# ECE 317

# Digital Signal Processing I (DSP I)

## Lecture 3

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* Based on Slides from Prof. Rashid Ansari.

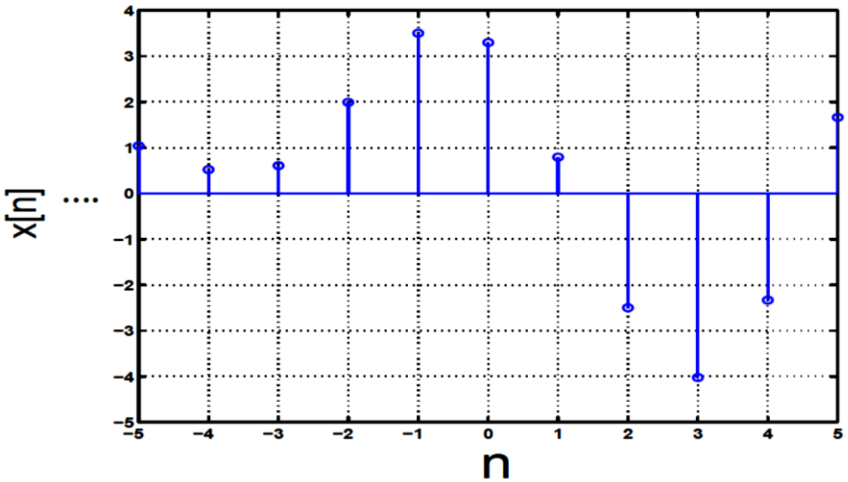
# Topics of last and today’s lectures

- **Last class:**
  - CT and DT signals, DT signal representation
- **Today’s class:**
  - DT signal representation
  - Basic DT signals
  - Complex numbers

# Discrete-Time (DT) Signals and Digital Signals

Discrete-time signal is a sequence that assumes real (or complex) values:

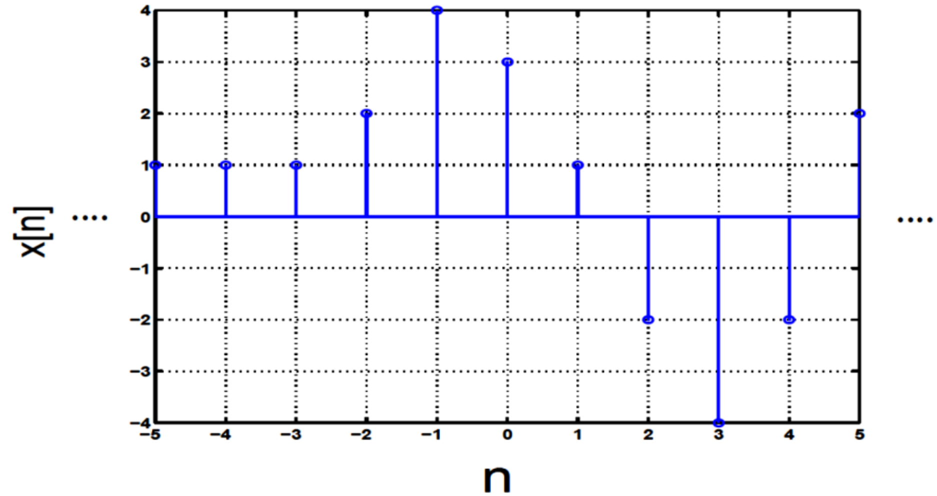
Discrete-time, Continuous-amplitude signal




Example of *discrete-time signal*

Digital signal is a sequence that assumes discrete values

Discrete-time, Discrete-amplitude signal



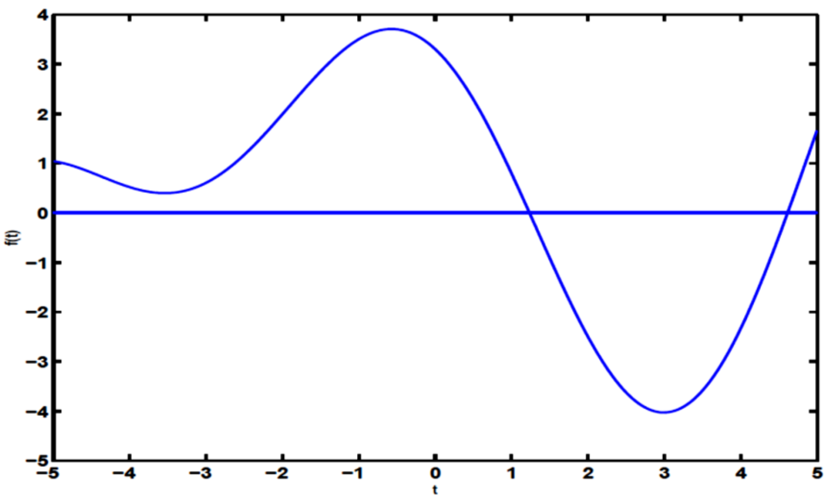

Example of *digital signal*

In practice the terms “discrete-time” and “digital” are loosely used, without making a distinction

# CT, Continuous-Amplitude Signals and CT, Discrete-Amplitude Signals

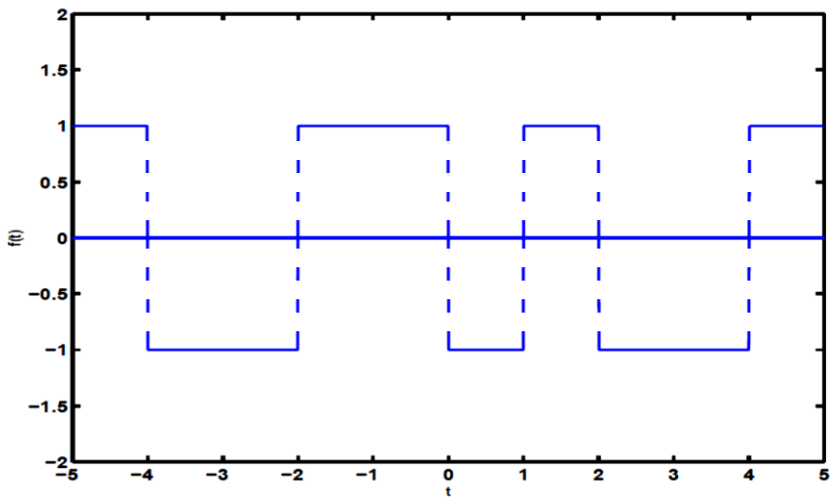
Continuous-time,

Continuous-amplitude signal




Continuous-time,

Discrete-amplitude signal

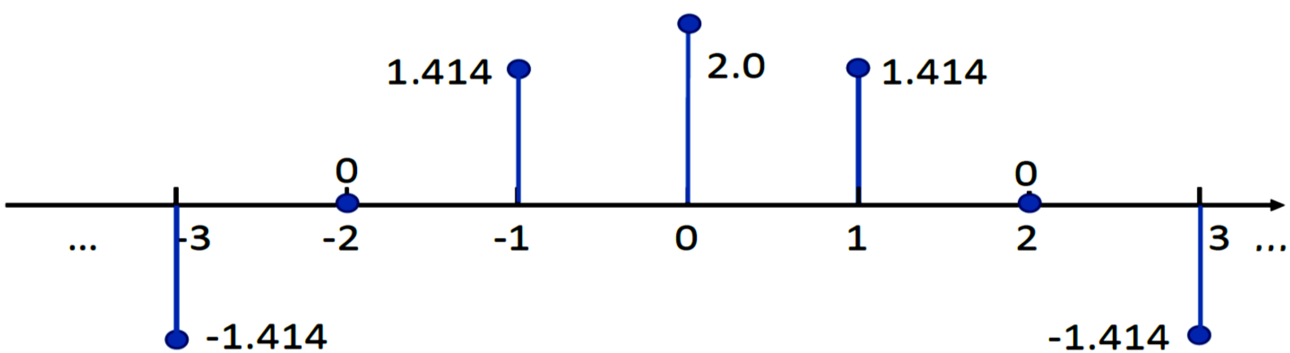


# Depicting a Discrete-Time (DT) Signal: $x[n] = 2\cos\frac{\pi}{4}n$

DT Signal $x: \mathbb{Z} \rightarrow \mathbb{R}$ $(or\ \mathbb{C}$, set of complex numbers$)$

| $\mathbb{Z}$ | ... | -3 | -2 | -1 | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|---|---|---|---|
| $\downarrow$ | | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | |
| $\mathbb{R}$ | ... | -1.414 | 0.0 | 1.414 | 2.0 | 1.414 | 0.0 | -1.414 | ... |

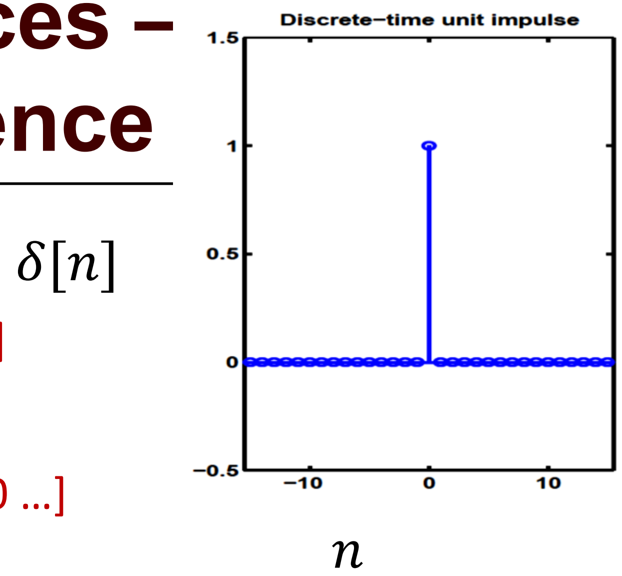
Plot (pictorial representation)




Another convenient representation (index 0 entry is underlined)

$x = [\ldots \quad -1.414 \quad 0.0 \quad 1.414 \quad \underline{2.0} \quad 1.414 \quad 0.0 \quad -1.414 \ldots]$

# Real-valued basic DT sequences – Unit Sample or Impulse sequence




- Unit sample (or unit impulse) sequence - $\delta[n]$

$$\delta[n] = \begin{cases} 1, n = 0, \\ 0, n \neq 0. \end{cases}$$ $\delta[n]$ = [… 0 0 0 0 $\underline{1}$ 0 0 0 0 …]

- $$\delta[n-3] = \begin{cases} 1, n = 3, \\ 0, n \neq 3. \end{cases}$$ $\delta[n-3]$ = [… 0 0 $\underline{0}$ 0 0 1 0 0 0 …]

- $$5\delta[n+2] = \begin{cases} 5, \quad n = -2, \\ 0, \quad n \neq -2. \end{cases}$$ $5\delta[n+2]$ = [… 0 0 5 0 $\underline{0}$ 0 0 …]

$\delta[n]$ is useful in representing a general signal $x[n]$:

- Let $x[n] = [\dots 0 \;\; 0 \;\; -1 \;\; \underline{2} \;\; 3 \;\; 0 \;\; 0 \;\; \dots]$

$\Rightarrow x[n] =$ -$\delta[n+1]$ +2 $\delta[n] + 3\delta[n-1]$.

(This expression illustrates the notion of shift and scaling of sequences).

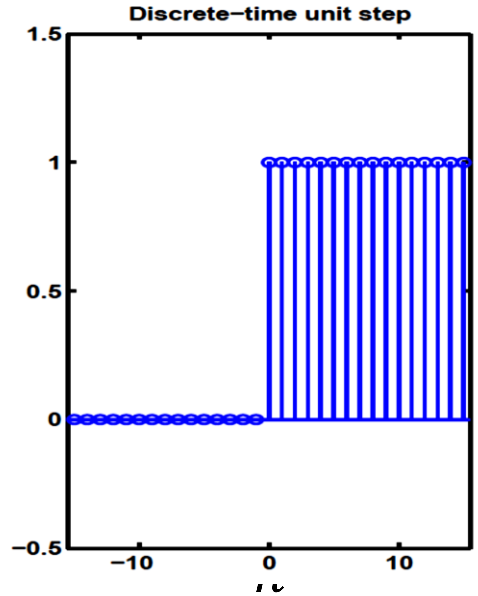
# Basic real-valued basic DT sequences – Unit Step sequence

- Unit step sequence – $u[n]$ or sometimes $\mu[n]$
- $u[n] = \begin{cases} 1, n \geq 0, \\ 0, n < 0. \end{cases}$

$u[n]$ has infinite duration – right-sided

- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$ .
- $\delta[n] = u[n] - u[n-1]$.
- What does $u[-n]$ look like?
- $u[n]$ is useful in restricting signal support:

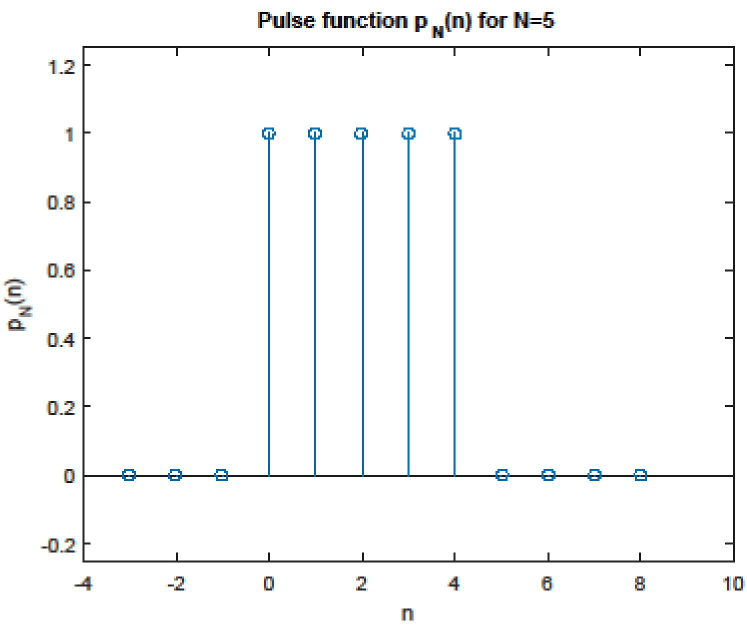
$u[n]$




$$x[n] = \cos(\omega_0 n) u[n] = \begin{cases} \cos(\omega_0 n), & n \geq 0, \\ 0, & n < 0. \end{cases}$$

# Basic real-valued basic DT sequences – Unit-amplitude pulse sequence

- (Unit-amplitude) Pulse sequence (causal rectangular window):
- Denoted as $p_N[n]$, sometimes as $r_N[n]$ or $R_N[n]$
- $p_N[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$
- Consider $N = 5$. Sketch $p_5[n]$.
- How would you represent $p_5[n]$ using unit step signals?
- $p_N[n] = u[n] - u[n-5]$.



# Basic real-valued basic DT sequences – Sinusoidal signals

- Sinusoidal signals
- $x[n] = A\cos(\omega_0 n + \phi)$. $\quad A > 0$
- What is the duration of this signal?
- Infinite duration – two-sided infinite duration
- Note that $\omega_0$ is expressed in radians and not in radians/sec as in the case of frequencies of CT signals. Also note $n$ is dimensionless.
- Note that if $\omega_1 = \omega_0 + 2k\pi$, for $k$ any integer, then

$$y[n] = A\cos(\omega_1 n + \phi) = A\cos(\omega_0 n + 2kn\pi + \phi) = x[n]$$

So sinusoidal signals with frequencies $\omega_1 = \omega_0 + 2k\pi$ are indistinguishable for integer values of $k$. We will re-visit this when we look at sampling and aliasing.

# Summary - Basic real-valued sequences

- **Unit sample (impulse) sequence**: $\delta[n] = \begin{cases} 1, n = 0, \\ 0, n \neq 0. \end{cases}$
- **Unit step sequence**: $u[n] = \begin{cases} 1, n \geq 0, \\ 0, n < 0 \end{cases} = \sum_{k=0}^{\infty} \delta[n-k]$.
- **Unit pulse sequence**: $p_N[n] = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$

  $p_N[n] = u[n] - u[n-N]$. Sometimes denoted as $r_N[n]$ or $R_N[n]$
- **Sinusoidal signals**: $x[n] = A\cos(\omega_0 n + \phi)$. ($\omega_0$ in radians)

  If $\omega_1 = \omega_0 + 2k\pi$, $k \in \mathbb{Z}$, then $A\cos(\omega_1 n + \phi) = A\cos(\omega_0 n + \phi)$

  Sinusoidal signals with frequencies $\omega_1 = \omega_0 + 2k\pi$ are indistinguishable

# Represent $x[n] = 1$ using basic real-valued sequences

- **Unit sample (impulse) sequence**: $\delta[n]$

  $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-k]$

- **Unit step sequence**: $u[n]$

  $x[n] = u[n] + u[-n-1]$

- **Unit pulse sequence**: $p_3[n] = \begin{cases} 1, & 0 \le n \le 2, \\ 0, & \text{otherwise.} \end{cases}$

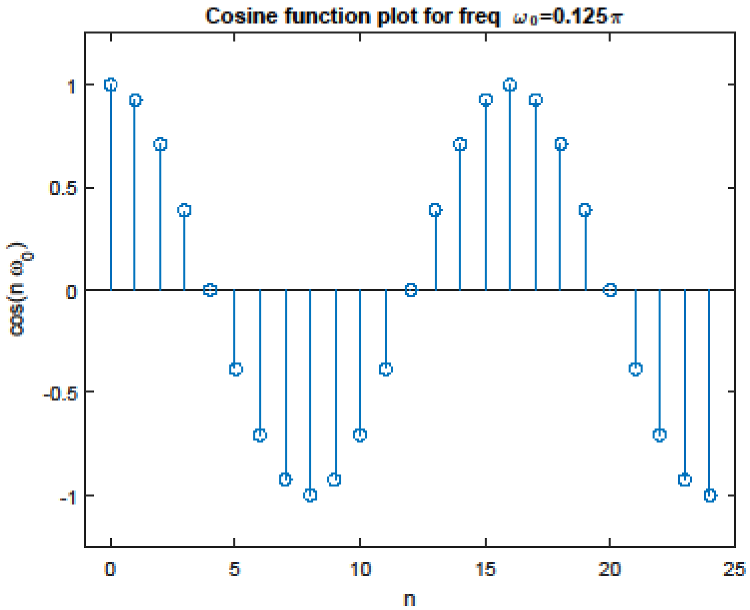
  $x[n] = \sum_{k=-\infty}^{\infty} p_3[n-3k]$

- **Sinusoidal signals**: $x[n] = A\cos(\omega_0 n + \phi)$.
- $x[n] = \cos(\omega_0 n),\ \omega_0 = 0 \rightarrow$ DC signal

# Examples of sinusoidal sequences: $x[n] = \cos(\omega_0 n),\ \omega_0 = 0.125\pi$

- **Sinusoidal signals**: $x[n] = \cos(\omega_0 n)$. ($\omega_0$ in radians)

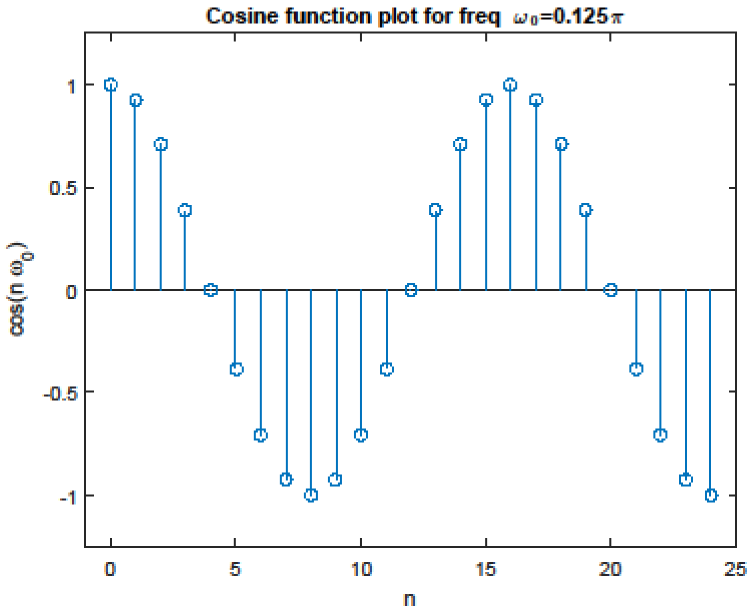
Sinusoidal signals with frequencies $\omega_1 = \omega_0 + 2k\pi$ are indistinguishable



# Examples of sinusoidal sequences: $x[n] = \cos(\omega_0 n),\ \omega_0 = 0.125\pi$

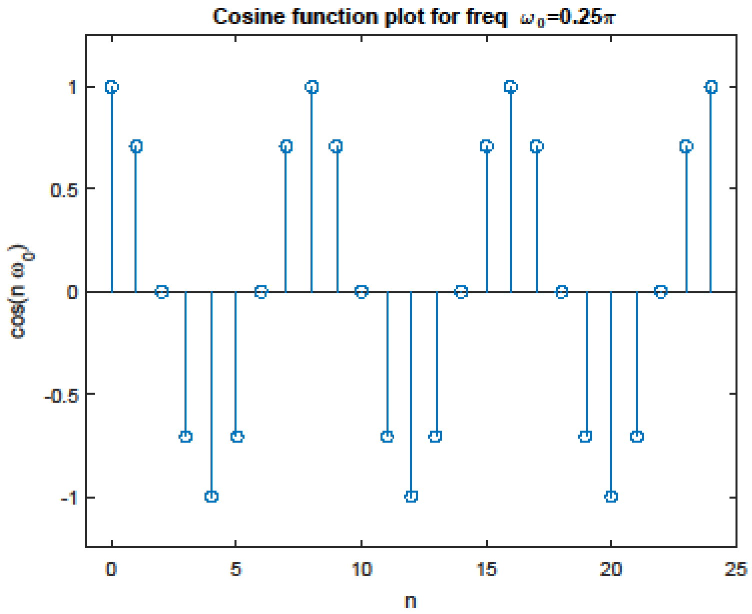
- **Sinusoidal signals**: $x[n] = \cos(\omega_0 n)$. ($\omega_0$ in radians)

Sinusoidal signals with frequencies $\omega_1 = \omega_0 + 2k\pi$ are indistinguishable



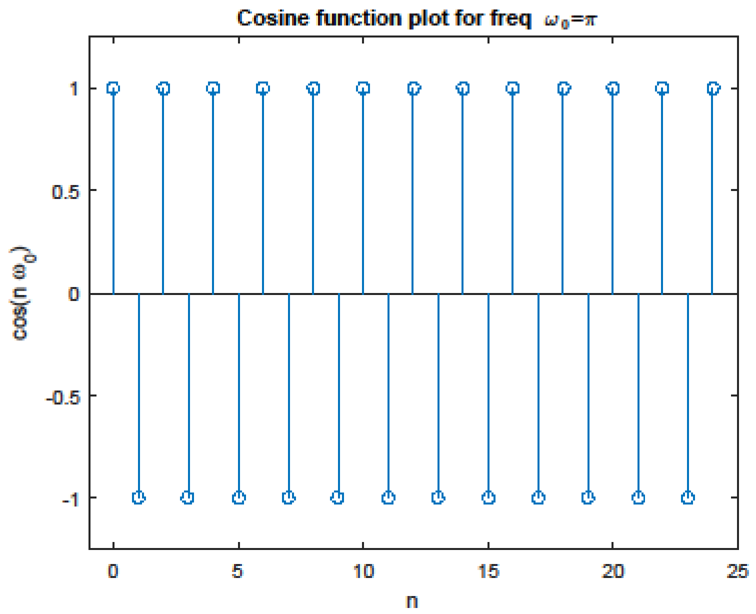
# Examples of sinusoidal sequences:

# $x[n] = \cos(\omega_0 n),\ \omega_0 = 0.25\pi$

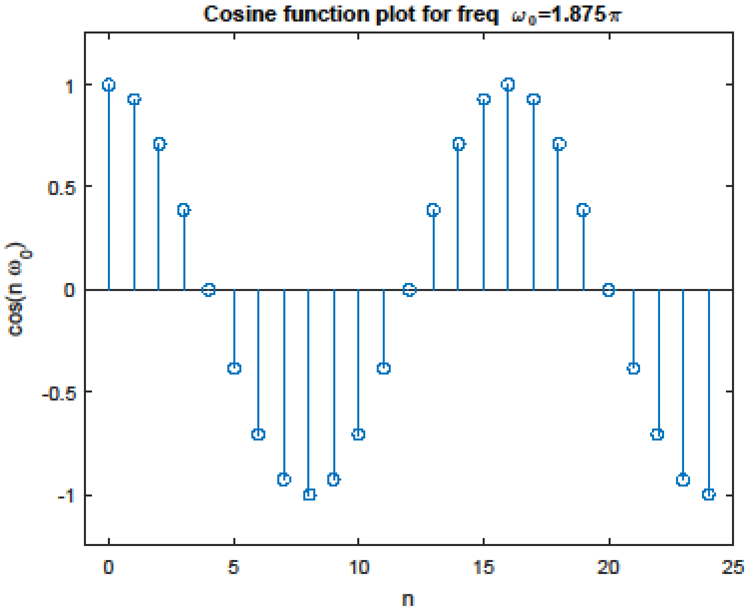


# Examples of sinusoidal sequences:

# $x[n] = \cos(\omega_0 n),\ \omega_0 = \pi$

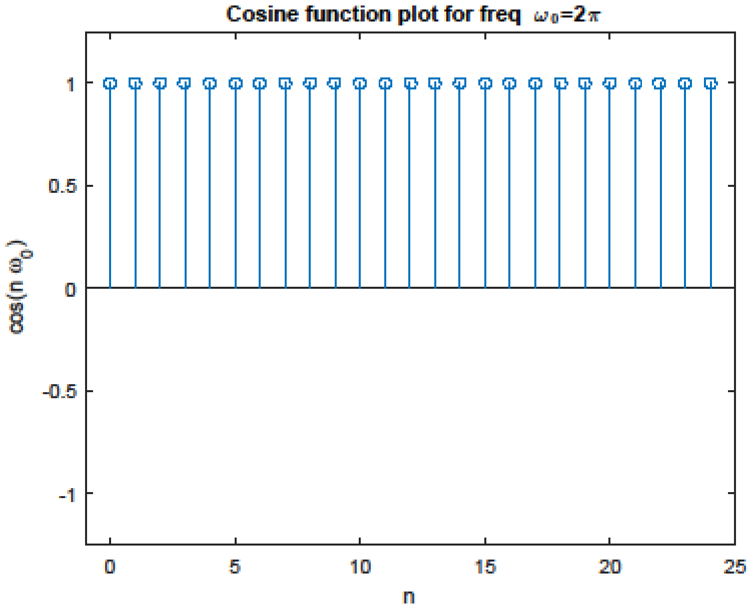


# Examples of sinusoidal sequences:
## $x[n] = \cos(\omega_0 n), \ \omega_0 = 1.875\pi$



# Examples of sinusoidal sequences:
## $x[n] = \cos(\omega_0 n),\ \omega_0 = 2\pi$

- Sinusoidal signals with frequencies $\omega_1 = \omega_0 + 2k\pi$ are indistinguishable: $\omega_0 = 0$ and $\omega_0 = 2\pi$



# Basic complex-valued DT sequences – complex-valued exponential signal

- A **discrete-time complex exponential signal** is defined as

$$x[n] = Az^n$$

  $z$ is complex and expressed as $z = re^{j\omega_0}$. Therefore

$$x[n] = Ar^n e^{j\omega_0 n}$$

  With $r = 1,\ A = 1, z = e^{j\omega_0}, x[n] = \ e^{j\omega_0 n}$.

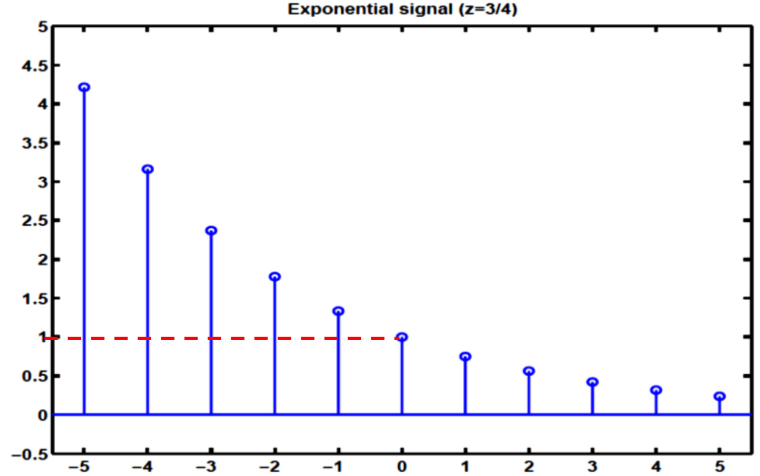
  We will refer to this as a unit-amplitude pure tone of frequency $\omega_0$.

- **Real exponential signal**: Consider the special case of the signal $x[n] = Az^n$, where $z = r$ is real, $A$ real.

  Then $x[n] = Ar^n$.

  DT real exponential signal is

  shown for $z\ =\ ¾$ and $A = 1$



# DT signal duration – Finite duration (assume non-zero $x[n]$)

- **Finite-duration signal:** Let $x[n]$ have at least one non-zero sample. Let the location of the 'left-most' non-zero sample be $n = N_1$ and the 'right-most'

  non-zero sample be $n = N_2$. $N_1, N_2$ are finite. What are $N_1, N_2$ in $x[n]$ below.
- $x[n]$ = [… 0 0 0 2 1 4 $\underline{2}$ 0 -1 -2 -3 -1 2 5 0 0 …]

  n= -3 0 7
- $N_1 = -3$ and $N_2 = 7$
- Duration is the total # of samples between the left-most and the right-most non-zero samples including those end samples.
- What is the duration in example?
- $7 - (-3) + 1 = 11$
- Duration in general: $N_2 - N_1 + 1$ where $x[n] = \begin{cases} 0, & x < N_1 \\ 0, & x > N_2 \end{cases}$ $N_2 \geq N_1$
- Exception: Sometimes you may pad zeros and define a longer "duration".

# DT signal duration – Infinite duration (assume non-zero $x[n]$)

- **Infinite-duration signal** is one that is not of finite duration.
- ➔ $x[n]$ is of infinite duration if no pair of two finite integers $N_1, N_2$ exist such that
- $x[n] = \begin{cases} 0, & x < N_1 \\ 0, & x > N_2 \end{cases} \quad N_2 \geq N_1$
  - Right-sided infinite-duration: $x[n] = 0$, for $n < N$, for some finite integer $N$
    - Example: $x[n] = \cos 0.125\pi n \, \mathrm{u[n+5]}$
  - Left-sided infinite-duration: $x[n] = 0$, for $n > N$, for some finite integer $N$
    - Example: $x[n] = 0.5^n \mathrm{u[-n]}$
  - Two-sided infinite sequences can be expressed as the sum of a left-sided infinite-duration and right-sided infinite-duration
    - $x[n] = 0.5^{|n|}$
    - $x[n] = 0.5^n u[n] + 0.5^{-n} u[-n-1]$

# “Switched” sinusoidal and exponential sequences

- Useful to define new useful sequences using sequences we have defined so far.
- “Switched” sinusoidal and exponential sequences are important in practice. The sequences are “switched on” by multiplying with a unit step function or sometimes its shifted version.
- A discrete-time switched complex exponential signal is defined as

  $x[n] = Az^n u[n]$, $z$ is a complex number.
- A discrete-time switched sinusoidal signal is defined as

  $$x[n] = A\cos(\omega_0 n + \phi)u[n]$$

# Complex Numbers and Euler's formula

- Euler's formula is often useful when dealing with complex numbers:

$$e^{j\omega} = \cos\omega + j\sin\omega \qquad j = \sqrt{-1}$$

- One proof is by using McLaurin series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}x + \dots$$

$e^{j\omega}$

# Complex Numbers and Euler's formula

- Apply McLaurin series expansion to $e^{j\omega}$

$$e^{j\omega} = 1 + j\omega + \frac{(j\omega)^2}{2!} + \frac{(j\omega)^3}{3!} + \frac{(j\omega)^4}{4!} + \frac{(j\omega)^5}{5!} + \frac{(j\omega)^6}{6!} + \frac{(j\omega)^7}{7!} + \frac{(j\omega)^8}{8!} + \ldots$$

$$j^0 = 1;\ j^1 = j;\ j^2 = -1;\ j^3 = -j;\ j^4 = 1;\ j^5 = j;\ j^6 = -1;\ j^7 = -j;\ j^8 = 1;$$

$$e^{j\omega} = 1 + j\omega - \frac{\omega^2}{2!} - j\frac{\omega^3}{3!} + \frac{\omega^4}{4!} + j\frac{\omega^5}{5!} - \frac{\omega^6}{6!} - j\frac{\omega^7}{7!} + \frac{\omega^8}{8!} + \ldots$$

Collect real and imaginary terms

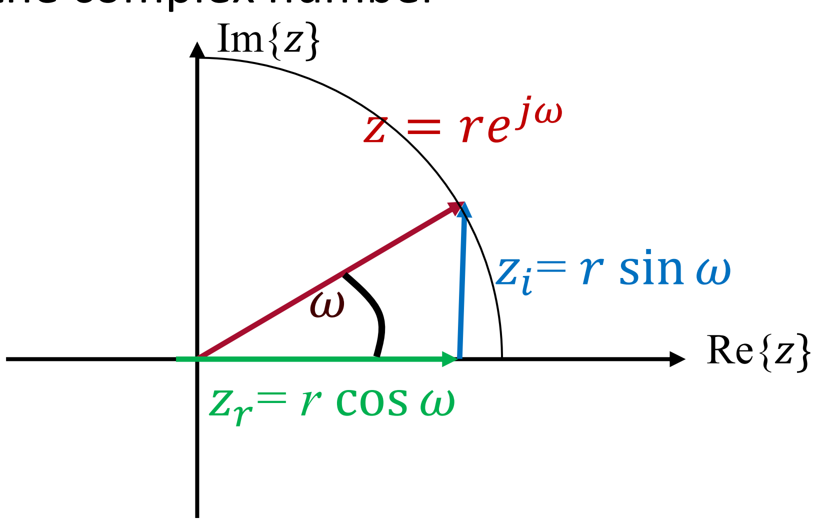
$$e^{j\omega} = \left(1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \frac{\omega^6}{6!} + \frac{\omega^8}{8!} - \ldots\right) + j\left(\omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \frac{\omega^7}{7!} + \frac{\omega^9}{9!} - \ldots\right)$$

$$\because \cos\omega = 1 - \tfrac{\omega^2}{2!} + \tfrac{\omega^4}{4!} - \tfrac{\omega^6}{6!} + \tfrac{\omega^8}{8!} - \ldots,\ \sin\omega = \omega - \tfrac{\omega^3}{3!} + \tfrac{\omega^5}{5!} - \tfrac{\omega^7}{7!} + \tfrac{\omega^9}{9!} - \ldots$$

$$e^{j\omega} = \cos\omega + j\sin\omega \quad \text{(Euler's formula)}$$

# Complex Numbers

- $z = z_r + jz_i$ → Complex number
- Euler's formula is useful in dealing with complex numbers.
- Euler's formula: $e^{j\omega} = \cos\omega + j\sin\omega$
- What is the magnitude of $e^{j\omega}$? That is, what is $|\ e^{j\omega}|$? =1
- $z = re^{j\omega} = r\cos\omega + j\ r\sin\omega$ $(z = z_r + jz_i)$
- $z_r = r\cos\omega$, $z_i = r\sin\omega$
- $|z| = r$ is the magnitude of the complex number

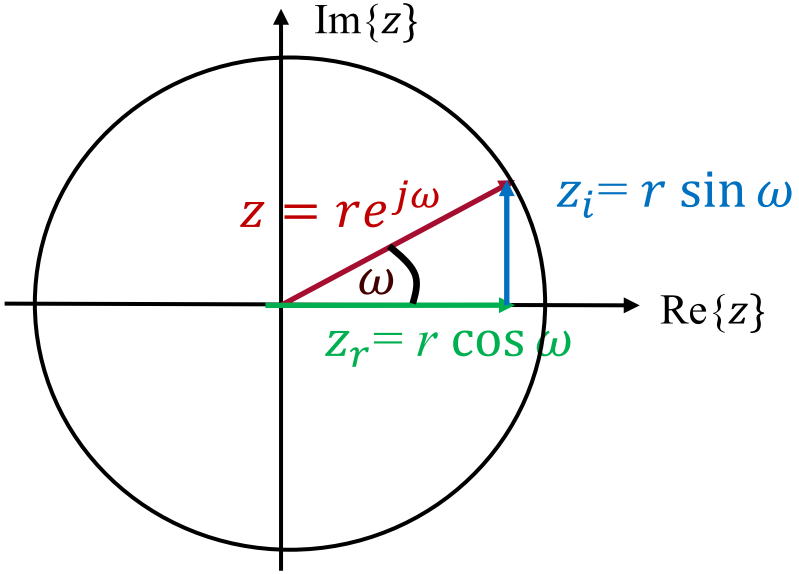


# Euler's formula

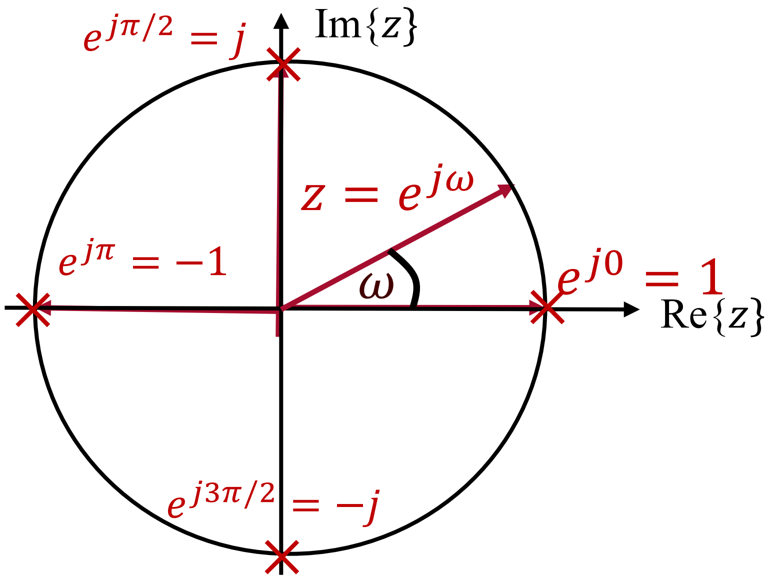
- $e^{j\omega} = \cos\omega + j\ \sin\omega$
- $e^{-j\omega} = \cos(-\omega) + j\ \sin(-\omega) = \cos\omega - j\ \sin\omega$
- $e^{j\omega} + e^{-j\omega} = 2\cos\omega$
- $e^{j\omega} - e^{-j\omega} = 2j\sin\omega$
- $\cos\omega = \frac{1}{2}(e^{j\omega} + e^{-j\omega})$
- $\sin\omega = \frac{1}{2j}(e^{j\omega} - e^{-j\omega})$

# Complex numbers and unit circle

- $z = re^{j\omega} = r\ \cos\omega + j\,r\sin\omega$
- As angle $\omega$ is increased from 0 to $2\pi$, $z$ sweeps a circle of radius $r$
- When $r = 1$, we get the unit circle




Circle of radius $r$




Unit circle: $z = e^{j\omega}, r = 1$