ECE 317

Digital Signal Processing I (DSP I)

Lecture 8

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* Based on 2023 Slides from Prof. Rashid Ansari.

Topics of last and today’s lectures

- **Last class:**
  - Review of LSI systems and convolution
  - LSI system configurations – Series and Parallel connections
  - Frequency analysis of LSI systems
- **Today’s class:**
  - DTFT computation and examples
  - Ideal filter impulse responses
  - Magnitude and Phase of DTFT
  - Properties of DTFT

DTFT Key definitions

DTFT and Inverse DTFT

- $x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad = \sum_{n=-\infty}^{\infty} x[n]\left(e^{j\omega}\right)^{-n}$
- $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
- z-Transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n](z)^{-n}$

Example of DTFT computation:
Digression for an important basic Math result

- What is the sum $S_N = \sum_{n=0}^{N-1} \alpha^n = ?$
- What is the sum $S = S_\infty = \sum_{n=0}^{\infty} \alpha^n = ?$
- If $\alpha = 1,\ S_N = \sum_{n=0}^{N-1} 1 = 1 + 1 + 1 + \cdots + 1 = N$
- If $\alpha \neq 1,\ S_N = \sum_{n=0}^{N-1} \alpha^n = 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1}$

$$\alpha S_N = \alpha \sum_{n=0}^{N-1} \alpha^n = \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{N-1} + \alpha^N$$

$$S_N - \alpha S_N = 1 - \alpha^N \Rightarrow (1-\alpha) S_N = 1 - \alpha^N$$

$$S_N = \frac{1-\alpha^N}{1-\alpha}, \alpha \neq 1$$

Now, if $|\alpha| < 1$, then $\lim_{N\to\infty} \alpha^N = 0$. Therefore,

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \qquad |\alpha| < 1$$

Example of DTFT computation

- Consider a signal x$[n] = ((\frac{1}{2})^n + (\frac{1}{3})^n)u[n]$
- ↑ Exponential sequences are key building blocks of impulse responses of commonly used filters
- Find the DTFT $X(e^{j\omega})$ of $x[n]$
- $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty}((\frac{1}{2})^n + (\frac{1}{3})^n)u[n]e^{-j\omega n}$

$$= \sum_{n=0}^{\infty}((\frac{1}{2})^n + (\frac{1}{3})^n)e^{-j\omega n} = \sum_{n=0}^{\infty}(\frac{1}{2})^n e^{-j\omega n} + \sum_{n=0}^{\infty}(\frac{1}{3})^n e^{-j\omega n}$$

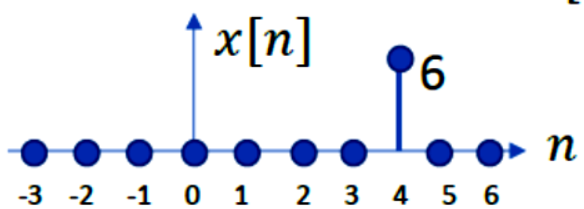
$$= \sum_{n=0}^{\infty}(\frac{1}{2}e^{-j\omega})^n + \sum_{n=0}^{\infty}(\frac{1}{3}e^{-j\omega})^n$$

- Recall: $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ if $|\alpha| < 1$
- Therefore, $X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{1-\frac{1}{3}e^{-j\omega}}$

Inverse DTFT: Examples

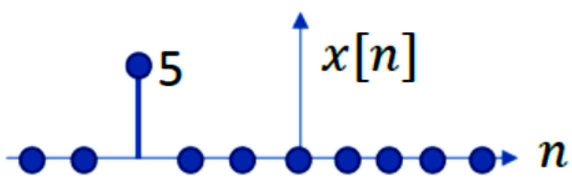
1. $X\left(e^{j\omega}\right) = 6e^{-j4\omega}$ $\Rightarrow x[n] = 6\delta[n-4]$

Equivalent representation of signal: $x[n] = [\,\ldots 0\ \ 0\ \ \underline{0}\ \ 0\ \ 0\ \ 0\ \ 6\ \ 0\ \ 0\ \ \ldots] = [\underline{0}\ \ 0\ \ 0\ \ 0\ \ 6]$



2. $X\left(e^{j\omega}\right) = 5e^{j3\omega} \Rightarrow x[n] = 5\delta[n+3]$

Equivalent representation of signal: $x[n] = [\,\ldots 0\ \ 5\ \ 0\ \ 0\ \ \underline{0}\ \ 0\ \ 0\ \ \ldots] = [5\ \ 0\ \ 0\ \ \underline{0}]$

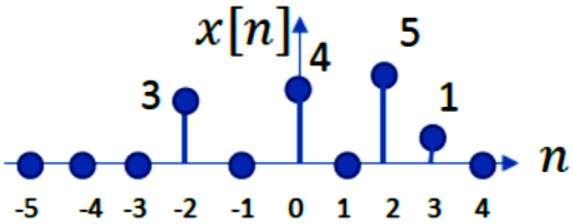


Inverse DTFT: Examples

3. $X(e^{j\omega}) = 3e^{j2\omega} + 4 + 5e^{-j2\omega} + e^{-j3\omega}$

$$\Rightarrow x[n] = 3\delta[n+2] + 4\delta[n] + 5\delta[n-2] + \delta[n-3]$$

Equivalent representation of signal: $x[n] = [\ \dots 0\ \ 0\ \ 3\ \ 0\ \ \underline{4}\ \ 0\ \ 5\ \ 1\ \ 0\ \ 0\ \dots\ ] = [3\ \ 0\ \ \underline{4}\ \ 0\ \ 5\ \ 1]$

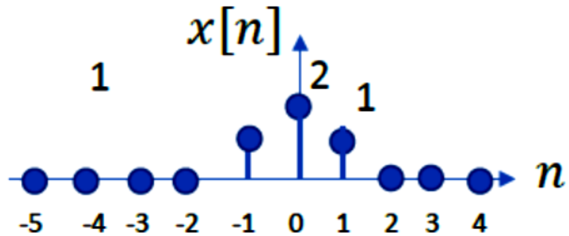


Inverse DTFT: Examples

4. $X(e^{j\omega}) = 2 + 2\cos\omega = 2 + e^{j\omega} + e^{-j\omega}$

$$\Rightarrow x[n] = 2\delta[n] + \delta[n+1] + \delta[n-1]$$

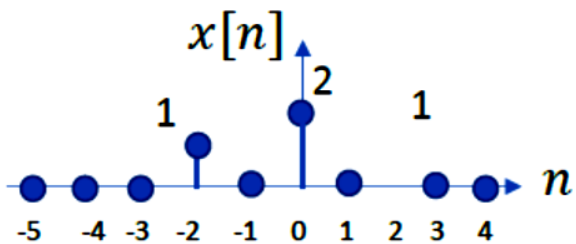
Equivalent representation of signal: $x[n] = [\,\ldots 0\ \ 0\ \ 0\ \ 1\ \ \underline{2}\ \ 1\ \ 0\ \ 0\ \ 0\ \ \ldots] = [1\ \ \underline{2}\ \ 1]$



5. $X(e^{j\omega}) = 2 + 2\cos 2\omega = 2 + e^{j2\omega} + e^{-j2\omega}$

$$\Rightarrow x[n] = 2\delta[n] + \delta[n+2] + \delta[n-2]$$

Equivalent representation of signal: $x[n] = [\,\ldots 0\ \ 0\ \ 1\ \ 0\ \ \underline{2}\ \ 0\ \ 1\ \ 0\ \ 0\ \ \ldots] = [1\ \ 0\ \ \underline{2}\ \ 0\ \ 1]$



Inverse DTFT

Inverse DTFT

Given the DTFT of a signal, the signal can be recovered from it using the so-called inverse DTFT given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

We will use the fact that

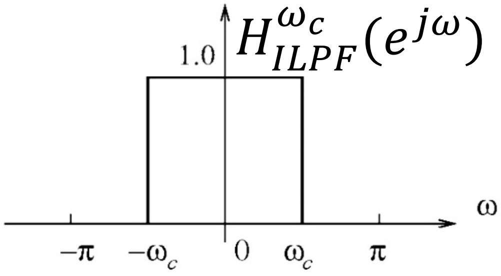
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega = \delta[n-m] = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

The validity of the above relation can be seen from

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m})e^{j\omega n}d\omega$$

$$= \sum_{m=-\infty}^{\infty} x[m]\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] = x[n].$$

Inverse DTFT of an ideal lowpass filter frequency response



$$h_{ILPF}^{\omega_c}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{ILPF}^{\omega_c}(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1.\, e^{j\omega n} d\omega$$

$$n = 0: \qquad \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1.\ e^{j\omega n} d\omega = \frac{\omega_c}{\pi}$$

$$n \neq 0: \qquad \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\frac{e^{j\omega n}}{jn}\Big|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi jn}$$

$$= \frac{2j \sin \omega_c n}{2\pi jn} = \frac{\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n}$$

$$h_{ILPF}^{\omega_c}[n] = \frac{\omega_c}{\pi}\operatorname{sinc}(\frac{\omega_c}{\pi} n)$$

Sinc function notation: Inverse DTFT of ILPF frequency response

$$n = 0: \quad h_{ILPF}^{\omega_c}[n] = \frac{\omega_c}{\pi}$$

$$n \neq 0: \quad h_{ILPF}^{\omega_c}[n] = \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}$$

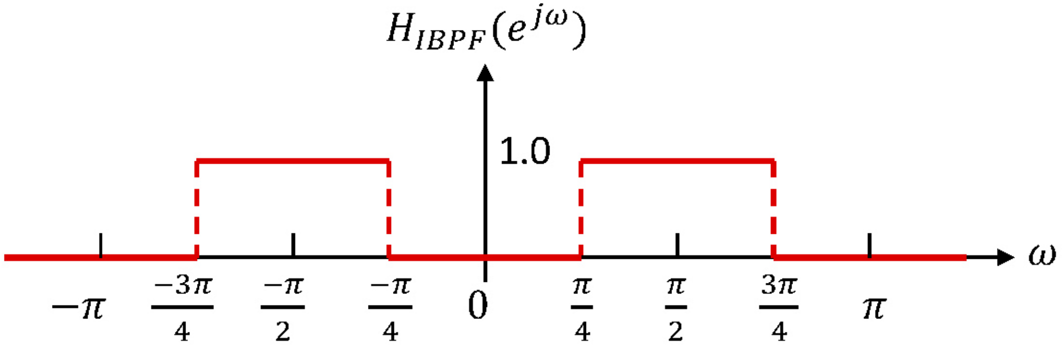
$$\text{sinc}\theta \triangleq \begin{cases} 1, & \theta = 0, \\ \frac{\sin(\pi\theta)}{\pi\theta}, & \theta \neq 0. \end{cases}$$

$$h_{ILPF}^{\omega_c}[n] = \frac{\omega_c}{\pi} \text{sinc}(\frac{\omega_c}{\pi} n)$$

Convenient to use the notation $H_{ILPF}^{\omega_c}(e^{j\omega})$ or $H_{LPF}^{\omega_c}\left(e^{j\omega}\right)$, and $h_{ILPF}^{\omega_c}[n]$ or $h_{LPF}^{\omega_c}[n]$

Inverse DTFT example: Ideal bandpass filter

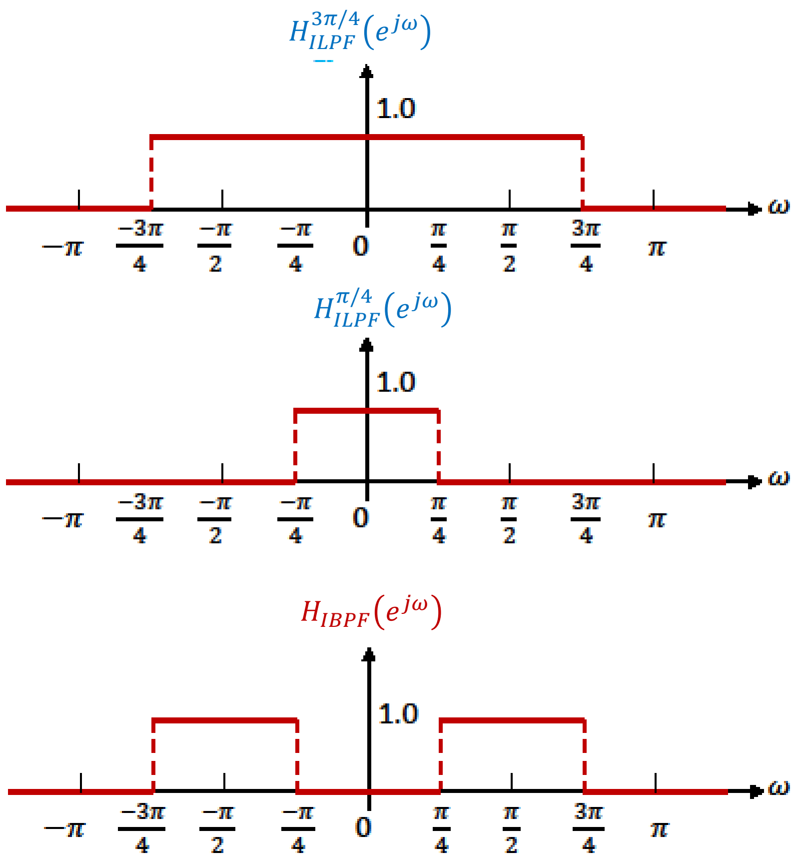
An ideal bandpass filter (LSI system) has the frequency response $H_{IBPF}(e^{j\omega})$ shown below:



(a) Express $H_{IBPF}(e^{j\omega})$ in terms of ideal lowpass filter frequency responses $H_{ILPF}^{\omega_c}(e^{j\omega})$ with suitable cut-off frequencies ω_c

(b) Determine the ideal bandpass filter impulse response.

Ideal bandpass filter frequency response using LPF frequency response



$$H_{IBPF}(e^{j\omega}) = H_{ILPF}^{3\pi/4}(e^{j\omega}) - H_{ILPF}^{\pi/4}(e^{j\omega})$$

Ideal bandpass filter impulse response using LPF impulse response: Windowing

The given $H_{IBPF}\left(e^{j\omega}\right)$ can be expressed as

(a) $$H_{IBPF}\left(e^{j\omega}\right) = H_{ILPF}^{3\pi/4}\left(e^{j\omega}\right) - H_{ILPF}^{\pi/4}\left(e^{j\omega}\right)$$

$$h_{ILPF}^{\omega_c}[n] = \frac{\omega_c}{\pi}\operatorname{sinc}\frac{n\omega_c}{\pi} \leftrightarrow H_{ILPF}^{\omega_c}\left(e^{j\omega}\right) = \begin{cases} 1, |\omega| < \omega_c \\ 0, \omega_c < |\omega| \leq \pi \end{cases}$$

(b) $$\Rightarrow h_{IBPF}[n] = h_{LP}^{3\pi/4}[n] - h_{LP}^{\pi/4}[n]$$

$$\Rightarrow h_{IBPF}[n] = \frac{3}{4}\operatorname{sinc}\frac{3n}{4} - \frac{1}{4}\operatorname{sinc}\frac{n}{4}$$

```
% Finite-duration approximation: Truncated/windowed ideal BPF impulse response
M = 20;
n = -M:M;
wc1=pi/4;
wc2=3*pi/4;
hibpf = (wc2/pi)*sinc(n*wc2/pi)-(wc1/pi)*sinc(n*wc1/pi);
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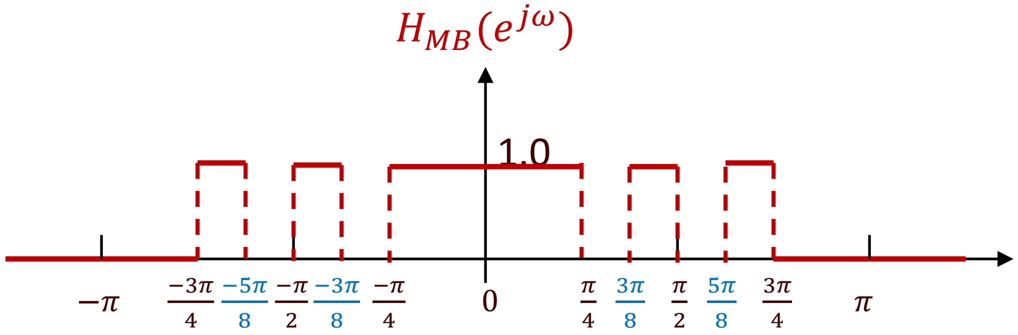
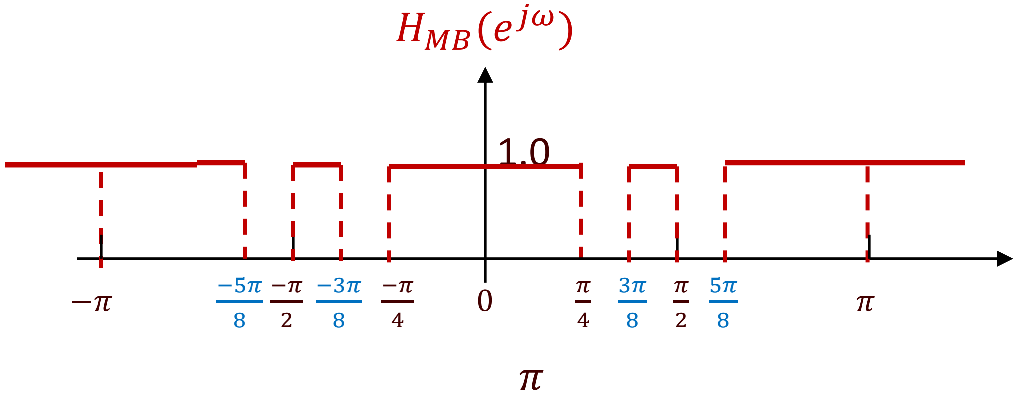
Another Example: Multi-band filter 1



$$H_{MB}(e^{j\omega}) = H_{ILPF}^{\frac{3\pi}{4}}(e^{j\omega}) - H_{ILPF}^{\frac{5\pi}{8}}(e^{j\omega}) + H_{ILPF}^{\frac{\pi}{2}}(e^{j\omega}) - H_{ILPF}^{\frac{3\pi}{8}}(e^{j\omega}) + H_{ILPF}^{\frac{\pi}{4}}(e^{j\omega})$$

$$h_{MB}[n] = \frac{3}{4}\operatorname{sinc}\frac{3n}{4} - \frac{5}{8}\operatorname{sinc}\frac{5n}{8} + \frac{1}{2}\operatorname{sinc}\frac{n}{2} \ldots$$

Another Example: Multi-band filter 2



$$H_{MB}(e^{j\omega}) = 1 - H_{ILPF}^{\frac{5\pi}{8}}(e^{j\omega}) + H_{ILPF}^{\frac{\pi}{2}}(e^{j\omega}) - H_{ILPF}^{\frac{3\pi}{8}}(e^{j\omega}) + H_{ILPF}^{\frac{\pi}{4}}(e^{j\omega})$$

$$h_{MB}[n] = \delta[n] - \frac{5}{8}\mathrm{sinc}\frac{5n}{8} + \frac{1}{2}\mathrm{sinc}\frac{n}{2} \ldots$$

Magnitude and Phase of DTFT

- $X(e^{j\omega})$ is complex in general.
- A complex number z can be expressed as $z = re^{j\theta}$, where r and θ are real, $r \geq 0$
- So $X(e^{j\omega})$ can be expressed in the form
$$X\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right| e^{j\phi(\omega)}$$
- $\left|X\left(e^{j\omega}\right)\right|$ is called the magnitude of the DTFT.
- $\phi(\omega)$ is called the phase of the DTFT

Frequency response $H\left(e^{j\omega}\right)$: Magnitude and Phase response

- $h[n] = \frac{1}{4}[\underline{1} \;\; 2 \;\; 1]$
- $H\left(e^{j\omega}\right) = \frac{1}{4}\left(1 + 2e^{-j\omega} + e^{-j2\omega}\right) = \frac{1}{4}e^{-j\omega}\left(e^{j\omega} + 2 + e^{-j\omega}\right)$
- So $H\left(e^{j\omega}\right) = \frac{1}{4}e^{-j\omega}(2 + 2\cos\omega)$.
- Now note: $(2 + 2\cos\omega)$ is real and non-negative, and $|e^{-j\omega}| = 1$
- So $|H\left(e^{j\omega}\right)| = \frac{1}{4}(2 + 2\cos\omega)$
- Therefore, $H\left(e^{j\omega}\right) = |H\left(e^{j\omega}\right)|e^{j\phi(\omega)} = \frac{1}{4}(2 + 2\cos\omega)e^{-j\omega}$
- Magnitude response: $|H\left(e^{j\omega}\right)| = \frac{1}{4}(2 + 2\cos\omega)$
- Phase response: $\phi(\omega) = -\omega$
- Phase response $\phi(\omega) = -\omega$ is linear in ω (of the form $\phi(\omega) = \beta\omega$).
- This system has a linear phase response.