ECE 317

Digital Signal Processing I (DSP I)

Lecture 7

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* Based on 2023 Slides from Prof. Rashid Ansari.

Topics of last and today's lectures

- **Last class:**
  - Linear shift-invariant (LSI) systems
  - Convolution and examples

- **Today's class:**
  - Review of LSI systems
  - LSI system configurations – Series and Parallel connections
  - Frequency analysis of LSI systems

Key observations about Linear Shift-Invariant (LSI) Systems

$$\delta[n] \longrightarrow \overset{\text{LSI}}{\boxed{T}} \longrightarrow h[n] \qquad x[n] \longrightarrow \overset{\text{LSI}}{\boxed{T}} \longrightarrow y[n] = T(x[n])$$

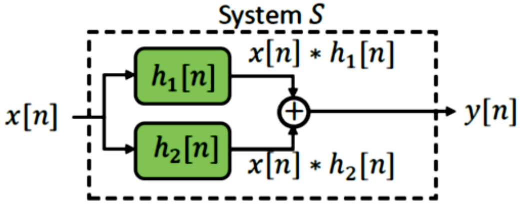
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \overset{\text{LSI}}{\boxed{h[n]}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$\boxed{h[n] \text{ completely characterizes an LSI system}}$

- Requirement for LSI system to be causal: $h[n] = 0$ for $n < 0$
- LSI system stable $\Leftrightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

Systems configured with multiple LSI systems: Example of parallel connection

Consider a system S configured by interconnecting two LSI systems with impulse response $h_1[n]$ and $h_2[n]$:



We can show that this system is LSI. Let $x_1[n]$ and $x_2[n]$ be arbitrary inputs to S. The outputs are:
$y_1[n] = h_1[n] * x_1[n] + h_2[n] * x_1[n]$ and $y_2[n] = h_1[n] * x_2[n] + h_2[n] * x_2[n]$.

Examine the output for input $x[n] = ax_1[n] + bx_2[n]$:
$y[n] = h_1[n] * (ax_1[n] + bx_2[n]) + h_2[n] * (ax_1[n] + bx_2[n])$.

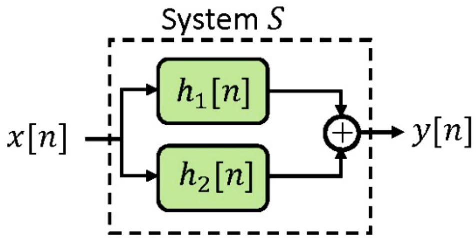
Rearranging terms, we get $y[n] = ay_1[n] + by_2[n]$.
Therefore, the system is linear.

Similarly we can show it is shift-invariant.
Hence the system S is LSI.

LSI systems in parallel

LSI systems in parallel

What is the system impulse response $h[n]$?



Here $h[n]$ is the impulse response of the overall system S

Let $x[n]$ be the input to S.
The output is $y[n] = h_1[n] * x[n] + h_2[n] * x[n]$.

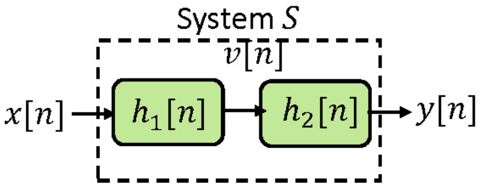
To find the impulse response set $x[n] = \delta[n]$.
The output is the impulse response:
$h[n] = h_1[n] * \delta[n] + h_2[n] * \delta[n]$.

$h[n] = h_1[n] + h_2[n]$.

LSI systems in series

LSI systems in series

What is the system impulse response $h[n]$?



Let $x[n]$ be the input to S.
The output is $y[n] = h_2[n] * v[n]$
$v[n] = h_1[n] * x[n]$.
$y[n] = h_2[n] * v[n] = h_2[n] * h_1[n] * x[n]$ Can be shown to be LSI.

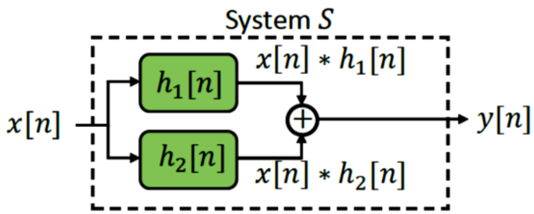
To find the impulse response $h[n]$, set $x[n] = \delta[n]$.
The output is the impulse response:
$h[n] = h_2[n] * h_1[n] * \delta[n]$.
$h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n]$.

Can be extended to other parallel and series configurations.

Summary of Parallel/Series connection of two LSI systems: Overall Impulse response

Consider a system S configured by interconnecting two LSI systems with impulse response $h_1[n]$ and $h_2[n]$:

Series (cascade):

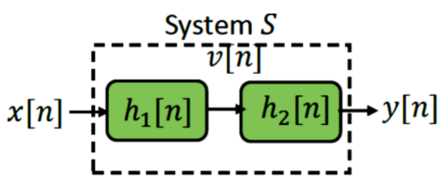


$$x \longrightarrow \boxed{\quad h \quad} \longrightarrow y = x * h$$

Equivalent simplified system with overall impulse response h

$$\boxed{h[n] = h_1[n] + h_2[n].}$$

Parallel:



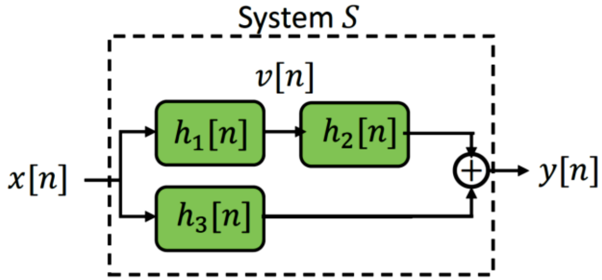
$$x \longrightarrow \boxed{\quad h \quad} \longrightarrow y = x * h$$

Equivalent simplified system with impulse response h

$$\boxed{h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n].}$$

Concepts can be extended to other configurations using different combinations of parallel and series connections.

Example - Parallel+Series connection of 3 LSI systems: Impulse response



Equivalent simplified system with impulse response h

What is the impulse response h of the equivalent system?

$$h[n] = h_1[n] * h_2[n] + h_3[n]$$

Result can be generalized to get the impulse response h of the equivalent system of a series/parallel configuration of any number of LSI systems

LSI systems: Frequency analysis

Recall the signal $x[n] = e^{j\omega n}$ (pure tone of frequency ω). It plays an important role in the frequency analysis of LSI systems.

- **Frequency analysis of LSI systems**

$$x[n] = e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] \quad = \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)} h[k]$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} = H(e^{j\omega}) x[n]$$

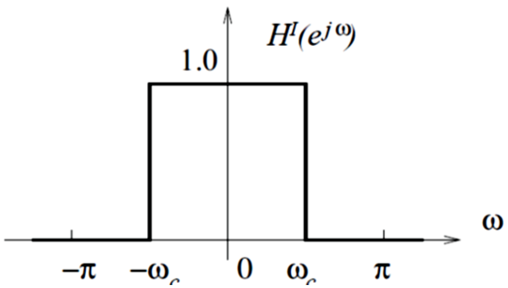
- $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$ is the frequency response

  (also the discrete-time Fourier Transform of $h[n]$)

➡ $e^{j\omega n}$ is called an eigenfunction of the LSI system. It comes out of the LSI system simply scaled by H($e^{j\omega}$). Analogous to an eigenvector.

An important LSI system: Ideal lowpass filter

Frequency response of an ideal lowpass filter: Real-valued with cutoff at $\omega = \omega_c$



We denote the system frequency response by $H(e^{j\omega})$ or $H^I(e^{j\omega})$ to indicate that the response is ideal

$$x[n] = e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(e^{j\omega})x[n]$$

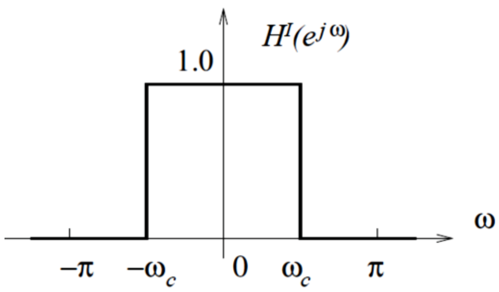
This system passes all pure tones of the form $e^{j\omega_0 n}$ with gain 1 if $|\omega_0| \le \omega_c$ and completely suppresses all pure tones if $\omega_c < |\omega_0| \le \pi$.

What if the input to the system is $\cos n\omega_0$? How do you find the output?

Use Euler's formula: $\cos n\omega = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n})$, a sum of scaled eigenfunctions. Also use the fact that the system is linear.

Ideal lowpass filter with input $x[n] = \cos n\omega_0$

$$x[n] = \cos n\omega_0 \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



Here the input to the system is $x[n] = \cos n\omega_0 = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n})$, the sum of scaled eigenfunctions. Since the system is linear, the output is:

$$y[n] = \frac{1}{2}\left(H\left(e^{j\omega_0}\right)e^{j\omega_0 n} + H\left(e^{-j\omega_0}\right)e^{-j\omega_0 n}\right)$$

Since $H\left(e^{j\omega_0}\right)$ is symmetric, $H\left(e^{j\omega_0}\right) = H\left(e^{-j\omega_0}\right)$. So

$$y[n] = H\left(e^{j\omega_0}\right)\frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) = H\left(e^{j\omega_0}\right)\cos n\omega_0$$

Discrete-time Fourier transform (DTFT)

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Frequency response was defined as

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

This is a special case of the Discrete-time Fourier transform (DTFT) that can be extended to general sequences, not necessarily the impulse response.

DTFT and Inverse DTFT

- $x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
- $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$ ← We will prove this later.

Examples of DTFT

(1) $x[n] = \delta[n]$.

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \delta[0] e^{-j\omega 0} = 1$

(2) $x[n] = 3\delta[n-7]$.

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} 3\delta[n-7] e^{-j\omega n} = 3\delta[0] e^{-j\omega 7} = 3e^{-j\omega 7}$

(3) $x[n] = p_3[n] = \sum_{n=0}^{2} \delta[n-k] = \delta[n] + \delta[n-1] + \delta[n-2] = [\underline{1} \;\; 1 \;\; 1]$.

$X(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega}$.

Formally, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (\sum_{k=0}^{2} \delta[n-k]) e^{-jn\omega} =$

$\sum_{k=0}^{2} (\sum_{n=-\infty}^{\infty} \delta[n-k] e^{-jn\omega}) = \sum_{k=0}^{2} e^{-jk\omega} = 1 + e^{-j\omega} + e^{-j2\omega}$.