ECE 317

Digital Signal Processing I (DSP I)

Lecture 9

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* Based on 2023 Slides from Prof. Rashid Ansari.

Topics of last and today's lectures

- **Last class:**
  - DTFFT computation and examples
  - Ideal filter impulse responses
  - Magnitude and Phase of DTFT
- **Today's class:**
- Properties of DTFT
- Periodicity, linearity, symmetry, signal shift, modulation, DTFT of convolution, DTFT of signal products, Parseval's relation
- DTFT of power signals: $x[n] = e^{j\omega_0 n}, \cos \omega_0 n$

Properties of DTFT

Properties of DTFT

- Sufficient condition for the existence of DTFT
- Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$
- Symmetry
- Linearity property of DTFT
- Shift: If $x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \;\Rightarrow x[n-n_o] \leftrightarrow e^{-j\omega n_o}X(e^{j\omega})$
- Modulation: $e^{j\omega_o n}x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega-\omega_o})$
- Convolution:

  $$(x_1 * x_2)[n] = \text{“}x_1[n] * x_2[n]\text{''} \xleftrightarrow{\text{DTFT}} X_1(e^{j\omega})X_2(e^{j\omega})$$
- Product: $x_1[n]x_2[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega-\theta})X_2(e^{j\theta})d\theta$
- Parseval's relation: $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

Properties of DTFT: Sufficient condition for the existence of DTFT

The discrete-time Fourier Transform (DTFT) is defined for an arbitrary discrete-time signal. The issue of its convergence is briefly examined.

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We assume a sufficient condition for the existence of DTFT. This is done by considering the signal $x[n]$ to be absolutely summable, i.e. $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$. In this case

$$|X(e^{j\omega})| = |\sum_{n=-\infty}^{\infty} e^{-j\omega n} x[k]| \leq \sum_{n=-\infty}^{\infty} |e^{-j\omega n}||x[k]| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Therefore, if the signal $x[n]$ is absolutely summable the series converges and the DTFT exists.

Properties of DTFT: Existence condition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Implication: Sufficient condition for the existence of DTFT

If the signal $x[n]$ is absolutely summable then DTFT $X(e^{j\omega})$ converges to a continuous **function** of ω (result from Math)

If $X(e^{j\omega})$ is not continuous, then $x[n]$ is not absolutely summable!

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Properties of DTFT: Periodicity

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Properties of DTFT: Periodicity

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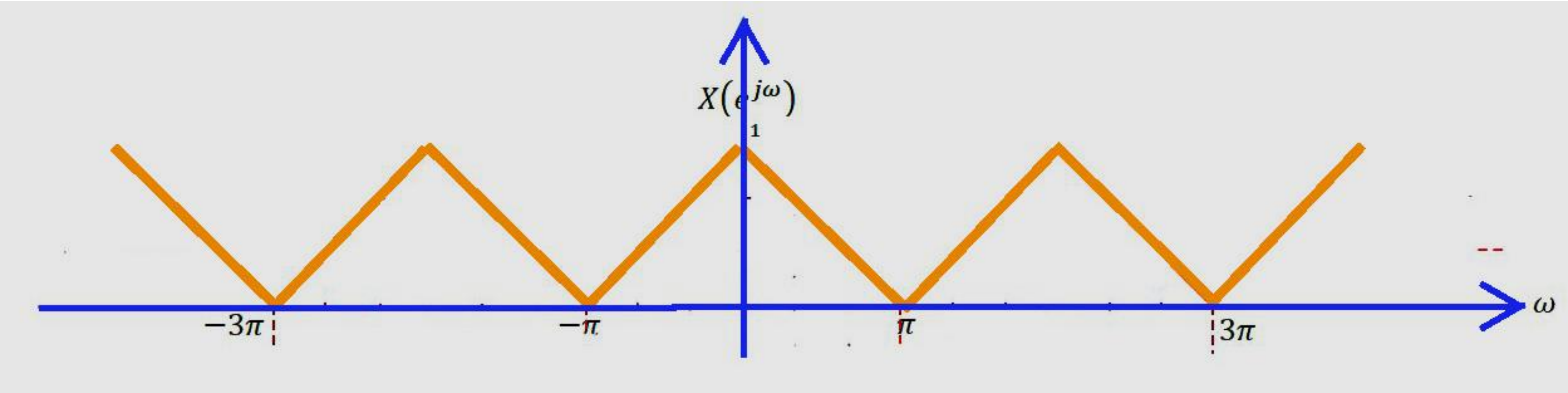
The DTFT is a periodic function in ω with period 2π, since

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi n} = X(e^{j\omega}).$$

(since $e^{-j2\pi n} = \cos 2\pi n + j \sin 2\pi n = 1 + j0 = 1$)

Periodicity of DTFT



Properties of DTFT: Linearity

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A property that follows from the definition is that the DTFT of a linear combination of two or more signals is equal to the same linear combination of the DTFTs of each signal.

$$y[n] = ax_1[n] + bx_2[n] \Leftrightarrow Y(e^{j\omega}) = aX_1(e^{j\omega}) + bX_2(e^{j\omega}).$$

Let $x_1[n] \xleftrightarrow{\text{DTFT}} X_1(e^{j\omega})$ and $x_2[n] \xleftrightarrow{\text{DTFT}} X_2(e^{j\omega})$

$$y[n] = ax_1[n] + bx_2[n] \Leftrightarrow Y(e^{j\omega}) = aX_1(e^{j\omega}) + bX_2(e^{j\omega}).$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} ax_1[n]e^{-j\omega n} + \sum_{n=-\infty}^{\infty} bx_2[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = aX_1(e^{j\omega}) + bX_2(e^{j\omega}).$$

Properties of DTFT: Symmetry of DTFT for real signals

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$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Properties of DTFT: Symmetry of DTFT for real signals

If the signal $\boxed{x[n] \text{ is real } (x[n] = x^*[n])}$, then $\boxed{X(e^{j\omega}) = X^*(e^{-j\omega})}$.
This follows from:

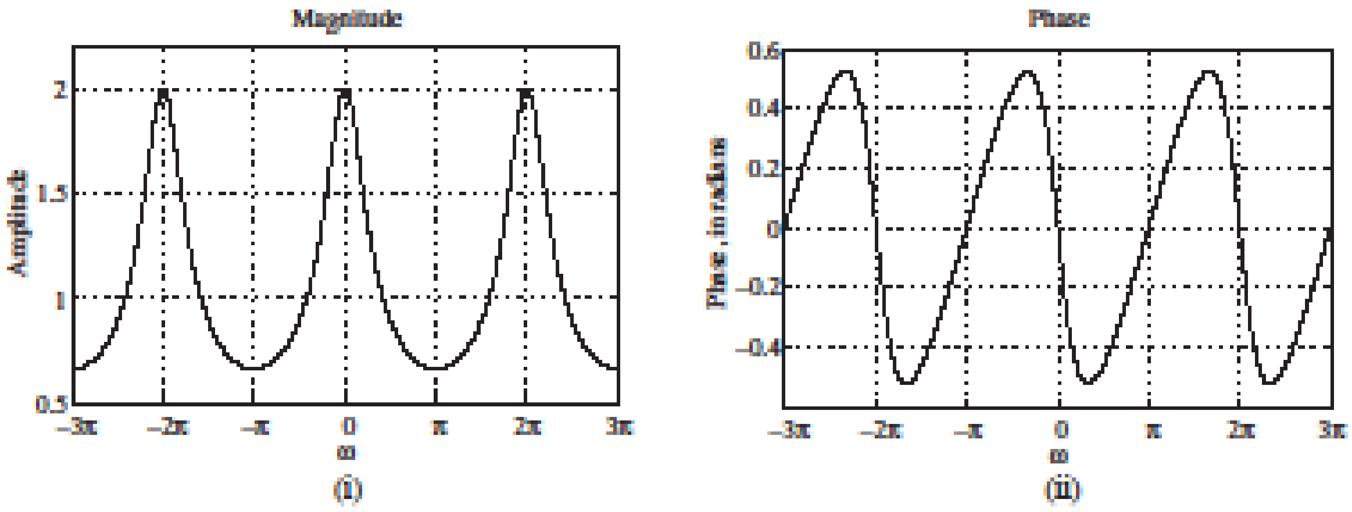
$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}.$$

$$X^*(e^{-j\omega}) = (\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n})^* = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}).$$

If $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\phi(\omega)}$, then $X^*(e^{-j\omega}) = |X(e^{-j\omega})|e^{-j\phi(-\omega)}$.
Therefore for real $x[n]$, $X(e^{j\omega}) = X(e^{-j\omega})$ and $\phi(\omega) = -\phi(-\omega)$, that is the magnitude of the DTFT is an even function of ω, and the phase of the DTFT is an odd function of ω.

Implication of symmetry of DTFT for real-valued sequence $x[n] = 0.5^n u[n]$

- $X(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}}$
- $|X(e^{j\omega})| = |X(e^{-j\omega})|, \quad \phi(\omega) = -\phi(-\omega)$



(i) Magnitude and (ii) Phase of $X(e^{j\omega}) = 1/(1-0.5e^{-j\omega})$.

Properties of DTFT: Time shift of signals

A shift of the signal $x[n]$ in the time domain does not affect the magnitude of the DTFT but produces phase shift that is linear in ω and in proportion to the time shift.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

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This follow from

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n - n_0]e^{-j\omega n}$$

$$= e^{-j\omega n_0} \textstyle\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} = e^{-j\omega n_0} X(e^{j\omega}).$$

Frequency response $H(e^{j\omega})$ with time-shifted impulse response

- $h[n] = \frac{1}{8}[-1 \;\; 2 \;\; \underline{6} \;\; 2 \;\; -1]$, real and $h[-n] = h[n]$
- $H(e^{j\omega}) = \frac{1}{8}(6 + 4\cos\omega - 2\cos 2\omega)$. Note $H(e^{j\omega})$ is real.
- Is the above system causal?
- No.
- This system has a zero-phase frequency response.
- Let $g[n] = h[n-2]$.
- $g[n] = \frac{1}{8}[\underline{-1} \;\; 2 \;\; 6 \;\; 2 \;\; -1]$.
- A system with impulse response $g[n]$ is causal.
- $G(e^{j\omega}) = e^{-j2\omega}\frac{1}{8}(6 + 4\cos\omega - 2\cos 2\omega)$
- This system has linear phase but not zero phase.

Properties of DTFT: Modulation

A modulation of the signal in the time domain by a complex exponential signal corresponds to a shift in the frequency domain. That is

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$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

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This is seen from

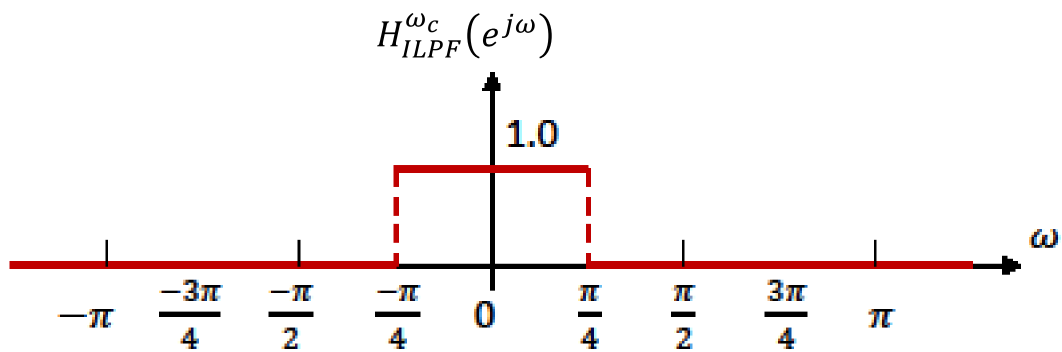
$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega_0 n}e^{j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} = X(e^{j(\omega-\omega_0)}).$$

Modulation property of DTFT: Useful in filter design

- Let us consider a filter with an impulse response $h[n]$
- Corresponding frequency response: $h[n] \leftrightarrow H\left(e^{j\omega}\right)$
- We will modulate $h[n]$ with a cosine function as shown later
- We will not use a complex function $e^{j\omega_0 n}$. Use $\cos \omega_0 n$ instead.
- Let $g[n] = 2\cos \omega_0 n \, h[n] = (e^{j\omega_0 n} + e^{j\omega_0 n})h[n]$
- $G\left(e^{j\omega}\right) = H\left(e^{j(\omega-\omega_0)}\right) + H\left(e^{j(\omega+\omega_0)}\right)$
- Consider the use of modulation designing a bandpass filter starting with a lowpass filter.

Modulation property of DTFT: Useful in filter design

- $h_{ILPF}^{\omega_c}[n] \leftrightarrow H_{ILPF}^{\omega_c}(e^{j\omega})$
- $h_{IBPF}[n] = 2\cos\omega_0 n\, h_{ILPF}^{\omega_c}[n] = (e^{j\omega_0 n} + e^{j\omega_0 n})h_{ILPF}^{\omega_c}[n]$
- $H_{IBPF}(e^{j\omega}) = H_{ILPF}^{\omega_c}(e^{j(\omega-\omega_0)}) + H_{ILPF}^{\omega_c}(e^{j(\omega+\omega_0)})$



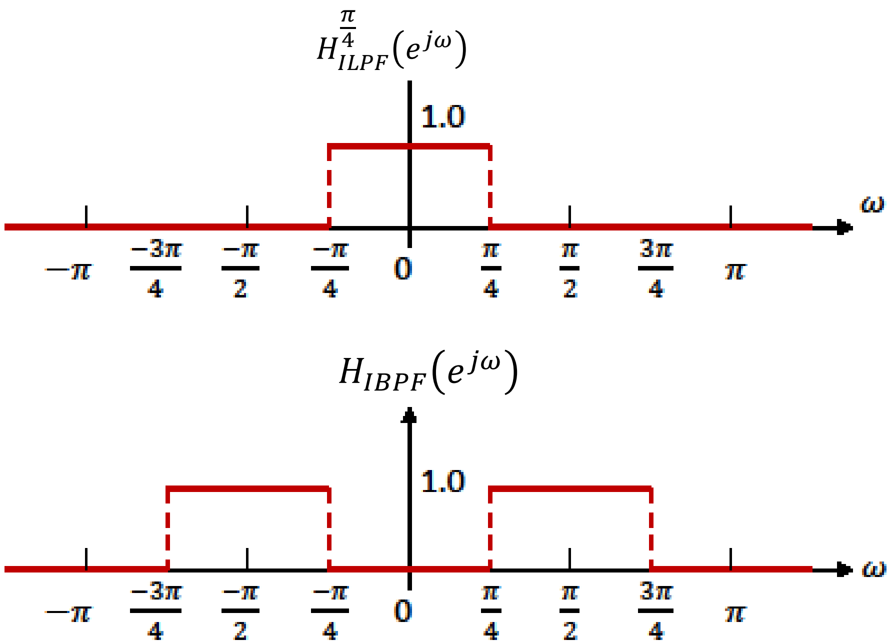
Here $\omega_c = \frac{\pi}{4}$

Now shift this response to $\pm\,\omega_0$ where $\omega_0 = \frac{\pi}{2}$ to get a bandpass frequency response.

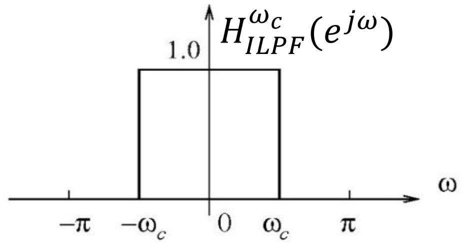
Modulation property of DTFT: Useful in filter design

- $h_{ILPF}^{\frac{\pi}{4}}[n] \leftrightarrow H_{ILPF}^{\frac{\pi}{4}}\left(e^{j\omega}\right)$
- $h_{IBPF}[n] = 2\cos(\omega_0 n)\, h_{ILPF}^{\frac{\pi}{4}}[n] = (e^{j\omega_0 n} + e^{j\omega_0 n})h_{ILPF}^{\frac{\pi}{4}}[n]$
- $H_{IBPF}\left(e^{j\omega}\right) = H_{ILPF}^{\frac{\pi}{4}}\left(e^{j(\omega-\omega_0)}\right) + H_{ILPF}^{\frac{\pi}{4}}\left(e^{j(\omega+\omega_0)}\right)$

Shift response to $\omega_0 = \frac{\pi}{2}$



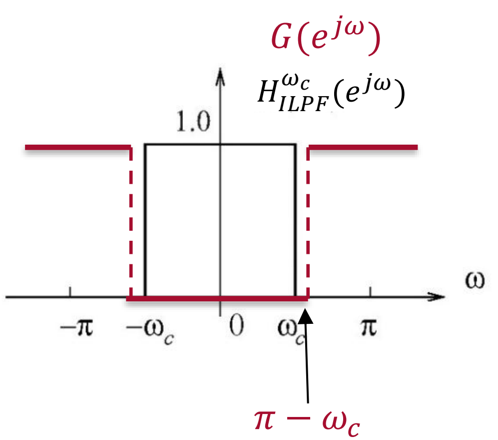
DTFT and properties: Example of modulation



- Example: LPF impulse response $h_{ILPF}^{\omega_c}[n]$. Frequency response shown above.
- Define $g[n] = (-1)^n h_{ILPF}^{\omega_c}[n]$
- In the above expression, the sign of the impulse response samples $h_{ILPF}^{\omega_c}[n]$ with odd indices is reversed.
- Sketch the DTFT $G(e^{j\omega})$ of $g[n]$.
- $G\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty}(-1)^n h_{ILPF}^{\omega_c}[n]e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} h_{ILPF}^{\omega_c}[n]\left(-e^{j\omega}\right)^{-n} = H_{ILPF}^{\omega_c}\left(-e^{j\omega}\right)$$

- Note that $-1 = e^{-j\pi}$, $(-1)^n = e^{-j\pi n}$
- Therefore, $G\left(e^{j\omega}\right) = H_{ILPF}^{\omega_c}(e^{j(\omega-\pi)})$
- Here an LPF is mapped to an HPF



Properties of DTFT: Convolution

The DTFT of the convolution sum of two signals $x_1[n]$ and $x_2[n]$ is the product of their DTFTs, $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$. That is

$$\boxed{y[n] = x_1[n] * x_2[n] \Leftrightarrow Y(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})}$$

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The DTFT of $y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$ is given by

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]e^{-j\omega n} =$$

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$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]e^{-j\omega n} =$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]e^{-j\omega k} \sum_{n=-\infty}^{\infty} x_2[n-k]e^{-j\omega(n-k)} =$$

$$X_1(e^{j\omega})X_2(e^{j\omega}).$$

Convolution property: Input-Output relation of LSI system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- $y[n] = x[n] * h[n]$
- $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- Explains how the LSI system acts on the input frequency content.
- If the system is an ideal lowpass filter, then only the signal content in the passband is passed to the output.

Convolution property: Example

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

- $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- Let $X(e^{j\omega})$ be as defined below, with $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

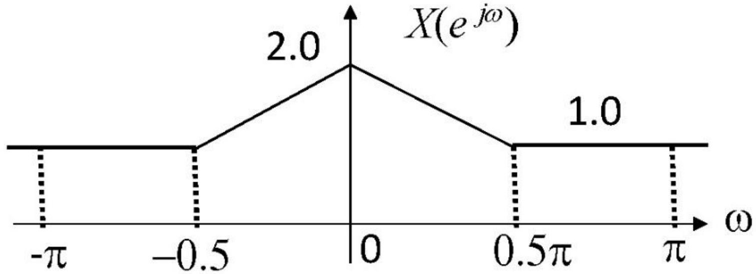
$$X(e^{j\omega}) = \begin{cases} 2\left(1 - \dfrac{|\omega|}{\pi}\right), & |\omega| \leq \dfrac{\pi}{2} \\ 1, & \dfrac{\pi}{2} < |\omega| \leq \pi \end{cases}$$

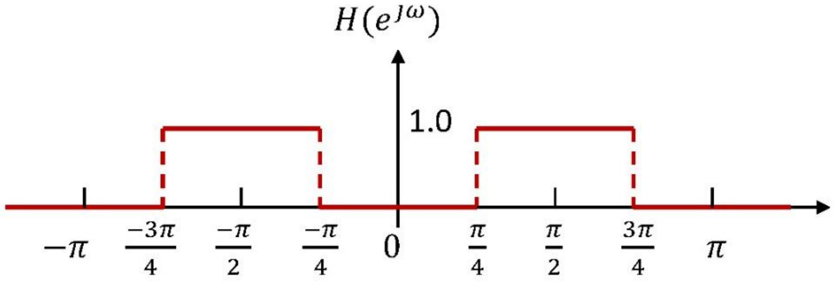
- Let $h[n] = \frac{3}{4}\,\mathrm{sinc}\left(\frac{3}{4}n\right) - \frac{1}{4}\,\mathrm{sinc}\left(\frac{1}{4}n\right)$. Sketch $Y(e^{j\omega})$.
- Now, with our class notation, $H(e^{j\omega}) = H_{ILPF}^{\frac{3\pi}{4}}(e^{j\omega}) - H_{ILPF}^{\frac{\pi}{4}}(e^{j\omega})$

Convolution property: Example

- Let $X(e^{j\omega})$ be as defined below, with $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

- $X(e^{j\omega}) = \begin{cases} 2(1-\frac{|\omega|}{\pi}), |\omega| \le \frac{\pi}{2} \\ 1, \frac{\pi}{2} < |\omega| \le \pi \end{cases}$





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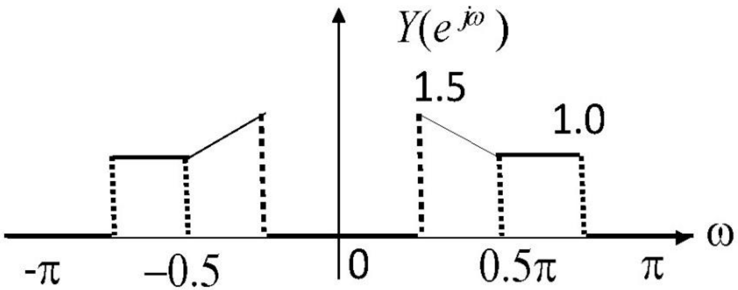

- $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$