# ECE 317

# Digital Signal Processing I (DSP I)

## Lecture 5

Prof. Mojtaba Soltanalian

* Based on 2023 Slides from Prof. Rashid Ansari.

# Topics of last and today's lectures

- **Last class:**
  - Two paradigms for signal processing
  - CT vs DT signal processing
- **Today's class:**
  - DT systems and properties: linearity, shift-invariance, memoryless, causality, stability
  - Linear shift-invariant (LSI) systems

# System Properties (To provide structure on systems)

1. Linearity
2. Shift-Invariance
3. Memoryless (System)
4. Causality
5. Stability

# System Property – Linearity

Consider two inputs $x_1[n]$, $x_2[n]$ to a system.

Let $T\{x_1[n]\} = y_1[n], T\{x_2[n]\} = y_2[n]$.

$x_1[n] \longrightarrow \boxed{T} \longrightarrow y_1[n]$ $\qquad$ $x_2[n] \longrightarrow \boxed{T} \longrightarrow y_2[n]$

The system is linear if the following is true for arbitrary scalars $a$ and $b$ and arbitrary input signals $x_1[n]$, $x_2[n]$.

$$T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n]$$

$x[n] = ax_1[n] + bx_2[n] \longrightarrow \boxed{T} \longrightarrow y[n] = ay_1[n] + by_2[n]$

If input $x[n]$ to a linear system is zero, the output $y[n]$ should be zero.

# Linearity Example 1 with details

$x_1[n] \longrightarrow \boxed{T} \longrightarrow y_1[n]$ $\qquad$ $x_2[n] \longrightarrow \boxed{T} \longrightarrow y_2[n]$

---

$x[n] = ax_1[n] + bx_2[n] \longrightarrow \boxed{T} \longrightarrow y[n] = ay_1[n] + by_2[n]?$

# Linearity Example 1 with details

$x_1[n] \longrightarrow \boxed{T} \longrightarrow y_1[n]$ $\qquad$ $x_2[n] \longrightarrow \boxed{T} \longrightarrow y_2[n]$

---

$x[n] = ax_1[n] + bx_2[n] \longrightarrow \boxed{T} \longrightarrow y[n] = ay_1[n] + by_2[n]?$

Checking Linearity of a System: Example 1

Consider a system with input $x[n]$ and output $y[n] = e^{x[n]}$:

Check if $T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n](?)$

$x_1[n]$, $x_2[n]$ are two arbitrary input signals to a system.

$T\{x_1[n]\} = y_1[n] = e^{x_1[n]}$, $T\{x_2[n]\} = y_2[n] = e^{x_2[n]}$.

Next consider an input $x[n] = ax_1[n] + bx_2[n]$.

The output is $y[n] = e^{x[n]} = e^{ax_1[n]+bx_2[n]} = e^{ax_1[n]}e^{bx_2[n]}$

$\Rightarrow y[n] = (e^{x_1[n]})^a(e^{x_2[n]})^b = (y_1[n])^a(y_2[n])^b$

$\therefore y[n] \neq ay_1[n] + by_2[n]$

So $T\{ax_1[n] + bx_2[n]\} \neq ay_1[n] + by_2[n]$

System in not linear.

# Linearity Example 2 with details

$$x_1[n] \longrightarrow \boxed{T} \longrightarrow y_1[n] \qquad x_2[n] \longrightarrow \boxed{T} \longrightarrow y_2[n]$$

$$x[n] = ax_1[n] + bx_2[n] \longrightarrow \boxed{T} \longrightarrow y[n] = ay_1[n] + by_2[n]?$$

Checking Linearity of a System: Example 2

# Linearity Example 2

$x_1[n] \rightarrow \boxed{T} \rightarrow y_1[n]$ $\qquad$ $x_2[n] \rightarrow \boxed{T} \rightarrow y_2[n]$

---

$x[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{T} \rightarrow y[n] = ay_1[n] + by_2[n]$?

Checking Linearity of a System: Example 2

Consider a system with input $x[n]$ and output $y[n] = x[n] - x[n-1]$:

Check if $T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n]$(?)

$x_1[n]$, $x_2[n]$ are two arbitrary input signals to a system.

Outputs are

$y_1[n] = T\{x_1[n]\} = x_1[n] - x_1[n-1]$

$y_2[n] = T\{x_2[n]\} = x_2[n] - x_2[n-1]$.

Next consider an input $x[n] = ax_1[n] + bx_2[n]$.

Output $y[n] = x[n] - x[n-1]$.

So $y[n] = ax_1[n] + bx_2[n] - ax_1[n-1] - bx_2[n-1]$.

$\therefore y[n] = a(x_1[n] - x_1[n-1]) + b(x_2[n] - x_2[n-1])$.

$\therefore y[n] = ay_1[n] + by_2[n]$

So $T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n]$

System in linear.

# Linearity: Often easier to check with additivity and homogeneity (scaling)

- For a system to be linear, it must satisfy both the additivity and homogeneity properties:
- Additivity:

  If $T\{x_1[n]\} = y_1[n]$ and $T\{x_2[n]\} = y_2[n]$

  $\rightarrow T\{x_1[n] + x_2[n]\} = y_1[n] + y_2[n]$

  means that the system satisfies the additivity property.
- Homogeneity or Scaling:

  If $T\{x[n]\} = y[n] \rightarrow T\{ax[n]\} = ay[n]$

  means that a system satisfies the homogeneity (scaling) property.
- If system is linear, then homogeneity with $a = 0$ implies:

  If $T\{x[n]\} = y[n]$ then $T\{0.x[n]\} = 0.y[n]$

  that is, an input sequence that is zero produces a zero output.

# Linearity: Often easier to check with additivity and homogeneity (scaling)

- So one implication of homogeneity: If input to a linear system is

  $x[n] = 0,$ the output should be $y[n] = 0$

- Consider the example $y[n] = e^{x[n]}$.

  If $x[n] = 0$ for all $n$, then $y[n] = 1$ for all $n$.

  So the system is not linear

- Consider the example $y[n] = x[n] + 3$.

  If $x[n] = 0$ for all $n$, then $y[n] = 3$ for all $n$.

  So the system is not linear

# Linearity – Intuitive understanding

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

Consider some example transformations $T$

$y[n]$ can be expressed in terms of different functions of $x[n], x[n-1], x[n-2], \ldots, x[n+1], x\{n+2], \ldots$ In addition, it may be expressed in terms of $y[n-1], y[n-2], \ldots,$

1. $y[n] = ax[n]$
2. $y[n] = \frac{1}{4}x[n-3] + \frac{1}{2}x[n] + \frac{1}{4}x[n+3]$
3. $y[n] = 5x^3[n+2]$
4. $y[n] = e^{x[n]}$
5. $y[n] = x[n] - x[n-1]$
6. $y[n] = n^2 x[n]$
7. $y[n] = x[n] + 3$
8. $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

1. *Yes*
2. *Yes*
3. *No*
4. *No*
5. *Yes*
6. *Yes*
7. *No*
8. *Yes*

# System Property – Shift-Invariance

Let $x[n]$ be an input to a system and $y[n]$ be the corresponding output: $y[n] = T\{x[n]\}$.

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

Next consider an input $x_1[n] = x[n - n_0]$ to the system and $y_1[n]$ be the corresponding output: $y_1[n] = T\{x_1[n]\}$.
The system is shift-invariant if the following is true for an arbitrary integer $n_0$ and arbitrary input signals $x[n]$:

$$x_1[n] = x[n - n_0] \longrightarrow \boxed{T} \longrightarrow y_1[n] = y[n - n_0]$$

$y_1[n] = y[n - n_0]$, i.e. $T\{x[n - n_0]\} = y[n - n_0]$.

# Shift Invariance: Example

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

---

$$x_1[n] = x[n - n_0] \longrightarrow \boxed{T} \longrightarrow y_1[n] = y[n - n_0]$$

# Shift Invariance: Example

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

$$x_1[n] = x[n-n_0] \longrightarrow \boxed{T} \longrightarrow y_1[n] = y[n-n_0]$$

Checking Shift Invariance of a System: Example

Consider a system with input $x[n]$ and output $y[n] = nx[n]$:

Check if $T\{x[n-n_0]\} = y[n-n_0]$.(?)

Let $x[n]$ be an an arbitrary input to a system.

The corresponding output is $y[n] = T\{x[n]\} = nx[n]$.

Now consider an input $x_1[n] = x[n-n_0]$ to the system.

The corresponding output is $y_1[n] = nx_1[n] = nx[n-n_0]$.

Question: Is $y_1[n] = y[n-n_0]$?

Since $y[n] = nx[n]$, $y[n-n_0] = [n-n_0]x[n-n_0]$.

So $y_1[n] \neq y[n-n_0]$

$\Rightarrow$ System in not shift-invariant (It is shift-varying).

# Shift Invariance: Example (Cont'd)

# Shift Invariance: Example (Cont'd)

Checking Shift Invariance of a System: Example (continued)

Again consider a system with input $x[n]$ and output $y[n] = nx[n]$:

Observation: Output depends on a time-varying coefficient $n$ in $y[n] = nx[n]$.

Sometimes a simple counter-example may be used.
Consider an input $x[n] = \delta[n]$.
Then $y[n] = T(\delta[n]) = n\delta[n] = 0$.
Next consider an input $x_1[n] = \delta[n-1]$
Now the output is $y_1[n] = T(\delta[n-1]) = n\delta[n-1] = \delta[n-1]$

So $y_1[n] \neq y[n-n_0]$
$\Rightarrow$ System in not shift-invariant (It is shift-varying).

# Shift Invariance – Intuitive understanding (system parameters change with time)

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

Consider some example transformations $T$

$y[n]$ can be expressed in terms of different functions of $x[n], x[n-1], x[n-2], \ldots, x[n+1], x\{n+2], \ldots$ In addition, it may be expressed in terms of $y[n-1], y[n-2], \ldots,$

1. $y[n] = ax[n]$, $a$ is a constant
2. $y[n] = nx[n]$
3. $y[n] = \frac{1}{4}x[n-3] + \frac{1}{2}x[n] + \frac{1}{4}x[n+3]$
4. $y[n] = 5x^3[n]$
5. $y[n] = e^{x[n]}$
6. $y[n] = e^{nx[n]}$
7. $y[n] = x[n] + 3$
8. $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

1. *Yes*
2. *No*
3. *Yes*
4. *Yes*
5. *Yes*
6. *No*
7. *Yes*
8. *Yes*

# System Property – Memoryless and Causal systems

# System Property – Memoryless and Causal systems

Memoryless system:

A system is memoryless if the output $y[n_0]$ at "instant" $n_0$ depends only on input sample $x[n_0]$.

- $y[n] = x^2[n] + (n+1)^2$. Is the system memoryless?

Yes.

- $y[n] = \frac{1}{2}(x[n-1] + x[n+1])$. Is the system memoryless?

No.

Causality:

Output $y[n_0]$ depends on input samples $\{x[n_0], x[n_0 - 1], x[n_0 - 2], ....\}$ and not on "future" samples $\{x[n_0 + 1], x[n_0 + 2], ....\}$

- $y[n] = \frac{1}{2}(x[n-1] + x[n+1])$. Is the system causal?

No. $y[n_0]$ depends on future sample $x[n_0 + 1]$

- $y[n] = \cos(n+1)x[n]$. Is the system causal?

Yes.

# System Property: Bounded-Input Bounded-Output (BIBO) Stability

- What is a bounded sequence (or DT signal)?
- $x[n]$ is said to be bounded if there exists a positive number (bound) $B$ such that $|x[n]| \leq B$ for all integers $n$
- Question: Is $x[n] = \cos 0.25\pi n$ bounded?
- Question: Is $x[n] = n$ bounded?

- Notation:
  - $B_I$ for bound on input signal
  - $B_O$ for bound on output signal

# System Property – BIBO Stability and example

# System Property – BIBO Stability and example

- Bounded-Input Bounded-Output (BIBO)

(BIBO) Stability:

A system is BIBO stable if the following is true:

If the system input $x[n]$ is bounded for all $n \in \mathbb{Z}$, i.e $|x[n]| \leq B_I < \infty, \forall n \in \mathbb{Z}$, for a given real positive $B_I$

then the output $y[n]$ is bounded for all $n \in \mathbb{Z}$, i.e. $|y[n]| \leq B_O < \infty, \forall n \in \mathbb{Z}$, for some real positive $B_O$

Example 1 (stability)

- $y[n] = e^{x[n]}$, $x[n]$ real. Is the system stable?

Let $|x[n]| \leq B_I < \infty, \forall n \in \mathbb{Z}$.

Now $|y[n]| = |e^{x[n]}| \leq e^{|x[n]|} \leq e^{B_I} \triangleq B_O$.

We found a $B_O$ bounding the output $\Rightarrow$ System is stable.

# BIBO stability example 2

# BIBO stability example 2

(BIBO) Stability:

Example 2 (stability)

- $y[n] = n2^{x[n]}$. Is the system stable?

Let $\mid x[n] \mid \leq B_I < \infty, \forall n \in \mathbb{Z}$.
Now $\mid y[n] \mid = |n2^{x[n]}| = |n||2^{x[n]}|$.
Let $x[n] = B_I$ for all $n \in \mathbb{Z}$. Then $\mid y[n] \mid = |n||2^{B_I}|$.
Suppose we are given any positive real $B_O$, however large.
We can find an $N$ large enough so that $|N||2^{B_I}| > B_O$.
For all $|n| \geq N$, $\mid y[n] \mid = |n||2^{B_I}| \geq |N||2^{B_I}| > B_O$.
System is not stable.

# DT systems: Implication of Linearity and Shift-Invariance

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

Linearity implies:

If $x_1[n] \longrightarrow \boxed{T} \longrightarrow y_1[n]$ and $x_2[n] \longrightarrow \boxed{T} \longrightarrow y_2[n]$

Then $x[n] = ax_1[n] + bx_2[n] \longrightarrow \boxed{T} \longrightarrow$

$$\begin{aligned} y[n] &= T(ax_1[n] + bx_2[n]) \\ &= aT(x_1[n]) + bT(x_2[n]) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Generalization to more sum of than two sequences applied to a linear system:

If the input is $x[n] = \sum_{k=-\infty}^{\infty} a_k x_k[n]$, and $T(x_k[n]) = y_k[n]$, then the output is

$$y[n] = T\left(\sum_{k=-\infty}^{\infty} a_k x_k[n]\right) = \sum_{k=-\infty}^{\infty} a_k T(x_k[n]) = \sum_{k=-\infty}^{\infty} a_k y_k[n]$$

Shift Invariance implies:

If $x_k[n] \longrightarrow \boxed{T} \longrightarrow y_k[n]$ then $x_k[n-n_0] \longrightarrow \boxed{T} \longrightarrow y_k[n-n_0]$

# Recall: Signal representation as a sum of scaled and shifted unit impulse signals

Any signal $x[n]$ can be represented as a sum of scaled and shifted unit impulse signals

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$x[k]$ is a scalar, not a signal/sequence

$\delta[n-k]$ is a signal – an impulse signal with a shift by an integer $k$

$x[n]$ is applied to a linear shift-invariant system:

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

In the special case where the input is an impulse signal:

$$\delta[n] \longrightarrow \boxed{T} \longrightarrow h[n]$$

we will denote the output as $h[n]$, and call $h[n]$ the system impulse response.

# Linear Shift-Invariant (LSI) System

# Linear Shift-Invariant (LSI) System

We will deal primarily with LSI systems in this course. That is, the system is both linear and shift-invariant. Consider a LSI system $T$ characterized by impulse response $h[n]$. That is, $T(\delta[n]) = h[n]$.

Now consider an arbitrary input $x[n]$ to the system:

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n] = T(x[n])$$

Now $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ where $x[k]$ are scalars and $\delta[n-k]$ are shifted imppulse sequences.

So $y[n] \ = T(x[n]) = T(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k])$.

Since $T$ is linear, $y[n] \ = \sum_{k=-\infty}^{\infty} x[k]T(\delta[n-k])$.

Since $T$ is shift-invariant $T(\delta[n-k]) = h[n-k]$.

Therefore $y[n] \ = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.



**System response is completely characterized by its impulse response $h[n]$**

# Linear Shift-Invariant (LSI) System

LSI system is characterized by its impulse response $h[n]$

# Linear Shift-Invariant (LSI) System

LSI system is characterized by its impulse response $h[n]$

- Output of LSI system $y[n]$ is the convolution sum of input $x[n]$ and $h[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$$

Once you know the impulse response $h[n]$, you can determine the output $y[n]$ for any input $x[n]$.

Stability and causality of an LSI system can be inferred from $h[n]$.

- Stable LSI system $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

- Causal LSI system $\Leftrightarrow h[n] = 0, \quad n < 0$

Later, we will see why these two conditions are valid.

- Impulse response duration: FIR (finite-duration impulse response), IIR (infinite-duration impulse response)



Next, we will review how convolution is computed.