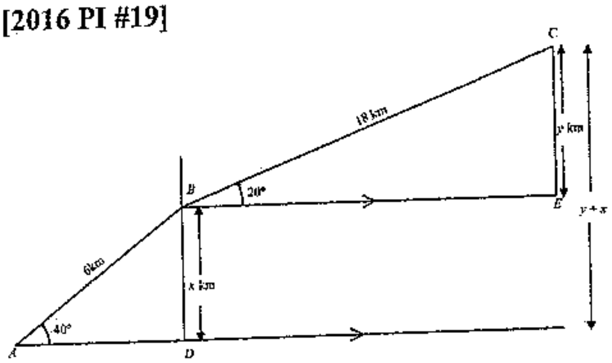
Let $\angle ACB$ be θ

$\cos\ \theta = \frac{Adj}{Hyp} = \frac{5}{\sqrt{41}} = \frac{5}{6.4031} = 0.7809$

$\theta = \cos^{-1} 0.7809 \Rightarrow \theta = 38.6569^\circ$

$\therefore \angle ACB = 39^\circ$ (to the the nearest degree)

11. **[2016 PI #19]**



Using right ΔABD

$$\sin 40° = \frac{BD}{AB} \Rightarrow \sin 40° = \frac{x}{6}$$

$$\Rightarrow 6\sin 40° = x \therefore x = 3.8567\,km$$

Using ΔBCE,

$$\sin 20° = \frac{CE}{BC} \Rightarrow \sin 20° = \frac{y}{18}$$

$$\Rightarrow 18\sin 20° = y$$

$$\therefore y = 6.15635\,km$$

The height of the plane above the ground at point C $= x + y$

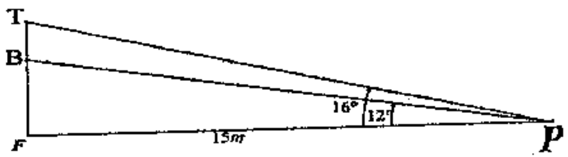
$= 3.8567\,km + 6.15635\,km$

$= 10.013\,km$

$= 10\,km$ (to the nearest degree)

12. **[2016 PII #6a]**

Let the foot of the wall be point F



The height of the window BT = FT − FB

Using ΔFBP.

$$\tan 12° = \frac{FB}{FP} = \frac{FB}{15}$$

$$\Rightarrow FB = 15\tan 12° \Rightarrow BF = 3.188\,m$$

Using ΔFTP,

$$\text{Tan } 16° = \frac{FT}{FP} \Rightarrow \text{Tan } 16° = \frac{FT}{15}$$

$$\Rightarrow FT = 15\tan 16° \Rightarrow FT = 4.301\,m$$

$BT = 4.301\,m - 3.188\,m$

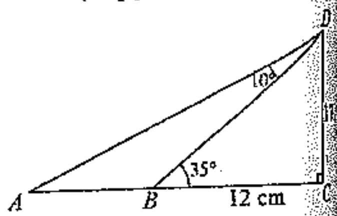
$BT = 1.113\,m$

$\therefore BT = 1\,m$ (to the nearest m)

13. **[2017 PI #20]**

ABC is a straight line

$\angle DAB + \angle ADB = \angle DBC$ (2opp.int $\angle s$ = ext.$\angle$)



$\angle DAB + 10° = 35°$

$\angle DAB = 35° - 10°$

$\angle DAB = 25°$

In ΔDAC,

$$\text{Tan } \angle DAC = \frac{DC}{AC}$$

But $\angle DAC = \angle DAB = 25°$ (same $\angle$)

$$\text{Tan } 25° = \frac{17.1}{AC}$$

$$AC = \frac{17.1}{\text{Tan } 25°}$$

$$= \frac{17.1}{0.4663}$$

$= 36.6717$

$= 36.67$

But $AC = AB + BC$

$AB = AC - BC$

$AB = 36.67 - 12$

$AB = 24.67$

Alternatively,

In right ΔBCD, Sum of angles $= 180°$

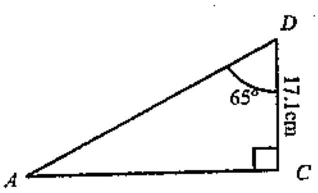
$\Rightarrow \angle BCD + \angle BCD + \angle CDB = 180°$

$\angle BCD = 180° - 90° - 10° \Rightarrow \angle BCD = 55°$

Using ΔACD

$\angle ADC = \angle ADB + \angle BDC \Rightarrow \angle ADC = 10° + 55°$

$\therefore \angle ADC = 65°$



$\text{Tan } 65° = \frac{Opp}{Adj} = \frac{AC}{DC} = \ldots$

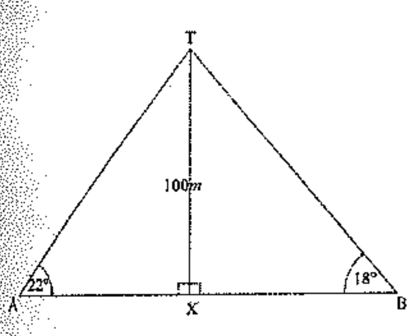
$AC = 17.1 \times \text{Tan } 65°$

$AC = 17.1 \times 2.1445$

$AC = 36.67$

$\therefore AB = AC - BC = 36.67 - 12 = 24.67$ cm

14. **[2019 PI #14]**



$$Tan\theta = \frac{Opposite}{Adjacent}$$

$$Tan\ 22° = \frac{100}{AX}$$

$$\therefore AX = \frac{100}{Tan\ 22°}$$

$$= \frac{100}{0.4040}$$

$$AX = 247.51m$$

$$Tan\ 18° = \frac{100}{XB}$$

$$\therefore XB = \frac{100}{Tan\ 18°}$$

$$= \frac{100}{0.3249}$$

$$XB = 307.77m$$

The distance between them is AB

But AB=AX+XB

=$307.77m + 247.51m$

=555.28

≈ 555 (to the nearest whole number)

The girls are 555m apart

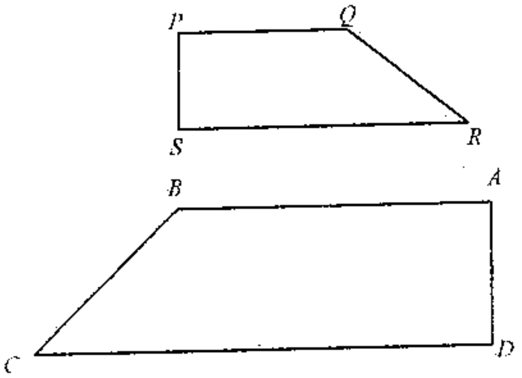
CH 23 SIMILARITY

Chapter Highlights

In this chapter, we will solve problems regarding similarity. These involve figures which can be obtained from another figure by multiplying the dimensions with a scale factor. The scale factor is the ratio of the corresponding sides of the similar figures.

Corresponding Sides

This are pairs of matching sides that are in the same location of two similar figures. In the figure below, for example, QR corresponds to BC; PS corresponds to AD; RS corresponds to CD; and AB corresponds to PQ.



$$\text{Scale factor} = \frac{\textit{Length of Big Figure}}{\textit{Length of Small Figure}}.$$

Area factor

This is the ratio of area of the image to the area of the object.

$$\text{Area factor} = (\text{scale factor})^2$$

$$\text{Area factor} = \frac{\textit{Area of Big Figure}}{\textit{Area of Small Figure}}$$

Relationship between the Scale Factor and Area Factor in one formula

$$\frac{\textit{Area of Big Figure}}{\textit{Area of Small Figure}} = \left(\frac{\textit{Length of Big Figure}}{\textit{Length of Small Figure}}\right)^2$$

Volume factor

$$\text{The volume factor} = \frac{\text{Volume of the image}}{\text{Volume of the object}}$$

The Scale Factor and Volume Factor are connected in the formula:

$$\text{Volume factor} = (\text{scale factor})^3$$

$$\frac{\text{Volume of Big Figure}}{\text{Volume of Small Figure}} = \left(\frac{\text{Length of Big Figure}}{\text{Length of Small Figure}}\right)^3$$

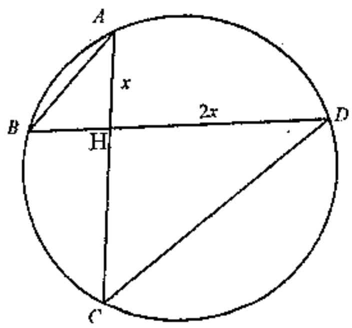
1. The areas of two similar parallelograms are 72 cm^2 and 54cm^2. The height of the larger parallelogram is 8 cm. calculate the corresponding height of the smaller parallelogram. **[2003 PII #12a]**

2. The areas of two similar triangles **ABC** and **HKL** are 100cm^2 and 256cm^2 respectively. If the length of **AB** is 5cm. Calculate the length of **HK**. **[2004 PI #10]**

3. A trapezium has a height of 3cm and its area is $6cm^2$. Calculate the area of a similar trapezium with the height of 12cm. **[2005 P1 #11]**

4. Two triangles **LMN** and **ABC** are similar. In triangle **LMN, LM**=4cm, **MN**= 5cm and **LN**=6cm. the longest side of a triangle **ABC** is 5cm longer than its shortest side. Find the ratio of the areas of the two triangles. **[2005 PII #6a]**

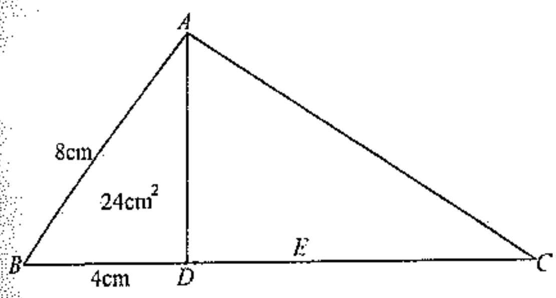
5. In **Figure 1,** circle **ABCD** has a chord **AC** and **BD** intersecting at **H**. length of a line **HD** is twice that of **HA**.



Find the ratio of areas of triangle **ABH** to triangle **CDH**. **[2006 P1 #17]**

6. The areas of two similar triangles are $64cm^2$ and $36cm^2$. the base of the bigger triangle is 16cm. calculate the corresponding height of the smaller triangle. **[2006 PII #7a]**

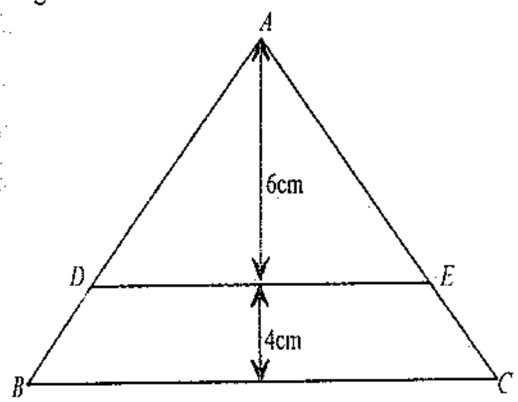
7. In **Figure 2,** triangle **ABC** is similar to triangle **DBA**. The area of triangles **DBA** is $24cm^2$, AB=8cm and DB=4cm



Calculate the area of a triangle ABC. **[2007 PI #23]**

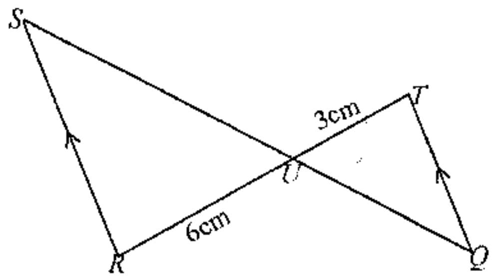
8. Triangles ABC and XYZ are similar. The sides of a triangle ABC are 6cm, 7cm and 8cm. the shortest side of a triangle XYZ is 2cm. given that the area of triangle XYZ is $4.5cm^2$. Calculate the area of triangle ABC. **[2007 PII #2b]**

9. **Figure 3** shows two similar triangles **ADE and ABC** in which **DE** is parallel to **BC.** The area of triangle **ADE** $= 12\,cm^2$



If the heights of triangle **ADE** and trapezium **DECB** are 6cm and 4 cm respectively, calculate the length of **BC.** **[2008 P2 #8a]**

10. **Figure 4** shows **two** similar triangles **TQU** and **RSU**. **TU**=3cm, **UR**=6cm and **RS** is parallel to **QT.**

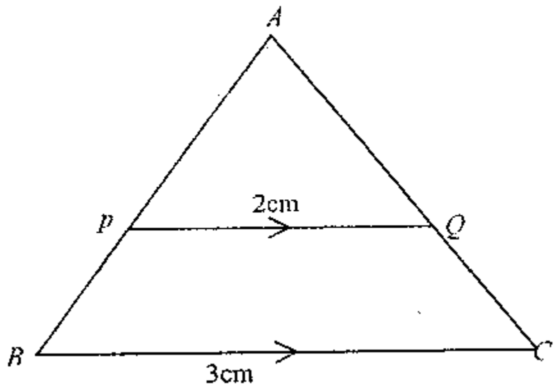


Calculate the ratio of the area of triangle **TQU** to the area of triangle **RSU**, leaving your answer in its simplest form. **[2010 P1 #13]**

11. The area of rectangle **ABCD** is $72cm^2$ and its width is 6cm. calculate the area of a similar rectangle **KLMN** that has a width 9cm. **[2010 PII #2a]**

12. The ratio of-area of two circles is 4:9. Given that the radius of the bigger circle is 18cm, find the radius of the smaller circle. **[2011 PI #5]**

13. In **Figure 5, PQ** = 2cm,**BC** = 3 cm and the area of trapezium **PQCB** $= 10\,cm^2$



If the area of triangle **APQ** $= x\ cm^2$. Calculate the value of x . **[2011 PII #7a]**

14. The areas of two similar triangles are $75cm^2$ and $243cm^2$. if the base of the smaller triangle is $30cm$ and that of the bigger triangle is b cm, calculate the value of b . **[2012 P1 #8]**

15. **Figure 6** shows two similar triangles ABC and ADE. BC = 6 cm and DE = x cm

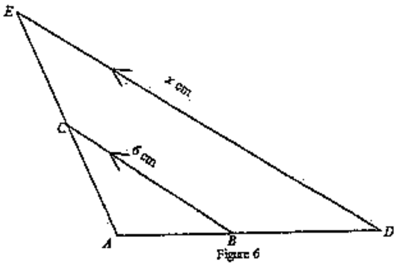


Figure 6

Given that the ratio of areas of $\Delta ABC : \Delta ADE$ is 2:5, calculate the value of x giving your answer correct to 1 decimal place. **[2013 P1 #18]**

16. **Figure 7**, shows two similar triangles ABC and AQS. AB = 6cm, BQ = 8cm, AC = 9cm and CS = 12cm.

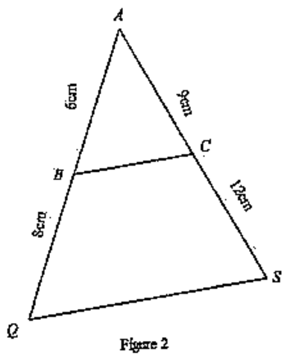


Figure 2

Calculate the ratio of the area of the triangle ABC to the area of the quadrilateral BCSQ. **[2014 PII #5b]**

17. **Figure 8,** shows two similar triangles RST and VSU. VS=5cm, VT=7cm, angle RST = 90° and angle SRT= angle SVU.

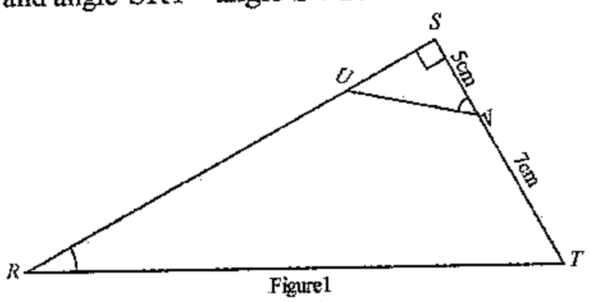


Figure1

If the areas of triangles RST and VSU are $90cm^2$ and $10cm^2$ respectively, calculate the length of SU . **[2015 PII #4a]**

18. A cylinder with base radius r has an area of 77 cm². Calculate the area of a similar cylinder with base radius $2r$. **[2016 PII #4a]**

19. **Figure 9** shows two similar rectangles **ABCD** and **WXYZ.**
WZ = 6cm, **ZY** =10cm and **BC** = 12cm.

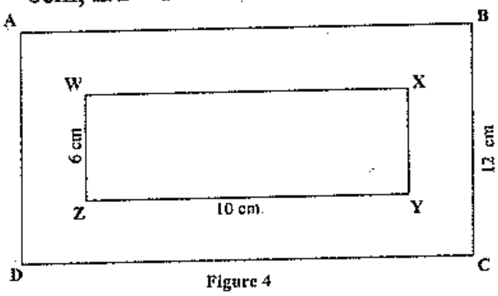


Figure 4

Calculate the area of rectangle **ABCD.** **[2018 P1 #15]**

20. Triangles **ABC** and **PQR** are similar. If the area of triangle **ABC** is four times the area of triangle **PQR,** calculate the ratio of their sides. **[2018 PII #9]**

21. A container of height 6 cm has a volume of 16 cm³ calculate the height h of a similar container whose volume is 54cm³. **[2019 PII #3b]**

22. At his house, Kenneth discovered that a carpet with an area of 13.5m² fitted exactly on the floor of a room 4.5m long. When the boy moved the carpet to a similar room which is 1.5m longer, how much floor area remained uncovered? **[2020 Mock PII 4a]**

23. The areas of two triangles are 6cm² and 24cm². The base of the smaller triangle is 4cm while the base of the bigger triangle is $2b$. Find the height of the bigger triangle. **[2021 Mock PI #18]**

1. **[2003 PII #12a]**

Let the height of the smaller //gram be h

Since the two parallelograms are similar, then,

$Area\ factor = (scale\ factor)^2$

$$\frac{54cm^2}{72cm^2}=\left(\frac{h}{8}\right)^2$$

$$\frac{3}{4}=\frac{h^2}{64}$$

$$\frac{3\times \cancel{64}}{\cancel{4}}=h^2$$

$$3\times16=h^2$$

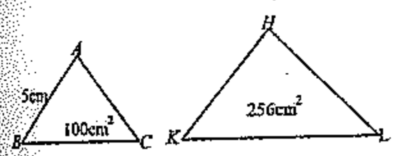
$$48=h^2$$

$$h=\sqrt{48}$$

$h=6.9cm$ (to 1 decimal place)

∴the height of the smaller parallelogram is 6.9cm.

2. **[2004 PI #10]**



Since the two triangles are similar then

Area factor = (scale factor)^2

$$\frac{area\ of\ \triangle HKL}{area\ of\ \triangle ABC}=\left(\frac{HK}{AB}\right)^2$$

$$\frac{256}{100}=\left(\frac{HK}{5}\right)^2$$

$$\frac{64}{25}=\left(\frac{HK}{5}\right)^2$$

$$\sqrt{\frac{64}{25}}=\frac{HK}{5}$$

$$\frac{8}{5}=\frac{HK}{5}$$

∴ HK = 8cm

3. **[2005 P1 #11]**

Let the Area of the bigger Trapezium be A

Since the trapeziums are similar, then

Area factor = (scale factor)^2

$$\frac{A}{6}=\left(\frac{12}{3}\right)^2$$

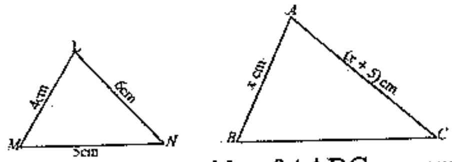
$$\frac{A}{6}=(4)^2$$

$$A=16\times6$$

$A=96cm^2$

∴the area of the trapezium with height 12cm is 96cm^2.

4. **[2005 PII #6a]**



Let the shortest side of ΔABC = x cm

Then the longest side = $(x + 5)$ cm

Since ΔLMN /// ΔABC

$$\Rightarrow \frac{LM}{AB}=\frac{MN}{BC}=\frac{LN}{AC}$$

$$\frac{4}{x}=\frac{6}{x+5}$$

$4(x + 5) = 6x$ *(cross-multiplying)*

$4x + 20 = 6x$

$4x - 6x = -20$

$$\frac{-2x}{-2}=\frac{-20}{-2}$$

$x = 10$

∴AB = 10cm

Now,

$$\frac{area\ of\ \triangle LMN}{area\ of\ \triangle ABC}=\left(\frac{LM}{AB}\right)^2\ (area\ factor)$$

$$=\left(\frac{4}{10}\right)^2$$

$$=\frac{16}{100}=\frac{4}{25}$$

∴ΔLMN:ΔABC = 4:25

∴the ratio of the areas of the two triangles is 4:25.

5. **[2006 P1 #17**

$\angle A = \angle D$ ($\angle s$ in same seg)

$\angle B = \angle C$ ($\angle s$ in same seg)

$\angle AHB = \angle DHC$ (vert. opp. $\angle s$)

$\therefore \Delta ABH\ ///\ \Delta DCH\ (AAA)$

$$So, \frac{AB}{DC}=\frac{BH}{CH}=\frac{AH}{DH}$$

Area factor=(scale factor)^2

$=\left(\dfrac{AH}{DH}\right)^2$

$=\left(\dfrac{x}{2x}\right)^2$

$=\left(\dfrac{1}{2}\right)^2$

$=\dfrac{1}{4}$

$\therefore$ the ratio of areas of ΔsABH:DCH=1:4.

6. **[2006 PII #7a]**

Area of big$\Delta = 64cm^2$

$\frac{1}{2}b \times H = 64cm^2$

$H = \dfrac{\cancel{64} \times 2}{\cancel{16}}$

$H = 4 \times 2$

$H = 8cm$

To find height of smaller Δ

Area factor=(scale factor)^2

$\left(\dfrac{36}{64}\right) = \left(\dfrac{h}{8}\right)^2$

$\sqrt{\dfrac{36}{64}} = \dfrac{h}{8}$

$\sqrt{\dfrac{9}{16}} = \dfrac{h}{8}$

$\dfrac{3 \times 8}{4} = h$

$h = 6cm$

$\therefore$corresponding height of smaller Δ is 6cm.

7. **[2007 PI #23]**

Given $\Delta ABC /// \Delta DBA$

Area factor = (scale factor)^2

$\dfrac{\Delta ABC}{\Delta DBA} = \left(\dfrac{AB}{DB}\right)^2$

$\dfrac{\Delta ABC}{24cm^2} = \left(\dfrac{8}{4}\right)^2$

Area $\Delta ABC = 24cm^2 \times (2)^2$

$= 24 \times 4$

$= 96cm^2$

$\therefore$ the area ΔABC is 96cm^2

8. **[2007 PII #2b]**

The shortest side of ΔXYZ corresponds to the shortest side of ΔABC which is 6cm (similar Δs)

Hence, *Area Factor=(Scale Factor)*^{2}

$\dfrac{\Delta ABC}{\Delta XYZ} = \left(\dfrac{\text{shortest side } \Delta ABC}{\text{shortest side } \Delta XYZ}\right)^2$

$\dfrac{\Delta ABC}{4.5} = \left(\dfrac{6}{2}\right)^2$

$\dfrac{\Delta ABC}{4.5} = (3)^2$

$\Delta ABC = 9 \times 4.5$

$\Delta ABC = 40.5cm^2$

9. **[2008 P2 #8a]**

Let height of ΔADE=h
Height of ΔABC=H
So, Area of ΔADE=12cm^2

$12cm^2 = \frac{1}{2}b \times h$

$12 = \frac{1}{\cancel{2}} \times DE \times \cancel{6}$

$\dfrac{\cancel{12}}{\cancel{3}} = \dfrac{\cancel{3}}{\cancel{3}}DE$

$DE = 4cm$

$\dfrac{BC}{DE} = \dfrac{H}{h}$

$\dfrac{BC}{4} = \dfrac{10}{6}$

$BC = \dfrac{10 \times \cancel{4}}{\cancel{6}}$

So, $BC = \dfrac{20}{3}$

$BC = 6.67cm$ (to 2dp)

10. **[2010 P1 #13]**

Given the triangles are simillar,

$scale\ factor = \Delta s\dfrac{TQU}{RSU}$

$= \dfrac{TU}{RU}$

$= \dfrac{3}{6}$

$= \dfrac{1}{2}$

$$\text{area factor} = (scale\ factor)^2$$
$$= \left(\frac{TU}{RU}\right)^2$$
$$= \left(\frac{3}{6}\right)^2$$
$$= \left(\frac{1}{2}\right)^2$$
$$= \frac{1}{4}$$

11. **[2010 PII #2a]**

$$\text{Area factor} = (scale\ factor)^2$$
$$\frac{\text{Area KLMN}}{\text{Area ABCD}} = \left(\frac{3}{2}\right)^2$$
$$\frac{\text{Area KLMN}}{72cm^2} = \frac{9}{4}$$
$$\text{Area KLMN} = \frac{\cancel{72}cm^2 \times 9}{\cancel{4}}$$
$$\text{Area of rectangle KLMN} = 162cm^2$$

12. **[2011 PI #5]**

Area Factor=(Scale Factor)^2

$$\frac{4}{9} = \left(\frac{r}{18}\right)^2$$
$$\sqrt{\frac{4}{9}} = \frac{r}{18}$$
$$\frac{2}{3} = \frac{r}{18}$$
$$\frac{2 \times \cancel{18}}{\cancel{3}} = r$$
$$r = 2 \times 6$$
$$r = 12cm$$

$\therefore$ the radius of the smaller circle is $12cm$

13. **[2011 PII #7a]**

$\angle APQ = \angle ABC$ (corr. $\angle$s PQ//BC)

$\angle AQP = \angle ACB$ (corr. $\angle$s PQ//BC)

$\angle PAQ = \angle BAC$ (same $\angle$)

$\therefore \Delta PAQ\ /// \ \Delta BAC\ (AAA)$

$$\therefore \frac{PQ}{BC} = \frac{AP}{AB} = \frac{AQ}{AC}$$

So, Area factor=(scale factor)^2

$$\frac{Area\ \Delta APQ}{Area\ \Delta ABC} = \left(\frac{PQ}{BC}\right)^2$$
$$\frac{\Delta APQ}{\Delta APQ + Trap\ PQCB} = \left(\frac{PQ}{BC}\right)^2$$
$$\frac{x}{x+10} = \left(\frac{2}{3}\right)^2$$
$$\frac{x}{x+10} = \frac{4}{9}$$

$9x = 4(10+x)$ (cross-multiplying)

$$9x - 4x = 40$$
$$\frac{\cancel{5}}{\cancel{5}}x = \frac{4\cancel{0}}{\cancel{5}}$$
$$\therefore x = 8$$

14. **[2012 P1 #8]**

Since two triangles are similar:

Area factor=(scale factor)^2

$$\frac{243}{75} = \left(\frac{b}{30}\right)^2$$
$$\frac{243}{75} = \frac{b^2}{900}$$
$$b^2 = \frac{243 \times 900}{75}$$
$$b^2 = 2916$$
$$b = \sqrt{2916}$$
$$b = 54$$

$\therefore$ base of bigger triangle=54cm

15. **[2013 P1 #18]**

In Δs $\frac{ABC}{ADE}$

$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE}$$

Area Factor=(Scale factor)^2

$$\left(\frac{BC}{DE}\right)^2 = \frac{2}{5}$$
$$\frac{6^2}{x^2} = \frac{2}{5}$$
$$\frac{36}{x^2} = \frac{2}{5}$$

$$36 \times 5 = 2x^2$$
$$2x^2 = 180$$
$$x^2 = 90$$
$$x = \sqrt{90}$$
$$x = 9.5 \text{(to 1 decimal place)}$$

16. **[2014 PII #5b]**

$\Delta s \dfrac{ABC}{AQS}$ *are* similar

Thus, $\dfrac{AB}{AQ} = \dfrac{BC}{QS} = \dfrac{AC}{AS}$

and, area factor=(scale factor)^2

Ratio of the areas for Δs:

$$\frac{\Delta ABC}{\Delta AQS} = \left(\frac{AB}{AQ}\right)^2$$
$$= \left(\frac{6}{6+8}\right)^2$$
$$= \left(\frac{6}{14}\right)^2$$
$$= \left(\frac{3}{7}\right)^2$$
$$= \frac{9}{49}$$

Ratio of the quad BCSQ

=ΔAQS-ΔABC

=49-9

=40

$\therefore$ The ratio of area ΔABC to area quad BCSQ is 9:40

17. **[2015 PII #4a]**

Since $\Delta RST /// \Delta VSU$, then

Area factor=(scale factor)^2

$$\frac{\Delta VSU}{\Delta RST} = \left(\frac{SU}{ST}\right)^2$$
$$\frac{10}{90} = \left(\frac{SU}{5+7}\right)^2$$
$$\sqrt{\frac{10}{90}} = \frac{SU}{12}$$
$$\sqrt{\frac{1}{9}} = \frac{SU}{12}$$
$$\frac{1}{3} = \frac{SU}{12}$$
$$\frac{\cancel{12}}{\cancel{3}} = SU$$
$$4 = SU$$
$$\therefore SU = 4cm$$

18. **[2016 PII #4a]**

Let Area of large cylinder be A

Area factor=(scale factor)^2

$$\frac{A}{77} = \left(\frac{2r}{r}\right)^2$$
$$\frac{A}{77} = \left(\frac{2}{1}\right)^2$$
$$\frac{A}{77} = 4$$
$$A = 4 \times 77$$
$$= 308cm^2$$

$\therefore$ The area of the cylinder is $308cm^2$

19. **[2018 P1 #15]**

ABCD is similar to WXYZ

Area factor=(scale factor)^2

Area factor=(ratio of widths)^2

$$\frac{\text{ABCD}}{\text{WXYZ}} = \left(\frac{12}{6}\right)^2$$

$\therefore$ Area of rectange WXYZ =Length $\times$ Width

$= 10\text{cm} \times 6\text{cm}$

$=60\text{cm}^2$

$$So,\ \frac{\text{ABCD}}{60} = (2)^2$$
$$\cancel{60} \times \frac{\text{ABCD}}{\cancel{60}} = 4 \times 60$$

Area of ABCD = 240 cm^2

20. **[2018 PII #9]**

ABC and PQR are similar

$$\text{Area factor} = \frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta ABC}$$

$$= \frac{1}{4}$$

$$\text{Scale factor} = \sqrt{\text{Area factor}}$$

$$= \sqrt{\frac{1}{4}}$$

$$= \frac{1}{2}$$

The ratio of the sides of ΔPQR:ΔABC is 1:2

21. **[2019 PII #3b]**

$$\text{Scale factor} = \frac{6cm}{h}$$

$$\text{Volume factor} = \frac{16cm^3}{54cm^3}$$

$$= \frac{8}{27}$$

$$\text{Scale factor} = \sqrt[3]{\text{Volume factor}}$$

$$\frac{6cm}{h} = \sqrt[3]{\frac{8}{27}}$$

$$\therefore \frac{6cm}{h} = \frac{2}{3} \quad \left(\begin{array}{c}\text{cross multiply and} \\ \text{make } h \text{ subject}\end{array}\right)$$

$$h = \frac{3 \times 6cm}{2}$$

$$h = 9cm$$

22. **[2020 Mock PII 4a]**

$$\left(\frac{A_2}{A_1}\right) = \left(\frac{L_2}{L_1}\right)^2$$

$$\left(\frac{A_2}{13.5}\right) = \left(\frac{6}{4.5}\right)^2$$

$$A_2 = 13.5 \times \left(\frac{4}{3}\right)^2$$

$$A_2 = 13.5 \times \frac{16}{9}$$

$$A_2 = 24m^2$$

$\therefore$ uncovered area $= A_2 - A_1$

$$= 24 - 13.5$$

$$= 10.5m^2$$

23. **[2021 Mock PI #18]**

Area factor=(scale factor)^2

$$\left(\frac{6}{24}\right) = \left(\frac{4}{2b}\right)^2$$

$$\frac{1}{4} = \left(\frac{2}{b}\right)^2$$

$$\frac{1}{4} = \frac{2^2}{b^2}$$

$$\frac{1}{4} = \frac{4}{b^2}$$

$b^2 = 16$ (cross-multiplying)

$$b = \pm\sqrt{16}$$

$$b = 4 \text{ or } -4$$

Hence, base of bigger triangle is 4cm.

To find height:

$$\text{Area} = \frac{1}{2} base \times height$$

$$A = \frac{1}{2} bh$$

$$24 = \frac{1}{2} \times 4 \times h$$

$$24 = 2h$$

$$\frac{24}{2} = h$$

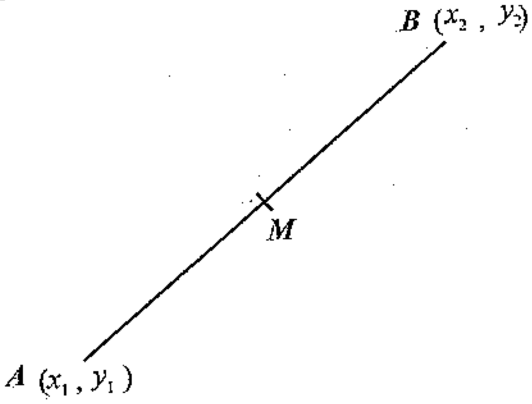
$$h = 12$$

$\therefore$ height of bigger triangle $= 12cm$

CH 14
COORDINATE GEOMETRY

Chapter Objectives

In this chapter, we will practice problem involving coordinate geometry. Before attempting the questions, you should know how to:



- Calculate the gradient of a straight-line
  m (gradient) $= \dfrac{y_2 - y_1}{x_2 - x_1}$
- Finding and generating the equation of a straight line
  $y = mx + c$
- Calculate the distance between two point
  Distance $= \sqrt{(x_2 - x_2) + (y_2 - y_1)}$
- The midpoint between (x_1, y_1) and (x_2, y_2) denoted M is given as:
  $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
- Note: two lines are parallel when they have the same gradient.
- On the $x-$axis, $y = 0$ and on the $y-$axis, $x = 0$. So given the equation $y = mx + c$, to find the $y-$intercept, let $x = 0$, and to find the $x-$intercept, let $y = 0$.

1. The line joining the points $A(3,\ q)$, $B(5-q, 8)$ has a gradient of $1/2$. Calculate the value of q.
[2003 P1 #12]

2. Suppose $y = (a+1)x + 5$ and $y + 2x = 0$ are two parallel straight lines, calculate the value of a.
[2003 PII #10a]

3. A straight line passes through the point (1, 6) and cuts the y-axis at 4. Calculate its gradient.
[2004 PI #2]

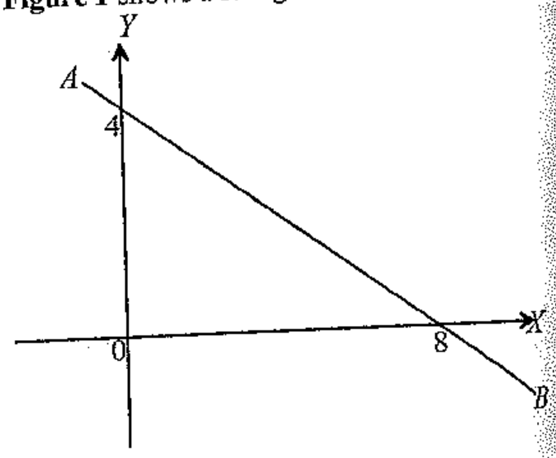
4. The distance between points M(3,5) and N(8, a) is 13. Find the values of a. **[2004 PII #1b]**

5. Find the equation of a straight line passing through the point (0,7) and parallel to the line of the equation y=2x + 5. **[2005 PI #12]**

6. A straight line passes through the points $(-1, -2)$ and $(3,\ 4)$. Find the equation of the straight line in form of y=mx + c. **[2005 PII #10a]**

7. Two lines **G** and **H** intersect at a point P. G passes through the points (-4, 0) and (0, 6). Given that H has the equation: $y = 4x - 4$. Find by calculation the coordinates of **P**. **[2006 PI #1**

8. **Figure 1** shows a straight-line graph.

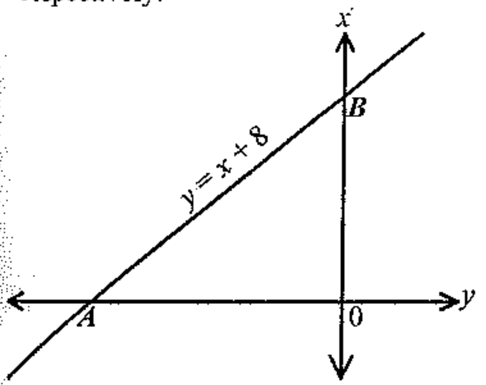


Find the equation of line **AB** and write it in the form of y=mx + c. **[2006 PII #**

9. The gradient of a straight line passing through point P (-2, 5) is $-\frac{1}{2}$. Find the equation of the line in the form of y= mx + c. **[2007 PI #**

10. Calculate the distance between the two points (-3, 1) and B (3, 9). **[2007 PII #**

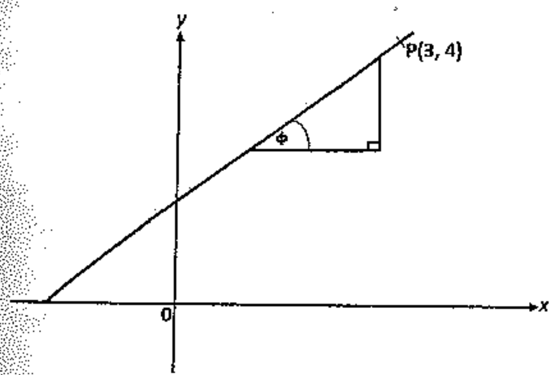
11. **Figure 2** shows a straight-line graph $y = x + 8$ crossing the x and y axes at A and B respectively.



Calculate the distance between points **A** and **B**, leaving your answer corrects to 2 decimal places. **[2008 P2 #9a]**

12. Given that $y = 2x - 3$ and $y = (b-1)x + 5$ are equations for graphs of two parallel straight lines. Calculate the value of b. **[2008 P1 #8]**

13. **Figure 3** Shows a straight line passing through a point P(3, 4).



Given that tan $\theta = \frac{2}{3}$. Find the equation of the line in the form $y = mx + c$. **[2010 P1 #9]**

14. Calculate the distance between x and y intercepts of the graph of $y = 12 - 2.4x$. **[2010 PII #4b]**

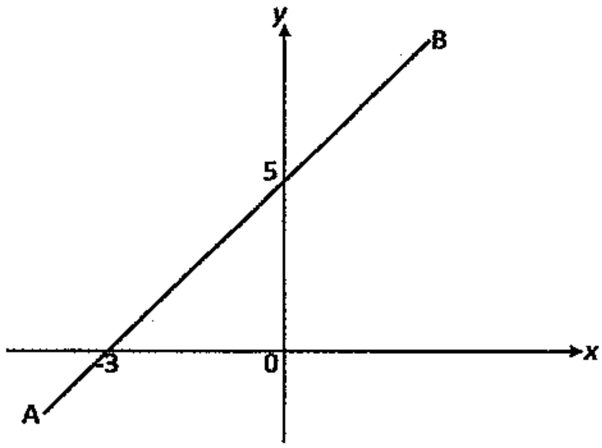
15. A straight line which passes through (3t, 7) and (t, -5) has gradient 3. Find the equation of the line. **[2011 PI #13]**

16. A line parallel to $2y = 3x - 4$ passes through y-axis at 3. Find it equation in form $y = mx + C$. **[2011 PII #9a]**

17. Find the gradient of a straight line whose equations is $y - 1 = mx$ and passes through $(-2, -5)$. **[2012 P1 #7]**

18. Find the distance between L (-2, 8) and P (4, -1), giving your answer correct to 2 decimal places. **[2012 P2 #4a]**

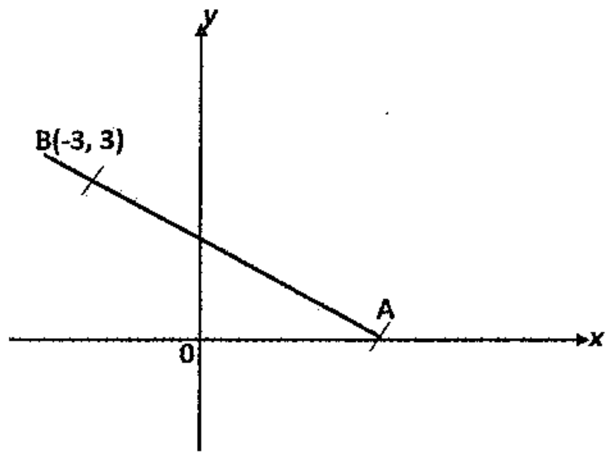
19. **Figure 4** shows a sketch of a straight-line graph AB.



Find the equation of the line AB in the form: $y = mx + c$. **[2013 P1 #2]**

20. Given that the gradient of a line passing through $A(3, 2a)$ and $B(4, -a)$ is 6, find the y coordinate for point B. **[2013 P2 #1b]**

21. **Figure 5** shows a graph of straight line **AB** whose gradient is $-\frac{1}{2}$. The coordinates of point **B** are (-3,3).



Find the coordinates of point A. **[2014 P1 #11]**

22. The distance between $M(4, y)$ and $N(10, -5)$ is 10 units. Find the negative value of y.
[2014 PII #9a]

23. Find the y-intercept of the line through the point $(1, 2)$ parallel to the line $3x + 4y = 7$.
[2015 PI #11]

24. The equation of line passing through a point $A(x, 2)$ is $y - 3x = -1$. Find the value of x.
[2015 PII #8a]

25. A straight line whose equation is $y = -3x + \frac{c}{2}$ passes through $x-axis$ at 4. Calculate the value of c.
[2016 PI #16]

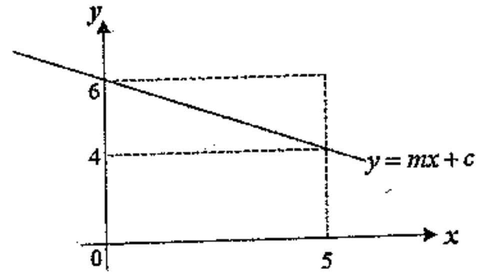
26. Straight lines whose equations are $3y - 3x = 4y + 2x$ and $y = (a-2)x$ are parallel. Calculate the value of a.
[2016 PII #9a]

27. A straight line passes through the points $P(1, 3)$ and $Q(2, 1)$. Calculate the x-intercept of the line.
[2017 PI #5]

28. Find the equation, in the form $y = mx + c$, of the straight-line **PQ** which crosses x-axis y at -5 and x-axis at 6.
[2018 PII #2]

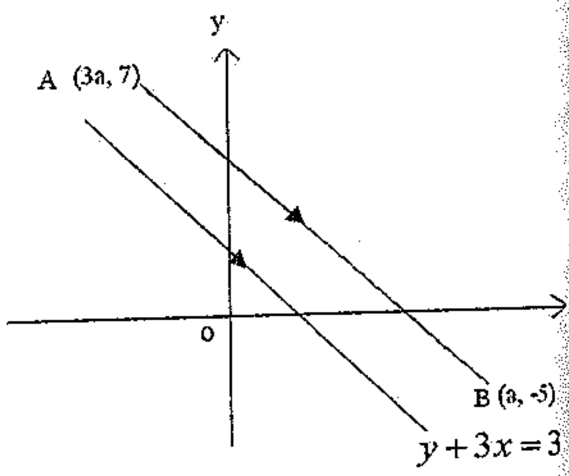
29. The angle between **AB** and the x-axis is 45°. Given that the coordinates of **A** and **B** are (2, 0) and (x, 4) respectively, calculate the value of x.
[2019 PI #6]

30. **Figure 6** shows a straight line $y = mx + c$ on a Cartesian plane



Find the values of m and c. **[2020 Mock PI #4]**

31. **Figure 7** shows a straight-line **AB** passing through the points $A(3a, 7)$ an $B(a, -5)$ and a parallel line $y + 3x = 3$.



Find the equation of the straight-line **AB** in the form of $y = mx + c$. **[2021 Mock PII 3b]**

1. **[2003 P1 #12]**

$$Gradient = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{8-q}{(5-q)-3}$$
$$\frac{1}{2} = \frac{8-q}{5-q-3}$$
$$\frac{1}{2} = \frac{8-q}{2-q}$$
$2-q = 2(8-q)$ *cross multiplying*
$2-q = 16-2q$
$-q+2q = 16-2$
$q = 14$
$\therefore$ the value of q is 14

2. **[2003 PII #10a]**

For $y = (a+1)x+5$
Gradient $= a+1$
For $y = -2x$ *(change subject)*
Gradient $= -2$
Since gradients of //str. lines are equal.
i.e. $a+1 = -2$
$a = -2-1$
$a = -3$
$\therefore$the value of a is -3.

3. **[2004 PI #2]**

$$Gradient = \frac{change\,in\,y}{change\,in\,x}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$
Let point (1,6) be (x_1, y_1)
Where a line cuts y-axis is x=0 so point is (0, 4)
Let (0, 4) be (x_2, y_2)
Then, gradient $= \frac{4-6}{0-1}$
$= \frac{-2}{-1}$
Gradient = 2

OR
Taking $y = mx + c$
$y-$intercept is 4
So, $y = mx + 4$
Taking point $(1,6)$ and substituting,
$6 = m(1) + 4$
$6 - 4 = m$
$m = 2$
$\therefore$ gradient =2

4. **[2004 PII #1b]**

distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

let M(3, 5) be (x_1, y_1)
let N(8, a) be (x_2, y_2)
so $\sqrt{(8-3)^2 + (a-5)^2} = 13$ *(given)*
then $(8-3)^2 + (a-5)^2 = 13^2$ *(squaring both sides)*
$5^2 + (a-5)^2 = 169$
$(a-5)^2 = 169 - 25$
$(a-5)^2 = 144$
$a - 5 = \pm\sqrt{144}$
$a - 5 = \pm 12$
Thus $a - 5 = 12$ or $a - 5 = -12$
$a = 12 + 5$ $\quad$ $a = -12 + 5$
$a = 17$ or $a = -7$

5. **[2005 PI #12]**

Form: $y = mx + c$
Line parallel to $y = 2x + 5$
$\therefore m = 2$ (parallel lines have equal to the slope)
So new line: $y = 2x + c$
At point (0, 7):
7 = 2(0) + c
7 = c
$\therefore$ Equation of the line is $y = 2x + 7$.

6. **[2005 PII #10a]**

Let $(x_1, y_1) = (-1, -2)$
$(x_2, y_2) = (3, 4)$
Gradient $= m = \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{4-(-2)}{3-(-1)}$

$=\frac{6}{4}$

$=\frac{3}{2}$

$y=\frac{3}{2}x+c$

$y=1.5x+c$

At $(-1,-2)$;

$-2=1.5\times(-1)+c$

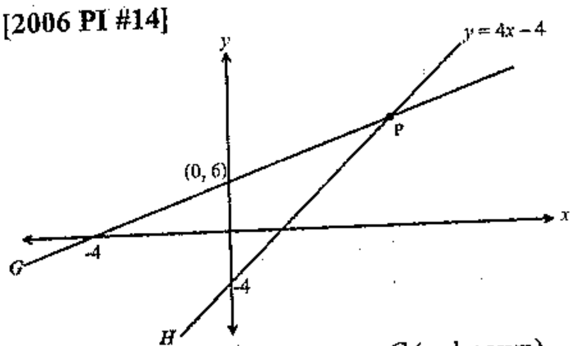
$-2=-1.5+c$

$-2+1.5=c$

$-0.5=c$

So, $y=1.5x-0.5$ or $y=\frac{3}{2}x-\frac{1}{2}$

7. **[2006 PI #14]**



Take the equation G: $y = mx + C$ (unknown)

At (0, 6) is y-intercept because $x = 0$

So $y = mx + 6$

Taking (-4, 0)

$0 = -4m + 6$

$\frac{-6}{-4}=\frac{-4m}{-4}$

$m=\frac{3}{2}$

So G; $y=\frac{3}{2}x+6$ (i)

We solve for intersection point with H

H: $y = 4x - 4$ (ii)

Substituting in (i) or equating the y's

$4x-4=\frac{3}{2}x+6$

$4x-\frac{3}{2}x=6+4$

$4x-1.5x=6+4$

$\frac{2.5}{2.5}x=\frac{10}{2.5}$

$x=4$

To solve for the $y-coordinate$:

$y=1.5(4)+6$

$=6+6$

$=12$

Therefore, point P is (4, 12).

8. **[2006 PII #2b]**

From the figure, $y-$intercept is $(0,4)$:

So, taking the form $y=mx+c$:

$y=mx+4$ (i)

From the fig, the other point is $(8,0)$:

Substituting in (i),

$0=m(8)+4$

$\frac{-4}{8}=\frac{8}{8}m$

$-\frac{1}{2}=m$

Hence, the equation of line AB is: $y=-\frac{1}{2}x+4$

9. **[2007 PI #7]**

point $p(-2,5)$ and gradient$=-\frac{1}{2}$,

Then using $y=mx+c$, one gets

$y=-\frac{1}{2}x+c$

at point $p(-2,5), x=-2$ and $y=5$

$\Rightarrow 5=-\frac{1}{2}(-2)+c$

$5=1+c$

$5-1=c$

$c=4$

Hence the equation is: $y=-\frac{1}{2}x+4$

10. **[2007 PII #3b]**

Taking $A(x_1,y_1)$ and $B(x_2,y_2)$,

distance between A and B,

$|AB|=\sqrt{(x_2-x_1)^2-(y_2-y_1)^2}$ (distance formula)

now: $|AB|=\sqrt{(3--3)^2+(9-1)^2}$

$AB=\sqrt{(3+3)^2+(9-1)^2}$

$AB=\sqrt{6^2+8^2}$

$AB=\sqrt{36+64}$

$AB=\sqrt{100}$

$AB=10$

$\therefore$ the distance btn points A and B is 10 units.

11. **[2008 P2 #9a]**

$y=x+8$

from the equation, y-intercept $=8$

At $y=8, x=0$

$\therefore B=(0,8)$

At point A, $y = 0$

substitute 0 for y in the equation to find x :

$0 = x + 8 \Rightarrow x = -8$

$\therefore A = (-8, 0)$

$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2}$

$|AB| = \sqrt{(0 - -8)^2 + (8 - 0)^2}$

$= \sqrt{8^2 + 8^2}$

$= \sqrt{64 + 64}$

$= \sqrt{128}$

$= 11.3137$

$\simeq 11.31$ (to 2 decimal places)

$\therefore$ The distance between A and B is 11.31 units

12. **[2008 P1 #8]**

Given, $y = 2x - 3$ (i)

$y = (b-1)x + 5$ (ii)

$\therefore b - 1 = 2$ (gradients of // lines are equal)

$b = 2 + 1$

$b = 3$

$\therefore$ the value of b is 3

13. **[2010 P1 #9]**

Gradient = $Tan\theta$ (by definition of Tan θ)

$m = \frac{2}{3}$

$y = mx + c$

$y = \frac{2}{3}x + c$

At point $P(3, 4)$, substitute in the eq.

$4 = \frac{2}{3}(3) + c$

$4 = 2 + c$

$c = 4 - 2$

$c = 2$

$\therefore$ The equation of the str. line is $y = \frac{2}{3}x + 2$

14. **[2010 PII #4b]**

$y = 12 - 2.4x \Rightarrow y = -2.4x + 12$

y – intercept = $(0,12)$

x – intercept occurs when $y = 0$

$0 = 12 - 2.4x$

$2.4x = 12$

$\frac{2.4x}{2.4} = \frac{12}{2.4}$

$x = 5$

Let the two points be $A(0,12)$ and $B(5,0)$

$|AB| = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

$= \sqrt{(0-12)^2 + (5-0)^2}$

$= \sqrt{(-12)^2 + 5^2}$

$= \sqrt{144 + 25}$

$= \sqrt{169}$

$= 13$

Distance btn x – and y – intercepts = 13 units

15. **[2011 PI #13]**

Given coordinates on straight line

$(3t, 7)$ and $(t, -5)$ and $y = 3x + c$

$Gradient = \frac{\text{change in } y}{\text{change in } x}$

$3 = \frac{7 - -5}{3t - t}$

$3 = \frac{7 + 5}{2t}$

$3(2t) = 12$

$\frac{\cancel{6}t}{\cancel{6}} = \frac{\cancel{12}}{\cancel{6}}$

$t = 2$

> Taking point $(2, -5)$
> Substitute in $y = 3x + c$
> $-5 = 3(2) + c$
> $-5 = 6 + c$
> $-5 - 6 = c$
> $-11 = c$
> $\therefore$ equation is: $y = 3x - 11$

Alternative

Take $y = 3x + c$

At $(3t, 7)$, $7 = 3(3t) + c$

$7 = 9t + c$ (i)

At $(t, -5)$, $-5 = 3(t) + 3$

$-5 = 3t + c$ (ii)

Taking (i) and (ii)

$9t + c = 7$ (i)

$\underline{3t + c = -5}$ (ii)

$\frac{6t}{6}=\frac{12}{6}$

$t=2$

taking (i)

$7=9(2)+c$

$7=18+c$

$7-18=c$

$-11=c$

so $y=3x+c$ becoming

$y=3x-11$

$\therefore$ equation is: $y=3x-11$

16. **[2011 PII #9a]**

y – intercept $=3$

So $y=mx+3$

To get m:

Taking the given eqtn,

$2y=3x-4$

$y=\frac{3}{2}x-4$

$m=\frac{3}{2}$ (//lines have same gradient)

Hence, eqtn is: $y=\frac{3}{2}x+3$

17. **[2012 P1 #7]**

Given $y-1=mx$

$y=mx+1$

Since the line passes through (-2,-5)

$-5=m(-2)+1$

$-5=-2m+1$

$-5-1=-2m$

$\frac{\cancel{-6}}{\cancel{-2}}=\frac{\cancel{-2}}{\cancel{-2}}m$

$m=3$

$\therefore$ gradient $=3$

18. **[2012 P2 #4a]**

Let L(-2, 8) be (x_1,y_1)

Let P(4, -1) be (x_2,y_2)

Distance between two points

$=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$|LP|=\sqrt{(-2-4)^2+(8--1)^2}$

$=\sqrt{36+81}$

$=\sqrt{117}$

$=10.8166$

≈ 10.82 (to 2dp.)

Distance between L and P is 10.82 units

19. **[2013 P1 #2]**

From the fig, y – intercept $=5$

So, $y=mx+5$

At $(-3,0)$,

$0=m(-3)+5$

$\frac{-5}{-3}=\frac{-3}{-3}m$

$\frac{5}{3}=m$

So the equation is: $y=\frac{5}{3}x+5$

20. **[2013 P2 #1b]**

Let $A(3,2a)$ be (x_1,y_1)

Let $B(4,-a)$ be (x_2,y_2)

$\frac{y_2-y_1}{x_2-x_1}=Gradient$

$\frac{-a-2a}{4-3}=6$

$\frac{-3a}{1}=6$

$\frac{-3a}{-3}=\frac{6}{-3}$

$a=-2$

Hence, the y – coordinate of point B

is $-a=-(-2)=2$

21. **[2014 P1 #11]**

Take $y=mx+c$

then, given $m=-\frac{1}{2}$

$y=-\frac{1}{2}x+c$ at $(-3,3)$

$3=-\frac{1}{2}(-3)+c$

$3=\frac{3}{2}+c$

$3-\frac{3}{2}=c$

$3-1.5=c$

$1.5=c$

$y=-\frac{1}{2}x+\frac{3}{2}$

At A, $y=0$

so, $0=-\frac{1}{2}x+\frac{3}{2}$

$-\frac{3}{2}=-\frac{1}{2}x$

$3=x$

$\therefore$ Point A is $(3,0)$

22. **[2014 PII #9a]**

The distance between M(4,y) and $N(10,-5)$ is 10

Distance $=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$10=\sqrt{(y--5)^2+(4-10)^2}$

$(10)^2=\left(\sqrt{(y+5)^2+(-6)^2}\right)^2$

$100=(y+5)^2+(-6)^2$

$(y+5)^2=100-36$

$y+5=\pm\sqrt{64}$

$y+5=\pm8$

$y=-5\pm8$

$y=-5+8$ or $y=-5-8$

$y=3$ or $-y=-13$

$\therefore y=3$ or $y=-13$

The negative value of y is -13

23. **[2015 PI #11]**

express lines as $y=mx+c$

Line 1: $3x+4y=7$

$4y=-3x+7$

$y=-\frac{3}{4}x+\frac{7}{4}$

Line 2: $y=-\frac{3}{4}x+c$ (//lines have equal gradient)

At (1,2),

$2=-\frac{3}{4}(1)+c$

$2+\frac{3}{4}=c$

$2\frac{3}{4}=c$

$\frac{11}{4}=c$

Hence, y-intercept is $\frac{11}{4}$.

24. **[2015 PII #8a]**

We solve for x where $y=2$.

$y-3x=-1$

At $A(x,2)$, $y=2$

$2-3x=-1$

$2+1=3x$

$\frac{3x}{3}=\frac{3}{3}$

$x=1$

25. **[2016 PI #16]**

$y=mx+c$

$y=-3x+\frac{c}{2}$

At $x-$axis 4, $y=0$ thus,

$0=-3(4)+\frac{c}{2}$

$0=-12+\frac{c}{2}$

$12=\frac{c}{2}$

$12\times2=c$

$\therefore c=24$

26. **[2016 PII #9a]**

Take $3y-3x=4y+2x$

Express as $y=mx+c$

$3y-4y=3x+2x$

$-y=5x$

$y=-5x+0$..............(i)

Take $y=(a-2)x$............(ii)

$\therefore a-2=-5$ (//lines have equal gradient)

$a=-5+2$

$a=-3$

27. **[2017 PI #5]**

$P(1,3)$ and $Q(2,1)$

Lets find the equation in the form $y=mx+c$

$m=\frac{y_2-y_1}{x_2-x_1}$

$m=\frac{1-3}{2-1}$

$m=-2$

So, $y=-2x+c$

Taking $P(1,3)$,

$3=-2(1)+c$

$3+2=c$

$\therefore c=5$

The equation of the line PQ is

$y=-2x+5$

At $x-$intercept, $y=0$

$0=-2x+5$

$2x=5$

$x = 2.5$

Hence $x-$intercept of the line is 2.5

28. **[2018 PII #2]**

coordinates at $x-$intercept $(6,0)$

coordinate at $y-$intercept: $(0,-5) \Rightarrow c = -5$

Thus gradient of P : $m = \dfrac{y_2 - y_1}{x_2 - x_2}$

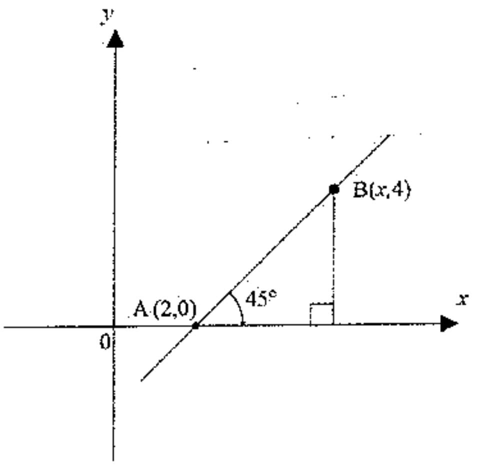
$\Rightarrow m = \dfrac{-5-0}{0-6} \Rightarrow m = \dfrac{-5}{-6}$

$\therefore m = \dfrac{5}{6}$

$y = mx + c$

$\therefore y = \dfrac{5}{6}x - 5$

29. **[2019 PI #6]**



The coordinates are $A(2,0)$ and $B(x,4)$

Gradient of the line, $m = \text{Tan } 45°$

$\therefore m = 1$

But $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$1 = \dfrac{4-0}{x-2}$

$x - 2 = 4$

$x = 4 + 2$

$x = 6$

30. **[2020 Mock PI #4]**

From the figure, there are two points,
the y-intercept (0,6) and a point (5,4)
So, $y = mx + 6$

Taking point (5, 4)

$4 = m(5) + 6$

$4 - 6 = 5m$

$-2 = 5m$

$\dfrac{-2}{5} = m$

Hence, $c = 6$ and $m = -\frac{2}{5}$.

31. **[2021 Mock PII #3b]**

Let Line AB be $y = mx + c$:

Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{7 - -5}{3a - a}$

$= \dfrac{12}{2a}$

$= \dfrac{6}{a}$

//line: $y = -3x + 3$

Slope of AB= -3 (//lines have equal slope)

$\dfrac{6}{a} = -3$

$6 = -3a$

$\dfrac{\cancel{6}}{\cancel{-3}} = a$

$a = -2$

Hence, point B is $(-2,-5)$

Letting AB be $y = -3x + c$ and using point $(-2,-5)$,

$-5 = -3(-2) + c$

$-5 = +6 + c$

$-5 - 6 = c$

$c = 11$

$\therefore y = -3x + 11$

CH 15 VARIATIONS

Chapter Highlights

In this chapter, we will solve problems on variations. Variations define a concept where one variable changes as a result of changes in other variables. The problems discussed here involve four types of variations:

1. Direct variation
   A relation whereas one variable (x) increases or decreases, the other variable e.g. y also increases or decreases at a constant rate.
   i.e. $y \propto x$ (read as "y varies directly as x") leads to the equation $y = kx$ where k is a constant.
2. Inverse Variation
   Given two variables e.g. r and q, we say r varies inversely as q when:
   - As q increases, r decreases. $p \propto \frac{1}{r}$
   - As q decreases, r increases.
3. Joint Variation
   Joint Variation occurs when one variable varies direct as with two other variables (The variable can also vary both directly and inversely to two or more variables). i.e.
   $y \propto xq$ means y varies directly as a product of x and q.
4. Partial Variation
   This type of variations consists of two or more parts added together. The various usually results in equations with two variables. We therefore urge you to have knowledge on solving simultaneous linear equations using either elimination or substitution method.

Direct Variation

1. Given that $q \, \alpha \sqrt{p}$ and $p = 4$ when $q = 3$, find the value of p when $q = 15$. **[2008 P1 #21]**

Joint Variation

2. Given that $x \propto \frac{y}{z}$. When $x = 10$, $y = 2$ and $z = 4$. Find value of x when $y = 1$ and $z = 5$. **[2006 P1 #13]**

3. A quantity P varies directly as q and inversely as q^2+1. When $P=1$, $q=2$. Express P in terms of q only. **[2006 PII #11a]**

4. P varies directly as x^3 and inversely as y. When $x=2$ and $y=4$, $P=3$. Find the value of x when $P=48$ and $y=2$. **[2007 PI #12]**

5. Given that x varies jointly as y and inversely as the square of z. Calculate the missing value in **Table 1** below given that it is positive.

x	y	z
3	1	2
1	3	

[2003 PP1 13]

6. x varies directly as y and inversely as the square of n. If $x = 15$, $y = 24$ and $n = 4$, calculate the values of n when $x = 8$ and $y = 20$. **[2008 PII #12a]**

7. A quantity b varies jointly with r and t, and $b=108$ when $r=3$ and $t=6$. Find an equation which expresses b in terms of r and t. **[2004 PI #4]**

8. Given that P varies as a product of q and r^2, and that $P=50$ when $q=1$ and $r=5$, find P when $q=3$ and $r=8$. **[2005 P1 #13]**

9. Given that $V \propto rd$ and $V = 54$ when $r = 2$ and $d = 3$. Find V when r = $\frac{1}{2}$ and $d = 6$. **[2010 P1 #20]**

10. p varies directly as r and inversely as the square root of q. Given that $p = 4$ when $q = 9$ and $r = 1$, calculate q when $r = 2$ and $p = 6$. **[2011 PI #14]**

11. Given that $x \; \alpha \; \frac{y}{z^2}$ and $x = 12$ when $y = 2$ and $z = 1$. Find y when $x = 15$ and $z = 2$. **[2012 PI #19]**

12. The quantity p varies directly as the square of q and inversely as r. When $p = 18$, $q = 6$ and $r = 30$. Find r when $p = 27$ and $q = 12$. **[2013 P1 #17]**

13. A quantity w varies directly as the cube root of x and inversely as y. When $x = 27$ and $y = 2$, $w = 6$. Calculate the value of x when $w = 5$ and $y = 8$. **[2017 PI #13]**

14. A quantity x varies directly as the square of z and inversely as y. Given that $x=2$ when $y=2$ and $z=1$, find y when $x=5$ and $z=-2$. **[2015 PI #18]**

15. A quantity f varies directly with the square of g and with h. When $g=2$ and $h=9$ then $f=4$. Find f when $g=6$ and $h=20$.
[2014 P1 #17]

16. The cost (c) of hiring a minibus varies directly as the distance (d) travelled and the number of passengers (p). When the distance is 10km and the number of passengers is 8, the cost is k 10 000. Given that the cost of travelling 25 km is K46 875, calculate the number of passengers. **[2016 PI #14]**

17. A quantity p varies directly as the square root of r and inversely as t. When $p=2$ and $r=4$ and $t=3$. Calculate the value of t when $p=6$ and $r=16$ **[2018 P1 #9]**

Partial Variation

18. Given that y is partly constant and partly varies as x and $y=-3$ when $x=3$ and $y=22$ when $x=-2$, calculate the value of y when $x=2$. **[2003 PII 2a]**

19. A quantity q is the difference between two parts. The first part is constant, and second part varies inversely as the square of p. if $q=1$ then $p=2$ and $q=6$ when $p=3$. Find the positive value of p when $q=9$. **[2004 PII #7b]**

20. The cost (C) for an international call from Malawi to Europe, partly varies inversely as time, (t), and partly as the square of time.
A one-minute call costs K120 and a two-minute call costs K200. Find the cost of a five minute call.
[2005 PII #8a]

21. The bus fare per passenger (F) is partly constant and partly inversely proportional to the number (n) of passengers. The fare per passenger for 40 passengers is K240, and 50 passengers is K200. Calculate the fare per passenger when there are 100 passengers. **[2007 PII #7b]**

22. The quantity q is partly constant and partly varies as p^3. When $q=5$, $p=1$ and when $q=26$, $p=2$.
Find q when $p=-2$. **[2012 P2 #7b]**

23. The time, T, taken for students to finish discuss a topic in a group is partly constant and par varies as the number, N, of the students in t group. A group of 5 students takes 120 minu while a group of 9 students takes 180 minut Calculate the time a group of 12 students can ta to finish discussing the topic. **[2010 PII #1**

24. A quantity a is the sum of two parts, one of whi is constant while the other varies inversely as th square of n. When $n=1$, $a=-1$ and when $n=2$, $a=2$ Find the value of a when $n = 4$.
[2011 PII #1

25. p is the sum of two, one which varies directly a q and the other directly as q^2. When
when $q=4, p=52.8$ and when $q=5$
$p=81$, find p when $q=6$. **[2013 P2 #8**

26. The number, N of workers hired to construct a road is partly constant and partly varies with the length, L of the road. A 40 Km road requires 150 workers and a 54 Km road requires 192 workers. Calculate the number of workers of to be hired to complete a 73 Km road over the same period.
[2014 PII #7

27. The distance (d) covered by a moving object is partly constant and partly varies with the square of its velocity (v). When the velocity is 2m/hr, th distance covered is 15 m, and when the velocity i 3m/hr, the distance covered is 20 m. If the distance covered is 36 m, find the velocity of the object. **[2017 PII #11b]**

28. The cost (p) for hiring a taxi is partly constant an partly varies as the distance (d) travelled. A distance of 30 km costs K10,800 and that of 50 km costs K13,500. Calculate the cost for a distance of 40 km. **[2015 PII #9a]**

29. A variable y is partly constant and partly varies as x. When $x=2$, $y=16$ and when $x=7$, $y=31$.
Find y in terms of x. **[2016 PII #4b]**

30. Takondwa, Mphatso and Chikondi share bananas. The number of bananas (t) that Takondwa gets varies directly with the square root of the number of bananas (m) that Mphatso gets and inversely with the number of bananas (c) Chikondi gets. When Takondwa gets 3 bananas, Mphatso gets 4 bananas and Chikondi gets 8 bananas. Calculate the number of bananas that Chikondi gets when Takondwa and Mphatso gets 6 bananas and 25 bananas respectively. **[2019 PI #13]**

Direct Variation

1. **[2008 P1 #21]**

Given $q \propto \sqrt{p}$
We introduce the variation constant k:
$q = k\sqrt{p}$
$p = 4$ when $q = 3 \Rightarrow 3 = k\sqrt{4}$
$3 = k \times 2 \Rightarrow 2k = 3$
$\Rightarrow k = \frac{3}{2}$ (divide both sides by 2)
substitute $\frac{3}{2}$ for k in the equation:
$q = \frac{3}{2}\sqrt{p}$
now, need to find p when $q = 15$
$\Rightarrow 15 = \frac{3}{2}\sqrt{p}$
$\frac{15 \times 2}{3} = \sqrt{p}$ (multiply by $\frac{3}{2}$ both sides)
$\Rightarrow \sqrt{p} = 10$ (sq.root both sides)
$(\sqrt{p})^2 = 10^2 \Rightarrow p = 100$
$\therefore$ The value of p is 100

Joint Variation

2. **[2006 P1 #13]**

Introducing k as the constant of variation.
$x \propto \frac{y}{z}$ then $x = \frac{ky}{z}$
Substituting $x = 10, y = 2$ and $z = 4$ in $x = k\frac{y}{z}$ to find k
$10 = \frac{1}{2}k \times$ (simplifying)
$\Rightarrow 20 = k$ (cross multiplying)
$\therefore k = 20$
Substitute 20 for k in $x = k\frac{y}{z}$
$\Rightarrow x = 20\frac{y}{z}$
Now we need to find x when $y = 1$ and $z = 5$
$x = 20 \times \frac{1}{5} = 4$
$\therefore x = 4$

3. **[2006 PII #11a]**

$P \propto \frac{q}{(q^2+1)}$
$P = \frac{kq}{q^2+1}$ (introduce contant k)
Substituting P = 1 and $q = 2$ to find k.
$\Rightarrow 1 = \frac{k \times 2}{2^2+1}$
$1 = \frac{2k}{4+1}$
$1 = \frac{2k}{5}$
$5 = 2k$ (multiply 5 both sides)
$\frac{5}{2} = \frac{2k}{2}$ (divide by 2 both sides)
$\Rightarrow k = 2.5$
Substitute 2.5 for k in the equation to set
$P = \frac{2.5q}{q^2+1}$
$\therefore$ In terms of q only; $P = \frac{2.5q}{q^2+1}$

4. **[2007 PI #12]**

P varies directly as x^3 and inversely as y.
$\Rightarrow p \propto \frac{x^3}{y}$
Introducing k as the constant of variation;
$p = \frac{kx^3}{y}$
when $x = 2, y = 4, p = 3$
substitute in the equation to find k.
$3 = \frac{8k}{4}$
$3 = 2k$ (simplifying)
$\frac{3}{2} = k$ (divide by 2 both sides)
$k = \frac{3}{2}$.

Substitute k into equation.
$\therefore p = \frac{\frac{3}{2}x^3}{y}$

We now find x when $p = 48$ and $y = 2$

$48=\frac{\frac{3}{2}x^2}{2}$ (substituting p and y)

$48\times 2=\frac{3}{2}x^3$

$96=\frac{3}{2}x^3$

$\frac{96\times 2}{3}=\frac{3x^3}{2}\times\frac{2}{3}$ (multiply $\frac{2}{3}$ both sides)

$\Rightarrow 64=x^3$

$x=\sqrt[3]{64}$ (take cube root both sides)

$\therefore x=4$

---

5. **[2003 PP1 13]**

Given that x varies jointly as y and inversely as the square of z:

$\Rightarrow x\propto\frac{y}{z^2}$

Introducing constant of variation k:

$\Rightarrow x=\frac{ky}{z^2}$.........................(i)

From the table, $x=3$ when $y=1$ and $z=2$.

We substitute in (i) to find x

$3=\frac{k\times 1}{2^2}$

$\Rightarrow 3=\frac{k}{4}$

$12=k$ (multiply by 4 both sides)

$\Rightarrow k=12$

$\therefore$ Equation (i) now becomes $x=\frac{12y}{z^2}$

Now we need to find z when $x=1$ and $y=3$

$1=\frac{12\times 3}{z^2}$

$z^2=36$

$z=\pm\sqrt{36}$ (take square root both sides)

$z=\pm 6$

$\therefore$ The missing value in Table 1 is 6.

---

6. **[2008 PII #12a]**

Given that x varies directly as y and inversely as the square of n

$\Rightarrow\ x\propto\frac{y}{n^2}$

Introduce the constant k then

$x=\frac{ky}{n^2}$.................(i)

Given $x=15, y=24$ and $n=4$, we find k

$15=\frac{k\times 24}{4^2}$

$15=\frac{24k}{16}$

$15=\frac{3}{2}k$ (simplifying)

$\frac{2\times 15}{3}=k$

$2\times 5=k$

$k=10$

$\because$ the (i) becomes $x=\frac{10y}{n^2}$

now we need to find n when $x=8, y=20$

$\Rightarrow 8=\frac{10\times 20}{n^2}$ (make n subject of formula)

$8\times n^2=\frac{200}{n^2}\times n^2$

$8\times n^2=200$ (divide both sides by 8)

$n^2=25$

$n=\pm\sqrt{25}$

$n=\pm 5$

$\therefore$ n is either 5 or -5.

---

7. **[2004 PI #4]**

b varies jointly with r and t

$\Rightarrow b\propto rt$

Introducing the constant of variation k

$b=krt$.......................(i)

Substitute b=108, r=3 and t=6

$\Rightarrow 108 = k \times 3 \times 6 \Rightarrow 108 = 18k$

$\Rightarrow k = \frac{108}{18} \quad \therefore k = 6$

Substitute k in (i)

$\Rightarrow b = 6rt$

$\therefore$ In terms of r and t, $b = 6rt$.

8. **[2005 P1 #13]**

$P \propto qr^2$

Introducing a constant k

$P = kqr^2$ ....(i)

Given that $P = 50, q = 1$ and $r = 5$,

$\Rightarrow 50 = k(1)(5)^2$

$\Rightarrow 50 = 25k$ (divide by 25 both sides)

$\Rightarrow k = 2$

Substituting $k = 2$ in (i):

$P = 2qr^2$

When q=3 and r =8

$\Rightarrow$ P=$2 \times 3 \times 8^2$

$\therefore P = 384$

9. **[2010 P1 #20]**

Given that $V \propto rd$

$V = krd$ (where k is a constant)

Given that V=54 when $r = 2$ and $d = 3$

$54 = k \times 2 \times 3 \Rightarrow 54 = 6k$

$k = \frac{54}{6} \qquad \therefore k = 9$

$\Rightarrow V = 9rd$

$\Rightarrow$ When $r = \frac{1}{2}, d = 6$.

$V = 9 \times \frac{1}{2} \times 6$

$V = 9 \times 3$

$V = 27$

10. **[2011 PI #14]**

$p \propto \frac{r}{\sqrt{q}}$

$\Rightarrow p = \frac{kr}{\sqrt{q}}$, where k is constant

Given $p = 4, q = 9, r = 1$

$\Rightarrow 4 = k\frac{1}{\sqrt{9}}$

$4 = k \times \frac{1}{3}$

$4 \times 3 = k \times 1$ (multiply by 3 both sides)

$\Rightarrow k = 4 \times 3 = 12$

Substituting k into our equation: $p = \frac{12r}{\sqrt{q}}$

We now find q, when r =2, p=6.

$6 = \frac{12 \times 2}{\sqrt{q}}$

$6\sqrt{q} = 24$ (multiply $\sqrt{q}$ both sides)

$\sqrt{q} = \frac{24}{6}$ (divide 6 both sides)

$\sqrt{q} = 4$

$\left(\sqrt{q}\right)^2 = (4)^2$ (squaring both sides)

$\therefore q = 16$.

11. **[2012 P1 #19]**

Given $x \propto \frac{y}{z^2}$

$\Rightarrow x = \frac{ky}{z^2}$ (where k is a constant)

when $x = 12, y = 2, z = 1$.

$12 = \frac{k(2)}{(1)^2} \Rightarrow k = \frac{12}{2}$

$\therefore k = 6$.

$x = \frac{6y}{z^2}$. Now we find y when x = 15 and z=2

$15 = \frac{6y}{2^2} \Rightarrow 15 = \frac{6y}{4}$

$6y = 60$ (multiply 4 both sides)

$\therefore y = 10$ (divide by 6 both sides)

12. **[2013 P1 #17]**

$p \propto \frac{q^2}{r}$

$p = \frac{kq^2}{r}$..........(i) (where k is a constant)

given that $p = 18, q = 6$ and $r = 30$.

$\Rightarrow 18 = \frac{k \times 6^2}{30}$

$\Rightarrow 18 = \frac{36k}{30}$ (multiply 30 both sides)

$18 \times 30 = 36k$

$\Rightarrow 540 = 36k$ (divide by 36 both sides)

$\frac{540}{36} = \frac{36k}{36} \Rightarrow 15 = k$

$\therefore k = 15$

Substituting k in (i)

$\Rightarrow p = \frac{15q^2}{r}$

Find r when $p = 27$ and $q = 12$.

$27 = \frac{15(12)^2}{r}$ (make r subject)

$r = \frac{2160}{27}$

$\therefore r = 80.$

13. **[2017 PI #13]**

$w \propto \frac{\sqrt[3]{x}}{y}$ (introduce constant k)

$w = \frac{k\sqrt[3]{x}}{y}$

Given that $x = 27, y = 2$ and $w = 6$.

$6 = \frac{k\sqrt[3]{27}}{2} \Rightarrow 6 = \frac{3k}{2}$

$12 = 3k$

$\frac{12}{3} = \frac{3k}{3}$ (divide 3 both sides)

$k = 4$

$\therefore$ The equation is $w = \frac{4\sqrt[3]{x}}{y}$

Given that $w = 5$ and $y = 8$, we can find x:

$5 = \frac{\sqrt[3]{x}}{2}$

$5 \times 2 = \sqrt[3]{x}$ (cross multiplying)

$10 = \sqrt[3]{x}$

$(10)^3 = \left(\sqrt[3]{x}\right)^3$

$\Rightarrow 1000 = x$ (cubing both sides)

$\therefore x = 1000$

14. **[2015 PI #18]**

$x \propto \frac{z^2}{y}$

$x = \frac{kz^2}{y}$....................(i)(where k is a constant)

When $x = 2$, $y=2$ and $z = 1$

$2 = \frac{k(1)^2}{2}$ (multiply 2 both sides)

$k = 4$

(i) become: $x = \frac{4z^2}{y}$

When $x = 5$ and $z=-2$

$5=\frac{4(-2)^2}{y} \Rightarrow y = \frac{4(4)}{5}$

$\Rightarrow y = \frac{16}{5}$

$\therefore y = 3\frac{1}{5}$ or 3.2

15. **[2014 P1 #17]**

$f \alpha g^2 h$

$f = kg^2 h$

$4 = k(2)^2(9)$

$\frac{4}{36} = \frac{36k}{36}$ (Divide by 36 both sides)

$\Rightarrow k = \frac{1}{9}$

$\Rightarrow f = \frac{1}{9}g^2 h$

Given $g = 6$ and $h = 20$, we find f:

$f = \dfrac{6^2 \times 20}{9} \Rightarrow f = \dfrac{36 \times 20}{9}$

$\therefore f = 80$

16. **[2016 PI #14]**

$c \propto dp$

$c = kdp$..........(i) (where k is a constant)

When $d = 10$, $p = 8$ and $c = 10{,}000$

$\therefore 10{,}000 = k(10)(8)$

$10000 = 80k$

$\dfrac{10000}{80} = k \qquad \Rightarrow k = 125$

Equation (i) becomes :

$c = 125dp$

Finding p when d=25 c= 46875

$46875 = 125(25)p$

$46875 = 3125p$ (divide 3125 both sides)

$p = \dfrac{46875}{3125}$

$p = 15$

15 passengers are transported for a distance of 25 km and cost of K46,875.

17. **[2018 P1 #9]**

$p \propto \dfrac{\sqrt{r}}{t}$

$p = \dfrac{k\sqrt{r}}{t}$..............(i)

when $p = 2$, $r = 4$ and $t = 3$

$2 = \dfrac{k\sqrt{4}}{3}$

$2 \times 3 = k \times 2$

$6 = 2k$

$3 = k$

$\therefore k = 3$

[Substitute k in (i)]

$\Rightarrow p = \dfrac{3\sqrt{r}}{t}$.

Given that $p = 6$ and $r = 16$, we calculating the value of t:

$6 = \dfrac{3\sqrt{16}}{t}$ ....multiplying t both sides.

$6t = 3 \times 4 \qquad \Rightarrow 6t = 12$

$\dfrac{\cancel{6}^{1} t}{\cancel{6}} = \dfrac{12}{6}$........dividing 6 on each side

$t = \dfrac{\cancel{12}^{2}}{\cancel{6}_{1}}$

$\therefore t = 2.$

PARTIAL VARIATION

18. **[2003 PII 2a]**

$y = C + kx$..................(1)

$y = c + kx$ (where c and k are constants)

given that $y = -3$ when x=3

$-3 = c + k(3)$

$\Rightarrow c + 3k = -3$......................(i)

Also given thatwhen $y = 22$ when $x = -2$

substitute in $y = c + kx$:

$22 = c - 2k$

$\Rightarrow c - 2k = 22$.................(ii)

Solving the two equations simultanously:

$$
\begin{array}{r}
c + 3k = -3 \\
- \quad \underline{c - 2k = 22} \\
\underline{5k = -25}
\end{array}
$$

$\Rightarrow k = \dfrac{-25}{5}$ (divide both sides by 5)

$k = -5$

Substitute into the equation (i)

$c + 3k = -3 \qquad \Rightarrow c + 3(-5) = -3$

$c - 15 = -3 \qquad \Rightarrow c = -3 + 15$

$\therefore c = 12$

$\Rightarrow y = c + kx$ becomes $y = 12 - 5x$

When $x = 2$,

$y = 12 - 5(2) \qquad \Rightarrow y = 12 - 10$

$\Rightarrow y = 2$

$\therefore$ The value of $y = 2$ when $x = 2$

19. **[2004 PII #7b]**

Since q varies as a difference of two parts, we subtract the parts: $q = c - \frac{b}{p^2}$.........(i)

(Where c and b are constants)

When q=1, p=2 substitute into equation (i)

$1 = c - \frac{b}{2^2} \Rightarrow 1 = c - \frac{b}{4}$

$\Rightarrow 4 = 4c - b$ (multiply each term by 4)...(1)

When q=6, p=3 Substituting into equation (i)

$\Rightarrow 6 = c - \frac{b}{3^2} \Rightarrow 6 = c - \frac{b}{9}$

$6 \times 9 = c \times 9 - \frac{b}{9} \times 9$ (multiply each term by 9)

$\Rightarrow 54 = 9c - b$.........(2)

To find c and b we solve (1) and (2) simultanously:

$$\begin{array}{l} 4 = 4c - b \\ -\underline{(54 = 9c - b)} \\ -50 = -5c \end{array}$$

$\Rightarrow \frac{-50}{-5} = \frac{-5c}{-5}$ (divide by -5) $\Rightarrow 10 = c$

$\therefore c = 10$

Substituting $c = 10$ in (1)

$4 = 4c - b$

$4 = 4(10) - b \Rightarrow 4 = 40 - b$ (make b subject)

$b = 40 - 4$

$\therefore b = 36$

Substituting c and b in equation (i):

$q = 10 - \frac{36}{p^2}$

When $q = 9$

$\Rightarrow 9 = 10 - \frac{36}{p^2}$

$\Rightarrow 9 - 10 = -\frac{36}{p^2}$

$\Rightarrow -1 = -\frac{36}{p^2}$

$\Rightarrow -p^2 = -36 \Rightarrow p^2 = 36$

$p = \sqrt[2]{36} \Rightarrow p = -6$ or 6

$\therefore$ The positive value of p is 6

---

20. **[2005 PII #8a]**

$C = \frac{a}{t} + bt^2$

When t=1 min, C=120:

$\Rightarrow 120 = \frac{a}{1} + b(1)^2 \Rightarrow 120 = a + b$......(1)

When $t = 2$ min, $C = 200$

$\Rightarrow 200 = \frac{a}{2} + b(2)^2 \Rightarrow 200 = \frac{a}{2} + 4b$

$400 = a + 8b$ (multiply by 2)......................(2)

Solve (1) and (2) simultanously;

$$\begin{array}{l} 400 = a + 8b \\ -\underline{(120 = a + b)} \\ 280 = 7b \end{array}$$ (divide by 7)

$\Rightarrow b = \frac{280}{7} \quad \therefore b = 40$

Substituting $b = 40$ in (1)

$120 = a + 40 \Rightarrow 120 - 40 = a$

$\therefore a = 80$

thus $C = \frac{80}{t} + 40t^2$

The cost for 5 min call ($t = 5$):

$C = \frac{80}{5} + 40(5)^2 \Rightarrow C = 16 + 40(25)$

$C = 16 + 1000$

$\therefore C = $ K1,016.

---

21. **[2007 PII #7b]**

F is partly constant and partly inversely proportional to n.

$\Rightarrow F = k + \frac{c}{n}$

Given that F =240 when n = 40; Substituting in the equation:

$240 = k + \frac{c}{40}$

$240 \times 40 = k \times 40 + \frac{c}{40} \times 40$

$9600 = 40k + c$

$\Rightarrow 40k + c = 9600$ (i)

We are also given that $F = 200$, when $n = 50$.

Substituting into $F = k + \frac{c}{n} \Rightarrow 200 = k + \frac{c}{50}$

$200 \times 50 = k \times 50 + \frac{c}{50} \times 50$

$10000 = 50k + c$

$\Rightarrow 50k + c = 10000 \qquad \text{(ii)}$

We solve (i) and (ii) simultanously:

$$\begin{array}{rl} 40k + c = 9600 & \text{(i)} \\ -(50k + c = 10000) & \text{(ii)} \\ \hline -10k = -400 & \end{array}$$

$k = 40$ (divide both sides by -10)

substitute $k = 40$ in (i):

$40 \times 40 + c = 9600$

$1600 + c = 9600$

$c = 9600 - 1600$

$c = 8000$

substituting $k = 40$ and $c = 8000$ in $F = k + \frac{c}{n}$:

$F = 40 + \frac{8000}{n}$

Finding F when $n = 100$:

$F = 40 + \frac{8000}{100}$

$F = 40 + 80$

$= 120$

The fare per passenger when there are a 100 passengers is K120

22. **[2012 P2 #7b]**

Given that q is partly constant and varies as p^3

$\Rightarrow q = k + cp^3$, where c and k are constants.

when $q = 5$, $p = 1$

$\Rightarrow 5 = k + (1)^3 c$

$\Rightarrow 5 = k + c \qquad \text{(i)}$

when $q = 26$, $p = 2$

$\Rightarrow 26 = k + (2)^3 c$

$\Rightarrow 26 = k + 8c \qquad \text{(ii)}$

subtracting (i) from (ii)

$$\begin{array}{r} k + 8c = 26 \\ -(k + c = 5) \\ \hline 7c = 21 \end{array} \qquad \therefore c = 3$$

substituting c in (i)

$5 = k + 3$

$5 - 3 = k$

$\therefore k = 2$

The equation is $q = 2 + 3p^3$.

When $p = -2$

$q = 2 + 3(-2)^3$

$= 2 - 24$

$\therefore q = -22$

23. **[2010 PII #12a]**

$T = a + kN$, where a and k are constants

Given that N=5, T=120

$\Rightarrow 120 = a + 5k$

$\Rightarrow a + 5k = 120 \ldots\ldots\ldots\ldots\text{(i)}$

We are also given that when N =9, T = 180.

$\Rightarrow 180 = a + 9k$

$\Rightarrow a + 9k = 180 \ldots\ldots\ldots\ldots\text{(ii)}$

Subtracting (i) from (ii):

$$\begin{array}{r} a + 9k = 180 \\ -(a + 5k = 120) \\ \hline 4k = 60 \\ \hline \end{array}$$

$\Rightarrow \frac{4k}{4} = \frac{60}{4}$

$\therefore k = 15$

Substitute in equation (i)

$a + 5k = 129$

$a + 5(15) = 120$

$a = 120 - 75$

$a = 45$

The equation becomes $T = 45 + 15N$

When N=12

$T = 45 + 15(12)$

$T = 45 + 180$

$T = 225$

$\therefore$ 12 students will take 225 minutes to finish a topic

24. **[2011 PII #10a]**

$a = k + \frac{c}{n^2}$ (k and c are constants)

Given $n = 1$ when $a = -1$, substitute in: $a = k + \frac{c}{n^2}$

$-1 = k + \frac{c}{(1)^2}$

$k + c = -1$ (i)

when $n = 2, a = 2$, again substite in $a = k + \frac{c}{n^2}$

$2 = k + \frac{c}{(2)^2}$

$4k + c = 8$ (multiply by 4) (ii)

We simultaneously solve equations (i) and (ii)

$k + c = -1$
$-(4k + c = 8)$
$-3k = -9$ (divide each side by -3)

$\Rightarrow k = 3$

substituting $k = 3$ in (i)

$3 + c = -1$

$c = -1 - 3$

$\therefore c = -4$

Substituting $c = -4$ and $k = 3$ in $a = k + \frac{c}{n^2}$.

$\therefore a = 3 - \frac{4}{n^2}$

$\Rightarrow$ when $n = 4$

$a = 3 - \frac{4}{4^2} \Rightarrow a = 3 - \frac{1}{4}$

$\therefore a = 2\frac{3}{4}$

$\therefore$ The value of a is $2\frac{3}{4}$ when $n = 4$.

25. **[2013 P2 #8a]**

$p = aq + kq^2$

when $q = 4, p = 52.8$

$\Rightarrow 52.8 = a(4) + k(4^2)$

$4a + 16k = 52.8$..............(i)

when $q = 5, p = 81$

$81 = a(5) + k(5^2)$

$81 = 5a + 25k$

$\therefore 5a + 25k = 81$..............(ii)

The two equations are:

$(5a + 25k = 81) \quad \times 4$
$-(4a + 16k = 52.8) \quad \times 5$

$20a + 100k = 324$
$-\ 20a + 80k = 264$
$20k = 60$

Divide both sides by 20

$k = \frac{60}{20} \Rightarrow k = 3$

Substituting $k = 3$ in (ii)

5a+25(3)=81

5a+75=81 $\Rightarrow$ 5a=81-75

5a=6 $\Rightarrow \frac{5a}{5} = \frac{6}{5}$

$a = \frac{6}{5}$

$\Rightarrow$ the equation becomes $p = \frac{6}{5}q + 3q^2$

when $q = 6$,

$p = \frac{6}{5}(6) + 3(6)^2$

$p = \frac{36}{5} + 108$

$p = \frac{576}{5}$

$\therefore p = 115.2$

26. **[2014 PII #7a]**

$N = a + kL$; where a and k are constants

150=a+40k.................(i)

192=a+54k.................(ii)

subtract (i) from (ii) to eliminate a

$192 = a + 54k$
$-\ 150 = a + 40k$
$42 = 14k$

$42 = 14k$ (divide 14 both sides)

$\frac{42}{14} = \frac{14k}{14}$

$k = 3$

To find a, we substitute $k = 3$ in (i)

$a + 40k = 150$

$a + 40(3) = 150$

$a = 150 - 120$

$a = 30$

When $k = 3$, $a = 30$.

The relationship is $N = 30 + 3L$

$N = 30 + 3(73)$

$N = 30 + 219$

$N = 249$

$\therefore$ 249 workers will be hired to complete a 73 km road

27. **[2017 PII #11b]**

$d = a + kv^2$ where a and k are constants

when $v = 2$ and $d = 15$

$15 = a + 2^2k$

$a + 4k = 15$.............(i)

when $v = 3$ and $d = 20$

$20 = a + 3^2k$

$a + 9k = 20$.............(ii)

By elimination, subtract (ii) from (i)

$$\begin{array}{r} a + 9k = 20 \\ -\ a + 4k = 15 \\ \hline 5k = 5 \\ \hline \end{array}$$

$\frac{5k}{5} = \frac{5}{5}$

$\therefore k = 1$

Substituting in equation (i)

$a + 4k = 15$

$a + 4(1) = 15$

$a = 15 - 4$

$a = 11$

The relationship is $11 + v^2 = d$

If distance is 36m

$36 - 11 = v^2$

$25 = v^2$

$v = \sqrt{25}$

$\sqrt{v^2} = \sqrt{36 - 11}$

$v = \pm\sqrt{25}$

$v = 5$ or $v = -5$

since velocity is a

$v = 5m/hr$ or $-5\,km/hr$ (velocity can be $\pm$)

28. **[2015 PII #9a]**

$p \alpha a + d$

$p = a + kd$, where a and k are constants

when d=30, p= 10800

$10800 = a + 30k$................(i)

when d=50, p=13500

$13500 = a + 50k$...............(ii)

Solve the equations simultanously, subtract (i) from (ii)

$13500 = a + 50k$

$-(10800 = a + 30k)$

$2700 = 20k$

$\frac{2700}{20} = \frac{20k}{20}$

$k = 135$

Substitute $k = 135$ in any equation

$10800 = a + 30(135)$

$10800 - 4050 = a$

$a = 6750$

$\therefore$ The relationship is $p = 6750 + 135d$

when d=40, price is

$p = 6750 + 135(40)$

$p = 6750 + 5400$

$p = K12150$

29. **[2016 PII #4b]**

$y = a + kx$ (where a and k are constants)

When $x = 2$ $y = 16$

$16 = a + k(2)$

$16 = a + 2k$..................(i)

When $x = 7$, $y = 31$

$31 = a + 7k$..............(ii)

Solve by elimination

$31 = a + 7k$

$\underline{-(16 = a + 2k)}$

$15 = 5k$

$\frac{15}{5} = \frac{5k}{5}$

$k = 3$

Substitute in equation (i)

$16 = a + 2k$

$16 = a + 2(3)$

$16 - 6 = a$

$a = 10$

$\therefore$ The relationship is $y = 10 + 3x$

---

30. **[2019 PI #13]**

$t \propto \frac{\sqrt{m}}{c} \Rightarrow t = \frac{k\sqrt{m}}{c}$ (introduce k as constant)

When $t = 3$, $m = 4$ and $c = 8$

$3 = \frac{k\sqrt{4}}{8} \Rightarrow 2k = 24 \therefore k = 12$

$\Rightarrow$ The relationship is; $t = \frac{12m}{c}$

When $t = 6$ and $m = 25 \Rightarrow 6 = \frac{12\sqrt{25}}{c}$

$c = \frac{12 \times 5}{6} \Rightarrow c = 2 \times 5$

$c = 10$

$\therefore$ Chikondi got 10 bananas

CH 16
GRAPHS OF QUADRATIC FUNCTIONS

Chapter Highlights

In this chapter, we are presented with problems that require us to draw and interpret graphs of quadratic functions. These graphs are called parabolas with the general form:

- $y = ax^2 + bx + c$ .... (*)

  where a, b and c are constants and $a > 0$.

  Using completing the square method, equation (*) can be turned into the standard form $y = a(x-h)^2 + k$, where coordinate (h, k) is the turning point.

  To find the equation of line of symmetry of a quadratic equation you will use the formula:

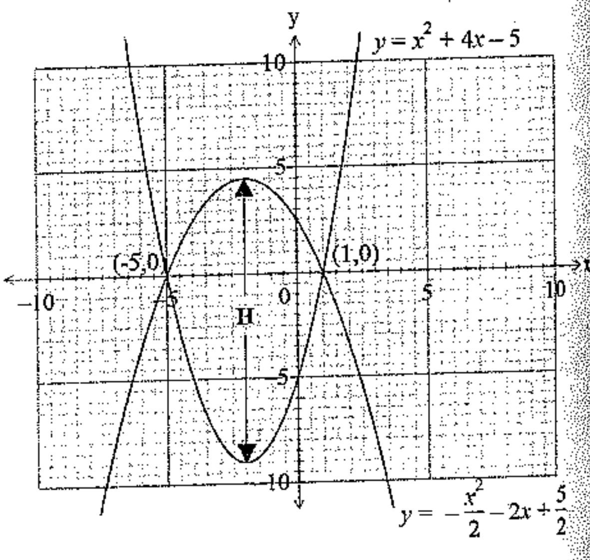
- $x = -\dfrac{b}{2a}$ where a and b are from (*)

  The equation line of symmetry is the x-coordinate of the turning point. To find the the y-coordinate, we will substitute the value $x = -\dfrac{b}{2a}$ into the (*).

  We will draw table of values and graphs of quadratic functions. The graphs will then be used to solve quadratic equations or both linear and quadratic equations simultaneously.

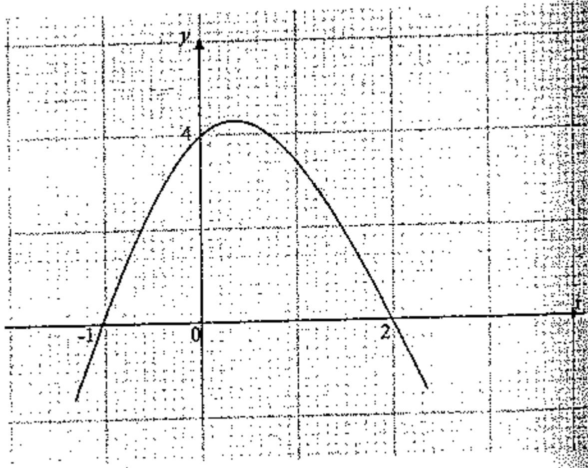
1. Calculate the coordinates of the turning point on the curve $y = x^2 + 4x$. **[2003 P1 #9]**

2. In **Figure 1** two quadratic graphs $y = x^2 + 4x - 5$ and $y = -\dfrac{x^2}{2} - 2x + \dfrac{5}{2}$ are crossing each other at (-5,0) and (1,0). **H** is the distance between the maximum $y = -\dfrac{x^2}{2} - 2x + \dfrac{5}{2}$ and the minimum $y = x^2 + 4x - 5$.



Calculate the value of **H**. **[2003 PII 8b]**

3. **Figure 2** shows a graph of $y = ax^2 + bx + c$ where a, b and c are constants



Find the equation of;

i. the curve in the form of $y = ax^2 + bx + c$.

ii. the line of symmetry

[2004 PII #4a]

4. The table below shows some of the values for the equation; $y = x^2 + x - 2$.

x	-4	-3	-2	-1	0	1
y	10		0	-2	-2	0

i. Copy and complete the table.
ii. Using a scale of 2cm to represent 2 units on the y-axis and 2cm to represent 1 unit

on the x-axis, draw the graph of $y=x^2+x-2$

iii. Use the graph to find the minimum value of y. **[2004 PII #4b]**

5. Copy and complete the table of values for the equation $y=x^2-3x+10$

x	-1	0	1	2	3	4	5
y	14	10	8		10		20

i. Using a scale 2cm to represent I unit on the horizontal axis and 2cm to represent 2 units on the vertical axis, draw the graph of $y=x^2+3x+10$.

ii. Use your graph to solve the simultaneous equation:

$y=x^2+3x+10$

$y=x+7$

[2008 P2 #7b]

6. The table below shows the values of x and y of the equation $y=x^2+x-2$.

x	-4	-3	-2	-1	0	1
y	10		0	-2	-2	0

i. Copy and complete the table and draw the graph of $y=x^2+x-2$.

ii. Using a scale of 2 cm to represent 2 units on the y-axis and 2 cm to represent 1 unit on the x-axis, draw the graph of $y=x^2+x-2$.

iii. Use the graph to find the minimum value of y. **[2006 PII #11b]**

7. The table below shows some values for the equation $y=1+5x-x^2$.

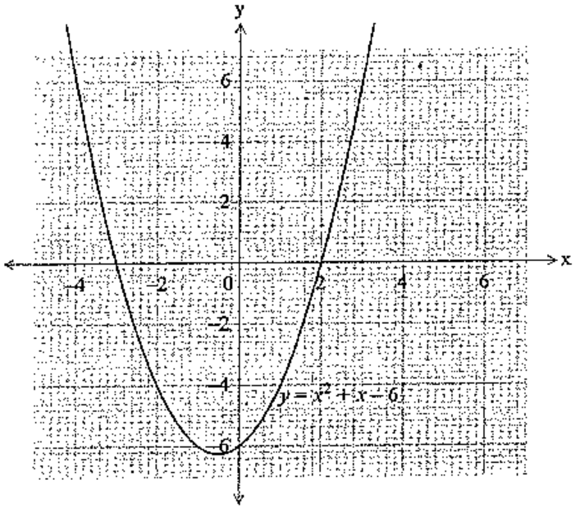
X	-2	-1	0	1	2	3	4	5	6
y	-13		1	5		7	5	1	-5

i. Complete the table of values

ii. Using a scale of 2 cm to represent 1 unit on the horizontal axis and 2 cm to represent 2 units on the vertical axis, draw the graph of $y=1+5x-x^2$

iii. Use your graph to solve the equation $1+5x-x^2=1$. **[2012 P2 #11b]**

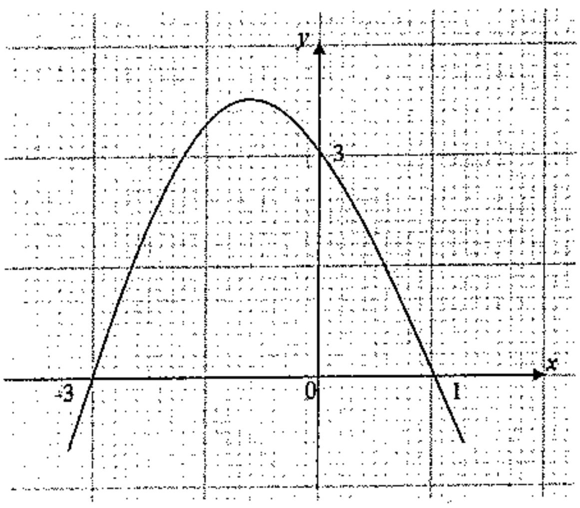
8. **Figure 3,** shows a graph of $y=x^2+x-6$



Use the figure to solve the equation $x^2=5-x$

[2005 P1 #15]

9. **Figure 4** is a graph of a quadratic equation.



Formulate an equation of the graph in the form of $y=ax^2+bx+c$. **[2010 P1 #16]**

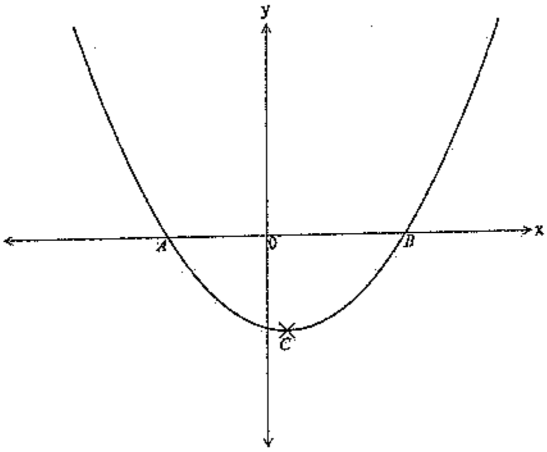
10. i. Copy and complete the table of values for the equation $y=2x^2-5x-5$.

x	-2	-1	0	1	2	3	4	5
y	13	2	-5		-7		7	20

ii. Using a scale of 2cm to represent 1 unit on horizontal axis and 2cm to represent 5 units on the vertical axis, draw the graph of $y=2x^2-5x-5$

iii. Use your graph to solve the equation $2x^2 - 3x - 10 = 0$. **[2010 PII #7b]**

11. **Figure 5** shows a sketch of a graph of a quadratic function $y = x^2 - 2x - 3$. The graph cuts the x-axis at A and B. C is the turning point.



i. Find the coordinates of points A and B.
ii. Find the equation of the line of symmetry of the graph.
iii. Find the minimum value of y for the equation $y = x^2 - 2x - 3$. **[2013 P2 #11b]**

12. **Figure 6** shows a straight line intersecting the graph of $y = x^2 + 1$ at point A and B. The x value at point A and B are 0 and 2 respectively.

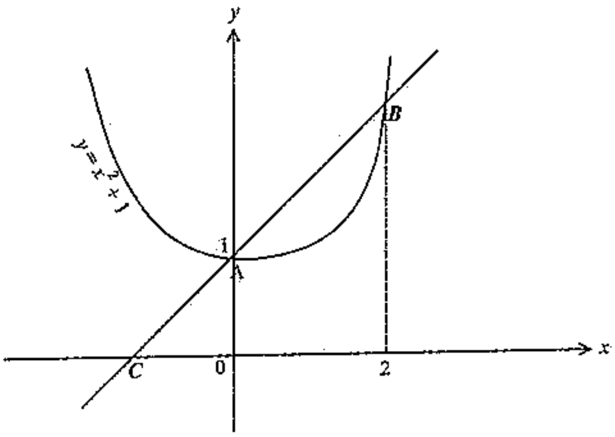


Figure 4

Find x-coordinate of point C.

[2015 PI #20]

13. The table below shows some values of x and y for the equation $y = 5 + x - 2x^2$.

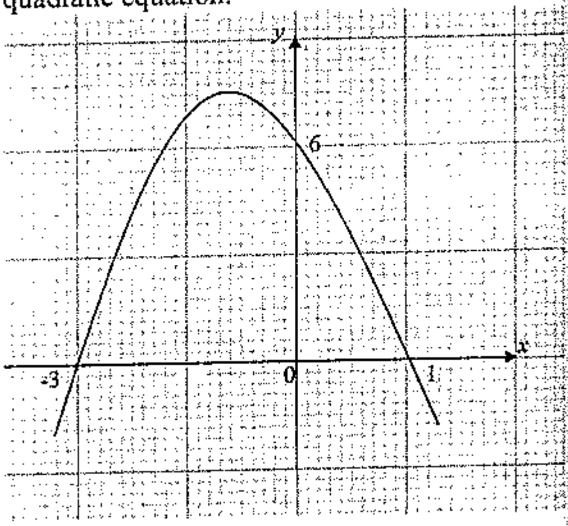
x	-3	-2	-1	0	1	2	3
y	-16		2	5		-1	-10

i. Complete the table of values
ii. Using a scale of 2cm to represent 1 unit on the horizontal axis and 2cm to represent 2 units on the vertical axis, draw the graph of $y = 5 + x - 2x^2$.
iii. Use your graph to solve the equation $3 - 2x - 2x^2 = 0$. **[2015 PII #7b]**

14. A curve $y = 4 - x - x^2$ meets a line M at point $P(x, y)$. If the equation of the line M is $y = -3x - 4$, find the y coordinate of point P when x coordinate is negative. **[2015 PII #11b]**

15. The curve of the equation $y = ax^2 + ax + ac$ passes through points (2,0),(-3,0) and (3,12). Calculate the value of a. **[2016 PI #9]**

16. **Figure 7** shows a sketch of the graph for a quadratic equation.



Find the equation of the graph in the form $y = ax^2 + bx + c$. **[2017 PI #11]**

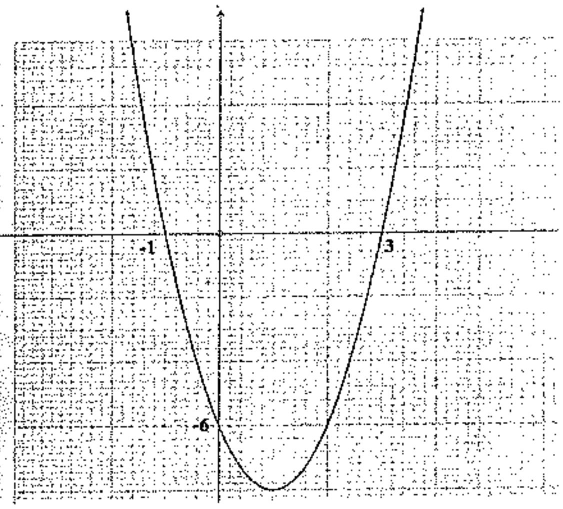
17. The **table below** shows some values of x and y for the equation $y = x^2 + 2x - 4$.

x	-4	-3	-2	-1	0	1	2	3
y	4	-1		0		-1	4	11

i. Complete the table.
ii. Use the scale of 2 cm to represent one unit on the horizontal axis and 2 cm to represent 2 units on the vertical axis, draw a graph of $y = x^2 + 2x - 4$.

iii. Use the graph to find the minimum value of y. **[2018 PII #14]**

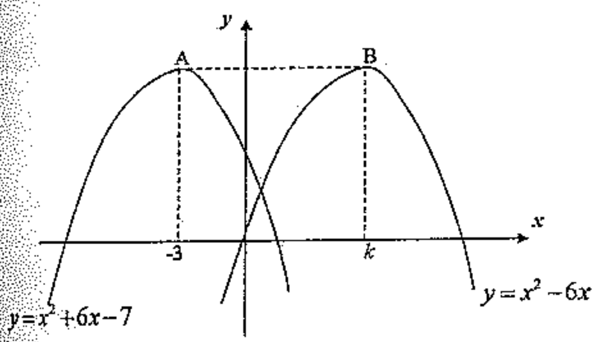
18. **Figure 8** shows a graph of $y = ax^2 + bx + c$ where a, b and c are constants.



Find the equations of:

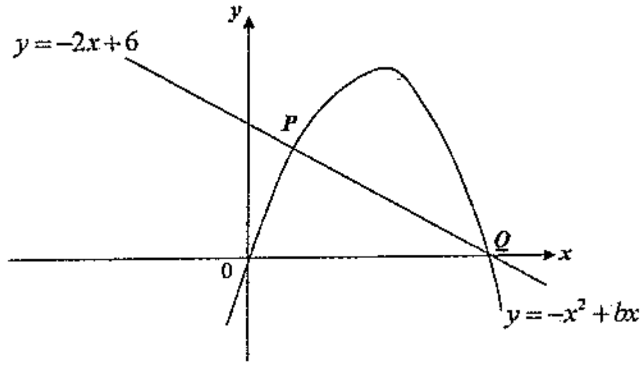
a. Curve
b. Line of symmetry. **[2019 PII #8]**

19. **Figure 9** shows a sketch of two graphs of quadratic equations $y = x^2 + 6x - 7$ and $y = x^2 - 6x$. The horizontal line AB is the distance between the turning points of the two graphs.



By solving for the value of k, calculate the distance AB. **[2020 Mock PI #8]**

20. **Figure 10** shows the graph of $y = -2x + 6$ intersecting with the graph of $y = -x^2 + bx$ at points P and Q.



Calculate the value of b. **[2021 Mock PI #8]**

1. **[2003 P1 #9]**

The equation $y=x^2+4x$ is of the form $y=ax^2+bx+c$ where $a=1, b=4, c=0$.

x coordinate of the turning point $=-\dfrac{b}{2a}$

$$\Rightarrow x=-\frac{b}{2a}$$
$$=\frac{-4}{2\times 1}=-2$$

Substituting $x=-2$ in $y=x^2+4x$

$y=(-2)^2+4(-2)$

$y=4-8$

$\therefore y=-4$ $\therefore$ turning point is $(-2,-4)$

Alternative,

By completeting tghe square method:

$y=x^2+4x$

let $x^2+4x=0$

$x^2+4x+\left(\frac{4}{2}\right)^2=\left(\frac{4}{2}\right)^2$

$x^2+4x+(2)^2=(2)^2$

$x^2+4x+4=4$

$(x+2)^2=4$

$(x+2)^2-4=0$

thus $y=(x+2)^2-4$

hence TP is $(-2,-4)$

2. **[2003 PII 8b]**

The equation of the two graphs are of the form $y=ax^2+bx+c$

$x-$coordinate of minimum $=\dfrac{-b}{2a}$

Taking $y=x^2+4x-5 \Rightarrow a=1$, b= 4

$\therefore x-$coordinate of minimum $=\dfrac{-4}{2\times 1}$

$$=\frac{-4}{2}=-2$$

$\Rightarrow$ To find the y-coordinate, we substitute -2 in

$y=x^2+4x-5$

$\Rightarrow y=(-2)^2+4(-2)-5 \Rightarrow y=4-8-5$

$\Rightarrow y=-9$

$\therefore$ the coordinate of minimum is (-2, -9)

For $y=-\dfrac{x^2}{2}-2x+5/2$, $a=-\dfrac{1}{2}$, b = −2,

$x-$coordinate of the equation $x=-\dfrac{b}{2a}$

$$=-\frac{(-2)}{2(-1/2)}$$

$\therefore x=-2$

To find the $y-$coordinate, we substitute $x=-2$

$y=-\dfrac{x^2}{2}-2x+5/2$

$\Rightarrow y=-\dfrac{(-2)^2}{2}-2(-2)+\dfrac{5}{2}$

$\therefore y=-2+4+\dfrac{5}{2}=4.5$

The coordinate of the minimum is (-2, 4.5)...

Using distance formula:

$H=\sqrt{(x_2-x_1)+(y_2-y_1)}$

$H=\sqrt{[-2-(-2)]+[4.5-(-9)]}$

$H=\sqrt{(0)^2+13.5^2}$

$H=\sqrt{182.25} \Rightarrow H=13.5$

or (from*)

since H is vertical or x-values are equal,

H = 4.5 − −9

= 13.5 units

3. **[2004 PII #4a]**

i. the roots of the required equation are $x=-1$ and $x=2$.

$\therefore y=a(x+1)(x-2)$ where a is a constant

$=a(x^2-x-2)$

$=ax^2-ax-2a$ .... (i)

From the graph the y-intercept is 4.

$\therefore -2a=4$

$a = -2$ *(divide both sides by 2)*

substitute -2 for a in (i)

$y = (-2)x^2 - (-2)x - 2(-2)$

$= -2x^2 + 2x + 4$

$\therefore$ The equation of the symmetry $y = -2x^2 + 2x + 4$

ii. The equation of the line of symmetry is

$x = \dfrac{-b}{2a}$

From the equation, $b = 2$ and $a = -2$

$\therefore x = \dfrac{-2}{2(-2)} = \dfrac{1}{2}$

$\therefore$ The equation of the line of symmetry is $x = \dfrac{1}{2}$

---

4. **[2004 PII #4b]**

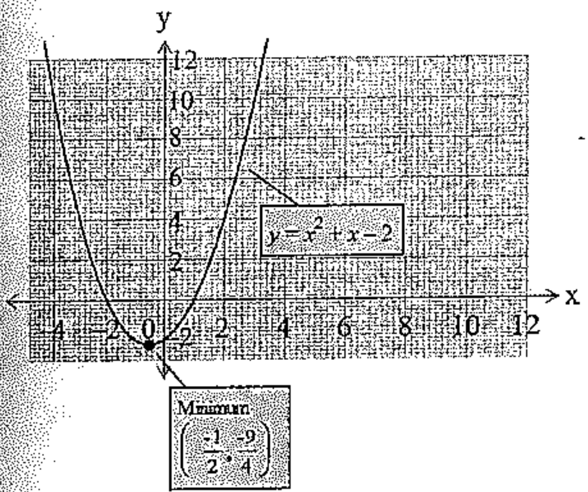
i. Completing table:

When $x = -3$,

$\Rightarrow y = -3^2 + (-3) - 2$

$= 9 - 3 - 2$

$\Rightarrow y = 4$



iii. The minimum value of $y = -\dfrac{9}{4}$.

---

5. **[2008 P2 #7b]**

Completing table of values.

$y = x^2 - 3x + 10$

When $x = 2$, then $y = 2^2 - 3(2) + 10$

$\Rightarrow y = 4 - 6 + 10$

$\Rightarrow y = 8$

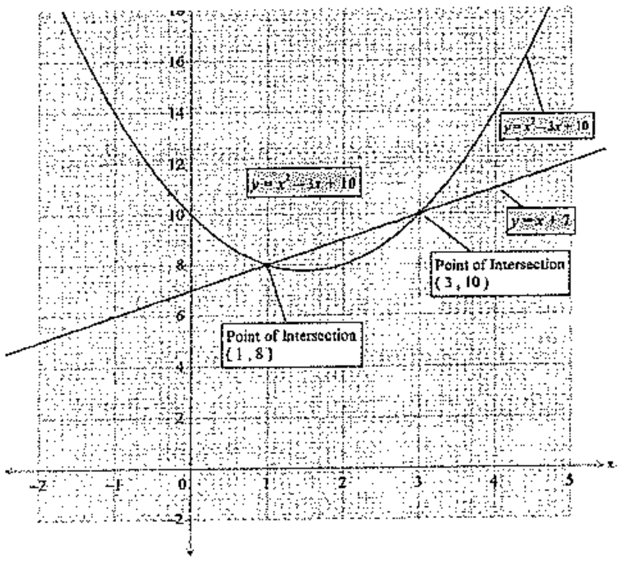
When $x = 4$, then $y = 4^2 - 3(4) + 10$

$y = 16 - 12 + 10$

$y = 14$

x	-1	0	1	2	3	4	5
y	14	10	8	**8**	10	**14**	20

i. Drawing the graph:



ii.

Table of values for $y = x + 7$:

x	-1	0	3
y	6	7	10

Solution to the simultaneous equations are where the two graphs intersect.

Thus $x = 3$ and $y = 10$

and $x = 1$ and $y = 8$

6. **[2006 PII #11b]**

i.

$y = x^2 + x - 2$

When $x = -3$, then $y = (-3)^2 + (-3) - 2$

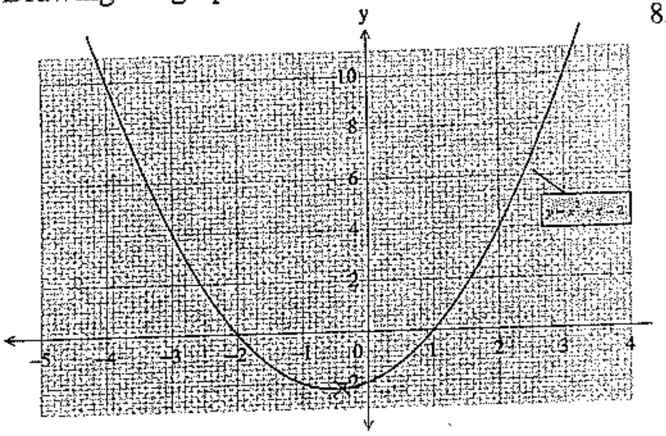
$y = 9 - 3 - 2 \quad \Rightarrow y = 4$

$\therefore y = 4.$

x	-4	-3	-2	-1	0	1
y	10	4	0	-2	-2	0

ii.

Drawing the graph:



iii. From the graph, the minimum value of y is -2.25.

7. **[2012 PII #11b]**

i. Completing the table

For the equation, $y = 1 + 5x - x^2$

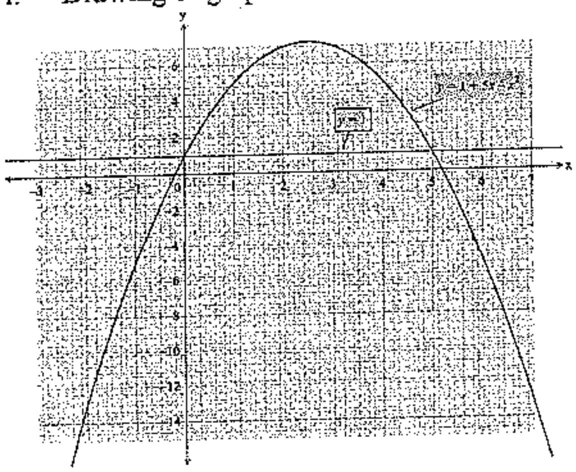
ii. when $x = -1$, then $y = 1 + 5(-1) - (-1)^2$

$y = 1 - 5 - 1$

$\Rightarrow y = -5$

x	-2	-1	0	1	2	3	4	5	6
y	-13	-5	1	5	7	7	5	1	-15

i. Drawing of graph:



ii.

$y = 1 + 5x - x^2$

$\underline{-(1 = 1 + 5x - x^2)}$

$\underline{y - 1 = 0}$

$\therefore y - 1 = 0, \Rightarrow y = 1$

The solution of the equation $1 + 5x - x^2 = 1$ is found where the line $y = 1$ cuts the parabola

The solution is $x = 0$ and $x = 5$.

8. **[2005 P1 #15]**

The equation to be solved: $x^2 = 5 - x$

Is the same as $x^2 + x - 5 = 0$

subtracted from the plotted graph:

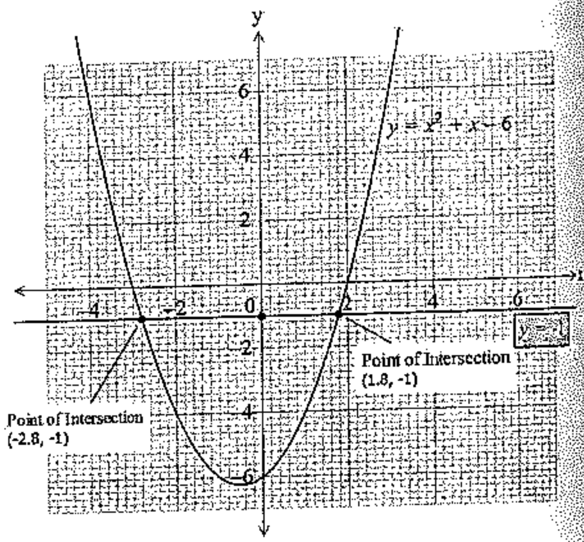
$y = x^2 + x - 6$

$\underline{-(0 = x^2 + x - 5)}$

$\underline{y = 0 \quad + 0 - 1}$

$\therefore y = -1$

We now plot the $y = -1$ on the figure:



The solution is where the graph intersects the line $y = -1$, the two values are:

$x = -2.8$ or $x = 1.8$.

9. **[2010 P1 #16]**

From the graph, the roots of the equation are $x = -3$ and $x = 1$

$\therefore y = a(x + 3)(x - 1)$ where a is a constant

$y = a(x+3)(x-1)$

$y = a(x^2 + 2x - 3)$

$y = ax^2 + 2ax - 3a$............(i)

From the graph the $y-$intercept is 3

$\Rightarrow -3a = 3 \;\; \Rightarrow a = -1$ (divide -3 both sides)

Substitute $a = 1$ in (i), we have:

$y = (-1)x^2 + 2(-1)x - 3(-1)$

$\Rightarrow y = -x^2 - 2x + 3$

---

10. **[2010 PII #7b]**

i. To complete the table of values

$y = 2x^2 - 5x - 5$

when $x = 1$

$y = 2(1)^2 - 5(1) - 5$

$y = -8$

when $x = 3$

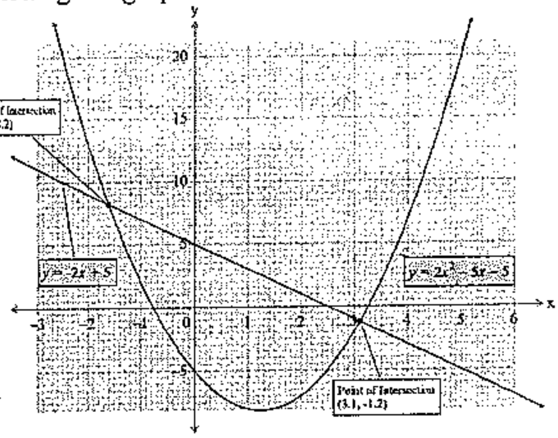
$y = 2(3)^2 - 5(3) - 5$

$y = 18 - 15 - 5$

$y = -2$

x	-2	-1	0	1	2	3	4	5
y	13	2	-5	**-8**	-7	**-2**	7	20

ii. Plotting the graph:



iii.

$$y = 2x^2 - 5x - 5$$
$$-(0 = 2x^2 - 3x - 10)$$

$y = -2x + 5$

Table of value for $y = -2x + 5$

x	-2	0	2
y	9	5	1

Plot the graph of $y = -2x + 5$ on the same graph paper.

$\therefore x = 3.1$ or -1.6

---

11. **[2013 P2 #11b]**

i.

Points A and B are x-intercepts where $y = 0$.

$\Rightarrow y = x^2 - 2x - 3$ becomes $0 = x^2 - 2x - 3$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = 3$ and $x = -1$

$\Rightarrow A = -1$ and $B = 3$

ii.

From the equation $a = 1, b = -2$

Equation of the line of symmetry $= -\dfrac{b}{2a}$

$$= -\frac{(-2)}{2(1)} = 1$$

iii. Equation of line of symmetry is $x-$value of the minimum point. In order to find the $y-$value, substitute $x = 1$ in $y = x^2 - 2x - 3$

$y = (1)^2 - 2(1) - 3 \Rightarrow y = 1 - 2 - 3$

$\Rightarrow y = -4$

$\therefore$ the minimum value of y is -4.

---

12. **[2015 PI #20]**

Lets find y coordinates for points A and B

Using the quadratic equation $y = x^2 + 1$;

At point A, $x = 0$

$y = 0^2 + 1$

$y = 1$

$\therefore$ A is $(0,1)$

At point B, $x = 2$

$y = 2^2 + 1 = 4 + 1$

$y = 5$

$\therefore$ B is $(2,5)$

The gradient of line using A and B

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{5-1}{2-0} \Rightarrow m = \frac{4}{2}$$

$$\therefore m = 2$$

Equation of line is $y = mx + c$

Using point A(0,1), equation of line is:

$1 = 2(0) + c \quad \Rightarrow c = 1$

$\therefore y = 2x + 1$.........(i)

Let point C be (x, 0). Substitute C in (i)

$0 = 2x + 1. \quad \Rightarrow x = \frac{-1}{2}$

$\therefore$The $x-$coordinate of C is $\frac{-1}{2}$.

---

13. **[2015 PII #7b]**

(i) Completed table of values

$y = 5 + x - 2x^2$.

when $x = -2$

$y = 5 + (-2) - 2(-2)^2$

$y = -5$

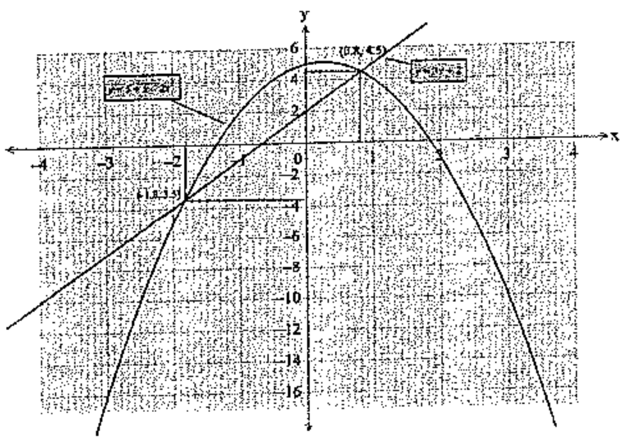
when $x = 1$

$y = 5 + 1 - 2(1)^2$

$y = 4$

x	-3	-2	-1	0	1	2	3
y	-16	-5	2	5	4	-1	-10

(ii) Plotting the graph



(iii) $3 - 2x - 2x^2 = 0$

subtract it from the main equation

$y = 5 + x - 2x^2$

$\underline{-(0 = 3 - 2x - 2x^2)}$

$y = 3x + 2$

The point of intersection of the two groups are at points $x = 0.8$ and $x = -1.8$.

---

14. **[2015 PII #11b]**

Point P is the point of intersection; we solve the equations simultanously.

$y = 4 - x - x^2$ ....(i)

$y = -3x - 4$ ....(ii)

Substituting (*ii*) in (*i*)

$-3x - 4 = 4 - x - x^2$

$4 + 4 + 3x - x - x^2 = 0$ (like terms together)

$8 + 2x - x^2 = 0$

$-\left(x^2 - 2x - 8\right) = 0$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

Either $(x-4) = 0$ or $(x+2) = 0$

$\Rightarrow x = 4$ or $x = -2$

Substituting x in (ii)

When $x = -2$ value of $y = -3(-2) - 4$

$y = 6 - 4 \quad \Rightarrow y = 2$

$\therefore y = 2$ when x is negative

---

15. **[2016 PI #9]**

roots are (2, 0) and (-3, 0)

so $x = 2$ *or* $x = -3$

$x - 2 = 0$ *or* $x + 3 = 0$

so $y = a(x-2)(x+3)$

$y = a(x^2 + 3x - 2x - 6)$

$= a(x^2 + x - 6)$

$y = ax^2 + ax - 6a$

taking point point (3, 12)

$12 = a(3)^2 + a(3) - 6a$

$12 = 9a + 3a - 6a$

$c = -6$

Taking point $(3,12)$ and substituting c=-6

$12 = a(3)^2 + 3a + a(-6)$

$12 = 9a + 3a - 6a$

$\Rightarrow 12 = 6a$

$\therefore a = 2$ (divide a both sides)

---

16. [2017 PI #11]

$y = ax^2 + bx + c$

Either $x = -3$ or $x = 1$

$y = (x+3)(x-1)$

$y = x^2 - x + 3x - 3$

$y = x^2 + 2x - 3$

$y = a(x^2 + 2x - 3)$

$y = ax^2 + 2xa - 3a$

y-intercept $= (0, 6)$

$6 = a(0)^2 + 2(0)a - 3a$

$6 = -3a$

$a = -2$

$\therefore y = -2x^2 - 4x + 6$

Alternatively,

the roots of the required equation are $x = -3$ and $x = 1$

$\therefore y = a(x+3)(x-1)$ where a is a constant

$= a(x^2 + 2x - 3)$

$= ax^2 + 2ax - 3a$ .... (i)

From the graph the y-intercept is 6.

$\therefore -3a = 6$

$a = -2$ *(divide both sides by -3)*

Substitute -2 for a in (i)

$\Rightarrow y = (-2)x^2 + 2(-2)x - 3(-2)$

$\therefore y = -2x^2 - 4x + 6$

---

17. [2018 PII #14]

i.

when $x = -2, y = (-2)^2 + 2(-2) - 4$

$y = 4 - 4 - 4 \Rightarrow y = -4$

$\therefore$ when $x = -2, y = -4$

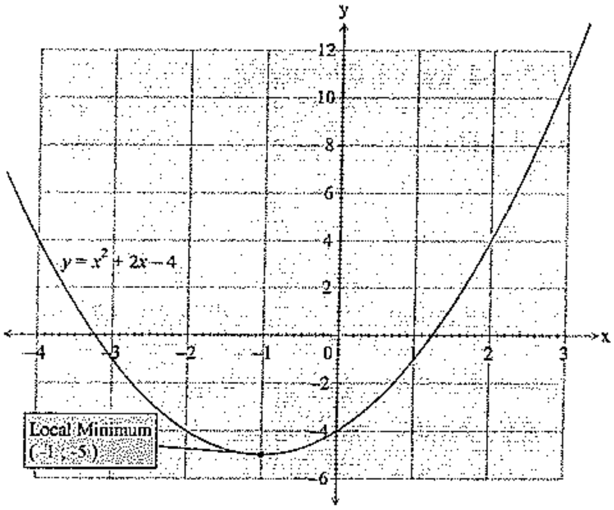
when $x = 0, y = (0)^2 + 2(0) - 4$

$y = 0 + -0 - 4 \Rightarrow y = -4$

$\therefore$ when $x = 0, y = -4$

x	-4	-3	-2	-1	0	1	2	3
y	4	-1	-4	0	-4	-1	4	11

ii.



iii. From the graph, the minimum value of y is -5.

---

18. [2019 PII #8]

a.

$x = -1$ or $x = 3$

$(x+1)(x-3) = y$

$x(x-3) + 1(x-3) = y$

$x^2 - 3x + x - 3 = y$

$x^2 - 2x - 3 = y$

$a(x^2 - 2x - 3) = y$

At the y-intercept $(0, -6)$

$y = a(x^2 - 2x - 3)$

$-6 = a(0^2 - 2(0) - 3)$

$-6 = -3a$

$$\frac{\cancel{-6}^{2}}{\cancel{-3}} = \frac{\cancel{-3}a}{\cancel{-3}}$$

$a = 2$

$\therefore$ The equation is $y = 2(x^2 - 2x - 3)$

$$y = 2x^2 - 4x - 6$$

b. By completing the square, we find the equation of the line of symmetry.

Let $2x^2 - 4x - 6 = 0$

$x^2 - 2x - 3 = 0$

$x^2 - 2x = 3$

$x^2 - 2x + (-1)^2 = 3 + (-1)^2$

$(x-1)^2 - 4 = 0$

letting $x - 1 = 0$

$x = 1$ is the line of symmetry.

Alternatively

Equation of line of symmetry: $x = -\dfrac{b}{2a}$

From the equation, $2x^2 - 4x - 6 \equiv ax^2 + bx + c$

$\Rightarrow a = 2; b = -4$

line of symmetry is:

$x = -\dfrac{-4}{2(2)} = \dfrac{4}{4} = 1$

19. **[2020 Mock PI #8]**

To solve for k, we solve for the equation of the line of symmetry, for the equation:

$y = x^2 - 6x$ as follows:

Let $x^2 - 6x = 0$

$x^2 - 6x + (-3)^2 = (-3)^2$

$(x-3)^2 = 9$

$(x-3)^2 - 9 = 0$

By letting $x - 3 = 0$

$x = 3$ is the line of symmetry

$\therefore k = 3$

Since AB is a horizontal line, then,

$|AB| = B - A = 3 - -3$

$|AB| = 3 + 3$

$|AB| = 6$ units

20. **[2021 Mock PI #8]**

The roots of the quadratic equation $y = -x^2 + bx$ lie at $x = 0$ and at point Q.

We solve for the coordinate of the root at point Q by equating the equation to zero as follows:

Let $-x^2 + bx = 0$

$x(-x + b) = 0$

either $x = 0$ or $-x + b = 0$

$x = 0$ or $x = b$

Hence at point Q, $x = b$

But at Q, the straight line $y = -2x + 6$ has x – intercept

At x – intercept, $y = 0$

So, $-2x + 6 = 0$

$-2x = -6$

$x = \frac{-6}{-2}$

$x = 3$

Since at point Q, $x = b$ and $x = 3$

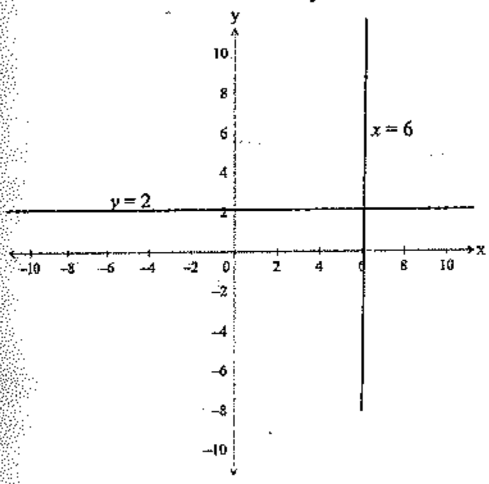
$\therefore b = 3$

CH 17
INEQUALITIES

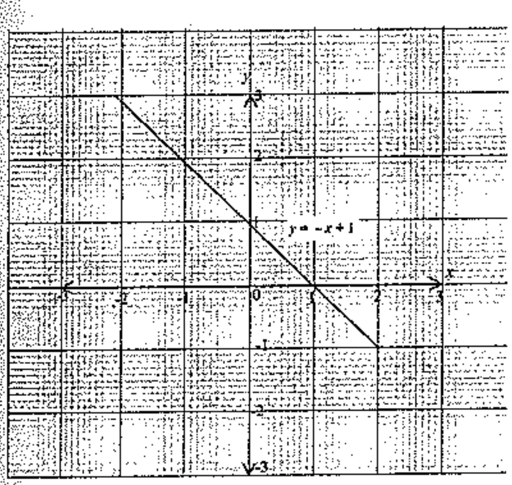
Chapter Highlights

In this chapter, we will solve problem on Inequalities.

- You should know how to represent a line on a graph:
  1. Lines $x = 6$ and $y = 2$.



  2. Slanted lines $y = -x + 1$

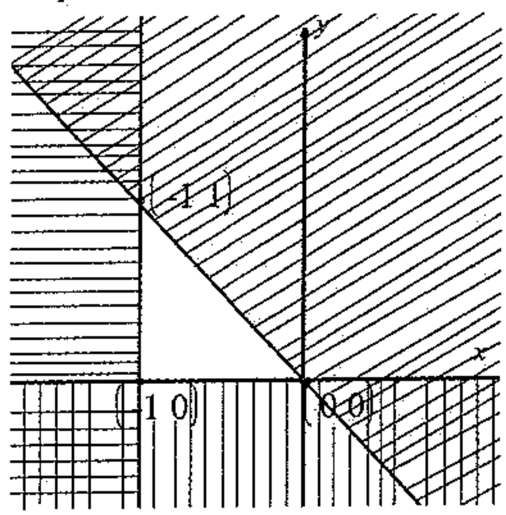


- Procedure for presenting inequalities on a graph:
1. Make y the subject of the formula
2. Replace the inequality symbol by "=" and draw line of the equation
3. Draw a solid line for "$\leq$" and "$\geq$" and a dashed/dotted line for "$<$" and "$>$".
4. Shade below the line for "$>$and $\geq$" or else shade above the line for "$<$ and $\leq$".

- You should also be able to write down inequalities from the unshaded region of a graph:
  1. Determine the equation for the line
  2. Note whether the line is dotted or solid.
  3. If a line is dotted, the inequality is "$<$ or $>$". If line is solid the inequality is "$\leq$ or $\geq$".
  4. To determine the inequality symbol, use the graph's shading:
     - Plug in a point into the equation from the unshaded part of the inequality.
     - Check if the left hand side of the equation is less or greater than the right hand side.
     - Replace the equals sign with the relevant inequality symbol. OR:
       Simply make sure y is the subject of the formula. If the inequality is shaded above then the wanted region is below thus "$<$ or $\leq$". If the inequality is shaded below then the wanted region is above, thus "$<$or $\leq$".

1. P is a set of points (x, y) which satisfies the three inequalities:
$x > -1$;
$y > -2$;
$x + y < 2$.
Using a scale of 2cm to represent 1 unit on the x-axis and y-axis draw region P. **[2003 PI #18]**

2. **Figure 1** shows an unshaded region bounded by three inequalities.



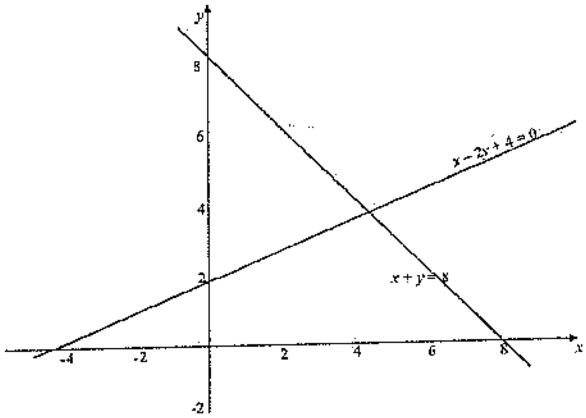
Write down the **three** inequalities.

[2004 PP1 #11]

3. A farmer is selling at most 70 chickens out of which less than 30 are hens. Using x to represent the number of hens and y to represent the number of cocks, write down **four** inequalities involving x and y.

[2005 PI #14]

4. **Figure 2,** shows a graph of $x + y = 8$ and $x - 2y + 4 = 0$.



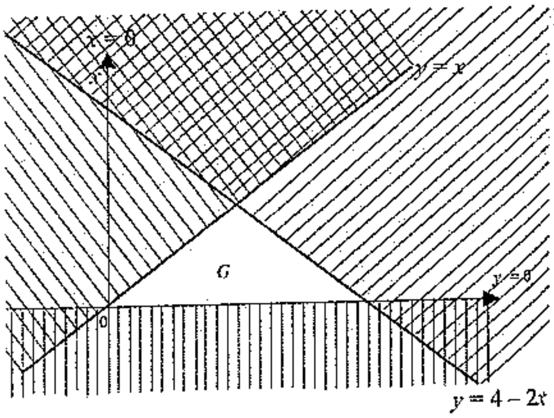
Copy the figure on a graph paper and show the region bounded by the following inequalities.

$$x \geq 0;$$
$$x + y \leq 8;$$
$$x - 2y + 4 \geq 0.$$

by shading the unwanted region. **[2006 PI #18]**

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

5. **Figure 3** shows the region **G** bounded by inequalities.



Write the **three** inequalities that describe the region **G**. **[2007 PI #20]**

6. On the same axes, sketch the graph of the region described by the inequalities $y < 3$ and $y \geq -x$. **[2008 PI #20]**

7. On the same axes, sketch the graphs of the region described by the following inequalities:

$$x \geq 0$$
$$y \geq 0$$
$$y \leq 3x + 2$$
$$y + 4x < 8$$

Shade the unwanted region. **[2010 PI #19]**

8. Sketch the region represented by the following inequalities by shading the unwanted region:

$x \geq -1$

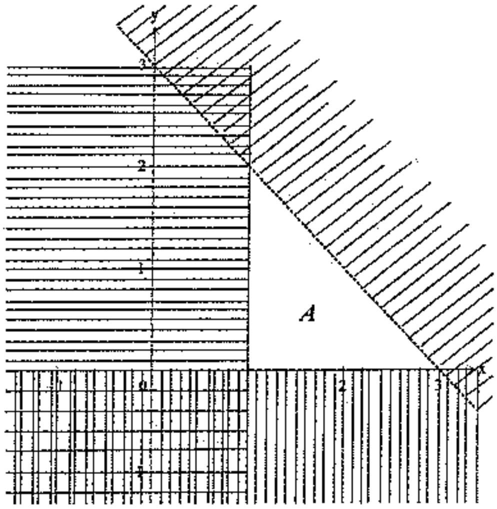
$y \geq -4$

$y < 2x + 4$

$y + 2x \leq 6$

[2011 PI #18]

9. **Figure 4** shows the unshaded region A on a Cartesian plane.



Write down the three inequalities that describe the unshaded region **A.** **[2012 P1 #18]**

10. Sketch the region described by the following inequalities by shading the unwanted region.

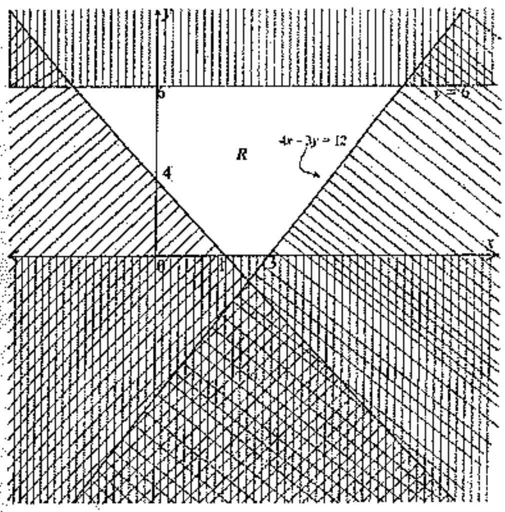
$y \leq 3$

$2x + y \geq 3$

$x - y < 4$

[2013 P1 #20]

11. **Figure 5** shows region **R** bounded by four inequalities.



Write down the **four** inequalities which define the region. **[2014 P1 #20]**

12. On the same axes, sketch the graph to show the region described by the following inequalities by shading the unwanted region

$x \geq 0$

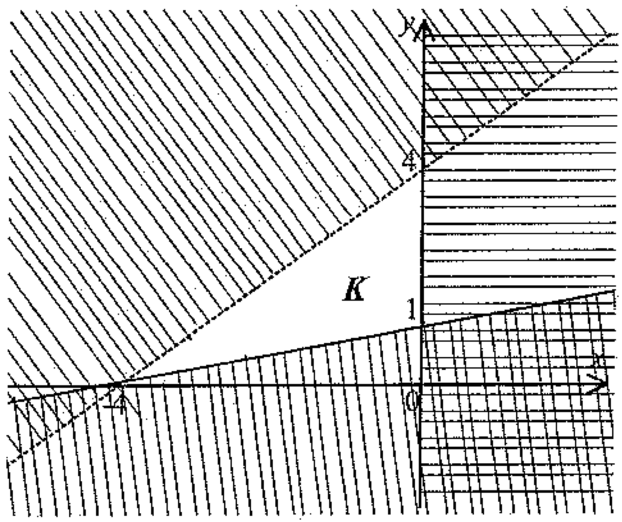
$y > -2$

$2x + y < 4$

$y + 4 \geq 2x$

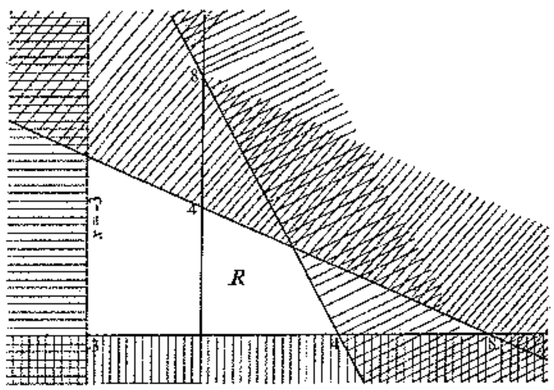
[2015 PI #16]

13. **Figure 6** shows the region **K** bounded by three inequalities.



Write down the three inequalities that define the region. **[2016 PI #18]**

14. **Figure 7** shows unshaded region **R** described by four inequalities.



Write down three inequalities in addition to $x > -3$ which defines the region.

[2017 PI #17]

15. On the same axes, sketch the graph of the region bounded by the following inequalities, shade the unwanted region.

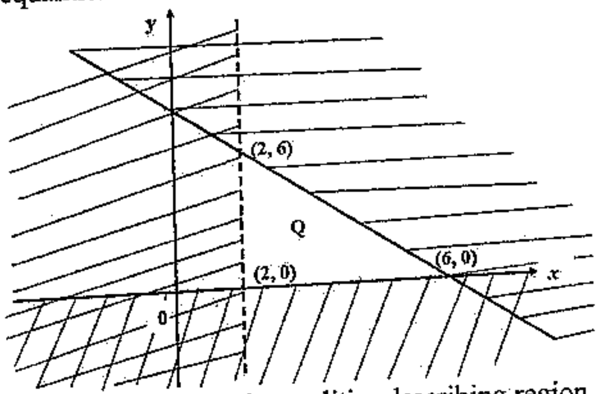
$x - y \leq 2$

$x + y < 4$

$x \geq 0$

[2019 PI #16]

16. **Figure 8** below shows region **Q** bounded by three inequalities.



Write down the **three** inequalities describing region Q. **[2021 Mock PI #18]**

1. **[2003 PI #18]**

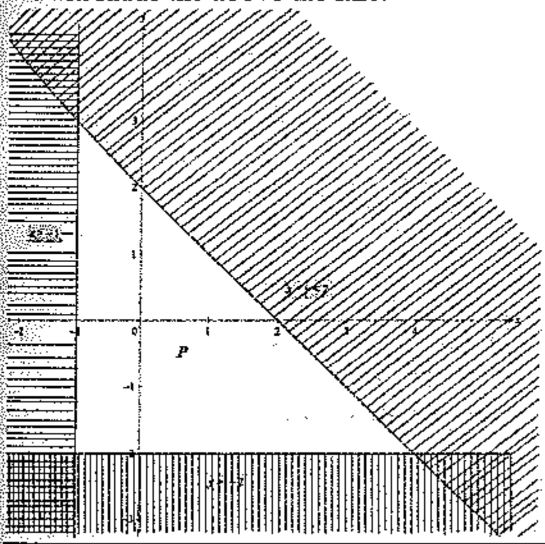
For $x > -1$, we draw a dotted line $x = -1$ and shade the left side since it is the unwanted region.

For $y > -2$, we draw a dotted line $y = -2$ and shade the below the line since it is the unwanted region.

For $x + y < -2$, we draw dotted line $y = 2 - x$:

x	2	0
y	0	2

We will shade the above the line.



2. **[2004 PP1 #11]**

Let (-1, 1) be A
Let (-1, 0) be B
Let (0, 0) be C
i. AB is a vertical line so it is an equation in X only:
AB is x = -1
Since the unshaded part is the left side then
$x \geq -1$
ii. BC is a horizontal line, so it is an equation in y only
BC; y = 0
The unwanted part is below the line, the inequality is $y \geq 0$.
iii. AC is a slanted line then it is of the form $y = mx + c$

$$m = \frac{-1-0}{1-0}$$

$$= \frac{-1}{1} \Rightarrow m = -1$$

y-intercept = 0 $\Rightarrow$ c = 0
So, $y = -x + 0$
$\therefore y = -x$
The unshaded part (the wanted region) is below the line, then $y \leq -x$ or $y + x \leq 0$.

3. **[2005 PI #14]**

The farmer's chickens are hens and cocks
Let x represents the number of hens and
y represents the number of cocks.

$x \geq 0$ and $y \geq 0$ (since the number of hens and chickens will be zero or more)

The farmer has less than 30 hens $\therefore x < 30$

Since the farmer is selling at most 70 chickens:
$x + y \leq 70$

$\therefore$ the four inequalities are as follows:

(i) $x \geq 0$
(ii) $y \geq 0$
(iii) $x < 30$
(iv) $x + y \leq 70$

4. **[2006 PI #18]**

For $x \geq 0$, shade the left side.

For $x + y \leq 8 \Rightarrow y \leq 8 - x$. We then shade above the line.

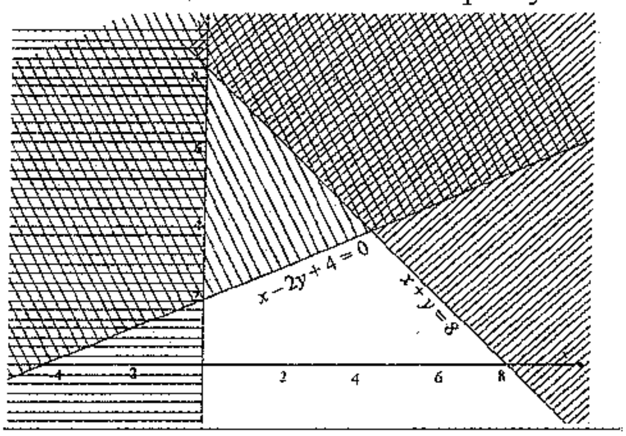
For $x - 2y + 4 \geq 0$, make y subject:

$-2y \geq -x - 4$

$-2y \geq -x - 4$ $\begin{pmatrix} \text{divide both sides by} \\ -2 \text{ and change sign} \end{pmatrix}$

$$y \leq \frac{1}{2}x + 2.$$

Shade above the line since the inequality is "$\leq$".



5. **[2007 PI #20]**

The three inequlities bonding the region G are:

(*i*) $y \leq 4 - 2x$ (Since the unshaded part is below the line)

(*ii*) $y \leq x$ (Unshaded part below the line)

(*iii*) $y \geq 0$ (Unshaded above line y=0)

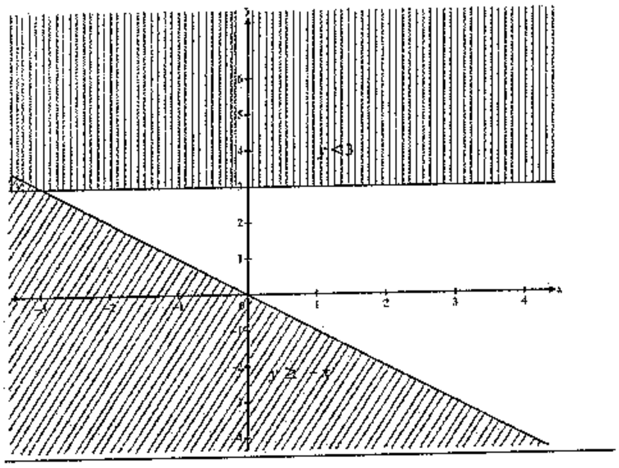
6. **[2008 P1 #20]**

For $y < 3$, first draw the line is $y = 3$ (since it is '<', we draw a dashed line.

For $y \geq -x$, draw the line is $y = -x$ using table of values:

$y = -x$

x	-2	0	2
y	2	0	-2



7. **[2010 P1 #19]**

(i) For $x \geq 0$, The equation is $x = 0$.
(draw solid line and shade the left side of line)
(ii) For $y \geq 0$, the line is $y = 0$. draw solid line and shade below the line.)
(iii) For $y \leq 3x + 2$; the the equation is $y = 3x + 2$.
(draw solid line and shade above since inequality is '$\leq$'). Table of values:

$$y = 3x + 2$$

x	-2	0	2
y	-4	2	8

(iv) $y + 4x < 8 \Rightarrow y < 8 - 4x$.

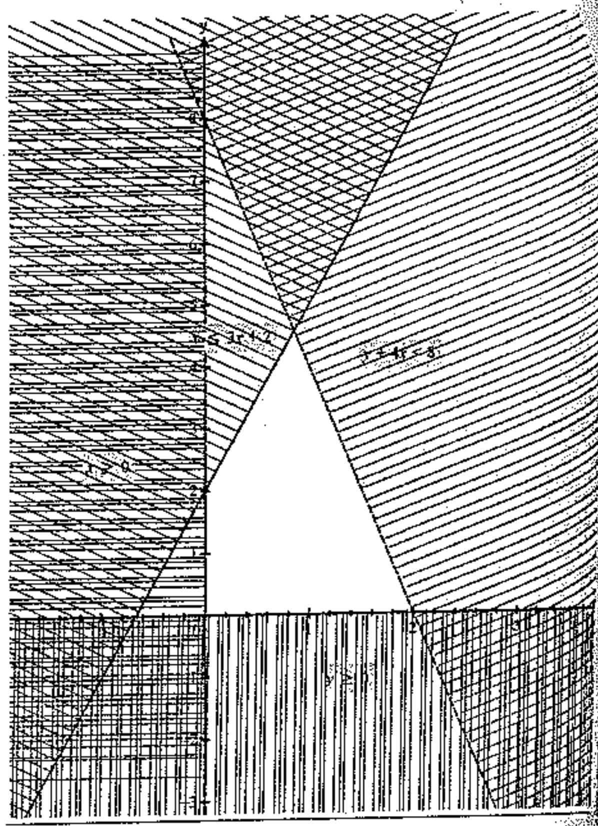
Equation is $y = 8 - 4x$

(Draw dotted line and shade above the line).

Table of values :

$y = 8 - 4x$

x	-2	0	2
y	16	8	0



8. **[2011 PI #18]**

→ For $x \geq -1$, the line is $x = -1$. The line will b solid ($\geq$) and we will shade the left side.

→ For $y \geq -4$, the line is y=4. The line will be Solid ($\geq$) and we will shade below the line.

→ $y < 2x + 4$, the line is y=$2x + 4$ and we will shade above the line. The table of values is:

y=$2x$+4

x	-2	0	1
y	0	4	6

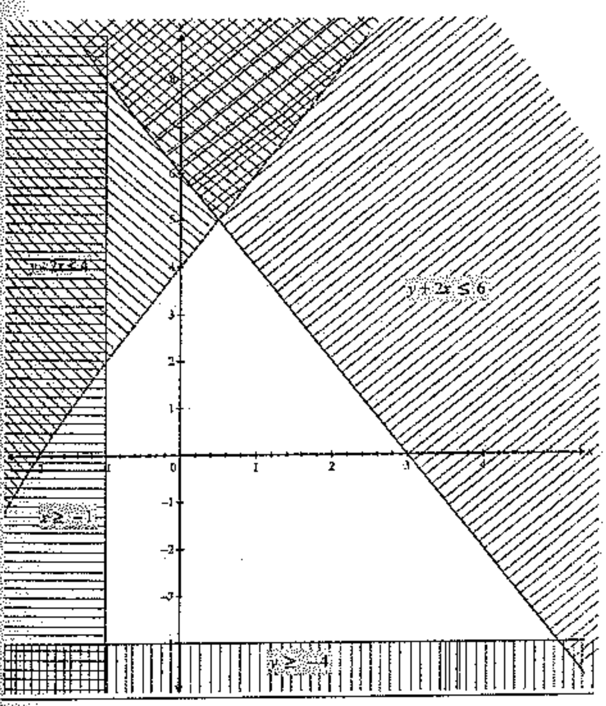
→ For $y + 2x \leq 6$, the line is $y = -2x - 6$

The table of values is:

$$y = -2x - 6$$

x	-2	0	2
y	-2	-6	-10

For this line, we will shade above the line.



9. **[2012 PI #18]**

The line $y = 0$ gives the inequality

a. $y \geq 0$ (since it shaded below)

b. The line $x = 1$ gives the inequality $x \geq 1$ (since the wanted region is to the right)

c. For the line cutting the y-axis at (0,3) and x-axis at (3,0):

$$\text{gradient} = \frac{3-0}{0-3} = -1$$

equation of line: $y = -x + c$

the line passes through (0,3)

$3 = -(0) + c \quad \Rightarrow 3 = 0 + c$

$\therefore c = 3 \therefore$ equation of line is $y = -x + 3$

Since it is shaded above, the wanted region is below, $y < -x + 3$

10. **[2013 P1 #20]**

$\rightarrow$ For $y \leq 3$, the line is $y = 3$ (solid line)

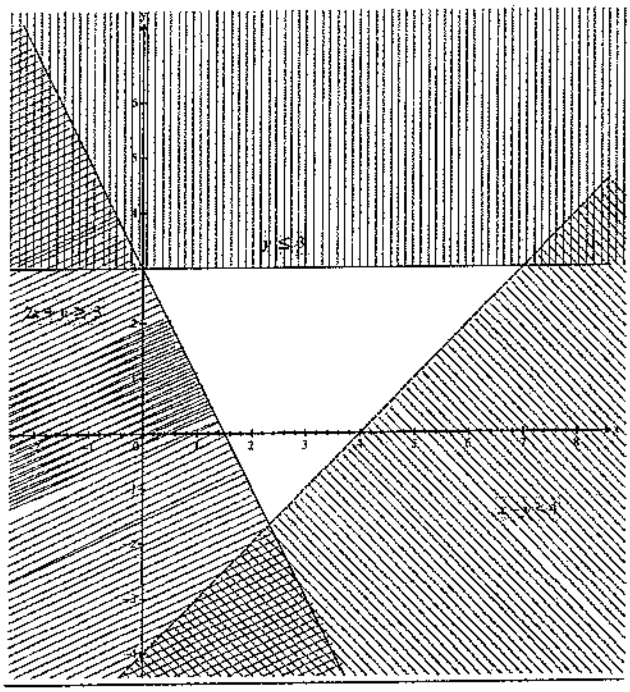
$\rightarrow$ For $2x + y \geq 3$, the line is $y = 3 - 2x$. It is a solid line (since inequality is $\geq$)

With table of values:

x	-2	0	2
y	7	3	-1

$\rightarrow$ For $x - y > 4$, the line is $x - y = 4$

$\Rightarrow y = x - 4$. We draw a dotted line (<)



11. **[2014 P1 #20]**

The line $y = 6$ is dotted and shaded above

$\therefore y < 6$ ....(i)

The line $y = 0$ is solid and shaded below

$\therefore y \geq 0$..(ii)

iii. The line $4x - 3y = 12$ is solid.

We substitute point (0,0) in inequality,

$\Rightarrow$ Left hand side: 4(0)-3(0)=1,

Right hand side: 12 $\therefore$ $1 \leq 12$

$\Rightarrow$ inequality is: $4x - 3y \leq 12$.........(iii)

For line passing throught (1,0) and (0, 4)

$$m = \frac{4-0}{0-1} \Rightarrow m = -4$$

$y = mx + c$, taking point (0, 4) $\Rightarrow 4 = (-4)(0) + c$

$\Rightarrow c = 4 \therefore y = -4x + 4$

substituting point (0, 6) in the unshaded region,

Left hand side: 6, Right hand side: -4(0)+4=4

$\Rightarrow 6 \geq 4 \therefore y \geq -4x + 4$.................(iv)

12. **[2015 PI #16]**

$\rightarrow$ For $x \geq 0$. draw a solid line $x = 0$. Shade the left side since thats the unwanted region.

$\rightarrow$ For $y > -2$, draw slanted line $y = -2$. Shade below the line.

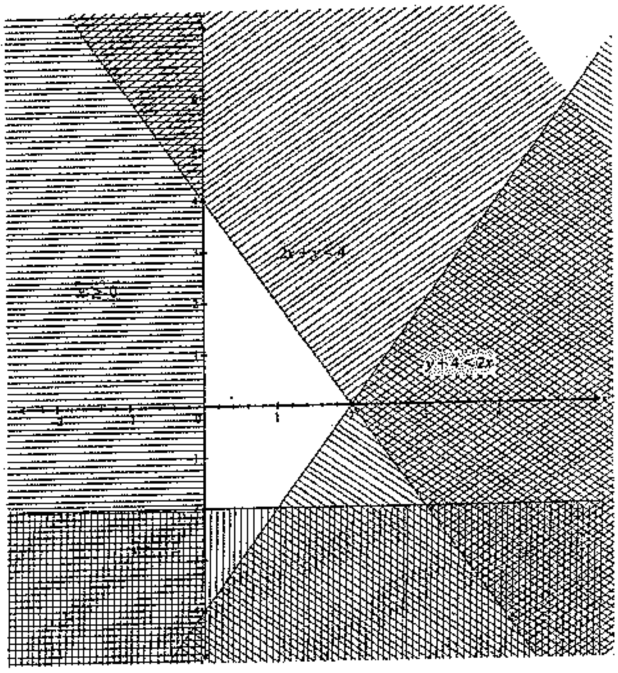
$\rightarrow$ For $2x + y < 4$, draw dotted line $2x + y = 4$
$\Rightarrow y = -2x + 4$ and shade above the line.

Table of values:

x	-1	0	2
y	6	4	0

$\rightarrow$ For $y + 4 \geq 2x$, draw solid line $y+4=2x$
$\Rightarrow y = 2x - 4$ and shade below the line

x	-1	0	2
y	-6	-4	0



13. **[2016 PI #18]**

The three inequalities that define the feasible region are:

(i)The line $x = 0$ is solid and not shaded to the left $\therefore x \leq 0$

(ii) The line passing through $(0,1)$ and $(-4,0)$ is dotted. We find the equation of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \Rightarrow m = \frac{0-1}{-4-0}$$

$$m = \frac{-1}{-4} \Rightarrow m = \frac{1}{4}$$

c, the y intercept is at point $(0,1) \Rightarrow$ c=1

$\therefore y = \frac{1}{4}x + 1$

Substitute any point in unshaded region i.e $(0,2)$

$\Rightarrow 2 = \frac{1}{4}(0) + 1$

Left Hand Side: 2, Right Hand Side: 1

$\Rightarrow 2 \geq 1$

$\therefore y \geq \frac{1}{4}x + 1.$

(iii)The line passing through $(-4,0)$ and and $(0,4)$

$$m = \frac{4-0}{0-(-4)} \quad \Rightarrow m = \frac{4}{4} \qquad \therefore m = 1$$

c, the y intercept is at ponit $(0, 4) \Rightarrow$ c=4

$\therefore y = x + 4$

Substitute any point in unshaded region i.e $(0,0)$

$\Rightarrow 0 = 0 + 4$

Left Hand Side: 0, Right Hand Side: 4

$\Rightarrow 0 < 4$

$\therefore y < x + 4$

14. **[2017 PI #17]**

(i) Line $y = 0$ is solid and unshaded above.

$\Rightarrow y \geq 0$

(ii) Line passing through $(4,0)$ and $(0,8)$ is solid. We then find the equation of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \Rightarrow m = \frac{8-0}{0-4} \quad m = \frac{8}{-4}$$

$\therefore m = -2.$

c, the y intercept is at point $(0,8)$ $\Rightarrow c=8$

$\therefore y = -2x + 8$

Taking point (0, 0) $\Rightarrow$ 0= -2(0)+8

Left Hand Side: 0 and Right Hand Side: 8

$\Rightarrow 0 \leq 8$

$\therefore y \leq -2x + 8$

(iii) line passing through $(0,4)$ and $(8,0)$ is solid. We then find the equation of the line:

$$m = \frac{4-0}{0-8} \quad \Rightarrow m = \frac{4}{-8} = -\frac{1}{2}$$

The y-intercept is at point (0, 4) $\Rightarrow c = 4$

$\therefore y = -\frac{1}{2}x + 4$

Taking point (0,0) $\Rightarrow 0 = -\frac{1}{2}(0) + 4$

Left Hand Side : 0 and Right Hand Side: 4

$\Rightarrow 0 \leq 4$

$\therefore y \leq -\frac{1}{2}x + 4$

(iv) $y \geq -3$ (given)

---

15. [2019 PI #16]

$x - y \leq 2$, can be rewritten as :

$-y \leq -x + 2$ (Divide by -1 and change direction of inequality)

$\Rightarrow y \geq x - 2$

We draw the solid line $y = x - 2$. and shade below the line. Table of values for $y = x - 2$:

x	-2	0	2
y	-4	-2	0

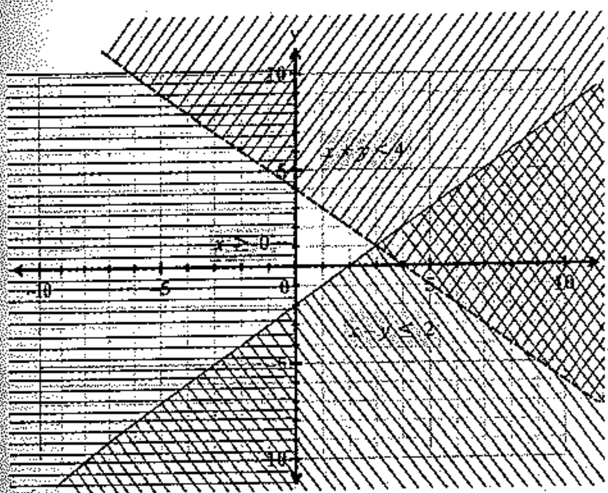
For $x + y < 4 \Rightarrow y < 4 - x$. Draw dotted line $y = 4 - x$ and shade above the line.

Table of values:

$$y = 4 - x$$

x	-5	0	4
y	9	4	0

The graph showing the region bounded by the three inequalities.



---

16. **[2021 Mock PI #18]**

Three inequalities describing region Q:

Inequality 1: vertical through (2,0)

dotted : $x > 2$ (part less than 2 is shaded)

Inequality 2: horizontal through (0,0)

bold : $y \geq 0$ (part less than 0 is shaded)

Inequality 3: slanted through (2,6) and (6,0)

$Slope : m = \frac{6-0}{2-6} = \frac{6}{-4} = -\frac{3}{2}$

$y = -\frac{3}{2}x + c$

Taking (6,0)

$0 = -\frac{3}{2}(6) + c$

$0 = -9 + c$

$9 = c$

equation : $y = -\frac{3}{2}x + 9$

Inequality: $y \leq -\frac{3}{2}x + 9$

CH 18
STATISTICS 1

Chapter Highlights

In this section you will practice problems in statistics. Before attempting these questions you should be able to:

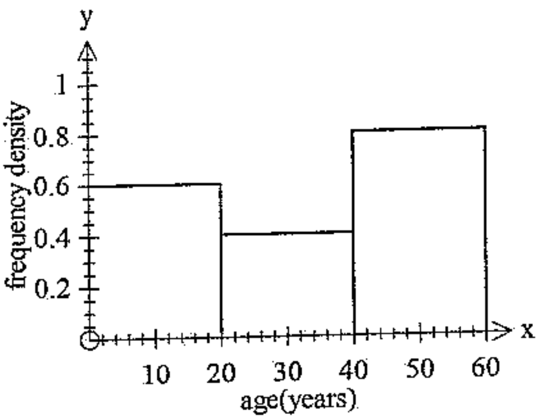
- Present discrete and continuous data in form of tables and charts such as pie charts, histograms, bar graphs, frequency tables, frequency polygons,
- Be able to find the variance and standard deviation.
- Calculate measures of central tendency such as mean, median, mode.
    - The mean is average.
    - The mode is the most frequently appearing number
    - The median is the middle number when the numbers are sorted in either ascending or descending order.

1. **Table 1** shows the distribution of the number of employees in 43 factories in a town.

Number of employees	0-39	40-59	60-79	80-99
Number of factories	5	15	13	10

Draw a histogram. **[2003 PP1 #15]**

2. In **Figure 1** the histogram is representing the ages of the people who attended a meeting



Calculate the number of people who were between 20 and 40 years old at the meeting. **[2004 PI #20]**

3. **Table 2** shows the results of the test which 30 pupils sat for.

2	9	14	7	12	3	19	7	13	19
8	14	23	7	18	12	9	14	8	22
17	9	12	18	14	18	13	12	24	4

a. Using class intervals of the marks as 1-5, 6-10, 11-15, 16-20, 21-25, construct a frequency table for the marks.

b. Using your frequency table, draw a frequency polygon. **[2005 PII 7a]**

4. **Table 3** shows marks that student A and student B got from tests. Student A sat for 5 tests while student B sat for 4 tests. Student B has x marks missing.

Student A	55	70	80	30	65
Student B	67	60	x	53	

Given that the mean mark of the student A is the same as the mean mark of a student B, calculate the value of x. **[2006 PI #7]**

5. **Table 4** shows tallies of students' marks in a test.

Marks (m)	tally
$0 \leq M \leq 10$	
$10 \leq M \leq 20$	//
$20 \leq M \leq 30$	///
$30 \leq M \leq 40$	///
$40 \leq M \leq 50$	~~////~~ ~~////~~
$50 \leq M \leq 60$	//
$60 \leq M \leq 70$	~~////~~ ////
$70 \leq M \leq 80$	~~////~~ /
$80 \leq M \leq 90$	////
$90 \leq M \leq 100$	/

a. Find the modal class.

b. If 50% of the students were to pass, find the passing mark of the test. **[2006 PII #10a]**

6. Three data values x, y and z have the following relationship:

$$x = a^2 - a$$
$$y = 2 - a$$
$$z = 7 + 5a - a^2$$

Calculate the mean of x, y and z in terms of a in the simplest form. **[2007 PI #22]**

7. **Table 5** shows a frequency distribution of marks scored by students in a test.

Marks	frequency
20-29	2
30-39	3
40-49	4
50-59	8
60-69	5
70-79	3
80-89	4
90-100	1

a. How many students scored between 30 and 59 marks inclusive?

b. On the graph paper provided, draw a histogram to represent the distribution of the marks. **[2007 PII #9b]**

8. In a survey conducted at Chitsa village, 20 people responded **'YES'** 30 responded **'NO'** and 10 responded **'DON'T KNOW'**.

a. Draw a pie chart to represent the information. **[2008 P1 #24]**

9. The results of a test marked out of 25 written by 20 learners were as follows:

1	7	13	12	14
12	18	17	19	17
17	19	22	23	24
22	22	24	23	22

Using class intervals of 1-5, 6-10, 11-15---, construct a frequency table for the results. **[2010 P1 #11]**

10. **Table 6** below shows the marks scored by learners in a test.

Marks	10	11	12	13	14	15
frequency	2	3	4	12	5	4

Calculate the mean mark. **[2010 PII #1b]**

11. The mean of $(x-1), (x+2)$ and $(x+5)$ is $2x$, find the value of x. **[2011 PI #7]**

12. Given that two sets of numbers 2, 3, 5, 7, 7, 8, 10 and x, 4, 6, 6, 8, and 8 have equal mean, calculate the value of x. **[2011 PII #2a]**

13. **Table 7** below shows the age group of passengers and their frequencies.

Age group	3-7	8-12	13-17	18-22
frequency	2	5	3	1

Using a scale of 2cm to represent 1 unit on the vertical axis, draw a histogram to represent the information in the table. **[2013 P1 #13]**

14. The number $2, 3, 6, 8, 9, x$ are arranged in ascending order. If the mean of the numbers is equal to the median of the numbers, find value of x. **[2013 P2 #5a]**

15. **Table 8** shows a frequency distribution of marks 20 students.

Marks	1 - 10	11 - 20	21 - 30
Frequency	4	10	6

Calculate the mean mark of the students. **[2014 PII #6a]**

16. **Table 9** below shows frequencies of different class intervals of ages of people in years.

Class interval	1-10	11-20	21-30	31-40	41-50	51-60
Frequency	10	5	17	25	20	13

Using a scale of 2cm to represent 5 units on the vertical axis and 2cm to represent a class interval on the horizontal axis, draw a histogram to represent the information in the table. **[2015 PI #19]**

17. **Table 10** shows marks obtained by students in a Mathematics test, marked out of 60.

Marks	1 - 15	16 - 30	31 - 45	46 - 60
Frequency	7	12	6	4

Using 2cm to represent 2 units on vertical axis, draw a histogram to represent the information in **Table 10.** **[2016 PII #5a]**

18. **Table 11** shows goals of scores by teams during league matches.

Number of Goals	0	1	2	3	4	5
Number of matches	3	6	x	2	2	2

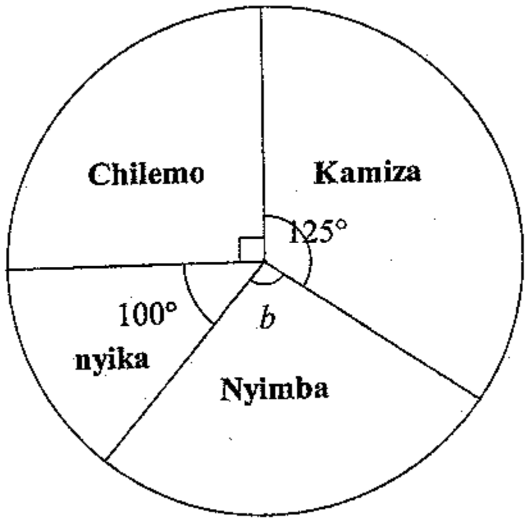
If the total number of matches is 17, calculate the mean number of goals per match. **[2017 PI #10]**

19. The following marks we obtained by a class of 20 students in a test marked out of 30:

22	30	12	15	10
15	23	19	26	16
3	20	28	10	11
9	17	18	24	14

Taking 0 as the lowest class limit and using a class width of 10, construct a grouped frequency distribution table for the marks. **[2018 P1 #10]**

20. **Figure 2** is a pie chart showing number of students who sleep in different hostels at a certain boarding school.



Given the total number of students for the hostels is 288, calculate the number of students who sleep in Nyimba hostel. **[2019 PI #12]**

21. The following were results of a mathematics test obtained by students at a certain secondary school.

3	17	8	11	26	23	18	28	33	38
12	38	22	50	5	35	39	30	31	43
27	34	9	25	39	14	27	16	33	49

a. Using a class interval of marks 1-10, 11-20, 21-30. . ., construct a frequency distribution table for the marks.

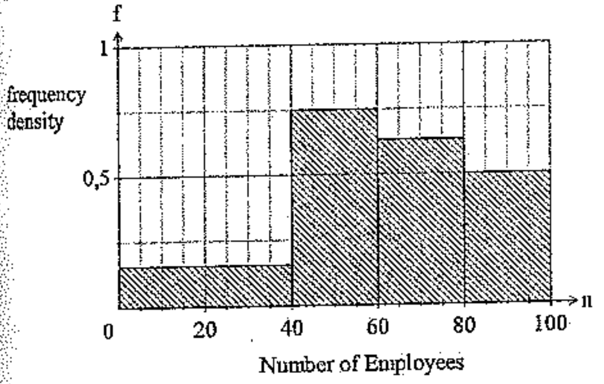
b. Using the frequency distribution table, draw a frequency polygon on a graph paper.

[2019 PII #10]

22. The mean of five numbers is 7. When two numbers x and $2x+1$ are added to the five numbers, the mean becomes 9. Find the value of x. **[2020 Mock PI #10]**

1. **[2003 PP1 #15]**

#employees	0-39	40-59	60-79	80-99
class-width	40	20	20	20
freq	5	15	13	10
density	40/5=0,125	20/15=0,75	20/13=0,65	20/10= 0,5



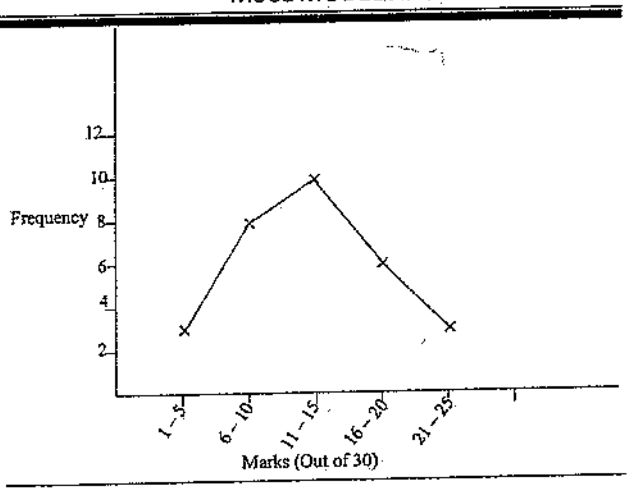
2. **[2004 PI #20]**

In a histogram, area of rectangles is the frequency.
Number of people between 20 and 40 years
= area of the middle rectangle
= (40 – 20) × (0.4 – 0)
= 20 × 0.4
= 8
∴the No. of people between 20 and 40 years old at the meeting were 8.

3. **[2005 PII 7a]**
(i)Develop frequency table using tally

Class interval	Tally	Frequency (f)	Mid class
1-5	///	3	$\frac{5+1}{2}=3$
6-10	//// ///	8	$\frac{6+10}{2}=8$
11-15	//// ////	10	$\frac{11+15}{2}=13$
16-20	//// /	6	$\frac{16+20}{2}=18$
21-25	///	3	$\frac{21+25}{2}=23$

Step 1: column for class interval
Step 2: go by each number in the box of numbers, cross it out and tally against its interval in the frequency table.
ii. frequency polygon
Use the mid class
Because the class width is the same e.g 5 – 1 = 4, 10 – 6 = 4, then there is no need for frequency density. You can just use frequencies and mid class.



4. **[2006 P1 #7]**

Mean student A = mean student B

$$\frac{sumA}{n_A}=\frac{sumB}{n_B}$$

$$\frac{55+70+80+30+65}{5}=\frac{67+60+x+53}{4}$$

$$\frac{300}{5}=\frac{180+x}{4}$$

$$60=\frac{180+x}{4}$$

$60 \times 4 = 180 + x$ (cross multiply)
$240 - 180 = x$ (solving for x)
x = 60

5. **[2006 PII #10a]**

Marks (m)	tally	Freq	Cum.freq
$10\leq M\leq 20$	//	2	2
$20\leq M\leq 30$	///	3	5
$30\leq M\leq 40$	///	3	8
$40\leq M\leq 50$	~~////~~ ~~////~~	10	18
$50\leq M\leq 60$	//	2	20
$60\leq M\leq 70$	~~////~~ ////	9	29
$70\leq M\leq 80$	~~////~~ /	6	35
$80\leq M\leq 90$	////	4	39
$90\leq M\leq 100$	/	1	40

i. the modal class is the class with the highest frequencies. So modal class is $40\leq M\leq 50$.
ii. *the number of students* = 2 + 3 + 3 + 10 + 2 + 9 + 6 + 4 + 1 = 40
if 50% of the class were to pass then,

$$\Rightarrow \frac{50\%}{100\%}\times 40 = 20$$

∴ 20 students were to pass the test.

Passing Mark: Using cumulative frequencies, the first 20 students who **do not pass** the test [marked in a bold box], get scores in the range $50 \leq M \leq 60$. So, to pass the test, a student must score beyond this range.
$\therefore$ the passing mark for the test should be greater that 60%.

6. **[2007 PI #22]**

Given, $x = a^2 - a$

$y = 2 - a$

$z = 7 + 5a - a^2$

$$\text{mean} = \frac{x+y+z}{3}$$

$$= \frac{(a^2 - a) + (2 - a) + (7 + 5a + a^2)}{3}$$

$$= \frac{a^2 - a + 2 + a + 7 + 5a + a^2}{3}$$

$$= \frac{3a + 9}{3}$$

$$= a + 3$$

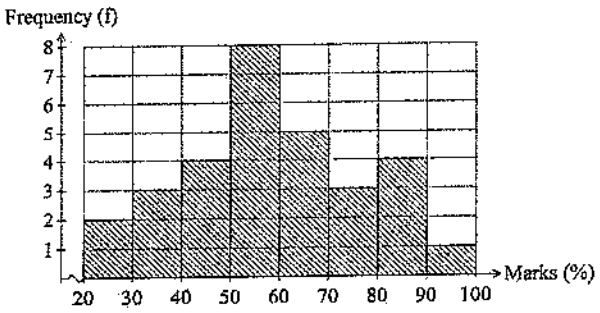
$\therefore$ the mean of x, y and z in terms of a is $a + 3$

7. **[2007 PII #9b]**

Marks	Number of students
30-39	3
40-49	4
50-59	8

Total number of students of students =3 + 4 + 8 =15
$\therefore$15 students scored between 30 and 59 marks inclusive.
ii. Draw a histogram to represent the distribution of the marks.

Histogram



8. **[2008 P1 #24]**

Total number of people in the survey = 20 + 30 + 10 = 60
Angular share of those who responded 'YES':

$$\frac{20}{60} \times 360^\circ = 120^\circ$$

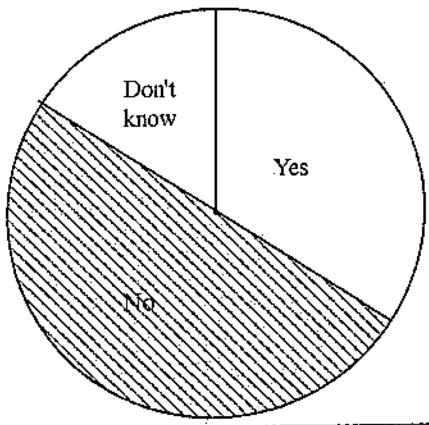
Angular share of those who responded 'NO'

$$\frac{30}{60} \times 360^\circ = 180^\circ$$

Angular share of those who responded 'DON'T KNOW'

$$\frac{10}{60} \times 360^\circ = 60^\circ$$

The pie chart will look as follows:



9. **[2010 P1 #11]**

CLASS INTERVAL	TALLY	FREQUENCY
1-5	/	1
6-10	/	1
11-15	////	4
16-20	//// /	6
21-25	~~////~~ ///	8
TOTAL		**20**

10. **[2010 PII #1b]**

Marks (x)	*Frequency(f)*	*fx*
10	2	$10 \times 2 = 20$
11	3	$11 \times 3 = 33$
12	4	$12 \times 4 = 48$
13	12	$13 \times 12 = 156$
14	5	$14 \times 5 = 70$
15	4	$15 \times 4 = 60$
Sum	**30**	**387**

$$mean\left(\bar{x}\right) = \frac{\sum fx}{\sum f}$$

$$= \frac{387}{30}$$

$$= 12.9$$

11. **[2011 PI #7]**

mean of $(x-1),(x+2),(x+5)$

$sum = x-1+x+2+x+5$

$=3x+6$

No of terms $=3$

$mean = \dfrac{3x+6}{3} = 2x$

$3x+6=6x$ (cross-multiplying)

$6=6x-3x$

$6=3x$

$\therefore x=2$

---

12. **[2011 PII #2a]**

Let $\overline{x}_A = \dfrac{x+4+6+6+8+8}{6}$

Let $\overline{x}_B = \dfrac{2+3+5+7+7+8+10}{7}$

Then, $\overline{x}_A = \overline{x}_B$

$\dfrac{x+32}{6} = \dfrac{42}{7}$

$\dfrac{x+32}{6} = 6$

$x+32=36$ (cross multiply)

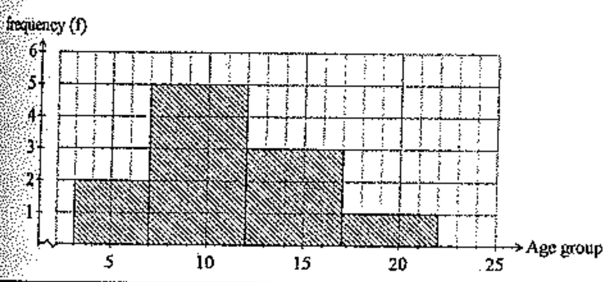
$x=36-32$

$\therefore x=4$

---

13. **[2013 P1 #13]**

Age group	0-2	3-7	8-12	13-17	18-22
Class boundary	0.5	7.5	12.5	17.5	22.5
frequency	0	2	5	3	1



---

14. **[2013 P2 #5a]**

$mean = \dfrac{Sum\ of\ numbers}{numbers}$

$= \dfrac{2+3+6+8+9+x}{6}$

$= \left(\dfrac{x+28}{6}\right)$

Median is the middle number

$= \dfrac{6+8}{2}$

$= \dfrac{14}{2}$

$=7$

Given that mean $=$ *median*

$\dfrac{x+28}{6} = 7$

$x+28=42$ (cross-multipying)

$x=42-28$

$x=14$

---

15. **[2014 PII #6a]**

marks	*mid-mark*	*f*	*fx*
1-10	$\frac{1+10}{2}=\frac{11}{2}=5.5$	4	$5.5\times4=22$
11-20	$\frac{11+20}{2}=\frac{31}{2}=15.5$	10	$15.5\times10=155$
21-30	$\frac{21+30}{2}=\frac{51}{2}=25.5$	6	$25.5\times6=153$
Sum		**20**	**330**

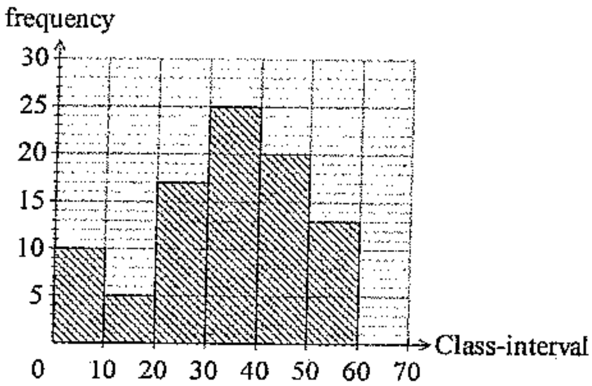
Mean mark$=\dfrac{\sum fx}{\sum f}$

$=\dfrac{330}{20}$

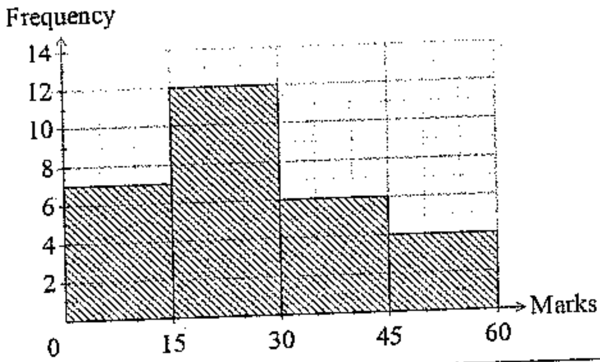
$=16.5$

---

16. **[2015 PI #19]**



17. **[2016 PII #5a]**



18. **[2017 PI #10]**

Total number of matches $= 17$

$3+6+x+2+2+2=17$

$15+x=17$

$x=17-15$

$x=2$

x	f	fx
0	3	$0\times3=0$
1	6	$1\times6=6$
2	2	$2\times2=4$
3	2	$3\times2=6$
4	2	$4\times2=8$
5	2	$5\times2=10$
Sum	**17**	**34**

Mean number of goals

$=\frac{\sum fx}{n}$

$=\frac{34}{17}$

$=2$

19. **[2018 P1 #10]**

Class interval	tally	frequency
0-9	//	2
10-19	~~////~~ ~~////~~ /	11
20-29	~~////~~ /	6
30-39	/	1
Sum		**20**

20. **[2019 PI #12]**

Nyimba+Kamiza+Chilema+Nyika=360°

$b+125°+90°+100°=360°$

$b+315°=360°$

$b=360°-315°$

$b=45°$

To find number of students who sleep in Nyimba

Number of students $=\frac{45°}{360°}\times288$

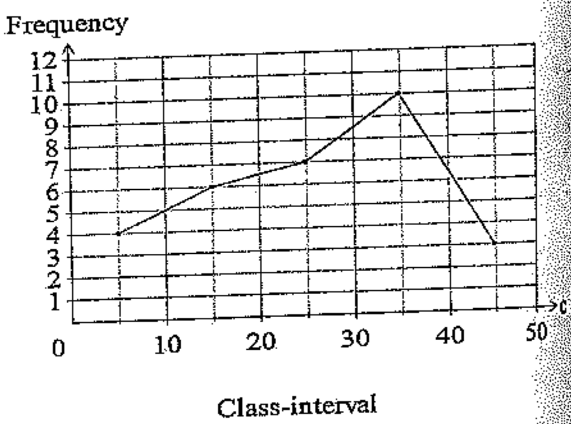
$=36$

$\therefore$ 36 students were sleeping in Nyimba hostel

21. **[2019 PII #10]**

Class interval	Tally	Frequency	Mid-class
1 – 10	////	4	11/2=5.5
11 – 20	~~////~~/	6	31/2=15.5
21 – 30	~~////~~//	7	51/2=25.5
31 – 40	~~////~~ ~~////~~	10	71/2=35.5
41 – 50	///	3	91/2=45.5

Note: Mid-class=(upper+lower intervals)/2



22. **[2020 Mock PI #10]**

$\frac{\sum z}{5}=7 \quad (i)$

Then two numbers added:

$\frac{\sum z+x+2x+1}{5+2}=9 \quad (ii)$

$\frac{\sum z+3x+1}{7}=9$

$\sum z+3x+1=7\times9$ (cross-multiplying)

$\sum z+3x+1=63 \quad (iii)$

But, from (i), $\sum z=35$ (cross-multiplying)

Substituting in
$35+3x+1=63$
$3x=63-36$
$3x=27$
$x=\frac{27}{3}$
$\therefore x=9$

CH 19 MATRICES

Chapter Highlights

A matrix is a rectangular array of numbers or expressions arranged in rows and columns.

The order of a matrix represents the number of row or columns it has. The rows are listed first and the column second. e.g. The matrix A is of 2×3 means it has 2 rows and 3 columns.

$$A = \begin{bmatrix} a & b & k \\ h & m & n \end{bmatrix}$$

When adding or subtracting two matrices make sure they have the same order. Make sure you add and subtract only for corresponding elements.

Multiplying matrices

1. Multiplying a matrix by a scalar

Multiply each and every element of matrix by the scalar. i.e.

$$k\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} kp & kq \\ kr & ks \end{bmatrix}$$

2. Multiplying matrix and a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

3. Multiplying matrix and matrix

Multiplying the two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

This is similar to multiplying a vector and a matrix only that there are two column vectors within the matrix, that is $\begin{bmatrix} e \\ g \end{bmatrix}$ and $\begin{bmatrix} f \\ h \end{bmatrix}$ whose operation must be carried out one column vector at a time, as illustrated in 2. above.

1. Given that $A = \begin{pmatrix} 3 & 0 \\ -4 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$ calculate AB. **[2003 PI #4]**

2. A and B are two matrices. If A=$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, find B given that $A^2 = A + B$. **[2003 PII #2b]**

3. Given that $\begin{pmatrix} 3c & c \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 28 \\ 22 \end{pmatrix}$ find c. **[2004 PI #8]**

4. Given that A=$\begin{pmatrix} 0 & 1 \\ 5 & 1 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$, find A + 2B. **[2004 PII #12a]**

5. Calculate the values of a and b if:

$$\begin{pmatrix} 3a & 18 \\ 4 & 6 \end{pmatrix} - \begin{pmatrix} 2a & 2 \\ 2 & -6 \end{pmatrix} = \begin{pmatrix} 4 & 16 \\ 2 & b \end{pmatrix}$$

[2005 PII #2b]

6. Find the values of x and y in the following matrix equation:

$$\begin{pmatrix} 6 & -3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \end{pmatrix}$$

[2006 P1 #11]

7. Blouses and shirts for school uniform were selling at K500 and K250 respectively. Phiri family bought two blouses and four shirts, while Mwale family bought three blouses and six shirts for their children.
   i. Present this information in two matrices.
   ii. Using matrix multiplication, calculate the amount of money each family spent on the clothes. **[2006 PII #5a]**

8. Given that A=$\begin{pmatrix} 2 & 6 \\ 1 & -4 \end{pmatrix}$, B=$\begin{pmatrix} -3 & 2 \\ 0 & -5 \end{pmatrix}$ and C= $\begin{pmatrix} -1 & 8 \\ -5 & 7 \end{pmatrix}$, simplify ¼ (A-B+C). **[2007 PI #9]**

9. Show that $\begin{pmatrix}4 & 0\\1 & 2\end{pmatrix}\begin{pmatrix}-2 & 3\\5 & -1\end{pmatrix}+\begin{pmatrix}8 & -12\\-8 & -1\end{pmatrix}$ is a zero matrix. **[2007 PII #8a]**

10. Given that matrix V=$\begin{pmatrix}3 & 5\\-1 & 1\end{pmatrix}$ and W=$\begin{pmatrix}-1 & 1\\0 & 2\end{pmatrix}$. Find **VW-V.** **[2008 P2 #3a]**

11. Given that $\begin{pmatrix}6 & 8\\2 & 3\end{pmatrix}\begin{pmatrix}w & -4\\-1 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, find the value of w. **[2008 P1 #11]**

12. Given that matrix P= $\begin{pmatrix}7 & 5\\2 & 4\end{pmatrix}$ and Q=$\begin{pmatrix}3 & 10\\5 & 1\end{pmatrix}$, find PQ. **[2010 P1 #7]**

13. Given the matrix $P=\begin{pmatrix}3 & 4\\1 & -2\end{pmatrix}, Q=\begin{pmatrix}1 & 3\\6 & 1\end{pmatrix}$ and $R=\begin{pmatrix}2 & 0\\-1 & 4\end{pmatrix}$. Find 3(Q – PR). **[2010 PII #3b]**

14. T and R are two matrices. Given that $T=\begin{pmatrix}2 & 1\\-1 & 3\end{pmatrix}$ and R =$\begin{pmatrix}0 & 3\\-1 & 1\end{pmatrix}$, find 3R-T^2. **[2011 PI #12]**

15. Given that A=$\begin{pmatrix}2 & 1\\-3 & 4\end{pmatrix}$ and B=$\begin{pmatrix}0 & -10\\8 & 14\end{pmatrix}$. Calculate $A + \frac{1}{2}B$. **[2011 PII #3a]**

16. Given that $M=\begin{pmatrix}3 & 0\\1 & -2\end{pmatrix}$, $N=\begin{pmatrix}-1 & 2\\0 & K\end{pmatrix}$ and $MN=\begin{pmatrix}-3 & 6\\-1 & -1\end{pmatrix}$, find the value of k. **[2012 P1 #10]**

17. Given that $M=\begin{pmatrix}a & d\\1 & 2\end{pmatrix}$ and N=$\begin{pmatrix}5 & 2\\1 & 3\end{pmatrix}$, find MN. **[2012 P2 #3a]**

18. Given that matrix M = $\begin{pmatrix}3 & 3\\-1 & -1\end{pmatrix}$ and $M^2 = kM$ where k is a constant, Calculate the value of k. **[2013 P1 #12]**

19. Given that $Y=\begin{pmatrix}-1 & 0\\2 & 1\end{pmatrix}$. Find the value Y^2-2Y. **[2013 P2 #11a]**

20. Find the matrix M which satisfies the equation $5M-2\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}=\begin{pmatrix}18 & 17\\29 & 11\end{pmatrix}$. **[2014 P1 #13]**

21. Given that $A=\begin{pmatrix}1 & 0\\3 & 2\end{pmatrix}$, $B=\begin{pmatrix}x & 0\\1 & 3\end{pmatrix}$ and AB= BA, find the value of x. **[2014 P1 #2b]**

22. Given that matrix $A=\begin{bmatrix}1 & 2\\-3 & 4\end{bmatrix}$, B=$\begin{bmatrix}2 & 3\\2 & 4\end{bmatrix}$,and C=$\begin{bmatrix}2 & 4\\2 & -3\end{bmatrix}$, find $A(B+C)$. **[2015 PI #6]**

23. Solve the matrix equation $\begin{pmatrix}2 & 1\\3 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\-2\end{pmatrix}$ **[2015 PII #11a]**

24. If $A=\begin{pmatrix}y & 2\\2 & -y\end{pmatrix}$ *and* $A^2=\begin{pmatrix}5 & 0\\0 & 5\end{pmatrix}$, calculate the values of y. **[2016 PI #5]**

25. If P=$\begin{pmatrix}2 & 4\\0 & -1\end{pmatrix}$ and Q=$\begin{pmatrix}-4 & 2\\3 & 1\end{pmatrix}$, calculate $PQ-\frac{1}{2}P$. **[2016 PII #2b]**

26. Given that **M**=$\begin{pmatrix}3 & 1\\6 & 0\end{pmatrix}$ and **N** =$\begin{pmatrix}5 & -7\\0 & 4\end{pmatrix}$, find $\frac{1}{2}(\mathbf{M}^2+\mathbf{N})$. **[2017 PI #8]**

27. John bought 3 lemons and 4 oranges while Mary bought 4 lemons and 2 oranges. Lemons cost K20 each while oranges cost K70 each.
   i. Represent this information in **two** matrices.
   ii. Using matrix multiplication, find the amount of money spent by each person.

[2017 PII #7b]

28. Given that $5\begin{pmatrix} 1 & 4 \\ 3-n & 8 \end{pmatrix} = \begin{pmatrix} 5 & 20 \\ -15 & 40 \end{pmatrix}$, calculate the value of n. **[2018 PI #13]**

29. Given that ***A***, ***B*** and ***M*** are matrices such that $B = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 6 \\ -1 & 3 \end{pmatrix}$ and $2M = B + C$.
Calculate matrix ***M.*** **[2018 PII #19]**

30. Given that matrix K$=\begin{pmatrix} 2 & 4 \\ 3 & 0 \end{pmatrix}$ and matrix **M**$=$ $\begin{pmatrix} 1 & -2 \\ 6 & 4 \end{pmatrix}$, find the matrix $\mathbf{K} - \mathbf{M}^2$. **[2019 PI #10]**

1. **[2003 PI #4]**

$$A=\begin{pmatrix}3 & 0\\ -4 & 4\end{pmatrix}, B=\begin{pmatrix}2 & -1\\ -1 & 0\end{pmatrix}$$

$$AB=\begin{pmatrix}3 & 0\\ -4 & 4\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 0\end{pmatrix}$$

$$\begin{pmatrix}3\times2+0\times-1 & 3\times-1+0\times0\\ -4\times2+4\times-1 & -4\times-1+4\times0\end{pmatrix}$$

$$\begin{pmatrix}6+0 & -3+0\\ -8+-4 & 4+0\end{pmatrix}$$

$$\begin{pmatrix}6 & -3\\ -12 & 4\end{pmatrix}$$

2. **[2003 PII #2b]**

$$A^2=A+B$$

$$\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}^2=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}+B$$

$$\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}+B$$

$$\begin{pmatrix}1\times1+2\times4 & 1\times2+2\times4\\ 4\times1+3\times4 & 4\times2+3\times3\end{pmatrix}=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}+B$$

$$\begin{pmatrix}1+8 & 2+6\\ 4+12 & 8+9\end{pmatrix}=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}+B$$

$$\begin{pmatrix}9 & 8\\ 16 & 17\end{pmatrix}-\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}=B$$

$$\begin{pmatrix}9-1 & 8-2\\ 16-4 & 17-3\end{pmatrix}=B$$

$$\begin{pmatrix}8 & 6\\ 12 & 14\end{pmatrix}=B$$

3. **[2004 PI #8]**

$$\begin{pmatrix}3c\times4+c\times2\\ 5\times4+1\times2\end{pmatrix}=\begin{pmatrix}28\\ 22\end{pmatrix}$$

$$\begin{pmatrix}12c+2c\\ 20+2\end{pmatrix}=\begin{pmatrix}28\\ 22\end{pmatrix}$$

$$\begin{pmatrix}14c\\ 22\end{pmatrix}=\begin{pmatrix}28\\ 22\end{pmatrix}$$

$$14c=28$$

$$\frac{14c}{14}=\frac{28}{14}$$

$$\therefore c=2$$

4. **[2004 PII #12a]**

$$A+2B$$

$$=\begin{pmatrix}0 & 1\\ 5 & 6\end{pmatrix}+2\begin{pmatrix}2 & 3\\ -1 & 4\end{pmatrix}$$

$$=\begin{pmatrix}0 & 1\\ 5 & 6\end{pmatrix}+\begin{pmatrix}2\times2 & 2\times3\\ 2\times(-1) & 2\times4\end{pmatrix}$$

$$=\begin{pmatrix}0 & 1\\ 5 & 6\end{pmatrix}+\begin{pmatrix}4 & 6\\ -2 & 8\end{pmatrix}$$

$$=\begin{pmatrix}0+4 & 1+6\\ 5+(-2) & 6+8\end{pmatrix}$$

$$=\begin{pmatrix}4 & 7\\ 3 & 14\end{pmatrix}$$

5. **[2005 PII #2b]**

$$\begin{pmatrix}3a & 18\\ 4 & 6\end{pmatrix}-\begin{pmatrix}2a & 2\\ 2 & -6\end{pmatrix}=\begin{pmatrix}4 & 16\\ 2 & b\end{pmatrix}$$

$$\begin{pmatrix}3a-2a & 18-2\\ 4-2 & 6-(-6)\end{pmatrix}=\begin{pmatrix}4 & 16\\ 2 & b\end{pmatrix}$$

$$\begin{pmatrix}a & 16\\ 2 & 12\end{pmatrix}=\begin{pmatrix}4 & 16\\ 2 & b\end{pmatrix}$$

$\therefore a=4$ and $b=12$ (equating corresponding terms)

6. **[2006 P1 #11]**

$$\begin{aligned}&6x-3y=12\\ +\;&2x+3y=-4\end{aligned}$$

$$\frac{8x}{8}=\frac{8}{8}$$

$$x=1$$

Substitute in (i)

$$6(1)-3y=12$$

$$6-3y=12$$

$$-3y=12-6$$

$$\frac{-3y}{-3}=\frac{6}{-3}$$

$$y=-2$$

7. **[2006 PII #5a]**

i.Presenting two matrices

Family	Blouse	Skirt
Phiri	2	4
Mwale	3	1

Matrix of quantities:

$$\text{Quantity Matrix}=\begin{pmatrix}2 & 4\\ 3 & 1\end{pmatrix}$$

Matrix of costs:

$$\text{Cost Matrix}=\begin{bmatrix}500\\ 250\end{bmatrix}$$

ii. amount of money each family spent on clothes:

$$\begin{array}{l}\textit{phiri}\\ \textit{mwale}\end{array}\begin{pmatrix}2 & 4\\ 3 & 1\end{pmatrix}\begin{pmatrix}500\\ 250\end{pmatrix}$$

$$\begin{pmatrix}2\times 500+4\times 250\\ 3\times 500+6\times 250\end{pmatrix}=\begin{pmatrix}1000+1000\\ 1500+1500\end{pmatrix}=\begin{pmatrix}2000\\ 3000\end{pmatrix}$$

∴the phiri family spent K2000 and the mwale family spent K3000 on clothes.

8. **[2007 PI #9]**

Step 1: find A - B

Step 2: A - B + C

Step 3: $\frac{1}{4}(A-B+C)$

$$A-B=\begin{pmatrix}2 & 6\\ 1 & -4\end{pmatrix}-\begin{pmatrix}-3 & 2\\ 0 & -5\end{pmatrix}$$

$$=\begin{pmatrix}2 & 6\\ 1 & -4\end{pmatrix}+\begin{pmatrix}3 & -2\\ 0 & 5\end{pmatrix}$$ Removing negatives

$$=\begin{pmatrix}2+3 & 3-2\\ 1+0 & -4+5\end{pmatrix}$$ adding terms

$$=\begin{pmatrix}5 & 4\\ 1 & 1\end{pmatrix}$$

$$A-B+C$$

$$=\begin{pmatrix}5 & 4\\ 1 & 1\end{pmatrix}+\begin{pmatrix}-1 & 8\\ -5 & 7\end{pmatrix}$$

$$=\begin{pmatrix}5-1 & 4+8\\ 1-5 & 1+7\end{pmatrix}$$

$$=\begin{pmatrix}4 & 12\\ -4 & 8\end{pmatrix}$$

So, $\frac{1}{4}(A-B+C)$

$$=\tfrac{1}{4}\begin{pmatrix}4 & 12\\ -4 & 8\end{pmatrix}$$

$$=\begin{pmatrix}\tfrac{1}{4}\times 4 & \tfrac{1}{4}\times 12\\ \tfrac{1}{4}\times -4 & \tfrac{1}{4}\times 8\end{pmatrix}$$

$$=\begin{pmatrix}1 & 3\\ -1 & 2\end{pmatrix}$$

9. **[2007 PII #8a]**

$$\begin{pmatrix}4 & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}-2 & 3\\ 5 & -1\end{pmatrix}+\begin{pmatrix}8 & -12\\ -8 & -1\end{pmatrix}$$

$$=\begin{pmatrix}4\times(-2)+0\times 5 & 4\times 3+0\times(-1)\\ 1\times(-2)+2\times 5 & 1\times 3+2\times(-1)\end{pmatrix}+\begin{pmatrix}8 & -12\\ -8 & -1\end{pmatrix}$$

$$=\begin{pmatrix}-8+0 & 12+0\\ -2+10 & 3-2\end{pmatrix}+\begin{pmatrix}8 & -12\\ -8 & -1\end{pmatrix}$$

$$=\begin{pmatrix}-8 & 12\\ 8 & 1\end{pmatrix}+\begin{pmatrix}8 & -12\\ -8 & -1\end{pmatrix}$$

$$=\begin{pmatrix}-8+8 & 12+(-12)\\ 8+(-8) & 1+(-1)\end{pmatrix}$$

$$=\begin{pmatrix}-8+8 & 12-12\\ 8-8 & 1-1\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

∴ indeed a zero matrix.

10. **[2008 P2 #3a]**

Step 1: VW

Step 2: $VW-V$

$$VW-V=\begin{pmatrix}3 & 5\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 0 & 2\end{pmatrix}-\begin{pmatrix}3 & 5\\ -1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}3\times(-1)+5\times 0 & 3\times 1+5\times 2\\ -1\times(-1)+1 & -1\times 1+1\times 2\end{pmatrix}-\begin{pmatrix}3 & 5\\ -1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-3+0 & 3+10\\ 1+0 & -1+2\end{pmatrix}-\begin{pmatrix}3 & 5\\ -1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 13\\ 1 & 1\end{pmatrix}-\begin{pmatrix}3 & 5\\ -1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-3-3 & 13-5\\ 1-(-1) & 1-1\end{pmatrix}$$

$$=\begin{pmatrix}-6 & 8\\ 2 & 0\end{pmatrix}$$

11. **[2008 P1 #11]**

$$\begin{pmatrix}6 & 8\\ 2 & 3\end{pmatrix}\begin{pmatrix}w & -4\\ -1 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}6\times w+8\times(-1) & 6\times(-4)+8\times 3\\ 2\times w+3\times(-1) & 2\times(-4)+3\times 3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}6w-8 & -24+24\\ 2w-3 & -8+9\end{pmatrix}$$

$$\begin{pmatrix} 6w-8 & 0 \\ 2w-3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$2w-3=0$ (equating corresponding terms)

$2w=3$

$w=\frac{3}{2}$

$\therefore w=1.5$

12. **[2010 P1 #7]**

Given $P=\begin{pmatrix} 7 & 5 \\ 2 & 4 \end{pmatrix}$ and $Q=\begin{pmatrix} 3 & 10 \\ 5 & 1 \end{pmatrix}$

$$PQ=\begin{pmatrix} 7 & 5 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 10 \\ 5 & 1 \end{pmatrix}$$

$$PQ=\begin{pmatrix} 7\times3+5\times5 & 7\times10+5\times1 \\ 2\times3+4\times5 & 2\times10+4\times1 \end{pmatrix}$$

$$PQ=\begin{pmatrix} 21+25 & 70+5 \\ 6+20 & 20+4 \end{pmatrix}$$

$$PQ=\begin{pmatrix} 46 & 75 \\ 26 & 24 \end{pmatrix}$$

13. **[2010 PII #3b]**

Step 1: PR

Step 2: $Q-PR$

Step 3: $3(Q-PR)$

$$So, PR=\begin{pmatrix} 3 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix}$$

$$PR=\begin{pmatrix} 3\times2+4\times-1 & 3\times0+4\times4 \\ 1\times2+-2\times-1 & 1\times0+-2\times4 \end{pmatrix}$$

$$PR=\begin{pmatrix} 6+-4 & 0+14 \\ 2+2 & 0+-8 \end{pmatrix}$$

$$PR=\begin{pmatrix} 2 & 14 \\ 4 & -8 \end{pmatrix}$$

$$Q-PR=\begin{pmatrix} 1 & 3 \\ 6 & 1 \end{pmatrix}-\begin{pmatrix} 2 & 14 \\ 4 & -8 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 3 \\ 6 & 1 \end{pmatrix}+\begin{pmatrix} -2 & -14 \\ -4 & +8 \end{pmatrix}$$

$$=\begin{pmatrix} 1-2 & 3-14 \\ 6-4 & 1+8 \end{pmatrix}$$

$$=\begin{pmatrix} -1 & -11 \\ 2 & 9 \end{pmatrix}$$

$$3(Q-PR)=3\begin{pmatrix} -1 & -11 \\ 2 & 9 \end{pmatrix}$$

$$=\begin{pmatrix} 3\times-1 & 3\times-11 \\ 3\times2 & 3\times9 \end{pmatrix}$$

$$=\begin{pmatrix} -3 & -33 \\ 6 & 27 \end{pmatrix}$$

14. **[2011 PI #12]**

Step 1: 3R

Step 2: T^2

Step 3: $3R-T^2$

$$3R=3\begin{bmatrix} 0 & 3 \\ -1 & 1 \end{bmatrix}=\begin{bmatrix} 0\times3 & 3\times3 \\ -1\times3 & 1\times3 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 9 \\ -3 & 3 \end{bmatrix}$$

$$T^2=\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 2\times2+1\times-1 & 2\times1+1\times3 \\ -1\times2+3\times-1 & -1\times1+3\times3 \end{bmatrix}$$

$$=\begin{bmatrix} 4-1 & 2+3 \\ -2-3 & -1+9 \end{bmatrix}$$

$$=\begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$$

$$3R-T^2=\begin{bmatrix} 0 & 9 \\ -3 & 3 \end{bmatrix}-\begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$$

$$=\begin{bmatrix} 0-3 & 9-5 \\ -1--5 & 3-8 \end{bmatrix}$$

$$=\begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$$

15. **[2011 PII #3a]**

Step 1: $\frac{1}{2}B$

Step 2: $A-\frac{1}{2}B$

$$\tfrac{1}{2}B=\tfrac{1}{2}\begin{bmatrix} 0 & -10 \\ 8 & 14 \end{bmatrix}$$

$$=\begin{bmatrix} 0\times\frac{1}{2} & -10\times\frac{1}{2} \\ 8\times\frac{1}{2} & 14\times\frac{1}{2} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -5 \\ 4 & 7 \end{bmatrix}$$

$$A+\tfrac{1}{2}B=\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}+\begin{bmatrix} 0 & -5 \\ 4 & 7 \end{bmatrix}$$

$$=\begin{bmatrix} 2+0 & 1+-5 \\ -3+4 & 4+7 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -4 \\ 1 & 11 \end{bmatrix}$$

16. **[2012 P1 #10]**

$$MN=\begin{pmatrix}3\times(-1)+0\times 0 & 3\times 2+0\times K\\ 1\times(-1)+(-2)\times 0 & 1\times 2+(-2)\times K\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 6\\ -1 & 2-2k\end{pmatrix}$$

But; $\begin{pmatrix}-3 & 6\\ -1 & 2-2k\end{pmatrix}=\begin{pmatrix}-3 & 6\\ -1 & -1\end{pmatrix}$

$\therefore 2-2K=-1$ (equating corresponding terms)

$\therefore 2K=2+1$

$K=\dfrac{3}{2}$

17. **[2012 P2 #3a]**

$$\text{MN}=\begin{pmatrix}a & d\\ 1 & 2\end{pmatrix}\begin{pmatrix}5 & 2\\ 1 & 3\end{pmatrix}$$

$$=\begin{pmatrix}a\times 5+d\times 1 & a\times 2+d\times 3\\ 1\times 5+2\times 1 & 1\times 2+2\times 3\end{pmatrix}$$

$$\text{MN}=\begin{pmatrix}5a+d & 2a+3d\\ 7 & 8\end{pmatrix}$$

18. **[2013 P1 #12]**

$M^2=KM$

$$=\begin{pmatrix}3 & 3\\ -1 & -1\end{pmatrix}^2=k\begin{pmatrix}3 & 3\\ -1 & -1\end{pmatrix}$$

$$=\begin{pmatrix}3 & 3\\ -1 & -1\end{pmatrix}\begin{pmatrix}3 & 3\\ -1 & -1\end{pmatrix}=k\begin{pmatrix}3 & 3\\ -1 & -1\end{pmatrix}$$

$$=\begin{pmatrix}3\times 3+3\times -1 & 3\times 3+3\times -1\\ -1\times 3+-1\times -1 & -1\times 3+-1\times -1\end{pmatrix}=k\begin{pmatrix}3 & 3\\ -1 & -1\end{pmatrix}$$

$$\begin{pmatrix}9-3 & 9-3\\ -3+1 & -3+1\end{pmatrix}=k\begin{pmatrix}3 & 3\\ -1 & -1\end{pmatrix}$$

$$\begin{pmatrix}6 & 6\\ -2 & -2\end{pmatrix}=k\begin{pmatrix}3 & 3\\ -1 & -1\end{pmatrix}$$

$$\begin{pmatrix}6 & 6\\ -2 & -2\end{pmatrix}=\begin{pmatrix}3k & 3k\\ -k & -k\end{pmatrix}$$

$-2=-k$ (equating corresponding terms)

$\therefore k=2$

19. **[2013 P2 #11a]**

Step 1: Y^2

Step 2: $2Y$

Step 3: Y^2-2Y

$Y^2=(Y)(Y)$

$$=\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-1\times -1+0\times 2 & -1\times 0+\times 0\times 1\\ 2\times -1+1\times 2 & 2\times 0+1\times 1\end{pmatrix}$$

$$=\begin{pmatrix}1+0 & 0+0\\ -2+2 & 0+1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$2Y=2\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2\times -1 & 2\times 0\\ 2\times 2 & 2\times 1\end{pmatrix}$$

$$2Y=\begin{pmatrix}-2 & 0\\ 4 & 2\end{pmatrix}$$

Y^2-2Y

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}-2 & 0\\ 4 & 2\end{pmatrix}$$

$$=\begin{pmatrix}1--2 & 0-0\\ 0-4 & 1-2\end{pmatrix}$$

$$=\begin{pmatrix}3 & 0\\ -4 & -1\end{pmatrix}$$

$$\therefore Y^2-2Y=\begin{pmatrix}3 & 0\\ -4 & -1\end{pmatrix}$$

20. **[2014 P1 #13]**

$$5M-2\begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}=\begin{pmatrix}18 & 17\\ 29 & 11\end{pmatrix}$$

$$5M-\begin{pmatrix}2\times 1 & 2\times 4\\ 2\times 3 & 2\times 2\end{pmatrix}=\begin{pmatrix}18 & 17\\ 29 & 11\end{pmatrix}$$

$$5M-\begin{pmatrix}2 & 8\\ 6 & 4\end{pmatrix}=\begin{pmatrix}18 & 17\\ 29 & 11\end{pmatrix}$$

$$5M=\begin{pmatrix}18 & 17\\ 29 & 11\end{pmatrix}+\begin{pmatrix}2 & 8\\ 6 & 4\end{pmatrix}$$

$$5M=\begin{pmatrix}20 & 25\\ 35 & 15\end{pmatrix}$$

$$M=\frac{1}{5}\begin{pmatrix}20 & 25\\ 35 & 15\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{5}\times 20 & \frac{1}{5}\times 25\\ \frac{1}{5}\times 35 & \frac{1}{5}\times 15\end{pmatrix}$$

$$M=\begin{pmatrix}4 & 5\\ 7 & 3\end{pmatrix}$$

21. **[2014 P1 #2b]**

Step1: AB

Step 2: BA

Step 3: AB=BA to find x.

$$AB=\begin{pmatrix}1&0\\3&2\end{pmatrix}\begin{pmatrix}x&0\\1&3\end{pmatrix}$$

$$=\begin{pmatrix}1\times x+0\times1 & 1\times0+0\times3\\3\times x+2\times1 & 3\times0+2\times3\end{pmatrix}$$

$$=\begin{pmatrix}x+0 & 0+0\\3x+2 & 0+6\end{pmatrix}$$

$$=\begin{pmatrix}x & 0\\3x+2 & 6\end{pmatrix}$$

$$BA=\begin{pmatrix}x&0\\1&3\end{pmatrix}\begin{pmatrix}1&0\\3&2\end{pmatrix}$$

$$=\begin{pmatrix}x\times1+0\times3 & x\times0+0\times2\\1\times1+3\times3 & 1\times0+3\times2\end{pmatrix}$$

$$=\begin{pmatrix}x+0 & 0+0\\1+9 & 0+6\end{pmatrix}$$

$$=\begin{pmatrix}x & 0\\10 & 6\end{pmatrix}$$

So AB=BA,

$$\begin{pmatrix}x & 0\\3x+2 & 6\end{pmatrix}=\begin{pmatrix}x & 0\\10 & 6\end{pmatrix}$$

equating corresponding terms

$\therefore 3x+2=10$

$3x=10-2$

$\frac{3x}{3}=\frac{8}{3}$

$x=2\frac{2}{3}$

22. **[2015 PI #6]**

Step 1: B+C

Step 2: A[B+C]

$B+C$

$$=\begin{bmatrix}2&3\\2&4\end{bmatrix}+\begin{bmatrix}2&4\\2&-3\end{bmatrix}$$

$$=\begin{bmatrix}2+2 & 3+4\\2+2 & 4+-3\end{bmatrix}$$

$$=\begin{bmatrix}4&7\\4&1\end{bmatrix}$$

$A[B+C]$

$$=\begin{bmatrix}1&2\\-3&4\end{bmatrix}\begin{bmatrix}4&7\\4&1\end{bmatrix}$$

$$=\begin{bmatrix}1\times4+2\times4 & 1\times7+2\times1\\-3\times4+4\times4 & -3\times7+4\times1\end{bmatrix}$$

$$=\begin{bmatrix}4+8 & 7+2\\-12+16 & -21+4\end{bmatrix}$$

$$=\begin{bmatrix}12 & 9\\4 & -17\end{bmatrix}$$

23. **[2015 PII #11a]**

$$\begin{pmatrix}2x+y\\3x-2y\end{pmatrix}=\begin{pmatrix}8\\-2\end{pmatrix}$$

$2x+y=8$..............(i)

$3x-2y=-2$..............(ii)

making y in (i) subject:

$y=8-2x$.................(iii)

substituting in equation (ii)

$3x-2(8-2x)=-2$

$3x-16+4x=-2$

$3x+4x=-2+16$

$7x=14$

$\frac{7x}{7}=\frac{14}{7}$

$x=2$

Substitute in equation (iii)

$y=8-2(2)$

$y=8-4$

$\therefore y=4$

24. **[2016 PI #5]**

Step 1: $A\times A$

Step 2: $A^2=A\times A$

$A\times A$

$$=\begin{pmatrix}y&2\\2&-y\end{pmatrix}\begin{pmatrix}y&2\\2&-y\end{pmatrix}$$

$$=\begin{pmatrix} y\times y+2\times 2 & y\times 2+2\times -y \\ 2\times y+-y\times 2 & 2\times 2+-y\times -y \end{pmatrix}$$

$$=\begin{pmatrix} y^2+4 & 0 \\ 0 & y^2+4 \end{pmatrix}=\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$y^2+4=5$

$y^2=5-4$

$y^2=1$

$y=\pm\sqrt{1}$

$y=\pm 1$

$y=1$ or $y=-1$

25. **[2016 PII 2b]**

Step 1: PQ

Step 2: $\frac{1}{2}P$

Step 3: PQ-$\frac{1}{2}P$

$PQ=$

$$=\begin{pmatrix} 2 & 4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -4 & 2 \\ 3 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 2\times -4+4\times 3 & 2\times 2+4\times 1 \\ 0\times -4+-1\times 3 & 0\times 2+-1\times 1 \end{pmatrix}$$

$$=\begin{pmatrix} -8+12 & 4+4 \\ 0-3 & 0-1 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 8 \\ -3 & -1 \end{pmatrix}$$

$\frac{1}{2}P$

$$=\frac{1}{2}\begin{pmatrix} 2 & 4 \\ 0 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} 0.5\times 2 & 0.5\times 4 \\ 0.5\times 0 & 0.5\times -1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 2 \\ 0 & -0.5 \end{pmatrix}$$

$PQ-\frac{1}{2}P$

$$=\begin{pmatrix} 4 & 8 \\ -3 & -1 \end{pmatrix}-\begin{pmatrix} 1 & 2 \\ 0 & -0.5 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 8 \\ -3 & -1 \end{pmatrix}+\begin{pmatrix} -1 & -2 \\ -0 & +0.5 \end{pmatrix}$$

$$=\begin{pmatrix} 4-1 & 8-2 \\ -3-0 & -1+0.5 \end{pmatrix}$$

$$PQ-\tfrac{1}{2}P=\begin{pmatrix} 3 & 6 \\ -3 & -0.5 \end{pmatrix}$$

26. **[2017 PI #15]**

Step 1: M^2

Step 2: M^2+N

Step 3: $\frac{1}{2}(M^2+N)$

$$M^2=\begin{pmatrix} 3 & 1 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 6 & 0 \end{pmatrix}$$

$$M^2=\begin{pmatrix} 3\times 3+1\times 6 & 3\times 1+1\times 0 \\ 6\times 3+0\times 6 & 6\times 1+0\times 0 \end{pmatrix}$$

$$M^2=\begin{pmatrix} 9+6 & 3+0 \\ 18+0 & 6+0 \end{pmatrix}$$

$$M^2=\begin{pmatrix} 15 & 3 \\ 18 & 6 \end{pmatrix}$$

$$\mathbf{M}^2+\mathbf{N}=\begin{pmatrix} 15 & 3 \\ 18 & 6 \end{pmatrix}+\begin{pmatrix} 5 & -7 \\ 0 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 15+5 & 3+-7 \\ 18+0 & 6+4 \end{pmatrix}$$

$$=\begin{pmatrix} 20 & -4 \\ 18 & 10 \end{pmatrix}$$

$$\therefore \frac{1}{2}(\mathbf{M}^2+\mathbf{N})=\frac{1}{2}\begin{pmatrix} 20 & -4 \\ 18 & 10 \end{pmatrix}$$

$$=\begin{pmatrix} 20\times\frac{1}{2} & -4\times\frac{1}{2} \\ 18\times\frac{1}{2} & 10\times\frac{1}{2} \end{pmatrix}$$

$$=\begin{pmatrix} 10 & -2 \\ 9 & 5 \end{pmatrix}$$

$$\therefore \frac{1}{2}(\mathbf{M}^2+\mathbf{N})=\begin{pmatrix} 10 & -2 \\ 9 & 5 \end{pmatrix}$$

27. **[2017 PII #7b]**

	Lemon	Orange
John	3	4
Mary	4	2

Quantity matrix$=\begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}$

Cost matrix$=\begin{pmatrix} 20 \\ 70 \end{pmatrix}$

(i) *John* and Mary$=\begin{pmatrix}3 & 4\\4 & 2\end{pmatrix}\begin{pmatrix}20\\70\end{pmatrix}$

(ii) To find the amount of money, multiply the matrices

$$\begin{pmatrix}3 & 4\\4 & 2\end{pmatrix}\begin{pmatrix}20\\70\end{pmatrix}=\begin{pmatrix}3\times20+4\times70\\4\times20+2\times70\end{pmatrix}$$

$$=\begin{pmatrix}60+280\\80+140\end{pmatrix}$$

$$=\begin{pmatrix}340\\220\end{pmatrix}$$

$\therefore$ John spent K340.00 and Mary spent K220.00

28. **[2018 P1 #13]**

$$5\begin{pmatrix}1 & 4\\3-n & 8\end{pmatrix}=\begin{pmatrix}5 & 20\\-15 & 40\end{pmatrix}$$

$$\begin{pmatrix}5 & 20\\5(3-n) & 40\end{pmatrix}=\begin{pmatrix}5 & 20\\-15 & 40\end{pmatrix}$$

$$\begin{pmatrix}5 & 20\\15-5n & 40\end{pmatrix}=\begin{pmatrix}5 & 20\\-15 & 40\end{pmatrix}$$

$$15-5n=-15$$

$$-5n=-15-15$$

$$\frac{\cancel{-5}^{1}n}{\cancel{-5}_{1}}=\frac{\cancel{-30}^{6}}{\cancel{-5}_{1}}$$

$$n=6$$

29. **[2018 PII #19]**

$$B=\begin{pmatrix}1 & 2\\-3 & -1\end{pmatrix}\quad C=\begin{pmatrix}2 & 6\\-1 & 3\end{pmatrix}$$

$$2M=B+C$$

$$2M=\begin{pmatrix}1 & 2\\-3 & -1\end{pmatrix}+\begin{pmatrix}2 & 6\\-1 & 3\end{pmatrix}$$

$$2M=\begin{pmatrix}3 & 8\\-4 & 2\end{pmatrix}\text{ (multiply by half both sides)}$$

$$M=\frac{1}{2}\begin{pmatrix}3 & 8\\-4 & 2\end{pmatrix}$$

$$M=\begin{pmatrix}1.5 & 4\\-2 & 1\end{pmatrix}$$

30. **[2019 PI #10]**

$$K=\begin{pmatrix}2 & 4\\3 & 0\end{pmatrix}\text{ and }M=\begin{pmatrix}1 & -1\\6 & 4\end{pmatrix}$$

$$K-M^2$$

$$M^2=\begin{pmatrix}1 & -1\\6 & 4\end{pmatrix}\begin{pmatrix}1 & -1\\6 & 4\end{pmatrix}$$

$$M^2=\begin{pmatrix}1\times1+-1\times6 & 1\times-1+-1\times4\\6\times1+4\times6 & 6\times-1+4\times4\end{pmatrix}$$

$$M^2=\begin{pmatrix}1-6 & -1-4\\6+24 & -6+16\end{pmatrix}$$

$$M^2=\begin{pmatrix}-5 & -5\\30 & 10\end{pmatrix}$$

$$\therefore K-M^2=\begin{pmatrix}2 & 4\\3 & 0\end{pmatrix}-\begin{pmatrix}-5 & -5\\30 & 10\end{pmatrix}$$

$$=\begin{pmatrix}2--5 & 4--5\\3-30 & 0-10\end{pmatrix}$$

$$=\begin{pmatrix}2+5 & 4+5\\3-30 & 0\text{-}10\end{pmatrix}$$

$$=\begin{pmatrix}7 & 9\\-27 & -10\end{pmatrix}$$

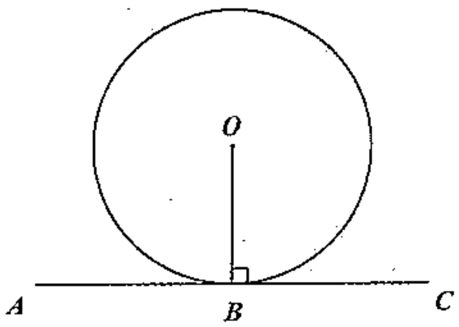
CH 20A
CIRCLE GEOMETRY III
(TANGENTS TO CIRCLES)

Chapter Highlights

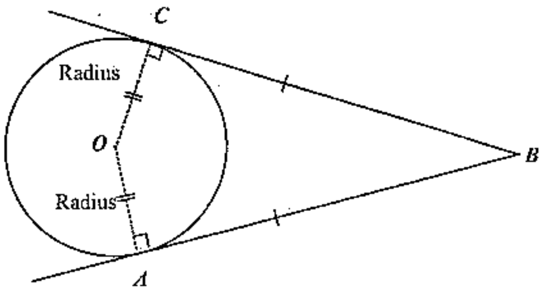
A tangent is line which has one and only one point in contact with a circle. This chapter, we solve problems requiring properties of tangents to circles.

Theorems for tangents to circles

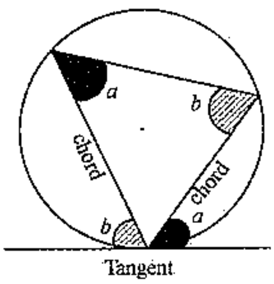
- A tangent to a circle is perpendicular to the radius/diameter



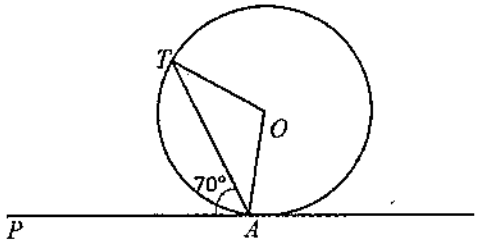
- Tangents from the same external point are equal.



- Angles in alternate segments – The angle between a tangent and a chord is equal to the angle the chord subtends in the alternate segment of the circle.

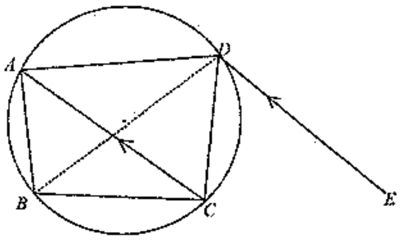


1. A circle center **O** has a tangent **PA** at a point **A. AT** is a chord such that angle **TAP** is acute.



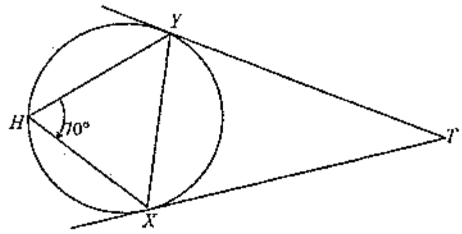
If angle **TAP** =70°, calculate the value of angle **OTA.** **[2003 PI #16]**

2. In **Figure 2, ABCD** is a circle and **DE** is a tangent to the circle at **D. AC** is parallel to the tangent **DE**



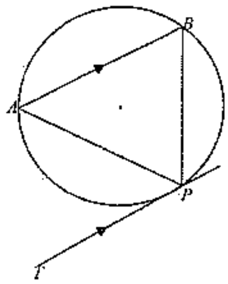
Prove that:
i. Triangle **ADC** is isosceles;
ii. Angle **ABC** is twice angle **DAC.**
[2003 PP II #5a]

3. In **Figure 3, TX** and **TY** are tangents to the circle **XHY** at **X** and **Y**



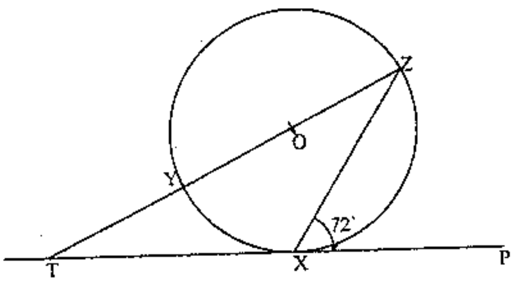
If angle **XHY**=70°, calculate angle **XTY.**
[2004 PP1 #17]

4. **Figure 4, TP** is a tangent to a circle **APB** at **P** and **AB** is parallel to **PT**.



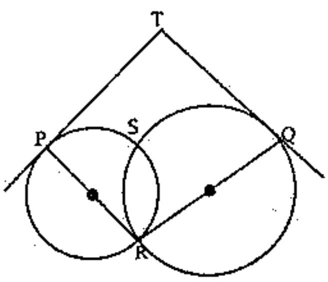
Prove that **AP=BP.** **[2005 P1 #3]**

5. In **Figure 5, XYZ** is a circle with center **O. TXP** is a tangent to the circle at **X**. The diameter **ZY** produced meets the tangent at **T.**



If angle **ZXP=72°** calculate the value of **XTY.**
[2005 PII #4a]

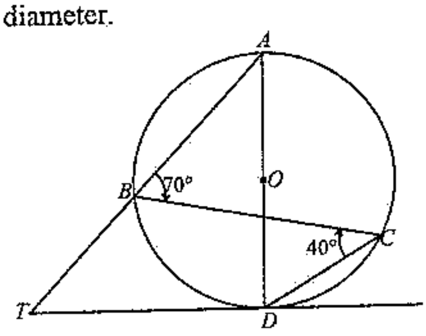
6. **Figure 6** shows two unequal circles **PRS** and **QRS** intersecting **R** and **S. TP** and **TQ** are tangents to the circle at **P** and **Q** respectively. **PR** and **QR** are diameters of the circles.



Prove that;

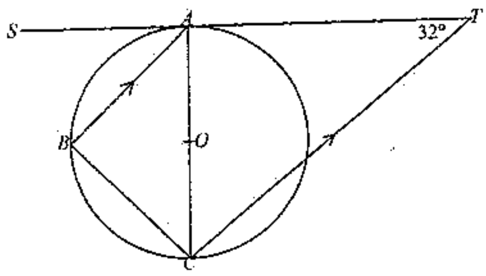
i. **PRQT** is a cyclic quadrilateral
ii. Angle **PRS** = angle **QRT.** **[2005 PII #11a]**

7. **Figure 7** is a circle **ABDC** center **O. TD** is a tangent, **TBA** is a straight line and **AD** is a diameter.



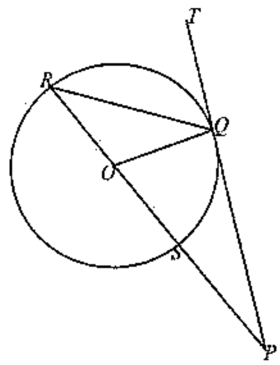
If angle **ABC** = 70° and angle **BCD**= 40°, calculate:
a. Angle **BDC;**
b. Angle **ATD.** **[2006 PII #6]**

8. **Figure 8** shows a tangent to a circle **ABC** with center **O.** line **CT** is parallel to **BA** and angle **ATC**=32°



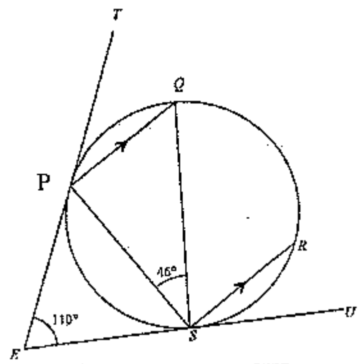
Calculate angle **ACB.** **[2007 PI #6]**

9. **Figure 9** shows a circle QRS with center O. PQT is a tangent to the circle at Q and PSOR is a straight line.



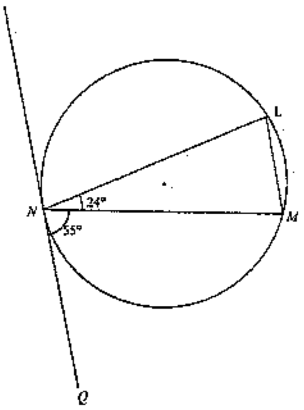
If angle RQO=28°. Calculate angle RPQ.
[2007 PII #5a]

10. In **Figure 10, ET** and **EU** are tangents to the circle **PQRS at P and S** respectively. Angle **TEU** =110° , angle **PSQ** =46° and **PQ** is parallel to **SR.**



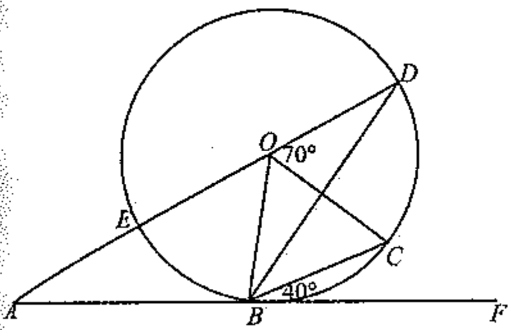
Calculate the value of angle **RSU.**
[2008 P2 #2b]

11. In **Figure 11**, **QN** is a tangent to the circle **LMN** at **N.**



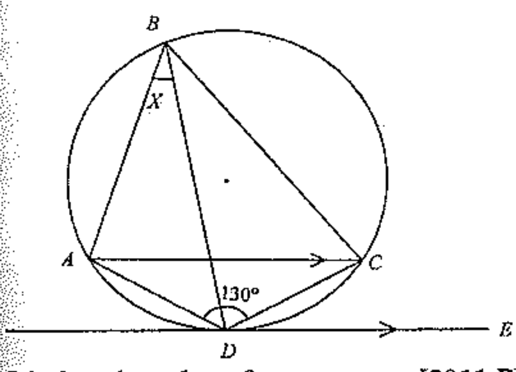
If angle QNM= $55°$ and angle **LNM** $= 24°$, calculate angle **NML.** **[2008 P1 #5]**

12. **Figure 12** shows a circle BCDE with center O. ABF is a tangent to the circle at B, angle CBF =40°, angle DOC= 70° and AEOD is a straight line.



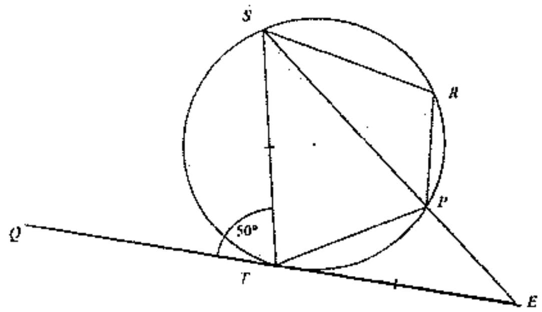
Calculate angle **OAB.** **[2010 PII #6b]**

13. **Figure 13** shows a tangent **DE** to a circle **ABCD** at **D. AC** is parallel to **DE** and angle **ADC**=$130°$



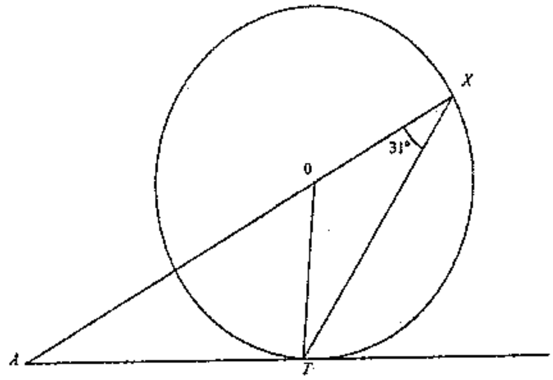
Calculate the value of x. **[2011 PI #15]**

14. **Figure 14** shows a tangent **QTE** to a circle **TSRP** at **T**. Angle **STQ=50°** and **SPE** is a straight line.



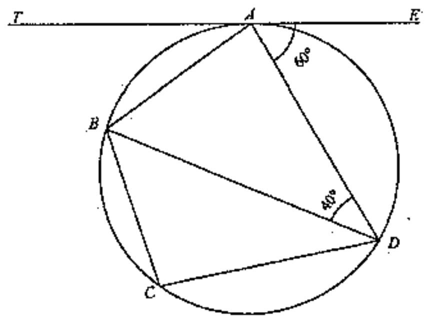
If **TE=TS,** calculate the value of angle **SRP.** **[2012 P1 #9]**

15. In **Figure 15**, AT is a tangent to the circle Centre O. Angle OXT $=31°$ and AOX is a straight line.



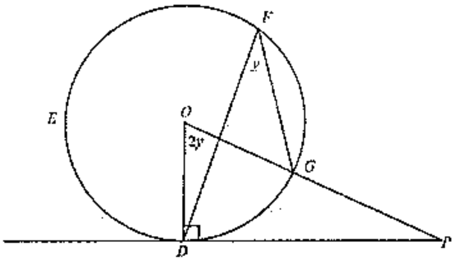
Calculate angle **OAT.** **[2012 P2 #5a]**

16. **Figure 16** shows a circle ABCD, tangent TAE at A, angle EAD $= 60°$ and angle ADB $= 40°$



Calculate angle BCD. **[2014 P1 #14]**

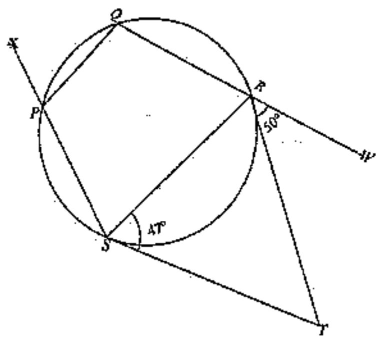
17. In **Figure 17**, DP is a tangent to the circle DEFG centre O.



Given that OGP is a straight line and angle DEF = y°, show that angle DPO = $2(45-y)$.

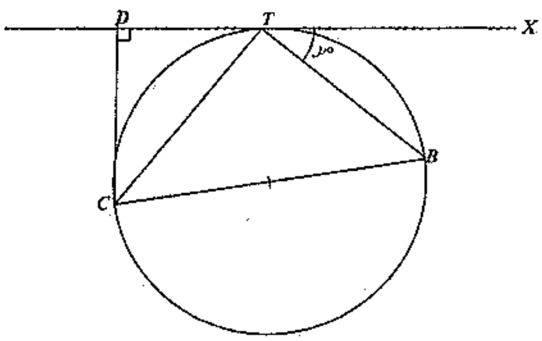
[2014 P1 #4b]

18. **Figure 18** shows a cyclic quadrilateral PQRS .TR and TS are tangents to the circle PQRS at R and S respectively. QRW and XPS are straight lines, angle TSR=47° and angle WRT=50°.



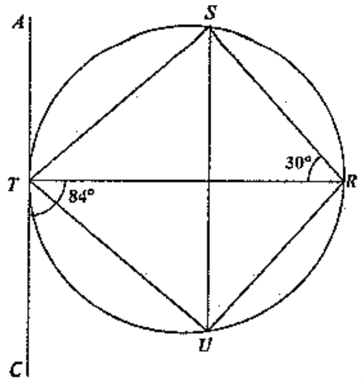
Calculate angle **XPQ**. **[2015 PI #15]**

19. In **Figure 19**, **BOC** is a diameter and angle **TDC** = 90° **DTX** is a tangent to the circle **BCT** at **T**.



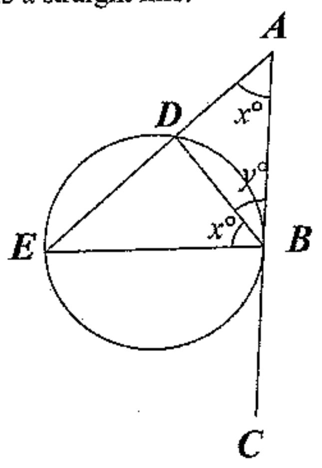
If angle **BTX** $= y^\circ$, show that angle **DCT** = angle **BCT**. **[2015 PII #6a]**

20. In **Figure 20, ATC** is a tangent to circle **RSTU** at **T** angle **TRS** = 30° and angle **CTR** 84°



Calculate angle **RUS**. **[2016 PI #13]**

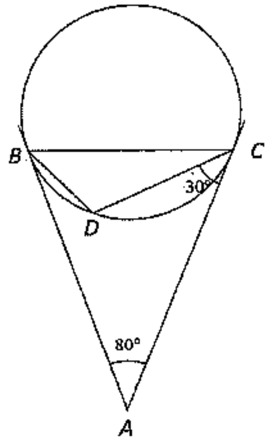
21. In **Figure 21, ABC** is a tangent to the circle **BDE** at **B**. **ADE** is a straight line.



Given that angle **ABC**= y° and angle **BAE** = angle **DBE** = x°, show that **EB** is a diameter.

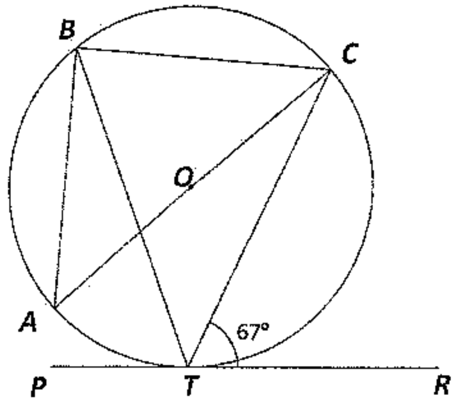
[2016 PII #5b]

22. **Figure 22** shows a circle **BCD** in which **AB** and **AC** are tangents at Band **C** respectively. Angle BAC = 80° and angle **ACD** = 30°.



Calculate angle **ABD.** **[2017 PI #14]**

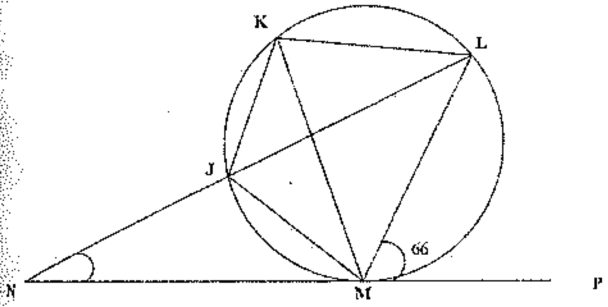
23. **Figure 23** shows a circle **ABCT** with center **O**, and tangent **PRT** to the circle at **T**.



If angle **CTR**= 67°, calculate angle **ACT**.

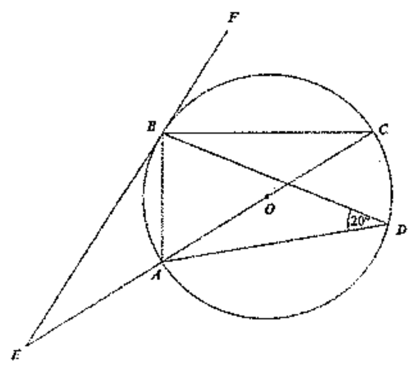
[2017 PII #5b]

24. **Figure 24** shows a circle **JKLM. NMP** is a tangent to the circle at **M** and **NJL** is a straight line.



If angle **LNP** = 30 and angle **LMP** = 66, calculate the value of angle **JML**. **[2018 P1 #14]**

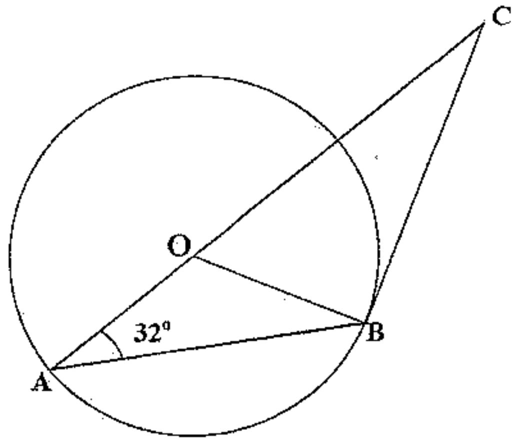
25. **Figure 25** shows a circle **ABCD** with center **O** with **BC** and **AD** as chords, **EBF** is a tangent at **B** and **COA** is a diameter extended to meet **FBE** at **E**.



If angle **ADB** = 20°, find the value of **BEC.**

[2019 PI #9]

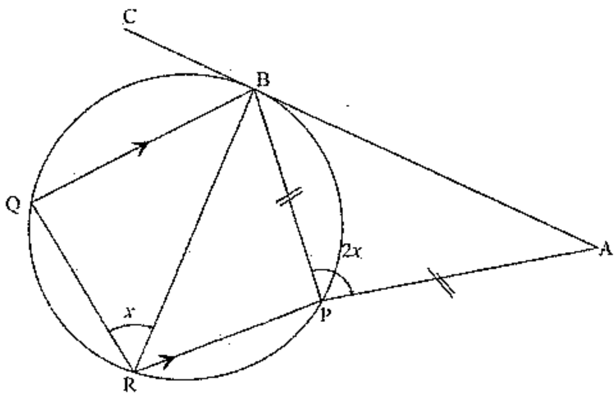
26. **Figure 26** shows a circle centre O, tangent CB to the circle at B. Angle BAO=32°



Calculate the value of Angle **BCA**.

[2020 Mock PI #19]

27. **Figure 27** shows a Circle with tangent ABC at B. PB=PA and QB is parallel to RP. Angle BPA=$2x$ and Angle BQR=x.



Using the figure:

i. Find angle ABP in terms of x.
ii. Find angle QBR in terms of x.
iii. Show that RB is a diameter.

[2021 Mock PII #12]

1. **[2003 PI #16]**

$\angle OAP = 90°$ *radius* $\perp$ *to* tangent

$\angle OAT = \angle OAP - \angle TAP$

$= 90° - 70°$

$= 20°$

In $\triangle OTA$

$OT = OA$ *(radii)*

$\angle OTA = \angle OAT$ *($\angle$s opposite equal sides)*

$= 20°$

$\therefore$ *the value of* $\angle OTA$ *is* $20°$

2. **[2003 PP II #5a]**

given: circle ABCD in which DE is tangent to the circle at D. AC || DE

to prove: **i.** ΔADC is isosceles
ii. $\angle$ABC is twice $\angle$DAC

proof:
i. $\angle$DCA=$\angle$EDC *(alt. $\angle$s, AC//DE)*
$\angle$DAC = $\angle$EDC *($\angle$s in alt. seg.)*
$\therefore\angle$DCA=$\angle$DAC *(both= $\angle$EDC)*
$\therefore\Delta$ADC is isosceles.
Construction: Join BD
$\angle$CBD = $\angle$DAC (1) *($\angle$s in same seg.)*
$\angle$ ABD = $\angle$ DCA (2) *(same reason)*
(1) + (2):
$\angle$CBD +$\angle$ ABD = $\angle$DAC + $\angle$DCA
$\angle$ABC = $\angle$DAC + $\angle$DCA
But $\angle$DCA = $\angle$DAC
$\angle$ABC = 2$\angle$DAC *(proved in (i))*
$\therefore$ $\angle$ABC is twice $\angle$DAC.

3. **[2004 PP1 #17]**

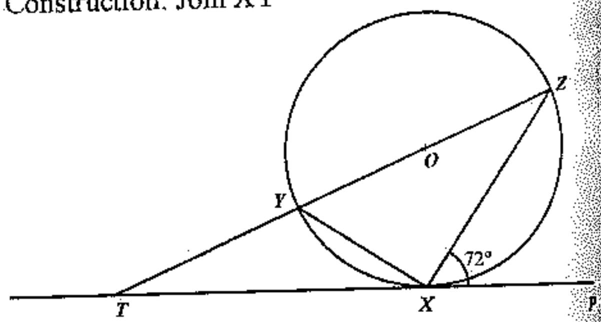
Given: TX and TY tangents to circle XHY at X and Y.
Required to find; $\angle$XTY
Now $\angle$YXT = $\angle$XHY *($\angle$s in alt. seg.)*
$\angle$YXT = 70°
In ΔTYX:
TX=TY
$\angle$XYT = $\angle$YXT *($\angle$s opposite equal sides)*
$\angle$XTY = 70°
$\angle$XTY + $\angle$YXT + $\angle$XYT = 180° *($\angle$ sum of a Δ)*
$\angle$XTY = 180° - (70° - 70°)
= 180° – 140°
$\therefore\angle$XTY = 40°

4. **[2005 P1 #3]**

Given: TP tangent to circle APB at P and AB//PT
Required to prove: AP = BP
Proof: $\angle$BAP = $\angle$BPT *($\angle$s in alt. segment)*
$\angle$ABP = $\angle$BPT *(alt. $\angle$s: AB//PT)*
$\angle$BAP = $\angle$ABP *(both equal to $\angle$BPT)*
$\therefore\Delta$APB is isosceles *($\angle$BAP and $\angle$ABP are both base angles)*
$\therefore$AP = BP *(sides opp. equal angles)*

5. **[2005 PII #4a]**

Given XYZ center O with TXP tangent at X. diameter ZY produced to T meeting the tangent at T.
To calculate $\angle$XTY:
Construction: Join XY



$\angle$YXZ = 90° *($\angle$ in a semicircle)*
$\angle$YXT + 90° + 72° = 180° *($\angle$s on a str. line)*
$\angle$YXT = 180° – 90° – 72°
= 18°
$\angle$XYZ = 72° *($\angle$s in alt seg)*
In ΔXTY:
$\angle$YXT + $\angle$XTY = $\angle$XYZ *(ext $\angle$ of a ΔXTY)*
$\Rightarrow$18° + $\angle$XTY = 72°
$\angle$XTY = 72° - 18°
= 54°
$\therefore\angle$XTY = 54°
$\therefore$the value of angle XTY is 54°

6. **[2005 PII #11a]**

Given: 2 circles PRS and QRS intersecting at R and S; TP tangent
To circle PRS at P and TQ tangent to circle QRS at Q; PR diameter of a circle PRS and RQ diameter of circle QRS.
(i) $\angle$RPT = 90° *(diameter to tangent)*
$\angle$RQT = 90° *(diameter RQ to tangent)*
$\angle RPT + \angle RQT = 90° + 90° = 180°$
$\therefore$ PRQT is a cyclic quadrilateral *(opp $\angle$s suppl)*
(ii) join RS, RT and PQ
PRQT is a cyclic quadrilateral *(proved in part (i))*
$\therefore \angle QRT = \angle QPT$ *($\angle$s in the same seg)*
But $\angle$PRS = $\angle$QPT *($\angle$s in alt. seg)*
$\therefore \angle QRT = \angle PRS$ *(both $\angle$s equal to QPT)*

7. [2006 PII #6]

a. angle BDC

join BD

$\angle ADC = \angle ABC$ (*$\angle$s in the same segment*)

$\therefore \angle ADC = 70°$

$\angle ACD = 90°$ (*$\angle$ in a semi circle*)

$\angle ACB + \angle BCD = 90°$

$\angle ACB + 40° = 90°$

$\therefore \angle ACB = 90° - 40°$

$= 50°$

But $\angle BDA = \angle ACB$ (*$\angle$s in a same segment.*)

$\therefore \angle BDA = 50°$

$\angle BDC = \angle BDA + \angle ADC$

$\angle BDC = 50° + 70°$

$\therefore \angle BDC = 120°$

b. angle ATD

$\angle TAD = \angle BCD$ (*$\angle$s in same segment*)

$\angle TAD = 40°$

$\angle TDA = 90°$ (*AD is $\perp$ to tangent TD*)

In ΔATD:

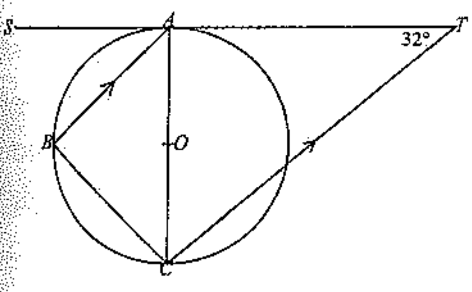
$\angle ATD + \angle TDA + \angle TAD = 180°$ (*$\angle$ sum of a Δ*)

$\angle ATD + 90° + 40° = 180°$

$\angle ATD = 180° - 90° - 40°$

$\therefore \angle ATD = 50°$

8. [2007 PI #6]



Given : Circle ABC centre O with TA tangent at A, $CT // BA, \angle ATC = 32°$

To calculate angle ACB :

$\angle SAB = 32°$ (corr.$\angle$s; $CT // BA$)

$\angle ACB = \angle SAB$ ($\angle s$ in alt. seg)

$\therefore \angle ACB = 32°$

9. [2007 PII #5a]

Given: circle QRS centre O.

Tangent PQT at Q

To find $\angle RPQ$

$\angle ORQ = 28°$ (*$\angle$s opposite equal sides*)

$\angle OQR = 90°$ (tangent $\perp$ radius)

in ΔPQR,

$\angle ORQ + \angle RQP + \angle RPQ = 180°$ ($\angle$ sum of Δ)

$28° + 28° + 90° + \angle RPQ = 180°$

$146° + \angle RPQ = 180°$

$\angle RPQ = 180° - 146°$

$\angle RPQ = 34°$

10. [2008 P2 #2b]

Given: ET and EU tangents to the circle $PQRS$ at P and S respectively,

$\angle TEU = 110°, \angle PSQ = 46°, PQ // SR$

To find: Angle RSU

now, $EP = ES$ (tangent from external pt.)

$\therefore \Delta EPS$ is isoscles.

$\Rightarrow \angle ESP = \angle EPS$ (*$\angle$s opposite equal sides*)

$\angle ESP = \dfrac{180° - 110°}{2}$ (*$\angle$sum of a Δ*)

$= 35°$

$\angle RSQ = \angle PQS = 35°$ (alt.$\angle s, PQ \parallel SR$)

on straight line ESU,

$35° + 46° + 35° + R\hat{S}U = 180°$

($\angle s$ on a straight line)

$R\hat{S}U = 180° - 116°$

$R\hat{S}U = 64°$

$\therefore$ The value of angle RSU is $64°$

11. [2008 P1 #5]

Given: QN tangent to the circle LMN at $N, \angle QNM = 55°, \angle LNM = 24°$

Required to find: $\angle NML$

Now, $\angle NLM = \angle QNM = 55°$ ($\angle s$ in alt.seg)

$\angle NML = 180^\circ - (55^\circ + 24^\circ)$ $(\angle sum\ of\ \Delta)$

$= 180^\circ - 79^\circ$

$= 101^\circ$

$\therefore$ the value of angle NML is 101°

12. [2010 PII #6b]

In ΔOBC

$OB = OC = OD$ (radii)

$< OBF = 90^\circ$ (radius $\perp$ tangent)

$< OBC = 90^\circ - 40^\circ$ (Complementary angles)

$< OBC = 50^\circ$

$< OCB = 50^\circ$ (*$\angle s$ opposite equal sides*)

$< BOC = 180^\circ - (50^\circ + 50^\circ)$ ($\angle$ sum of a Δ)

$< BOC = 180^\circ - 100^\circ$

$< BOC = 80^\circ$

In ΔABO

$< AOB = 180^\circ - (80^\circ + 70^\circ)$ (Adj $\measuredangle$s)

$< AOB = 30^\circ$

$< ABO = 90^\circ$ (radius $\perp$ tangent)

$< OAB = 180^\circ - (30^\circ + 90^\circ)$ ($\angle$sum of a Δ)

$< OAB = 180^\circ - 120^\circ$

$< OAB = 60^\circ$

13. [2011 PI #15]

$\angle ABD = \angle ACD = x$ ($\angle s$ in the same seg.)

$\angle ACD = \angle CDE = x$ (alt $\angle$s)

$\angle CDE = \angle DAC = x$ ($\angle$s in alt.seg)

$\angle DAC + \angle ACD + \angle ADC = 180^\circ$ ($\angle$ sum of $\triangle$)

$x + x + 130^\circ = 180^\circ$

$2x = 180^\circ - 130^\circ$

$2x = 50^\circ$

$x = 25^\circ$

14. [2012 P1 #9]

Given $TE = TS$, $\angle QTS = 50^\circ$

$\angle TSE = TES$ (*base $\angle s$ of isosΔ*)

$\angle QTS = \angle TSE + \angle TES$

(*ext. $\angle$ = sum of opp.* int.*$\angle s$ of Δ*)

$\angle TPS = \angle TSE + \angle TES$ $(\angle QTS = \angle TPS) = 50^\circ$

$\therefore \angle TSE = \frac{50}{2} = 25^\circ$

$\angle TSE = \angle PTE$ (*$\angle s$ in alt.seg*)

$\angle PTE = 25^\circ$

$\angle STP = 180^\circ - (50 + 25)$ (*$\angle s$ on st.line*)

$\angle STP = 105^\circ$

$\angle STP + \angle SRP = 180^\circ$ (*opp.$\angle s$ of cyclic quad*)

$\angle SRP = 180^\circ - 105^\circ$

$\angle SRP = 75^\circ$

15. [2012 P2 #5a]

$\angle OXT = 31^\circ$ (*given*)

$\angle OTA = 90^\circ$ (radius $\perp$ to tangent)

$\angle OTX = 31^\circ$ (*$\angle s$ opposite equal sides. radii $OT = OX$*)

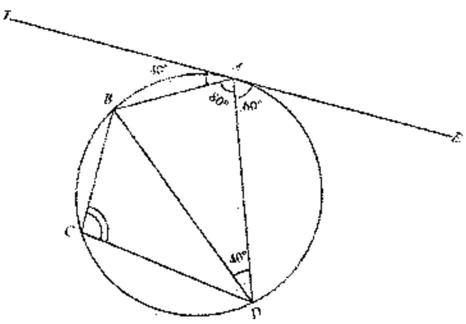
In ΔAXT,

$\angle OAT = 180^\circ - (31^\circ + 31^\circ + 90^\circ)$ ($\angle s$ sum of Δ)

$= 180^\circ - 152$

$\therefore \angle OAT = 28^\circ$

16. [2014 P1 #14]



$\angle TAB = 40^\circ$ (*alt$\angle s$*)

$\angle BAD = 180^\circ - (60^\circ + 40^\circ)$ (*adjacent angles*)

$= 180^\circ - 100^\circ$

$= 80^\circ$

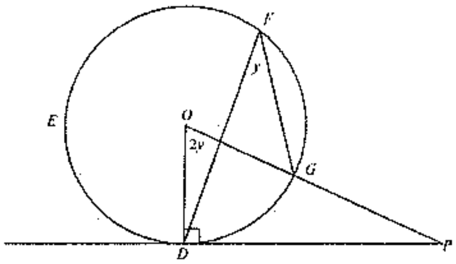
ABCD is a cyclic quad

$\angle BCD + \angle BAD = 180^\circ$ (*opp$\angle s$ of a cyc quad*)

$\angle BCD = 180^\circ - 80^\circ$

$\angle BCD = 100^\circ$

17. **[2014 PII #4b]**



Note: point E is optional

PROOF : show that $DPO = 2(45 - y)^\circ$

$\angle ODP = 90^\circ$ (*rad* $\perp$ *Tan*)

$\angle DOP = 2y^\circ$ ($\angle$*at centre* 2× *at* O^{ce})

in Δ *ODP*

$\angle DPO = 180^\circ - (90^\circ + 2y^\circ)$ ($\angle$*sum of* Δ)

$\angle DPO = 180^\circ - 90^\circ + 2y^\circ$

$\angle DPO = 90^\circ - 2y^\circ$

$\angle DPO = 2(45^\circ - y^\circ)$ (*factor out* 2)

$\therefore$ *angle* $DPO = 2(45 - y)^\circ$ *QED*

18. **[2015 PI #15]**

$TS = TR$ (Tangents from an ext. pt. are equal)

$\therefore \angle TSR = \angle TRS = 47^\circ$ ($\angle s$ opp. equal sides)

$\therefore \angle SRW = 47^\circ + 50^\circ$

$\angle SRW = 97^\circ$

$\angle QRS + \angle SRW = 180^\circ$ ($adj.\angle s$)

$\angle QRS = 180^\circ - 97^\circ$

$\angle QRS = 83^\circ$

But in the cyclic quad PQRS;

$\angle XPQ = \angle QRS$

(ext. $\angle$ of a cyclic quad=int. opp. $\angle$)

$\therefore \angle XPQ = 83^\circ$

19. **[2015 PII #6a]**

$B\hat{C}T = B\hat{T}X$ ($\angle s$ in opp. seg)

$\therefore B\hat{C}T = y^\circ$

$B\hat{T}C = 90^\circ$ ($\angle s$ in a semi circle)

$\therefore C\hat{T}X = 90^\circ + y^\circ$

$D\hat{T}C = 180^\circ - C\hat{T}X$ ($adj. \angle s$)

$D\hat{T}C = 180^\circ - (90^\circ + y^\circ)$

$D\hat{T}C = 90^\circ - y^\circ$

But $T\hat{D}C = 90^\circ$ (*given*)

In a ΔDTC

$D\hat{T}C + T\hat{D}C + D\hat{C}T = 180^\circ$ ($\angle$ sum in a Δ)

$(90^\circ - y^\circ) + 90^\circ + D\hat{C}T = 180^\circ$

$D\hat{C}T = 180^\circ - (90^\circ - y^\circ) - 90^\circ$

$D\hat{C}T = y^\circ$

$\therefore D\hat{C}T = B\hat{C}T = y^\circ$

20. **[2016 PI #13]**

$\angle ATS = \angle TRS$ ($\angle s$ in alt. seg.)

$\angle ATS = 30^\circ$

So $30^\circ + \angle SRT + 84^\circ = 180^\circ$

($\angle s$ on a str. line)

$\angle SRT = 180^\circ - (84^\circ + 30^\circ)$

$\angle SRT = 180^\circ - 114^\circ$

$= 66^\circ$

$\angle RUS = \angle SRT$ ($\angle$s in alt. segment)

$\therefore \angle RUS = 66^\circ$

21. **[2016 PII #5b]**

$\angle ABD = \angle AEB = y^\circ$ (*angles* in alt. seg)

In ΔABD

$\angle$EDB=$\angle$DAB+$\angle$DBA

(sum 2 int. $\angle s$=ext. opp angle)

$\angle$EDB=$x^\circ + y^\circ$

In ΔDBE

$\hat{D}+\hat{B}+\hat{E}=180^\circ$ ($\angle$ sum of a Δ)

$(x^\circ + y^\circ) + x^\circ + y^\circ = 180^\circ$

$2x^\circ + 2y^\circ = 180^\circ$

$2(x^\circ + y^\circ) = 180^\circ$

$x^\circ + y^\circ = 90^\circ$

$\angle EDB = 90^\circ$

$\therefore$ *ED* is a diameter ($\angle$ *in a* semicircle=90°)

22. **[2017 PI #14]**

$AB = AC$ (*Tangent from* the same external point)

$\therefore \angle ABC = \angle ACB$ ($\angle s$ opp. equal sides)

Let each angle be x

$x + x + 80^\circ = 180^\circ$ ($\angle$ sum of a triangle)

$2x = 180^\circ - 80^\circ$

$2x = 100^\circ$

$x = \frac{100}{2}$

$x = 50^\circ$

$\angle ABC = 50^\circ$

$\angle BCD = 50^\circ - 30^\circ$ (*given*)

$= 20^\circ$

$\angle ABD = \angle BCD$ (*Angles* in *alt. seg*)

$\angle ABD = 20^\circ$

23. **[2017 PII #5b]**

$\angle ABC = 90^\circ$ ($\angle s$ in semicircle)

$\angle CBT = 67^\circ$ ($alt.\angle s$)

$\angle ABT = 90^\circ - 67^\circ$

$\angle ABT = 23^\circ$

But $\angle ACT = \angle ABT$ ($\angle s$ subt by the same arc AT)

$\therefore \angle ACT = 23^\circ$

24. **[2018 P1 #14]**

$\angle LJM = 66^\circ$ ($\angle$s in alt. seg)

but $\angle JMN + \angle JNM = \angle LJM$

(2int. opp. $\angle s$ = ext.$\angle$)

$\angle JMN + 30^\circ = 66^\circ$

$\angle JMN = 66^\circ - 30^\circ$

$= 36^\circ$

$\angle JMN + \angle JML + \angle LMP = 180^\circ$ ($\angle s$ on a str. line)

$36^\circ + \angle JML + 66^\circ = 180^\circ$

$\angle JML + 102^\circ = 180^\circ$

$\angle JML = 180^\circ - 102^\circ$

$\angle JML = 78^\circ$

25. **[2019 PI #9]**

$\angle BCA = \angle ADB$ ($\angle s$ in the same seg)

$\angle BCA = 20^\circ$

$\angle BCE = 20^\circ$ (same $\angle$)

$\angle ABE = \angle BCA$ ($\angle s$ in alt. seg)

$\angle ABE = 20^\circ$

But $\angle ABC = 90^\circ$ ($\angle$ in a semicircle)

In ΔEBC,

$\angle BEC + \angle EBC + \angle BCE = 180^\circ$ ($\angle$sum of a Δ)

$\angle BEC + 20^\circ + 90^\circ + 20^\circ = 180^\circ$

(Since $\angle EBC = 20^\circ + 90^\circ$)

$\angle BEC + 130^\circ = 180^\circ$

$\angle BEC = 180^\circ - 130^\circ$

$\angle BEC = 50^\circ$

26. **[2020 Mock PI #19]**

In ΔOBA,

OB=OA (radii)

so, $\angle OBA = 32^\circ$ ($\angle s$ opp. equal sides)

$\angle OBC = 90^\circ$ (radius $\perp$ tangent)

In ΔABC,

$32^\circ + A\hat{B}C + B\hat{C}A = 180^\circ$ ($\angle$sum of a Δ)

$32^\circ + (90^\circ + 32^\circ) + B\hat{C}A = 180^\circ$

(since $A\hat{B}C = A\hat{B}O + O\hat{B}C$)

$B\hat{C}A + 154^\circ = 180^\circ$

$B\hat{C}A = 180^\circ - 154^\circ$

$B\hat{C}A = 26^\circ$

27. **[2021 Mock PII #12]**

i. ABP in terms of x.

In ΔAPB,

$AB = PA$ (given)

$A\hat{B}P = P\hat{A}B$ ($\angle$s opp. equal sides)

$2x + A\hat{B}P + P\hat{A}B = 180^\circ$ ($\angle sum$ of a Δ)

$2x + 2A\hat{B}P = 180^\circ$ (since $A\hat{B}P = P\hat{A}B$)

$2A\hat{B}P = 180^\circ - 2x$

$A\hat{B}P = \frac{\cancel{2}(90^\circ - x)}{\cancel{2}}$

$A\hat{B}P = 90^o - x$

ii. QBR in terms of x.

AB is tangent (given)

so, BRP=ABP ($\angle$s in alt.seg)

$B\hat{R}P = 90^o - x$

But, $Q\hat{B}R = B\hat{R}P$ (alt.$\angle$s. QB//RP)

$Q\hat{B}R = 90^o - x$

iii. To show RB is diameter.

Show that $\angle RQB = 90^o$

In ΔRQB,

$Q\hat{R}B + R\hat{B}Q + R\hat{Q}B = 180^o$ ($\angle$sum of a Δ)

$x + 90^o - x + R\hat{Q}B = 180^o$

$90^o + R\hat{Q}B = 180^o$

$R\hat{Q}B = 180^o - 90^o$

$R\hat{Q}B = 90^o$

$\therefore RB$ is diameter

($\angle$ in a semi-circle is a right angle)

Alternatively:

$\angle QBC = x$ ($\angle$s in alt. seg)

Then,

$\angle RBC = \angle QBC + \angle RBQ$

$\angle RBC = 90^o - x + x$

$\angle RBC = 90^o$

$\therefore RB$ is diameter (radius/diameter $\perp$ tangent)

NB: Kindly note that where RPA is not given as a straight line (like in this case), the exterior angle of a cyclic quad RQBP does not exist.

CH 20 B
CIRCLE GEOMETRY III (CONSTRUCTION)

Study Objectives

A student must be able to:

- Construct Angles of 90°, 60°, 30°, 15°, and 75°, 120°, 150°.
- Construct Circles and Intersected Circles.
- Construct tangents to a point on the circumference of a circle.
- Construct Tangents from an external point.
- Measure and State various lengths and angles.

1. Draw a circle center **O** with radius 3 cm. Construct another circle radius 4 cm passing through point **O.** Label its center **C.** Label one of the intersection points of the two circles **A.** Using a ruler only, construct a tangent to the circle center **O** at point **A.**

   Measure and stage angle **AOC.** **[2003 PI #24]**

2. Using a ruler and a pair of compasses only, construct on the same diagram:
   i. A circle center **O** radius of 4cm.
   ii. A tangent **PQ** at P such that **PQ**=6cm
   iii. A line **OR** 10cm long such that angle **POR** =120 ° and **PR** is longer than **QR.**
   iv. a point **S** on the circumference of a circle closer to **Q** such that **RS** is a tangent to the circle.
   v. Measure and state length of **QS.** **[2004 PII #6b]**

3. Using a ruler and a pair of compasses, draw a line **AB**=10cm and construct a circle with the line **AB** as a diameter. Mark a point **C** on the circle such that **AC**= 6cm. join **AC** and **BC.**

   Construct a tangent **CP** such that angle **BCP** is acute. **[2006 P1 #16]**

4. Using a ruler and a pair of compasses only, construct in the same diagram:
   i. A circle Centre **O** of radius 3cm.
   ii. Two tangents to the circle from a point **T** which is 8 cm from **O**, touching the circle at points **M** and **N.**
   iii. Measure and state angle **MTN.**

   [2006 PII #5b]

5. Using a ruler and pairs of compasses only, construct in the same diagram: a circle center O of radius 3cm, a tangent to the circle at **P** from a point T which is 8cm from the circle. **[2008 P1 #22]**

6. Using a ruler and a pair of compass only, construct in the same diagram
   i. A circle Centre **O** of radius 4cm.
   ii. A diameter **PT** and a tangent to the circle a **T.**
   iii. A point **R** on a tangent such that **TR** = 9cm measure and state **PR.** **[2008 P2 #11a]**

7. Using a ruler and a pair of compass only. Construct in the same diagram;
   i. A circle centre **O** with a radius of 4cm
   ii. A tangent **TP** to the circle at any point **P** such that angle **POT=60°.**
   iii. Measure and state the length of **PT.**

   [2010 P1 #24]

8. Using a ruler and pair of compasses only, construct in the same diagram:
   i. A circle centre O of radius 3 cm.
   ii. A point **R** outside the circle such that **OR** =10cm.
   iii. A tangent **TR** to the circle at **T.**
   iv. Measure and state the length of **TR.**

   [2012 P2 #6b]

9. Using a ruler and a pair of compasses only, construct a circle Centre O of diameter 5cm. Through a point X on the circle construct a tangent PX such that $PX = 6cm$. Measure and state the length of **PO.** **[2014 P1 #5a]**

10. Using a ruler and a pair of compasses only, construct the same diagram:
    i. A circle center **O** with radius 3 cm.
    ii. A tangent **TZ** at **Z** such that **OT** = 5cm.

    In the diagram, measure and state the length of **TZ.** **[2017 PII #4b]**

11.
12.
13.
14.

11. Using a ruler and pair of compasses only, construct in the same diagram:
   i. A circle centre **O** of radius 4 cm.
   ii. A point T outside the circle such that OT= 8cm.
   iii. A tangent, **TA** to the circle at **A.**
   iv. Produce radius **OA** to point C such that **OC** = 10 cm and join **CT**.

   In the diagram, measure, and state angle **OCT.**

   [2016 PII #6b]

12. Using a ruler and a pair of compasses only, construct in the same diagram:
   a. A circle centre **O** of radius 4 cm.
   b. Two tangents to the circle at **A** and at **B** from an external point **P** which is 9cm away from the center **O**

   Measure and state the value of angle **APB.**

   [2018 PII #6]

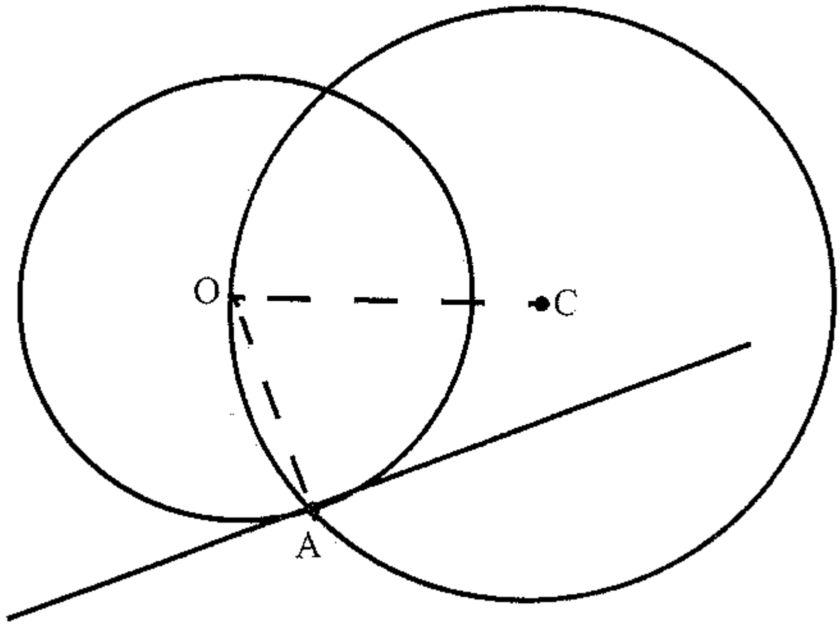
13. Using a ruler and a pair of compasses only, construct in the same diagram:
   - a circle center **O** with radius 2.5 cm.
   - a point **T** outside the circle and 7 cm from the center.
   - Tangent **TP** and **TQ** at **P** and **Q** respectively.
   - Measure and state the size of angle **POQ.**

   [2019 PII #5a]

14. Using a ruler and a pair of compasses only, construct in the same diagram:
   - A triangle **ABC** in which **AB**=8cm, AC=6cm and angle **BAC**=60°.
   - A circle Centre O with AB as diameter.
   - Measure and state the length of **OC.**

   [2021 Mock PII #5a]

1. [2003 PI #24]

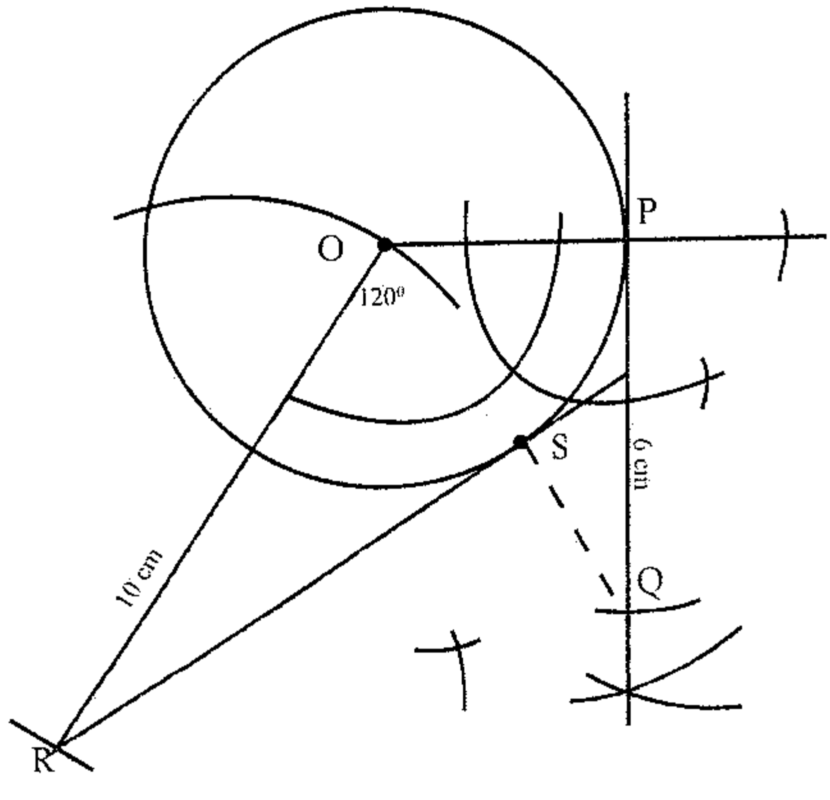


STEP 1 : Circle centre O
Radius = 3cm
STEP 2 : Circle through O
Radius = 4cm
Centre = C
STEP 3 : Label lower intersection point A
STEP 4 : Draw tangent at A
STEP 5 : Measure and state $\angle AOC$

$\angle AOC = 68°$

---

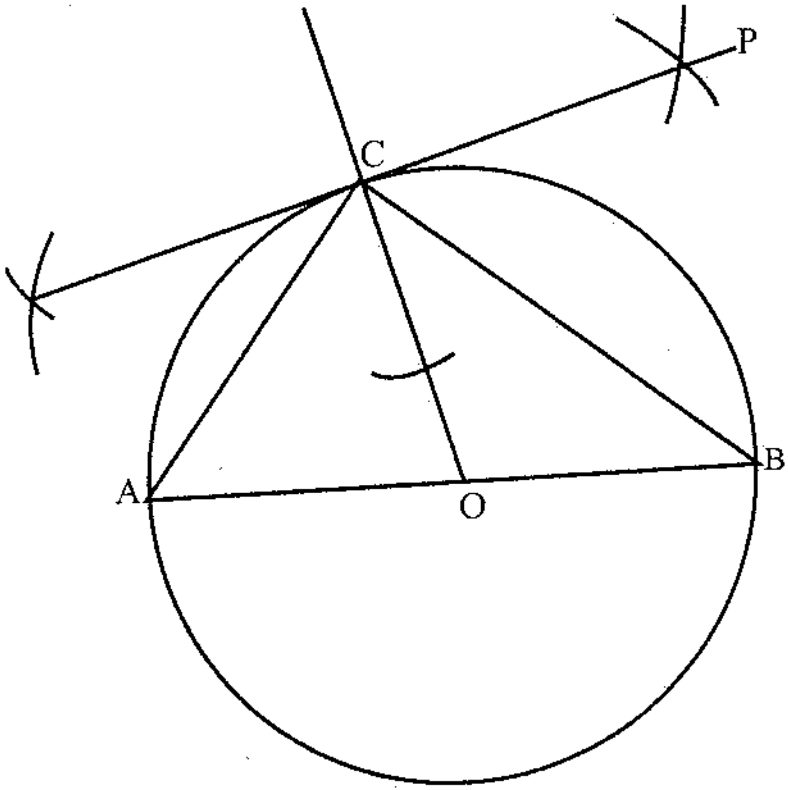
2. [2004 PII #6b]



STEP 1 : Circle centre O
radius = 4cm
STEP 2 : Project OP draw perpendicular through P
STEP 3 : Tangent PQ = 6cm
STEP 4 : $P\hat{O}R$ draw two 60° angles in succession
STEP 5 : OR = 10cm
STEP 6 : Tangent RS
STEP 7 : Join QS
STEP 8 : Measure and state QS

QS = 3.2cm±0.1cm

3. [2006PAPER 1 #16]



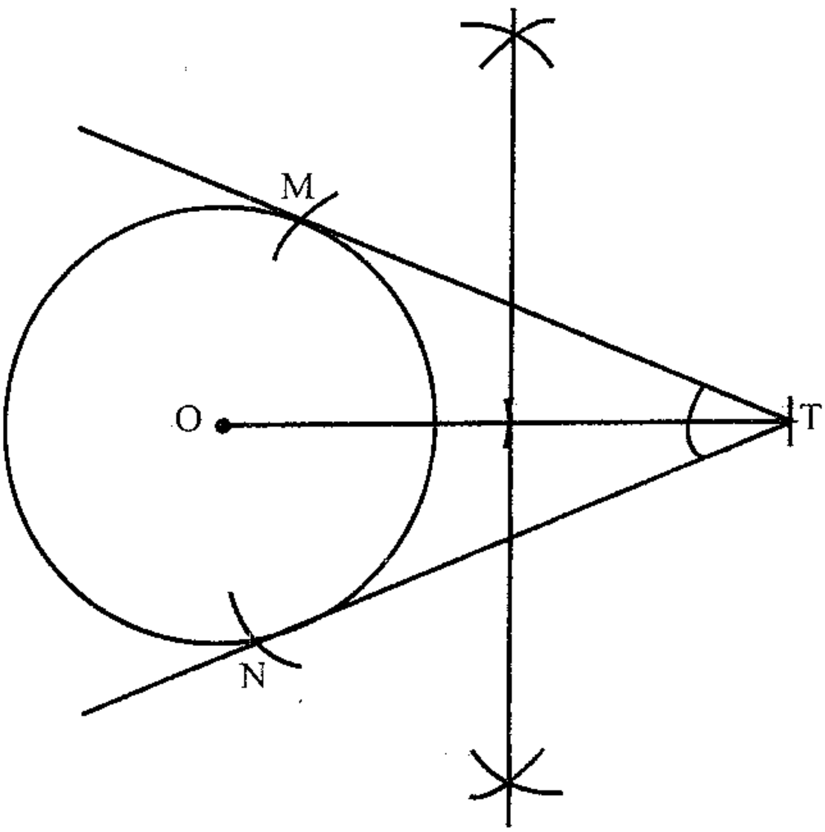
STEP 1 : Draw AB = 10cm
STEP 2 : Bisect AB mark Center O

OA = OB = 5cm

STEP 3 : Mark with campus AC=6cm
STEP 4 : Join AC, BC
STEP 5 : To construct tangent CP (recall tangent is ⊥ radius) so draw line from O through C

STEP 6: Draw a ⊥ for this line

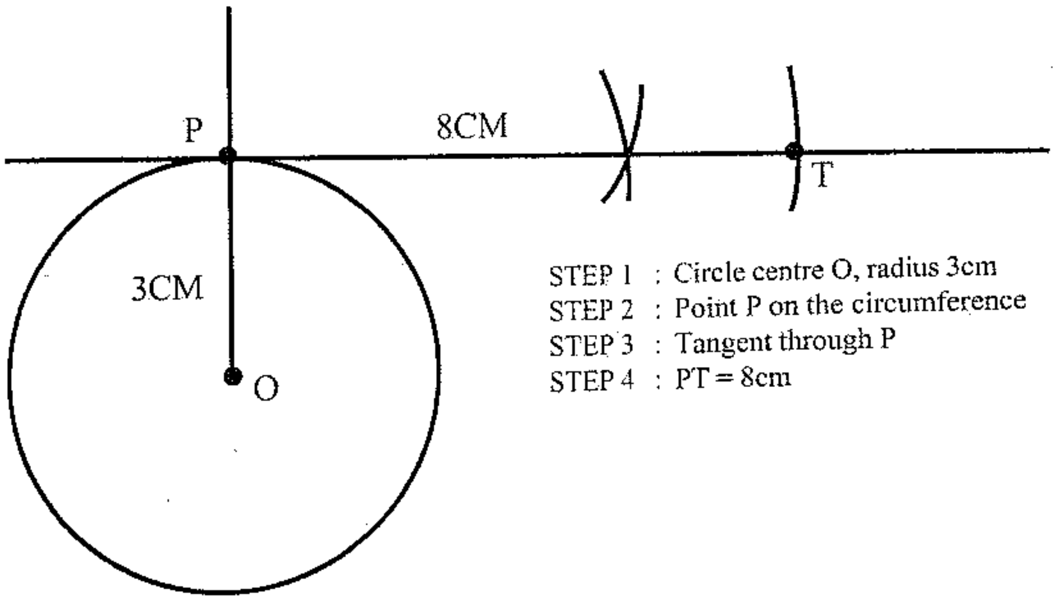
4. [2006PII #5b]



STEP 1 : CIRCLE centre O radius 3cm
STEP 2 : OT = 8cm
STEP 3 : Bisect OT
STEP 4 : From centre of OT mark two points on circle centre O as M and N
STEP 5 : Join TM, TN
STEP 6 : Measure and state ∡MTN

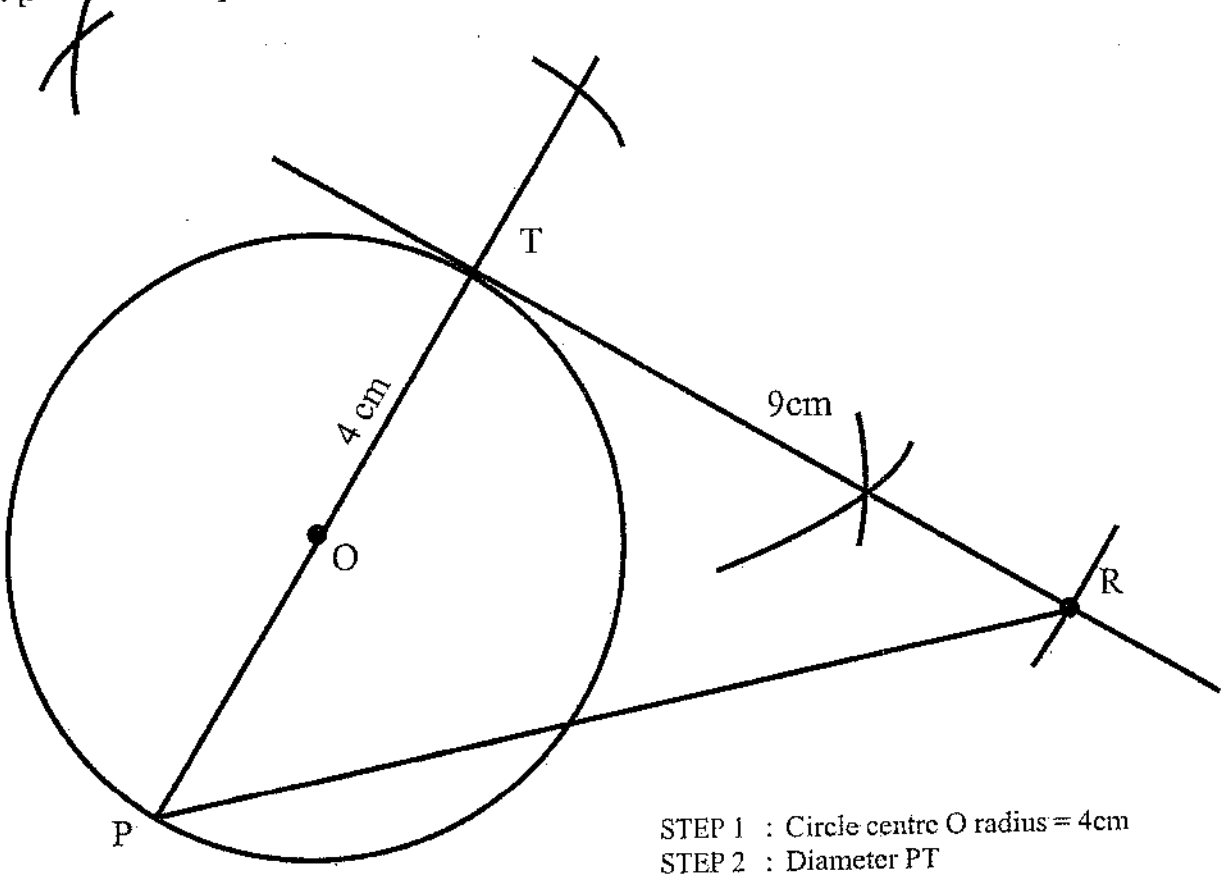
$\angle MTN = 44^0$

5. 2008 PI #22]



STEP 1 : Circle centre O, radius 3cm
STEP 2 : Point P on the circumference
STEP 3 : Tangent through P
STEP 4 : PT = 8cm

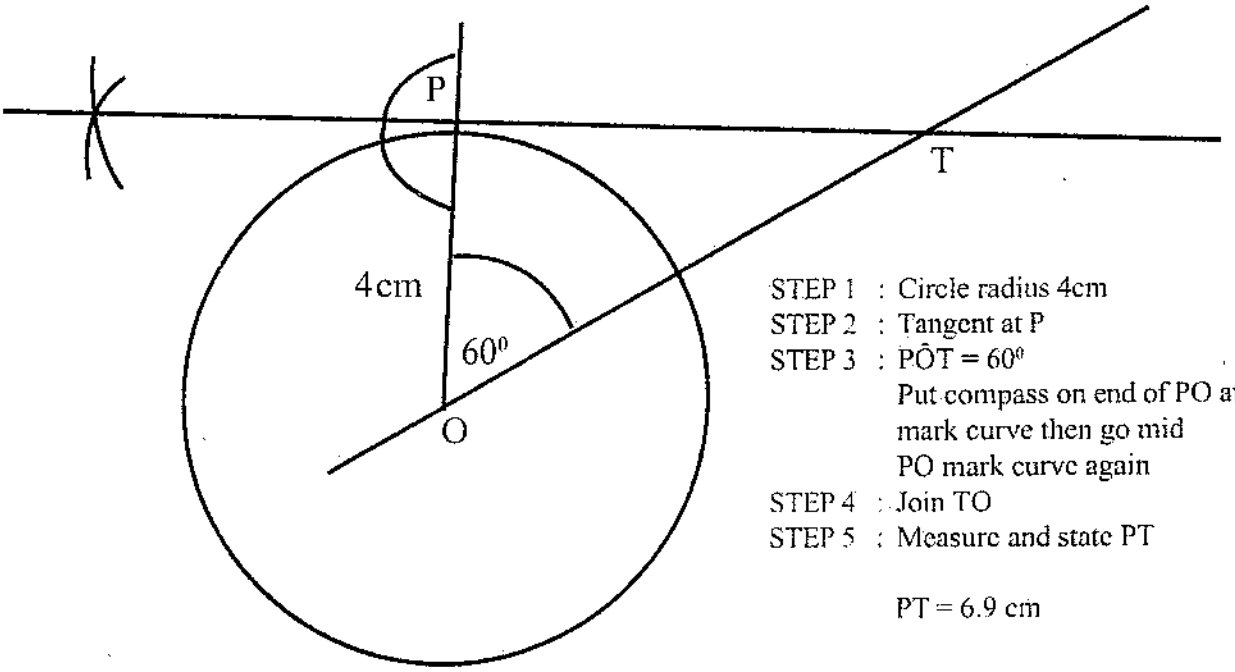
6. [2008 PII #11a]



STEP 1 : Circle centre O radius = 4cm
STEP 2 : Diameter PT
STEP 3 : Tangent RT through T
STEP 4 : Measure and state PR

PR=12 cm

7. [2010 PI # 24]

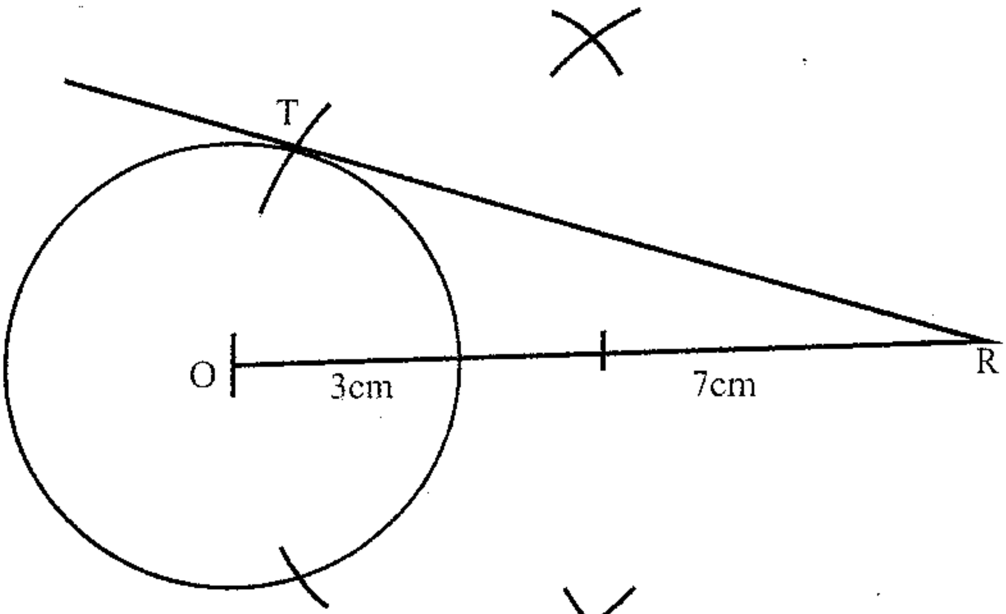


STEP 1 : Circle radius 4cm
STEP 2 : Tangent at P
STEP 3 : $P\hat{O}T = 60^0$
Put compass on end of PO at O, mark curve then go mid PO mark curve again
STEP 4 : Join TO
STEP 5 : Measure and state PT

PT = 6.9 cm

---

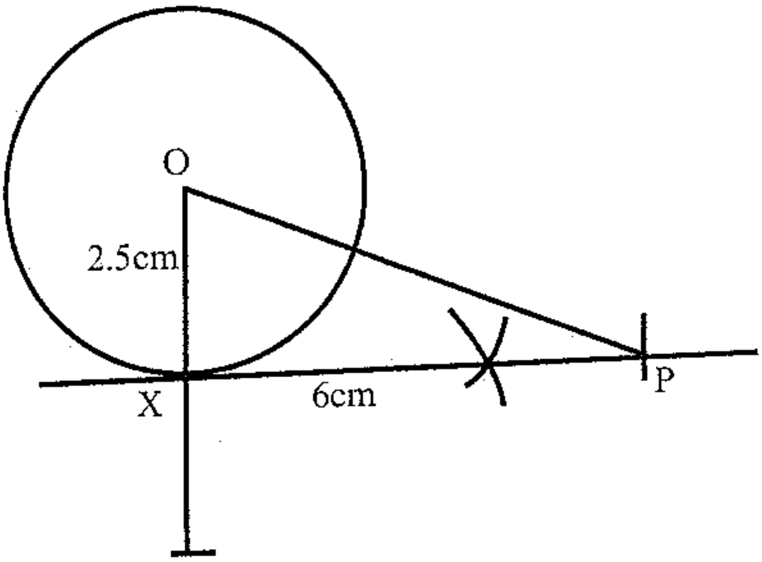
8. [2012 PII #6b]



STEP 1 : Circle radius 3cm
STEP 2 : OR = 10cm
STEP 3 : Bisect OR to find centre
STEP 4 : Use centre of OR to mark T on circle

STEP 5 : Join TR
STEP 6 : Measure and state TR
TR = 9.9cm

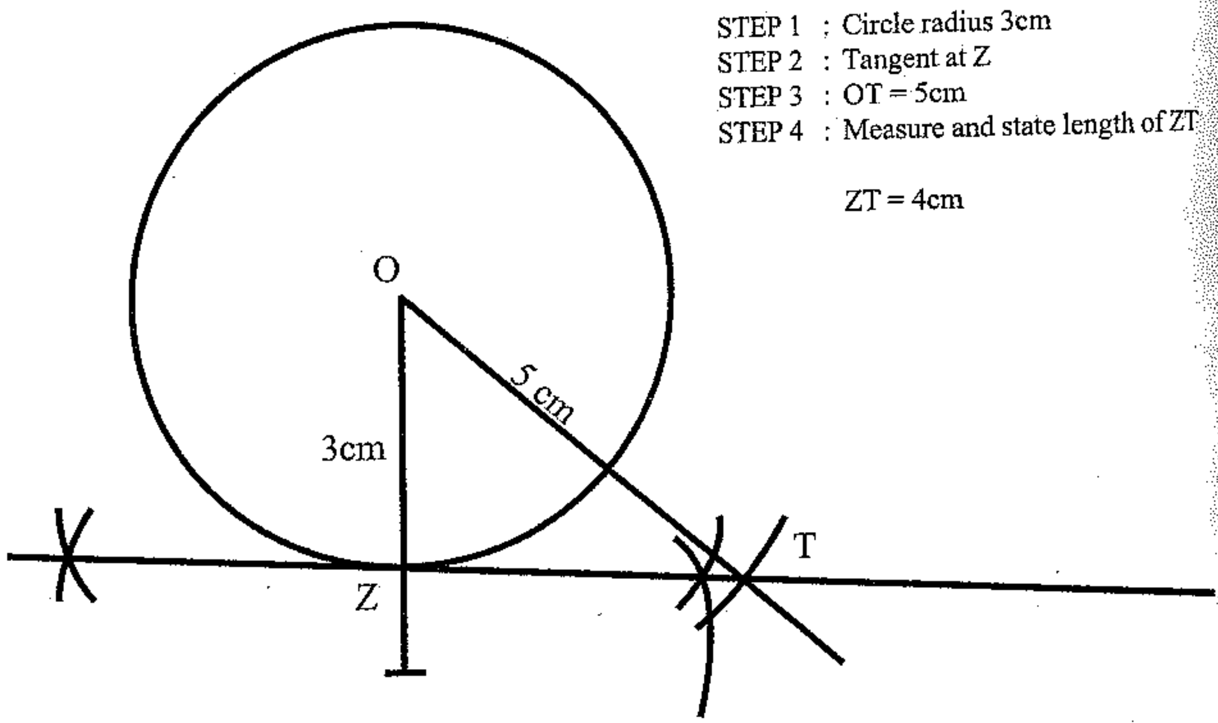
9. [2014 P 1 #5a]



STEP 1 : Circle radius 2.5cm
STEP 2 : Tangent at X
STEP 3 : PX = 6cm
STEP 4 : Measure and state PO

PO = 6.5cm

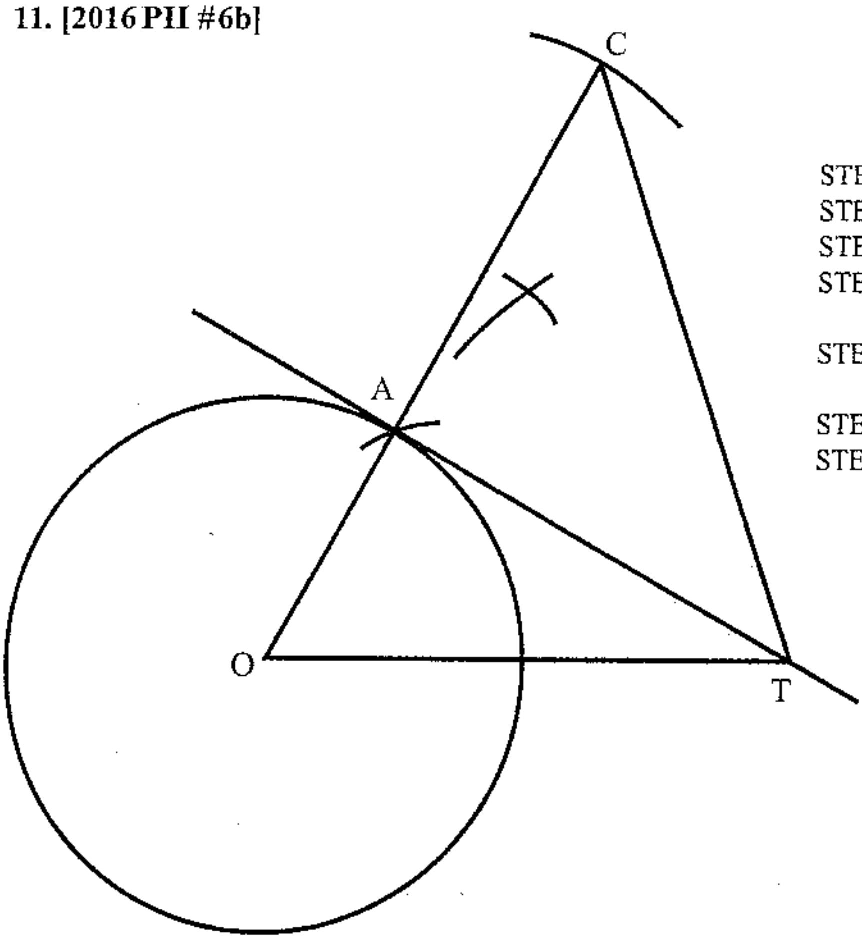
10. [2017 PII #4b]



STEP 1 : Circle radius 3cm
STEP 2 : Tangent at Z
STEP 3 : OT = 5cm
STEP 4 : Measure and state length of ZT

ZT = 4cm

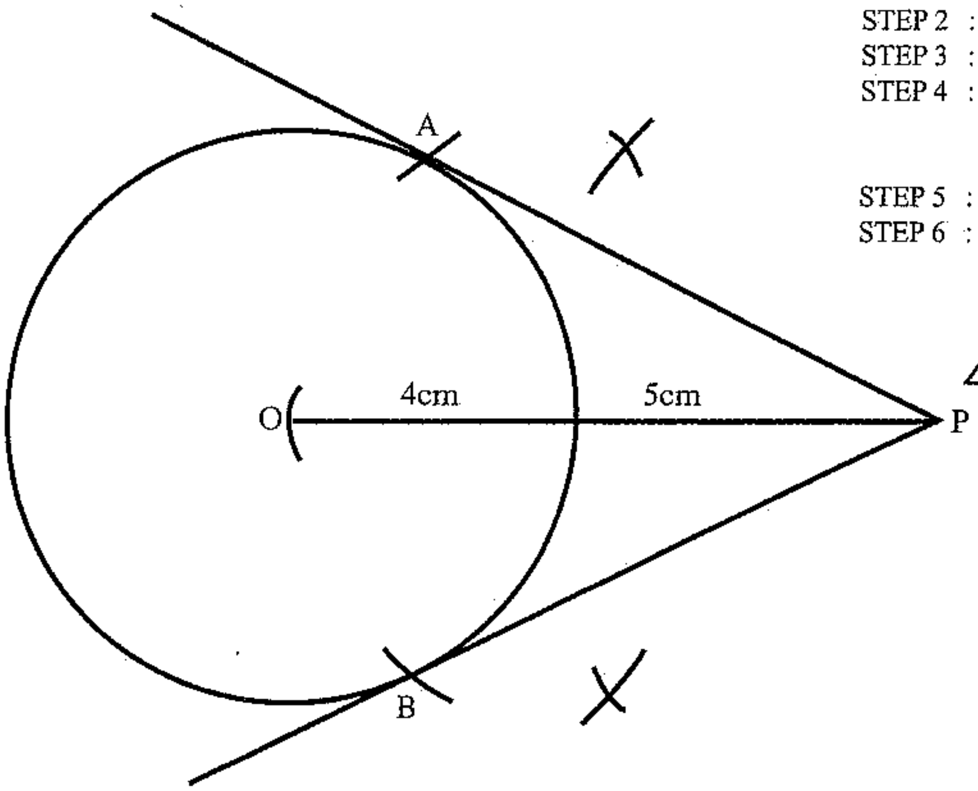
11. [2016 PII #6b]



STEP 1 : Circle radius 4cm
STEP 2 : OT = 8cm
STEP 3 : Bisect OT
STEP 4 : On centre of OT mark point A on circle
STEP 5 : Point is A. Join TA as Tangent
STEP 6 : OC = 10cm
STEP 7 : Measure and state angle OCT

$\angle OCT = 50°$

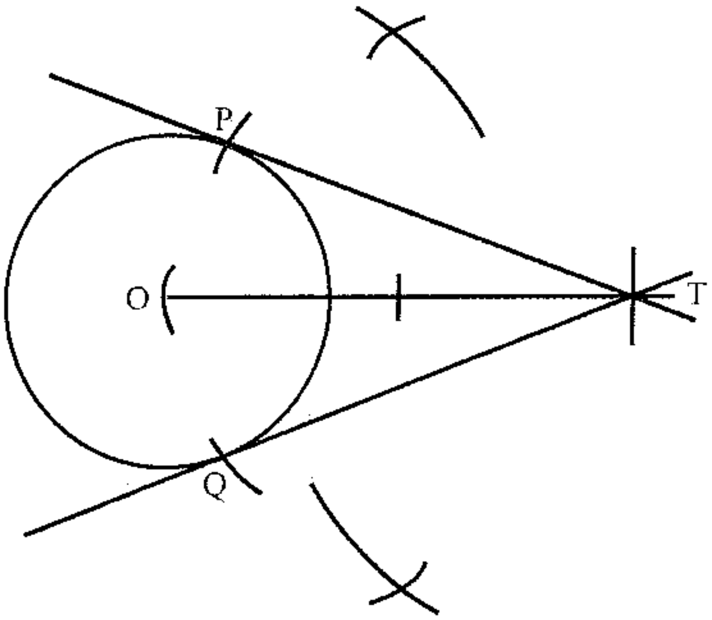
12. [2018 PII #6]



STEP 1 : Circle centre O radius 4cm
STEP 2 : OT = 9cm
STEP 3 : Bisect OP
STEP 4 : From centre of OP mark two points on circle as A and B
STEP 5 : Join PA, PB
STEP 6 : Measure and state $\angle APB$

$\angle APB = 50°$

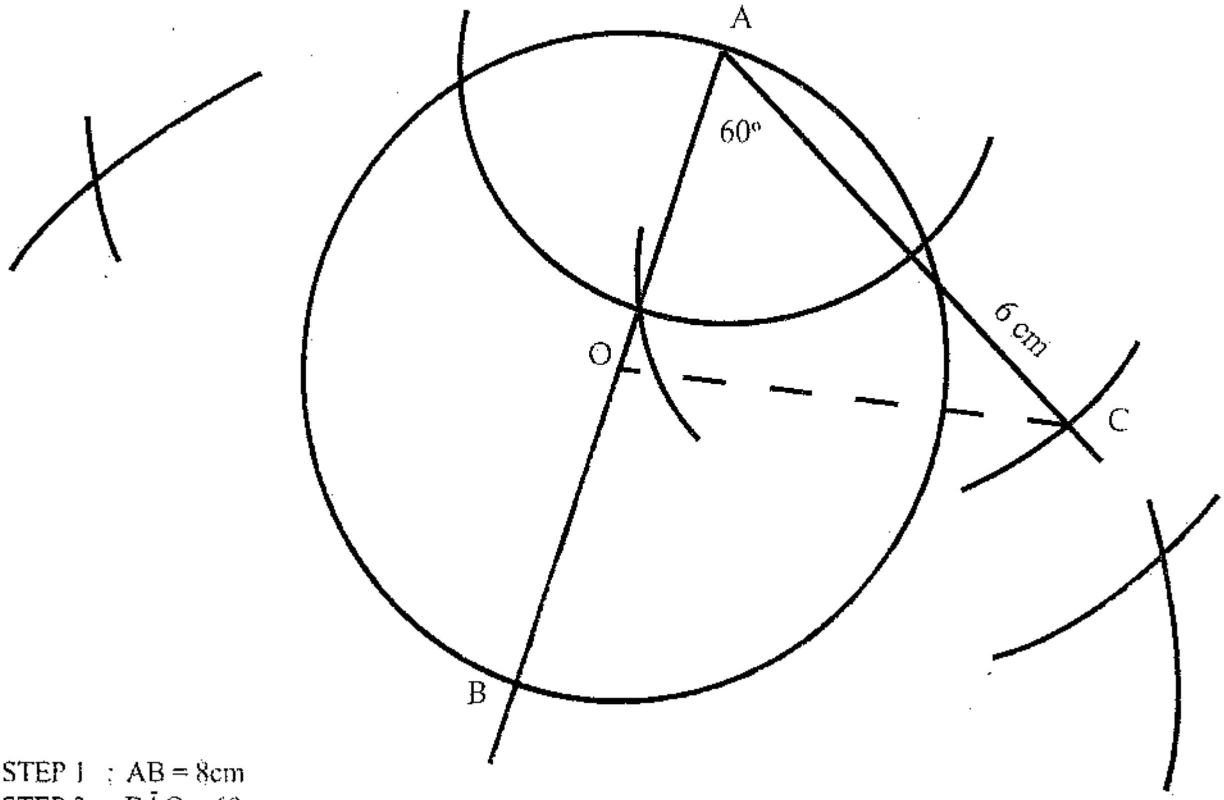
13. [2019 PII #5a]



STEP 1 : Circle centre O radius 2.5cm
STEP 2 : OT = 7cm
STEP 3 : Bisect OT
STEP 4 : From centre of OT mark two points on circle and label the points as P and Q
STEP 5 : Joint TP and TQ
STEP 6 : Join PO, QO
STEP 7 : Measure and state $\angle POQ$

$\angle POQ = 140°$

14. [2021 Mock PII #5a]



STEP 1 : AB = 8cm
STEP 2 : $B\hat{A}C = 60°$
STEP 3 : Mark AC = 6cm

STEP 4 : Bisect AB mark centre as O
STEP 5 : Use centre O and point A or B to draw circle
STEP 6 : Measure and state OC, join OC

OC = 5.4cm

CH 21
STATISTICS II

Chapter Highlights

This chapter contains further problems and solutions on statistics. We will focus on solutions to problems on calculating and interpreting data using measures of dispersion such as range, mean deviations, variance and standard deviation. Measures of dispersion are statistical measures that help us understand how scattered or spread a distribution of data distribution is.

Range: This is the difference between the largest and the smallest value in the data set.

Range = Largest value – Smallest value.

Mean deviation: The mean of the absolute values (positive values) of deviations from the mean.

$$\text{Mean deviation} = \frac{\sum f\left|x-\bar{x}\right|}{\sum f}$$

Variance – This measures the square of the deviations from the mean.

$$\text{Variance} = \frac{\sum f\left(x-\bar{x}\right)^2}{\sum f} \quad \text{or}$$

$$\text{Variance} = \frac{\sum fx^2}{\sum f} - \bar{x}^2$$ (When expanded – useful for easy calculations when mean is a fraction)

Standard deviation: This indicates how data values are grouped around the mean. The smaller the standard deviation the closer the set of data is to the mean. This is found by taking the positive square root of the variance.

$$\text{Standard deviation} = \sqrt{\frac{\sum f\left(x-\bar{x}\right)^2}{\sum f}}$$

$$\text{Standard deviation} = \sqrt{\text{Variance}}$$

1. The variance of two temperature measurements in degrees Celsius, 2 and 2a is 9. Calculate the possible values of a. **[2004 PII #6b]**

2. The table below shows ages of 5 pupils with the mean of 12.6 years.

Age (years)	**Deviation from the mean**	**Square of deviation**
10	-2.6	6.76
11	-1.6	2.56
13	0.4	0.16
14	y	z
15	2.4	5.76
Total	0	

Copy and complete the table to calculate the variance of the ages. **[2005 PI #16]**

3. The table below shows the deviations (d) from the mean of marks and the frequencies (f) of the marks pupils scored in a test.

Mark	**f**	**D**	d^2	fd^2
20	1	-11	121	121
23	1	-8	64	64
26	2	-5	25	50
27	1	-4	16	3
30	3	-1	1	16
34	2	3	9	18
35	3	4	16	48
40	2	9	81	162

Using information from the table, calculate:

a. total number of pupils
b. mean
c. standard deviation to 3 significant figures **[2008 P2 #9b]**

4. Find the standard deviation for a set of data whose deviations from the mean are -2, -3, 4, and 1. **[2012 P2 #5b]**

5. The mean of the data is 16 and its sum is 80. If the square of deviations of the data is 551.25, calculate the standard deviation of the data. **[2015 PII #7a]**

6. Deviations from the mean of two sets of data are: -5, -3, 1 and 3, 4 and x. Given that the sets have the same variance, calculate the values of x. (leaving your answer in surd form). **[2017 PII #11a]**

7. The score of a mathematics test for 10 students were 1, 5, 4, 9, 4, 6, 5, 7, 1, 8. Using a formula, calculate the standard deviation for the scores. **[2018 PII #16]**

1. **[2004 PII #6b]**

Denote the mean by $\overline{x}$
since we have two values 2 and $2a$, then:

mean = $\overline{x} = \dfrac{2+2a}{2}$

$= \overline{x} = \dfrac{2(1+a)}{2}$

$= a+1$

x	Frequency (f)	$x-\overline{x}$	$(x-\overline{x})^2$	$f(x-\overline{x})^2$
2	1	$1-a$	$(1-a)^2$	$(1-a)^2$
2a	1	$a-1$	$(a-1)^2$	$(a-1)^2$

$\sum f = 2$

$\sum(x-\overline{x})^2$

$=(1-a)^2+(a-1)^2$

$=\left[(-1)^2(a-1)^2\right]+(a-1)^2$ (factoring out -1)

$=(a-1)^2+(a-1)^2$

$=2(a-1)^2$

Variance= $\dfrac{\sum f(x-\overline{x})^2}{\sum f}$

$\dfrac{2(a-1)^2}{2}=9$ (given)

$(a-1)^2=9$

$a-1=\pm\sqrt{9}$

$a=1\pm3$

$a=1+3$ or $a=1-3$

$a=4$ or $a=-2$

2. **[2005 PI #16]**

Age (years)	Deviation from the mean	Square of deviation
10	-2.6	6.76
11	-1.6	2.56
13	0.4	0.16
14	***y* =1.4**	***z* =1.96**
15	2.4	5.76
Total	**0**	**17.2**

Deviations from the mean add up to zero, so;

$-2.6-1.6+0.4+y+2.4=0$

$-1.4+y=0$

$y=1.4$

$y^2=z$

So, $1.4^2=z$

$1.96=z$

So, variance= $\dfrac{\sum(x-\overline{x})^2}{n}$

$=\dfrac{17.2}{5}$

$=3.44$

3. **[2008 P2 #9b]**

Mark	f	D	d²	fd²	fx
20	1	-11	121	121	20x1=20
23	1	-8	64	64	23x1=23
26	2	-5	25	50	26x2=52
27	1	-4	16	3	27x1=27
30	3	-1	1	16	30x3=90
34	2	3	9	18	34x2=68
35	3	4	16	48	35x3=105
40	2	9	81	162	40x2=80
Sum	**15**	0		**482**	**465**

i) from the table, $\sum f=15$

ii) Mean $=\dfrac{\sum fx}{\sum f}$

$\therefore$ Mean $=\dfrac{465}{15}=31$

iii) standard deviation $=\sqrt{\text{variance}}$

now: Variance $=\dfrac{\sum fd^2}{\sum f}$

$=\dfrac{482}{15}$ (from the table)

$\therefore$ standard deviation $=\sqrt{\dfrac{482}{15}}$

$=5.6686277$

$=5.67$(to 3 significant figures)

4. **[2012 P2 #5b]**

Given deviation from mean as follows -2, -3, 4, 1

$x-\overline{x}$	$(x-\overline{x})^2$
-2	4
-3	9
4	16
1	1

$\sum(x-\bar{x})^2 = 30$

n = 4 (given 4 items)

$$SD = \sqrt{\frac{\sum(x-\bar{x})^2}{\sum f}}$$

$$SD = \sqrt{\frac{30}{4}}$$

SD =2.7386

The standard deviation is 2.7

5. **[2015 PII #7a]**

$$s.d = \sqrt{\frac{\sum\left(x-\bar{x}\right)^2}{n}}$$

Since n is missing, we have to first find n

using, $\bar{x} = \frac{\sum x}{n}$, we solve for n:

$$\Rightarrow n = \frac{\sum x}{\bar{x}}$$

$$n = \frac{80}{16} = 5$$

$$s.d = \sqrt{\frac{551.25}{5}}$$

$$s.d = \sqrt{110.25}$$

standard deviation=10.5

6. **[2017 PII #11a]**

First set

$(x-\bar{x})$	$(x-\bar{x})^2$
-5	$(-5)^2 = 25$
-3	$(-3)^2 = 9$
1	$(1)^2 = 1$
Total	35

$Variance(V_1) = 35$

Second set

$(x-\bar{x})$	$(x-\bar{x})^2$
3	$3^2 = 9$
4	$4^2 = 16$
x	$x^2 = x^2$
Total	$x^2 + 25$

$$Variance(V_2) = \frac{x^2+25}{3}$$

$V_2 = V_1$ (given)

$$\frac{x^2+25}{3} = \frac{35}{3}$$

$x^2 + 25 = 35$(multiplying both sides by 3)

$x^2 = 35 - 25$

$\therefore x^2 = 10$

$x = \pm\sqrt{10}$

$x = \sqrt{10}$ or $-\sqrt{10}$

$x = 3.16$ or -3.16 (to 2 dec. places)

7. **[2018 PII #16]**

x	$\bar{x}$	$x-\bar{x}$	$(x-\bar{x})^2$
1	5	-4	16
5	5	0	0
4	5	-1	1
9	5	4	16
4	5	-1	1
6	5	1	1
5	5	0	0
7	5	2	4
1	5	-4	16
8	5	3	9
Sum=50		**Sum=0**	**Sum=64**

Standard deviation= $\sqrt{\text{variance}}$

$$\text{Variance} = \frac{\sum(x-\bar{x})^2}{n}$$

variance=64/10=6.4

Standard deviation $= \sqrt{6.4}$

$= 2.53$ (to 2 dp)

CH 22 SIMULTANEOUS LINEAR AND QUADRATIC EQUATIONS

Chapter Highlights

In this chapters, we will solve problems involving linear and quadratic equations at the same time. You will be required to find the values of the two variables in the equations.

We will mostly use the substitution method of solving simultaneous equation.

1. Solve the simultaneous equations: **[2003 PII #8a]**
$$x^2 - y - 5 = 0$$
$$\tfrac{1}{2}y - x = -1$$

2. Solve the simultaneous equations: **[2004 PI #24]**
$$y = x^2$$
$$y = 5x - 6$$

3. Solve the simultaneous equations: **[2005 PI #24]**
$$xy = -9$$
$$y = x + 6$$

4. A rectangular garden has a perimeter of 40 meters. If its area is 91 m ^{2} , calculate the length of the garden. **[2005 PII #7b]**

5. Solve the simultaneous equations: **[2007 PI #15]**
$$y = x + 2$$
$$x^2 - xy = 4$$

6. Solve the simultaneous equations; **[2007 PII #6b]**
$$m - n = 5$$
$$m^2 - n^2 = 35$$

7. Solve the following simultaneous equation: **[2008 P2 #11b]**
$$x = \frac{8}{y}$$
$$y = x + 2$$

8. Solve the simultaneous equations: **[2008 PI #14]**
$$y = x - 2$$
$$xy + 1 = 0$$

9. Solve the following silmutenous equations **[2010 PII #9b]**
$$x - y = 3$$
$$x + x/y = 8$$

10. Solve the following simultaneous equations: **[2011 PII #8a]**
$$x^2 - y^2 = 0$$
$$x - 2y = 1$$

11. Solve the following simultaneous equations: **[2012 PII #3b]**
$$x - y = 0$$
$$xy = 9$$

12. Solve the following simultaneous equations: **[2013 PI #15]**
$$y = x^2 - 1$$
$$4x = y - 4$$

13. Solve the following simultaneous equations: **[2013 PII #7a]**
$$9x^2 - y^2 = 15$$
$$3x - y = 5$$

14. Solve the following equations simultaneously: **[2019 PII #3a]**
$$6r - 9 = t$$
$$r^2 = t$$

15. Given that $\begin{pmatrix} x^2 & y \\ y & x \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, find the values of x and y. **[2020 Mock PII #7a]**

1. **[2003 PII #8a]**

$x^2-y-5=0$ (1)

$\frac{1}{2}y-x=1$ (2)

Make y the subject of the formula in (2)

$y=2x-2$ *(using Ch.10 techniques)*

substitute $2x-2$ for y in (2)

$x^2-(2x-2)-5=0$

$x^2-2x+2-5=0$

$x-2x-3=0$

$(x-3)(x+1)=0$

Either $(x-3)=0$ or $(x+1)=0$

$x=3$ or $x=-1$

Using equation (2); $y=2x-2$

At $x=3$; $y=2(3)-2=6-2=4$

At $x=-1$; $y=2(-1)-2=-2-2=-4$

$\therefore$ When $x=-1$, $y=-4$

When $x=3$, $y=4$

2. **[2004 PI #24]**

$y=x^2$ (i)

$y=5x-6$ (ii)

Substitute x^2 for y in (ii)

$x^2=5x-6$

$x^2-5x+6=0$

$(x-2)(x-3)=0$ (*factorizing by inspection*)

Either $(x-2)=0$ or $x-3=0$

$x=2$ or $x=3$

if $x=2$, then substitute 2 for x in (i)

$y=2^2$

$y=4$

if $x=3$ then substitute 3 for x in (i)

$y=3^2$

$y=9$

the solutions of the simultaneous equations are as follows

when $x=2$; $y=4$

when $x=3$; $=3$; $y=9$

3. **[2005 PI #24]**

$xy=-9$ (i)

$y=x+6$ (ii)

substitute $x+6$ for y in (i):

$x(x+6)=-9$

$x^2+6x=-9$

$x^2+6x+9=0$

$(x+3)(x+6)=0$

$\therefore x=-3$ or $x=-3$

Substitute -3 for x in (ii)

$y=-3+6$

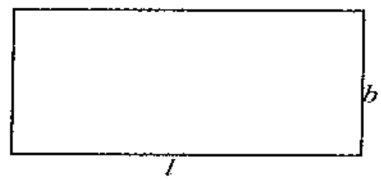
$=3$

$\therefore x=-3$; $y=3$

4. **[2005 PII #7b]**

Let length=l

Let breadth=b



Perimeter = $2(l+b)$

Area = lb

Thus $2(l+b)=40$ (given) (i)

$lb=91$ (given) (ii)

taking (i)

$2(l+b)=40$

$l+b=20$ *(divide both side 2)*

$b=20-l$ (iii)

substitute (iii) in (ii)

$l(20-l)=91$

$20l-l^2=91$

$l^2-20l+91=0$ *(forming a quadratic equation)*

$(l-7)(l-13)=0$

Either $l-7=0$

$=7$

Or $l-13=0$

$=13$

When $l=7$, $b=20-7$

$=13$

When $l=13$, $b=20-13$

$=7$

But since the length must be bigger that width;

Length = 13m and width = 7m.

5. **[2007 PI #15]**

$y=x+2$.........(*i*)

$x^2-xy=4$......(*ii*)

substuting $x+2$ for y in (ii) we get

$x^2-x(x+2)=4$

$x^2-x^2-2x=4$

$-2x=4$

$x=-2$ (divide both sides by -2)

substituting -2 for y in (i), we get

$y = -2 + 2$

$= 0$

$\therefore x = -2; y = 0$

---

6. **[2007 PII #6b]**

$m - n = 5 \quad (i)$

$m^2 - n^2 = 35 \quad (ii)$

using the difference of 2 squares in (ii)

$(m-n)(m+n) = 35$

$5(m+n) = 35$ (since from (i), $m - n = 5$)

$m + n = \frac{35}{5}$

$m + n = 7.....(iii)$

Taking (i) and (iii):

$m - n = 5 \quad (i)$

$m + n = 7.....(iii)$

$2m = 12$ (adding the two and eliminating n)

$m = \frac{12}{2}$

$m = 6$

Substituting in (ii),

$6^2 - n^2 = 35$

$-n^2 = 35 - 36$

$-n^2 = -1$

$n^2 = 1$

$n = \pm\sqrt{1}$

$\therefore n = 1$ or -1

When $m = 6, n = 1$

OR

make m the subject in (i):

$m = 5 + n$

substituting $5+n$ for m in (ii) one gets

$(5+n)^2 - n^2 = 35$

$5(5+n) + n(5+n) - n^2 = 35$

$25 + 5n + 5n + n^2 - n^2 = 35$

$25 + 10n = 35$

$10n = 35 - 25$

$\frac{10}{10}n = \frac{10}{10}$

$\therefore n = 1$

Using (ii),

$m^2 - (1)^2 = 35$

$m^2 = 35 + 1$

$m^2 = 36$

$m = \pm\sqrt{36}$

$m = 6$ or -6

When $n = 1$; $m = 6$

---

7. **[2008 P2 #11b]**

$x = \frac{8}{y} \quad$ (i)

$y = x + 2 \quad$ (ii)

from eqtn (i),

$xy = 8$

Substituting (ii) in (iii):

$x(x+2) = 8$

$x^2 + 2x = 8$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$ (by inspection)

either $(x+4) = 0$ or $(x-2) = 0$

$x = -4$ or $x = 2$

Using eqtn (ii),

At $x = -4$, $y = -4 + 2 = -2$

At $x = 2$, $y = 2 + 2 = 4$

when $x = -4$; $y = -2$

when $x = 2$; $y = 4$

---

8. **[2008 PI #14]**

$y = x - 2 \quad$ (i)

$xy + 1 = 0 \quad$ (ii)

Substituting $x - 2$ for y in (ii):

$x(x-2) + 1 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$ (factorisation)

$\Rightarrow x = 1$

substituting 1 for x in(i):

$y = 1 - 2$

$y = -1$

When $x = 1$; $y = -1$

9. **[2010 PII #9b]**

$x - y = 3$ ....(i)

$x + x/y = 8$ ....(ii)

Make x in equation (i) the subject [Ch.10]

$x - y = 3$

$x = 3 + y$ ....(iii)

Substitute in equation (ii)

$x + x/y = 8$ ....(ii)

$(3+y) + (3+y)/y = 8$

$y(3+y) + (3+y) = 8y$ multiplying through by y

$3y + y^2 + 3 + y = 8y$

$y^2 + 3y + y - 8y + 3 = 0$

$y^2 - 4y + 3 = 0$

$(y-3)(y-1) = 0$ (by inspection)

Either $(y-3) = 0$ or $(y-1) = 0$

$y = 3$ or $y = 1$

Substitute in equation (iii)

When $y = 3$;

$x = 3 + y$

$x = 3 + 3$

$x = 6$

When $y = 1$;

$x = 3 + y$

$x = 3 + 1$

$x = 4$

<u>Solutions:</u>

When $x = 6$, $y = 3$

and

When $x = 4$, $y = 1$

---

10. **[2011 PII #8a]**

$x^2 - y^2 = 0$ (i)

$x - 2y = 1$ (ii)

making x subject in (ii)

$x = 2y + 1$

$(2y+1)^2 - y^2 = 0$ substituting for x in (i)

$(2y+1)(2y+1) - y^2 = 0$

$2y(2y+1) + 1(2y+1)$ split and spread method of expansion

$4y^2 + 2y + 2y + 1 - y^2 = 0$

$4y^2 + 4y + 1 - y^2 = 0$

$3y^2 + 4y + 1 = 0$

$3y^2 + 3y + y + 1 = 0$

$3y(y+1) + 1(y+1) = 0$

$(y+1) + (3y+1) = 0$

either $y + 1 = 0$ or $3y + 1 = 0$

$y = -1$ or $3y = -1$

$y = -1$ or $\frac{3y}{3} = \frac{-1}{3}$

$y = -1$ or $y = -\frac{1}{3}$

when $y = -1$

$x = 2(-1) + 1 = -1$

when $y = \frac{-1}{3}$

$x = 2(-1/3) + 1 = 1/3$

<u>Solutions</u>

When $x = -1$, $y = -1$

When $x = \frac{1}{3}$, $y = -\frac{1}{3}$

---

11. **[2012 PII #3b]**

$x - y = 0$ (i)

$xy = 9$ (ii)

making x subject of formula in (i)

$x = y$

substituting x in (ii)

$y \times y = 9$

$y^2 = 9$

$y = \pm\sqrt{9}$

$y = \pm 3$

substituting $y = 3$ in (i)

$x - 3 = 0$

$\therefore x = 3$

and substituting $y = -3$ in (i) gives $x = -3$

<u>Solutions:</u>

When $x = -3$, $y = -3$

When $x = 3$, $y = 3$

---

12. **[2013 PI #15]**

$y = x^2 - 1 \ldots\ldots\ldots (i)$

$4x = y - 4 \ldots\ldots\ldots (ii)$

$4x = (x^2 - 1) - 4(\text{substitution})$

$4x = x^2 - 5$

$x^2 - 4x - 5 = 0$

$(x+1)(x-5) = 0$ (by inspection)

either $(x+1) = 0$ or $(x-5) = 0$

$x = -1$ or $x = 5$

when $x = -1$

$y = (-1)^2 - 1$

$= 1 - 1$

$= 0$

when $x = 5$

$y = 5^2 - 1$

$y = 25 - 1$

$y = 24$

$\begin{pmatrix} \text{when } x = -1, y = 0 \\ \text{when } x = 5, y = 24 \end{pmatrix}$

---

13. **[2013 PII #7a]**

$9x^2 - y^2 = 15 \ldots\ldots\ldots (i)$

$3x - y = 5 \ldots\ldots\ldots (ii)$

Using the difference of 2 squares, equation (i) becomes:

$(3x)^2 - (y)^2 = 15$

$(3x - y)(3x + y) = 15$

$5(3x + y) = 15$ [Since in (ii), $3x - y = 5$]

$3x + y = \frac{15}{5}$

$3x + y = 3 \ldots (iii)$ but

$3x - y = 5 \ldots (ii)$

$6x = 8$ (by elimination)

$x = \frac{8}{6}$

$x = \frac{4}{3}$

To find y, using (iii):

$\cancel{3}(\frac{4}{\cancel{3}}) + y = 3$

$4 + y = 3$

$y = 3 - 4$

$y = -1$

Solutions:

when $x = \frac{4}{3}$; $y = -1$

OR

Making y subject in *(ii)*

$y = 3x - 5$

Substitute for y in *(i)*

$9x^2 - y^2 = 15$ becomes:

$9x^2 - (3x - 5)^2 = 15$

$9x^2 - [3x(3x-5) - 5(3x-5)] = 15$ split and spread

$9x^2 - [9x^2 - 15x - 15x + 25] - 15 = 0$

$9x^2 - [9x^2 - 30x + 25] - 15 = 0$

$9x^2 - 9x^2 + 30x - 25 - 15 = 0$

$30x - 40 = 0$

$30x = 40$

$x = \frac{40}{30}$

$x = \frac{4}{3}$

From equation *(ii)*, $y = 3x - 5$

So, taking $x = \frac{4}{3}$

$y = 3(\frac{4}{3}) - 5$

$= 4 - 5$

$= -1$

Solutions:

When $x = \frac{4}{3}$, $y = -1$

---

14. **[2019 PII #3a]**

$6r - 9 = t \ldots\ldots\ldots (i)$

$r^2 = t \ldots\ldots\ldots (ii)$

Equation (i) is equal to equation (ii) (both are t)

$\Leftrightarrow 6r - 9 = r^2$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$ by inspection

Either $(r-3) = 0$ or $(r-3) = 0$

$\therefore r = 3$, substituting in (ii)

$t = 3^2$

$t = 9$

When $r = 3$, and $t = 9$

---

15. **[2020 Mock PII #7a]**

$$\begin{pmatrix} x^2 & y \\ y & x \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Using Matrix Multiplication,

$2x^2 - y = 0.....(i)$

$2y - x = 3....(ii)$

From (i),

$2x^2 = y$

Substituting in (ii):

$2(2x^2) - x = 3$

$4x^2 - x - 3 = 0$

$4x^2 - 4x + 3x - 3 = 0$ by identifying factors

$4x(x-1) + 3(x-1) = 0$

$(4x+3)(x-1) = 0$

Either $4x + 3 = 0$ or $x - 1 = 0$

$4x = -3$ or $x = 1$

$x = -\frac{3}{4}$

when $x = -\frac{3}{4}$,

$y = 2(-\frac{3}{4})^2$

$y = \frac{9}{8}$

when $x = 1$,

$y = 2(1)^2$

$y = 2$

Solutions

When $x = -\frac{3}{4}$, $y = \frac{9}{8}$

When $x = 1$, $y = 2$

CH 23 PROGRESSIONS

Chapter Highlights

In this topic, we solve problem in arithmetic and geometric progression/series. The difference between an arithmetic and geometric progression is that in geometric progression we consider a series developed by multiplying a term by common ratio to obtain the consecutive while an arithmetic progression is a series where the consecutive terms have a common difference between them i.e. the same difference is added to a term to find the next term.

One of the skills needed to successfully solve the problems in this topic is the ability to find the n^{th} term of an arithmetic and geometric progression.

Arithmetic progression (AP):

n^{th} term $= a + d(n-1)$........n^{th} term

- The formula for the sum of n terms is:

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \text{ or}$$

$$S_n = \frac{n}{2}(a + l)$$

Where:

S_n is the sum of n terms

a is the first term

n is the position of the n^{th} term

d is the common difference

l is the last term

Geometric Progression (GP):

n^{th} term $= ar^{n-1}$ and $S_n = \dfrac{a(1-r^n)}{1-r}$

Arithmetic Progression (AP)

1. The sum of n terms of an arithmetic progression is $\frac{5n^2+3n}{2}$. Calculate the first two terms of the arithmetic progression. **[2004 P2 #3a]**

2. The nth term of an Arithmetic Progression is $5n$-3. Calculate the sum of the first 6 terms of the AP. **[2005 PI #18]**

3. The ratio of the 2nd term to the 7th term of an arithmetic progression is 1:3 and their sum is 20. Calculate the sum of the first 10 terms of the progression. **[2005 PII #10b]**

4. The fourth term of an arithmetic progression is 11 and the seventh term is 20. Calculate the first term. **[2007 PI #14]**

5. Find the sum of the first 20 terms of the arithmetic progression 4, 2, 0,............ **[2008 P1 #8]**

6. The first term of an arithmetic progression is 5 and the last term is 43. If the sum of the terms is 480. Calculate the number of terms. **[2010 PII #1a]**

7. The 3rd and 9th terms of an arithmetic progression (AP) are 29 and 8, respectively. Calculate the 20th term of the progression. **[2011 PI #16]**

8. Given that the first term of an AP is -8 and 11th term is 22, calculate the 7th term. **[2011 PII #5b]**

9. The second term of an arithmetic progression (AP) is 7 and 10th term is 39. Calculate the Common difference of the AP. **[2012 P2 #2a]**

10. The sum of the first 5 terms of an arithmetic progression (AP) is 35. If the common difference is 2, find the first term. **[2017 PI #16]**

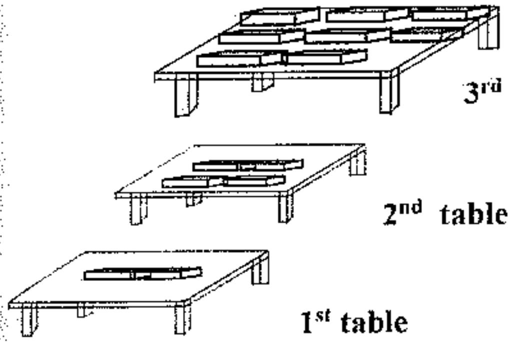
11. The initial speed of a car is 47km/hr. The speed increases by 12km every hour. If the car maintained the same pattern of speed, calculate the speed of the car in the 7th hour. **[2017 PII #9a]**

12. The 3rd term of an arithmetic progression (AP) is half the 6th term. If the sum of the first 20 terms is 420, find 15th term of the AP. **[2019 PII #11]**

Geometric Progression (GP)

13. The first three terms of a G.P are $x+1, x^2-1$ and $(x^2-1)(2x-4)$. Calculate the value of x. **[2003 PP1 #17]**

14. **Figure 1,** shows the display of the new types of bricks laid on down on the tables. On the 1st table there are 2 bricks, on the 2nd table there are 4 bricks, on the 3rd table there are 8 bricks and so on.



If the nth table there are 1024 bricks, calculate the value of n. **[2003 P1#11a]**

15. The sum of the two terms of a geometric progression (GP) is 4. If the first term is 3, find the common ratio. **[2004 PP1 #9]**

16. Find the sum of the first 12 terms of the following GP:

$$\frac{1}{2187}, \frac{1}{729}, \frac{1}{243}, \ldots\ldots$$

Give your answer correct to 2 decimal places. **[2006 PI #22]**

17. The sum of the first n terms of a geometric progression, GP is $2^{(n+2)}-4$.

Calculate;

i. The first term of the GP
ii. The common ratio of the GP.

[2006 PII #12a]

18. The first term of geometric progression (GP) is 81, and the common ratio is $\frac{1}{3}$. Calculate the fourth term. **[2007 PII #1b]**

19. The second term of geometric progression (GP) is -6 and the fourth term is -54. Calculate the common ratio, given that it is negative. **[2008 P2 #7a]**

20. A geometric progression (G.P.) has 6 terms. If its first term is 3 and the last term is 96, calculate the common ratio. **[2010 P1 #6]**

21. If the 4th and 7th terms of a geometric progression (GP) are 4 and 32 respectively calculate the first term. **[2012P1#17]**

22. Given that 5^{th} and the 10^{th} terms of a geometric progression (GP) are 4 and $\frac{1}{8}$ respectively, find the common ratio. **[2013 P1 #8]**

23. The last term of a geometric progression $3, 9, 27, \ldots\ldots\ldots\ldots$ is 729. Calculate the number of terms. **[2014 PI #3a]**

24. The third and sixth terms of a geometric progression (GP) are 27 and 1 respectively. Find the first term. **[2015 PII #3a]**

25. The third term of a geometric progression (GP) is 5 and its common ratio is -2. Calculate the sixth term of the GP. **[2018 P1 #6]**

26. The first, second and fifth terms on an arithmetic progression (AP), form a Geometric Progression (GP). If the first term is 3, find the common difference of the AP. **[2021 Mock PII #8a]**

1. **[2004 P2 #3a]**

Let the sum of n terms be S_n

Then $S_n = \frac{5n^2+3n}{2}$

$S_1 = \frac{5(1)^2+3(1)}{2}$ *(the sum of the first terms)*

$= \frac{5(1)+3}{2}$

$= \frac{5+3}{2}$

$= \frac{8}{2}$

$a = S_1 = 4$ *(by logic S_1 is the first term)*

$S_2 = \frac{5(2)^2+3(2)}{2}$

$= \frac{5(4)+6}{2}$

$= \frac{20+6}{2}$

$= \frac{26}{2}$

$S_2 = 13$

$S_2 = a_1 + a_2$ *(sum of first term and second term)*

$13 = 4 + a_2$ *(since $a_1 = S_1 = 4$)*

$a_2 = 13 - 4$

$a_2 = 9$

$\therefore$ the first two terms of the progression are 4 and 9.

2. **[2005 PI #18]**

nth term of the arithmetic progression (AP) $= 5n - 3$
1st term $= 5 \times 1 - 3 = 5 - 3 = 2$
2nd term $= 5 \times 2 - 3 = 10 - 3 = 7$
$\therefore$ the AP is 2, 7 ......
Sum of n terms of AP is given by the formula

$S_n = \frac{n}{2}[2a+(n-1)d]$

from the AP, a = 2, n = 6, d = 7 − 2 = 5

$\therefore S_6 = \frac{6}{2}[2\times 2+(6-1)5]$

$= 3[4+(5)5]$
$= 3(4+25)$
$= 3 \times 29$
$= 87$

$\therefore$ the sum of the first 6 terms is 87.

OR:

Because there are fewer terms from 1 to 6^{th} term, this alternative 'layman' method can be used:

nth term $=5n-3$

1st term $= 5(1)-3 = 5-3 = 2$

2nd term $= 5(2)-3 = 10-3 = 7$

3rd term $= 5(3)-3 = 15-3 = 12$

$4th$ term $= 5(4)-3 = 20-3 = 17$

$5th$ term $= 5(5)-3 = 25-3 = 22$

$6th$ term $= 5(6)-3 = 30-3 = 27$

So the sum of the first 6 terms is:

$2+7+12+17+22+27 = 87$

3. **[2005 PII #10b]**

nth term of an A.P $= a + (n-1)d$
2^{nd} term $= a + (2-1)d = a + d$
7^{th} term $= a + (7-1)d = a + 6d$
The ratio of 2^{nd} term to 7^{th} term = 1:3 *(given)*
$\therefore a + d : a + 6d = 1:3$

$\frac{a+d}{a+6d} = \frac{1}{3}$

$3(a+d) = 1(a+6d)$ cross-multipying

$3a + 3d = a + 6d$

$3a - a + 3d - 6d = 0$

$2a - 3d = 0$ .... (i)
2^{nd} term + 7^{th} term = 20
$2a + 7d = 20$ ......(ii)
Substituting $10d = 20 \Rightarrow d = 2$
Substitute 2 for d in (i)
$2a - 3(2) = 0$
$2a - 6 = 0$
$2a = 6 \Rightarrow a = 3$ *(divide both sides by 2)*
Now, the sum of the first 10 terms:
Sum of n terms of an A.P

$S_n = \frac{n}{2}[2a+(n-1)d]$

$n = 10, a = 3, d = 2$

$\therefore S_{10} = \frac{10}{2}[2(3)+(10-1)2]$

$= 5[6 + (9)2]$
$= 5[6 + 18]$
$= 5 \times 24$
$= 120$

$\therefore$ the sum of the first 10 terms is 120.

[2007 PI #14]

n^{th} term $= a+(n-1)d$,

So, 4^{th} term $= a+(4-1)d$

$a+(4-1)d = 11$ (given)

$a+3d = 11 \ldots\ldots\ldots(i)$

7^{th} term $= a+(7-1)d$

$a+(7-1)d = 20$ (given)

$a+6d = 20 \ldots\ldots\ldots\ldots\ldots(ii)$

we have the two equations:

$a+3d = 11 \ldots\ldots\ldots\ldots\ldots(i)$

$a+6d = 20 \ldots\ldots\ldots\ldots\ldots(ii)$

Eliminating d:

$-2a-6d = -22$

$\underline{a+6d = 20}$

$-a = -2$

$a = 2$

$\therefore$ the first term is 2.

---

5. **[2008 P1 #8]**

Given an arithmetic progression: 4, 2,0.....

$S_n = \frac{n}{2}[2a+(n-1)d]$,

Taking $n = 20, a = 4$,

$d = 2-4 = -2$

$\therefore S_{20} = \frac{20}{2}[2\times4+(20-1)(-2)]$

$= 10[8+(19)(-2)]$

$= 10(8-38)$

$= 10\times(-30)$

$= -300$

$\therefore$ The sum of the first 20 terms is -300.

[2010 PII #1a]

In AP

$a = 5,\ l = 43,\ s_n = 480$

$s_n = \frac{n}{2}(a+l)$

$480 = \frac{n}{2}(5+43)$

$480 = \frac{n}{2}(48)$

$480 = 24n$

$\frac{480}{24} = \frac{24n}{24}$

$n = 20$

$\therefore$ There are 20 terms

OR:

If you are not familiar with the Sum formula which uses the last term, here is an alternative.
To find the position of the last term, equate the nth term to the last term.

nth term $= a+d(n-1)$

$43 = a+d(n-1)$

But $a = 5$ (given)

$43 = 5+(n-1)d$

$43-5 = (n-1)d$

$38 = \boxed{(n-1)d}$

Kindly note that this term also appears in the Sum formula.

Take the Sum of n terms, S_n:

$S_n = \frac{n}{2}[2a+\boxed{(n-1)d}]$

$480 = \frac{n}{2}[2(5)+38]$

$480 = \frac{n}{2}[10+38]$

$2\times\frac{\cancel{480}}{\cancel{48}} = n(\frac{\cancel{48}}{\cancel{48}})$

$2\times10 = n$

$20 = n$

$\therefore$ There are 20 terms

7. **[2011 PI #16]**

nth term $= a+d(n-1)$

$3rd$ term $= a+d(3-1)$

$a+2d = 29 \quad \ldots(i)$

$9th$ term $= a+d(9-1)$

$a+8d = 8 \quad \ldots(ii)$

solve the simultaneous equation

$a+2d = 29$

$\underline{a+8d = 8}$

$-6d = 21$

$d = \frac{-21}{6} = -\frac{7}{2}$

Using eqtn (i), we find a:

$a+2(-\frac{7}{2}) = 29$

$a-7 = 29$

$a = 29+7$

$a = 36$

To find the 20th term:

20th term $= a+(20-1)d$

$= 36+19(-\frac{7}{2})$

$= 36-66.5$

$= -30.5$

8. **[2011 PII #5b]**

nth term $= a+(n-1)d$

$11th$ term $= a+(11-1)d$

$a+10d = 22$ given ...(i)

$-8+10d = 22$ also given that $a = -8$

$10d = 22+8$

$10d = 30$

$d = 3$

to find 7th term

$7th = a+(7-1)d$

$= -8+(7-1)\times 3$

$= -8+6\times 3$

$= -8+18$

$7th$ term $= 10$

9. **[2012 P2 #2a]**

For an AP: nth term $= a+(n-1)d$,

2^{nd} term $= a+(2-1)d$

$a+d = 7$ (i)

10^{th} term $= a+(10-1)d$

$a+9d = 39$ (ii)

subtracting (i) from (ii)

$$\begin{array}{r} a+9d = 39 \\ -\ \ a+d = 7 \\ \hline 8d = 32 \\ d = 4 \end{array}$$

$\therefore$ The common difference is 4

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

10. **[2017 PI #16]**

$Given: S_n = 35;\ n = 5; d = 2; a = ?$

$S_n = \frac{n}{2}[2a+d(n-1)]$

$35 = \frac{5}{2}[2a+2(5-1)]$

$70 = 5[2a+2(4)]$

$\frac{70}{5} = 2a+8$

$14-8 = 2a$

$\frac{6}{2} = \frac{2}{2}a$

$3 = a$

$\therefore$ The first term is 3.

11. **[2017 PII #9a]**

$sequence:$

$47 km/h,$

$47+12 km/h,$

$47+12+12 km/h, \ldots \therefore an\ AP.$

or

$47, 59, 71, \ldots$

$a = 47$

$d = 59-47 = 12$

$So,\ n^{th}\ term = a+(n-1)d$

$7^{th}\ term = 47+(7-1)12$

$= 47+6\times 12$

$= 47+72$

$= 119$

The speed of the car in the 7^{th} hour was 119 km/h.

12. **[2019 PII #11]**

$n^{th}\ term = a+(n-1)d$

$3rd\ term = a+(2-1)d$

$= a+2d$

$6th\ term = a+(6-1)d$

$= a+5d$

But $3rd\ term = \frac{1}{2}(6th\ term)$

$a+2d = \frac{1}{2}(a+5d)$

$2(a+2d)=a+5d$

$2a+4d=a+5d$

$2a-a=5d-4d$

$a=d$..................(i)

But $S_n=\frac{n}{2}(2a+(n-1)d)$

$S_{20}=\frac{20}{2}[2a+(20-1)d]=420$

$10(2a+19d)=420$

$\frac{10(2a+19d)}{10}=\frac{420}{10}$

$2a+19d=42$...............(ii)

Substitute equation (i) in equation (ii)

$2(d)+19d=42$

$21d=42$

$\frac{\cancel{21}d}{\cancel{21}}=\frac{\cancel{42}^2}{\cancel{21}_1}$

$d=2$

Thus, $a=2$ [from (i) above]

$15th\ term=a+14d$

$=2+14(2)$ (since $a=d=2$)

$=2+28$

$=30$

13. **[2003 PP1 #17]**

Given 3 terms of a GP:

$x+1, x^2-1, (x^2-1)(2x-4)$

Common ratio= $\frac{x^2-1}{x+1}=\frac{(x+1)(x-1)}{x+1}=x-1$

Or $\frac{(x^2-1)(2x-4)}{x^2-1}=2x-4$

Since the common ratio is constant, then

$x-1=2x-4$

$x-2x=-4+1$

$-x=-3$ *divide both sides by* -1

$\therefore x=3$

14. **[2003 P1#11a]**

nth table	Number of bricks
1	$2=2^1$
2	$4=2^2$
3	$8=2^3$
N	$1024=2^N$

The number of bricks increasing exponentially. We therefore have a geometrical progression (GP) as follows:

2, 4, 8......

Common ratio (r) = $4/2=2$

The nth term = ar^{n-1}

$a=2$, $r=2$, nth term=1024, $n=?$

$\therefore n$th term $=2\times 2^{n-1}$

$1024 = 2\times 2^{n-1}$

$512 = 2^{n-1}$

$2^9=2^{n-1}$

$9=n-1$ (equal bases)

$n=9+1$

$n=10$

$\therefore$the value of n is 10.

15. **[2004 PP1 #9]**

Sum of the first two terms of GP= 4

1st term= a_1; 2nd term = a_2

So $a_1+a_2=4$

So $3+a_2=4$ (given)

$a_2=4-3$

$a_2=1$

Common ratio = a_2/a_1

$\therefore r=1/3$

16. **[2006 PII #22]**

Sum of n terms, S_n, of a G.P.:

$$S_n=\frac{a(r^n-1)}{r-1}$$

Where: a=1st term; r=common ratio; and n =number of terms.

$a = \frac{1}{2187}; r = \frac{1}{729} \div \frac{1}{2187} = 3, n = 12$

$\therefore S_n = \dfrac{\frac{1}{2187}\left(3^{12}-1\right)}{3-1}$

$= \dfrac{1}{2187} \times \dfrac{531441-1}{2}$

$= \dfrac{1}{2187} \times \dfrac{531440}{2}$

$= \dfrac{1}{2187} \times 265720$

$= 121.49977$

$= 121.50$ (to 2dp)

17. **[2006 PII #12a]**

Given $S_n = 2^{n+2} - 4$ of a G.P

i. the 1st term of the GP

n	Sum (S_n)$(2^{n+2})-4$	n th term
1	$2^{1+2}-4=8-4=4$	4
2	$2^{2+2}-4=16-4=12$	$(12-4)=8$
3	$2^{3+2}-4=32-4=28$	$(28-12)=16$

$\therefore$ 1st term = 4

ii. Common ratio (r) = $\dfrac{8}{4} = \dfrac{16}{8} = 2$

18. **[2007 PII #1b]**

Given: $a = 81, r = \frac{1}{3}$

for GP: n^{th} term $= ar^{n-1}$

4^{th} term $= 81\left(\frac{1}{3}\right)^{4-1}$

$= 81\left(\frac{1}{3}\right)^3$

$= 81 \times \frac{1}{27}$

$= 3$

$\therefore$ the fourth term of the GP is 3.

19. **[2008 P2 #7a]**

n^{th} term of $GP = ar^{n-1}$

2^{nd} term $= ar^{2-1}$

$ar = -6 \quad ...(i)$

4^{th} term $= ar^{4-1}$

$ar^3 = -54 \quad ...(ii)$

$(ii) \div (i)$: $\dfrac{ar^3}{ar} = \dfrac{-54}{-6}$

$r^2 = 9$

$r = \pm\sqrt{9}$

$= \pm 3$

since only the negative number is required, then,

$r = -3$

$\therefore$ the common ratio is -3.

20. **[2010 P1 #6]**

Given $l = 96, \quad a = 3, \quad n = 6$

n^{th} $term = ar^{n-1}$

6^{th} $term = 3 \times r^{6-1}$

$96 = 3r^5$ (the last term=the 6th term)

$\dfrac{96}{3} = \dfrac{3r^5}{3}$

$32 = r^5$

$2^5 = r^5$

$r = 2$ (equal powers)

$\therefore$ The common ratio is 2

21. **[2012 P1 #17]**

The n^{th} term of a GP $= ar^{n-1}$,

$4th$ term $= ar^{4-1}$

$ar^3 = 4$ (given)....(i)

$7th$ term$=ar^{7-1}$

$ar^6 = 32$ (given)....(ii)

dividing the 7th term by the 4th term:

$\dfrac{ar^6}{ar^3} = \dfrac{32}{4}$

$r^3 = 8$

$r^3 = 2^3$

$\therefore r = 2$ (equal powers)

substituting r in (i):

$a(2)^3 = 4$

$8a = 4 \ \therefore a = \frac{1}{2}$

$\therefore$ The first term $= \frac{1}{2}$

22. **[2013 P1 #8]**

n^{th} term of a GP $= ar^{(n-1)}$

$5^{th} = ar^{5-1} = ar^4$

$10^{th} = ar^{10-1} = ar^9$

$ar^4 = 4......(i)$

$ar^9 = \frac{1}{8}......(ii)$

Divide (ii) by (i)

$\frac{ar^9}{ar^4} = \frac{1}{8} \div \frac{4}{1}$

$r^5 = \frac{1}{8} \times \frac{1}{4}$

$r^5 = \frac{1}{32}$

$\therefore r = \sqrt[5]{\frac{1}{32}}$

$\therefore r = \frac{1}{2}$

23. **[2014 PI #3a]**

$r = 9 \div 3 = 3$

$l = 729$

$ar^{n-1} = 729$

$3r^{n-1} = 729$

$\frac{\cancel{3}}{\cancel{3}}r^{n-1} = \frac{\cancel{729}}{\cancel{3}}$

$3^{n-1} = 243$

$3^{n-1} = 3^5$

$n - 1 = 5$

$\therefore n = 5 + 1$

$n = 6$

Since the last term is the 6th term,

$\therefore$ *there are 6 terms in the progression*

24. **[2015 PII #3a]**

n^{th} term of a GP $= ar^{(n-1)}$

$ar^{(3-1)} = 27$

$ar^2 = 27..................(i)$

$ar^{(6-1)} = 1$

$ar^5 = 1..................(ii)$

divide (ii) by (i)

$\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{1}{27}$

$r^3 = \frac{1}{27}$

$r = \sqrt[3]{\frac{1}{27}}$

$\therefore r = \frac{1}{3}$

Substitute $r = \frac{1}{3}$ in any eqtn to find the 1st term

$ar^2 = 27$

$a\left(\frac{1}{3}\right)^2 = 27$

$\frac{a}{\cancel{9}} \times \cancel{9} = 27 \times 9$

$a = 27 \times 9$

$a = 243$

$\therefore$ The first term of the GP is 243.

25. **[2018 P1 #6]**

$n^{th} term = ar^{n-1}$ ....(n^{th} term of a G.P)

$3^{rd}\ term = a(-2)^{3-1}$ (since $r = -2$)

$3^{rd}\ term = a(-2)^2$

$= 4a$

$3^{rd}\ term = 5$......................... (given)

Thus, $4a = 5$

$\therefore a = \frac{5}{4}$

To calculate the 6^{th} Term;

$6th$ term$=ar^{6-1}$

$= ar^5$

Taking $a = \frac{5}{4}$, $r = -2$

$6th$ term$=(\frac{5}{4})(-2)^5$

$= \frac{5}{4} \times -32$

$= 5 \times -8$

$= -40$

26. **[2021 Mock PII #8a]**

nth term of AP$=a+(n-1)d$

$1st$ term $=a=3$

$2nd$ term $=3+(2-1)d$

$=3+d$

$5th$ term $=3+(5-1)d$

$=3+4d$

But 3, $3+d$, $3+4d$ form a G.P.

So, $\frac{3}{3+d}=\frac{3+d}{3+4d}$ (common ratio of GP is equal)

$3(3+4d)=(3+d)(3+d)$

$9+12d=3(3+d)+d(3+d)$

$9+12d=9+3d+3d+d^2$

$d^2+6d-12d+9-9=0$

$d^2-6d=0$

$d(d-6)=0$

Either $d=0$ or $d-6=0$

$d=0$ or $d=6$

Thus, the common difference is 6.

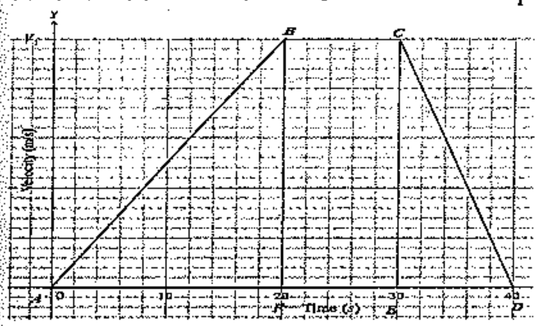
CH 24 TRAVEL GRAPHS

Chapter Highlights

Travel graphs are used to describe the motion of an object. In this chapter, we will solve problem on drawing and interpreting velocity-time graphs. We will also encounter problems that require us deduce velocity-time graphs from a given information and use such graphs to calculate acceleration/deceleration and distance.
We will also use the relationship between speed, time and distance in solving some problem:

$$\text{Average speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

Note:
The total distance travelled is found from a velocity-time graph by calculating the area under the graph. In the figure below, the total distance travelled is the area of **ABCD** which is a trapezium.



Distance covered from **A** to **B** is equal to the area of the triangle **ABF**, distance covered between **B** and **C** is the area of rectangle **BCEF.**

The acceleration of the object between two points is found from a velocity-time graph by calculating the gradient of the line passing through these points. In the figure, acceleration from **A** to **B** is the gradient of line segment **AB**.

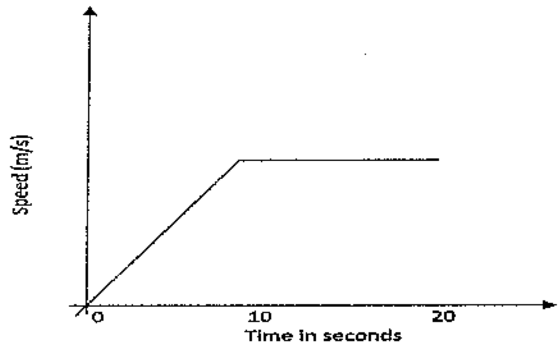
$$\text{Acceleration} = \frac{\text{Final velocity - Initial velocity}}{\text{Time taken}}$$

Gradient AB =

$$\frac{\text{Change in y}}{\text{Change in x}} = \frac{\text{Final velocity - Initial velocity}}{\text{Time taken}}$$

$\therefore$ Gradient = Acceleration
At **A** the object is at rest i.e. V= 0 m/s.
The object started from rest at **A** (V = 0 m\s) and accelerated with uniform velocity V_1 to **B** in 20 seconds.
From point **B** to **C**, the object is moving at constant motion for 10 seconds.
At point **C**, the object is moving at a velocity V_1 and uniformly decelerates to point **D** in 10 seconds.

1. **Figure 1** shows a speed-time graph for a particle during the first 20 seconds of its motion.

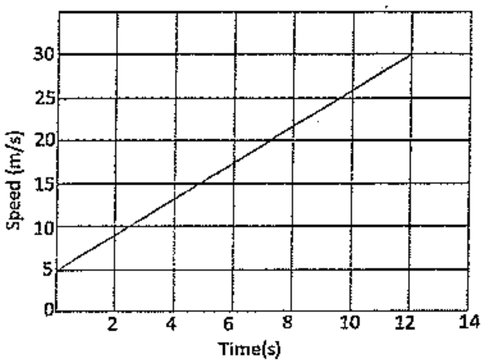


Calculate the particle's average speed during the 20 seconds. **[2003 PP1 #19]**

2. A vehicle travels from **P** to **Q** in 8 hours. It starts from rest at **P** increasing its speed steadily to 200km/h in 2 hours. It then travels at the speed steadily for the next 2 hours. Then it decelerated constantly to rest at **Q**, in 4hours.
i. sketch a speed-time graph of the vehicle.
ii. using the graph in **(i)**, calculate the distance the vehicle has travelled has travelled from **P** to **Q.**
[2003 P2 #9a]

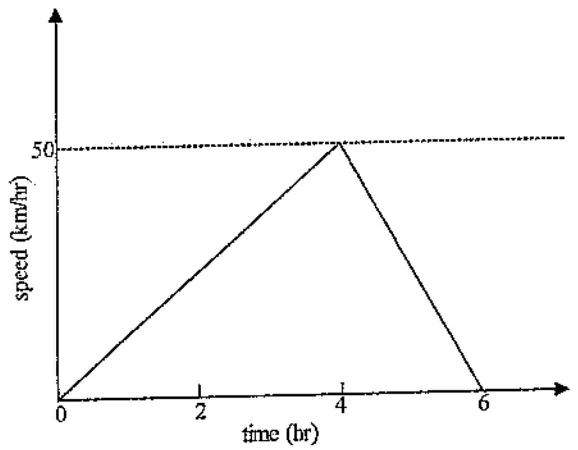
3. A minibus and a lorry left Dedza at the same time for Liwonde a distance of 160km. The minibus travelled at an average speed which is 10km/h faster than the lorry. It arrived at Liwonde 32 minutes earlier that the lorry.
i. Write down an expression for the time taken by the minibus.
ii. Write down an expression for the time taken by the lorry.
iii. Hence, form an equation in x and solve it to find the average speed of the minibus.
[2003 PII #12b]

4. **Figure 2** shows the speed time graph of a moving object.



Use the graph to find the acceleration of the object. **[2004 PI #12]**

5. **Figure 3,** shows a speed-time graph for a car during the first 6 hours.



Calculate the total distance travelled during the 6 hours. **[2005 P1 #19]**

6. A cyclist starts from rest and accelerates uniformly for 4 minutes to reach a speed of 300 meters/minute. She then maintains this speed for 6 minutes after which she decelerates uniformly for 5 minutes to come to a complete stop.

   Using a scale of 2cm to represent 50 meters/minute in the vertical axis and 2cm to represent 2 minutes in the horizontal axis. Draw a speed-time graph for the cyclist on a graph paper. **[2005 PII #5b]**

7. **Table 1** shows the speed of a train recorded every 10 seconds.

Speed (m/s)	200	400	600	800
Time (s)	10	20	30	40

Using a scale of 2cm to represent 100m/s on the vertical axis and 2m to represent 10 seconds on the horizontal axis, draw a speed-time graph and use it to calculate the acceleration of the train.

[2006 PI #12]

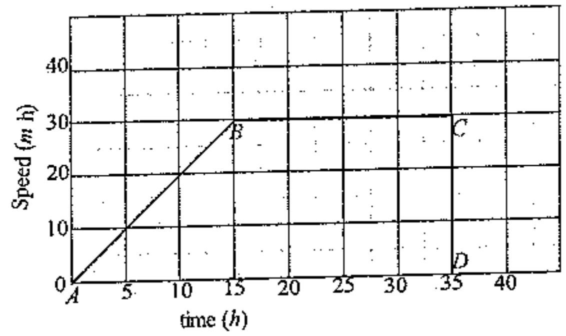
8. A person running at a speed by y meters per second passes a stationery train 50 meters long. If the person takes $\frac{25}{y-1}$ seconds to pass the train, calculate the value of y. **[2006 PII #3b]**

9. A motorist accelerates uniformly from the rest to 40m/s in 8 seconds. He then accelerate uniformly to 90m/s in the next 6 seconds.

   i. Using the scale of 2cm to represent 20m/s on the vertical axis and 2cm to represent 2 seconds on the horizontal axis, draw a speed-time graph for the first 14 seconds.

   ii. Calculate the acceleration for the last 6 seconds. **[2006 PII #4b]**

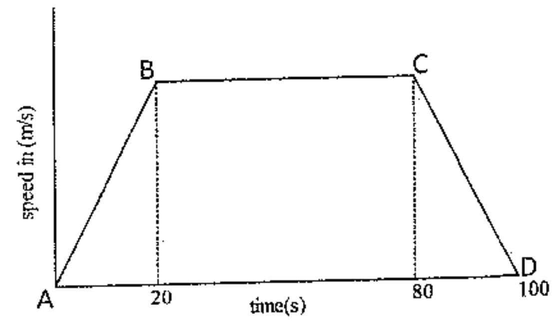
10. **Figure 4** shows the speed time graph of a moving object.



Use the graph to find the total distance travelled by the object in the first 35 seconds. **[2007 PII #19]**

11. Mr. Njinga cycled a distance of 42 km from his village. On his return journey he increased his speed by 2km/h and took half an hour less. Calculate the average speed on the journey from his village. **[2008 P2 #8b]**

12. **Figure 5** Shows a speed time graph of a car.



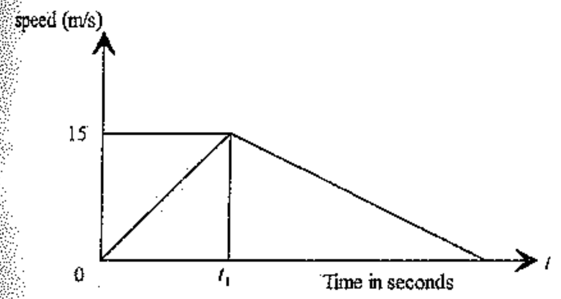
Given that the total distance travelled is 3200m, calculate the deceleration during the last 20 seconds. **[2008 P1 #16]**

13. An object starts from rest and accelerates uniformly at $5m/s^2$ in 4 seconds. If further accelerates uniformly to a velocity of 90m/s in the next 3 seconds. It maintains this velocity for 2 seconds and then it is brought to rest in another 5 seconds.

    i. Using a scale of 2cm to represent 20m/s on the vertical axis and 2 cm to represent 2 seconds on the horizontal axis. Draw a speed-time graph for the motion of the object.

    ii. Using your graph, calculate the acceleration in the last seconds. **[2010 PII #8b]**

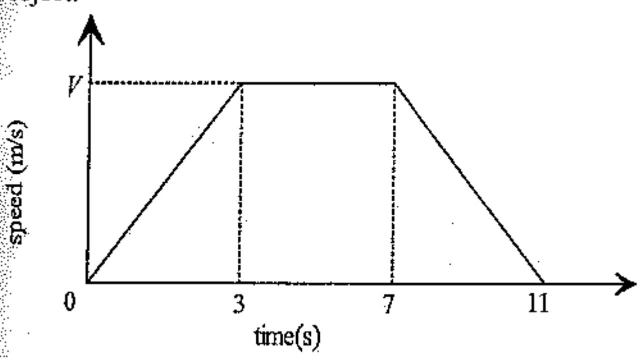
14. **Figure 6** is a speed-time graph of a moving object.



Given that the acceleration for the t_1 seconds is 0.3 ms^{-2}, calculate the value of t_1.

[2011 PI #9]

15. **Figure 7** shows the speed-time graph of a moving object.

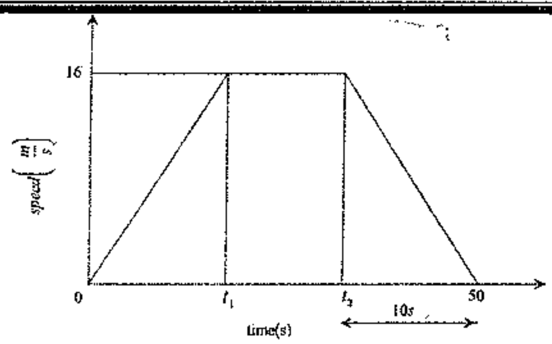


If the distance travelled by the object was 90m, Calculate the speed V of the object.

[2011 PII #8b]

16. A train accelerates uniformly from rest and reaches a speed of 20 m/s in 5 seconds. It then continues moving at the same speed for 3 seconds after which it accelerates constantly for 4 seconds and reaches a speed of 40 m/s. Using a scale of 2 cm to represent 10m/s on the vertical axis and 2 cm to represent 2 seconds on the horizontal axis, draw a speed-time graph to represent the motion of the train. **[2012 P1 #14]**

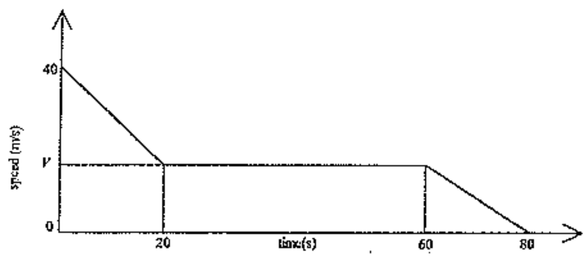
17. **Figure 8** is the speed time graph of a car. The car starts from rest and accelerates at $2m/s^2$ for t_1 seconds its speed reaching 16m/s. It then maintains this speed for some time after which it decelerates uniformly for 10 seconds and stops.



Calculate the distance that the car travelled from t_1 seconds to the t_2 when it started decelerating.

[2012 P2 #6a]

18. **Figure 9** shows a speed time graph of a train. The train decelerates from 40m/s in seconds. It then maintains this speed for the next 40 seconds after which it decelerates uniformly for 20 seconds and stops.



Given that the total distance travelled is 1600 m, calculate the

(i) Value of v.

(ii) Deceleration during the first 20 seconds.

[2013 P1 #16]

19. A cyclist accelerates uniformly from rest and reaches a speed of $30m/s$ in 2 seconds. He then continues moving at the same speed for 2 seconds after which he decelerates constantly to rest in 2.5 seconds. Using a scale of 2 cm to represents 1 unit on the horizontal axis and 2cm to represents 5 units on the vertical axis, draw a speed time graph to represent the race of the cyclist. **[2013 P2 #6b]**

20. **Figure 10** shows a speed time graph of a particle travelling in a straight line over a period of 90 seconds.

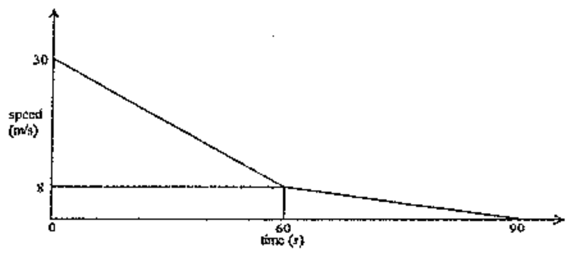
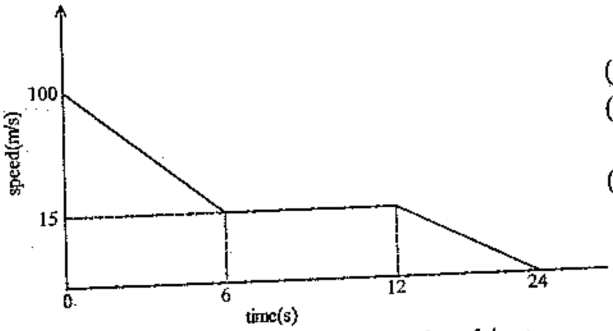


Figure 7

Calculate the distance travelled by the particle during the first 60 seconds. **[2014 P1 #16]**

21. **Figure 11** shows a speed – time graph of a moving object.



Find the total distance covered by the object. **[2015 PI #9]**

22. A bus travelled from stage A to stage B at speed of 50 km/h in one hour and from stage B to stage C at speed of 60 km/h for another one hour. It then slows down from stage C for 30 minutes before it stops at stage D. Using a scale of 4cm to represent 1 unit on the horizontal axis and 2cm to represent 10 units on the vertical axis, draw a graph to illustrate the motion of the bus stage A to stage D. **[2015 PII #8b]**

23. **Figure 12** shows a speed –time graph for the motion of a car over a period of 15 seconds.

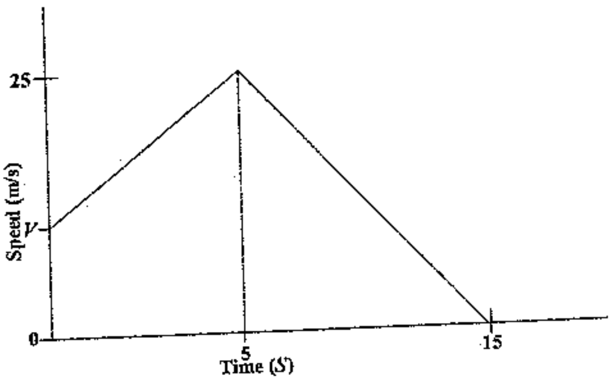
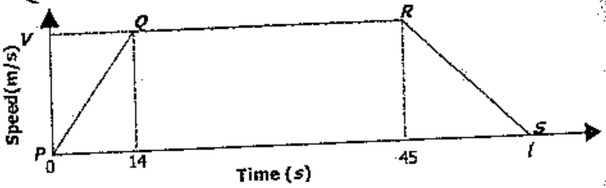


Figure 2

Given that the distance which the car covered before it started decelerating was 87.5 m, calculate the acceleration of the car. **[2016 PI #10]**

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

24. **Figure 13** shows a sketch of a speed-time graph **PQRS** of a car.

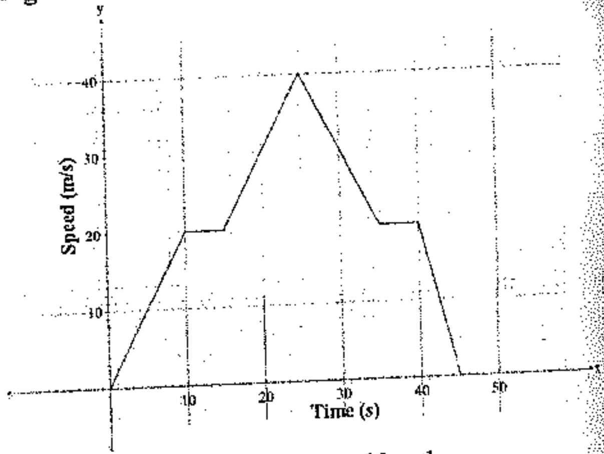


(i)
(ii)
(iii)

(i) If 175m is covered on the first 14 seconds, calculate the speed **V** of the car.

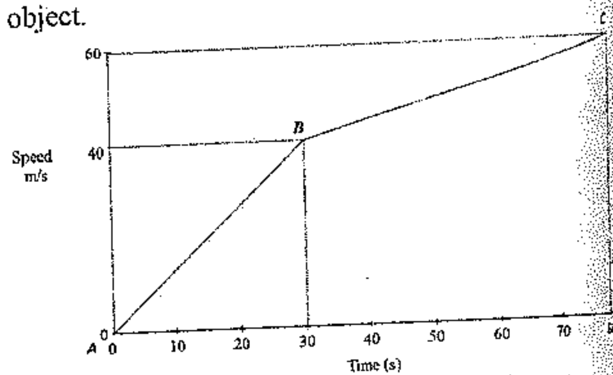
(ii) If the car decelerated at $5m/s^2$, calculate the total time taken for the whole journey, denoted as t in the figure. **[2016 PII #11a]**

25. **Figure 14** shows a speed time graph of a car.



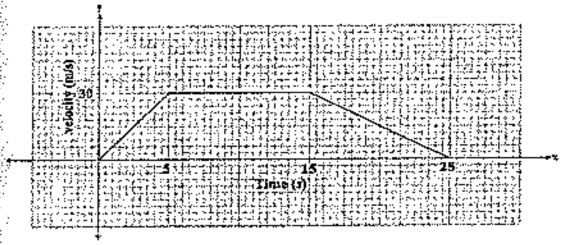
Calculate the distance covered by the car. **[2017 PI #8]**

26. **Figure 15** shows a speed-time graph of a moving object.



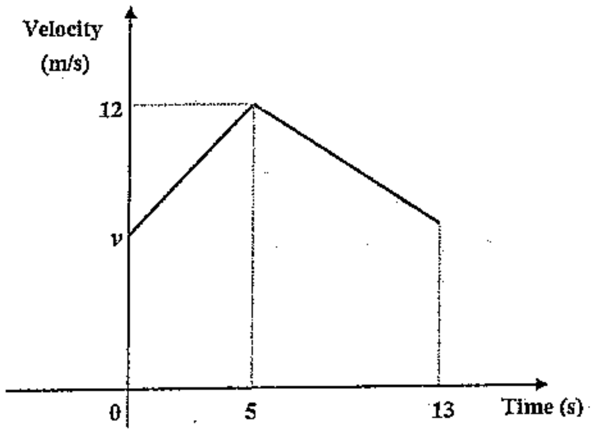
Calculate the total distance travelled by the object for the 80 seconds. **[2018 P1 #11]**

27. **Figure 16** is a sketch of velocity-time graph of a moving object



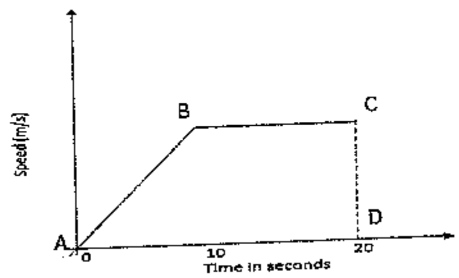
Calculate the average speed for the whole movement of the object. **[2019 PII #2b]**

28. **Figure 17** below is a velocity-time graph for a particle which starts at v m/s and stops with the same velocity v m/s.



If the total distance travelled by the particle is 117m, find the value of v. **[2021 Mock PI #20]**

1. **[2003 PP1 #19]**



$$Average\ speed = \frac{distance}{Time\ taken}$$

Now distance=Area of trapezium ABCD

Area of ABCD=$\frac{1}{2}$ *sum* //sides × height

$\frac{1}{2}(BC+AD)\times CD$

$=\frac{1}{2}(10+20)\times 10$

$=\frac{1}{2}\times 30\times 10$

$=150m$

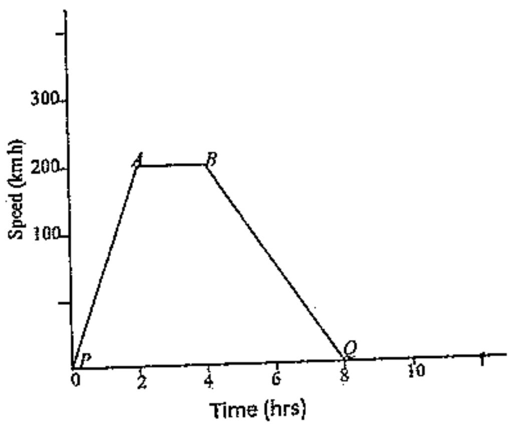
$time\,taken = 20$

$average\,speed = \frac{150m}{20s}$

$=7.5m/s$

The particles average speed during the 20 seconds is $7.5m/s$

2. **[2003 P2 #9a]**



ii. Distance= area of a trapezium PABQ

Area of PABQ=$\frac{1}{2}$ *sum* //sides × height

$=\frac{1}{2}(AB+PQ)\,h$

= ½ [(4-2) + (8-0)] × 200

= ½ ×(2+8)× 200

= ½ ×(10)× 200

= 5× 200

=1,000

∴ the distance travelled from P to Q is 1,000km.

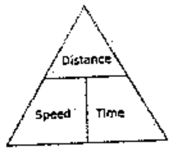
3. **[2003 PII #12b]**

Minibus & Lorry

Distance=160km

Let speed of lorry be x

then speed of minibus=$x+10$ (given)



$$\text{Time}=\frac{\text{Distance}}{\text{Speed}}$$

i.Minibus time=$\frac{160}{x+10}hr$

ii.Lorry time=$\frac{160}{x}hr$

iii.the minibus was faster by 32 minutes so,

Difference: $\frac{160}{x}-\frac{160}{x+10}=\frac{32}{60}$ (60minutes=1hr)

$\frac{160(x+10)-160x}{x(x+10)}=\frac{8}{15}$

$\frac{160x-160x+1600}{x(x+10)}=\frac{8}{15}$

$15(1600)=8x(x+10)$ cross-multiplying

$15\frac{(1600)}{8}=\frac{\cancel{8}}{\cancel{8}}x(x+10)$

$15\times 200 = x(x+10)$

$3000 = x^2+10x$

$x^2+10x-3000=0$

$(x-50)(x+60)=0$ by inspection

either $x-50=0$ or $x+60=0$

$x=50$ or $x=-60$

Hence, average speed of minibus

$=50+10=60km/hr$

4. **[2004 PI #12]**

Acceleration is the slope of the speed-time graph.

$$\text{Acceleration}=\frac{change\,in\,velocity}{change\,in\,time}$$

By reading off values from the graph

Acceleration= $\frac{30m/s-5m/s}{12-0s}$

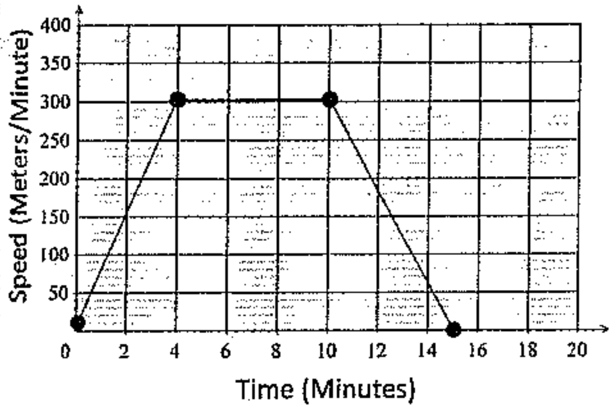
$=\frac{25m/s}{12s}=2.08333\ldots$

∴acceleration = 2. 08 m/s^2 (to 2 decimal places).

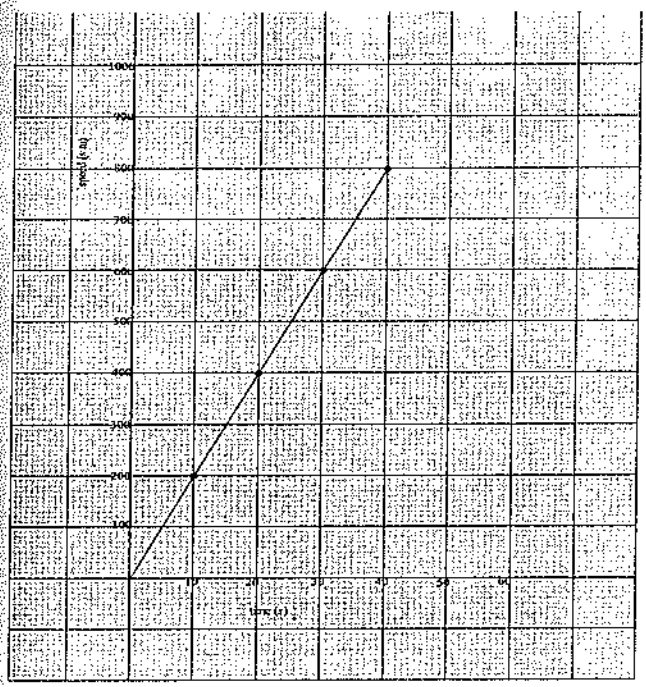
5. **[2005 P1 #19]**

Total distance is the area under the graph.
Since the graph is a triangular shape then;
Area = ½ × base × height (*area of the triangle*)
∴ distance = ½ × 6h × 50km/h
= 150km
∴ the distance travelled during the 6 hours is 150km

6. **[2005 PII #5b]**



7. **[2006 PI #12]**



Acceleration is the slope of a speed-time graph.

$$\text{Acceleration} = \frac{change\,in\,velocity}{change\,in\,time}$$

$$= \frac{800m/s - 200m/s}{40s - 10s}$$

$$= \frac{600m/s}{30s} = 20m/s^2$$

∴acceleration is $20m/s^2$

8. **[2006 PII #3b]**

length of the train = distance(D) = 50

speed, S = y time, T = $\frac{25}{y-1}$

using speed= $\frac{\text{distance}}{\text{time}}$

$$y = \frac{50}{25/y-1}$$

$$y = 50 \div \frac{25}{y-1}$$

$$y = \cancel{50} \times \frac{y-1}{\cancel{25}}$$

$$y = 2(y-1)$$

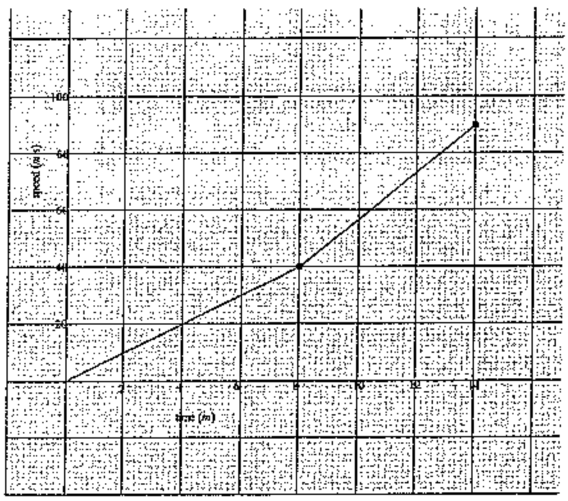
$$y = 2y - 2$$

$$y - 2y = -2$$

$$-y = -2$$

$\therefore y = 2$ (dividing both sides by −1)

9. **[2006 PII #4b]**



ii. Acceleration = gradient of speed time graph from 8 seconds to 14 seconds

=

$$\frac{change\,in\,speed}{change\,in\,time} = \frac{(90-40)m/s}{(14-8)s} = \frac{50}{6} = \frac{25}{3}$$

∴the acceleration of the motorist for the last 6 seconds is $\frac{25}{3}m/s^2$

10. **[2007 PII #19]**

In any speed-graph,

Distance travelled= Area of trapezium $ABCD$

$= \frac{1}{2}$ sum//sides $\times$ height

$\frac{1}{2}(BC+AD) \times CD$

$= \frac{1}{2}\left[(35-15)+(35-0)\right] \times (30-0)$

$= \frac{1}{2}(20+35) \times 30$

$= 55 \times 15$

$= 825m$

$\therefore$ The distance travelled by the object in 35 seconds is 825m.

11. **[2008 P2 #8b]**

Let initial average speed *be* x km/h

Time taken when going $= \dfrac{\text{Distance}}{\text{speed}}$

$= \dfrac{42}{x}$h

Mr Njinga's speed on return $= (x+2)$km/h.

Time taken on return

$= \dfrac{42}{x+2}$h

Mr Njinga took half an hour less:

$\therefore \dfrac{42}{x}\text{h} - \dfrac{42}{x+2}\text{h} = \frac{1}{2}\text{h}$

$\dfrac{42}{x} - \dfrac{42}{x+2} = \frac{1}{2}$

$\dfrac{42(x+2)-42x}{x(x+2)} = \dfrac{1}{2}$

$\dfrac{42x+84-42x}{x(x+2)} = \dfrac{1}{2}$

$2 \times 84 = x(x+2)$ *(cross-multipying)*

$168 = x^2 + 2x$

$x^2 + 2x - 168 = 0$

$(x+14)(x-12) = 0$

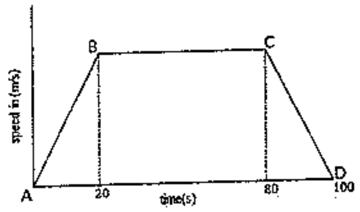
$x+14 = 0$ or $x-12 = 0$

$x = -14$ or $x = 12$

$\therefore x = 12$ (speed is always +ve)

$\therefore$ The average speed on the journey from his village is 12km/h

12. **[2008 P1 #16]**



Distance travelled =

= area of the trapezium $ABCD$

$= \frac{1}{2}$ sum of // sides $\times$ height

$= \frac{1}{2}(BC+AD)h$

$3200 = \frac{1}{2}[(80-20)+(100-0)]h$

$3200 = \frac{1}{2}(60+100)h$

$3200 - \dfrac{160h}{2}$

$80h = 3200$

$\dfrac{80h}{80} = \dfrac{3200}{80}$

$\therefore h = 40\text{m/s}$

Deceleration = gradient of CD

$\dfrac{0-40\text{m/s}}{100s-80s}$

$= \dfrac{-40\text{m/s}}{20\text{s}}$

$= -2\text{m/s}^2$

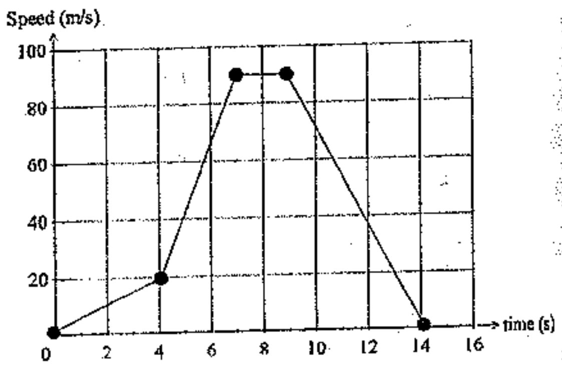
$\therefore$ deceleration during the last 20 seconds is -2m/s^2.

13. **[2010 PII #8b]**

First 4 seconds, velocity=acceleration x time

$= 5\text{m/s}^2 \times 4\text{s}$

$= 20\text{m/s}$



14. **[2011 PI #9]**

Acceleration=gradient of a speed-time graph

$$\text{Accelation} = \frac{\text{change in speed}}{\text{change in time}}$$

$$0.3 = \frac{15-0}{t_1 - 0}$$

$$0.3 = \frac{15}{t_1}$$

$$t_1 = \frac{15}{0.3}$$

$$= 50s$$

---

15. **[2011 PII #8b]**

Total distance = Area under graph

$$\text{Area} = \frac{1}{2}(sum\ of\ //\ \text{sides}) \times h$$

$$= \frac{1}{2}\left[(11-0)+(7-3)\right] \times (V-0)$$

$$= \frac{1}{2}(11+4) \times V$$

$$= \frac{15}{2}V$$

$but\ \frac{15}{2}V = 90$ (given distance=90m)

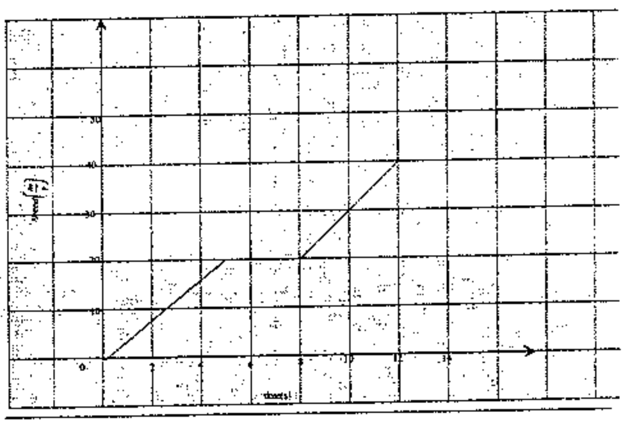
$$\therefore V = \frac{90}{15} \times 2 = 12$$

$\therefore speed\ \ V = 12m/s.$

---

16. **[2012 P1 #14]**

Speed-Time graph for a train



17. **[2012 P2 #6a]**

To find time between t_1 , and t_2 :

$$\text{Acceleration} = \frac{velocity}{time}$$

$$\text{Hence, } Time = \frac{velocity}{acceleration}$$

$$\text{Time from 0 to } t_1 = \frac{16m/s}{2m/s^2} = 8 \text{ seconds}$$

Time between t_1, and $t_2 = 50-(8+10)$

$= 32$ seconds

$\therefore$ Distance=Area under a graph(rectangle)

$= lenght \times breadth$

Distance between t_1,and $t_2 = 16m/s \times 32s$

$= 512m$

Distance travelled from time t_1 to time t_2

$= 512m$

---

18. **[2013 P1 #16]**

Distance=Area under of a graph

=Trapezium A+Trapezium B

$=\frac{1}{2}$ sum of//sides $\times$ height (for both A and B)

$= \frac{1}{2}[(40-0)+(V-0)] \times (20-0) +$

$\frac{1}{2}[(80-20)+(60-20)] \times (V-0)$

$= \frac{1}{2}(40+V)20 + \frac{1}{2}(60+40)V$

$= 10(40+V) + \frac{1}{2}(100)V$

$= 400 + 10V + 50V$

$= 400 + 60V$

$\therefore 400 + 60V = 1600$ (total distance given)

$60V = 1600 - 400$

$$\frac{\cancel{60}V}{\cancel{60}} = \frac{120\cancel{0}}{\cancel{6}0}$$

$\therefore V = 20$

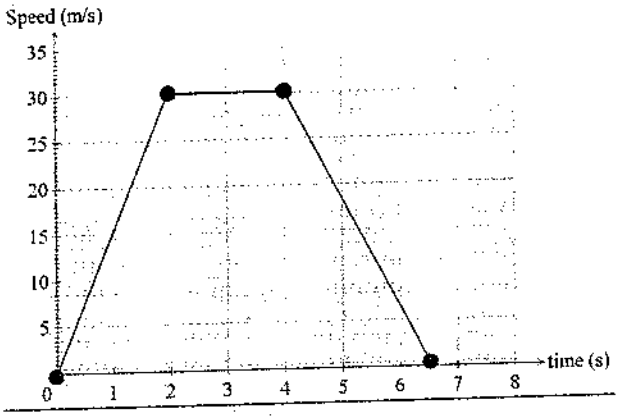
$(ii)\, Deceleration = \dfrac{v-u}{\Delta t}$

$= \dfrac{20m/s - 40m/s}{20s - 0s}$

$= \dfrac{-20m/s}{20s}$

$= -1m/s^2$

19. **[2013 P2 #6b]**



20. **[2014 P1 #16]**

1^{st} *area of triangle*

$\frac{1}{2}bh$

$= \frac{1}{2}\times(60-0)\times(30-8)$

$= 30\times 22$

$= 660m$

2^{nd} *area of* rectangle

$= (8-0)\times(60-0)$

$= 480m$

Total distance travelled by particle

$660m + 480m$

$1{,}140 metres$

Alternatively

Area of Trapezium

$= \frac{1}{2}\, sum\ //sides \times height$

$= \frac{1}{2}(30+8)\times 60$

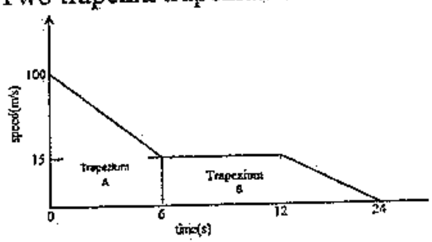
$= \frac{1}{2}\times 38\times 60$

$= 19\times 60$

$= 1{,}140m$

21. **[2015 PI #9]**

Two trapezia/trapeziums:



Distance covered for the first 6 seconds

Area of trapezium $A = \frac{1}{2}(\text{sum of } \parallel \text{ sides}) \times height$

$= \frac{1}{2}(100+15)\times 6$

$= \frac{1}{\cancel{2}}(115)\times \cancel{6}^{3}$

$= 3\times 115$

$= 345m$

Distance covered for the last 18 seconds $(6 \text{ to } 24)$

Area of trapezium $B = \frac{1}{2}(\text{sum of } \parallel \text{ sides}) \times$ height

$= \frac{1}{2}\left[(12-6)+(24-6)\right]\times 15$

$= \frac{1}{2}(6+18)\times 15$

$= \frac{1}{\cancel{2}}\cancel{(24)}^{12}\times 15$

$= 12\times 15$

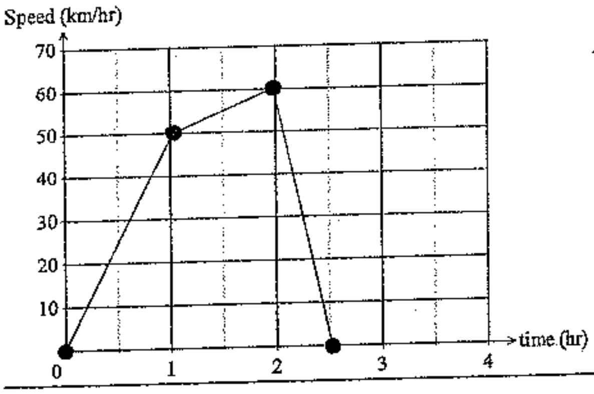
$= 180m$

Total distance covered

$= 345m + 180m$

$= 525m$

22. **[2015 PII #8b]**



23. **[2016 PI #10]**

Total distance before decelerating=87.5m

Area of trapezium=87.5m

$A = \frac{1}{2}(sum\ of \parallel sides)\times height$

$87.5 = \frac{1}{2}(25+v)\times 5$

$175 = (25+v)\times 5$

$\frac{175}{5} = 25 + v$

$35 = 25 + v$

$v = 35 - 25$

$v = 10 m/s$

$Acceleration = \frac{v-u}{t}$

$= \frac{(25-10) m/s}{5s}$

$= \frac{15 m/s}{5s}$

$= 3ms^{-2}$

---

24. **[2016 PII #11a]**

(i) distance=area

area of a $\Delta = \frac{1}{2} b \times h$

$175 = \frac{1}{2} \times (14 - 0) \times (V - 0)$

$175 \times 2 = 14V$

$\frac{350}{14} = V$

$25 = V$

$(ii) deceleration = slope$

so RS is:

$-5 = \frac{0-25}{t-45}$

$-5(t-45) = -25$

$t - 45 = \frac{\cancel{-25}}{\cancel{-5}}$

$t - 45 = 5$

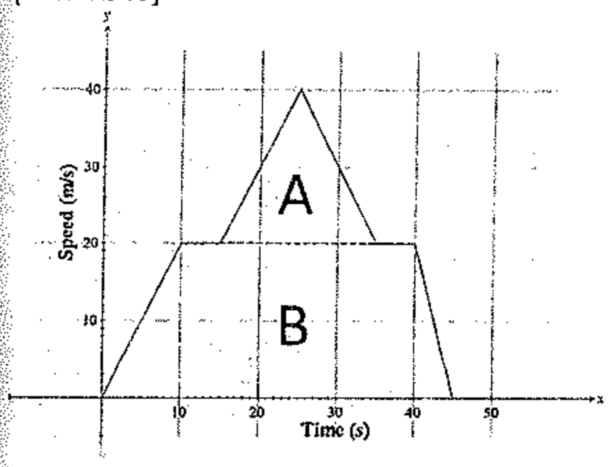
$t = 5 + 45$

$t = 50$

$\therefore$ Total time taken for the whole journey is 50seconds

---

25. **[2017 PI #8]**



Total distance covered

= Area under the graph

=Area of Δ + Area of trapezium

Area of triangle $= \frac{1}{2} bh$

$= \frac{1}{2}(35-15)(40-20)$

$= \frac{1}{2}(20)(20)$

$= 200m$

Area of trapezium $= \frac{1}{2}(sum\ of\ //\ sides) \times height$

$= \frac{1}{2}(45 + (40-10)) \times 20$

$= (45+30)10$

$= 10(75)$

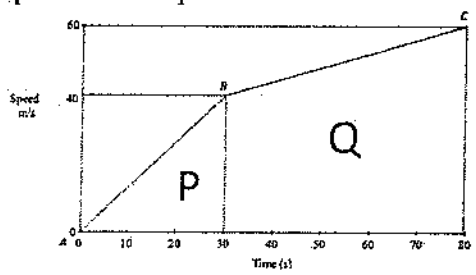
$= 750m$

Total distance travelled $= 200m + 750m$

$= 950m$

---

26. **[2018 P1 #11]**



Total distance = Area P + Area Q........

(Figure **P** is a triangle and Figure **Q** is a trapezium)

$= \left[\frac{1}{2}(base) \times height\right] + \left[\frac{1}{2}(sum\ of\ //\ sides) \times height\right]$

Total distance $= \left[\frac{1}{2}(30)(40)\right] + \left[\frac{1}{2}(40+60) \times (80-30)\right]$

Total distance $= \left[\frac{1}{2}(1200)\right] + \left[\frac{1}{2}(100) \times 50\right]$

Total distance = 600+2500

Total distance = 3100 m

27. **[2019 PII #2b]**

Average Speed $=\dfrac{\text{total distance}}{\text{total time}}$

Total time $=25s$

total distance = Area under the graph

=Area of trapezium

$=\dfrac{1}{2}\times(\text{sum of parallel sides})\times\text{height}$

(parallel sides are the bottom '25s' and the top side 15s-5s=10s)

$=\dfrac{1}{2}\times(25s+10s)\times 30m/s$

$=\dfrac{1}{2}\times 35s\times 30m/s \Rightarrow \dfrac{1}{2}\times 1050m$

$\therefore$ total distance $=525m$

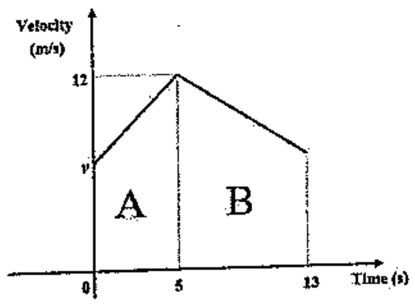
$\Rightarrow$ Average Speed $=\dfrac{525m}{25s}$

$\therefore$ the average speed is $21m/s$.

28. **[2021 Mock PI #20]**

Distance=Area under a Graph.

Method 1: Segment areas into two Trapezia (as given)



Total distance=Area of Trapezium A+Area of Trapezium B

Area of Trapezium=$\frac{1}{2}sum$ of //sides$\times$height

So,

Area of A=$\dfrac{1}{2}(v+12)\times(5-0)=\dfrac{5}{2}(v+12)$

Area of B=$\dfrac{1}{2}(v+12)\times(13-5)=\dfrac{8}{2}(v+12)$

$=4(v+12)$

$So,\ 117=\dfrac{5}{2}(v+12)+4(v+12)$

$234=5(v+12)+8(v+12)$ (multipying through by 2)

$234=5v+60+8v+96$

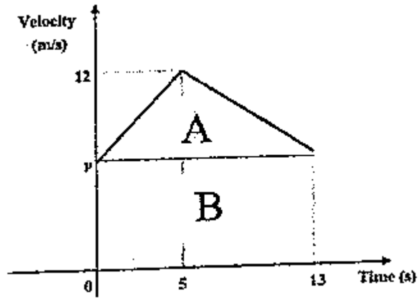
$234=13v+156$

$234-156=13v$

$\frac{\cancel{78}}{\cancel{13}}=\frac{\cancel{13}}{\cancel{13}}v$

$v=6$

Method 2: Segment areas into Triangle and Rectangle



Total distance=Area of Triangle A+Area of rectangle B

So,

Area of A=$\dfrac{1}{2}base\times height=\dfrac{1}{2}(13-0)(12-v)$

$=\dfrac{13}{2}(12-v)$

Area of B=$length\times width=(13-0)(v-0)=13v$

$So,\ 117=\dfrac{13}{2}(12-v)+13v$

$234=13(12-v)+26v$ (multipying through by 2)

$234=156-13v+26v$

$234-156=13v$

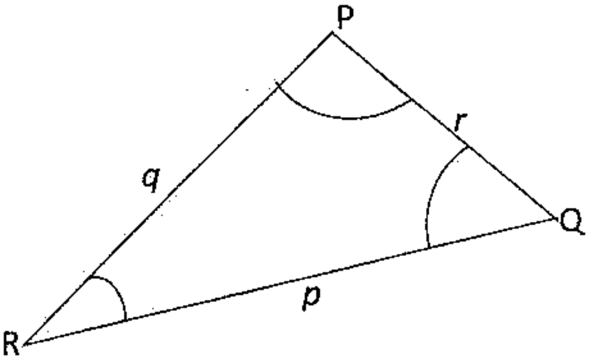
$\frac{\cancel{78}}{\cancel{13}}=\frac{\cancel{13}}{\cancel{13}}v$

$v=6$

CH 25
TRIGONOMETRY II

Chapter Highlights

This topic dwells emphasis on problems of trigonometry associated with oblique triangles. Oblique triangles are triangles with no right angle other than right-angled. We will solve these problems using the area rule, sine rule and cosine rule.



Triangle PRQ

The sine rule

In general, sine rule states that the sides of a triangle are proportional to the sines of their opposite angles i.e.

$$\frac{p}{\sin P}=\frac{q}{\sin Q}=\frac{r}{\sin R} \text{ or } \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Sine rule will be employed when given:

- Two angles and a side (to find unknown length).
- Two sides and an angle (to find an unknown angle)

Cosine rule

The cosine rule is:

$$p^2=q^2+r^2-2qr\cos P$$

$$q^2=p^2+r^2-2pr\cos Q$$

$$r^2=p^2+q^2-2pq\cos R$$

You can use the cosine rule to find an unknown aside in a triangle when you know the lengths of two sides and the angle between the sides.

The area rule

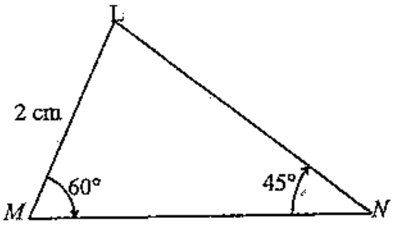
States that the area of a triangle is equal to half the product of the lengths of any two sides of the triangle multiplied by the sine of the angle between the two sides.

You can find the area of a triangle using the formula:

$$\text{Area of } \Delta PQR=\frac{1}{2}pq\sin R$$

$$or =\frac{1}{2}qr\sin P \text{ or } =\frac{1}{2}pr\sin Q$$

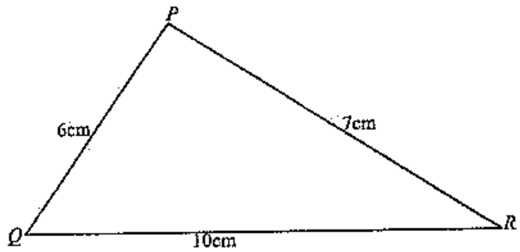
1. In **Figure 1**, triangle **LMN** in such that the angle **LMN**=60°, angle **MNL**=45° and **LM**=2 cm.



Find the length of **LN** leaving your answer in the simplest surd form. **[2004 PII #4b]**

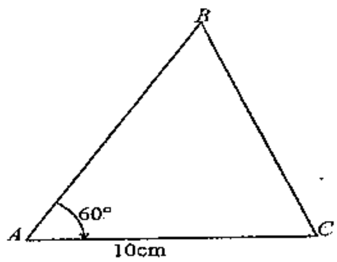
2. A boat sails 40km from point **A** to point **B**, east of **A** on a lake. Lodge **P** is south of point **A**. another lodge **Q** is south of point **B**. the bearing of **Q** from point **A** is S52°E and the bearing of **P** from **B** is S39°W. If **P** is 60km from **Q**, calculate the bearing of lodge **Q** from lodge **P**. **[2004 PII #9b]**

3. In **Figure 2, PQR** is a triangle such that **PQ**=6cm, **QR**=10cm and **RP**=7cm.



Calculate angle **PQR** to the nearest degree. **[2005 PI #10]**

4. **Figure 3** is a triangle **ABC** in which angle **BAC**=60°, **AC**= 10cm and the area of the triangle is $15\sqrt{6}cm^2$.

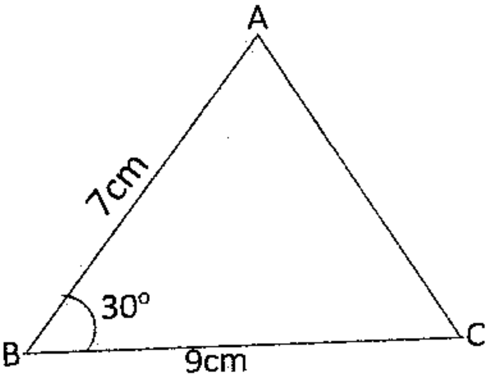


Calculate the length of **AB** leaving the answer in its simplest form. **[2006 P1 #5]**

5. An aeroplane leaves the airport **A** on a bearing of $N23^o E$ and flies for 340 km to another airport **B**. it then leaves airport **B** and flies on a bearing $N60^o W$ to another airport **C**. If the airports **A** and **C** are 680km apart, calculate the bearing of airport **A** from **C**. **[2006 PII #8b]**

6. Village **B** is on the bearing of 135° and a distance of 40km from village **A**. Village **C** is on bearing of 225° and a distance of 62 km from the village **A**.
   i. Show that **A, B** and **C** form a right-angled triangle.
   ii. Calculate angle ACB to the nearest degree. **[2008 P2 #12b]**

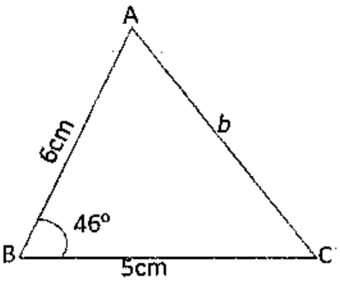
7. In **Figure 4, ABC** is a triangle such that $\angle$ **ABC**= 46°, **AB**= 7cm and **BC**= 9 cm.



Calculate the length of **AC** to one decimal place. **[2011 PI #17]**

8. Two cyclist **X** and **Y** are 10 kilometers apart. Cyclist **X** is due north of cyclist **Y**. A traffic officer at point **P** found that cyclist **X** is on bearing 031° and Y is on bearing 064°. how far is the traffic officer from cyclist **X?** **[2011 PII #6a]**

9. **Figure 5** shows a triangle ABC in which AB = 6 cm and BC = 5 cm.



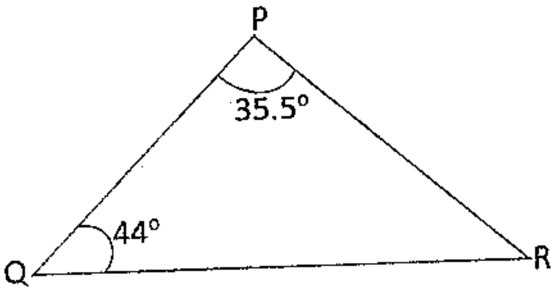
Given that the area of triangle **ABC** = 12cm². Find the value of angle **B** giving your answer correct to the nearest degree. **[2013 P1 #11]**

10. Two boats A and B are 57 meters are apart. The angle of elevation of a cliff (C) from boat A is 37 and from boat B is 25° .Calculate the distance of a boat B from the top of the cliff, giving your answer correct to nearest meter. **[2013 P2 #7b]**

To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: mscemodelmath@gmail.com.

11. Find the area of a triangle **PQR** in which **PQ**=12cm, **QR**=8cm, **PR**=5cm and angle **QPR**= 45° leaving your answer in surd form with a rationalized denominator. **[2014 P1 #18]**

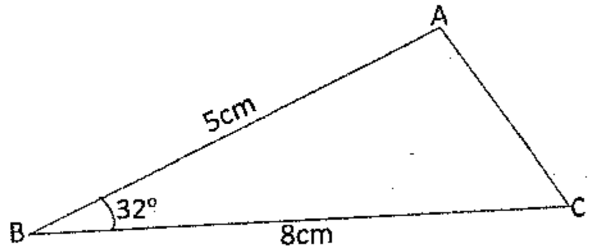
12. In **Figure 6**, Angle $P = 35.5^\circ$, Angle $Q = 44^\circ$ and $PR = 8cm$.



Find **QR**, leaving your answer correct to two significant figures. **[2014 PII #6b]**

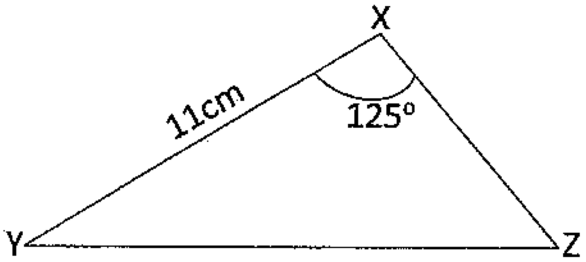
13. A tourist left city **A** and drove on a bearing 047° to a city **B**. She then drove 75 km on a bearing 117° to city **C** which is on a bearing of 100° from City **A**. Calculate the distance between cities **A** and **C**, giving your answer correct to two decimal places. **[2017 PII #10b]**

14. **Figure 7** shows a triangle **ABC** in which **AB**=5 *cm*, **BC**=8 cm, and angle **ABC** $= 32^\circ$.



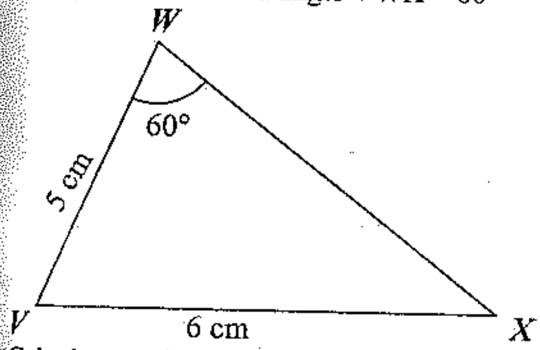
Calculate **AC**, leaving your answer correct to three significant figures. **[2015 PII #6b]**

15. **Figure 8** shows a triangle XYZ in which XY=11cm and angle YXZ=125°



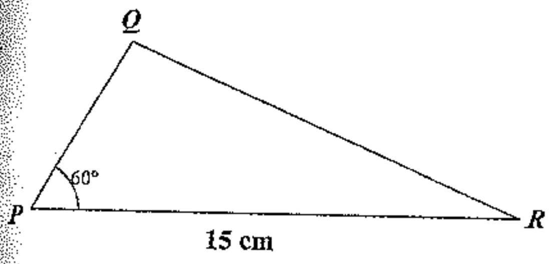
If the area of the triangle $XYZ = 36cm^2$, Calculate **XZ**, leaving your answer correct to the nearest whole number. **[2015 PI #12]**

16. **Figure 9** shows a triangle **VWX** in which **VW** = 5 cm, **VX** = 6 cm and angle **VWX** = 60°



Calculate angle **VXW**, giving your answer correct to one decimal place. **[2018 PII #12]**

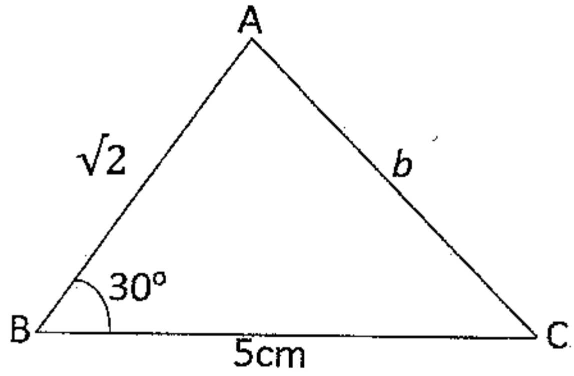
17. **Figure 10** is a triangle **PQR** in which **PR** = 15 cm and angle **QPR** = 60°



If the area of the triangle is $5\sqrt{6}$, without using a calculator, calculate the length of **PQ**.

[2019 PII #5b]

18. **Figure 11** is Triangle ABC with AB=$\sqrt{2}$ cm, BC=5 cm, and AC=b cm.



i. Find the Area of triangle ABC in surd form.
ii. Show that $AC^2 = 27 - 5\sqrt{6}$.
iii. Find Angle BAC, to the nearest degree.

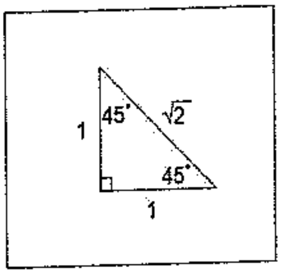
[2021 PII #11]

1. **[2004 PII #4b]**

Using the rule, $\dfrac{LN}{\sin M} = \dfrac{LM}{\sin N}$

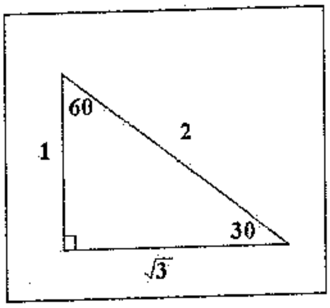
$$\frac{LN}{\sin 60°} = \frac{2cm}{\sin 45°}$$

$$LN = \frac{2cm \times \sin 60°}{\sin 45°}$$



$$= \frac{2cm \times \frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}}$$

(using special angles in Δs)



$$= \frac{\sqrt{3}cm}{\frac{1}{\sqrt{2}}}$$

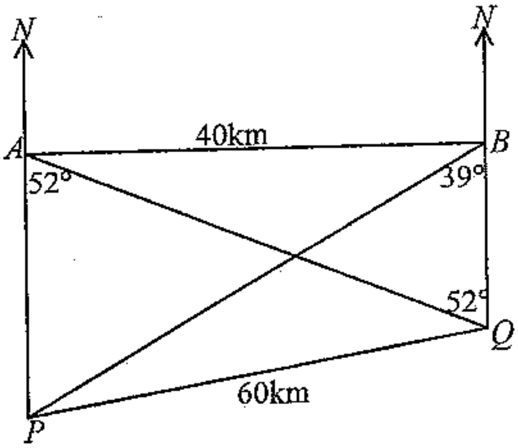
$$= \sqrt{3}cm \div \frac{1}{\sqrt{2}}$$

$$= \sqrt{3}cm \times \frac{\sqrt{2}}{1}$$

$$= \sqrt{3 \times 2}cm$$

$\therefore$LN = $\sqrt{6}$ cm

2. **[2004 PII #9b]**



We need to find $\angle$APQ

$\angle$AQB = $\angle$PAQ *(alt. $\angle$s PA // QB)*

$\angle$AQB = 52°

Since $\angle ABQ = 90°$

$$\text{Tan } 52° = \frac{40}{BQ}$$

$$BQ \text{ Tan } 52° = 40°$$

$$BQ = \frac{40}{\tan 52°}$$

$$BQ = \frac{40}{1.2.799}$$

$BQ = 31.25km$ (to $2dp$)

In ΔPBQ

$$\frac{\sin \angle BPQ}{BQ} = \frac{\sin \angle PBQ}{PQ}$$

$$\sin \angle BPQ = \frac{31.25 \times \sin 39°}{60}$$

$$\sin \angle BPQ = \sin^{-1}\left(\frac{31.25 \times 0.6293}{60}\right)$$

$\sin^{-1} 0.3278$

$\angle$BPQ = 19° *(to the nearest degree)*

$\angle$APB = $\angle$ PBQ *(alt. angles AP//BQ)*

$\angle$APB = 39°

$\angle$APQ = $\angle$APB + $\angle$BPQ

= 39° + 19°

= 58°

$\therefore$ the bearing of lodge Q from lodge P is N58°E.

3. **[2005 PI #10]**

$\angle$PQR = $\hat{R}$ so

$$r^2 = p^2 + q^2 - 2pq \cos R$$

$$6^2 = 10^2 + 7^2 - 2(10 \times 7)\cos R$$

$$36 = 100 + 49 - 2(70)\cos R$$

36 = 149 − 140 cos R

140cos R = 149 − 36

$$\frac{140}{140}\cos R = \frac{113}{140}$$

Cos R = 0.8071

$\hat{R}$ = $\cos^{-1}$0.8071

= 36.1823

$\therefore \angle PRQ \approx 36°$ *(to the nearest degree)*

4. **[2006 P1 #5]**

Let AB = c, AC = b = 10cm, A = 60°,

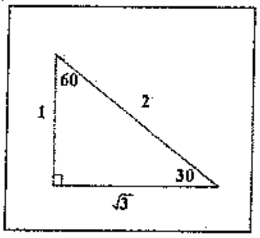
area = $15\sqrt{6}cm^2$

Area of triangle = ½ bcSin A

$$15\sqrt{6}cm^2 = \frac{10}{2}c\sin 60°$$

To find sin 60°, use special angles

$$\sin 60° = \frac{\sqrt{3}}{2}$$



So $15\sqrt{6} = 5c\frac{\sqrt{3}}{2}$

$$\frac{2\times 15\sqrt{6}}{\sqrt{3}} = 5c$$

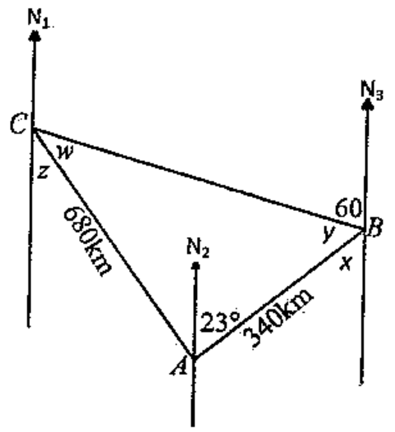
$$30\sqrt{\frac{6}{3}} = 5c$$

$$\frac{30\sqrt{2}}{5} = \frac{5c}{5}$$

$c = 6\sqrt{2}$ cm

Thus AB = $6\sqrt{2}$ cm

5. **[2006 PII #8b]**



Need to find the value of angle z:

x = 23° (*alt. ∠s; $CN_1 /\!/ BN_3$*)

60° + y + x = 180° (*∠s on a straight line*)

y = 180° - (60° + 23°)

= 180° - 83°

y = 97°

using sine rule,

$$\frac{\sin w}{340} = \frac{\sin y}{680}$$

$\Rightarrow \sin w$

$$= \frac{340\times \sin y}{640}$$

$$= \frac{340\times \sin 97°}{680}$$

$$= \frac{340\times 0.9925}{680}$$

$= 0.4963$

$\therefore w = \sin^{-1} 0.4963$

= 30° (*to the nearest degree*)

But $z + w$ = ∠CBN₃ (*alternate angles; $CN_1 /\!/ BN_3$*)

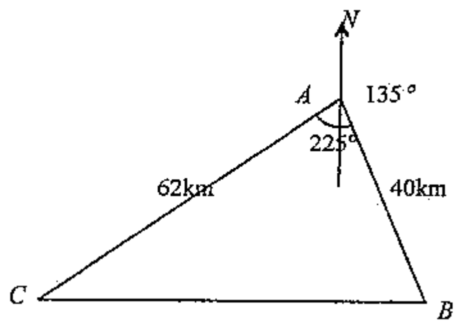
z + 30° = 60°

z = 60° - 30°

= 30°

∴ the bearing of airport A from C is S30°E

6. **[2008 P2 #12b]**



i) show that some angle in $\triangle ABC$ is a right angle.

$\angle BAC = \angle NAC - \angle NAB$ (split angle)

$= 225^\circ - 135^\circ$

$= 90^\circ$, Which is a right angle.

A, B, C from a right-angled triangle.

ii) Calculating $\angle ACB$:

now, $\tan \angle ACB = \frac{AB}{AC}$

$$\tan \angle ACB = \frac{40km}{62km}$$

$$\tan \angle ACB = \frac{40}{62}$$

$\angle ACB = \tan^{-1}(\frac{40}{62})$

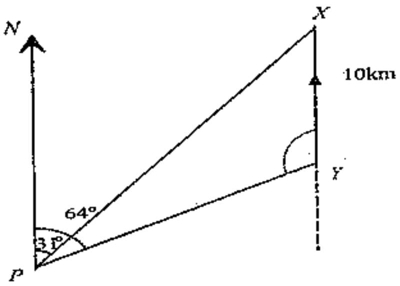
$\angle ACB = \tan^{-1} 0.6452$

$\angle ACB = 33^\circ$ (to the nearest degree)

$\therefore$ the value of angle ACB is 33°

7. **[2011 PI #17]**
Using cosine rule
$b^2 = a^2 + c^2 - 2ac \cos B$
$b^2 = 7^2 + 9^2 - 2\times7\times9 \cos 46°$
$b^2 = 7^2 + 9^2 - 126\ (0.6947)$
$b^2 = 130 - 87.53$
$b^2 = 42.47$
$b = \sqrt{42.47}$
$b = 6.52$
$\therefore AC = 6.5\ cm$ (to 1dp)

8. **[2011 PII #6a]**
Bearing of x =031°, $y = 064°$.
Bearing of x, y from P and xy=10 km

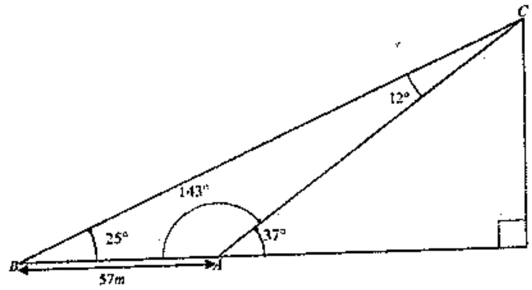


$\angle XPY = 64° - 31° = 33°$
$\angle XYP = 180° - 64°$ (allied.$\angle$s PN//XY)
$= 116°$
$$\frac{PX}{\text{Sin } 116°} = \frac{10}{\text{Sin } 33°}$$
$$PX = \frac{10 \text{ Sin } 116°}{\text{Sin } 33°}$$
$$PX = \frac{10 \times 0.8988}{0.5446}$$
$$PX = \frac{8.988}{0.5446}$$
$= 16.50$
$\therefore$ The traffic officer is 16.5 km from cyclist X

9. **[2013 P1 #11]**
Area$= \frac{1}{2}ac \sin B$
$12 = \frac{1}{2}\times5\times6 \sin B$ [Since $a = 5, c = 6$]
$\frac{12}{15} = \frac{\cancel{15} \sin B}{\cancel{15}}$
$B = \sin^{-1} 0.8$

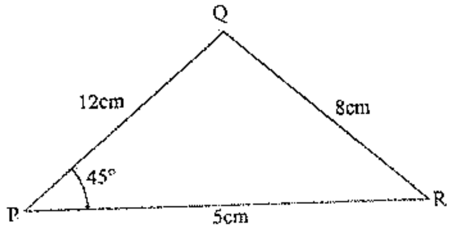
Angle B=53.13°
B=53° (to the nearest degree)

10. **[2013 P2 #7b]**
Sketch of the diagram



In triangle BAC
$$\frac{BC}{\sin 143°} = \frac{57m}{\sin \angle BDA} \quad \text{(sine rule)}$$
$\angle BDA = 180° - 25° - 143°$ (sum of angles in Δ)
$\Rightarrow \angle BDA = 12°$
$$\therefore \frac{BC}{\sin 143°} = \frac{57m}{\sin 12°}$$
$$\Rightarrow BC = \frac{57m}{\sin 12°} \times \sin 143°$$
$\Rightarrow BC = 164.9905m$
$\therefore BC = 165m$ (to the nearest meter)

11. **[2014 P1 #18]**



by area rule; $\frac{1}{2}qr \sin p$

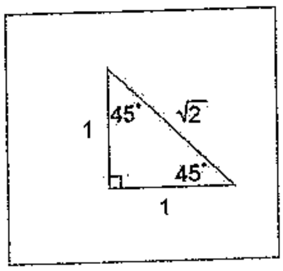
$\frac{1}{2}\times5\times12 \sin 45°$

$30\times\frac{1}{\sqrt{2}}$

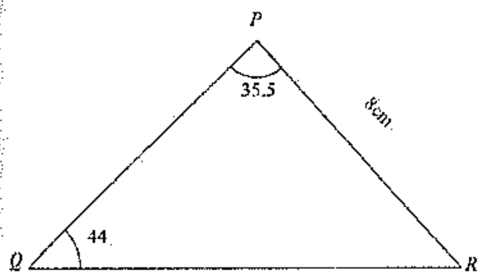
$\frac{30}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$

$\frac{30\sqrt{2}}{2}$

$15\sqrt{2}\ cm^2$



12. **[2014 PII #6b]**



Use the sine rule to find QR

$$\frac{r}{\sin\ R}=\frac{q}{\sin\ Q}=\frac{p}{\sin\ P}$$

$$\therefore \frac{q}{\sin\ Q}=\frac{p}{\sin\ P}$$

$$\frac{8}{\sin\ 44^\circ}=\frac{p}{\sin\ 35.5^\circ}$$

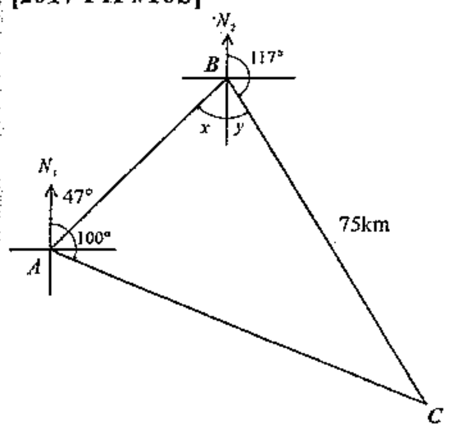
$$p=\frac{8\sin\ 35.5^\circ}{\sin\ 44^\circ}$$

$$p=\frac{8\times 0.5807}{0.6947}$$

$QR=6.687cm$

$QR\approx 6.7cm$ (to 2 sig figures)

13. **[2017 PII #10b]**



$\angle CAB=100^\circ-47^\circ=53^\circ$ (bisected angle)

$x=47^\circ (alt.\angle s\ AN_1//BN_2)$

$y+117^\circ=180^\circ (adj.\angle s)$

$y=180^\circ-117^\circ$

$y=63^\circ$

$\angle ABC=x+y$

$=47^\circ+63^\circ$

$\angle B=110^\circ$

Using the Sine rule;

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{75}{\sin 53^\circ}=\frac{b}{\sin 110^\circ}$$

$$b=\frac{75\sin 110^\circ}{\sin 53^\circ}$$

$$b=\frac{75\times 0.9397}{\sin 53^\circ}$$

$$b=\frac{70.4769}{0.7986}$$

$b=88.2467$

$\therefore b=88.25km(to\ 2dp)$

Cities A and C are 88.25km apart

14. **[2015 PII #6b]**

$a=8cm,\ b=?,\ c=5cm$

$b^2=a^2+c^2-2acCos\ \mathrm{B}$

$b^2=8^2+5^2-(2\times 8\times 5)Cos\ 32^\circ$

$b^2=64+25-(80)Cos\ 32^\circ$

$b^2=89-67.8438$

$\sqrt{b^2}=\sqrt{21.15615}$

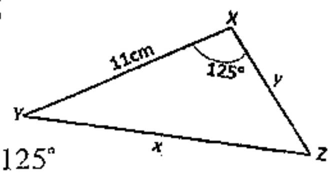
$b=4.59958$

$b\approx 4.60$

$AC=4.60(to\ 3\mathrm{s.f})$

15. **[2015 PI #12]**

Using the area rule;



Area$=\frac{1}{2}zy\sin X$

$$36=\frac{1}{2}\times 11\times y\times \sin\ 125^\circ$$

$$36=\frac{11\times y\times 0.8192}{2}$$

$$36\times 2=\frac{9.011y}{\cancel{2}}\times\cancel{2}$$

$72=9.011y$

$$\frac{72}{9.011}=\frac{\cancel{9.011}y}{\cancel{9.011}}$$

$y=7.9902$

$y\approx 8cm$(to the nearest whole number)

16. **[2018 PII #12]**

Using the sine rule

$$\frac{\sin X}{x}=\frac{\sin W}{w}$$

$$\frac{\sin X}{5}=\frac{\sin 60}{6}$$

$$\sin X=\frac{5\sin 60}{6} \quad \text{(multiply 5 both sides)}$$

$$\sin X=0.72168$$

$$X=\sin^{-1}(0.72168)$$

$$X=46.19°$$

$$X=46.2° \quad \text{(to one decimal place)}$$

17. **[2019 PII #5b]**

Using the area rule;

$$\text{Area of } \Delta PQR=\frac{1}{2}qr\sin P$$

$$5\sqrt{6}=\frac{1}{2}\times 15r\sin 60°$$

$$10\sqrt{6}=15r\sin 60° \text{ (cross-multiplying)}$$

$$\frac{10\sqrt{6}}{15}=r\sin 60°$$

Using the area rule;

$$\text{Area of } \Delta PQR=\frac{1}{2}qr\sin P$$

$$5\sqrt{6}=\frac{1}{2}\times 15r\sin 60°$$

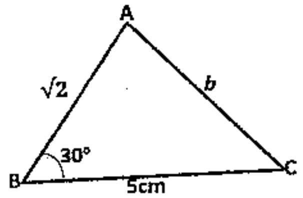
$$10\sqrt{6}=15r\sin 60° \text{ (cross-multiplying)}$$

$$\frac{10\sqrt{6}}{15}=r\sin 60°$$

$$r=\frac{\cancel{20\sqrt{3}\sqrt{2}}}{\cancel{15\sqrt{3}}}$$

$$r=\frac{4\sqrt{2}}{3}cm$$

18. **[2021 Mock PII #11]**



i. Area of ΔABC in surd form.

Area $\Delta ABC=\frac{1}{2}ac\,\text{Sin}\,B$

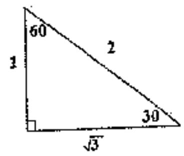
(formula depends on the available/given angle)

$$\Delta ABC=\tfrac{1}{2}\times 5\times\sqrt{2}\times \text{Sin}\,30°$$

$$=\tfrac{1}{2}\times 5\times\sqrt{2}\times\tfrac{1}{2}$$

$$=\tfrac{5\sqrt{2}}{4}cm^2$$

ii. Show that $AC^2=27-5\sqrt{6}$.



Using Cosine rule,

$$b^2=a^2+c^2-2ac\,\text{Cos}\,B$$

$$=5^2+\left(\sqrt{2}\right)^2-2\times5\times\sqrt{2}\,\text{Cos}\,30°$$

$$=25+2-10\sqrt{2}\left(\tfrac{\sqrt{3}}{2}\right) \text{ (using special } \angle s)$$

$$=27-10\tfrac{\sqrt{6}}{2}$$

$$b^2=27-5\sqrt{6}$$

$$AC^2=27-5\sqrt{6}$$

iii. Angle BAC

$$\frac{\text{Sin}\,A}{a}=\frac{\text{Sin}\,B}{b}$$

$$\frac{\text{Sin}\,A}{5}=\frac{\text{Sin}\,30°}{\sqrt{AC^2}}$$

$$\text{Sin}\,A=\frac{5\,\text{Sin}\,30°}{\sqrt{(27-5\sqrt{6})}}$$

$$\text{Sin}\,A=\frac{5\times0.5}{\sqrt{14.75}}$$

$$\text{Sin}\,A=\frac{2.5}{3.8409}$$

$$\text{Sin}\,A=0.6509$$

$$A=\text{Sin}^{-1}\,0.6509$$

$$A=40.6086$$

$$A=41° \text{(to the nearest degree)}$$

CH 26 POLYNOMIALS

Chapter Highlights

The **degree of a polynomial** is the highest power of x in the algebraic expression of a polynomial.

Polynomials of degree 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.

The **roots or zeros of a polynomial** are the x-coordinates of the points where the graph intersects the x-axis. The roots are also the solution of the polynomial when it is equated to zero.

In this chapter, will solve problems on polynomials. We will multiply and divide polynomial. Polynomials of a higher degree will be divided by binomials using long division method.

Multiplication of polynomial

- Involves the removal of blankets – expanding blankets.

Long Division

- The dividend and divisor must be written in descending powers of the variable
- For missing variables, insert a zero instead to arrange the polynomial.

The remainder theorem

- If a polynomial $f(x)$ is dividend by $ax-b$. To find the remainder, first find equate $ax-b$ to zero and find $x=\frac{b}{a}$, then our remainder $f\left(\frac{b}{a}\right)$.

Factor Theorem

- If $x-a$ is a factor of $f(x)$, then R = 0 when $f(x)$ is divided by $(x-a)$.
- $f(a)=0$ implies that $x-a$ is a factor.

Identities

Given two polynomials, $f(x)$ and $g(x)$, $f(x)=g(x)$ if:

They are of the same degree.
They have the same number of terms.

- The coefficient of the corresponding terms are equal.

---

1. Find the remainder when $2x^3-13x^2-8x+12$ is divided by $2x-1$. **[2003 PI #23]**

2. m^2-m-2 is a factor of m^3-2m^2-pm+c. When the polynomial is divided by $m+2$ the remainder is -12. Find the values of p, and c. **[2003 PII #9b]**

3. Use the remainder theorem to prove that $(x-y)$ is a factor of the polynomial $x^2(y-2)+y^2(2-x)$. **[2004 PI #18]**

4. Given that $(4x^2-9)(Bx+C)$ is identical to $16x^3+24x^2-36x-54$ calculate the values of B and C. **[2005 PI #20]**

5. Find the remainder when $t^3+2t^2-3(t+1)$ is divided by $(t+1)$. **[2005 PII #3b]**

6. When the polynomial ax^2+bx+c is divided by $(x+1)$ and $(x+3)$ it gives a remainder 2 in each case. Find the polynomial. **[2006 PI #6]**

7. Solve the equation $2x^3-5x^2+x+2=0$. **[2006 PII #7b]**

8. When the polynomial x^3+5x^2+Kx+3 is divided by $(x+2)$ it gives a remainder of 1. Find the value of K. **[2007 PI #13]**

9. If $(y+2)$ and $(y-3)$ are factors of $2y^3+by^2+cy-6$, find the values of b and c. **[2007 PII #11]**

10. Show that $k+3$ is a factor of $k^3+3k^2-4k-12$. **[2010 P1 #10]**

11. Solve the equation $a^3+5a^2-a-5=0$. **[2010 PII #12b]**

12. Given that $(x+1)$ and $(x-3)$ are two factors of the polynomial ax^3+bx-6. Calculate the values of a and b. **[2011 PI #10]**

13. Given that (x-2) is a factor of $x^3-6x^2+11x+p$, *find p*. **[2012 PII #9a]**

14. The polynomials x^3-x^2+x+9 and x^3-4x+3 leave the same remainder when divided by $(x-a)$. Calculate the values of a.
[2013 P1 #19]

15. Factorize completely $x^3+6x^2+3x-10$.
[2013 P2 #4a]

16. When x^3-7x+m is divided by $(x-1)$ the remainder is -1, find the value of m.
[2014 P1 #11a]

17. Show that $(x-2)$ is a factor of $x^3-9x+10$.
[2015 PII #3b]

18. Find the remainder when y^3+3y^2-2y-5 is divided by $(y-2)$. **[2016 PI #12]**

19. The remainder, when $2x^3+kx^2+7$ is divided by $(x-2)$ is half the remainder when the same expression is divided by $(2x-1)$. Calculate the value of k. **[2016 PII #7a]**

20. Factorize completely $2x^3+5x^2-4x-3$.
[2017 PII #2b]

21. Given that $(x+1)$ is a factor of $2x^3+qx^2-12x-9$, calculate the value of q.
[2018 P1 #17]

22. Given a polynomial $ax^3+3x^2-8x+d=0$ is identical to the polynomial whose roots are -3, $\frac{1}{2}$ and 1, calculate the value of a.
[2018 PII #4]

23. Solve the equations $2m^3-5m^2-4m+3=0$
[2019 PII #7]

1. **[2003 PI #23]**

Let the polynomial be a function, $f(x) = 2x^3 - 13x^2 - 8x + 12$

Let $2x - 1 = 0$

$\Rightarrow 2x = 1 \quad \therefore x = \frac{1}{2}$ substituting,

Then, $f(\frac{1}{2}) = 2(\frac{1}{2})^3 - 13(\frac{1}{2})^2 - 8(\frac{1}{2}) + 12$

$= 2 \times \frac{1}{8} - 13 \times \frac{1}{4} - 8 \times \frac{1}{2} + 12$

$= \frac{1}{4} - \frac{13}{4} - 4 + 12$

$= \frac{12}{4} + 8$

$= -3 + 8$

$= 5$

$\therefore$ the remainder is 5.

2. **[2003 P2 #9b]**

Let $m^2 - m - 2 = 0$

Then $(m+1)(m-2) = 0$

Either $(m+1) = 0$ or $(m-2) = 0$

$m = -1$ or $m = 2$

Substitute $m = -1$ in $m^3 - 2m^2 - pm + c$

$(-1)^3 - 2(-1)^2 - p(-1) + c = 0$

$-1 - 2 + p + c = 0$

$p + c = 3 \ldots\ldots\ldots\ldots\ldots\ldots$ (i)

Substitute $m = 2$ for m in $m^3 - 2m^2 - pm + c$

$(2)^3 - 2(2)^2 - p(2) + c = 0$

$8 - 8 - 2p + c = 0$

$c = 2p \ldots\ldots\ldots\ldots$ (ii)

Substituting (ii) in (i):

$p + c = 3$ becomes

$p + 2p = 3$

$\frac{3}{3}p = \frac{3}{3}$

$\therefore p = 1$

Recall, from (ii), $c = 2p$, so:

$c = 2(1)$

$\therefore c = 2$

$\therefore$ the value of p is 1 and the value of c is 2.

3. **[2004 PI #18]**

We use the remainder theorem:

Our polynomial is $x^2(y-2) + y^2(2-x)$

Let $x - y = 0$ then $x = y$

Substituting y for x, we have:

$y^2(y-2) + y^2(2-y)$

$= y^3 - 2y^2 + 2y^2 - y^3$

$= 0$

Since the polynomial $= 0$ then by the remainder theorem $(x - y)$ is a factor.

4. **[2005 PI #20]**

Expanding $(4x^2 - 9)(Bx + C)$, using **split and spread method [we split** $(4x^2 - 9)$ **and spread** $(Bx + C)$**]**, we have:

$(4x^2 - 9)(Bx + C)$

$= 4x^2(Bx + C) - 9(Bx + C)$

$= 4Bx^3 + 4Cx^2 - 9Bx - 9C$

But this is identical to $16x^3 + 24x^2 - 36x - 54$

$\therefore 4Bx^3 + 4Cx^2 - 9Bx - 9C \equiv 16x^3 + 24x^2 - 36x - 54$

$$\left.\begin{array}{l} 4B = 16 \Rightarrow B = 4 \\ 4C = 24 \Rightarrow C = 6 \end{array}\right\} \text{Equating coefficients of corresponding terms}$$

$\therefore$ the value of B is 4 and the value of C is 6.

5. **[2005 PII #3b]**

Let $p(t) = t^3 + 2t^2 - 3(t+1)$

Using the remainder theorem:

$t + 1 = 0$ then $t = -1$

By the remainder the remainder is $p(-1)$:

$p(-1) = (-1)^3 + 2(-1)^2 - 3(-1+1)$

$= -1 + 2 - 3(0)$

$= 1$

$\therefore$ the remainder is 1.

6. **[2006 PI #6]**

Let $f(x) = ax^2 + bx + c$

Letting $x + 1 = 0 \Rightarrow x = -1$

$f(-1) = a(-1)^2 + b(-1) + c$

$2 = a - b + c \ldots\ldots\ldots$(i) (remainder theorem)

Letting $x + 3 = 0 \Rightarrow x = -3$

$f(-3) = a(-3)^2 + b(-3) + c$

$2 = 9a - 3b + c$ (ii)

Since we cannot solve three unknowns we use another technique

$$\begin{array}{r|l} & 2 \\ \hline & 13 \\ 5 & -10 \\ \hline & 3 \end{array}$$

Example

$2 \times 5 + 3 = 13$

Thus quotient product + remainder = polynomial

So $(x+1)(x+3)+2=ax^2+bc+c$

$x(x+3)+1(x+3)+2=ax^2+bc+c$

$x^2+3x+x+3+2=ax^2+bc+c$

$x^2+4x+5=ax^2+bc+c$

$\therefore$ a = 1, b = 4 c = 5

7. **[2006 PII #7b]**

We try the factors of the constant 2 which are ±1, ±2.
Upon trying, we notice that $x = 1$ gives zero.

Thus $2(1)^3-5(1)^2+(1)+2=2-5+1+2$

= -3 + 3

= 0

$\therefore$This implies that $x-1$ is a factor of $2x^3-5x^2+x+2=0$.

Dividing the expression by $x-1$:

$$\begin{array}{r}
2x^2-3x-2 \\
x-1\overline{)\ 2x^3-5x^2+x+2} \\
-\underline{(2x^3-2x^2)} \\
-3x^2+x \\
-\underline{(-3x^2+3x)} \\
-2x+2 \\
\underline{-2x+2} \\
0
\end{array}$$

$\therefore 2x^3-5x^2+x+2=(x-1)(2x^2-3x-2)$

$(x-1)(2x^2-3x-2)=0$ (factorize the second bracket) factors: $2\times-2=-4 \Rightarrow -4+1=-3$

$(x-1)(2x^2-4x+x-2)=0$

$(x-1)[2x(x-2)+1(x-2)]=0$

$(x-1)(2x+1)(x-2)=0$

Either $x-1=0$ or $2x+1=0$ or $x-2=0$

$\therefore\ x=0$ or $x=-\frac{1}{2}$ or $x=2$

8. **[2007 PI #13]**

let $p(x)=x^3+5x^2+Kx+3$

Let $x+2=0$ then $x=-2$.

Then the remainder is $p(-2)$ by the remainder theorem. But we are given that when we divide by $x+2$, remainder is 1

$\therefore p(-2)=1$

$p(-2)=(-2)^3+5(-2)^2+K(-2)+3$

$-8+5\times4+(-2)\times K+3=1$

$-8+20-2K+3=1$ (arranging like terms)

$-2K+15=1$

$-2K=1-15$

$-2K=-14$ (divide both sides by -2)

$\therefore K=7$

9. **[2007 PII #11b]**

Let $p(y)=2y^3+by^2+cy-6$

Given that $(y+2)$ and $(y-3)$ are factors, then:

let $y+2=0$ and $y-3=0$

Then $y=-2$ and $y=3$

$\Rightarrow p(-2)=0$ and $p(3)=0$ (factor theorem)

$\Rightarrow p(-2)=2(-2)^3+b(-2)^2+c(-2)-6$

$=-16+4b-2c-6$

$\therefore -16+4b-2c-6=0$

$\Rightarrow 4b-2c=22$

$\Rightarrow 2b-c=11$............(i) (dividing throughout by 2)

Also $p(3)=2(3)^3+b(3)^2+c(3)-6$

$=9b+3c+48$

$\Rightarrow 9b+3c=-48$

$\Rightarrow 3b+c=-16$.........(ii)

We add equations (i) and (ii)

$2b-c=11$............(i)

$3b+c=-16$.........(ii)

$5b\quad=-5$

$b=-1$ (dividing both sides by 5)

But $3b+c=-16$, so

$3(-1)+c=-16$

$-3+c=-16$

$c=-16+3$

$\therefore c=-13$

10. **[2010 P1 #10]**

Let $f(k) = k^3 + 3k^2 - 4k - 12$

if $k + 3 = 0$

$k = -3 \Rightarrow$ Then by the remainder theorem, the remainder is $f(-3)$.

$\Rightarrow f(-3) = (-3)^3 + 3(-3)^2 - 4(-3) - 12$

$= -27 + 27 + 12 - 12$

$= 0$

Since the remainder is 0, then $k + 3$ is a factor of $k^3 + 3k^2 - 4k - 12$.

11. **[2010 PII #12b]**

$a^3 + 5a^2 - a - 5 = 0$

We use factors of the last term "-5" in a try and error method. The factors of -5 are ± 1 and ± 5.

Upon trying, we notice that $a = 1$

$f(a) = a^3 + 5a^2 - a - 5$

$f(1) = 1^3 + 5(1)^2 - 1 - 5$

$f(1) = 0$

$\therefore (a-1)$ is a factor

Since $(a-1)$ is a factor, the it leaves a remainder of 0.

We employ long divison method:

$$
\begin{array}{r|l}
 & a^2 + 6a + 5 \\
\hline
a - 1 & a^3 + 5a^2 - a - 5 \\
 & \underline{-(a^3 - a^2)} \\
 & \quad 6a^2 - a \\
 & \quad \underline{-(6a^2 - 6a)} \\
 & \qquad 5a - 5 \\
 & \qquad \underline{-(5a - 5)} \\
 & \qquad\quad 0
\end{array}
$$

Factorising the quotient

$a^2 + 6a + 5 = (a+5)(a+1)$ by inspection

$\Rightarrow a^3 + 5a^2 - a - 5 = 0$

$a^3 + 5a^2 - a - 5 = (a-1)(a+1)(a+5)$

Dividend = Divisor × Quotient

$\Rightarrow (a-1)(a+1)(a+5) = 0$

Either $(a-1) = 0$, $(a+1) = 0$ or $(a-5) = 0$

$\therefore a = 1$, or -1 or 5.

12. **[2011 PI #10]**

Let $f(x) = ax^3 + bx - 6$

let $x + 1 = 0$ and $x - 3 = 0$

$\Rightarrow x = -1$ and $x = 3$.

Given that $(x+1)$ and $(x-3)$ are factors,

It implies that the remainders $f(-1)$ and $f(3)$ are zeros.

Since $f(x) = ax^3 + bx - 6$,

then $f(-1) = a(-1)^3 + b(-1) - 6$

$= -a - b - 6.$

$\Rightarrow -a - b - 6 = 0$

$\therefore -a - b = 6$..................(i)

$f(3) = a(3)^3 + b(3) - 6$

$27a + 3b = 6$ (dividing by 3)

$9a + b = 2$..................(ii)

Solving by elimination

$$
\begin{array}{r}
9a + b = 2 \\
+\ \underline{-a - b = 6} \\
8a = 8
\end{array}
$$

$\therefore a = 1$

substituting in(ii)

$9(1) + b = 2 \quad \Rightarrow 9 + b = 2$

$\Rightarrow b = 2 - 9$

$\therefore b = -7$

13. **[2012 PII #9a]**

Let $f(x) = x^3 - 6x^2 + 11x + p$.

let $x - 2 = 0 \Rightarrow x = 2$.

Since $(x-2)$ is a factor of $x^3 - 6x^2 + 11x + p$,

then $f(2) = 0$

$f(2) = (2)^3 - 6(2)^2 + 11(2) + P$

$= 8 - 24 + 22 + P$

then $8-24+22+P=0$

$\Rightarrow 6+P=0$

$\therefore P=-6$

14. **[2013 P1 #19]**

Let $f(x)=x^3-x^2+x+9$

Let $x-a=0$, then $x=a$.

Using the remainder theorem, then the remainder is $f(a)$.

$f(a)=a^3-a^2+a+9$,

Let $g(x)=x^3-4x+3$, when divided by $x-a$, the remainder is $g(a)$.

$\Rightarrow g(a)=a^3-4a+3$

given the remainders are the same, then $f(a)=g(a)$.

$\Rightarrow a^3-a^2+a+9=a^3-4a+3$ (move *RHS* to LHS)

$a^3-a^2+a+9-a^3+4a-3=0$ (arrange like terms)

$a^3-a^3-a^2+4a+a+9-3=0$

$-a^2+5a+6=0$

$a^2-5a-6=0$ [multipying thru by -1 to make a^2 positive]

NB: This is only possible with an equation not an expression(without equals sign)!!!

$(a-6)(a+1)=0$ (by inspection)

either $a-6=0$ or $a+1=0$

$\therefore\ a=6$ or $a=-1$

15. **[2013 P2 #4a]**

To factorize $x^3+6x^2+3x-10$

We use factors of the last term "-10" in a try and error method. The factors of -10 are ±1, ±2, ±5 and ±10.

We will notice that $x=2$ gives zero.

Substitute $x=2$ into polynomial,

$x^3+6x^2+3x-10$

$-2^3+6(2)^3+3(2)-10$

$-8+24-6-10=0$

$0=0$

Then $(x+2)$ is a factor.

We now use long division method:

$$\begin{array}{r l}
 & x^2+4x-5 \\
x+2\,\big) & x^3+6x^2+3x-10 \\
 & \underline{-(x^3+2x^2)} \\
 & 4x^2+3x \\
 & \underline{-(4x^2+8x)} \\
 & -5x-10 \\
 & \underline{-(-5x-10)} \\
 & 0
\end{array}$$

$x^2+4x-5=(x-1)(x+5)$

$\Rightarrow x^3+6x^2+3x-10=(x+2)(x^2+4x-5)$

$=(x+2)(x-1)(x+5)$

16. **[2014 P1 #11]**

Let $f(x)=x^3-7x+m$ and let $x-1=0 \Rightarrow x=1$

$\therefore$ The remainder is $f(1)$.

$f(1)=(1)^3-7(1)+m$

$-1=1-7+m$ (Given that the remainder is -1)

$-1=-6+m$

$6-1=m$

$\therefore m=5$.

17. **[2015 PII #3b]**

Let $x-2=0, \Rightarrow$

$x=2$.

Let $f(x)=x^3-9x+10$

If $(x-2)$ is a factor, then $f(2)$ should give a remainder "0".

$f(2)=(2)^3-9(2)+10$

$=8-18+10$

$=-10+10$

$=0$

$\therefore (x-2)$ is a factor

18. **[2016 PI #12]**

Let $p(y) = y^3 + 3y^2 - 2y - 5$

Our divisor is $(y-2)$ so we

let $y - 2 = 0$

$\Rightarrow y = 2$

$p(2)$ is the remainder (remainder theorem)

since $p(y) = y^3 + 3y^2 - 2y - 5$

$$\begin{aligned} p(2) &= 2^3 + 3(2)^2 - 2(2) - 5 \\ &= 8 + 12 - 4 - 5 \\ &= 20 - 9 \\ &= 11 \end{aligned}$$

$\therefore$ The remainder 11.

19. **[2016 PII #7a]**

Let $f(x) = 2x^3 + kx^2 + 7$. Given the divisor

$x - 2 \quad \Rightarrow x = 2$ then when the divisor is

$x - 2$ the remainder is $f(2)$ – (remainder theorem)

$$\begin{aligned} \Rightarrow f(2) &= 2(2)^3 + k(2)^2 + 7 \\ &= 16 + 4k + 7 \\ &= 23 + 4k \end{aligned}$$

When the divisor is $2x - 1$, the remainder is $f\left(\frac{1}{2}\right)$

Letting $2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 + k\left(\frac{1}{2}\right)^2 + 7 = 2\left(\frac{1}{8}\right) + \frac{k}{4} + 7 \\ &= \tfrac{1}{4} + \tfrac{k}{4} + 7 \\ &= 7\tfrac{1}{4} + \tfrac{k}{4} \\ &= \tfrac{29}{4} + \tfrac{k}{4} \\ &= \tfrac{29+k}{4} \end{aligned}$$

$f(2) = \frac{1}{2}\left[f\left(\frac{1}{2}\right)\right]$

$(23 + 4k) = \frac{1}{2}\left(\frac{29+k}{4}\right)$

$23 + 4k = \frac{29+k}{8}$

$8(23 + 4k) = 29 + k$

$184 + 32k = 29 + k$

$32k - k = 29 - 184$

$31k = -155$

$\frac{31k}{31} = \frac{-155}{31}$

$k = -5$

20. **[2017 PII #2b]**

$2x^3 + 5x^2 - 4x - 3$

Factors of 3 are ± 1 and ± 3

Try $x = 1$

$f(x) = 2x^3 + 5x^2 - 4x - 3$

$$\begin{aligned} f(1) &= 2(1)^3 + 5(1)^2 - 4(1) - 3 \\ &= 2 + 5 - 4 - 3 \\ &= 0 \text{ works!!!} \end{aligned}$$

Since $f(1) = 0$, Let $x = 1 \Rightarrow x - 1 = 0$

$\therefore (x-1)$ is a factor of the polynomial

Using long division,

$$\begin{array}{r} 2x^2 + 7x + 3 \\ x-1\overline{)2x^3 + 5x^2 - 4x - 3} \\ \underline{-(2x^3 - 2x^2)} \\ 7x^2 - 4x \\ \underline{-(7x^2 - 7x)} \\ 3x - 3 \\ \underline{-(3x - 3)} \\ 0 \end{array}$$

$$\begin{aligned} \therefore 2x^3 + 5x^2 - 4x - 3 &= (x-1)(2x^2 + 7x + 3) \end{aligned}$$

Factors: $2 \times 3 = +6 \Rightarrow +6 + 1 = +7$

$$\begin{aligned} &= (x-1)(2x^2 + 6x + x + 3) \\ &= (x-1)[2x(x+3) + 1(x+3)] \\ &= (x-1)(x+3)(2x+1) \end{aligned}$$

21. **[2018 P1 #17]**

Let $x + 1 = 0$

$x = -1$

Since $x + 1$ is a factor, then,

substituting $x = -1$ in $2x^3 + qx^2 - 12x - 9$ gives 0.

$2(-1)^3 + q(-1)^2 - 12(-1) - 9 = 0$

$-2 + q + 12 - 9 = 0$

$q + 1 = 0$

$\therefore q = -1$

22. **[2018 PII #4]**

$ax^3 + 3x^2 - 8x + d = 0$ and the roots

are -3, $\frac{1}{2}$ and 1

$\therefore x = -3$ or $x = \frac{1}{2}$ or $x = 1$

(pay close attention to the middle equation)

$\Rightarrow x = -3$ or $2x = 1$ or $x = 1$

$\Rightarrow x + 3 = 0$ or $2x - 1 = 0$ or $x - 1 = 0$

Thus, $(x+3)(2x-1)(x-1) = 0$

Expanding the LHS

Starting with the first two brackets:

$[(x+3)(2x-1)](x-1) = 0$

$[x(2x-1) + 3(2x-1)](x-1) = 0$ (split and spread)

$(2x^2 - x + 6x - 3)(x-1) = 0$

$(2x^2 + 5x - 3)(x-1) = 0$

Now expand these two again:

$x(2x^2 + 5x - 3) - 1(2x^2 + 5x - 3) = 0$ (split and spread)

$2x^3 + 5x^2 - 3x - 2x^2 - 5x + 3 = 0$

$2x^3 + 5x^2 - 2x^2 - 3x - 5x + 3 = 0$ (grouping like terms)

$2x^3 + 3x^2 - 8x + 3 = 0$

Equating the two polynomials:

$ax^3 + 3x^2 - 8x + d \equiv 2x^3 + 3x^2 - 8x + 3$

$ax^3 = 2x^3$ (equating corresponding terms)

$$\frac{a\cancel{x^3}}{\cancel{x^3}} = \frac{2\cancel{x^3}}{\cancel{x^3}}$$

$a = 2$

23. **[2019 PII #7]**

$2m^3 - 5m^2 - 4m + 3 = 0$

Factors of 3 are; ± 1, ± 3

Trying $m = -1$

$f(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$

$f(-1) = 0$

$\therefore (m+1)$ is a factor

We proceed to use long division to identify other factors.

Using long division:

$$\begin{array}{r}
2m^2 - 7m + 3 \\
m+1 \overline{\smash{)}\, 2m^3 - 5m^2 - 4m + 3} \\
\underline{-(-2m^3 + 2m^2)} \\
-7m^2 - 4m \\
\underline{-(-7m^2 - 7m)} \\
3m + 3 \\
\underline{-(3m + 3)} \\
0
\end{array}$$

$\Leftrightarrow 2m^3 - 5m^2 - 4m + 3 = (m+1)(2m^2 - 7m + 3)$

Factorising the quotient

$2m^2 - 7m + 3 = 2m^2 - m - 6m + 3$

$= m(2m-1) - 3(2m-1)$

$= (m-3)(2m-1)$

$\therefore 2m^3 - 5m^2 - 4m + 3 = 0$

$(m+1)(m-3)(2m-1) = 0$

(pay close attention to the last equation below)

Either $m + 1 = 0$, $m - 3 = 0$ or $2m - 1 = 0$

Either $m = -1$, $m = 3$ or $2m = 1$

Either $m = -1$, $m = 3$ or $m = \frac{1}{2}$

Thus, $m = -1$, 3 or $\frac{1}{2}$

CH 27 PROBABILITY

Chapter Highlights

In this chapter, we will solve practical problems on probability involving two or more events. Probability of an event deals with how likely the event is to happen.

The probability of an event $P(\text{E})$ should be less than or equal to one and is found by the formula:

$$P(\text{E}) = \frac{\text{No. of outcomes in favour of E}}{\text{Total number of outcomes}}$$

Total Sum of Total Probability of all possibilities is 1. That is,

$$P(Event) + P(No\ Event) = 1$$

When constructing a tree diagram, the sum of probabilities for a branch must be 1.

1. A box contains 5 red balls, 8 white balls and 7 black balls. If one ball is selected at random, calculate the probability that it is white or black. **[2003 PI #14]**

2. The probability that it rains on a Monday is $\frac{1}{3}$, the probability that the teacher will be present on that day when it rains is $\frac{1}{6}$ and the probability that the teacher will be present when it does not rain is $\frac{7}{10}$. Draw a tree diagram and label all the probabilities for all the branches. **[2003 PII 4a]**

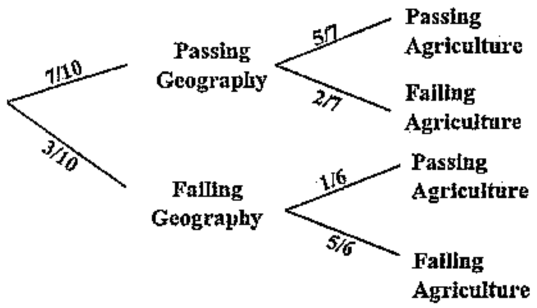
3. In a plastic bag there are x blue pens, 6 black pens, 4 red pens. If the probability of picking a red pen is $\frac{1}{5}$. Calculate the number of blue pens. **[2004 PI 23]**

4. A fair die and an unbiased coin are rolled and tossed respectively.

   a. Draw a probability space table for the possible outcomes of the die and the coin.

   b. Use the probability space table to find the probability of obtaining an even number on the die and a tail on the coin. **[2004 P2 #5a&b]**

5. **Figure 1** is a tree diagram illustrating the probability of a student passing Agriculture and Geography in an examination. The probability of passing Geography in the examination is $\frac{7}{10}$ and the probability of passing Agriculture after one has passed Geography is $\frac{5}{7}$. The probability of passing Agriculture after one has failed geography is $\frac{1}{6}$.



   Calculate the probability of a student passing Agriculture. **[2005 P1#21]**

6. A family has 3 children born at different times. Assuming it is equally likely to have a baby boy or a baby girl,
   a. Draw a tree diagram to show the possibility of having a baby boy or a girl on each of the three births.
   b. Calculate the probability that the family has 2 boys and 1 girl in any order of the births.

   [2005 PII #8b]

7. Coin A is tossed followed by coin B. the probability that coin A shows head ½ while the probability that coin B shows head is ¼. using a tree diagram, calculate the probability that coin A and B show tails. **[2006 P1 #21]**

8. A coin is tossed three times. Calculate the probability that the first- and third-coin tosses give heads as the outcomes, regardless of what the outcome of the second toss is. **[2006 PII #3a]**

9. The probability of having an early lunch at a boarding school is $\frac{2}{3}$. When the lunch is early the probability of having beef is $\frac{7}{10}$ and when late,

the probability of having beef is $\frac{1}{8}$. Draw a tree diagram to represent this information completing all branches. **[2007 PI #24]**

10. Two fair dice, each with six faces numbered 0, 1, 2, 3, 4, and 5 are rolled at the same time. **Table 1** shows the possible sums of the numbers on dice.

Die	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	7
3	3	4	5	6	7	8
4	4	5	6	7	8	9
5	5	6	7	8	9	10

Calculate the probability of getting;

a. a sum less than 4
b. a sum which is a multiple of 5.
[2007 PII #4a&4b]

11. The probability of the bus arriving early at depot is $\frac{1}{10}$ and arriving late is $\frac{3}{5}$. If 400 buses are expected at the depot during the day, calculate the number of buses that are likely to arrive at the depot on time. **[2008 PII #6a]**

12. **Table 2** shows numbers

16	17	3	13
5	11	10	8
9	7	6	12
4	14	15	17

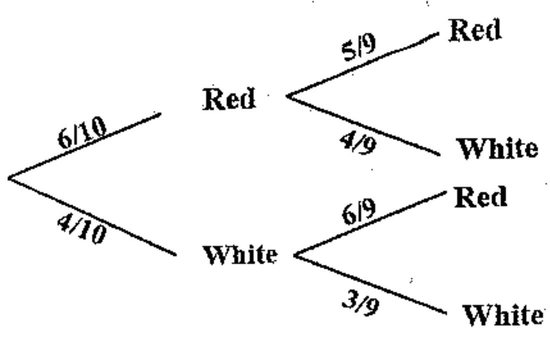
If a number is picked at random from the table, calculate the probability that it is prime or less than 9. **[2008 P1 #19]**

13. **Table 3** Shows the distribution of ages of learners in a form 2 class.

Age	14	15	16	17	18	19
Number of learners	2	10	8	4	9	3

What is the probability of picking at random a learner of 18 years of age? **[2009 P1 #9]**

14. **Figure 2** is a tree diagram which shows the probability of picking two balls one at a time without replacement from a bag containing 6 red and 4 white balls.



Using the tree diagram to calculate the probability of picking two balls of different colours, leaving your answer in its simplest form.
[2010 P1 #17]

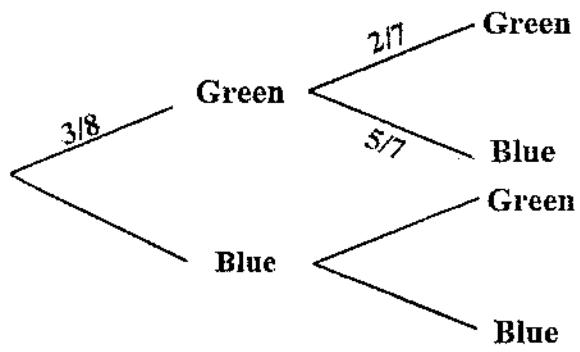
15. A bag contains beans, groundnuts and maize seeds. The probability of getting random bean seed is $\frac{1}{5}$, a groundnut seed is $\frac{x}{15}$ and a maize seed is $\frac{1}{3}$. Find the value represented by x.
[2010 PII #11a]

16. In ziweto village, families are allowed to keep three types of animals: chickens, goats and pigs. The probability that a family will keep a chicken, a goat and a pig is $\frac{1}{2}, \frac{1}{5}$, and $\frac{1}{10}$ respectively.
   i. Draw a tree diagram to illustrate this information.
   ii. Calculate the probability that a family will keep only one type of animal.
[2011 PII #10b]

17. A box has 5 cards 1, 2, 3, 4 and 5. Another box has 2 cards marked A and B. If a card is picked randomly from each box at the same time, calculate the probability of picking a card marked 3 from one and a card marked B from the other box. **[2012 P1 #16]**

18. In a classroom there are 10 boys and 16 girls. Two learners are chosen at random with replacement. Using a tree diagram, find the probability that the outcome is a boy and a girl, in any order.
[2013 PII #4b]

19. **Figure 3** is a tree diagram which shows the probability of picking two pens one at a time without replacing from the bag containing 3 green and 5 blue pens.



Complete the tree diagram and use it to calculate the probability of picking two blue pens.

[2014 PI #12]

20. A farmer's club committee has 4 women and 6 men. The club wants to form a sub-committee of 3 members, by electing a member at a time from the committee.
   i. Draw a tree diagram to show all possible outcomes of selecting the three members.
   ii. Use the tree diagram to calculate the probability of having one man and two women in the sub- committee.

[2014 PI #9b]

21. The whole numbers 3,4,5,6,7,8,9 were each written on a separate card. The cards were turned down on a table. Two cards were picked at random, one at a time without replacement. Calculate the probability that one card had an even number and the other had an odd number. **[2017 PI #18]**

22. A student has 6 grey and 4 black socks in his drawer. On a certain day he woke up late and quickly picked up 2 socks from a drawer at one time, put them on and rushed to school. Draw a tree diagram and use it to calculate the probability that the student wore socks of different colors.

[2017 PII #6a]

23. In order to be offered a place at college, a student must pass in three subjects: Mathematics, English and Science. If John has the probability of $\frac{1}{4}$ of failing Mathematics, $\frac{3}{4}$ of failing English and $\frac{1}{3}$ of failing Science, calculate the probability that he will be offered a place at college. **[2015 PII #4b]**

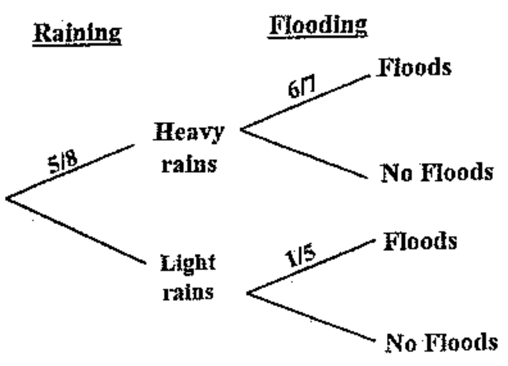
24. The probability of a football team winning a game at home is 0.6 and of winning away from home is 0.1. If the team played two games; home and away, calculate the probability of the team winning at least one game. **[2016 PI #17]**

25. A school library has 5 Mathematics and 7 Biology books. 3 of the Mathematics books and 4 of the Biology books have some pages missing. A student wants to borrow one Mathematics book and one Biology book from the library.
   a. Draw a tree diagram to show all the possible outcome of the book that the student can borrow.
   b. Use the tree diagram to find the probability that the student can borrow a mathematics book with all pages and a Biology book with some pages missing. **[2016 PII #11b]**

26. A wardrobe contains 5 reed shirts and 2 green ones. Mphatso picks a shirt at random and puts it on. The Chimwemwe picks another shirt at random and puts it on. Calculate the probability that Mphatso and Chimwemwe are wearing shirts of the same colour. **[2018 P1 #16]**

27. The probabilities of a team winning, drawing or losing a game are $\frac{5}{8}$, $\frac{1}{4}$ and $\frac{1}{8}$ respectively. If a team gets 2 points for winning and 1 point for a draw, calculate the probability that a team gets exactly 2 points in two games. **[2018 PII #11]**

28. **Figure 4** is a tree diagram showing probabilities of raining and flooding.



Given that the probability of raining heavily is $\frac{5}{8}$, the probability of flooding when it rains heavily is $\frac{6}{7}$ and the probability that of flooding when it rains lightly is $\frac{1}{5}$:

i. Complete the probability tree diagram in figure 4.

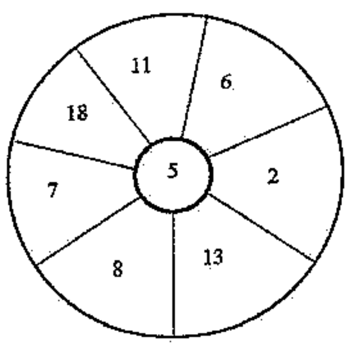
ii. Use the probability tree diagram to find the probability of flooding. **[2019 PII #6a]**

29. Two cards are chosen from the five cards shown in **Table 4** without replacement.

-2	-1	0	1	2

Find the probability that the product of the two numbers is negative. **[2020 Mock PII #3a]**

30. **Figure 5** is a dart board with various numbers and one can throw a dart and hit any spot.



What is the probability that when Ephraim throws a dart, he will hit:

i. A prime number.

ii. A multiple of 3. **[2021 Mock PII #5b]**

1. **[2003 PI #14]**

Let Red balls be R
Let White balls be W
Let Black balls be B
To find P(W or B)

$$P(W \text{ or } B) = \frac{n(W)+n(B)}{n(R+W+B)}$$

$$= \frac{8+7}{5+8+7}$$

$$= \frac{15}{20}$$

$$P(W \text{ or } B) = \frac{3}{4}$$

∴The probability that the ball selected is black or white is $\frac{3}{4}$.

2. **[2003 PII 4a]**

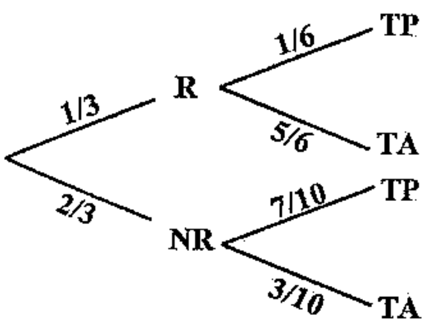
Let Rain=R, No Rain=NR
Let Teacher Present=TP
Let Teacher Absent=TA

P (R)= $\frac{1}{3}$ (given)
∴P (NR)= 1 - $\frac{1}{3}$ = $\frac{2}{3}$

P (TP, R)=$\frac{1}{6}$(given)
∴P (TA, R) = 1 – $\frac{1}{3}$ =$\frac{5}{6}$

P (TP, NR) = $\frac{7}{10}$ (given)
∴P (TA, NR)= 1 - $\frac{1}{3}$= $\frac{3}{10}$

The tree diagram will look like:



3. **[2004 PI 23]**

Probability of picking a red pen = $\frac{\textit{number of red pens}}{\textit{total number of pens}}$

So $\frac{1}{5} = \frac{4}{x+6+4}$ *(given)*

$$\frac{1}{5} = \frac{4}{x+10}$$

$1(x+10) = 5 \times 4$ *(cross multiplying)*
$x + 10 = 20$
$x = 20 - 10$
$x = 10$
∴there are ten blue pens.

4. **[2004 P2 #5a&b]**

Die

Coin	1	2	3	4	5	6
H	(H, 1)	(H, 2)	(H,3)	(H, 4)	(H, 5)	(H, 6)
T	(T,1)	(T, 2)	(T, 3)	(T, 4)	(T, 5)	(T, 6)

Where H is head T is tail and 1, 2, 3, 4, 5, 6 are the six faces of die
b. find P(Even and Tail)
from the table, we pick (T, 2), (T, 4) and (T, 6)

$$\text{P(Even and Tail)} = \frac{\textit{number of even and tail}}{\textit{total possibilities}}$$

$$= \frac{3}{12} \quad \textit{(from the table, 12 possibilities)}$$

$$= \frac{1}{4}$$

Alternatively, the probability space table can be drawn in a transposed manner as follows:

	H	T
1	(H, 1)	(T,1)
2	(H, 2)	(T, 2)
3	(H,3)	(T, 3)
4	(H, 4)	(T, 4)
5	(H, 5)	(T, 5)
6	(H, 6)	(T, 6)

Using this probability space table, the answers will be the same as previous.

5. **[2005 P1#21]**

Suppose passing agriculture is represented by A and passing geography is represented by G while failing geography is represented by G'
Using the tree diagram,
P(A) = P(G and A) + P(G' and A)

$$= \left(\frac{7}{10}\times\frac{5}{7}\right)+\left(\frac{3}{10}\times\frac{1}{6}\right)$$

$$= \frac{1}{2} + \frac{1}{20}$$

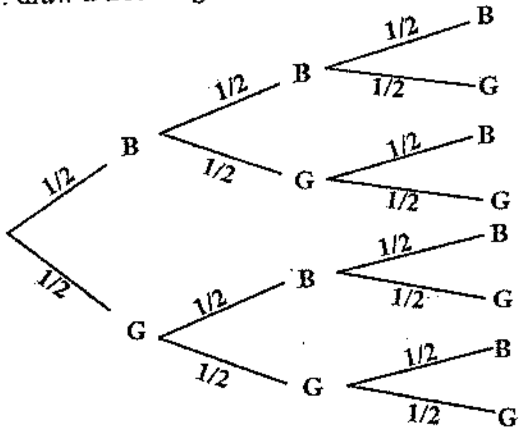
$$= \frac{10+1}{20}$$

$= \frac{11}{20}$

$\therefore$ The probability of passing agriculture is $\frac{11}{20}$.

6. **[2005 PII #8b]**

Let probability of having a baby boy = P(B)
Let probability of having a baby girl = P(G)
Recall total probability always equals to 1 so,
P(B) + P(G) =1
But P(B) = P(G) *(given)*
So P(B) + P(B) = 1
2P(B) = 1
P(B) = ½ hence P(G) = ½
i. draw a tree diagram



P(2 boys and 1 girl) = P(BBG) + P(BGB) + P(GBB)

$= (\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}) + (\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}) + (\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2})$

$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}$

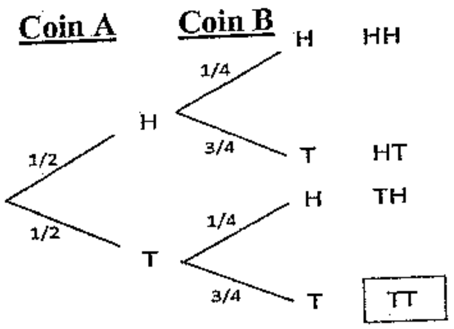
$= \frac{3}{8}$

Thus, in any order of births, the probability that the family has 2 boys and 1 girl is $\frac{3}{8}$.

7. **[2006 PI #21]**

Coin A: P(H)=$\frac{1}{2}$; P(T)=$1-\frac{1}{2}=\frac{1}{2}$;

Coin B: P(T)=$\frac{1}{4}$; P(T)=$1-\frac{1}{4}=\frac{3}{4}$;

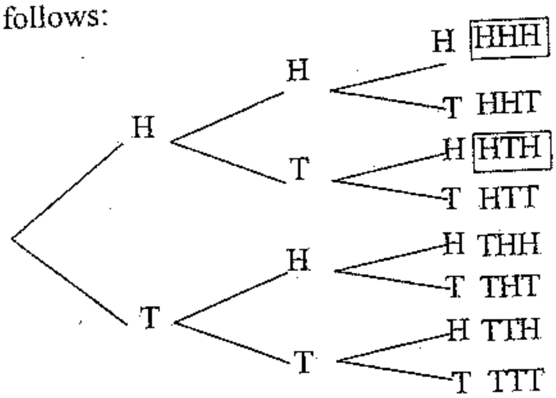


$P(TT) = \frac{1}{2}\times\frac{3}{4}$

$= \frac{3}{8}$

8. **[2006 PII #3a]**

Let H represent head and T represent tail. The tree diagram for the tosses is presented as follows:



There are two events that the first and the third tosses give heads out of a total of eight!

So $P(1st\text{ Head, }3rd\text{ Head})= \frac{2}{8}$

$= \frac{1}{4}$

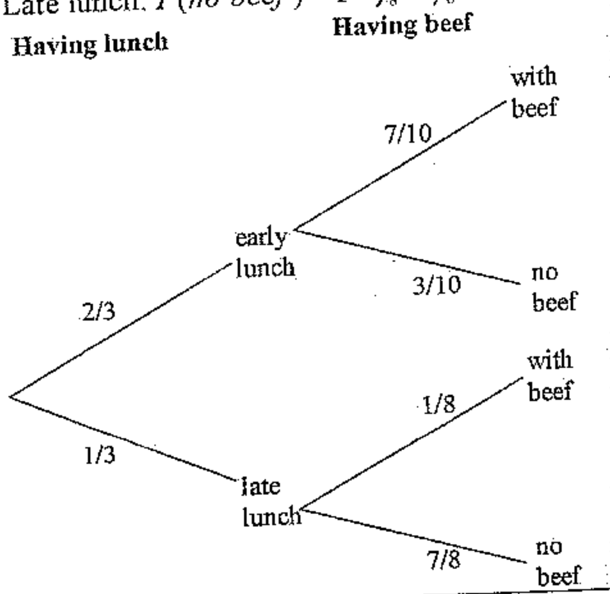
$\therefore$ the probability that the first and third tosses give heads is $\frac{1}{4}$.

9. **[2007 PI #24]**

$P(\text{late lunch}) = 1-\frac{2}{3}=\frac{1}{3}$

Early lunch: $P(\text{no beef}) = 1-\frac{7}{10}=\frac{3}{10}$

Late lunch: $P(\text{no beef}) = 1-\frac{1}{8}=\frac{7}{8}$



10. **[2007 PII #4a&4b]**

i. We count the numbers less than 4. These are 0, 1, 2, 3.

x	tally	freq
0	/	1
1	//	2
2	///	3
3	////	4
Sum		**10**

There are a total of 10 numbers less than 4 among a total of 36 numbers. So,

$$P(Sum < 4) = \frac{10}{36}$$
$$= \frac{5}{18}$$

ii. Multiple of 5

The multiples of 5 are: 5 and 10

x	tally	freq
5	###/	6
10	/	1
Sum		**7**

Thus, $P(\text{sum is multiple of 5}) = \frac{7}{36}$.

11. **[2008 PII #6a]**

Let $P(E)$ be probability of arriving *early*,

$P(L)$ be probability of arriving *late*

and $P(T)$ be probability of arriving *on time*.

$$P(E) + P(L) + P(T) = 1$$
$$\frac{1}{10} + \frac{3}{5} + P(T) = 1$$
$$P(T) = 1 - \frac{1}{10} - \frac{3}{5}$$
$$P(T) = \frac{10-1-6}{10}$$
$$P(T) = \frac{3}{10}$$

The total number of buses=400

buses *likely* to arrive on time:

$$= \frac{3}{10} \times 400 = 120$$

$\therefore$ 120 busses are likely to arrive on time.

12. **[2008 P1 #19]**

Prime={17,3,13,5,11,7,17}

Less than 9={3,5,8,7,6,4}

Prime or Less than 9

= Prime $\cup$ less than 9

={17,3,13,5,8,11,7,6,4,17}

$n(\text{Prime} \cup \text{less than 9}) = 10$

Total numbers=16

$\therefore P(\text{Prime} \cup \text{less than 9}) = \frac{10}{16} = \frac{5}{8}$

$\therefore$ The probability that the number picked is prime or less than 9 is $\frac{5}{8}$.

13. **[2009 P1 #9]**

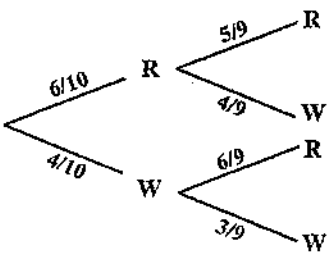
There are 9 learners who are 18 years old, so probability of 18year-old learners, $P(18\ years)$ is:

$$P(18\text{ years}) = \frac{\text{total no. of learners with 18 years}}{\text{total no. of learners}}$$
$$P(18\text{ years}) = \frac{9}{36}$$
$$P(18\text{ years}) = \frac{1}{4}$$

14. **[2010 P1 #17]**

Let Red be R

Let White be W



$$P(RW) = \frac{6}{10} \times \frac{4}{9} = \frac{4}{15}$$
$$P(WR) = \frac{4}{10} \times \frac{6}{9} = \frac{4}{15}$$
$$P(RW \text{ or } WR) = P(RW) + P(WR)$$
$$= \frac{4}{15} + \frac{4}{15}$$
$$= \frac{8}{15}$$

The probability of picking two balls of different colours is $\frac{8}{15}$

15. **[2010 PII #11a]**

$$P(beans) = \frac{1}{5};\ P(gnuts) = \frac{x}{15};\ P(maize) = \frac{1}{3}$$
$$\frac{1}{5} + \frac{x}{15} + \frac{1}{3} = 1 \text{ (Total probabilty} = 1)$$

$$\frac{3+x+5}{15}=1$$

$8+x=1(15)$

$8+x=15$

$x=15-8$

$\therefore x=7$

16. **[2011 PII #10b]**

Let keeping Chicken=C; No Chicken=NC

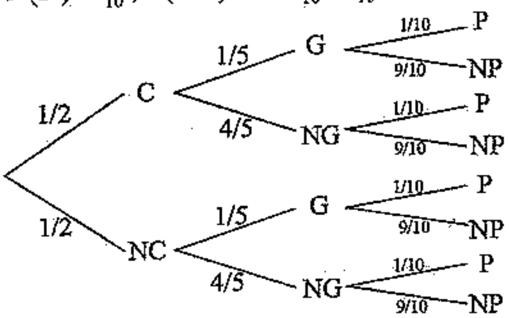
Let Keeping Goats=G; No Goats=NG

Let Keepin Pigs=P; No Pigs=NP

So, $P(C)=\frac{1}{2}$; $P(NC)=1-\frac{1}{2}=\frac{1}{2}$

$P(G)=\frac{1}{5}$; $P(NG)=1-\frac{1}{5}=\frac{4}{5}$

$P(P)=\frac{1}{10}$; $P(NP)=1-\frac{1}{10}=\frac{9}{10}$



Chicken Only: $P(C,NG,NP)=\frac{1}{2}\times\frac{4}{5}\times\frac{9}{10}=\frac{36}{100}$

Goat Only: $P(NC,G,NP)=\frac{1}{2}\times\frac{1}{5}\times\frac{9}{10}=\frac{9}{100}$

Pig Only: $P(NC,NG,P)=\frac{1}{2}\times\frac{4}{5}\times\frac{1}{10}=\frac{1}{100}$

$P(C,G,P \text{ only})=\frac{36}{100}+\frac{9}{100}+\frac{1}{100}=\frac{49}{100}$

The probability of keeping only one type of animal is $\frac{49}{100}$

17. **[2012 P1 #16]**

The sample possibility space is given as:

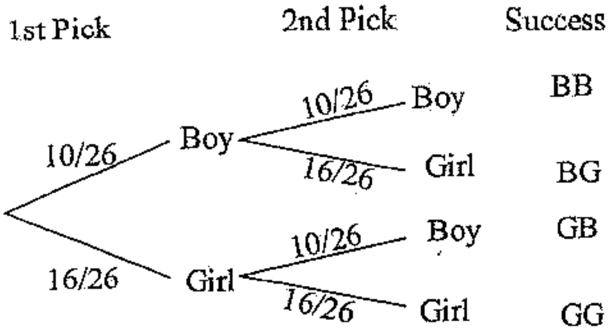
	1	**2**	**3**	**4**	**5**
A	A,1	A,2	A,3	A,4	A,5
B	B,1	B,2	B,3	B,4	B,5

So, $P(B,3)=\frac{1}{10}$ (one occurrence out of 10)

$\therefore$ probability of picking a card marked 3 from one box and a card marked B from the other box$=\frac{1}{10}$

18. **[2013 PII #4b]**

with replacement (given)



$P(\text{Boy \& Girl})=P(BG)+P(GB)$

$$=\left(\frac{10}{26}\times\frac{16}{26}\right)+\left(\frac{16}{26}\times\frac{10}{26}\right)$$

$$=\left(\frac{5}{13}\times\frac{8}{13}\right)+\left(\frac{8}{13}\times\frac{5}{13}\right)$$

$$=2\left(\frac{5}{13}\times\frac{8}{13}\right)$$

$$=2\left(\frac{40}{169}\right)$$

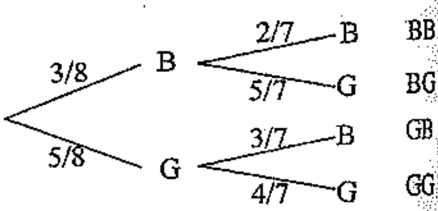
$$=\frac{80}{169}$$

19. **[2014 PI #12]**

$P(BB)$

$$=\frac{5}{\cancel{8}}\times\frac{\cancel{4}}{7}$$

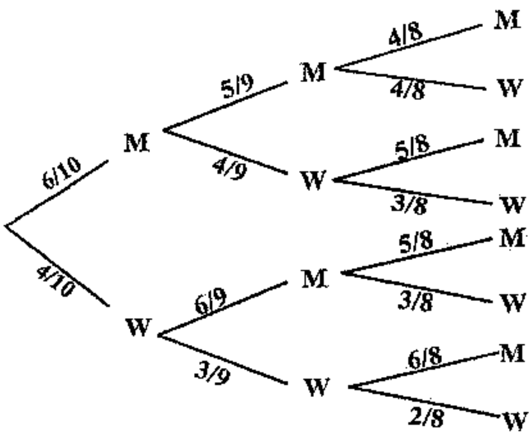
$$=\frac{5}{14}$$



$\therefore$ The prob of 2 pens is $\frac{5}{14}$.

20. **[2014 PI #9b]**

Formation of a sub-committee entails the selection from the main committee is *without replacement.*



iii.The probability of having one man and two women in the sub- committee is:

$$P(MWW)=\frac{6}{10}\times\frac{4}{9}\times\frac{3}{8}=\frac{72}{720}=\frac{1}{10}$$

$$P(WMW)=\frac{4}{10}\times\frac{6}{9}\times\frac{3}{8}=\frac{72}{720}=\frac{1}{10}$$

$$P(WWM)=\frac{4}{10}\times\frac{3}{9}\times\frac{6}{8}=\frac{72}{720}=\frac{1}{10}$$

$$P(M,2\text{W})=\frac{1}{10}+\frac{1}{10}+\frac{1}{10}$$

$$=\frac{1+1+1}{10}$$

$$=\frac{3}{10}$$

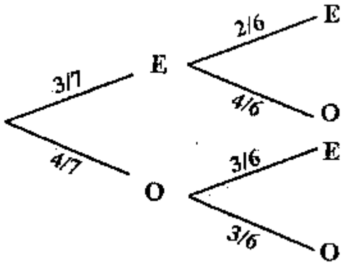
21. **[2017 PI #18]**

Let Even be E; E={4,6,8}

Let Odd be O; O={3,5,9,7}

So, $n(E)=3$; $P(E)=\frac{3}{7}$

So, $n(O)=4$; $P(O)=\frac{4}{7}$



$$P(EO \text{ and } OE)=P(EO)+P(OE)$$

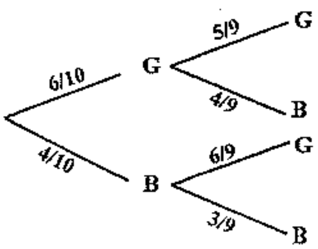
$$=\left(\frac{3}{7}\times\frac{4}{6}\right)+\left(\frac{4}{7}\times\frac{3}{6}\right)$$

$$=\frac{2}{7}+\frac{2}{7}$$

$$=\frac{4}{7}$$

22. **[2017 PII #6a]**

Let Grey=G; Let Black=B



$$P(\textit{Socks of different colours})=P(GB \text{ or BG})$$

$$=P(GB)+P(BG)$$

$$=\left(\frac{6}{10}\times\frac{4}{9}\right)+\left(\frac{4}{10}\times\frac{6}{9}\right)$$

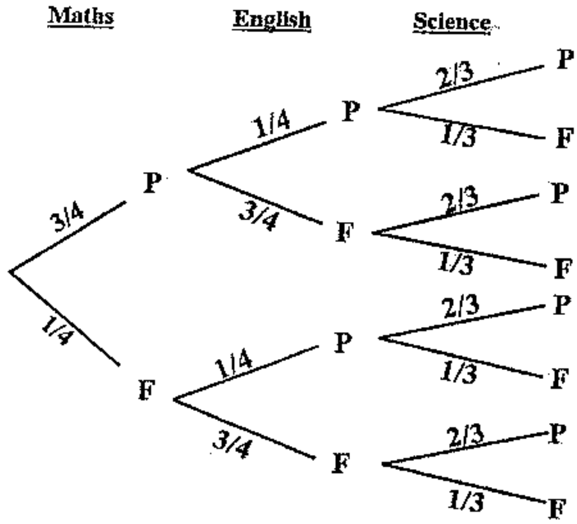
$$=\frac{4}{15}+\frac{4}{15}$$

$$=\frac{4+4}{15}$$

$$=\frac{8}{15}$$

23. **[2015 PII #4b]**

Let Pass=P; Fail=F.



In order to be offered a place at college, a student must pass in three subjects

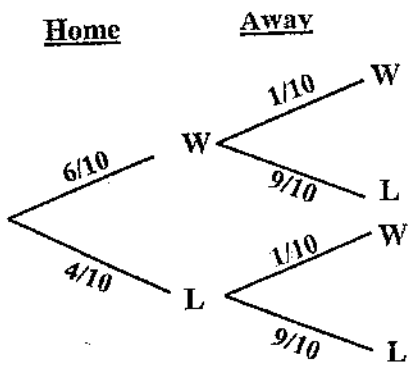
$$P(PPP)=\frac{3}{4}\times\frac{1}{4}\times\frac{2}{3}$$

$$P(PPP)=\frac{1}{8}$$

$\therefore$ The probability that he will be offered a place at college, $P(PPP)=\frac{1}{8}$

24. **[2016 PI #17]**

Let Win=W; Loss=W



P(at least one win)

$=1-P(\text{no win at all})$

$=1-P(LL)$

$=1-\left(\frac{4}{10}\times\frac{9}{10}\right)$

$=1-\frac{9}{25}$

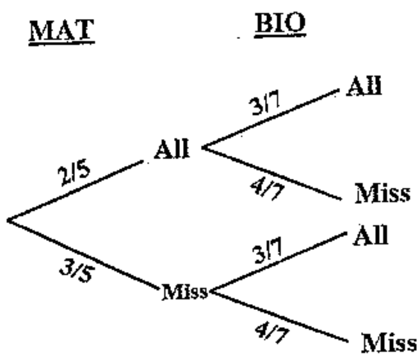
$=\frac{25-9}{25}$

$=\frac{16}{25}$

$\therefore$ The probability of winning at least one game $=\frac{16}{25}$ or 0.64

25. **[2016 PII #11b]**

(i) Let All pages available=All;
Let Some pages missing=Miss

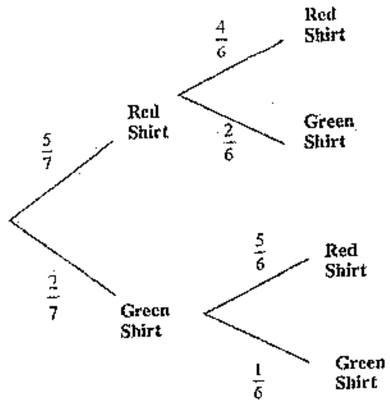


(ii) $P(All;Miss)=\frac{2}{5}\times\frac{4}{7}$
$=\frac{8}{35}$

The probability that the student can borrow a Mathematics book with *all pages* and a Biology book with some pages missing is

26. **[2018 P1 #16]**

Let Red shirt=R; Let Green shirt=G.



From the tree diagram:

$P(RR)+P(GG)$

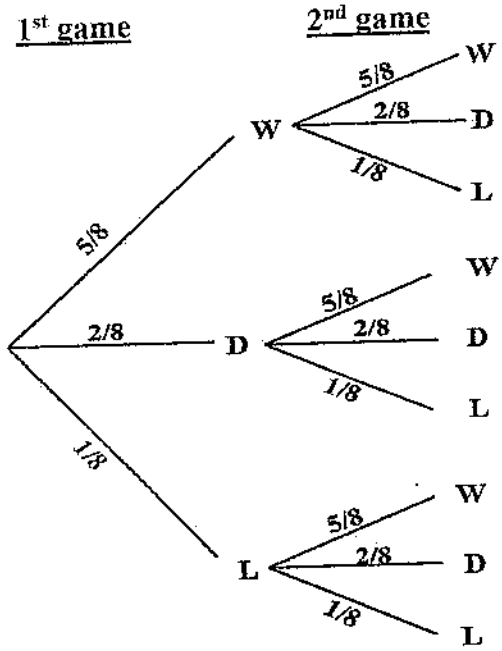
$=\left(\frac{5}{7}\times\frac{4}{6}\right)+\left(\frac{2}{7}\times\frac{1}{6}\right)$

$=\frac{10}{21}+\frac{1}{21}$

$=\frac{11}{21}$ 28.

$\therefore$ The probability that both are wearing a shirt of the same color is $\frac{11}{21}$

27. **[2018 PII #11]**



Exactly two points can be attained when:

(i) The team wins first game and lose the second

$P(WL)=\frac{5}{8}\times\frac{1}{8}=\frac{5}{64}$

or

(ii) When the team draws both games

$P(DD)=\frac{2}{8}\times\frac{2}{8}=\frac{1}{16}$

or

(iii) When they lose first game and win second

$P(LW)=\frac{1}{8}\times\frac{5}{8}=\frac{5}{64}$

$\therefore$ Probability of getting exactly 2 points

$=P(WL)+P(DD)+P(LW)$

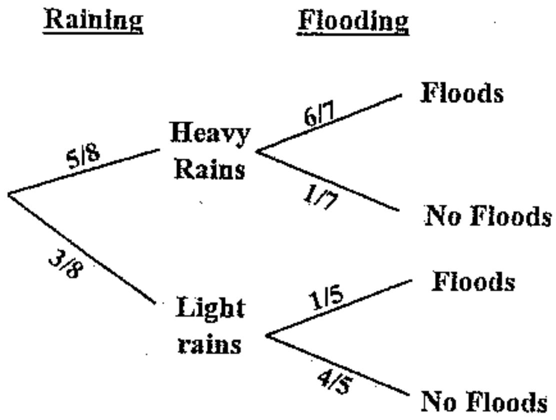
$=\frac{5}{64}+\frac{1}{16}+\frac{5}{64}$

$=\frac{5+4+5}{64}$

$=\frac{\cancel{14}^{7}}{\cancel{64}_{32}} \qquad =\frac{7}{32}$

28. **[2019 PII #6a]**

(i) The completed tree diagram



(ii) Probability of flooding.

$P(HF)=\frac{5}{8}\times\frac{6}{7}$

$=\frac{15}{28}$

$P(LF)=\frac{3}{8}\times\frac{1}{5}$

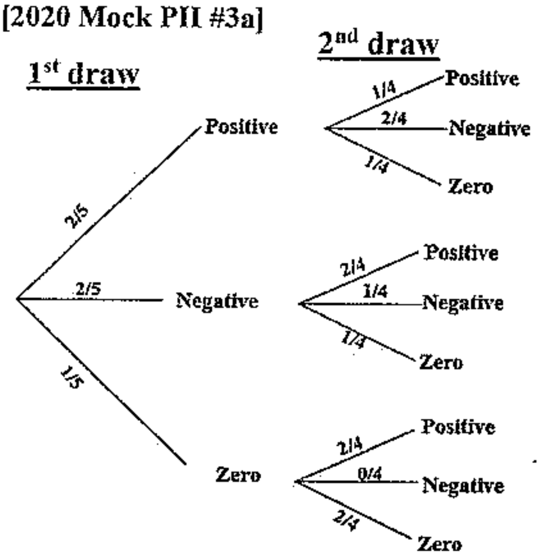
$=\frac{3}{40}$

$\therefore P(\text{Flooding})=P(HF)+P(LF)$

$=\frac{15}{28}+\frac{3}{40}$

$=\frac{171}{280}$

29. **[2020 Mock PII #3a]**



To find the probability that the product is negative. For a product to be negative, the numbers must be of different signs.
Let negative be N, Positive be P
P(negative product)$=(PN)+P(NP)$

$=\frac{2}{5}\times\frac{2}{4}+\frac{2}{5}\times\frac{2}{4}$

$=\frac{1}{5}+\frac{1}{5}$

$=\frac{2}{5}$

30. **[2021 Mock PII #5b]**

i.P(prime)

$P(prime\ number)=$

$P(2,5,7,11,13)$

$=\frac{5}{8}$

ii. P(multiple of 3)

$P(6,18)$

$=\frac{2}{8}$

$=\frac{1}{4}$

CH 28 VECTORS

Chapter Highlights

In this chapter, we will solve problems to do with vectors. Vectors include quantities like acceleration, velocity, displacement and all other quantities that require both magnitude and direction to be described.

Before solving these problems, you should be able to:

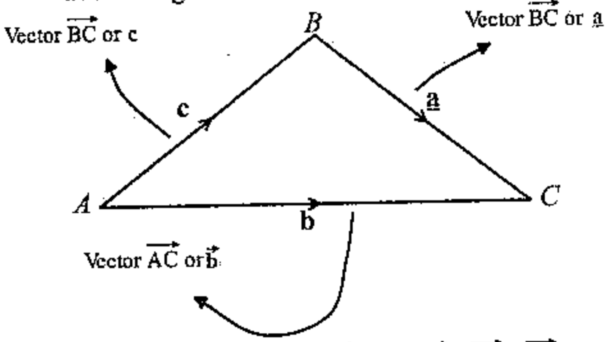
- Describe vectors, perform the basic operations of addition and subtraction as well as multiplying by a scalar on vectors.
- Calculate the magnitude and midpoint of a vector and show that points are collinear using the vector method.
- Use the parallelogram and triangle laws of vectors to solve some geometric problems on vectors.

Direction matters in describing vectors i.e. $\overrightarrow{AB} \neq \overrightarrow{BC}$.

Two conditions for collinear points are points:

- Vectors should be parallel
- Vectors should have a common point

The following notations will be useful in describing vectors:



$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{BC}$ $\quad$ $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$ $\quad$ $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{CA} = -\overrightarrow{AC} = -\vec{b}$

1. Calculate vector $\overrightarrow{AB}$ if vectors A$=\begin{pmatrix}3\\5\end{pmatrix}$ and B$=\begin{pmatrix}-4\\4\end{pmatrix}$ **[2003 PI #7]**

2. Solve for x and y in the vector equation. $\begin{pmatrix}x/3\\y/3\end{pmatrix} = 4\begin{pmatrix}-2\\1\end{pmatrix} - \begin{pmatrix}x\\y\end{pmatrix}$. **[2003 PII #7a]**

3. Given that $\overrightarrow{AB} = \begin{pmatrix}3\\5\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}9\\15\end{pmatrix}$, show that the points A, B and C are collinear. **[2004 PI #15]**

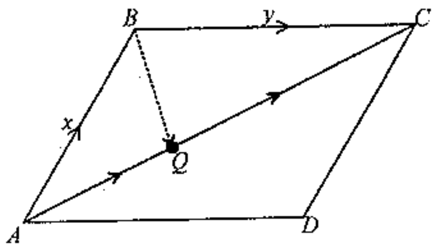
4. The position vector of points A, B, C and D are $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}4\\8\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix} and \begin{pmatrix}2\\-4\end{pmatrix}$, respectively. Show by vector method that ABCD is a parallelogram. **[2004 PII #3b]**

5. Given that $\overrightarrow{AB} = \begin{pmatrix}-9\\4\end{pmatrix}$. Calculate the length of $\overrightarrow{AB}$, leaving your answer correct to 3 significant figures. **[2005 PI #22]**

6. Let $\overrightarrow{AB} = \begin{pmatrix}6\\4\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}-4\\2\end{pmatrix}$. Calculate the midpoint of $\overrightarrow{BC}$. **[2005 PII #6b]**

7. Triangle **ABC** has vertices **A** (-1, 2), **B** (3, 3) and C (2, -1). Prove that angle **BAC**=angle **ACB**. **[2006 PI #10]**

8. **Figure 1** shows a parallelogram **ABCD** in which $\overrightarrow{AB} = \underline{x}$ and $\overrightarrow{BC} = \underline{y}$.



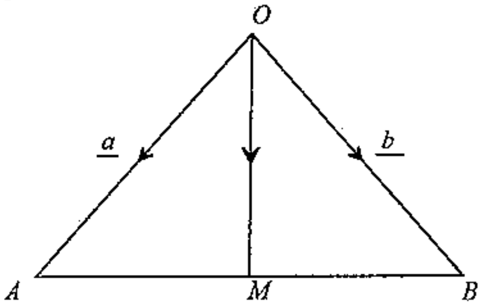
If $\overrightarrow{AQ} = 1/4\,\overrightarrow{AC}$ find $\overrightarrow{BQ}$ in terms of $\underline{x}$ and $\underline{y}$. **[2006 PI #24]**

9. Given that $\begin{pmatrix}a-1\\2\end{pmatrix} + \underline{b} = \begin{pmatrix}16\\7\end{pmatrix}$. Find $\underline{b}$. **[2006 PII #4a]**

10. Given $\underline{a} = \begin{pmatrix}-2\\-4\end{pmatrix}$ and $\underline{b} = \begin{pmatrix}-4\\0\end{pmatrix}$, find $\frac{1}{2}(\underline{b} - \underline{a})$. **[2007 PI #2]**

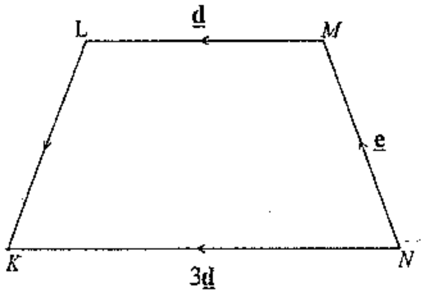
11. Points P and Q have a position vector (3, 7) and (9, 15) respectively. Find
    i. $\overrightarrow{PQ}$
    ii. $|\overrightarrow{PQ}|$ **[2007 PII #7a]**

12. In **Figure 2** $\overrightarrow{OA} = \underline{a}, \overrightarrow{OB} = \underline{b}$ and M is the mid-point $\overline{AB}$.



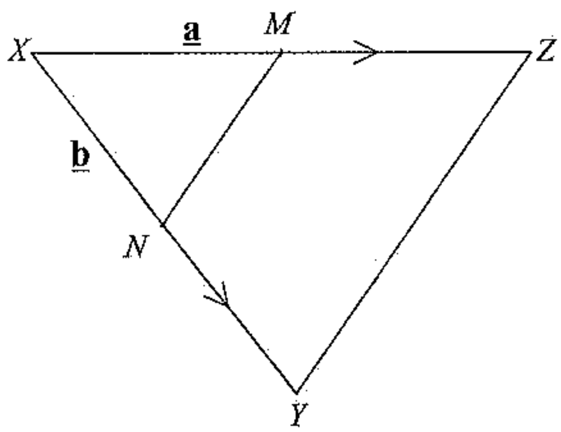
Find $\overline{OM}$ in terms of $\underline{a}$ and $\underline{b}$. **[2008 P1 #23]**

13. **Figure 3** shows a quadrilateral **KLMN** in which $\overline{NK} = 3\underline{d}$. $\overline{NM} = \underline{e}$ *and* $\overline{ML} = \underline{d}$



Express $\overline{LK}$ in terms of $\underline{d}$ and $\underline{e}$. **[2010 P1 #14]**

14. In **Figure 4, M** and **N** are midpoints of $\overline{XZ}$ *and* $\overline{XY}$ respectively. $\overrightarrow{XM} = \underline{a}$ and , $\overline{XN} = \underline{b}$.



Show that $\overline{MN}$ and $\overline{ZY}$ are parallel.

[2010 PII #5b]

15. Given that $\underline{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$, and calculate $|\underline{a} + \underline{b}|$. **[2011 PI #20]**

16. **A, B, C** and **D** are four points on the Cartesian plane such that **A(-6, 4), C (6, 2),** $\overline{AB} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$ and $\overline{AD} = \begin{pmatrix} 8 \\ -8 \end{pmatrix}$. Calculate the coordinates of point **B** and **D**. **[2011 PII #6b]**

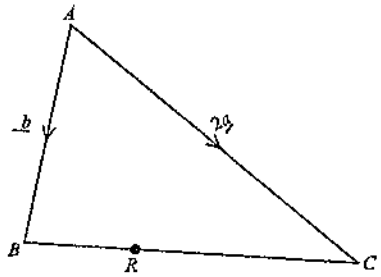
17. If $M = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $N = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ are position vectors, find $\overline{MN}$. **[2012 P1 #6]**

18. The points P (-1,-4), Q (1, 2), R (4, 4) and S (2,-2) form a quadrilateral. Show, by vector method, that **PQRS** is a parallelogram. **[2012 P2 #11a]**

19. Given that $\begin{pmatrix} -4 \\ 1 \end{pmatrix} - \underline{q} = \begin{pmatrix} -8 \\ 4, \end{pmatrix}$, calculate $\underline{q}$.

[2013 P1 #9]

20. **Figure 5** shows triangle ABC in which $\overline{AB} = \underline{b}$ and $\overline{AC}$=2$\underline{a}$. R is a point on BC such that $BR : RC = 1 : 3$.



(i). Express $\overline{BC}$ in terms of $\underline{a}$ and $\underline{b}$.

(ii).find $\overrightarrow{BR}$ in terms of $\underline{a}$ and $\underline{b}$.

[2013 P2 #6a]

21. **Figure 6** shows a triangle **ABC** with vertices, $A = (2,3)$ $B = (3,10)$ and $C = (8,7)$

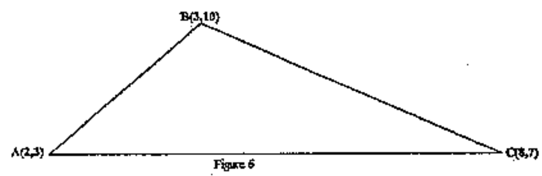


Figure 6

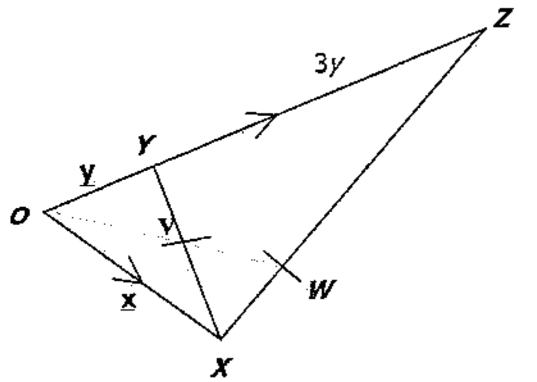
Show that $\vec{AB} + \vec{BC} = \vec{AC}$. **[2014 P1 #15]**

22. Given that point $A(2,5)$ and point $B=(10,-1)$. Find the position vector of point M on $\overrightarrow{AB}$ such that $\overrightarrow{AM} = \overrightarrow{MB}$. **[2014 PII #3b]**

23. The coordinates of point X are $(2,-4)$ and $\overrightarrow{XY} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the coordinates of point Y. **[2015 PI #8]**

24. Show that points $L(4,0)$, $M(14,11)$ and $N(19,16.5)$ are collinear. **[2015 PII #5b]**

25. **Figure 7** shows position vectors **OX** = $\underset{\sim}{x}$ and **OY** = $\underset{\sim}{y}$. **OY** is produced to a point **Z** where **OY:YZ** = 1:3. **V** is a point on **YX** such that **YV:VX**= 1:2 and **W** is a point on **XZ** such that **XW:WZ** =1:2.

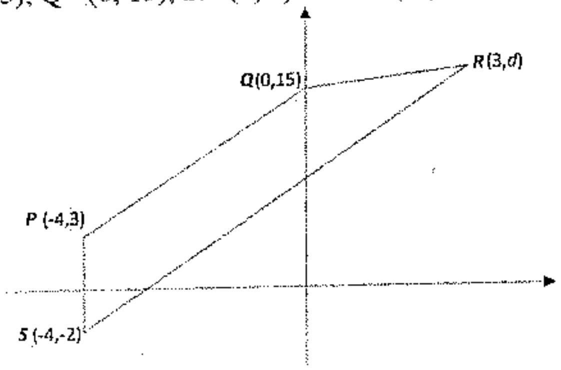


Show that **O, V** and **W** are collinear. **[2016 PII #8b]**

26. Given that $\left|\begin{pmatrix} -4 \\ w \end{pmatrix}\right| = 5$, calculate the values of w. **[2017 PI #6]**

27. Given that **M** and **N** are position vectors such that N= $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{\mathbf{MN}} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, find the position vector **M.** **[2017 PII #3b]**

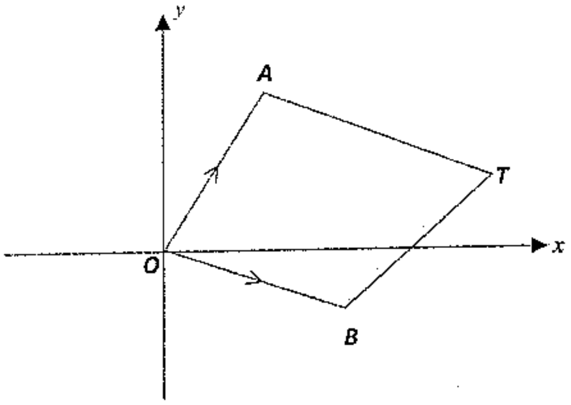
28. **Figure 8** shows a trapezium **PQRS** where **P** = (-4, 3), **Q**= (0, 15), **R**= (3, d) and **S**= (-4, -2).



Find the value of d. **[2017 PII #5a]**

29. Points **P** and **Q** have position vectors $\begin{pmatrix} 2 \\ b \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 13 \end{pmatrix}$ respectively. Given that $|\overline{\mathbf{PQ}}|$ =10, calculate the values of b. **[2018 P1 #8]**

30. In **Figure 9**, $\overline{\mathbf{OA}} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ and $\overline{\mathbf{OB}} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$.



If **OATB** is a parallelogram, find $\overrightarrow{\mathbf{OT}}$. **[2018 PII #8]**

31. Given that $\underset{\sim}{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ find $2\underset{\sim}{a} + 3\underset{\sim}{b}$. **[2019 PII #1b]**

1. **[2003 PI #7]**

Given $A=\begin{pmatrix}3\\5\end{pmatrix}$, $B=\begin{pmatrix}-4\\4\end{pmatrix}$, $\overrightarrow{AB}=B-A$

$$\overrightarrow{AB}=B-A \Rightarrow \overrightarrow{AB}=\begin{pmatrix}-4\\4\end{pmatrix}-\begin{pmatrix}3\\5\end{pmatrix}$$
$$=\begin{pmatrix}-4-3\\4-5\end{pmatrix}$$
$$\therefore \overrightarrow{AB}=\begin{pmatrix}-7\\-1\end{pmatrix}$$

2. **[2003 PII #7a]**

$$\begin{pmatrix}\frac{x}{3}\\\frac{y}{3}\end{pmatrix}=4\begin{pmatrix}-2\\1\end{pmatrix}-\begin{pmatrix}x\\y\end{pmatrix} \quad \left[\text{multiply 4 to }\begin{pmatrix}-2\\1\end{pmatrix}\right]$$

$$\begin{pmatrix}\frac{x}{3}\\\frac{y}{3}\end{pmatrix}=\begin{pmatrix}4\times-2\\4\times1\end{pmatrix}-\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}\frac{x}{3}\\\frac{y}{3}\end{pmatrix}=\begin{pmatrix}-8\\4\end{pmatrix}-\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}\frac{x}{3}\\\frac{y}{3}\end{pmatrix}=\begin{pmatrix}-8-x\\4-y\end{pmatrix} \quad \text{(equate corresponding terms)}$$

$\frac{x}{3}=-8-x$ (equating x values)

$$\left.\begin{aligned}x&=-24-3x\\x+3x&=-24\\4x&=-24\\x&=-\frac{24}{4}\\x&=-6\end{aligned}\right\}\text{Solving for } x$$

$$\left.\begin{aligned}\frac{y}{3}&=4-y\\y&=12-3y\\y+3y&=12\\4y&=12\\\therefore y&=3\end{aligned}\right\}\text{solving for } y$$

$\therefore$the value of x is -6 and the value of y is 3.

Alternative

Multiply throughout by 3:

$$3\begin{pmatrix}\frac{x}{3}\\\frac{y}{3}\end{pmatrix}=3\times4\begin{pmatrix}-2\\1\end{pmatrix}-3\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}3\times\frac{x}{3}\\3\times\frac{y}{3}\end{pmatrix}=\begin{pmatrix}12\times-2\\12\times1\end{pmatrix}-\begin{pmatrix}3x\\3y\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-24\\12\end{pmatrix}-\begin{pmatrix}3x\\3y\end{pmatrix}$$

So, $x=-24-3x$ (i)

$y=12-3y$ (ii)

From (i)	From (ii),
$x+3x=-24$	$y+3y=12$
$4x=-24$	$4y=12$
$x=-\frac{24}{4}$	$y=\frac{12}{4}$
$x=-6$	$y=3$

3. **[2004 PI #15]**

Two conditions for collinear points
i. Parallel lines
ii. Common point

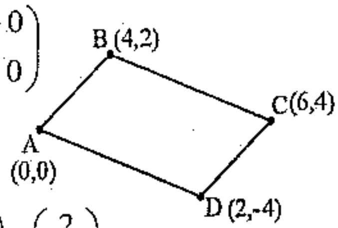
i. $\overrightarrow{AB}=\begin{pmatrix}3\\5\end{pmatrix}$, and $\overrightarrow{BC}=\begin{pmatrix}9\\15\end{pmatrix}$ taking $\overrightarrow{BC}$

$$\overrightarrow{BC}=3\begin{pmatrix}3\\5\end{pmatrix}=3\overrightarrow{AB}$$

So $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.
But B is a common point for $\overrightarrow{AB}$ and $\overrightarrow{BC}$
$\therefore$ points A, B, and C are collinear. Hence, ABC is a straight line.

4. **[2004 PII #3b]**

$$\overrightarrow{AB}=B-A \Rightarrow \overrightarrow{AB}=\begin{pmatrix}4\\8\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}$$
$$=\begin{pmatrix}4-0\\8-0\end{pmatrix}$$
$$=\begin{pmatrix}4\\8\end{pmatrix}$$



$$\overrightarrow{DC}=C-D \Rightarrow \overrightarrow{DC}=\begin{pmatrix}6\\4\end{pmatrix}-\begin{pmatrix}2\\-4\end{pmatrix}$$
$$\overrightarrow{DC}=\begin{pmatrix}6-2\\4--4\end{pmatrix}=\begin{pmatrix}4\\8\end{pmatrix}.$$

$\therefore \overrightarrow{AB} \parallel \overrightarrow{DC}$ and $|\overrightarrow{AB}| = |\overrightarrow{DC}|$

$\therefore$ABCD is a //gram (*opp. Sides are equal and //*)

5. **[2005 PI #22]**

Length of $\overrightarrow{AB} = |\overrightarrow{AB}|$

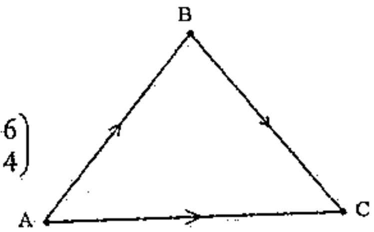
$$\therefore |\overrightarrow{AB}| = \sqrt{(-9)^2 + 4^2}$$
$$= \sqrt{81+16}$$
$$= \sqrt{97}$$
$$= 9.85 \qquad \textit{(to 3 significant figures)}$$

$\therefore$ the length of ($\overrightarrow{AB}$) is 9.85 units.

6. **[2005 PII #6b]**

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$$



$$\overrightarrow{BC} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} -4-6 \\ 2-4 \end{pmatrix}$$
$$= \begin{pmatrix} -10 \\ -2 \end{pmatrix}$$

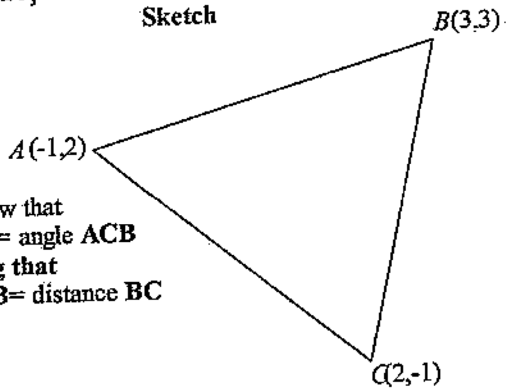
Midpoint of $\overrightarrow{BC} = \frac{1}{2}\,\overrightarrow{BC}$

$$= \frac{1}{2}\begin{pmatrix} -10 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} \times -10 \\ \frac{1}{2} \times -2 \end{pmatrix}$$
$$= \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

$\therefore$Coordinates of midpoint $\overrightarrow{BC}$ is (-5, -1)

7. **[2006 PI #10]**

Sketch



We can show that
angle **BAC**= angle **ACB**
by showing that
distance **AB**= distance **BC**

Compute $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$

$$|\overrightarrow{AB}| = |B - A|$$
$$= \left|\begin{pmatrix} 3 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} 3 \\ 3 \end{pmatrix} + \begin{pmatrix} +1 \\ -2 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} 3+1 \\ 3-2 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right|$$
$$= \sqrt{4^2 + 1^2}$$
$$= \sqrt{16+1}$$
$$= \sqrt{17}$$

$$|\overrightarrow{BC}| = |C - B|$$
$$= \left|\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} 2-3 \\ -1-3 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} -1 \\ 4 \end{pmatrix}\right|$$
$$= \sqrt{(-1)^2 + 4^2}$$
$$= \sqrt{1+16}$$
$$= \sqrt{17}$$

Hence, $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$.

$\therefore \angle BAC = \angle ACB$ (Angles opposite equal sides)

8. **[2006 PI #24]**

Given parallelogram ABCD in which $\overrightarrow{AB} = \underline{x}$ and $\overrightarrow{BC} = y$.

Asked to find $\overrightarrow{BQ}$:

$So, \overrightarrow{BQ} = \overrightarrow{BA} + \overrightarrow{AQ}$

$\overrightarrow{BA} = -\overrightarrow{AB}$

$\quad = -x$

$Thus, \overrightarrow{BQ} = -x + \overrightarrow{AQ}$

To find $\overrightarrow{AQ}$,

$\overrightarrow{AQ} = \frac{1}{4}\overrightarrow{AC}$ (given);

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{AC} = \underline{x} + \underline{y}$

Thus, $\overrightarrow{AQ} = \frac{1}{4}\overrightarrow{AC}$

$$\overrightarrow{AQ} = \frac{1}{4}(\underline{x} + \underline{y})$$
$$\overrightarrow{BQ} = -x + \frac{1}{4}(x + y)$$
$$\overrightarrow{BQ} = -x + \frac{1}{4}x + \frac{1}{4}y$$
$$\overrightarrow{BQ} = -\frac{3}{4}x + \frac{1}{4}y$$
$$\overrightarrow{BQ} = \frac{1}{4}(-3x + y).$$

9. **[2006 PII #4a]**

we make $\underline{b}$ subject:

$$\begin{pmatrix} a-1 \\ 2 \end{pmatrix} + \underline{b} = \begin{pmatrix} 16 \\ 7 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} 16 \\ 7 \end{pmatrix} - \begin{pmatrix} a-1 \\ 2 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} 16-(a-1) \\ 7-2 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} 16-a+1 \\ 7-2 \end{pmatrix}$$

$$\therefore \underline{b} = \begin{pmatrix} 17-a \\ 5 \end{pmatrix}$$

10. **[2007 PI #2]**

Given $\underline{a} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

Substitute a and b into the equation.

$$½(\underline{b}-\underline{a}) = ½\left[\begin{pmatrix} -4 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix}\right]$$

$$= ½\begin{pmatrix} -4--2 \\ 0--4 \end{pmatrix}$$

$$= ½\begin{pmatrix} -4+2 \\ 0+4 \end{pmatrix}$$

$$= ½\begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} ½\times -2 \\ ½\times 4 \end{pmatrix}$$

$$\therefore ½(\underline{b}-\underline{a}) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

11. **[2007 PII #7a]**

Given that P =(3,7) and Q =(9,15)

$$\overrightarrow{PQ} = Q - P$$

$$\overrightarrow{PQ} = \begin{pmatrix} 9 \\ 15 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 9-3 \\ 15-7 \end{pmatrix}$$

$$\therefore \overrightarrow{PQ} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

ii. $|\overrightarrow{PQ}|$

$$|\overrightarrow{PQ}| = \sqrt{x^2+y^2}$$

$$|\overrightarrow{PQ}| = \sqrt{6^2+8^2}$$

$$= \sqrt{36+64}$$

$$= \sqrt{100}$$

$$\therefore |\overrightarrow{PQ}| = 10$$

12. **[2008 P1 #23]**

Finding OM in terms of $\underline{a}$ and $\underline{b}$.

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$\overrightarrow{OA} = \underline{a}$ and

$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$ (since M is the midpoint of $\overrightarrow{AB}$)

But $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$AB = -\underline{a} + \underline{b}$ (Negate $\underline{a}$ since $\overrightarrow{AO} = -\overrightarrow{OA}$)

$\therefore \overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} \quad \Rightarrow \overrightarrow{AM} = \frac{1}{2}(-\underline{a}+\underline{b})$

$\therefore \overrightarrow{OM} = \underline{a} + \frac{1}{2}(-\underline{a}+\underline{b})$

$\Rightarrow \overrightarrow{OM} = \underline{a} - \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}, \quad \overrightarrow{OM} = \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$

$\therefore \overrightarrow{OM} = \frac{1}{2}(\underline{a}+\underline{b})$

13. **[2010 P1 #14]**

Finding $\overline{LK}$

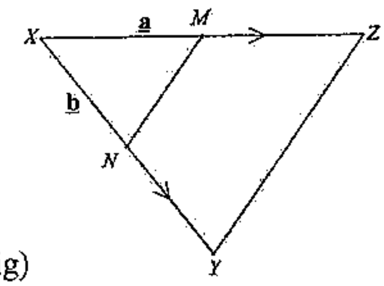
$\overrightarrow{LK} = \overrightarrow{LM} + \overrightarrow{MN} + \overrightarrow{NK}$ (addition law of vectors)

$= -\overrightarrow{ML} - \overrightarrow{NM} + \overrightarrow{NK}$ (by vector laws, $AB = -BA$)

$= -\underline{d} - \underline{e} + 3\underline{d}$

$\overline{LK} = 2\underline{d} - \underline{e}$

14. **[2010 PII #5b]**



Join $\overline{MN}$.

In ΔXMN

$\overrightarrow{MN} = \overrightarrow{MX} + \overrightarrow{XN}$

$\overrightarrow{MN} = \overrightarrow{XN} - \overrightarrow{XM}$

$\overrightarrow{MN} = \underline{b} - \underline{a}$ (from fig)

In ΔXZY

$\overrightarrow{ZY} = \overrightarrow{ZX} + \overrightarrow{XY}$

but $\overrightarrow{XM} = \frac{1}{2}\overrightarrow{XZ}$

$\Rightarrow 2\overrightarrow{XM} = \overrightarrow{XZ}$

$2(\underline{a}) = \overrightarrow{XZ}$

$\Rightarrow \overrightarrow{XZ} = 2\underline{a}$

$\therefore \overrightarrow{ZX} = -2\underline{a}$

$\overrightarrow{XN} = \frac{1}{2}\overrightarrow{XY}$

$\Rightarrow \overrightarrow{XY} = 2\overrightarrow{XN} \quad \Rightarrow \overrightarrow{XY} = 2\underline{b}$

$\overrightarrow{ZY} = 2\underline{a} + 2\underline{b}$

$\therefore \overrightarrow{ZY} = 2(\underline{-a} + \underline{b}) = 2(b - a)$

$\Rightarrow \overrightarrow{ZY} = 2\overrightarrow{MN}$

$\therefore \overrightarrow{ZY} \parallel \overrightarrow{MN}$

Since $\overrightarrow{ZY} = \lambda\overrightarrow{MN}$

where $\lambda = 2$

15. **[2011 PI #20]**

Given $\underline{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $b = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

$\underline{a} + \underline{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 9 \\ 4 \end{pmatrix} = \begin{pmatrix} 3+9 \\ 5+4 \end{pmatrix}$

$\underline{a} + \underline{b} = \begin{pmatrix} 12 \\ 9 \end{pmatrix}$

$|\underline{a} + \underline{b}| = \sqrt{12^2 + 9^2} = \sqrt{144 + 81} = \sqrt{225}$

$= 15$

16. **[2011 PII #6b]**

$A(-6,4)$, $B(x,y)$, $C(6,2)$, $D(x,y)$

$\overrightarrow{AB} = B - A$

$\Rightarrow \overrightarrow{AB} = B - \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ *But* $\overrightarrow{AB} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$ (given)

$\Rightarrow B - \begin{pmatrix} -6 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} \quad \Rightarrow B = \begin{pmatrix} 8 \\ 2 \end{pmatrix} + \begin{pmatrix} -6 \\ 4 \end{pmatrix}$

$\Rightarrow B = \begin{pmatrix} 8+-6 \\ 2+4 \end{pmatrix} \quad \therefore B = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$\therefore B(2,6)$ (in coordinate form)

$\overrightarrow{AD} = D - A$

$= D - \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ *But* $\overrightarrow{AD} = \begin{pmatrix} 8 \\ -8 \end{pmatrix}$ (given)

$\Rightarrow D - \begin{pmatrix} -6 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \end{pmatrix} \quad \Rightarrow D = \begin{pmatrix} 8 \\ -8 \end{pmatrix} + \begin{pmatrix} -6 \\ 4 \end{pmatrix}$

$\Rightarrow D = \begin{pmatrix} 8-6 \\ -8+4 \end{pmatrix} \quad \therefore D = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$\therefore D(2,-4)$ (in coordinate form)

17. **[2012 P1 #6]**

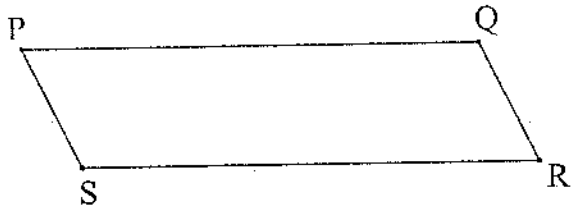
$\overrightarrow{MN} = N - M$

$M = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $N = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (given)

$\overrightarrow{MN} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} \Rightarrow \overrightarrow{MN} = \begin{pmatrix} 1-3 \\ 3-(-1) \end{pmatrix}$

$\therefore \overrightarrow{MN} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

18. **[2012 P2 #11a]**



Given that:

$P = \begin{pmatrix} -1 \\ -4 \end{pmatrix}, Q = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, R = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $S = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$\overrightarrow{PQ} = Q - P$

$\Rightarrow \overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad \Rightarrow \overrightarrow{PQ} = \begin{pmatrix} 1-(-1) \\ 2-(-4) \end{pmatrix}$

$\Rightarrow \overrightarrow{PQ} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$\overrightarrow{SR} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad \Rightarrow \overrightarrow{SR} = \begin{pmatrix} 4-2 \\ 4-(-2) \end{pmatrix}$

$\therefore \overrightarrow{SR} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, Thus, $|\overrightarrow{PQ}| = |\overrightarrow{SR}|$, and also $\overrightarrow{PQ} = 1\overrightarrow{SR}$.

Hence, $\overrightarrow{PQ} \parallel \overrightarrow{SR}$ (since $\overrightarrow{PQ} = \lambda\overrightarrow{SR}$ where $\lambda = 1$)

Therefore, PQRS is a parallelogram

(opposite sides equal and parallel)

---

19. **[2013 P1 #9]**

$$\begin{pmatrix}-4\\1\end{pmatrix} = \begin{pmatrix}-8\\4\end{pmatrix} + \underline{q}$$

$$\begin{pmatrix}-4\\1\end{pmatrix} - \begin{pmatrix}-8\\4\end{pmatrix} = \underline{q}$$

$$\Rightarrow \underline{q} = \begin{pmatrix}-4+8\\1-4\end{pmatrix}$$

$$\therefore \underline{q} = \begin{pmatrix}4\\-3\end{pmatrix}$$

So $|q| = \sqrt{4^2 + (-3)^2}$

$= \sqrt{16+9}$

$= \sqrt{25}$

$\therefore |q| = 5$ units

---

20. **[2013 P2 #6a]**

(i) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$\overrightarrow{BC} = -\underline{b} + 2\underline{a} = 2\underline{a} - \underline{b}$

(ii) $\overrightarrow{BR} : \overrightarrow{RC} = 1:3$

$\Rightarrow \dfrac{\overrightarrow{BR}}{\overrightarrow{RC}} = \dfrac{1}{3} \Rightarrow 3\overrightarrow{BR} = \overrightarrow{RC}$

So BC=$\overrightarrow{BR} + \overrightarrow{RC}$

$\overrightarrow{BC} = \overrightarrow{BR} + 3\overrightarrow{BR}$

$\overrightarrow{BC} = 4\overrightarrow{BR}$

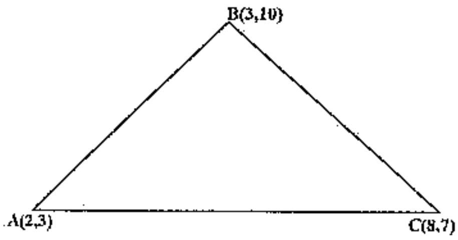
$\Rightarrow \overrightarrow{BR} = \dfrac{1}{4}\overrightarrow{BC}$

$= \dfrac{1}{4}(-\underline{b} + 2\underline{a})$

$= \dfrac{1}{4}(2\underline{a} - \underline{b})$

---

21. **[2014 P1 #15]**



To show that: $\vec{AB} + \vec{BC} = \vec{AC}$

$\overrightarrow{AB} = B - A$, $\overrightarrow{BC} = C - B$ and $\overrightarrow{AC} = C - A$.

$$\overrightarrow{AB} = \begin{pmatrix}3\\10\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}3-2\\10-3\end{pmatrix} = \begin{pmatrix}1\\7\end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix}8\\7\end{pmatrix} - \begin{pmatrix}3\\10\end{pmatrix} = \begin{pmatrix}8-3\\7-10\end{pmatrix} = \begin{pmatrix}5\\-3\end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix}8\\7\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}8-2\\7-3\end{pmatrix} = \begin{pmatrix}6\\4\end{pmatrix}$$

$$\overrightarrow{AB} + \overrightarrow{BC} = \begin{pmatrix}1\\7\end{pmatrix} + \begin{pmatrix}5\\-3\end{pmatrix} = \begin{pmatrix}1+5\\7-3\end{pmatrix} = \begin{pmatrix}6\\4\end{pmatrix}$$

$\therefore \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

---

22. **[2014 PII #3b]**

$A = \begin{pmatrix}2\\5\end{pmatrix}$ and $B = \begin{pmatrix}10\\-1\end{pmatrix}$

Let the coordinate M be (x, y).

$\overrightarrow{AM} = \overrightarrow{MB}$

$M - A = B - M$

$$M - \begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}10\\-1\end{pmatrix} - M \Rightarrow M + M = \begin{pmatrix}10\\-1\end{pmatrix} + \begin{pmatrix}2\\5\end{pmatrix}$$

$\Rightarrow 2M = \begin{pmatrix}12\\4\end{pmatrix}$ (Multiply $\frac{1}{2}$ both sides)

$\Rightarrow M = \dfrac{1}{2}\begin{pmatrix}12\\4\end{pmatrix} \qquad \Rightarrow M = \begin{pmatrix}6\\2\end{pmatrix}$

$\therefore$The position vector of point M, $\overrightarrow{OM} = \begin{pmatrix}6\\2\end{pmatrix}$

---

23. **[2015 PI #8]**

We know that $\overrightarrow{XY} = Y - X$

Given that $\overrightarrow{XY} = \begin{pmatrix}3\\2\end{pmatrix}$ and $X = \begin{pmatrix}2\\-4\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}3\\2\end{pmatrix} = Y - \begin{pmatrix}2\\-4\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}2\\-4\end{pmatrix} = Y$$

$$\Rightarrow \begin{pmatrix}3+2\\2-4\end{pmatrix} = Y$$

$$Y = \begin{pmatrix}5\\-2\end{pmatrix}$$

The coordinates of Y are (5, -2)

---

24. **[2015 PII #5b]**

$L=\begin{pmatrix}4\\0\end{pmatrix}, M=\begin{pmatrix}14\\11\end{pmatrix}$ and $N=\begin{pmatrix}19\\16.5\end{pmatrix}$

$\overrightarrow{LM}=L-M$

$=\begin{pmatrix}4\\0\end{pmatrix}-\begin{pmatrix}14\\11\end{pmatrix} \quad \Rightarrow \overrightarrow{LM}=\begin{pmatrix}-10\\-11\end{pmatrix}$

$\overrightarrow{MN}=N-M$

$=\begin{pmatrix}19\\16.5\end{pmatrix}-\begin{pmatrix}14\\11\end{pmatrix} \quad \Rightarrow \overrightarrow{MN}=\begin{pmatrix}5\\5.5\end{pmatrix}$

But $\overrightarrow{LM}=-2\begin{pmatrix}5\\5.5\end{pmatrix} \quad \Rightarrow \overrightarrow{LM}=-2\begin{pmatrix}5\\5.5\end{pmatrix}$

$\overrightarrow{LM}=-2\overrightarrow{MN} \Rightarrow \overrightarrow{\text{LM}} \parallel \overrightarrow{\text{MN}}$ and M is common.

$\therefore$ the points L, M and N are collinear.

25. **[2016 PII #8b]**

$\overrightarrow{YX}=\overrightarrow{YO}+\overrightarrow{OX}$

$=-\underline{y}+\underline{x}=\underline{x}-\underline{y}$

$\overrightarrow{YV}=\frac{1}{3}\overrightarrow{YX}$

$\overrightarrow{YV}=\frac{1}{3}(\underline{x}-\underline{y})$

$\therefore \overrightarrow{OV}=\overrightarrow{OY}+\overrightarrow{YV} \quad \Rightarrow \overrightarrow{OV}=\underline{y}+\frac{1}{3}(x-y)$

$\Rightarrow \overrightarrow{OV}=\underline{y}+\frac{1}{3}\underline{x}-\frac{1}{3}\underline{y} \Rightarrow \overrightarrow{OV}=\frac{1}{3}\underline{x}+\frac{2}{3}\underline{y}$

$\overrightarrow{XZ}=\overrightarrow{XY}+\overrightarrow{YZ}$

$\overrightarrow{XZ}=(\underline{y}-\underline{x})+3\underline{y} \quad \left(\text{Since } \overrightarrow{XY}=-\overrightarrow{YX}\right)$

$\overrightarrow{XZ}=\underline{y}+3\underline{y}-\underline{x} \quad \Rightarrow \overrightarrow{XZ}=4\underline{y}-\underline{x}$

$\overrightarrow{XW}=\frac{1}{3}\overrightarrow{XZ}$

$=\frac{1}{3}(4\underline{y}-\underline{x})$

$\overrightarrow{VX}=\overrightarrow{VO}+\overrightarrow{XO}$

$=-\overrightarrow{OV}+\overrightarrow{OX}$

$=-\frac{1}{3}\underline{x}-\frac{2}{3}\underline{y}+\underline{x}$

$=\frac{2}{3}\underline{x}-\frac{2}{3}\underline{y}$

$\overrightarrow{VW}=\overrightarrow{VX}+\overrightarrow{XW}$

$=\frac{2}{3}(\underline{x}-\underline{y})+\frac{1}{3}(4\underline{y}-\underline{x})$ (expand)

$\Rightarrow \overrightarrow{VW}=\frac{2}{3}\underline{x}-\frac{1}{3}\underline{x}-\frac{2}{3}\underline{y}+\frac{4}{3}\underline{y}$ (arrange like terms)

$\Rightarrow \overrightarrow{VW}=\frac{1}{3}\underline{x}+\frac{2}{3}\underline{y}$

thus $\overrightarrow{\text{OV}}=\lambda\overrightarrow{VW}$, where $\lambda=1$.

$\overrightarrow{OV}=\overrightarrow{VW}$, thus parallel and they have a common point V, therefore they are collinear.

26. **[2017 PI #6]**

$\left|\begin{pmatrix}-4\\w\end{pmatrix}\right|=5$

$\sqrt{(-4)^2+w^2}=5$

$\left(\sqrt{16+w^2}\right)^2=(5)^2$

$16+w^2=25 \quad \Rightarrow w^2=9$

$w^2=25-16 \quad \Rightarrow \sqrt{w^2}=\pm\sqrt{9}$

$\therefore w=-3$ or 3

27. **[2017 PII #3b]**

$N=\begin{pmatrix}1\\3\end{pmatrix}$; $\overrightarrow{MN}=\begin{pmatrix}-2\\4\end{pmatrix}$

Let $M=\begin{pmatrix}x\\y\end{pmatrix}$

$\therefore \overrightarrow{MN}=N-M$

$\begin{pmatrix}-2\\4\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}-M$

$M=\begin{pmatrix}1\\3\end{pmatrix}-\begin{pmatrix}-2\\4\end{pmatrix}$

$M=\begin{pmatrix}1--2\\3-4\end{pmatrix}$

$M=\begin{pmatrix}3\\-1\end{pmatrix}$

$\therefore$ The position vector $M=\begin{pmatrix}3\\-1\end{pmatrix}$

28. **[2017 PII #5a]**

Using law of vector addition

$\overrightarrow{SR} = \overrightarrow{SP} + \overrightarrow{PQ} + \overrightarrow{QR}$

$= (P - S) + (Q - P) + (R - Q)$

$= \begin{pmatrix} -4 \\ 3 \end{pmatrix} - \begin{pmatrix} -4 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 15 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ d \end{pmatrix} - \begin{pmatrix} 0 \\ 15 \end{pmatrix}$

$= \begin{pmatrix} -4 - (-4) + 0 - (-4) + 3 - 0 \\ 3 - (-2) + 15 - 3 + 15 - d \end{pmatrix}$

$\Rightarrow \overrightarrow{SR} = \begin{pmatrix} 7 \\ 32 - d \end{pmatrix}$

but $\overrightarrow{SR} = R - S \Rightarrow \overrightarrow{SR} = \begin{pmatrix} 3 \\ d \end{pmatrix} - \begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ d + 2 \end{pmatrix}$

$\therefore \begin{pmatrix} 7 \\ 32 - d \end{pmatrix} = \begin{pmatrix} 7 \\ d + 2 \end{pmatrix}$

$\Rightarrow 32 - d = d + 2$ (equating the x − values)

$\Rightarrow 32 - 2 = d + d$

$\Rightarrow \frac{30}{2} = \frac{2d}{2}$

$\Rightarrow 15 = d$

$\therefore d = 15$

---

29. **[2018 P1 #8]**

$\mathbf{Q} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$ *and* $\mathbf{P} = \begin{pmatrix} 2 \\ b \end{pmatrix}$

$\overrightarrow{\mathbf{PQ}} = \mathbf{Q} - \mathbf{P}$

$\overrightarrow{\mathbf{PQ}} = \begin{pmatrix} 8 \\ 13 \end{pmatrix} - \begin{pmatrix} 2 \\ b \end{pmatrix} \Rightarrow \overrightarrow{\mathbf{PQ}} = \begin{pmatrix} 8 - 2 \\ 13 - b \end{pmatrix}$

$\overrightarrow{\mathbf{PQ}} = \begin{pmatrix} 6 \\ 13 - b \end{pmatrix}$

$|\overrightarrow{\mathbf{PQ}}| = \sqrt{6^2 + (13 - b)^2}$

$|\overrightarrow{\mathbf{PQ}}| = \sqrt{36 + (13 - \mathrm{b})^2}$

Given $|\overrightarrow{\mathbf{PQ}}| = 10$,

$\therefore \sqrt{36 + (13 - b)^2} = 10$

$\left(\sqrt{36 + (13 - b)^2}\right)^2 = (10)^2$ (squaring both sides)

$(13 - \mathrm{b})^2 = 100 - 36$

$(13 - \mathrm{b})^2 = 64$

$13 - b = \pm\sqrt{64}$

$13 - b = \pm 8$

$13 \pm 8 = b$

$\Rightarrow$ either $b = 13 + 8$ or $b = 13 - 8$

$b = 21$ or $b = 5$

---

30. **[2018 PII #8]**

$OA = BT$ (*opp.* sides of a || gram)

$\therefore \overrightarrow{OA} = \overrightarrow{BT}$

$\Leftrightarrow \overrightarrow{BT} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

Using the triangle law of adding vectors

$\overrightarrow{OT} = \overrightarrow{OB} + \overrightarrow{BT}$

$\overrightarrow{OT} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$\overrightarrow{OT} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

OR

Using the parallelogram rule;

Provided OT is the diagonal of the parallelogram,

$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OT}$

$\overrightarrow{OT} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

$\overrightarrow{OT} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

---

31. **[2019 PII #1b]**

$\underline{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$2\underline{a} + 3\underline{b} = 2\begin{pmatrix} 3 \\ -2 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 2 \times 3 \\ 2 \times -2 \end{pmatrix} + \begin{pmatrix} 3 \times 2 \\ 3 \times 3 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ -4 \end{pmatrix} + \begin{pmatrix} 6 \\ 9 \end{pmatrix}$

$= \begin{pmatrix} 12 \\ 5 \end{pmatrix}$

CH 29 LINEAR PROGRAMMING

Chapter Highlights

In this topic, we will find solutions to real life problems using linear programming by creating and graphing linear inequalities in two variable, x and y usually, given detailed information in word problems.

In **Table 1** of commonly used terms below, assume unit cost of item $x = p$,
and unit cost of item $y = q$
Then, expenditure/spending on $x = px$,
and expenditure/spending on $y = qy$

Commonly used terms	Meaning
x is more than 100	$x > 100$
x is less than 20	$x < 20$
Spend more on x than y by K10	$px - qy > 10$
x is greater than or equal to half y	$x \geq \frac{1}{2}y$
y is less than twice x	$y < 2x$
y must be at least 5	$y \geq 5$
y must be at most 8	$y \leq 8$
Acquire up to 200 units of x and y	$x + y \leq 200$
Spend at least K25 more on x than y	$px - qy \geq 25$
x must not be more than 10	$x \leq 10$
x must not be less than 12	$x \geq 12$
Spend more on x than y by not more than 30	$px - qy \leq 30$
Spend less than half the money on x to buy y	$qy < \frac{1}{2}px$

The main goal of linear programming is to use given information to formulate an **objective function** and create inequalities. The objective function is a linear equation in two variables (x and y) whose value we want to maximize or minimize.

Before attempting these problems, you should be conversant with creating and graphing several inequalities on the same graph paper. This creates what is known as a **feasible region**.

The feasible region (the unshaded area) is usually a polygon determined by plotting all the inequalities. All the points in the feasible region are called **feasible solutions**. Within the feasible solution, one solution gives our desired maximum or minimum value to the objective function. Fortunately, this solution lies on one of the corner points.

1. A transport company has **three** 30-passenger buses and **nine** 15-passenger buses. The company contracts to transport more than 120 passengers a day to national parks. It costs K3000 per day to run each 30-passenger bus and K1000 per day to run each 15-passenger bus and the company must spend less than K12000 per day in order to meet costs. If x and y are respectively the numbers of 30-passenger and 15-passenger buses used each day,
   i. Show that $2x + y > 8$
   ii. Write down three other inequalities involving x and y.
   iii. Illustrate the solution set of the four inequalities on a graph paper and shade the unwanted regions.

[2003 PII #7b]

2. Suppose you have K1500 to spend in a shop. You have decided to;
   i. Divide the money between buying bottles of cooking oil at K150 each and tablets of soap at K60 each.
   ii. Buy at least five tablets of soap.
   iii. Spend more on cooking oil than soap by more than K300.

   a. If x and y represent the number of bottles of cooking oil and tablets of soap respectively, write down **three** inequalities which x and y must satisfy the above three conditions.
   b. Using a scale of 2cm to represent 2 units on the x-axis and 2cm to represent 5 units on the y-axis, draw a region bounded by the three inequalities obtained in (a) above by shading the unwanted area.
   c. For which values of x and y is $x + y$ the greatest? **[2004 PII #10b]**

3. A city assembly decides to construct a $1400m^2$ car park for lorries and minibuses. A minibus will fit on a $10m^2$ space while a lorry requires $15m^2$ of space.

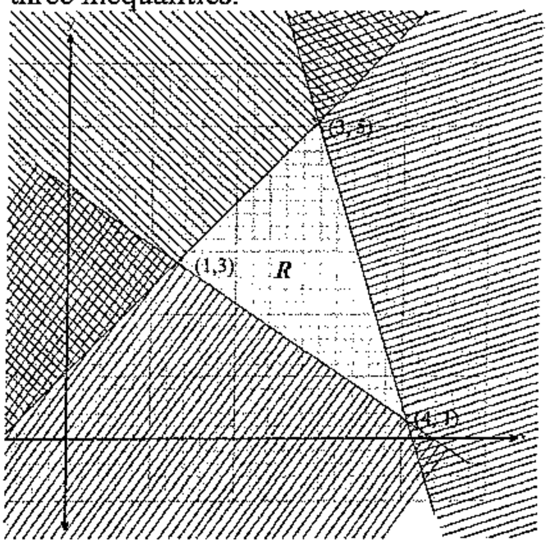
The number of Lorries has to be greater than or equal to half the number of minibuses. The number of lorries has to be less than twice the number of minibuses.

i. If x represents the number of minibuses and y represent the number of lorries, write down one inequality involving x and y in addition to $y \geq \frac{x}{2}$ and $y < 2x$.

ii. Using a scale of 2cm to represent 20 units on both axes, draw the region **R** bounded by the three inequalities.

iii. Use your graph to find the maximum number of vehicle (minibuses and lorries) that can be parked on the car park of such size. **[2005 PII #9b]**

4. **Figure 1** shows the region R bounded by three inequalities.



Calculate the maximum value of $5x - 4y + 8$ in this region. [2007 PI #17]

5. A farmer gets a loan of K50 000 to buy sheep and goats only. Sheep cost K5 000 each while goats cost K2 000 each. She would like to spend at least K10 000 more on the sheep than on the goats.

i. If x represents the number of sheep and y represents the number of goats, write down two inequalities in x and y in addition to $x \geq 0$ and $y \geq 0$.

ii. Using a scale of 2cm to represent 2 units on the horizontal axis and 2cm to represent 5 units on the vertical axis, draw the region which represents the four inequalities.

iii. How many sheep and goats can the farmer buy to have maximum number of animals with the loan.

[2006 PII #9b]

6. A business man wants to buy x meters of low quality cloth at K200 per meter and y meters of high quality cloth at K400 per meter. He decided to; Buy at most a total of 1000 meters of cloth; Spend at least a total sum K80 000.

i. Write down two inequalities in x and y in addition to $x \geq 0$ and $y \geq 0$.

ii. Using the scale of 2cm to represent 200 units on both axes, draw on a graph paper the region which represents the four inequalities.

iii. If the businessman will make a profit of K10 per meter on the cheaper cloth and K20 per meter on the expensive cloth. How many meters of each cloth must he but to get the maximum profit? **[2007 PII #12]**

7. Mrs. Mlimi has a piece of land on which she would like to plant maize and groundnuts. The cost of planting maize K300 per hectare while ground nuts cost K900 per hectare. Maize require 3 laborers per hectare while ground nuts requires 6 labourers per hectare. She hired 60 laborers and she has only K18000 for planting costs.

i. Write down two inequalities in addition to $x \geq 0$ and $y \geq 0$ by taking x to represent number of hectares of ground nuts and y to represent number of hectares of maize.

ii. Taking 2 cm to represent 5 units on the horizontal axis and 2cm to represent 10 units on the vertical axis, draw graphs to show the region represented by the inequalities in x and y. show by shading the unwanted area, the region bounded by the four inequalities in (i) above.

iii. Using the graph, find the maximum number of maize hectares if she plants 10 hectares of ground nuts. **[2008 P2 #10b]**

8. Mala has K6000 to buy skirts and blouses; skirts cost K600 each while a blouse is K300 each. She wants to spend at least K1200 more on skirts than on blouses. She would like to buy at least 4 skirts and 2 blouses.

i. i. If x represents the number of skirts and y represents the number of blouses. Write down four inequalities in x and y.

ii. ii. Using a scale of 2cm to represent on 2 units on both axes, draw the graphs to show the region which represents the four inequalities.

iii. iii. Using your graph find the maximum number of skirts and blouses she can buy.

[2010 PII #11b]

9. A vendor decides to spend up to K600 to buy packets of sweets and biscuits. She does not want to buy more than 10 packets altogether but wants to buy at least 2 packets of each. A packet of sweets costs K40, and that of biscuits costs K80. The profit of each packets of sweet is K10 and on each packet of biscuits is K15.

i. By taking x to represent the number of packets of sweets and y to represent the number of packets of biscuits, write down **four** inequalities that satisfy the above information.

ii. Using a scale of 2cm to represent 2 units on both axes, draw graphs to show the region represented by the inequalities.

iii. Use you graph to find the number of packets of sweets and biscuits respectively for the vendor to get the maximum profit. **[2011 PII #9b]**

10. A farmer had K5000 to buy axes and hoes only. An axe costs K500 while a hoe costs K200. She decided to spend the money as follows:

- Buy at least 3 axes and at least 5 hoes
- Spend more on axes than hoes by not more than K1000.

i. Taking x to represent number of axes and y to represent number of hoes to be bought, write down four inequalities in x and y that satisfy the above information

ii. Using a scale of 2cm to represent 2 units on the horizontal axis, and 2cm to represent 5 units on the vertical axis, draw a graph to show the region represented by the inequalities, shading the unwanted region.

iii. Use your graph to find the maximum number of hoes the farmer would buy if she bought 4 axes. **[2012 P2 #10b]**

11. A lady has k4 500 to buy two types of fish: micheni and chambo. Micheni costs k100 each and chambo cost k150 each. She plans to buy at least three of each type and not more than thirty micheni.

i. Taking x to represent number of micheni and y to represent number of chambo, write down four inequalities that satisfy the given information.

ii. Using a scale of 2cm to represent 10 units on both axis, draw a graph to show the region represented by the inequalities by shading the unwanted region.

iii. Use your graph to find the maximum number of fish she could buy.

[2013 P2 #9b]

12. A person wants to prepare 6 litres of local juice using two types of fruits: oranges and lemons. An orange produces 0.3 litres while lemon produces 0.2 litres. The person plans to use not less than 4 oranges to prepare the juice

i. Taking x to represent number of oranges and y to represent number of lemons, write down three inequalities that satisfy the above information.

ii. Using scale of 2cm to represent 5 units on both axes, draw graphs to show the region represented by the inequalities, shading the unwanted region.

iii. Use your graph to find the maximum number of lemons that can be used to make the juice.

[2014 PII #11b]

13.Tatsogola boarding secondary school would like to buy bed for its hostels. It has K960,000 for both steel and wooden beds. A steel bed cost K2,400 and a wooded bed costs K16,000. It would like to buy not more than 50 beds but at least 10 steel beds and 20 wooden beds.

i. i. Taking x to represent the number of steel beds and y to represent the number of wooden beds, write down three inequalities in x and y in addition to $x \geq 10$.

ii. ii. Using a scale of 2 cm to represent 10 units on both axes. Draw graphs to show the region represented by the **four** inequalities by shading the unwanted regions.

iii. iii. Using the graphs, find the maximum number of steel beds the school would buy. **[2017 PII #8b]**

14. A school plans to build two types of houses: one type with three bedrooms and the other with four bedrooms. The school is ready to spend at most K216 Million on at most 25 houses. Three bedroom houses cost at K8 Million each and four bedroom houses cost K12 Million each. It has decided to build at least 10 three bedroom houses.

i. Taking x to represent the number of three bedroom houses and y to represent the number of four bedroom houses, write down three inequalities that satisfy the information.

ii. Using a scale of 2 cm to represent 5 units on both axes, draw graphs to show the region represented by the **three** inequalities by shading the unwanted region.

iii. Using the graphs, find the maximum number of three bedroom houses the school can build. **[2015 PII #10b]**

15. Mrs Phiri employs a person to clean a house and to wash clothes. The person works almost 40 hours in week. He spends at least 16 hours cleaning the house and at least 6 hours washing clothes in a week. He also spends less than half the time in cleaning the house to wash clothes.

i. Taking x to represent the number of hours for cleaning the house and y to represent the number of hours for washing clothes, write down four inequalities, shade the unwanted region.

ii. Using a scale of 2 cm to represent 5 units on both axes, draw graphs to show the region bounded by the inequalities, shade the unwanted region

iii. Use the graph to find the maximum number of hours the person can spend in cleaning the house. **[2016 PII #9b]**

16. A company makes canoes and nets. A canoe take 2 hours of machine time and 6 hours of craftsman time. A net takes 4 hours of machine and 2 hours of craftsman time. The company has a maximum of 56 hours of machine time and 48 hours of craftsman time.

a. Taking x to represent the number of canoes and y to represent the number of nets, write down two inequalities in x and y in addition to $x \geq 0$ and $y \geq 0$ that satisfy the above information.

b. Using a scale of 2cm to represent 5 units on both axes, draw graphs to show the region represented by the four inequalities by shading the unwanted region.

c. Using the graphs, find the number of canoes and the company can work if it works in full capacity. **[2018 PII #20]**

17. A farmer wants to buy mango and guava seedlings. A mango seedling costs K600 and a guava seedling costs K300. He has K90 000 to spend for the seedlings. He wants to buy up to 200 seedlings and at least 60 of each type of seedlings.

a. If x represents the number of mango seedling and y represents the number of guava seedling, write down the three inequalities that satisfy the given information in addition to $x + y \leq 200$.

b. Using a scale of 2 cm to 50 units, on same axes, draw graphs on to show the region bounded by the **four** inequalities, shade the unwanted region.

c. Using the graphs, find the number of each type of seedlings that the farmer should buy to have the maximum number of seedlings at the lowest cost. **[2019 PII #9]**

1. **[2003 PII #7b]**

x represents the number of 30 passenger buses used.
y represents the number of 15 passenger buses.

i. Number of people that can be transported by 30-passenger buses = $30x$
Number of people that can be transported by 15-passenger buses = $15y$
Since the total number of passengers to be transported all by buses is to be more than 120, then:

$$30x + 15y > 120$$
$$2x + y > 8 \qquad \textit{(divide each term by 15)}$$

ii. Only three 30-passenger buses are available
$\therefore x \leq 3$
Only nine 30-passenger buses are available
$\therefore y \leq 9$
Total amount of money needed to run 30-passenger buses used each day = K3000x
Total amount of money needed to run 15-passenger buses used each day = K1000y
Since the company must spend less that K12000 to run all the buses then
$3000x + 1000y < 12000$
$3x + y < 12$ (divide both sides by 1000)
$\therefore$the three inequalities are:
$x \leq 3$
$y \leq 9$
$3x + y < 12$

iii. For $2x + y > 8$ as an equation is $y = 8 - 2x$.

Table of values:

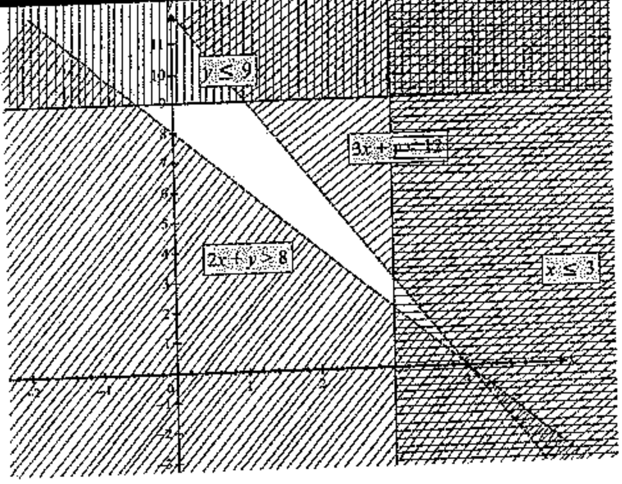
x	0	4
y	8	0

We draw a dotted line (since ">") and shade below the line.
For $x \leq 3$, we draw solid line $x = 3$ and shade the right side.
For $y \leq 9$, we draw solid line $y = 9$ and shade the upper side.
For $3x + y < 12$, we draw line $y = 12 - 3x$.

Table of values:

x	0	4
y	12	0

We draw a dotted line (since "<") and shade above the line.



2. **[2004 PII #10b]**

i. x = bottles of cooking oil
y = tablets of soap
Inequality on money available.

$150x + 60y \leq 1500$

Expenditure on cooking oil: since each bottle K150.
Expenditure on tablets of soap: since each soap is K60.
He can spend all the money or less.
He has K150

- $5x + 2y \leq 50$ (divide each term by 30)

Inequality on the expenditure on cooking oil should be K300 more than the expenditure spent on soap:

$150x - 60y \geq 300$

Money spent on cooing oil minus money spent on sop
you want to spend on cooking oil by "more than" K300

- $5x - 2y > 10$ (divide each term by 30)

Inequality on buying at least 5 tablets of soap:
- $y \geq 5$

$\therefore$the three inequalities are;
$5x + 2y \leq 50$
$5x - 2y > 10$
$y \geq 5$

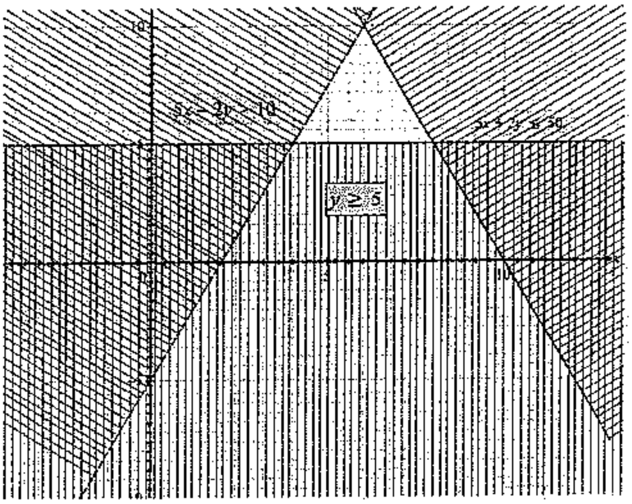
ii. $5x + 2y \leq 50$

x	-2	0
y	30	25

$5x - 2y > 10$

x	-2	0
y	-10	-5

$y = 5$ is the horizontal line that crosses the y-axis at 5



iii. Objective function $= x + y$
we need need to maximize $x + y$
From the graph, the feasible region has corners at the following coordinates:
(4, 5); (6, 10); (8, 5)
Substitute the coordinates in the objective function:
(4, 5): $4 + 5 = 9$
(6, 10): $6 + 10 = 16$
(8, 5): $8 + 5 = 13$
16 is the greatest.
$\therefore$When x=6 and y=10, the value of x+y is greatest.

3. **[2005 PII #9b]**
x represents the number of minibuses
y represents the number of lories

i. Since 1 lorry requires 15m^2 of space, then y lorries require $15y\,\text{m}^2$.

Since 1 minibus requires 10m^2 of space then x minibuses require $10x\text{m}^2$

With only 1400m^2 space available, then
$10x + 15y \le 1400$
$\Rightarrow 2x + 3y \le 280$ *(divide each term by 5)*
$\therefore$ the inequality is $2x + 3y \le 280$ ....(i)

The other given inequalities are $y \ge x/2$ and

$y < 2x$

ii. Table of values:
$2x + 3y = 280$ (when x=*0* and when y=*0*)

x	0	140
y	93	0

Draw a solid line and shade above the line.
$y = 2x$ (because of (0,0) we add any other point)

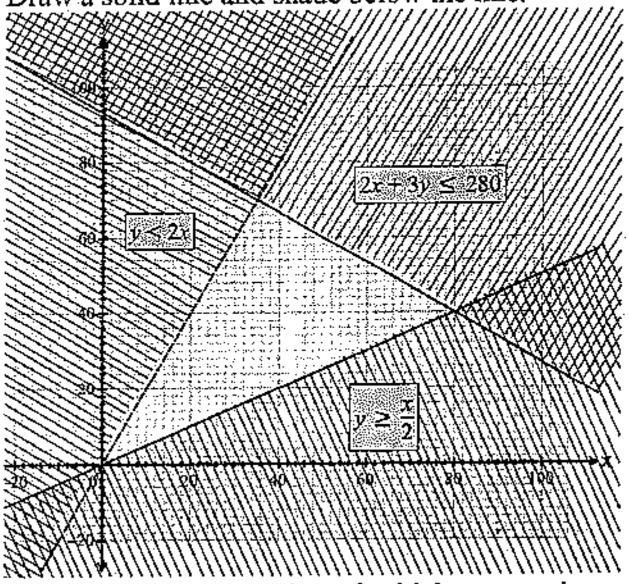
x	0	2
y	0	4

Draw a dotted line and shade above the line.

$y = \frac{x}{2}$ (because of $(0,0)$ we add any other point)

x	0	2
y	0	1

Draw a solid line and shade below the line.



iii. The maximum number of vehicles, $x + y$, is found at the corner point of the feasible region;
(0, 0) $= 0 + 0 = 0$
(80, 40) $= 80 + 40 = 120$
(35, 70) $= 35 + 70 = 105$
$\therefore$ The car park can accommodate a maximum number of 120 vehicles - 80 minibuses and 40 lorries.

4. **[2007 PI #17]**

The maximum value of $5x - 4y + 8$ is one of the corner points.

Corner Point (x, y)	Value when we substitute x and y in $5x - 5y + 8$
$(3,5)$	$5(3) - 4(5) + 8 = 15 - 20 + 8 = 3$
$(1,3)$	$5(1) - 4(3) + 8 = 5 - 15 + 8 = 1$
$(4,1)$	$5(4) - 4(1) + 8 = 20 - 4 + 8 = 24$

$\therefore$The maximum value is 24.

5. **[2006 PII #9b]**
x represents the number of sheep and y represents the number of goats.
i. Each sheep cost K5,000 each and each goat cost K2,000
$\therefore$ Total amount of money to be spent on buying sheep $= 5000x$

Total amount of money to be spent on buying goats $= 2000y$

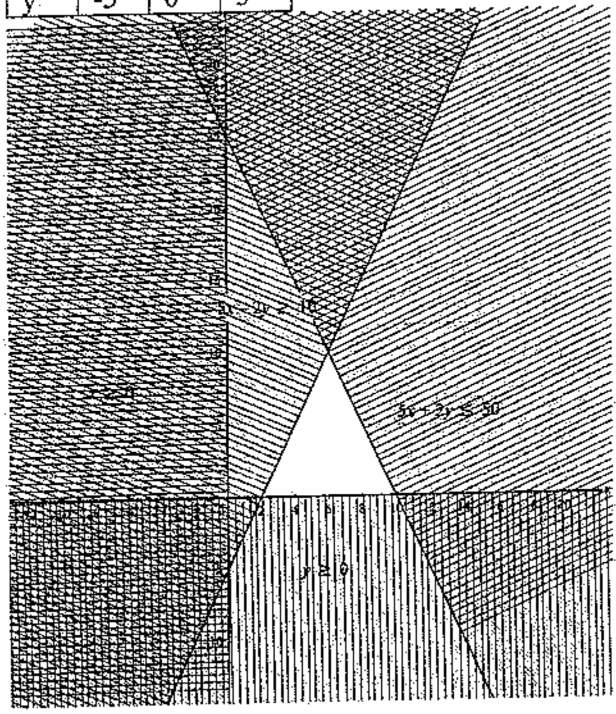
Since the farmer has an K50000 to be spent then,
$5000x + 2000y \leq 50000$
$5x + 2y \leq 50$ (divide each term by 1000)
The farmer intends to spend at least K10000 more on the sheep than on the goats then,
$5000x - 2000y \geq 10000$
$5x - 2y \geq 10$ (divide each term by 1000)
$\therefore$the two inequalities are as follows:
1. $5x + 2y \leq 50$
2. $5x - 2y \geq 10$

ii. $x = 0$ is the y-axis
$y = 0$ is the x-axis
$5x + 2y = 50$

x	0	6	10
y	25	10	0

$5x - 2y = 10$

x	0	2	4
y	-5	0	5



iii. Objective function = $x + y$
We need to maximize the objective function subject to:
Coordinates of the corner points are:
(2, 0), (10, 0) and (6, 10).
Substitute the coordinates in the objective function

Corner Point (x, y)	Profit made: $x + y$
(2, 0)	$2 + 0 = 2$
(10, 0)	$10 + 0 = 10$
(6, 10)	$6 + 10 = 16$

Since 16 is the maximum - the farmer should buy 6 sheep and 10 goats.

6. **[2007 PII #12]**

Let x represent metres of low quality cloth

Let y represent metres of high quality cloth

i. Since the businessman wants to buy at most 1000 meters of cloth, then $x + y \leq 1000$.

The total money to be spent on low quality cloth is $200x$ while on high quality cloth is $400y$. Since the businessman intends to spend at least a total sum of 80000, then:
$200x + 400y \geq 80000$ (divide each term by 200)

$x + 2y \geq 400$

$\therefore$ the two other inequalities are:

(i) $x + y \leq 1000$

(ii) $x + 2y \geq 400$

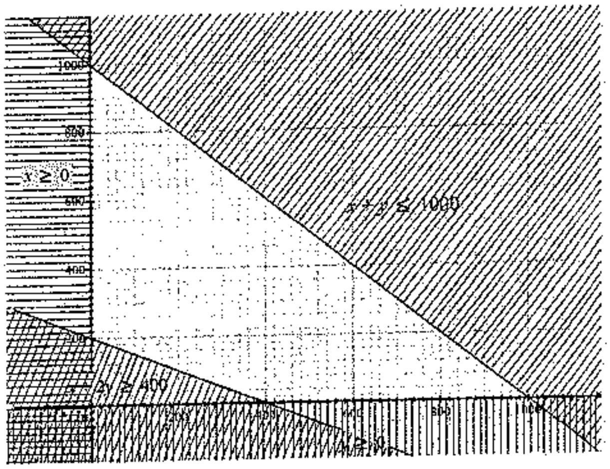
in addition to $x \geq 0; y \geq 0$

ii. The graph
$x + y = 1000$ or $y = 1000 - x$

x	0	100	1000
y	1000	900	0

$x + 2y = 400$ or $y = 200 - \frac{1}{2}x$

x	0	200	400
y	200	100	0



iii. The business will make a profit of K10/metre on cheaper cloth and K20/metre on expensive cloth
$\therefore$Total profit to be realized from cheaper cloth is K$10x$ while total profit to be realized from expensive cloth is K$20y$.
$\Rightarrow$ The objective function $= 10x + 20y$
The values for maximum profit will be found on the corner points. Coordinates of the corner points are:
(0, 1000) ;(0,200) ;(400,0) ;(1000, 0).

Corner Point (x, y)	Profit made: $10x + 20y$
(0, 200)	10(0)+20(200)= K4000
(400, 0)	10(400)+20(0)= K4000
(1000, 0)	10(1000)+20(0)=K10,000
(0, 1000)	10(0)+20(1000)=K20,000

The maximum is K20,000.

$\therefore$ The bussinessman must buy 1000 metres of high quality cloth and none (0) of low quality cloth in order to maximize profits

7. **[2008 P2 #10b]**

x represents hectares uses to plant groundnuts

y represents hectares used to plant maize

i.

- Total costs of planting maize =K$300y$
- Total costs of planting groundnuts =K$900x$

since Mrs Mlimi has only K18000 for planting costs,
then, $900x + 300y \leq 18000$
$3x + y \leq 60$ (divide each term by 300)
- Total number of laboures required for maize $= 3y$
- Total number of labourers required for groundnuts $= 6x$

Since Mrs Mlimi hired 60 labourers, then,
$6x + 3y \geq 60$ (labourers can be more)
$2x + y \geq 20$ (divide each term by 3)
$\therefore$ The two inequalities are
(1) $3x + y \leq 60$
(2) $2x + y \geq 20$

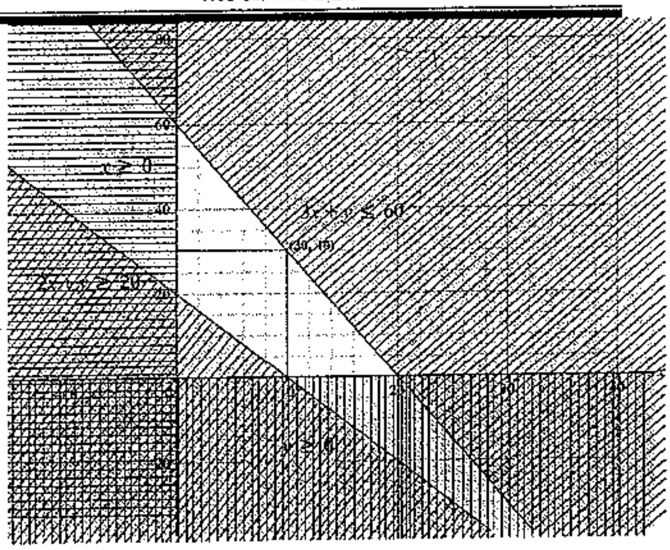
ii. Graph

$x = 0$ is the x-intercept

$y = 0$ is the y-intercept

$3x + y = 60$ crosses the x-axis at (20,0) and the y-axis at (60,0)

$2x + y = 20$ crosses the x-axis at (10,0) and the y- axis at (20,0)



iii.

From the graph, when groundnuts, x is 10, the largest y-coordinate is 30. $\therefore$ The maximum number of maize hectares is 30

8. **[2010 PII #11b]**

i.

The four inequalities

x represents number of skirts

y represents number of blouses

She would like to buy at least 4 skirts and 2 blouses

$\therefore x \geq 4$..............(i)

And $y \geq 2$...........(ii)

Total costs of skirts $= 600x$

Total cost of blouses $= 300y$

Mala has K6000 to spend

$\Leftrightarrow 600x + 300y \leq 6000$

$$\frac{600x}{300} + \frac{300y}{300} \leq \frac{6000}{300}$$

$\therefore 2x + y \leq 20$............$(iii)$

spend at least K1200 more on skirts than on blouses

$\Leftrightarrow 600x - 300y \geq 1200$

$\frac{600x}{300} - \frac{300y}{300} \geq \frac{1200}{300}$

$\therefore 2x - y \geq 4 \ldots\ldots\ldots\ldots (iv)$

The four inequalities are;

$x \geq 4 \ldots\ldots\ldots\ldots (i)$

$y \geq 2 \ldots\ldots\ldots (ii)$

$2x + y \leq 20 \ldots\ldots\ldots\ldots (iii)$

$2x - y \geq 4 \ldots\ldots\ldots\ldots (iv)$

(ii) To draw the graph of these inequalities;

The boundary lines are,

$x = 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

$y = 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

$2x + y = 20 \ldots\ldots\ldots\ldots\ldots\ldots (iii)$

$2x - y = 4 \ldots\ldots\ldots\ldots (iv)$

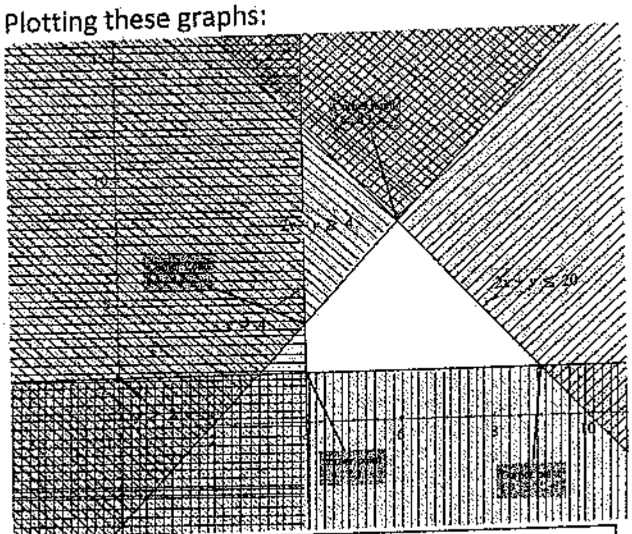
Table of values for $2x + y = 20$:

x	0	10
y	20	0

Table of values for $2x - y = 4$

x	0	2
y	-4	0

Plotting these graphs:



Corner Point	Total
(4, 4)	4+4=8
(4, 2)	4+2=6
(9, 2)	9+2=11
(6, 8)	6+8=14

The maximum number of skirts and blouses she can buy is 14.

9. **[2011 PII #9b]**

i. Let x be number of packets of sweets

Let y represent number of packets of biscuits

Item	**Amount**	**Cost**	**Profit**
sweets	x	40	10
biscuits	y	80	15
Total	10	K600	

Equations

$x \geq 2$ (i)

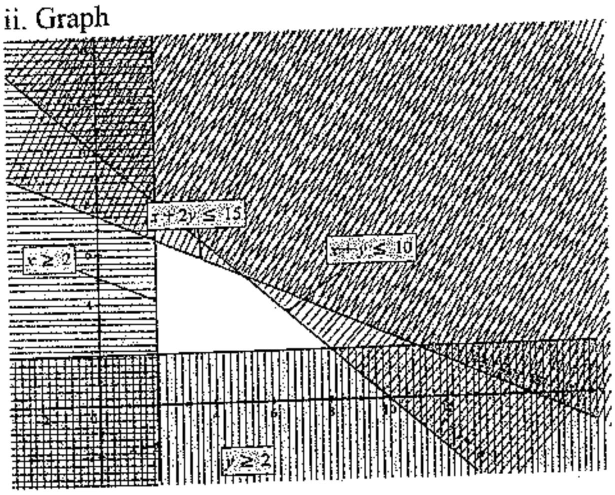
$y \geq 2$ (ii)

$x + y \leq 10$ (iii)

$40x + 80y \leq 600$ (iv)

$x + 2y \leq 15$ (dividing throughout by 40)

ii. Graph



iii. Calculating maximum profit:

Corner Point (x, y)	Profit made: $10x + 15y$
(2,2)	10(2) +15(2) =50
(2,6)	10(2) +15(6) =110
(5,5)	10(5) +15(5) =125
(8,2)	10(8) +15(2) =110

$\therefore$ The vendor should buy 5 packets of sweets and 5 packets of biscuits to make a maximum profit

10. **[2012 P2 #10b]**

Let x represent the number axes

Let y represent the number of hoes

i. Inequalities formed are

(1) At least 3 axes $\Rightarrow x \geq 3$

(2) At least 5 hoes $\Rightarrow y \geq 5$

(3) K5000 for axes & hoes at K500 and K200 respectively

$\Rightarrow 500x+200y \leq 5000$

$5x+2y \leq 50$

x	0	10
y	25	0

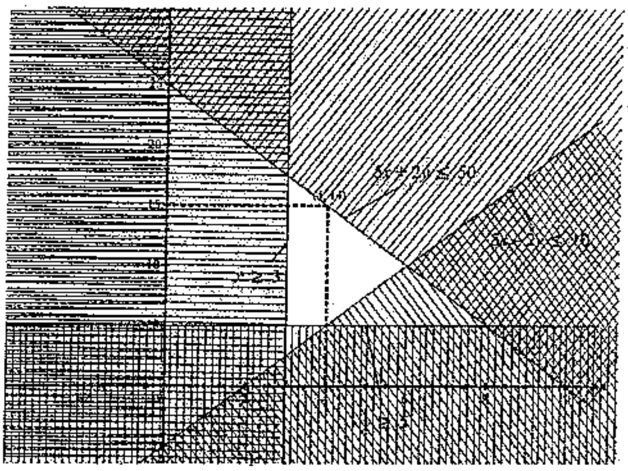
(4) Spend more on axes than hoes by not more than K1000

$\Rightarrow 500x-200y \leq 1000$

$5x-2y \leq 10$

x	0	2
y	-5	0

ii.



iii. From the graph, the maximum number of hoes the farmer would buy if she bought 4 axes is 15 hoes.

11. **[2013 P2 #9b]**

i.

Let x represent number of micheni

Let y represent number of chambo

She plans to buy at least 3 of each type of fish:

(1) $x \leq 3$

(ii) $y \leq 3$

She plans to buy not more than 30 micheni:

(iv) $x \leq 30$

Micheni costs K100 each $\Rightarrow$ total expenditure on micheni $=100x$

Chambo cost K150 each $\Rightarrow$ total expenditure on chambo $=150y$

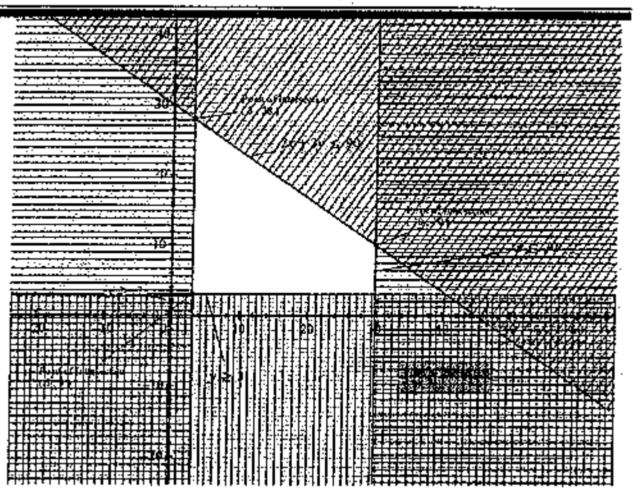
She has K4,500 $\Rightarrow 100x+150y \leq 4500$

Divide by 50:

(iii) $2x+3y \leq 90$

Table of values for $2x+3y=90$

x	0	45
y	30	0



iii.

The maximum number of fish she can by is found at one of the corner points:

Corner Point	Number of fish
(3, 3)	3+3=6
(3, 28)	3+28=31
(30, 10)	30+10=40
(30,3)	30+3=33

Therefore the maximum number of fish she can buy is 40.

12. **[2014 PII #11b]**

i.

The three inequalities are;

$0.3x+0.2y \leq 6$ we multiply through by 10

$3x+2y \leq 60$

$x \geq 4$

$y \geq 0$

ii.

For $3x+2y \leq 60$, equation is:

$3x+2y=60$......................(i)

Table of values:

x	0	20
y	30	0

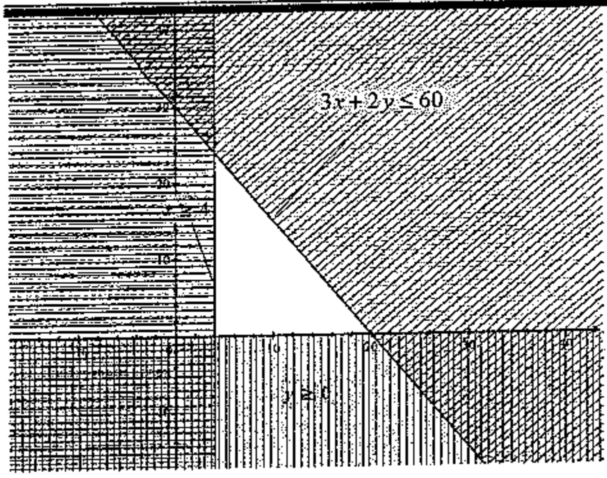
We plot a line passing through $(0,30)$ and $(20,0)$ and shade above the line.

For $x \geq 4$, the line is: $x=4$..................(ii)

We shade the left side of the line.

For $y \geq 0$, the equation is: $y=0$.............(iii)

We shade below the line.



iii.

The maximum amount of lemon that can be used are found at the highest y value in feasible region.

$\therefore$ the maximum number of lemons is 24.

13. **[2017 PII #8b]**

i.

The inequalities are

The school would like to buy at least 10 steel beds and 20 wooden beds:

$\therefore x \ge 10$............(i)

$y \ge 20$............(ii)

The school would like to buy not more than 50 beds

$\therefore x + y \le 50$.........(iii)

Total expenditure on steel beds = K24000x

Total expenditure on wooden beds = K16000y

$\Leftrightarrow$ $24000x + 16000y \le 960000$

$$\frac{24000x}{8000} + \frac{16000y}{8000} \le \frac{960000}{8000}$$

$\therefore 3x + 2y \le 120$................$(iv)$

ii.

To represent these inequalities on a graph,

Boundary lines are;

$x = 10$............(i)

draw solid line and shade the left side

$y = 20$.............(ii)

draw solid line and shade below line

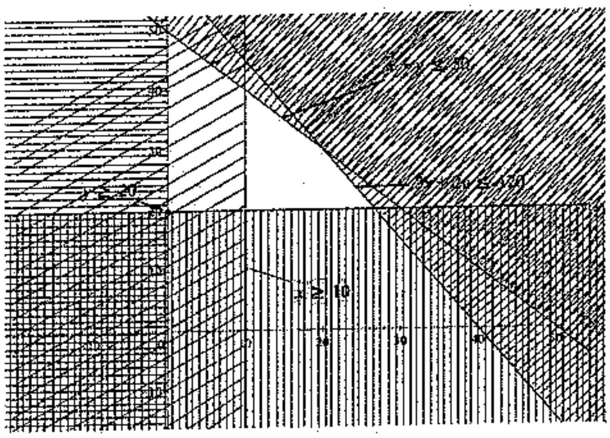
$3x + 2y = 120$..........(iii)

x	0	40
y	60	0

$x + y = 50$................(iv)

x	0	50
y	50	0

Below is the plotted graph:



iii.

The maximum number of steel beds is the maximum value of x on the feasible region:

The maximum value of x is at point $(27, 20)$

$\therefore$ The maximum number of steel beds is = 27 beds

14. **[2015 PII #10b]**

i.

Let x represent number of three bedroom house

Let y represent number of four bedroom house

The school decided to build atleast 10 three bedroom house

$\therefore x \ge 10$...............(i)

The school is ready to build at most 25 houses:

$x + y \le 25$.........(ii)

Total cost of three bedroom house = $8000000x$

Total cost of four bedroom house = $12000000y$

$\Leftrightarrow$ $8000000x + 12000000y \le 216000000$

$$\frac{8000000x}{4000000} + \frac{12000000y}{4000000} \le \frac{216000000}{4000000}$$

$\therefore 2x+3y \le 54 \ldots\ldots\ldots\ldots (iii)$

The inequalities are;

$x \ge 10 \ldots\ldots\ldots (i)$

$x + y \le 25 \ldots\ldots (ii)$

$2x + 3y \le 54 \ldots\ldots\ldots (iii)$

ii.

To plot the inequalities, turn them into equations and shade the relevant sides.

Turn the inequalities into equation to come up with boundary lines

$x = 10 \ldots\ldots (i)$

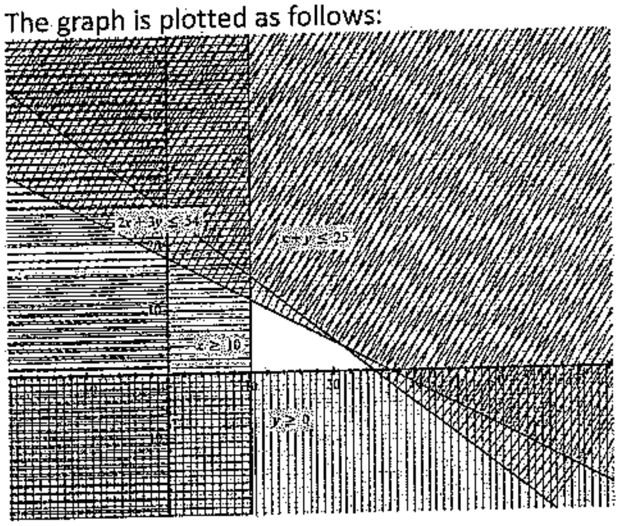
$x + y = 25 \ldots\ldots (ii)$

x	0	25
y	25	0

$2x + 3y = 54 \ldots\ldots (iii)$

x	0	27
y	18	0

The graph is plotted as follows:



The maximum number of three bedroomed house is found on the corners of the feasible region.

The corner points are: $(10, 0)$, $\left(10, \frac{34}{3}\right)$, $(21, 4)$ and $(25, 0)$.

The maximum value of x is 25.

$\therefore$ The maximum number of three bedroom houses that can be built is 25.

15. **[2016 PII #9b]**

(i) The four equations are

$x + y \le 40 \ldots\ldots\ldots\ldots (i)$

x	0	40
y	40	0

$x \ge 16 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

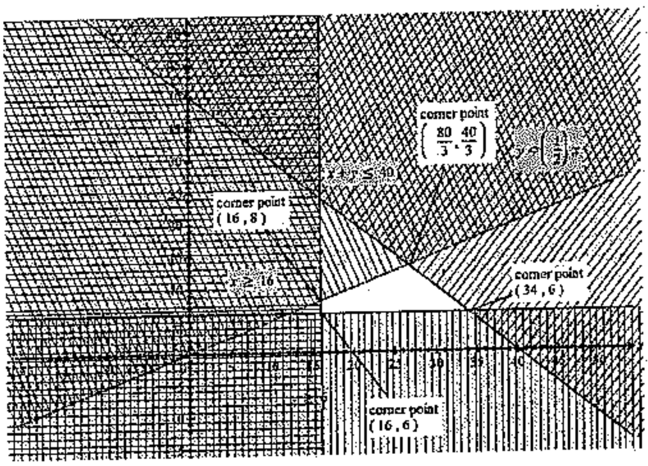
$y \ge 6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

Note that we use 20 here (not 2 or small numbers) because of the scale of the other equations (i) to (iii)

$y < \frac{1}{2}x \ldots\ldots\ldots\ldots\ldots\ldots (iv)$

x	0	20
y	0	10

ii. The plot is as follows:



iii.

The maximum number of hours can be found at the corner points: $(16, 6)$, $(16, 8)$, $\left(\frac{80}{3}, \frac{40}{3}\right)$ and $(34, 6)$. In the feasible region, the maximum value of x is 34. Therefore, maximum number of hours the person can spend in cleaning the house is 34 hours.

16. **[2018 PII #20]**

(i) The two inequalities

Total number of machine time for canoe $= 2x$

Total number of machine time for nets $= 4y$

$\Leftrightarrow 2x + 4y \le 56$

$\therefore x + 2y \le 28 \ldots\ldots\ldots\ldots\ldots\ldots (i)$

Total number of craftsman time for canoe $= 6x$

Total number of craftsman time for nets $= 2y$

$\Leftrightarrow 6x + 2y \le 48$

$\therefore 3x + y \le 24 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

The two additional inequalities are

$x + 2y \le 28 \ldots\ldots\ldots\ldots\ldots\ldots (i)$

$3x + y \le 24 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

(ii) To draw graphs change the inequalities into equation to have boundary lines

$x = 0$.....................(i)

$y = 0$.........................(ii)

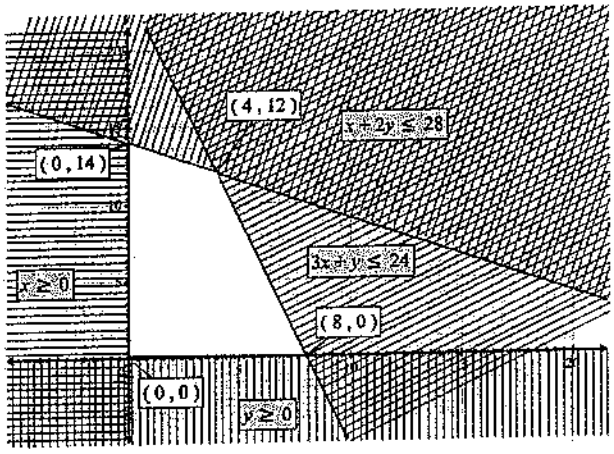
$x + 2y = 28$...............(iii) ⇨

x	0	28
y	14	0

$3x + y = 24$...................(iv) ⇨

x	0	8
y	24	0

Develop table of values and plot the graphs



iii. The maximum number of canoes and nets exist at one of the corner points:

Corner point	Number of canoes and nets
(0, 0)	0+0 = 0
(0, 14)	0+14 =14
(4, 12)	4+12 = 16
(8, 0)	8+0 = 8

Therefore, the maximum number of canoes and nets the company can make in full capacity is 16.

17. **[2019 PII #9]**

a. The inequalities are below

$x \geq 60$................(i)

$y \geq 60$.................(ii)

Total amount to spend on mango $= 600x$

Total amount to spend on guava $= 300y$

$\Leftrightarrow 600x + 300y \leq 90000$

$$\frac{\cancel{600}x^{2}}{\cancel{300}_1} + \frac{\cancel{300}^{1}y}{\cancel{300}_1} \leq \frac{\cancel{90000}^{300}}{\cancel{300}_1}$$

$\therefore 2x + y \leq 300$.......................$(iii)$

$x + y \leq 200$..........................(iv)

b. To plot a graph;

Change the inequalities into equations to have boundary lines

$x = 60$.............(i)

$y = 60$.................(ii)

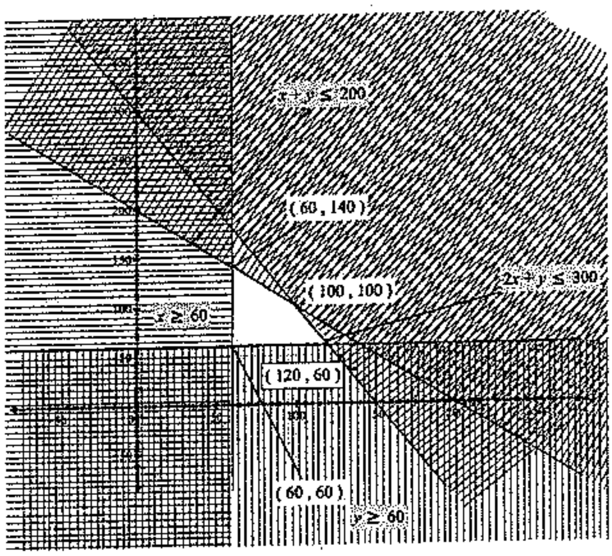
$2x + y = 300$.............(iii)

x	0	150
y	300	0

$x + y = 200$....................(iv) ⇨

x	0	200
y	200	0

Develop table of values and plot the graphs



c.

Maximum number of seedling exist at the corner point:

Corner Point	Cost=$600x$+$300y$
(60, 60)	600(60)+300(60) = K54,000
(60, 140)	600(60)+300(140) =K78,000
(100, 100)	600(100)+300(100) =K90,000
(120, 60)	600(120)+300(60) =K 90,000

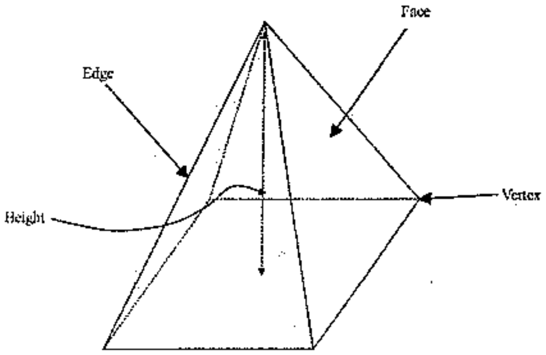
At lowest cost the farmer can use K54,000. Thus buying 60 mango seedlings and 60 guava seedlings.

CH 30
MENSURATION I
(SURFACE AREA AND VOLUME OF SOLIDS)

Chapter Highlights

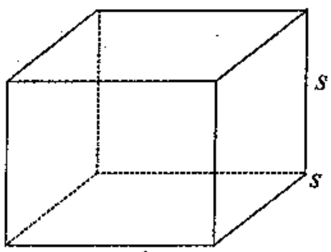
In this section, we will solve problems on mensuration. The main focus will be on 3-D shapes such as cubes, cuboids, cylinders, pyramids, cones, spheres and prisms.
To solve these problems you need to be familiar with formulas for surface area and volumes of 3-D shapes. The diagrams below show some important concepts in the discussion of 3D shapes.

PYRAMID



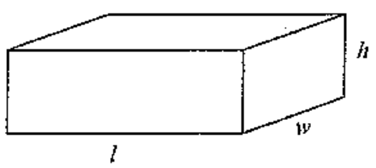
Face – a flat surface on a 3D shape.
Vertices – a corner where two or more faces meet.

CUBE



Volume = S^2
Surface Area = $6s^2$
Where S is the length of all sides

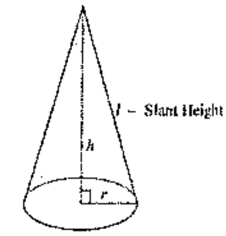
CUBOID



Total Surface Area = $2(lw + lh + wh)$

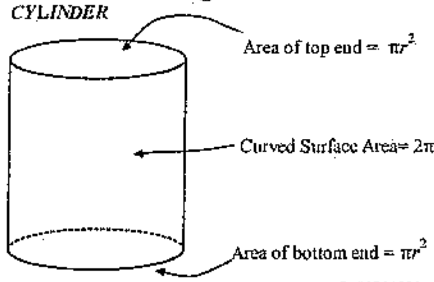
Volume = lwh

CONE



Volume = $\frac{1}{3}$ base area × height
Surface Area = $\pi rl + \pi r^2$ — Add this when the cone is closed on the bottom

CYLINDER



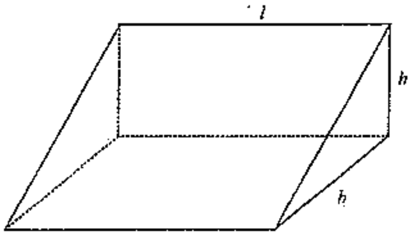
Note: a cyclinder could be closed on both sides, one side or neither of the sides. When finding total surface area of a cylinder, add area of the curved surface with the area of the bottom and top if it exists.

Total suface area of a cylinder closed on both ends = $\underbrace{2\pi h}_{\text{Area of curved surface}} + \underbrace{2(\pi r^2)}_{\text{Area of 2 closed ends}}$

Total suface area of a cylinder closed on one end = $2\pi h + \pi r^2$

Volume of a cylinder = $\pi r^2 h$

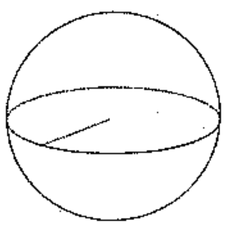
PRISM



Volume = Area of cross-section × Height
Volume = $\frac{1}{2}$bhl

Remark: A prism could have various shapes. These are all the figures with a unform thickness along a certain height e.g. cylinder, triangular prism, etc.

SPHERE

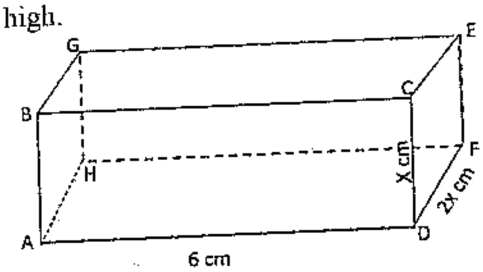


Volume = $\frac{4}{3}\pi r^3$
Surface Area = $4\pi r^2$

HEMISPHERE

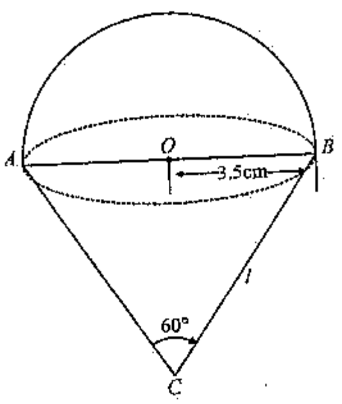
Volume = $\frac{2}{3}\pi r^3$
Surface Area = $3\pi r^2$

1. **Figure 1** shows a rectangular box with an open top. The box measures 6cm long, 2x cm wide and x cm high.



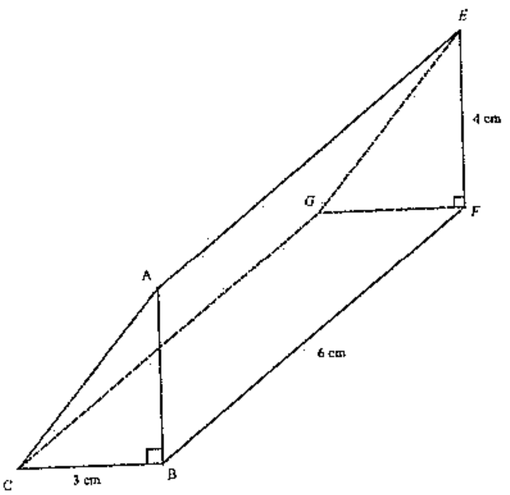
Given that the total outer surface area of the box is 108 cm². Form an equation in x and show that it simplifies to $x^2+6x-27=0$. **[2003 PP 1 #21]**

2. **Figure 2** represents a solid block made from a right cone and a hemisphere top. The radius of the hemisphere **OB**=3.5cm and angle **ACB**=60°



Calculate the surface area of the block. (Curved surface area of a cone= πrl, surface area of a sphere=$4\pi r^2$. Take $\pi = {}^{22}\!/_7$). **[2003 PII #10b]**

3. **Figure 3** shows a triangular prism, in which **BC**= 3cm **EF**= 6cm and angle **ABC**=90°.



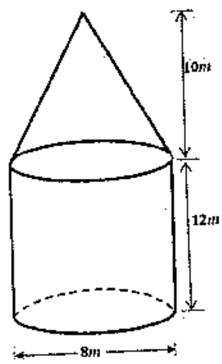
Calculate the volume of the prism. **[2004 PI #21]**

4. A metallic sphere of volume 770 cm³ is melted and made into a solid cylinder of length 5cm. Calculate the radius of the cylinder. (take $\pi = {}^{22}\!/_7$) **[2005 PII #5a]**

5. A cuboid is 76cm long, 50cm wide and 40cm high. Calculate the volume of the cuboid. **[2006 PI #3]**

6. Calculate the total surface area of solid hemisphere of radius 21 cm. (Area of a sphere $=4\pi r^2$; take $\pi = {}^{22}\!/_7$). **[2007 PI #18]**

7. **Figure 4** shows a cone placed on top of a cylinder. The height of the cone is 10m and that a cylinder is 12m. The diameter of both cone and cylinder is 8cm.



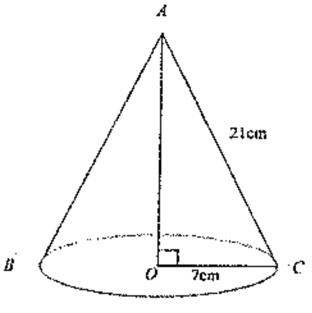
Calculate the total volume of the shape to two decimal places. (Volume of a cone $=\frac{1}{2}\pi r^2 h$, take π=3.14). **[2008 P1 #10a]**

8. A cylindrical metal bar whose volume is $594cm^3$ is melted and cast into a sphere. Assuming no loss of metal, calculate the radius of the sphere, leaving your answer correct to 2 decimal places. (Volume of a sphere $=\frac{4}{3}\pi r^3$, Take $\pi = 3.142$) **[2008 P1 #15]**

9. A right cone with radius 8cm is 15cm high. Calculate the total surface area of the cone. (take π=3.14). **[2010 PII #10a]**

10. A pond 12m in diameter, has a shape of hemisphere and is full of water. The pond is emptied, and all water poured into a cylindrical tank of radius 5m. Assuming there is no loss of water, calculate the height of water in the tank. (Volume of sphere= ${}^4\!/_3\pi r^3$). **[2011 PI #19]**

11. **Figure 5** shows a solid cone of base radius **OC** = 7cm and the slant height **AC**= 21cm.



Calculate the total surface area of the cone. (Curved surface Area of cone = $\pi r\ell$, take $\pi = \frac{22}{7}$) **[2012 P1 #20]**

12. A pyramid has a volume of $62.4\,\text{m}^3$. If its base is a square of side $6cm$, find the height. **[2013 P2 #5b]**

13. **Figure 6** is a diagram of a cylindrical flask of radius 5cm and height 25 cm, which has hemispherical cup.

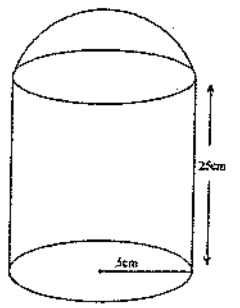
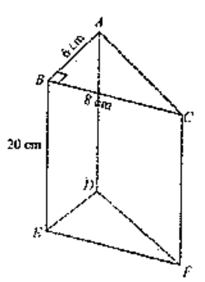


Figure 8

Calculate the total surface area of the flask. (Take $\pi = 3.14$ and surface area of the a sphere $= 4\pi r^2$) **[2014 P1 #19]**

14. The volume of a cone is $4950\,cm^3$. If the height of the cone is 21 cm, find its radius. $\left(Take \pi = \frac{22}{7} \text{ and volume} = \frac{1}{3}\pi r^2 h\right)$. **[2015 PI #12]**

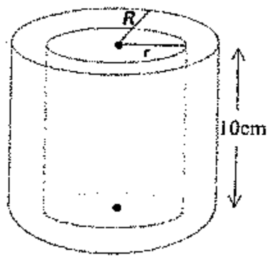
15. **Figure 7** is a prism with a triangular cross section. Angle **ABC** $= 90°$, **BE** $= 20cm$, **BA** = 6 cm and **BC**= 8 cm.



Calculate the total surface area of the prism. **[2016 PI #20]**

16. A metal cone whose volume is 1188cm^3 is recast into a sphere, assuming that there is no loss of metal, calculate the radius of the sphere, giving your answer correct to three significant figures. (Take $\pi = 3.14$ and volume of a sphere $= \frac{4}{3}\pi r^3$). **[2017 PI #19]**

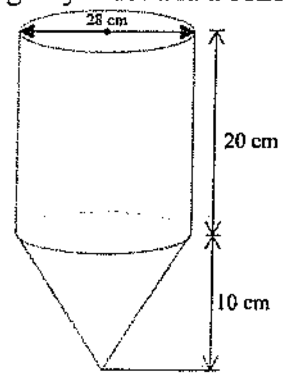
17. **Figure 8** shows a cylindrical metal with a cylindrical hole. The radius (R) of the cylinder metal is 14 cm and the radius (r) of its hole is 3 cm.



If both have a height of 10 cm, calculate the volume of the material of the metal, giving your answer correct to two decimal places.

(Take $\pi = 3.142$). **[2018 PI #18]**

18. **Figure 9** is a diagram of a drinking trough for chickens made by joining a cylinder and a cone and it is open at the top. The diameter of the trough is 28 cm while the height of the cylinder and trough are 20cm and 10 cm respectively.



If the trough was painted outside only, calculate the total surface area that was painted. (Take $\pi = \frac{22}{7}$ and area of curved surface of cone $= \pi rl$ where l is slant height. **[2019 PI #15]**

19. The volume of a cone is 462 cm^3. If the height of the cone is 9 cm, find the radius of the base. **[2020 Mock PI #11]**

1. **[2003 PP 1 #21]**
Since the rectangular box is open on top, it has 5 faces.
Area of $ABCD$ = Area of $ABGH$
Area of $C\Delta F$ = Area of $ABGH$ } Opposite sides of a cuboid.
Total surface area = area of 5 faces
$= 2(\text{Area of ABCD}) + 2(\text{Area of CDFE})$
$+\text{Area of ADFH}$
$=2(6 \times x) + 2(2x \times x) + 2x \times 6$
$= 2(6x) + 2(2x^2) + 12x$
$= 12x + 4x^2 + 12x$ (like terms together)
$= 4x^2 + 24x$
Total outter surface area= 108cm^2 (given)
$\Rightarrow 4x^2 + 24x = 108 \Rightarrow 4x^2 + 24x - 108 = 0$
Dividing each term by 4:
$$\frac{4x^2}{4} + \frac{24x}{4} - \frac{108}{4} = \frac{0}{4}$$
$\therefore x^2 + 6x - 27 = 0$

2. **[2003 PII #10b]**
$OB = 3.5cm$
$AB = 3.5 \times 2 = 7cm$ (diameter)
$CA = CB$ (right cone)
$\therefore \angle CAB = \angle CBA$ ($\angle$s opposite equal sides)
Since $\angle ACB = 60^\circ$, then $\angle CAB = \frac{180-60}{2}$
$\Rightarrow \angle CAB = 60^\circ$
$\therefore \Delta ACB$ is an equilateral $\Delta \Rightarrow$ all sides=$7cm$
$\Rightarrow l = 7cm$
Area=cone curved surface+area of hemisphere
$=\pi rl + 2\pi r^2$
$= \frac{22}{7} \times 3.5 \times 7 + 2 \times \frac{22}{7} \times (3.5)^2$
$= 77cm^2 + 77cm^2$
$\therefore$ total surface area of the block is $154cm^2$

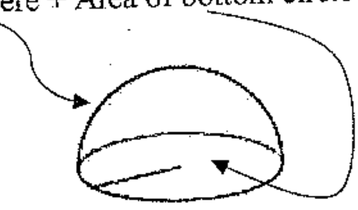
To buy or place an order for this book, kindly call or whatsapp 0995822298 or 0887616933. To order in bulk, email: msccmodelmath@gmail.com.

3. **[2004 PI #21]**
Volume = Cross-section area $\times$ height
Height of prism =6cm
Cross-section area = Area of triangle ABC
$= \frac{1}{2}\text{base} \times \text{height}$
$= \frac{1}{2} \times 3cm \times 4cm = 6cm^2$
$\Rightarrow \text{Volume} = 6cm^2 \times 6cm = 36cm^3$
$\therefore$ Volume of the prism is $36cm^3$.

4. **[2005 PII #5a]**
Since the volume of sphere is made into a cylinder, then volume of a cylinder =volume of a sphere
Volume of sphere =$770cm^3$ (given)
Volume of cylinder= $\pi r^2 h$
$\Rightarrow \pi r^2 h = 770$
$\Rightarrow \frac{22}{7} \times r^2 \times 5 = 770$ (make r subject)
$\Rightarrow r = \sqrt{\frac{770 \times 7}{22 \times 5}} \Rightarrow r = \sqrt{49} \Rightarrow r = 7$
$\therefore$ the radius of the cylinder is $7cm$

5. **[2006 P1 #3]**
Volume = length × width ×height
= 76 × 50 × 40
= $152,000\ \text{cm}^3$
$= \frac{152000}{1000} = 152\,litres$

6. **[2007 PI #18]**
Surface area of solid hemisphere
=Area of hemisphere + Area of bottom circle
$=2\pi r^2 + \pi r^2$
$= 3\pi r^2$
$= 3 \times \frac{22}{7} \times 21^2$
$= 9 \times 22 \times 21$
$= 4158cm^2$



7. [2008 P1 #10a]

Volume of figure

= cylinder volume +cone volume

raduus $=\frac{8m}{3}=4m$

Volume of cylinder $=\pi r^2h$

$$=3.14\times(4)^2\times12$$
$$=602.88m^3$$

Volume of cone $=\frac{1}{3}\pi r^2h$

$$=\frac{1}{3}\times3.14\times(4)^2\times10$$
$$=\frac{1}{3}\times3.14\times16\times10$$
$$=\frac{502.4}{3}m^3$$

Volume of figure $=602.88m^3+\frac{502.4}{3}m^3$

$$=770.35m^3\ (\text{to 2 d.p})$$

---

8. [2008 P1 #15]

since the volume of the molten cylindrical metal bar and cast into a sphere, then

Volume of cylinder = Volume of a sphere

Volume of cylinder = 594 cm^3 (given)

Volume of a sphere $=\frac{4}{3}\pi r^3$ (given)

$$\Rightarrow\frac{4}{3}\pi r^3=594\text{ cm}^3$$

$$\frac{4}{3}\times3.142\times r^3=594$$

$$r^3=\frac{594\times3}{4\times3.142}$$

$$r^3=\frac{1782}{12.568}$$

$$r=\sqrt[3]{\frac{1782}{12.568}}\text{ cm}^3$$

$r=5.21$ cm (to 2 decimal places)

$\therefore$ the radius of the sphere is 5.12 cm

---

9. [2010 PII #10a]

$r=8cm,\ \pi=3.14,\ h=15cm, slant\ height=l$

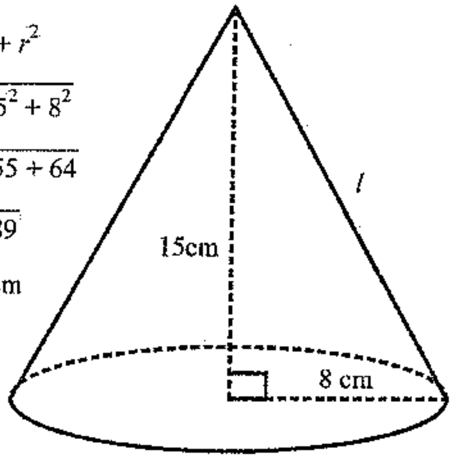
$l^2=h^2+r^2$

$l=\sqrt{15^2+8^2}$

$l=\sqrt{255+64}$

$l=\sqrt{289}$

$l=17$cm



Total surface Area $=\pi r^2+\pi rl$

$$=3.14\times8^2+3.14\times8\times17$$
$$=628cm^2$$

---

10. [2011 PI #19]

Volume of sphere $=\frac{4}{3}\pi r^3$

Volume of hemisphere$=\frac{2}{3}\pi r^3$

Since diameter=12cm, radius $=\frac{12m}{2}=6m$

volume of water in hemisphere

$$=\frac{2}{3}\times3.14\times6\times6\times6$$
$$=452.16m^3$$

volume of cyclinder $=\pi r^2h$

Since hemesphere volume is poured into cylinder

$$\Rightarrow\ \pi r^2h=452.16m^3$$

$$\Rightarrow h=\frac{452.16}{\pi r^2}\text{(making } h \text{ subject)}$$

$$h=\frac{452.16}{3.14\times5\times5}\Rightarrow h=5.76m$$

$\therefore$ the height of the water tank is $5.76m$.

11. [2012 P1 #20]

Since the cone is solid,

Total surface area = area of base + curved surface area

Surface area of a base $=\pi r^2$

$=\frac{22}{7}\times 7\times 7$

$=154cm^2$

curved surface area $=\pi rl$

$\frac{22}{7}\times 7\times 21$

$=462cm^2$

Total surface area $=154\text{cm}^2+462cm^2$

$=616cm^2$

12. [2013 P2 #5b]

Volume of pyramid $=\frac{1}{3}\text{base area}\times h$

$h=\frac{3(\text{volume of pyramid})}{\text{base area}}$

Base area (square) $=6\text{m}\times 6\text{m}$

$=36m^2$

$\Rightarrow h=\frac{3(62.4)}{36}=\frac{187.2}{36}$

$\therefore$ height $=5.2m$

13. [2014 P1 #19]

surface area of hemisphere $=2\pi r^2$

surface area of cylinder $=2\pi rh+\pi r^2$

Surface area of flask = area of cylinder + area of hemisphere

$=2\pi rh+\pi r^2+2\pi r^2$

$=2\pi rh+3\pi r^2$

$=2(3.14)(5)(25)+3(3.14)(5)^2$

$=785+235.5$

$=1020.5cm^2$

14. [2015 PI #13]

Volume of the cone=4950cm^3

height of the cone=21cm

$V=\frac{1}{3}\pi r^2h$ (substitute values to find radius)

$4950=\frac{1}{3}\times\frac{22}{7}\times 21\times r^2$

$4950=\frac{22\times\cancel{21}\times r^2}{\cancel{21}}$

$4950=22r^2$

$\frac{\cancel{22}r^2}{\cancel{22}}=\frac{4950}{22}$

$r^2=225$

$\sqrt{r^2}=\sqrt{225}$

$r=15$

$\therefore$ The radius of the cone is 15cm

15. [2016 PI #20]

Total surface area= sum of areas of all the faces

Area ABC=Area DEF (cross section)

Total surface area=2(area ABC) + Area BCFE +Area ABED + Area ACFD

In ΔABC

AC=$\sqrt{6^2+8^2}$ (pythagoras theorem)

AC = $\sqrt{100}$ $\Rightarrow AC=10$

Area of ΔABC = $\frac{1}{2}\times base\times height$

$=\frac{1}{2}\times 6\times 8$

$=24cm^2$

Total surface area $=2(24)+(20\times 8)$

$=+(20\times 6)+(10\times 20)$

$=48+208+120+200$

$=576cm^2$

16. [2017 PI #19]

Volume of sphere = 1188cm^3 (given)

Volume of cone = Volume of sphere (since no metal is lost)

$\Rightarrow\frac{4}{3}\pi r^3=1188\text{cm}^3$

$4\pi r^3=1188\times 3=3564$

$r^3=\frac{3564}{4\pi}=\frac{3564}{4\times 3.14}=\frac{3564}{12.56}$

$r^3=283.7580$

$\Rightarrow r=\sqrt[3]{283.7580}$

$r=6.5713$

$r=6.57$ (to 3 sig. fig)

17. [2018 P1 #18]

Volume of material of metal used
= Volume of cylinder with radius **R** – Volume of cylinder radius **r**

$= \pi R^2 h - \pi r^2 h$

$= 3.142(14)^2(10) - 3.142(3)^2(10)$

$= 3.142(196)(10) - 3.142(9)(10)$

$= 6158.32 - 282.78$

$= 5875.54 cm^3$

18. [2019 PI #15]

Total Surface Area=SA of Cylinder+SA of Cone

SA of Cylinder=$2\pi rh$

$= 2 \times \frac{22}{7} \times 14 \times 20$

$= 88 \times 20$

$= 1,760 cm^2$

Curved SA of Cone=πrl

$= \frac{22}{7} \times 14 \times 10$

$= 44 \times 10$

$= 440 cm^2$

Total Surface Area=$1,760 + 440$

$= 2,200 cm^2$

19. [2020 Mock PI #11]

Volume of a Cone $= \frac{1}{3}$*base* area × height

$V = \frac{1}{3}\pi r^2 h$

$462 = \times \frac{22}{7} \times r^2$

$462 \times 7 = 66r^2$

$\frac{3234}{66} = \frac{66}{66} r^2$

$r^2 = 49$

$r = \sqrt{49}$

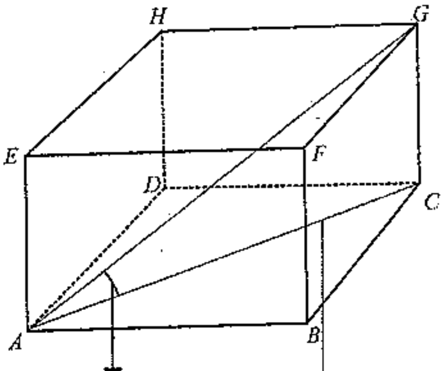
$r = 7cm$

CH 31 MENSURATION II
(THREE-DIMENSIONAL GEOMETRY)

Chapter Highlights

The problems in this chapter focus on calculating angles between two lines, two planes and angles between planes and lines in 3D shapes. We will also encounter problems that require us to use trigonometry to calculate the lengths of the sides of 3D shapes.

Note: The angle between a line and a plane is the angle between the line and its **projection** onto the plane. The projection of a line on a plane is created by rewriting the line on the plane as if it is a shadow created by a lamp directly illuminating the line perpendicular to the plane.

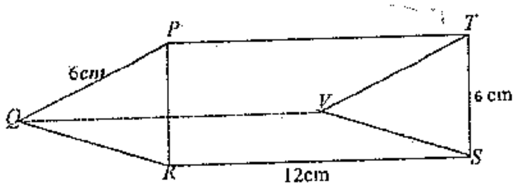


Angle between line AG and plane $ABCD$

Projection of AG on plane $ABCD$

Important Definitions:
Vertex - A point where two or more lines meet. A, B, C, D, E, F, G and H are vertices.
Edge - a line where two faces of a 3D figure meet e.g. AB, BF, FG, HG, etc.
Face – is a flat surface that make up the outside of the shape. e.g. ABFE, EFGH and FBCG.
Plane - is a flat 2D surface that extends in all directions. e.g. ABCD, EFGH, AGC.

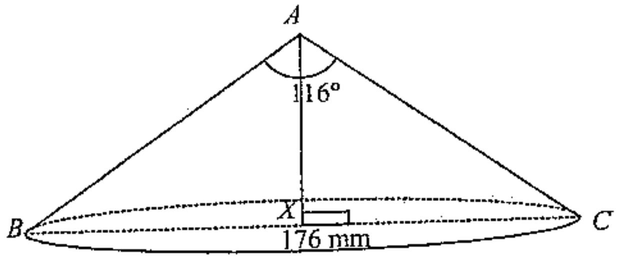
1. **Figure 1** is a prism with a cross section of an equilateral triangle **PQR. PRST, PQVT** and **QRSV** are rectangles, such that **RS**=12cm, **QP**=6cm and **X** is midpoint of **QR**.



Calculate:

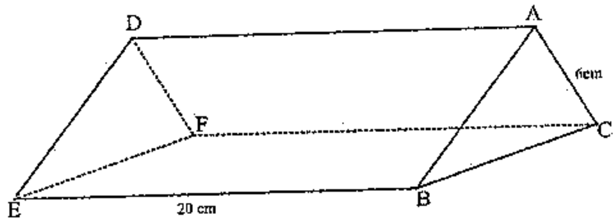
i. The length of **TX** giving your answer to correct 2 decimal places.
ii. The angle between **TX** and the base **QRSV** giving your answer to correct 2 decimal places. **[2004 PII #12b]**

2. **Figure 2** shows a right cone whose vertical angle **BAC**=116°, the diameter of its base **BC**=176mm and **AX** is the height.



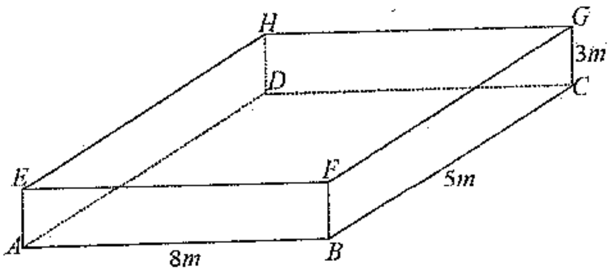
Calculate the length of **AX**. **[2005 PI #23]**

3. **Figure 3** shows a wooden prism with a triangular cross-section **ABC**. **BE**=20cm, **AB=BC=AC**=6cm, **ACFD, BCFE** and **ADEB** are rectangles.



Calculate angle **DCE**. **[2006 PII #10b]**

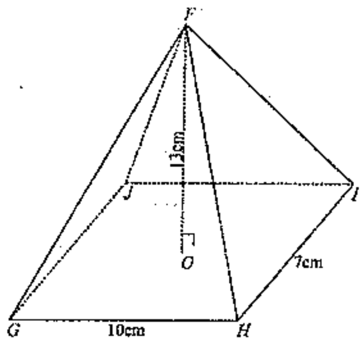
4. **Figure 4** shows a water tank with rectangular faces. **AB**=8m **BC**=5m and **CG**=3m.



Calculate:

i. The length of **AG** leaving your answer in its simplest surd form.
ii. The angle which **AG** makes with the plane **ABCD.** **[2007 PII #10b]**

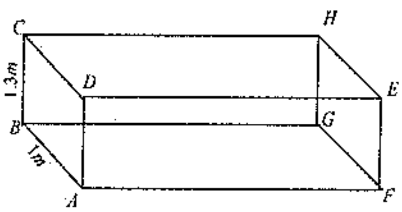
5. **Figure 5** shows a rectangular based right pyramid **FGHIJ** with **GH**=10cm, **HI**=7cm



If the height of the pyramid is 13cm, calculate the angle between **FHI** and the base.

[2010 P1 #18]

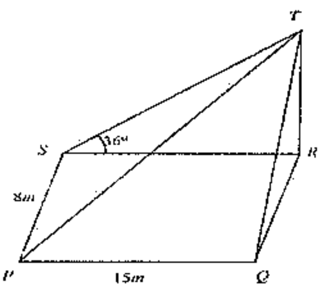
6. **Figure 6** shows a cuboid which is 2.3m long, 1m wide and 1,3m high.



Calculate:

i. The length **EB.**
ii. The angle between **EB** and the base of the cuboid. (giving your answer correct to 1 decimal place in both cases) **[2011 PII #5b]**

7. In **Figure 7, PQRS** is a rectangle, angle **TSR** =36°, angle **SRT**= angle **TRQ**=90°, **PQ**=15m and PS = 8m.



Calculate:

i. The length of **TR**
ii. The angle between **PT** and the rectangle **PQRS,** giving your answer correct to one decimal place. **[2012 PII #8b]**

8. **Figure 8** shows a prism ABCDEF , 5m long , with a triangular cross-section. AB = AC = 3m and BC = 4cm.

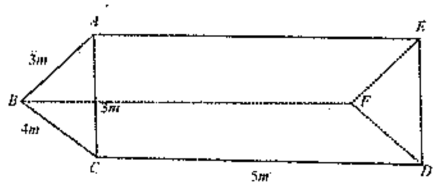


Figure 4

Calculate the:

i. Length of BE
ii. Angle between line ED and plane BCDF giving your answer correct to the nearest degree. **[2014 PII #10b]**

9. **Figure 9** is a cuboid which is 7 cm long, 2cm wide and 4cm high.

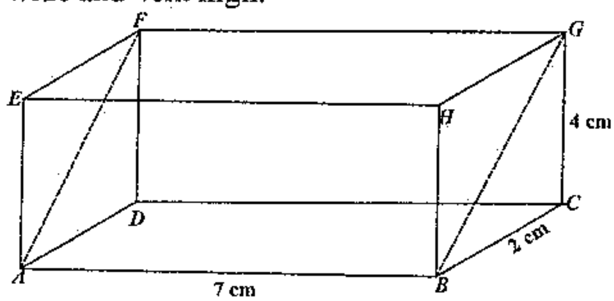


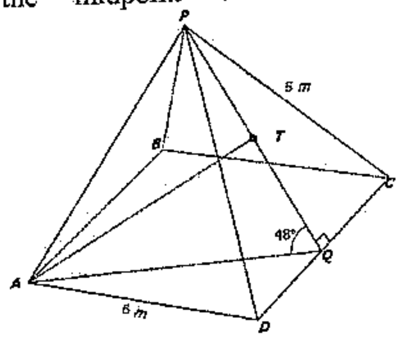
Figure 4

Calculate the:

i. Length of **AF.**
ii. Angle between **EFGH** and **AFGB,** giving your answer correct to 2 decimal places.

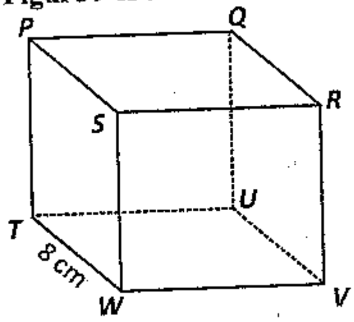
[2015 PII #9b]

10. **Figure 11** is a square based pyramid **PABCD. T** is the midpoint of the slant height **PQ.**



If angle **AQP**=48°,**PC**=5 m and **AD**= 6m, calculate the length of **AT,** giving the answer correct to 2 decimal places. **[2016 PII #7b]**

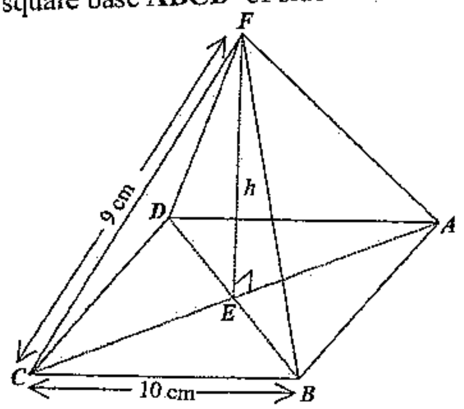
11. **Figure 9** shows a cube of sides 8 cm.



Calculate the angle, to the nearest degree, that **RT** makes with the plane **TUVW.**

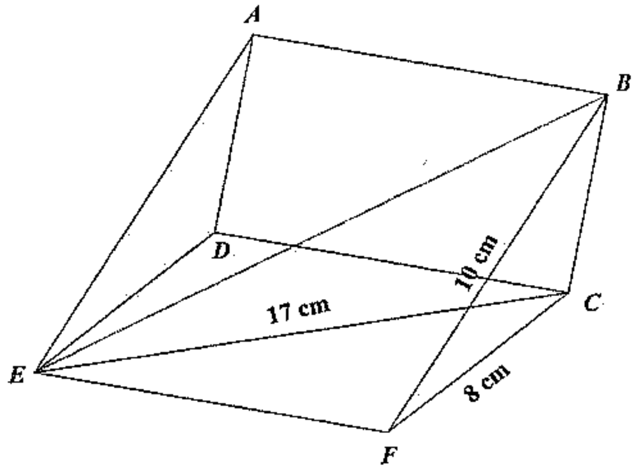
[2017 PII #6b]

12. **Figure 12** shows a right pyramid **ABCDF** on a square base **ABCD** of side 10 cm.



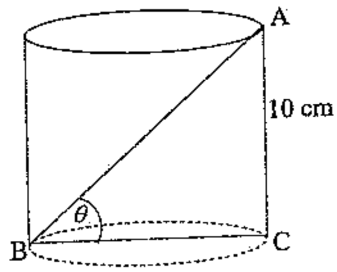
Calculate the height h of the pyramid to one decimal place. **[2018 PII #13]**

13. **Figure 13** is a wedge such that **ABCD, DCFE** and **ABFE** are rectangles.



If **CE** = 17 cm **CF** = 8 cm **BF**= 10 cm, calculate the length of **BE,** giving the answer correct to 2 decimal places. **[2019 PII #4b]**

14. **Figure 14** is a cylinder of height **AC**=10 cm



If the volume of the cylinder is 385 cm^3, find the value of Angle ABC labeled θ to the nearest degree. **[2021 Mock PII #7b]**

1. **[2004 PII #12b]**

(i). In ΔPQR, $PQ = QR$ *(equilateral triangle)*
$QR = 6cm$
$XR = \frac{1}{2} QR = \frac{1}{2}(6cm) = 3cm$ (X is the midpoint)
In ΔPXR, $PX^2 + XR^2 = PR^2$ (Pythagoras theorem)

$PX^2 + XR^2 = PR^2$ *(Pythagoras theorem)*

$\Rightarrow PX^2 = PR^2 - XR^2$

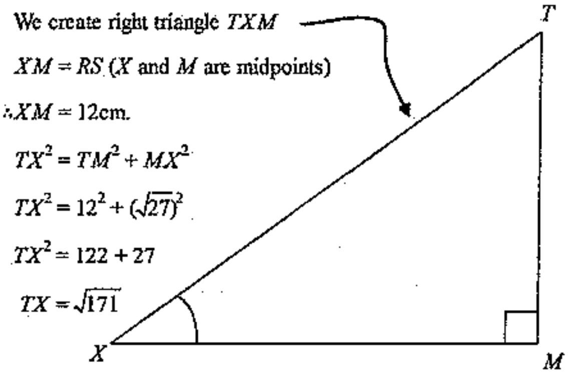
$\Rightarrow PX^2 = 6^2 - 3^2 \Rightarrow PX^2 = 36 - 9$

$\Rightarrow PX^2 = 27 \Rightarrow PX = \sqrt{27}$

$\Delta PQR = \Delta TVS$ *(ΔPQR is cross section)*

let M be the midpoint in VS in ΔTVS

$\Rightarrow PX = TM \therefore TM = \sqrt{27}$



$\therefore TX = 13.08$ (to 2 decimal places)

(ii). The projection of TX on base QRSV is line XM. $\therefore$ The angle TX makes with base QRSV is angle TXM in triangle TXM.

$$\cos\theta = \frac{XM}{TX} = \frac{12}{\sqrt{171}}$$

$$= 0.9174$$

$$\theta = \cos^{-1}\left(\frac{12}{\sqrt{171}}\right)$$

$\theta = 23.41°$ (to two decimal places)

$\therefore$the angle between TX and base QRSV is 23.41°

2. **[2005 PI #23]**

Consider ΔABC

$AB = AC$ *(right cone)* $\therefore$ ABX is isoscles triangle

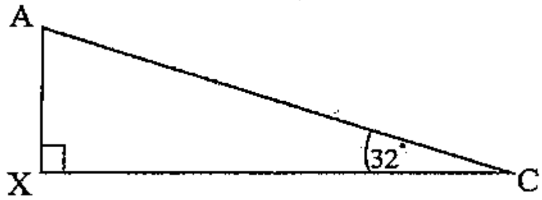
$\Rightarrow \angle ABC = \angle ACB$

$$\Rightarrow \angle ACB = \frac{180 - 116}{2}$$

$\Rightarrow \angle ACB = 32°$

AX bisects BC at X and $AX \perp BC$*(right cone)*

$\therefore$ AXC is a right angles Δ



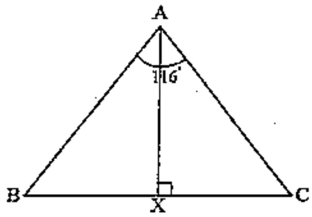
$XC = \frac{1}{2}BC$ *(BX is diameter)*

$\Rightarrow XC = \frac{1}{2}(176mm) \Rightarrow XC = 88mm$

$\Rightarrow \tan 32° = \frac{AX}{88} \qquad \Rightarrow AX = 88\tan 32°$

$\therefore AX = 54.98 \approx 55$mm *(to nearest whole number)*

Alternatively:
In Δs ABX and ACX



AX is common
$\angle AXB = \angle AXC$ *(both eq to 90°)*
$BX = CX$ *(radii of t circular base)*
$\therefore \Delta ABX = \Delta ACX$ *(SA*
$\therefore \angle BAX = \angle CAX$
(corresponding $\angle$s congruent Δs)

$$\therefore \angle BAX = \frac{116°}{2} = 58°$$

$BX = \frac{1}{2}(176) = 88$ *(radius is ½ diameter)*

In ΔABX

$$\frac{BX}{AX} = \tan\angle BAX$$

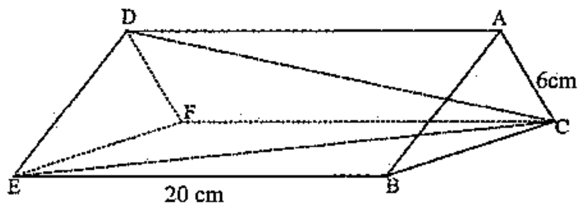
$$\frac{88mm}{AX} = \tan 58° \Rightarrow AX = \frac{88mm}{\tan 58°}$$

$$AX = \frac{88mm}{1.6003}$$

$= 55$mm (to the nearest whole number)

$\therefore$the length of AX is 55mm

3. **[2006 PII #10b]**



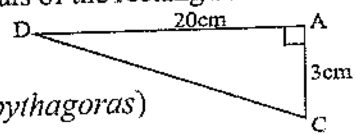
We construct DC and EC.

Rectangle ACFD and CBEF are equal

(20cm by 6 cm)

DC and EC are diagonals of the rectangles

$\therefore$ DC=EC

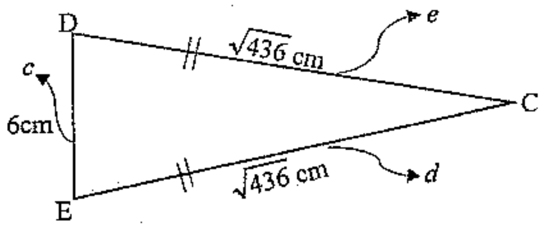


$DC^2 = DA^2 + AC^2$ (*pythagoras*)

$\Rightarrow DC^2 = 20^2 + 6^2$

$\Rightarrow DC^2 = 436 \Rightarrow DC = \sqrt{436}$

$\therefore EC = \sqrt{436}$ (Since $DC = EC$)



In ΔDEC

$c^2 = d^2 + e^2 - 2de\,\text{Cos}\,C$ ***(Cosine rule)***

$\Rightarrow 6^2 = \left(\sqrt{436}\right)^2 + \left(\sqrt{436}\right)^2 - 2\left(\sqrt{436}\right)\left(\sqrt{436}\right) \times \cos C$

$\Rightarrow 36 = 436 + 436 - 2(436) \times \text{Cos}\,C$

$\Rightarrow 36 = 872 - 872 \times \text{Cos}\,C$

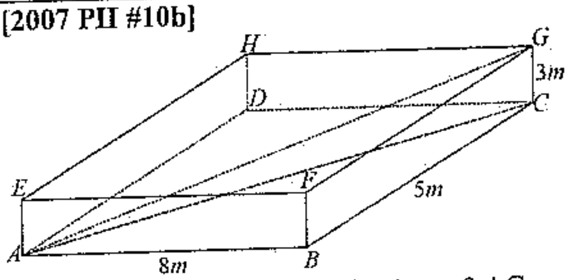
$\Rightarrow 36 - 872 = -872 \times \text{Cos}\,C$

$\Rightarrow -836 = -872 \times \text{Cos}\,C$

$\Rightarrow \dfrac{-836}{-872} = \text{Cos}C \Rightarrow \angle C = \text{Cos}^{-1}\dfrac{836}{872}$

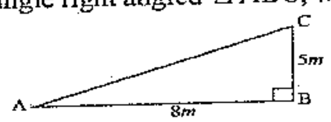
$\therefore \angle C = \angle DCE = 16.52^\circ$

4. **[2007 PII #10b]**



We draw line AG. The projection of AG on plane ABCD is line AC. We then create right angled Δ AGC.

Using triangle right angled Δ ABC, we find AC

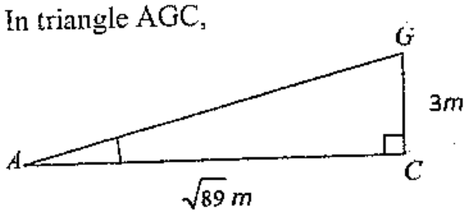


$AC^2 = AB^2 + BC^2$ (pythagoras theory)

$AC = \sqrt{8^2 + 5^2} \Rightarrow AC = \sqrt{64 + 25}$

$\therefore AC = \sqrt{89}\ m$

In triangle AGC,



$AG^2 = AC^2 + CG^2$ (*pythagoras theorem*)

$\Rightarrow AG = \sqrt{\left(\sqrt{89}\right)^2 + 3^2} \Rightarrow AG = \sqrt{89 + 9}$

$\Rightarrow AG = \sqrt{98}$ $\Rightarrow AG = \sqrt{49 \times 2}$

$\therefore AG = 7\sqrt{2}m$

ii. The angle AG make with the plane ABCD is the angle between AG and its projection on ABCD.

Thus we need to find $\angle GAC$

$\tan \angle GAC = \dfrac{3}{\sqrt{89}}$

$\Rightarrow \angle GAC = \tan^{-1}\dfrac{3}{\sqrt{89}}$

$\Rightarrow \angle GAC = 17.64^\circ$

$\therefore$ The angle which AG makes with the plane ABCD is 17.64°

5. **[2010 P1 #18]**

- Create a midpoint be M on IH by bisecting it.
- Join M to the bottom of the height O
- Join FM

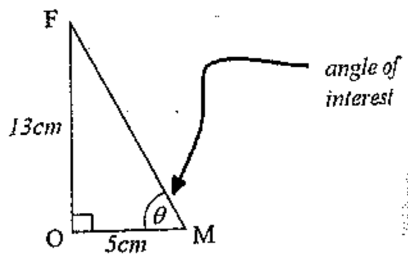
The angle between FHI and GHIJ=Angle FMO

But $MO = \dfrac{1}{2}GH$

$MO = \dfrac{1}{2} \times 10cm$

$MO = 5cm$

$Tan\ \angle FMO = \dfrac{opp}{adj}$



$Tan\ \angle FMO = \dfrac{13}{5} = 2.6$

$\angle FMO = \tan^{-1} 2.6$

$\angle FMO = 68.96^\circ$

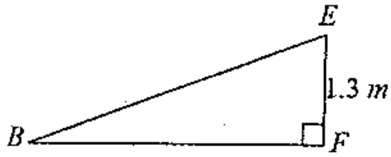
$\angle FMO = 69^\circ$ (to the nearest degrees)

$\therefore$ The angle between FHI and the base $= 69^\circ$

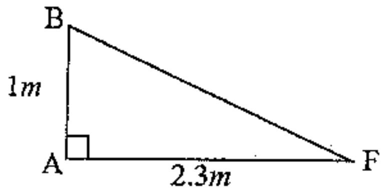
---

6. **[2011 PII #5b]**

i. To find EB, we first draw line EB. The projection of EB on plane ABCD is line BF. Draw line BF and create right angled Δ EBF.



We need BF to find EB. So using Δ ABF, we find BF.



$BF^2 = AB^2 + AF^2$ (pythagoras theorem)

$BF^2 = \sqrt{1^2 + 2.3^2} \Rightarrow BD^2 = \sqrt{1 + 1.69}$

$BF = \sqrt{1 + 5.29}$

$BF = \sqrt{6.29} m.$

In Δ EBF above (right angle at F)

$EB^2 = BF^2 + EF^2$

$\Rightarrow EB = \sqrt{\left(\sqrt{6.29}\right)^2 + 1.3^2}$

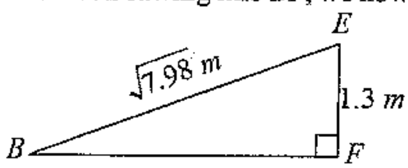
$\Rightarrow EB = \sqrt{6.29 + 1.69}$

$EB = \sqrt{7.98}$

$EB = 2.824889$

$\therefore EB = 2.8m$

ii. The projection of EB on the base of cuboid is line BF. Drawing line BF, we now have Δ EBF.



The angle EB makes with the base of cuboid is the like angle EB makes with the projection BF. Thus, $\angle EBF$ given as:

$\sin \angle EBF = \dfrac{1.3}{\sqrt{7.98}}$

$\angle EBF = \sin^{-1} \dfrac{1.3}{\sqrt{7.98}} \Rightarrow \angle EBF = 27.4^\circ$

The angle EB makes with the base of cuboid is 27.4°.

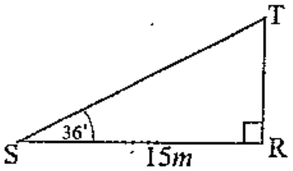
---

7. **[2012 P2 #8b]**

i. Since TRQ=90°, then TR is a vertical pole.

$\therefore \angle TRS = 90^\circ$

In rectangle SRQP, PQ=SR $\therefore$ SR=15cm



In ΔSTR, $\tan 36^\circ = \dfrac{TR}{15} \Rightarrow TR = 15 \tan 36^\circ$

length of $TR = 10.898$

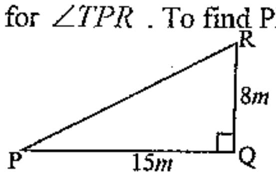
$\therefore TR = 10.90\,cm$ (to 2dec.places)

ii. The projection of TP on PQRS is PR. In Δ TPR, we are looking for $\angle TPR$. To find PR:

In ΔPRQ,

$PR^2 = PQ^2 + QR^2$

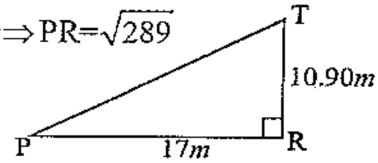
$PR^2 = 15^2 + 8^2$



$\therefore PR = \sqrt{225 + 64} \Rightarrow PR = \sqrt{289}$

$\therefore$ PR=17m

In ΔTPR



$\text{Tan}\angle TPR = \dfrac{TR}{PR} = \dfrac{10.90}{17}$

$\text{Tan}\angle TPR = 0.6412$

$\angle TPR = 32.7^\circ$

---

8. **[2014 P1 #10b]**

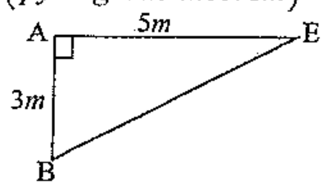
i. On rectangle ABFE, BE is the diagonal.

$\Rightarrow BE = \sqrt{5^2 + 3^2}$ (*pythagoras theorem*)

$= \sqrt{25 + 9}$

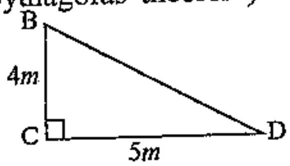
$= \sqrt{34}$

$= 5.83m$



ii. To work out an angle between line ED and plane BCDF thus angle BDE:

$BD^2 = BC^2 + CD^2$ (pythagoras theorem)

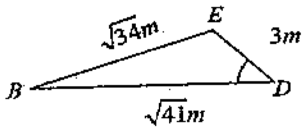


$BD^2 = 4^2 + 5^2$

$\sqrt{BD^2} = \sqrt{16+25}$

$BD = \sqrt{41}$

$\therefore BD = 6.4cm$



by cosine rule: $d^2 = e^2 + b^2 - 2eb\cos D$

$$\cos D = \frac{e^2+b^2-d^2}{2eb}$$

$$= \frac{\left(\sqrt{41}\right)^2 + 3^2 - \left(\sqrt{34}\right)^2}{2\times\left(\sqrt{41}\right)\left(\sqrt{34}\right)}$$

$$= \frac{41+9-34}{2(\sqrt{41})(3)}$$

$$= \frac{16}{6\sqrt{41}}$$

$$= 0.4165$$

$\Rightarrow \angle D = \angle BDE = \cos^{-1} 0.4165$

$\angle BDE = 65.39^\circ$ (to two decimal places)

9. **[2017 PII #6b]**

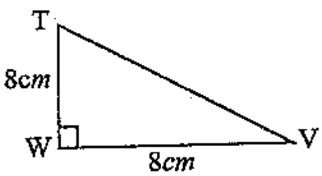
We draw line RT. The projection of RT on plane TUVW is line TV. We then create right angled triangle RTV to find angle RTV. However, we first solve for TV.

TUVW is a square, (cube). In ΔTWV, $TV^2 = TW^2 + WV^2$ (Pythagoras theorem).

$TV = \sqrt{8^2+8^2}$

$TV = \sqrt{64+64}$

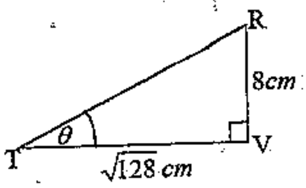
$\therefore TV = \sqrt{128}\ cm$



In Δ RTV, we calculate angle RTV.

$$\text{Tan}\,\theta = \frac{Opp}{Adj} = \frac{RV}{TV}$$

$$\text{Tan}\,\theta = \frac{8}{\sqrt{128}}$$



$\text{Tan}\,\theta = 0.7071$

$\theta = \text{Tan}^{-1} 0.7071$

$\theta = 35.2644^o$

Thus, $\angle$RTV=35°(to the nearest degree)

Therefore, RT makes an angle of 35° with the plane TUVW.

10. **[2015 PII #9b]**

i. face HGCB= face EFDA (Cuboid)

$\Rightarrow AD = 2cm$ and $FD = 4cm$

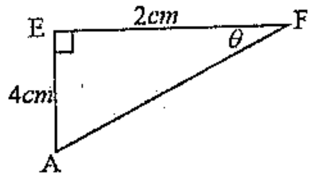
$\Rightarrow AF^2 = EA^2 + EF^2$ (*Pythagoras theorem*)

$AF = \sqrt{4^2+2^2}$

$AF = \sqrt{16+4}$

$AF = \sqrt{20}$

$\therefore AF = 4.47cm$



ii. The angle between EFGH and AFGB is angle between EF and AF, $\angle EFA$. Let $\angle EFA = \theta$

$$\tan\ \theta = \frac{EA}{EF}$$

$$\tan\ \theta = \frac{4cm}{2cm}$$

$\tan\ \theta = 2$

$\theta = \tan^{-1} 2 \Rightarrow \theta = 63.4349^\circ$

$\Rightarrow \angle EFA = 63.43^\circ$ (to 2 decimal places)

$\therefore$ The angle between EFGH and AFGB is 63.43°. (NB: same answer if you find $\angle HGB$)

11. **[2016 PII #7b]**

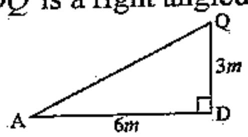
$AD = DC = 6cm$ (square – base)

$DQ = QC = \frac{1}{2}DC$

$\therefore DQ = 3cm$

$\angle ADQ = 90^\circ$ (angle of a square base)

$\therefore \Delta ADQ$ is a right angled triangle



$AQ^2 = AD^2 + DQ^2$ (*Pythagoras theorem*)

$AQ^2 = 6^2 + 3^2$

$AQ^2 = 36 + 9$

$AQ^2 = 45$

$\sqrt{AQ^2} = \sqrt{45}$

$AQ = 6.71cm$

$AQ \approx 7cm(1sf)$

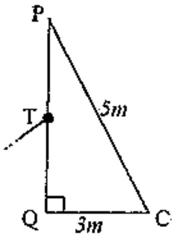
In ΔPCQ

$PQ^2 = PC^2 - QC^2$ (*Pythagoras theorem*)

$= 5^2 - 3^2$

$= 25 - 9$

$= 16$



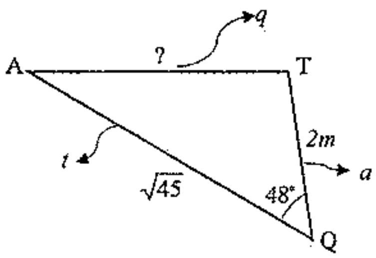
$\sqrt{PQ^2} = \sqrt{16}$

$PQ = 4cm$

$PT = TQ = \frac{1}{2}PQ$ (AT bisector at T)

$TQ = 2cm$

We now use ΔATQ to find AT:



We use Cosine rule to find AT=q *(Note that we have not been told that AT is perpendicular to PQ, hence Angle ATQ is not a right angle):*

$q^2 = a^2 + t^2 - 2at\,\mathrm{Cos}\,Q$

$q^2 = (2)^2 + (\sqrt{45})^2 - 2(2 \times \sqrt{45})\mathrm{Cos}\,48°$

$q^2 = 4 + 45 - 4\sqrt{45} \times 0.6691$

$q^2 = 49 - 17.9547$

$q^2 = 31.0453$

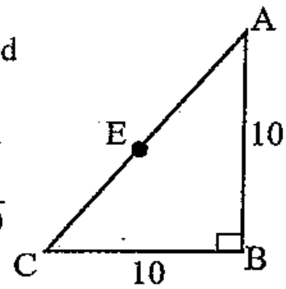
$q = \sqrt{31.0453}$

$q = 5.5718$

$\therefore AT = 5.57$(to 2decimal places)

12. **[2018 PII #13]**

ΔCAB is right angled



$\therefore$ CA=$\sqrt{CB^2 + BA^2}$

$= \sqrt{10^2 + 10^2}$

$= \sqrt{100 + 100}$

$= \sqrt{200}$

E is the midpoint of CA

CE=$\frac{1}{2}CA$

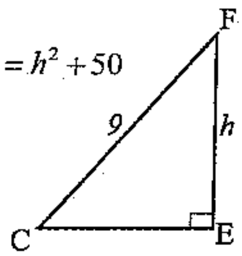
$CE = \frac{1}{2}(\sqrt{200})$

$= \frac{\sqrt{200}}{2}$

ΔCEF is right angled

CF$^2 = FE^2 + $CE^2 (pythagoras)

$9^2 = h^2 + \left(\frac{\sqrt{200}}{2}\right)^2$



$81 = h^2 + \frac{200}{4} \Rightarrow 81 = h^2 + 50$

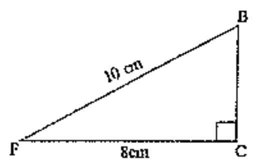
$h^2 = 81 - 50$

$h = \sqrt{31}$

$h = 5.5677$

$h = 5.6cm$ (to one decimal place)

13. **[2019 PII #4b]**



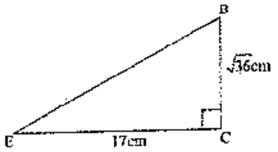
$BC^2 + 8^2 = 10^2$ (pythagoras)

$BC^2 = 100 - 64$

$BC^2 = 36$

$BC = \sqrt{36}$

$BC = 6cm$



$BE^2 = BC^2 + EC^2$ (pythagoras)

$BE^2 = 6^2 + 17^2$

$BE^2 = 36 + 289$ (from above)

$BE^2 = 325$

$BE = \sqrt{325}$

$BE = 18.02777$

$BE = 18.03$(to 2dp)

14. **[2021 Mock PII #7b]**
Use the given volume and height to find radius:

$V = \pi r^2 h$

$385 = \frac{22}{7} \times r^2 \times 10$

$385 \times 7 = 220r^2$

$2695 = 220r^2$

$\frac{220r^2}{220} = \frac{2695}{220}$

$r^2 = 12.25$ $\Rightarrow$ $r = \sqrt{12.25}$

$r = 3.5$

But diameter is twice radius so,
$BC = 2 \times 3.5 = 7cm$

Then using ΔABC, we find angle θ

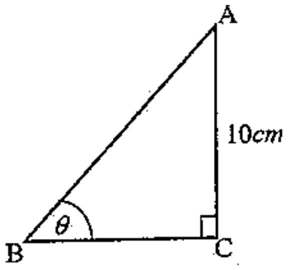
$\text{Tan}\,\theta = \frac{AC}{BC}$

$\text{Tan}\,\theta = \frac{10}{7}$

$\text{Tan}\,\theta = 1.4286$

$\theta = \tan^{-1} 1.4286$

$\theta = 55.01$



Hence, $\angle ABC = 55^o$ (to the nearest degree)

CH 32
GRAPHS OF CUBIC FUNCTIONS

Chapter Highlights

In this chapter, we will plot graphs of cubic functions i.e. $y = ax^3 + bx^2 + cx + d$ where the highest power of x is three.

In solving the problems that follow, knowledge on drawing graphs and solving equations graphically will be useful. Depending on the nature of the problem, the plotted graphs will be used to solve cubic equations or simultaneous cubic and linear functions.

The solutions to simultaneous cubic and linear equation will be found where the two graphs intersect.

To use our graph to solve an equation, we subtract the equation to be solved from our graphed equation. We then plot the graph of the resulting equation on the same axis - the solutions lies where the graphs of the two equations intersect.

1. The table below shows some values for equation $y = x^3 - 2x^2 - 5x + 6$. Copy and complete the table.

Table: $y = x^3 - 2x^2 - 5x + 6$

X	-3	-2	-1	0	1	2	3	4
Y	-24		8		0	-4	0	18

ii. Using a scale of 2cm to represent 1 unit on the x-axis and 2cm to represent 5 units on the y – axis. Draw the graph of $y = x^3 - 2x^2 - 5x + 6$.

iii. Use your graph to solve the equation $x^3 - 2x^2 - 5x + 4 = 0$

[2005 PII #12b]

2. Copy and complete the table of values for the equation $y = x(x^2 - x - 6)$

X	-3	-2	-1	0	1	2	3	4
Y	-18	0		0	-6		0	24

ii. Using the scale of 2cm to represent 1 unit on the x-axis and 2cm to represent 5cm on the y-axis, draw the graph of $y = x(x^2 - x - 6)$

iii. Use your graph to solve the equation $y = x(x^2 - x - 6) + 5 = 0$

[2004 PII #8b]

3. Complete the table of values for the equation $y = x^3 - 4x$:

x	-3	-2	-1	0	1	2	3
y	-15	0		0	-3		15

Using a scale of 2cm to represent 2 units on x-axis and 2cm to represent 5 units on y-axis, draw the graph of $y = x^3 - 4x$.

Use the graph to solve the equation $x^3 - 7x - 6 = 0$.

[2011 PII 11a]

4. The table below shows some values of x and y for the equation $y = x^3 - x^2 - 6x$.

x	-3	-2	-1	0	1	2	3	4
y	-18	0		0		-8	0	24

i. complete the table of values

ii. Using a scale of 2 cm to represent 1 unit on the horizontal axes and 2 cm to represent 5 units on the vertical axis, draw the graph of $y = x^3 - x^2 - 6x$.

iii. Use your graph to solve the equation $x^3 - x^2 - 4x + 4 = 0$. **[2017 PII #9b]**

5. The table below shows some values of x and y for the equation $y = 6x + x^2 - x^3$.

x	-3	-2	-1	0	1	2	3	4
y	18	0		0	6	8		-24

i. Complete the table of values.

ii. Using a scale of 2 cm to represent 1 unit on the horizontal axis and 2 cm to represent 5 units on the vertical axis, draw the graph of $y = 6x + x^2 - x^3$.

iii. Use the graph to solve the equation $y = 6x + x^2 - x^3 - 5 = 0$.

[2016 PII #10b]

6. Table 2, shows some values of x and y for the equation $y = x^3 - 2x^2 - 6x + 3$.

x	-3	-2	-1	0	1	2	3	4
y	-24		6	3	-4	-9		11

i. Complete the table of values.

ii. Using a scale of 2cm to represent 1 unit on the horizontal axis and 2 cm to represent 5 units on the vertical axis, draw a graph of $y = x^3 - 2x^2 - 6x + 3$.

iii. Use your graph to solve the equation $x^3 - 2x^2 - 7x + 5 = 0$.

[2014 PII #8b]

7. Copy and complete the table of values for the equation $y = (x+1)(x^2 + x - 6)$

x	-4	-3	-2	-1	0	1	2	3
Y	-18	0	4		-6	-8	0	24

ii. Using a scale of 2cm to represent 1 unit on the x-axis and 2cm to represent 5 units on the y-axis, draw the graph of $y = (x+1)(x^2 + x - 6)$

iii. Use your graph to solve the simultaneous equations;

$$y = x - 4$$
$$y = x^3 + 2x^2 - 5x - 6$$

[2007 PII 8b]

1. **[2005 PI #12b]**

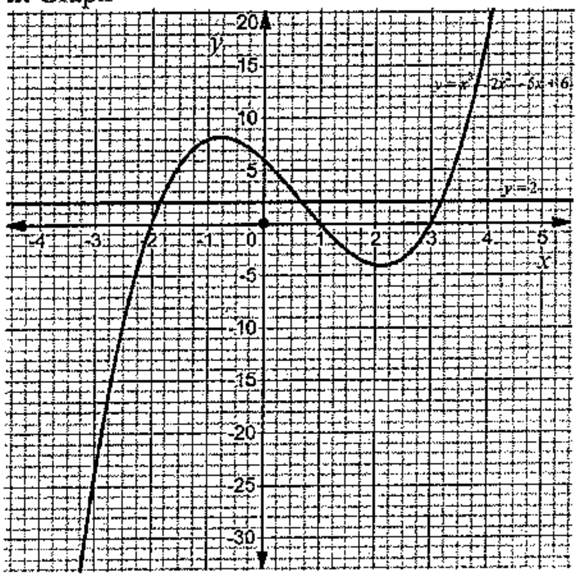
i.

If $x=-2$, then $y=(-2)^3-2(-2)^2-5(-2)+6$
$=-8-8+10+6$
$=0$

If $x=0$, then $y=(0)^3-2(0)^2-5(0)+6$
$=0-0-0+6$
$=6$

x	-3	-2	-1	0	1	2	3	4
y	-24	**0**	8	**6**	0	-4	0	18

ii. Graph



iii.

$y=x^3-2x^2-5x+6 \longrightarrow$ Graphed equation
$-\;0=x^3-2x^2-5x+4 \longrightarrow$ Equation to be solved
$y=0+0+0+2$

$\therefore y=2$

We then draw the graph of $y = 2$ on the same axes as the graph of $y=x^3-2x^2-5x+6$.
According to the graph, the intersections are at $x=-1.9, x=0.7$ or $x=3.2$.

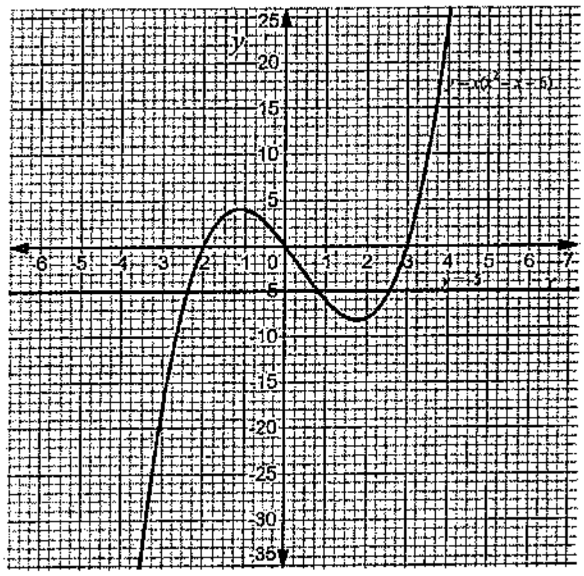
2. **[2004 PII #8b]**

i. Given: $y=x(x^2-x-6)$
if $x=-1$ then
$y=\left[-1(-1)^2-(-1)-6\right]$
$y=-1(1+1-6)$
$y=-1(-4)$
$y=4$

if $x=2$ then $y=2(2^2-(2)-6)$
$=2(4-2-6)$
$=2(-4)$
$=-8$

x	-3		-2	-1	0	1	2	3	4
y	-18		0	**4**	0	-6	**-8**	0	24

ii.



iii.

$y=x(x^2-x-6)$ becomes:
$y=x^3-x^2-6x$
and $x(x^2-x-6)+5=0$ becomes:
$x^3-x^2-6x+5=0$
Subtracting the equations :

$y=x^3-x^2-6x \longrightarrow$ Graphed equation
$-\;0=x^3-x^2-6x+5 \longrightarrow$ Equation to be solved
$y=0+0\quad-5$
$\therefore y=-5$

We now plot the graph of $y=-5$ on the same axis check graph above. According to the graph, the intersections are at $x=-2.4$, $x=0.8$ or $x=2.5$

3. **[2011 PII 11a]**

i. $y=x^3-4x$
If $x=-1$, then
$y=(-1)^3-4(-1)$

$y = -1 + 4 = 3$

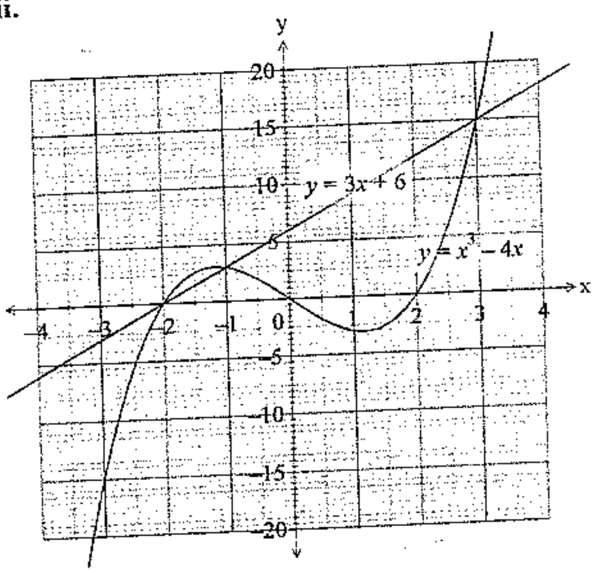
When $x = 2$, then

$y = (2)^3 - 4(2)$

$y = 8 - 8 = 0$

x	-3	-2	-1	0	1	2	3
y	-15	0	**3**	0	-3	**0**	15

ii.



iii.

$y = x^3 - 4x$

$\underline{-\; 0 = x^3 - 7x - 6}$

$\underline{y = 0 + 3x + 6}$

$\therefore y = 3x + 6$

We then draw the graph of $y = 3x + 6$ on the same axes as the graph of $y = x^3 - 4x$.

$y = 3x + 6$

x	0	-2
y	6	0

According to the graph, the intersections are at $x = -2, x = -1$ or $x = 3$.

4. **[2017 PII #9b]**

i.

To complete the table,

when $x = -1$; $y = (-1)^3 - (-1)^2 - 6(-1)$

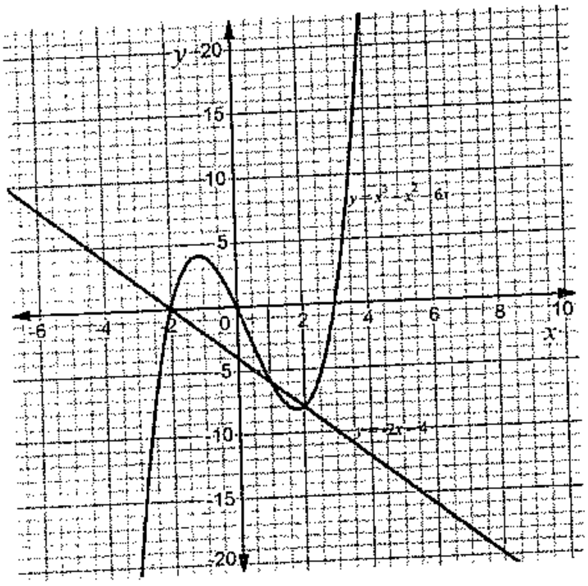
$y = 4$

when $x = 1$; $y = 1^3 - 1^2 - 6(1)$

$y = -6$

x	-3	-2	-1	0	1	2	3	4
y	-18	0	4	0	**-6**	-8	0	24

ii.



iii.

$y = x^3 - x^2 - 6x$

$-\underline{(0 = x^3 - x^2 - 4x + 4)}$

$\underline{y = 0 + 0 - 2x - 4}$

$\therefore y = -2x - 4$

We then draw the graph of $y = -2x - 4$ on the same axis.

$y = -2x - 4$

x	-2	0
y	0	-4

According to the graph, it intersection are at:

$y = -2, x = 1$ or $x = 2$.

5. **[2016 PII #10b]**

i. To complete the table, $y = 6x + x^2 - x^3$

When $x = -1$, $y = 6(-1) + (-1)^2 - (-1)^3$

$y = -6 + 2$

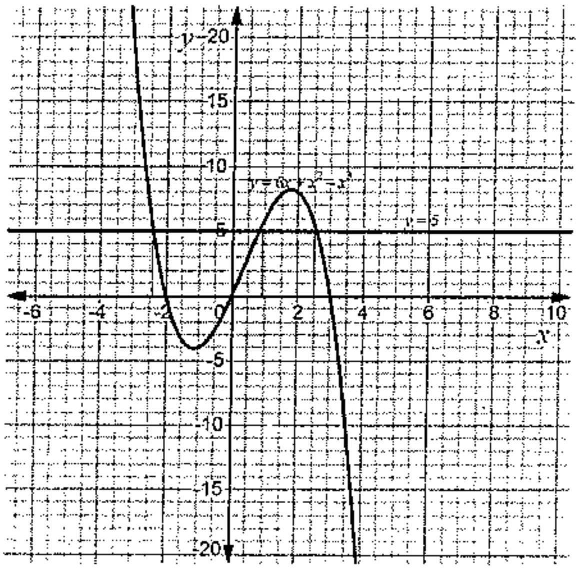
$y = -4$

When $x = 3$, $y = 6(-3) + (-3)^2 - (-3)^3$

$y = 0$.

x	-3	-2	-1	0	1	2	3	4
y	18	0	-4	0	6	8	**0**	-24

ii.



iii.

$$y = \quad 6x + x^2 - x^3$$

$$-(0 = -5 + 6x + x^2 - x^3)$$

$$y = +5 + 0 \;\; +0 \;\; +0$$

$$\therefore y = 5$$

We draw the line y=-5. The solution, thus where two equations intersect:

x=-2.4, x=0.8 or x= 2.5.

6. **[2014 P1 #8b]**

$y = x^3 - 2x^2 - 6x + 3$

i. Completing table of values:
-when x=-2

$y = x^3 - 2x^2 - 6x + 3$

$y = (-2)^3 - 2(-2)^2 - 6(-2) + 3$

$y = -8 - 8 + 12 + 3 \quad \therefore y = -1$

When x=3

$y = (3)^3 - 2(3)^2 - 6(3) + 3$

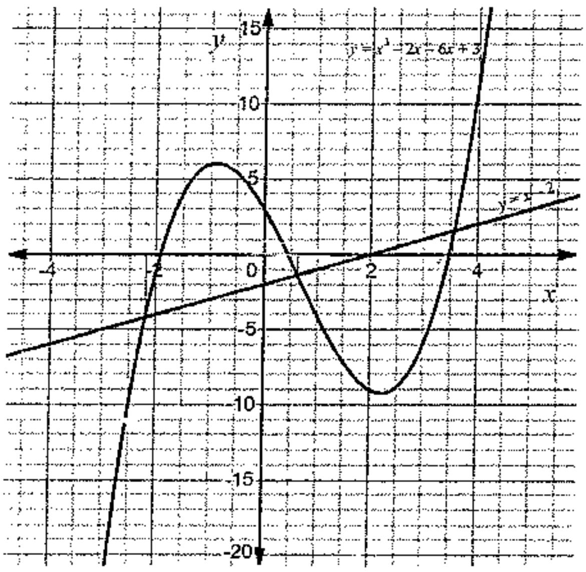
$= 27 - 18 - 18 + 3$

$= 27 - 36 + 3$

$\therefore y = -6$

x	-3	-2	-1	0	1	2	3	4
y	-24	-1	6	3	-4	-9	-6	11

(ii)



(iii) $x^3 - 2x^2 - 7x + 5 = 0$

subtract it from the main equation

$y = x^3 - 2x^2 - 6x + 3$

$-(0 = x^3 - 2x^2 - 7x + 5)$

$y = x - 2$

x	0	2
y	-2	0

The points of intersection of $y = x - 2$ with the main equation provides the solutions to

$x^3 - 2x^2 - 7x + 5 = 0$

The points of intersection give solutions to the equation: $x^3 - 2x^2 - 7x + 5 = 0$.

$\therefore x = -2.2,\ x = 0.6,\ \text{or } x = 3.5$

7. **[2007 PII 8b]**

i. $y = (x+1)(x^2 + x - 6)$

When $x = -1$, then,

$y = [(-1)+1][(-1)^2 + (-1) - 6]$

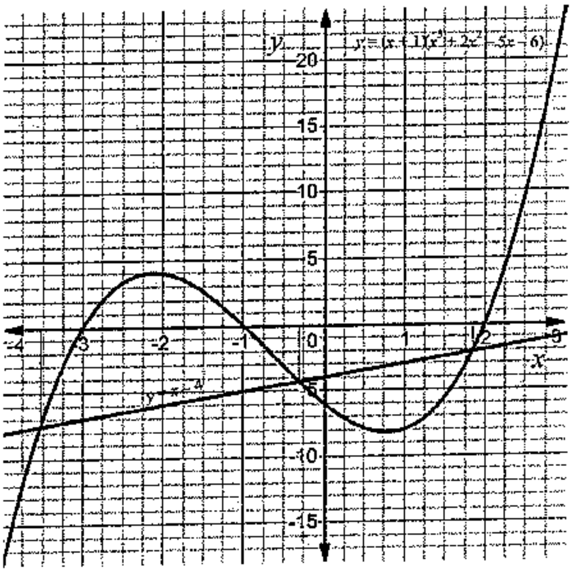
$y = (0)(1 - 1 - 6)$

$y = 0$

x	-4	-3	-2	-1	0	1	2	3
y	-18	0	4	0	-6	-8	0	24

ii.

Graph



iii. Using our graph to solve the simultaneous equations;

$y = x - 4$

$y = x^3 + 2x^2 - 5x - 6$

We first expand $y = (x+1)(x^2 + x - 6)$ in order to compare it with the given expanded equation in question (iii) above:

$$y = (x+1)(x^2 + x - 6)$$

$\Uparrow$ $\Uparrow$

Using **split** and **spread** method;

$= x(x^2 + x - 6) + 1(x^2 + x - 6)$

$= x^3 + x^2 - 6x + x^2 + x - 6$

$= x^3 + 2x^2 - 5x - 6$

We note that the given equation in (iii) $y = x^3 + 2x^2 - 5x - 6$ is the expanded version of the equation $y = (x+1)(x^2 + x - 6)$.

Hence, draw the graph of $y = x - 4$ on the same axis as the graph of $y = x^3 + 2x^2 - 5x - 6$.

The table of values of $y = x - 4$ is given by

x	-2	0	2
y	-6	-4	0

According to the graph above, the two equations intersect at $x = -3.5$, $x = -0.3$, or $x = 1.8$.