

Appendix: Proof of recursive description of Gaussian wavefunction

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This appendix offers a proof (missing from the original paper <https://arxiv.org/abs/0801.0342>) for the recursive description of a Gaussian state. Specifically, we wish to prove that

$$|\xi_{\sigma,\mu,N}\rangle = \sum_{i=0}^{2^N-1} \xi_{\sigma,\mu,N}(i) |i\rangle,$$

where

$$\begin{aligned} \xi_{\sigma,\mu,k}^2(i) &= \sum_{j=-\infty}^{\infty} \tilde{\psi}_{\sigma,\mu}^2(i + j \cdot 2^k) \\ \tilde{\psi}_{\sigma,\mu}(i) &= \frac{1}{\sqrt{f(\sigma,\mu)}} e^{-\frac{(i-\mu)^2}{2\sigma^2}} \\ f(\sigma,\mu) &= \sum_{n=-\infty}^{\infty} e^{-\frac{(n-\mu)^2}{\sigma^2}} \end{aligned}$$

satisfies the recursion relation

$$|\xi_{\sigma,\mu,N}\rangle = \begin{cases} \cos \alpha |0\rangle + \sin \alpha |1\rangle & , N = 1 \\ |\xi_{\frac{\sigma}{2}, \frac{\mu}{2}, N-1}\rangle \otimes \cos \alpha |0\rangle + |\xi_{\frac{\sigma}{2}, \frac{\mu-1}{2}, N-1}\rangle \otimes \sin \alpha |1\rangle & , N > 1 \end{cases} \quad (1)$$

where

$$\alpha = \cos^{-1} \sqrt{f\left(\frac{\sigma}{2}, \frac{\mu}{2}\right) / f(\sigma, \mu)}$$

Useful identities

Before we delve any further, it is useful to first note that

$$\begin{aligned} f\left(\frac{\sigma}{2}, \frac{\mu}{2}\right) + f\left(\frac{\sigma}{2}, \frac{\mu-1}{2}\right) &= \sum_{n=-\infty}^{\infty} e^{-\frac{(n-\frac{\mu}{2})^2}{(\sigma/2)^2}} + \sum_{n=-\infty}^{\infty} e^{-\frac{(n-\frac{\mu-1}{2})^2}{(\sigma/2)^2}} \\ &= \sum_{n=-\infty}^{\infty} e^{-\frac{(2n-\mu)^2}{\sigma^2}} + \sum_{n=-\infty}^{\infty} e^{-\frac{(2n+1-\mu)^2}{\sigma^2}} \\ &= \sum_{\text{even } n} e^{-\frac{(n-\mu)^2}{\sigma^2}} + \sum_{\text{odd } n} e^{-\frac{(n-\mu)^2}{\sigma^2}} \\ &= \sum_{n=-\infty}^{\infty} e^{-\frac{(n-\mu)^2}{\sigma^2}} \\ &= f(\sigma, \mu) \end{aligned} \quad (2)$$

and also that

$$\begin{aligned}
\sum_{i=0}^{2^{N-1}-1} g(2i)|i\rangle \otimes |0\rangle + \sum_{i=0}^{2^{N-1}-1} g(2i+1)|i\rangle \otimes |1\rangle &= \sum_{i=0,2,4,\dots}^{2^N-2} g(i)|i\rangle + \sum_{i=1,3,5,\dots}^{2^N-1} g(i)|i\rangle \\
&= \sum_{i=0}^{2^N-1} g(i)|i\rangle
\end{aligned} \tag{3}$$

since the index of $|i\rangle \otimes |0\rangle$ is $\dots + 0 \times 2^0$, an even number; while the index of $|i\rangle \otimes |1\rangle$ is $\dots + 1 \times 2^0$, an odd number.

Base Case

First, we wish to show that (for $N=1$)

$$\xi_{\sigma,\mu,1}(0)|0\rangle + \xi_{\sigma,\mu,1}(1)|1\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle$$

To do this, it is sufficient to demonstrate that the coefficients of $|0\rangle$ and $|1\rangle$ match on either side of the above equation. For the coefficient of $|0\rangle$, we can see that

$$\begin{aligned}
\xi_{\sigma,\mu,1}(0) &= \left[\sum_{j=-\infty}^{\infty} \tilde{\psi}^2(2j) \right]^{1/2} \\
&= \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(2j-\mu)^2}{\sigma^2}}}{f(\sigma, \mu)} \right]^{1/2} \\
&= \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(j-\mu/2)^2}{(\sigma/2)^2}}}{f(\sigma, \mu)} \right]^{1/2} \\
&= \left[\sum_{j=-\infty}^{\infty} \frac{f(\sigma/2, \mu/2)}{f(\sigma, \mu)} \right]^{1/2} \\
&= \cos \alpha
\end{aligned}$$

while for the coefficient of $|1\rangle$, we can see that

$$\begin{aligned}
\xi_{\sigma,\mu,1}(1) &= \left[\sum_{j=-\infty}^{\infty} \tilde{\psi}^2(2j+1) \right]^{1/2} \\
&= \left[\sum_{j=-\infty}^{\infty} \frac{e^{\frac{-(2j+1-\mu)^2}{\sigma^2}}}{f(\sigma,\mu)} \right]^{1/2} \\
&= \left[\frac{\sum_{j=-\infty}^{\infty} e^{\frac{-(j-\mu)^2}{\sigma^2}} - \sum_{j=-\infty}^{\infty} e^{\frac{-(2j-\mu)^2}{\sigma^2}}}{f(\sigma,\mu)} \right]^{1/2} \\
&= \left[\frac{f(\sigma,\mu) - f(\sigma/2,\mu/2)}{f(\sigma,\mu)} \right]^{1/2} \\
&= \left[1 - \frac{f(\sigma/2,\mu/2)}{f(\sigma,\mu)} \right]^{1/2} \\
&= [1 - \cos^2 \alpha]^{1/2} \\
&= \sin \alpha
\end{aligned}$$

where in the third line, we have used the fact that an infinite integer sum can be split into its even and odd parts ($\sum_{\text{all } n}(\cdot) = \sum_{\text{even } n}(\cdot) + \sum_{\text{odd } n}(\cdot)$).

Inductive Step

Next, we assume that $N > 1$ and $|\xi_{\sigma',\mu',N-1}\rangle = \sum_{i=0}^{2^{N-1}-1} \xi_{\sigma',\mu',N-1}(i)|i\rangle$, and using the recurrence relation (??), we wish to show that

$$|\xi_{\sigma,\mu,N}\rangle = \sum_{i=0}^{2^N-1} \xi_{\sigma,\mu,N}(i)|i\rangle$$

Starting with (??), we have

$$\begin{aligned}
|\xi_{\sigma,\mu,N}\rangle &= \sum_{i=0}^{2^{N-1}-1} \xi_{\frac{\sigma}{2}, \frac{\mu}{2}, N-1}(i) |i\rangle \otimes \cos \alpha |0\rangle + \sum_{i=0}^{2^{N-1}-1} \xi_{\frac{\sigma}{2}, \frac{\mu-1}{2}, N-1}(i) |i\rangle \otimes \sin \alpha |1\rangle \\
&= \left(\sum_{i=0}^{2^{N-1}-1} \left[\frac{f\left(\frac{\sigma}{2}, \frac{\mu}{2}\right)}{f(\sigma, \mu)} \right]^{1/2} \cdot \left[\sum_{j=-\infty}^{\infty} \tilde{\psi}_{\frac{\sigma}{2}, \frac{\mu}{2}}^2(i + j \cdot 2^{N-1}) \right]^{1/2} |i\rangle \otimes |0\rangle \right) \\
&\quad + \left(\sum_{i=0}^{2^{N-1}-1} \left[\frac{f(\sigma, \mu) - f\left(\frac{\sigma}{2}, \frac{\mu}{2}\right)}{f(\sigma, \mu)} \right]^{1/2} \cdot \left[\sum_{j=-\infty}^{\infty} \tilde{\psi}_{\frac{\sigma}{2}, \frac{\mu-1}{2}}^2(i + j \cdot 2^{N-1}) \right]^{1/2} |i\rangle \otimes |1\rangle \right) \\
&= \left(\sum_{i=0}^{2^{N-1}-1} \left[\frac{f\left(\frac{\sigma}{2}, \frac{\mu}{2}\right)}{f(\sigma, \mu)} \right]^{1/2} \cdot \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(i+j \cdot 2^{N-1} - \frac{\mu}{2})^2}{(\sigma/2)^2}}}{f\left(\frac{\sigma}{2}, \frac{\mu}{2}\right)} \right]^{1/2} |i\rangle \otimes |0\rangle \right) \\
&\quad + \left(\sum_{i=0}^{2^{N-1}-1} \left[\frac{f\left(\frac{\sigma}{2}, \frac{\mu-1}{2}\right)}{f(\sigma, \mu)} \right]^{1/2} \cdot \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(i+j \cdot 2^{N-1} - \frac{\mu-1}{2})^2}{(\sigma/2)^2}}}{f\left(\frac{\sigma}{2}, \frac{\mu-1}{2}\right)} \right]^{1/2} |i\rangle \otimes |1\rangle \right) \quad (\text{using (??)}) \\
&= \left(\sum_{i=0}^{2^{N-1}-1} \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(2i+j \cdot 2^N - \mu)^2}{\sigma^2}}}{f(\sigma, \mu)} \right]^{1/2} |i\rangle \otimes |0\rangle \right) \\
&\quad + \left(\sum_{i=0}^{2^{N-1}-1} \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(2i+1+j \cdot 2^N - \mu)^2}{\sigma^2}}}{f(\sigma, \mu)} \right]^{1/2} |i\rangle \otimes |1\rangle \right) \\
&= \sum_{i=0}^{2^N-1} \left[\sum_{j=-\infty}^{\infty} \frac{e^{-\frac{(i+j \cdot 2^N - \mu)^2}{\sigma^2}}}{f(\sigma, \mu)} \right]^{1/2} \quad (\text{using (??)}) \\
&= \sum_{i=0}^{2^N-1} \xi_{\sigma,\mu,N}(i) |i\rangle
\end{aligned}$$

QED.