# Calcul numérique des solides et structures non linéaires

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# Numerical solutions of non linear problems in structural dynamics

# Overview

1. Equations of motion
2. Variational formulation
3. Discretisation
4. Solution by implicit methods
5. Solution by explicit methods
6. Convergence of the methods
7. Conclusion

# 1. Equations of motion

# Different possibilities

- Variational principles as for non linear elasticity

  Hamilton's principle

  (also principle of least action, principle of stationnary action, ...)

$$S(\underline{q})=\int_{t_1}^{t_2} L(\underline{q}(t),\dot{\underline{q}}(t),t)\,dt \qquad \left(L=W_{\text{ext}}+W_{\text{kin}}-W_{\text{int}}\right)$$

$$\frac{\delta S}{\delta \underline{q}(t)}=0$$

- Principle of virtual power

$$P_{\text{int}}+P_{\text{ext}}=P_{\text{a}}$$

**In the following, we will use this principle of virtual power**

Can be easier for non conservative forces

# conservation laws and partial differential equations

## Mass conservation (Lagrangian)

$$\int_{D(t)} \rho(\underline{x},t)\,dV = \int_{D(0)} \rho(\underline{X},t)\,J(\underline{X},t)\,dV_0$$

$$= \int_{D(0)} \rho_0(\underline{X})\,dV_0$$

$$\rho(X,t)\,J(X,t) = \rho_0(X)$$

Current value Reference value

# Conservation of momentum (Lagrangian)

$$\underline{f}(t) = \frac{D\underline{p}}{Dt}(t) \quad (Newton's\ \ law)$$

with
$$\underline{f}(t) = \int_{D(0)} \rho_0(\underline{X})\underline{f}(\underline{X},t)dV_0 + \int_{\partial D(0)} \underline{\underline{P}}(\underline{X},t).Nda_0$$
$$= \int_{D(0)} \rho_0(\underline{X})\underline{f}(\underline{X},t)dV_0 + \int_{D(0)} div\,\underline{\underline{P}}(\underline{X},t)dV_0$$

and
$$\frac{D\underline{p}}{Dt}(t) = \frac{D}{Dt}\int_{D(0)} \rho_0(\underline{X})\underline{v}(\underline{X},t)dV_0$$
$$= \int_{D(0)} \rho_0(\underline{X})\frac{\partial \underline{v}}{\partial t}(\underline{X},t)dV_0$$

$$\rho_0(\underline{X})\frac{\partial \underline{v}}{\partial t}(\underline{X},t) = \rho_0(\underline{X})\underline{f}(\underline{X},t) + div\,\underline{\underline{P}}(\underline{X},t)$$

# Static or dynamic ?

By making the equation dimensionless

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = \rho_0 \underline{f} + div\, \underline{\underline{P}}$$

$$\frac{\rho_0 L^2}{E\, T^2} \frac{\partial \underline{\tilde{v}}}{\partial \tilde{t}} = \frac{L}{E} \rho_0 \underline{f} + div\, \underline{\underline{\tilde{P}}}$$

The problem can be considered as static when

$$\frac{\rho_0 L^2}{E\, T^2} \ll 1$$

Or also

$$\left(\frac{L}{c\, T}\right)^2 \ll 1$$

So when the wave propagation is very fast over the distance L

# 2. Variational formulation

Two types of boundary conditions

Displacement conditions $u_i = u_i^d \quad on \quad \partial D_{u_i}$

Cinematically admissible displacement fields

Force conditions $T_i = T_i^d \quad on \quad \partial D_{T_i}$

At each point of the boundary and in each direction only one condition

# Principle of virtual power

Vector space of virtual velocities

$$V = \left\{ \delta \underline{v} \;\middle|\; \delta v_i = 0 \;\; on \;\; \partial D_{u_i} \right\}$$

$$P_{\text{int}} + P_{\text{ext}} = P_a$$

And for all rigid motions $$P_{\text{int}} = 0$$

# Lagrangian description

$$P_a = \int_{D(0)} \rho_0 \frac{\partial \underline{v}}{\partial t} . \delta \underline{v} \, dV_0$$

$$= \int_{D(0)} \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} . \delta \underline{v} \, dV_0$$

Power of acceleration

$$P_{\text{ext}} = \int_{D(t)} \rho \underline{f} . \delta \underline{v} \, dV + \int_{\partial D_T(t)} \underline{T}^d . \delta \underline{v} \, da$$

$$= \int_{D(0)} \rho_0 \underline{f} . \delta \underline{v} \, dV_0 + \int_{\partial D_T(0)} \underline{T}^d . \delta \underline{v} \, dA_0$$

Power of external forces

$$P_{\text{int}} = -\int_{D(t)} \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(\delta \underline{v}) \, dV$$

$$= -\int_{D(0)} \underline{\underline{P}} : \underline{\underline{\nabla \delta \underline{v}}} \, dV_0$$

Power of internal forces

# Principle of virtual power

$$P_{\text{int}} + P_{\text{ext}} = P_a$$

$$\forall\, \delta \underline{v} \in V$$

$$\int_{D(0)} \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} . \delta \underline{v}\, dV_0 \quad = \quad \int_{D(0)} \rho_0 \underline{f} . \delta \underline{v}\, dV_0 + \int_{\partial D_T(0)} \underline{T}^d . \delta \underline{v}\, dA_0$$
$$- \int_{D(0)} \underline{\underline{P}} : \underline{\underline{\nabla \delta v}}\, dV_0$$

Equivalent to the partial differential equation with the boundary conditions

Find $\underline{u}$ regular enough such that

$$P_{\text{int}} + P_{\text{ext}} = P_a$$

$$\forall \delta \underline{v} \in V \qquad -K(\underline{u}, \delta \underline{v}) + F(\delta \underline{v}) = M(\underline{\ddot{u}}, \delta \underline{v})$$

with

$$K(\underline{u}, \delta \underline{v}) = \int_{D(0)} \underline{\underline{P}} : \underline{\underline{\nabla \delta v}} \, dV_0$$

$$F(\delta \underline{v}) = \int_{D(0)} \rho_0 \underline{f} . \delta \underline{v} \, dV_0 + \int_{\partial D_T(0)} \underline{T}^d . \delta \underline{v} \, dA_0$$

$$M(\underline{\ddot{u}}, \delta \underline{v}) = \int_{D(0)} \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} . \delta \underline{v} \, dV_0$$

K bilinear for a linear problem, otherwise only linear relatively to $\delta \underline{v}$

F and M linear in $\delta \underline{v}$

# 3. Discretisation

# Discrete functions

Choose a basis $\{N_i\}_{i=1}^{i=N}$ then

$$\underline{u}_h(\underline{x},t)=\sum_{j=1}^{j=N} \underline{u}_j(t) N_j(\underline{x}), \qquad \underline{u}_j(t) \text{ unknowns to be found}$$

$$\delta \underline{v}_h(\underline{x})=\sum_{i=1}^{i=N} \delta \underline{v}_i N_i(\underline{x}), \qquad \delta \underline{v}_i \text{ arbitrary parameters, virtual velocity}$$

Insert this into the variational formulation

$$-K(\underline{u}_h, \delta \underline{v}_h)+F(\delta \underline{v}_h)=M(\ddot{\underline{u}}_h, \delta \underline{v}_h) \qquad \forall \delta \underline{v}_h \in V_h$$

# Form of the discrete equation

$$
\begin{aligned}
-K\left(\underline{u}_h,\delta\underline{v}_h\right)+F\left(\delta\underline{v}_h\right) &= M\left(\underline{\ddot{u}}_h,\delta\underline{v}_h\right)\\
-K\left(\underline{u}_h,\sum_{i=1}^{i=N}\delta\underline{v}_i N_i(\underline{x})\right)+F\left(\sum_{i=1}^{i=N}\delta\underline{v}_i N_i(\underline{x})\right) &= M\left(\sum_{j=1}^{j=N}\underline{\ddot{u}}_j N_j(\underline{x}),\sum_{i=1}^{i=N}\delta\underline{v}_i N_i(\underline{x})\right)\\
-\sum_{i=1}^{i=N}\delta\underline{v}_i K\left(\underline{u}_h,N_i(\underline{x})\right)+\sum_{i=1}^{i=N}\delta\underline{v}_i F\left(N_i(\underline{x})\right) &= \sum_{i=1}^{i=N}\sum_{j=1}^{j=N}\delta\underline{v}_i\underline{\ddot{u}}_j M\left(N_j(\underline{x}),N_i(\underline{x})\right)\\
\sum_{i=1}^{i=N}\delta\underline{v}_i\underline{F}_i^{\text{int}}+\sum_{i=1}^{i=N}\delta\underline{v}_i\underline{F}_i^{\text{ext}} &= \sum_{i=1}^{i=N}\sum_{j=1}^{j=N}\delta\underline{v}_i\underline{\ddot{u}}_j M_{ij}
\end{aligned}
$$

$$\underline{F}^{\text{int}}+\underline{F}^{\text{ext}}=\underline{\underline{M}}\,\underline{\ddot{u}}$$

$$\underline{F}^{\text{int}}+\underline{F}^{\text{ext}}=0 \qquad \textit{for static problems}$$

# Computation of interior forces

$$
\begin{aligned}
-\int_{D(0)} \underline{\underline{P}} : \underline{\underline{\nabla \delta v}}\, dV_0 &= -\int_{D(0)} P_{ik} \frac{\partial \left( \sum_{j=1}^{j=n} \delta v_{kj} N_j \right)}{\partial X_i} dV_0 \\
&= -\sum_{j=1}^{j=n} \int_{D(0)} P_{ik} \frac{\partial \left( \delta v_{kj} N_j \right)}{\partial X_i} dV_0 \\
&= -\sum_{j=1}^{j=n} \int_{D(0)} B_{ji} P_{ik} \delta v_{kj}\, dV_0 \\
&= \sum_{j=1}^{j=n} f_{kj}^{\text{int}} \delta v_{kj}
\end{aligned}
$$

with

$$
f_{kj}^{\text{int}} = -\int_{D(0)} P_{ik} \frac{\partial N_j}{\partial X_i} dV_0 = -\int_{D(0)} B_{ji} P_{ik}\, dV_0
$$

$$
B_{ji} = \frac{\partial N_j}{\partial X_i}
$$

# Computation of external forces

$$\int_{D(0)} \rho_0 \underline{f} \,.\, \delta \underline{v} \, dV_0 + \int_{\partial D_T(0)} \underline{T}^d \,.\, \delta \underline{v} \, dA_0$$

$$= \int_{D(0)} \rho_0 f_k \sum_{j=1}^{j=n} \delta v_{kj} N_j(\underline{X}) \, dV_0 + \int_{\partial D_T(0)} T_k^d \sum_{j=1}^{j=n} \delta v_{kj} N_j(\underline{X}) \, dA_0$$

$$= \sum_{j=1}^{n} f_{kj}^{ext} \delta v_{kj}$$

*with*

$$f_{kj}^{ext} = \int_{D(0)} \rho_0 f_k N_j(\underline{X}) \, dV_0 + \int_{\partial D_T(0)} T_k^d N_j(\underline{X}) \, dA_0$$

# Computation of acceleration forces

$$\int_{D(0)} \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} . \delta \underline{v}\, dV_0 \quad = \quad \int_{D(0)} \rho_0 \sum_{l=1}^{l=n} \ddot{u}_{kl} N_l(\underline{X}) \sum_{j=1}^{j=n} \delta v_{kj} N_j(\underline{X})\, dV_0$$

$$= \quad \sum_{j=1}^{n} f_{kj}^{a} \delta v_{kj}$$

with

$$f_{kj}^{a} \quad = \quad \int_{D(0)} \rho_0 \sum_{l=1}^{l=n} \ddot{u}_{kl} N_l(\underline{X}) N_j(\underline{X})\, dV_0$$

$$= \quad \sum_{l=1}^{l=n} \int_{D(0)} \rho_0 N_l(\underline{X}) N_j(\underline{X})\, dV_0\, \ddot{u}_{kl} = \sum_{l=1}^{l=n} M_{jl} \ddot{u}_{kl}$$

# Damping

- Can be included into interior forces
- Many possibilities
- For linear viscous damping the force is

$$\underline{F}^{damp} = \underline{\underline{C}}\,\underline{\dot{u}}$$

Several possibilities for the damping matrix $\underline{\underline{C}}$

One of them is the Rayleigh damping

$$\underline{\underline{C}} = a\,\underline{\underline{M}} + b\,\underline{\underline{K}}$$

a and b are constants computed from the values of the damping at the two extremities of the frequency band of interest

# Final equations

$$\underline{F}^{int} + \underline{F}^{ext} = \underline{\underline{M}}\,\ddot{\underline{u}}$$

$$\underline{u}(0) = \underline{u}_0$$

$$\dot{\underline{u}}(0) = \dot{\underline{u}}_0$$

The damping force has been added to the interior force

# Remark

- The mass matrix is constant

$$
\begin{aligned}
M_{jl} &= \int_{D(t)} \rho \, N_l(\underline{x}) N_j(\underline{x}) dV \\
&= \int_{D(0)} \rho_0 N_l(\underline{X}) N_j(\underline{X}) dV_0
\end{aligned}
$$

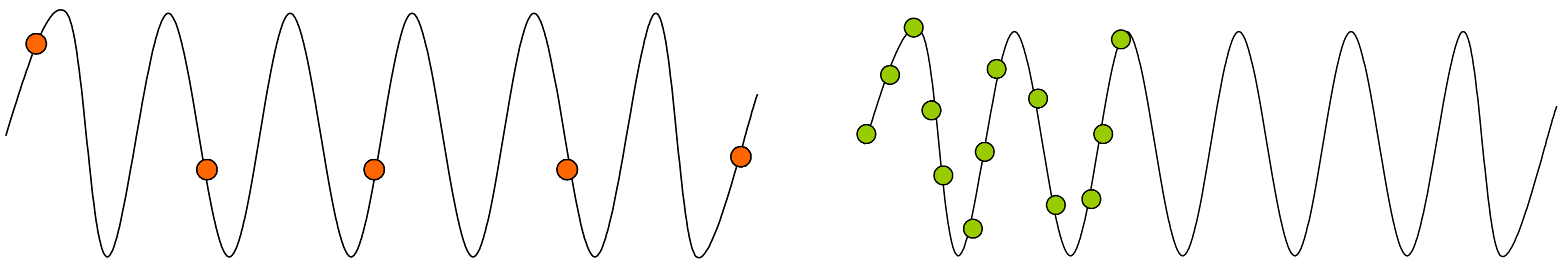
- Calculation of derivatives of shape functions

$$
\frac{\partial N_j}{\partial X_i} = \frac{\partial N_j}{\partial \xi_k} \cdot \frac{\partial \xi_k}{\partial X_i} = \frac{\partial N_j}{\partial \xi_k} \cdot G_{ki}^{-1}
$$

$$
with \quad G_{ki} = \frac{\partial X_k}{\partial \xi_i}
$$

# Mesh refinement

One needs between 5 and 10 nodes by wavelength



Not dense enough

Correct

# 4. Solution by implicit methods

## Integrate equations like

$$\underline{F}^{int} + \underline{F}^{ext} = \underline{\underline{M}}\,\underline{\ddot{u}}$$

$$\underline{u}(0) = \underline{u}_0$$

$$\underline{\dot{u}}(0) = \underline{\dot{u}}_0$$

Discretise in time $\underline{u}_n = \underline{u}(n\Delta t)$

How to go from $\left\{\underline{u}_n, \underline{\dot{u}}_n, \underline{\ddot{u}}_n\right\}$ to $\left\{\underline{u}_{n+1}, \underline{\dot{u}}_{n+1}, \underline{\ddot{u}}_{n+1}\right\}$ ?

The system to solve can be written as

$$0 = \underline{r}\left(\underline{u}^{n+1}, t^{n+1}\right) = s\,\underline{\underline{M}}\,\underline{a}^{n+1} - \underline{f}^{int}\left(\underline{u}^{n+1}, t^{n+1}\right) - \underline{f}^{ext}\left(\underline{u}^{n+1}, t^{n+1}\right)$$

$$\textit{with}$$

$$s = \begin{cases} 0 \;\; \textit{for a statical problem} \\ 1 \;\; \textit{for a dynamical problem (and the damping is included} \\ \textit{in the interior force)} \end{cases}$$

$\underline{r}\left(\underline{u}^{n+1}, t^{n+1}\right)$ is the residue which must equal zero for the solution

# Integration scheme

- There are many possibilities for the integration scheme of these equations
- A popular method is the Newmark scheme

$$\underline{u}^{n+1} = \underline{\tilde{u}}^{n+1} + \beta \Delta t^2 \underline{a}^{n+1}$$

$$\underline{v}^{n+1} = \underline{\tilde{v}}^{n+1} + \gamma \Delta t \, \underline{a}^{n+1}$$

$$\Delta t = t^{n+1} - t^n$$

$$\underline{\tilde{u}}^{n+1} = \underline{u}^n + \Delta t \, \underline{v}^n + \Delta t^2 \left( \frac{1}{2} - \beta \right) \underline{a}^n$$

$$0 \le \beta \le 1/2$$

$$0 \le \gamma \le 1$$

$$\underline{\tilde{v}}^{n+1} = \underline{v}^n + (1 - \gamma) \Delta t \, \underline{a}^n$$

$$if \;\; \beta = 0 \,, \gamma = \frac{1}{2} \;\; central\ difference$$

$$if \;\; \beta = \frac{1}{4} \,, \gamma = \frac{1}{2} \;\; non\ damped\ trapezoidal\ rule$$

$$if \;\; \gamma > \frac{1}{2} \;\; numerical\ damping\ proportionnal\ to \;\; \gamma - \frac{1}{2}$$

$$If \;\; \beta \geq \frac{\gamma}{2} \text{ and } \gamma \geq \frac{1}{2}\ unconditionnaly\ stable$$

One can solve the new accelerations by

$$\underline{u}^{n+1} = \underline{\tilde{u}}^{n+1} + \beta \Delta t^2 \underline{a}^{n+1}$$

$$\underline{a}^{n+1} = \frac{1}{\beta \Delta t^2} \left( \underline{u}^{n+1} - \underline{\tilde{u}}^{n+1} \right)$$

What leads to

$$\begin{aligned} 0 &= \underline{r}\left(\underline{u}^{n+1}, t^{n+1}\right) \\ &= \frac{s}{\beta \Delta t^2} \underline{\underline{M}} \left(\underline{u}^{n+1} - \underline{\tilde{u}}^{n+1}\right) - \underline{f}^{int}\left(\underline{u}^{n+1}, t^{n+1}\right) - \underline{f}^{ext}\left(\underline{u}^{n+1}, t^{n+1}\right) \end{aligned}$$

Non linear system of equations relative to $\underline{u}^{n+1}$

# Solving by the Newton's method

(also Newton-Raphson)

Iterative method on the displacement between $t^n \ and \ t^{n+1}$

very close to the non linear elastic case

And starting at $\underline{u}_0 = \underline{u}^n$

Development of the residue relatively to the displacement around its current value

$$\underline{r}(u_v) + \frac{\partial \underline{r}}{\partial \underline{u}}(u_v)\Delta \underline{u} + O\left(|\Delta \underline{u}|^2\right) = 0$$

$$\underline{\underline{A}}(u_v) = \frac{\partial \underline{r}}{\partial \underline{u}}(u_v)$$

Jacobian matrix

(tangent stiffness)

# Solving by the Newton's method

(also Newton-Raphson)

Iterative method on the displacement between $t^n$ *and* $t^{n+1}$

very close to the non linear elastic case

And starting at $\underline{u}_0 = \underline{u}^n$, then $\underline{u}_{\nu+1} = \underline{u}_\nu + \Delta \underline{u}_\nu$

Development of the residue relatively to the displacement around its current value

$$\underline{r}(\underline{u}_\nu) + \frac{\partial \underline{r}}{\partial \underline{u}}(\underline{u}_\nu)\Delta \underline{u}_\nu + O\left(\left|\Delta \underline{u}_\nu\right|^2\right) = 0$$

$$\underline{\underline{A}}(\underline{u}_\nu) = \frac{\partial \underline{r}}{\partial \underline{u}}(\underline{u}_\nu)$$

Jacobian matrix

(tangent stiffness)

Ignoring the second order terms

$$
\begin{aligned}
\underline{r}\left(\underline{u}_v\right)+\underline{\underline{A}}\left(\underline{u}_v\right)\Delta\,\underline{u}_v &= 0 \\
\underline{\underline{A}}\left(\underline{u}_v\right)\Delta\,\underline{u}_v &= -\underline{r}\left(\underline{u}_v\right) \\
\Delta\,\underline{u}_v &= -\underline{\underline{A}}^{-1}\left(\underline{u}_v\right)\underline{r}\left(\underline{u}_v\right) \\
\underline{u}_{v+1} &= \underline{u}_v-\underline{\underline{A}}^{-1}\left(\underline{u}_v\right)\underline{r}\left(\underline{u}_v\right)
\end{aligned}
$$

One goes on until convergence

# Computation of the jacobian matrix

$$\underline{\underline{A}}\left(\underline{u}_v\right) = \frac{\partial \underline{r}}{\partial \underline{u}}\left(\underline{u}_v\right)$$

$$= \frac{s}{\beta\left(\Delta t\right)^2}\underline{\underline{M}} - \frac{\partial \underline{f}^{int}}{\partial \underline{u}} - \frac{\partial \underline{f}^{ext}}{\partial \underline{u}}$$

$$\underline{\underline{K}}^{int}\left(\underline{u}_v\right) = \frac{\partial \underline{f}^{int}}{\partial \underline{u}} \quad \text{tangent stiffness matrix}$$

$$\underline{\underline{K}}^{ext}\left(\underline{u}_v\right) = \frac{\partial \underline{f}^{ext}}{\partial \underline{u}} \quad \text{loading stiffness matrix}$$

$$\underline{\underline{A}}\left(\underline{u}_v\right) = \frac{s}{\beta\left(\Delta t\right)^2}\underline{\underline{M}} - \underline{\underline{K}}^{int}\left(\underline{u}_v\right) - \underline{\underline{K}}^{ext}\left(\underline{u}_v\right)$$

# Use implicit methods for slow dynamics

Slow dynamics : Time scale of same order as the periods of first modes

- Vibration
- Seismic
- Large deformation of elasto-plastic structures
- Response involving a small number of modes

# 5. Solution by explicit methods

# Central difference method

Simulation time $0 \leq t \leq t_E$

Divided in time steps $\Delta t\,, \quad t^n = n\,\Delta t$

The displacement at time step *n* is denoted $\underline{u}^n = \underline{u}\left(t^n\right)$

For the velocity

$$\dot{\underline{u}}^{n+1/2} = \underline{v}^{n+1/2} = \frac{\underline{u}^{n+1} - \underline{u}^n}{\Delta t}$$

What is equivalent to

$$\underline{u}^{n+1} = \underline{u}^n + \Delta t\,\underline{v}^{n+1/2}$$

For the acceleration

$$\ddot{\underline{u}}^n = \underline{a}^n = \frac{\underline{v}^{n+1/2} - \underline{v}^{n-1/2}}{\Delta t}$$

$$\underline{v}^{n+1/2} = \underline{v}^{n-1/2} + \Delta t\, \underline{a}^n$$

$$\ddot{\underline{u}}^n = \underline{a}^n = \frac{\underline{u}^{n+1} - 2\,\underline{u}^n + \underline{u}^{n-1}}{\Delta t^2}$$

Formula of the central difference for the second derivative

# Integration of the equation of motion

$$\underline{\underline{M}}\,\underline{a}^n = \underline{f}^{ext}\left(\underline{u}^n, t^n\right) + \underline{f}^{int}\left(\underline{u}^n, t^n\right) = \underline{f}^n$$

With the displacement boundary conditions

$$\underline{g}_I\left(\underline{u}^n\right) = 0,\ \ I = 1..n_c$$

Updating the velocity

$$\underline{v}^{n+1/2} = \underline{v}^{n-1/2} + \Delta t \, \underline{\underline{M}}^{-1} \underline{f}^{n}$$

At each time step the velocity is known, then

$$\underline{u}^{n+1} = \underline{u}^{n} + \Delta t \, \underline{v}^{n+1/2}$$

The nodal forces $\underline{f}^{n}$ can be computed from the constitutive relation and the external forces and obtained from $\underline{u}^{n}$

$$\underline{\underline{M}}\,\underline{a}^n = \underline{f}^{ext}\left(\underline{u}^n, t^n\right) + \underline{f}^{int}\left(\underline{u}^n, t^n\right) = \underline{f}^n$$

- If the mass matrix is diagonal one can compute the acceleration without solving any equation
- Idem for the updating of the velocity and the displacement
- The price to pay is that for the method to be stable one has to satisfy

$$\Delta t \leq \Delta t_{crit}$$

Otherwise the solution increases without limit

One has to take $\Delta t \leq \alpha \, \Delta t_{crit}$ with for instance

$$\Delta t_{crit} = \frac{2}{\omega_{\max}} \leq \min_{e,I} \frac{2}{\omega_I^e} = \min_e \frac{l_e}{c_e}$$

With :

- $\omega_{\max}$ maximal pulsation of the linearized system
- $l_e$ characteristic length of element e
- $c_e$ current wave velocity in element e

$0.8 \leq \alpha \leq 0.98$

To take into account the destabilizing effect of non linearities

# summary

$$\underline{f}^{n} = \underline{f}^{ext}\left(\underline{u}^{n}, t^{n}\right) + \underline{f}^{int}\left(\underline{u}^{n}, t^{n}\right)$$

$$\underline{v}^{n+1/2} = \underline{v}^{n-1/2} + \Delta t\, \underline{\underline{M}}^{-1} \underline{f}^{n}$$

$$\underline{u}^{n+1} = \underline{u}^{n} + \Delta t\, \underline{v}^{n+1/2}$$

# Use explicit methods for fast dynamics

Fast dynamics : Time scale much shorter than the periods of first modes

- Shock
- Wave propagation
- Response involving medium and high frequencies

Simulation
of explosion

# Vibration of building with contact

# 6. Convergence of the methods

# Conservation of energy

$$\underline{\underline{M}}\,\underline{\ddot{u}} + \underline{\underline{K}}\,\underline{u} = 0$$

Case of a linear problem

$${}^t\underline{\dot{u}}\,\underline{\underline{M}}\,\underline{\ddot{u}} + {}^t\underline{\dot{u}}\,\underline{\underline{K}}\,\underline{u} = 0$$

This can be written as

$$\frac{d}{dt}\left[\mathcal{K}(t) + \mathcal{W}(t)\right] = 0$$

**Is the conservation satisfied for the discrete system ?**

with

$$\mathcal{K}(t) = \frac{1}{2}\,{}^t\underline{\dot{u}}\,\underline{\underline{M}}\,\underline{\dot{u}}$$

$$\mathcal{W}(t) = \frac{1}{2}\,{}^t\underline{u}\,\underline{\underline{K}}\,\underline{u}$$

From the relations

$$
\begin{aligned}
\mathcal{K}(t) &= \frac{1}{2}\, {}^t\underline{\dot{u}}\, \underline{\underline{M}}\, \underline{\dot{u}} \\
\mathcal{W}(t) &= \frac{1}{2}\, {}^t\underline{u}\, \underline{\underline{K}}\, \underline{u}
\end{aligned}
$$

At the discrete level one gets

$$
\begin{aligned}
\mathcal{K}(t_{n+1}) - \mathcal{K}(t_n) &= 2\underline{\dot{u}}_n^s\, \underline{\underline{M}}\, \underline{\dot{u}}_n^d \\
\mathcal{W}(t_{n+1}) - \mathcal{W}(t_n) &= 2\underline{u}_n^s\, \underline{\underline{K}}\, \underline{u}_n^d
\end{aligned}
$$

with

$$
\begin{aligned}
\underline{u}_n^s &= \frac{1}{2}\left(\underline{u}_{n+1} + \underline{u}_n\right) \\
\underline{u}_n^d &= \frac{1}{2}\left(\underline{u}_{n+1} - \underline{u}_n\right)
\end{aligned}
$$

Relations for the Newmark's scheme

$$
\begin{aligned}
0 &= -2\,\underline{u}_n^d + \Delta t\,\underline{\dot{u}}_n^s - \Delta t\,\underline{\dot{u}}_n^d + \frac{1}{2}\Delta t^2 [\underline{\ddot{u}}_n^s + (4\beta - 1)\underline{\ddot{u}}_n^d] \\
0 &= -2\,\underline{\dot{u}}_n^d + \Delta t [\underline{\ddot{u}}_n^s + (2\gamma - 1)\underline{\ddot{u}}_n^d]
\end{aligned}
$$

By defining the energy like

$$\mathcal{E}(t) = \mathcal{K}(t) + \mathcal{W}(t) + \frac{1}{2}\Delta t^2 \left(\beta - \frac{\gamma}{2}\right) {}^t\underline{\ddot{u}}\,\underline{\underline{M}}\,\underline{\ddot{u}}$$

And after some calculations (exercise), one gets

$$\mathcal{E}(t_{n+1}) - \mathcal{E}(t_n) = 2(1 - 2\gamma)\,{}^t\underline{u}_n^d\,\underline{\underline{K}}\,\underline{u}_n^d + \Delta t^2 (\gamma - 2\beta)(2\gamma - 1)\,{}^t\underline{\ddot{u}}_n^d\,\underline{\underline{M}}\,\underline{\ddot{u}}_n^d$$

# Stability of the discrete scheme

$$\mathcal{E}(t_{n+1}) - \mathcal{E}(t_n) = 2(1-2\gamma)\,{}^t\underline{u}_n^d\,\underline{\underline{K}}\,\underline{u}_n^d + \Delta t^2(\gamma-2\beta)(2\gamma-1)\,{}^t\underline{\ddot{u}}_n^d\,\underline{\underline{M}}\,\underline{\ddot{u}}_n^d$$

- If $\gamma = 1/2$, the energy $\varepsilon(t)$ is constant
- If $\gamma \geq \frac{1}{2}$ and $2\beta - \gamma = 0$, the total energy $\mathcal{K}(t) + \mathcal{W}(t)$ is decreasing
- If $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ the total energy is conserved
- If $\gamma \geq \frac{1}{2}$ and $2\beta > \gamma$ the energy $\varepsilon(t)$ is decreasing
- If $\gamma < \frac{1}{2}$ the energy is increasing if $\gamma \leq 2\beta$
  and the scheme can diverge

# 7. Conclusion

Essentially two possibilities:

- Implicit methods:
  - ✔can be inconditionaly stable
  - ✔large time steps
  - ✔complex in each iteration
  - ✔for slow dynamics

- Explicit methods:
  - ✔conditionaly stable
  - ✔small time steps
  - ✔simple in each iteration
  - ✔for fast dynamics

- Several algorithms in each family

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