Appendix: Proof of recursive description of Gaussian wavefunction

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This appendix offers a proof (missing from the original paper https://arxiv.org/abs/0801.0342) for the recursive description of a Gaussian state. Specifically, we wish to prove that

$$|\xi_{\sigma,\mu,N}\rangle = \sum_{i=0}^{2^N-1} \xi_{\sigma,\mu,N}(i)|i\rangle \,,$$

where

$$\begin{aligned}
\xi^2_{\sigma,\mu,k}(i) &= \sum_{j=-\infty}^{\infty} \tilde{\psi}^2_{\sigma,\mu}(i + j \cdot 2^k) \\
\tilde{\psi}_{\sigma,\mu}(i) &= \frac{1}{\sqrt{f(\sigma,\mu)}} e^{-\frac{(i-\mu)^2}{2\sigma^2}} \\
f(\sigma,\mu) &= \sum_{n=-\infty}^{\infty} e^{-\frac{(n-\mu)^2}{\sigma^2}}
\end{aligned}$$

satisfies the recursion relation

$$|\xi_{\sigma,\mu,N}\rangle = \begin{cases} \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle & ,\mathrm{N}=1 \\ |\xi_{\frac{\sigma}{2},\frac{\mu}{2},N-1}\rangle \otimes \cos\alpha\,|0\rangle \;+\; |\xi_{\frac{\sigma}{2},\frac{\mu-1}{2},N-1}\rangle \otimes \sin\alpha\,|1\rangle & ,\mathrm{N}>1 \end{cases} \tag{1}$$

where

$$\alpha = \cos^{-1}\sqrt{f\left(\frac{\sigma}{2},\frac{\mu}{2}\right)/f(\sigma,\mu)}$$

Useful identities

Before we delve any further, it is useful to first note that

$$\begin{aligned}
f\left(\frac{\sigma}{2},\frac{\mu}{2}\right) + f\left(\frac{\sigma}{2},\frac{\mu-1}{2}\right) &= \sum_{n=-\infty}^{\infty} e^{-\frac{(n-\frac{\mu}{2})^2}{(\sigma/2)^2}} + \sum_{n=-\infty}^{\infty} e^{-\frac{(n-\frac{\mu-1}{2})^2}{(\sigma/2)^2}} \\
&= \sum_{n=-\infty}^{\infty} e^{-\frac{(2n-\mu)^2}{\sigma^2}} + \sum_{n=-\infty}^{\infty} e^{-\frac{(2n+1-\mu)^2}{\sigma^2}} \\
&= \sum_{\text{even } n} e^{-\frac{(n-\mu)^2}{\sigma^2}} + \sum_{\text{odd } n} e^{-\frac{(n-\mu)^2}{\sigma^2}} \\
&= \sum_{n=-\infty}^{\infty} e^{-\frac{(n-\mu)^2}{\sigma^2}} \\
&= f(\sigma,\mu)
\end{aligned} \tag{2}$$

and also that

$$\sum_{i=0}^{2^{N-1}-1} g(2i)|i\rangle \otimes |0\rangle + \sum_{i=0}^{2^{N-1}-1} g(2i+1)|i\rangle \otimes |1\rangle = \sum_{i=0,2,4,...}^{2^N-2} g(i)|i\rangle + \sum_{i=1,3,5,...}^{2^N-1} g(i)|i\rangle$$

$$= \sum_{i=0}^{2^N-1} g(i)|i\rangle \qquad (3)$$

since the index of $|i\rangle \otimes |0\rangle$ is $... + 0 \times 2^0$, an even number; while the index of $|i\rangle \otimes |1\rangle$ is $... + 1 \times 2^0$, an odd number.

Base Case

First, we wish to show that (for N=1)

$$\xi_{\sigma,\mu,1}(0)|0\rangle + \xi_{\sigma,\mu,1}(1)|1\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle$$

To do this, it is sufficient to demonstrate that the coefficients of $|0\rangle$ and $|1\rangle$ match on either side of the above equation. For the coefficient of $|0\rangle$, we can see that

$$\begin{aligned}
\xi_{\sigma,\mu,1}(0) &= \left[\sum_{j=-\infty}^{\infty} \tilde{\psi}^2(2j)\right]^{1/2} \\
&= \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(2j-\mu)^2}{\sigma^2}}}{f(\sigma,\mu)}\right]^{1/2} \\
&= \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(j-\mu/2)^2}{(\sigma/2)^2}}}{f(\sigma,\mu)}\right]^{1/2} \\
&= \left[\sum_{j=-\infty}^{\infty} \frac{f(\sigma/2,\mu/2)}{f(\sigma,\mu)}\right]^{1/2} \\
&= \cos\alpha
\end{aligned}$$

while for the coefficient of $|1\rangle$, we can see that

$$\begin{aligned}
\xi_{\sigma,\mu,1}(1) &= \left[\sum_{j=-\infty}^{\infty} \tilde{\psi}^2(2j+1)\right]^{1/2} \\
&= \left[\sum_{j=-\infty}^{\infty} \frac{e^{\frac{-(2j+1-\mu)^2}{\sigma^2}}}{f(\sigma,\mu)}\right]^{1/2} \\
&= \left[\frac{\sum_{j=-\infty}^{\infty} e^{\frac{-(j-\mu)^2}{\sigma^2}} - \sum_{j=-\infty}^{\infty} e^{\frac{-(2j-\mu)^2}{\sigma^2}}}{f(\sigma,\mu)}\right]^{1/2} \\
&= \left[\frac{f(\sigma,\mu) - f(\sigma/2,\mu/2)}{f(\sigma,\mu)}\right]^{1/2} \\
&= \left[1 - \frac{f(\sigma/2,\mu/2)}{f(\sigma,\mu)}\right]^{1/2} \\
&= \left[1 - \cos^2\alpha\right]^{1/2} \\
&= \sin\alpha
\end{aligned}$$

where in the third line, we have used the fact that an infinite integer sum can be split into its even and odd parts ($\sum_{\text{all } n}(..) = \sum_{\text{even } n}(..) + \sum_{\text{odd } n}(..)$).

Inductive Step

Next, we assume that N > 1 and $|\xi_{\sigma',\mu',N-1}\rangle = \sum_{i=0}^{2^{N-1}-1} \xi_{\sigma',\mu',N-1}(i)|i\rangle$, and using the recurrence relation (1), we wish to show that

$$|\xi_{\sigma,\mu,N}\rangle = \sum_{i=0}^{2^N-1} \xi_{\sigma,\mu,N}(i)|i\rangle$$

Starting with (1), we have

$$
\begin{aligned}
|\xi_{\sigma,\mu,N}\rangle &= \sum_{i=0}^{2^{N-1}-1} \xi_{\frac{\sigma}{2},\frac{\mu}{2},N-1}(i)|i\rangle \otimes \cos\alpha|0\rangle + \sum_{i=0}^{2^{N-1}-1} \xi_{\frac{\sigma}{2},\frac{\mu-1}{2},N-1}(i)|i\rangle \otimes \sin\alpha|1\rangle \\
&= \left(\sum_{i=0}^{2^{N-1}-1} \left[\frac{f\left(\frac{\sigma}{2},\frac{\mu}{2}\right)}{f(\sigma,\mu)} \right]^{1/2} \cdot \left[\sum_{j=-\infty}^{\infty} \tilde{\psi}^2_{\frac{\sigma}{2},\frac{\mu}{2}}(i + j\cdot 2^{N-1}) \right]^{1/2} |i\rangle \otimes |0\rangle \right) \\
&\quad + \left(\sum_{i=0}^{2^{N-1}-1} \left[\frac{f(\sigma,\mu) - f\left(\frac{\sigma}{2},\frac{\mu}{2}\right)}{f(\sigma,\mu)} \right]^{1/2} \cdot \left[\sum_{j=-\infty}^{\infty} \tilde{\psi}^2_{\frac{\sigma}{2},\frac{\mu-1}{2}}(i + j\cdot 2^{N-1}) \right]^{1/2} |i\rangle \otimes |1\rangle \right) \\
&= \left(\sum_{i=0}^{2^{N-1}-1} \left[\frac{f\left(\frac{\sigma}{2},\frac{\mu}{2}\right)}{f(\sigma,\mu)} \right]^{1/2} \cdot \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{\left(i+j\cdot 2^{N-1}-\frac{\mu}{2}\right)^2}{(\sigma/2)^2}}}{f\left(\frac{\sigma}{2},\frac{\mu}{2}\right)} \right]^{1/2} |i\rangle \otimes |0\rangle \right) \\
&\quad + \left(\sum_{i=0}^{2^{N-1}-1} \left[\frac{f\left(\frac{\sigma}{2},\frac{\mu-1}{2}\right)}{f(\sigma,\mu)} \right]^{1/2} \cdot \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{\left(i+j\cdot 2^{N-1}-\frac{\mu-1}{2}\right)^2}{(\sigma/2)^2}}}{f\left(\frac{\sigma}{2},\frac{\mu-1}{2}\right)} \right]^{1/2} |i\rangle \otimes |1\rangle \right) \quad \text{(using (2))} \\
&= \left(\sum_{i=0}^{2^{N-1}-1} \left[\sum_{j=-\infty}^{\infty} \frac{e^{\frac{-(2i+j\cdot 2^N-\mu)^2}{\sigma^2}}}{f(\sigma,\mu)} \right]^{1/2} |i\rangle \otimes |0\rangle \right) \\
&\quad + \left(\sum_{i=0}^{2^{N-1}-1} \left[\sum_{j=-\infty}^{\infty} \frac{e^{\frac{-(2i+1+j\cdot 2^N-\mu)^2}{\sigma^2}}}{f(\sigma,\mu)} \right]^{1/2} |i\rangle \otimes |1\rangle \right) \\
&= \sum_{i=0}^{2^N-1} \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(i+j\cdot 2^N-\mu)^2}{\sigma^2}}}{f(\sigma,\mu)} \right]^{1/2} \quad \text{(using (3))} \\
&= \sum_{i=0}^{2^N-1} \xi_{\sigma,\mu,N}(i)|i\rangle
\end{aligned}
$$

QED.