



ECE 317

Digital Signal Processing I (DSP I)

Lecture 5

Prof. Mojtaba Soltanalian

* Based on 2023 Slides from Prof. Rashid Ansari.

Topics of last and today's lectures

□ Last class:

- Two paradigms for signal processing
- CT vs DT signal processing

□ Today's class:

- DT systems and properties: linearity, shift-invariance, memoryless, causality, stability
- Linear shift-invariant (LSI) systems

System Properties (To provide structure on systems)

1. Linearity
2. Shift-Invariance
3. Memoryless (System)
4. Causality
5. Stability

System Property – Linearity

Consider two inputs $x_1[n]$, $x_2[n]$ to a system.

Let $T\{x_1[n]\} = y_1[n]$, $T\{x_2[n]\} = y_2[n]$.

$$x_1[n] \rightarrow \boxed{T} \rightarrow y_1[n] \quad x_2[n] \rightarrow \boxed{T} \rightarrow y_2[n]$$

The system is linear if the following is true for arbitrary scalars a and b and arbitrary input signals $x_1[n]$, $x_2[n]$.

$$T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n]$$

$$x[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{T} \rightarrow y[n] = ay_1[n] + by_2[n]$$

If input $x[n]$ to a linear system is zero, the output $y[n]$ should be zero.



Linearity Example 1 with details

$$x_1[n] \rightarrow \boxed{T} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{T} \rightarrow y_2[n]$$

$$x[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{T} \rightarrow y[n] = ay_1[n] + by_2[n]?$$

Linearity Example 1 with details

$$x_1[n] \rightarrow \boxed{T} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{T} \rightarrow y_2[n]$$

$$x[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{T} \rightarrow y[n] = ay_1[n] + by_2[n]?$$

Checking Linearity of a System: Example 1

Consider a system with input $x[n]$ and output $y[n] = e^{x[n]}$:

Check if $T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n](?)$

$x_1[n], x_2[n]$ are two arbitrary input signals to a system.

$T\{x_1[n]\} = y_1[n] = e^{x_1[n]}, T\{x_2[n]\} = y_2[n] = e^{x_2[n]}$.

Next consider an input $x[n] = ax_1[n] + bx_2[n]$.

The output is $y[n] = e^{x[n]} = e^{ax_1[n]+bx_2[n]} = e^{ax_1[n]}e^{bx_2[n]}$

$$\Rightarrow y[n] = (e^{x_1[n]})^a(e^{x_2[n]})^b = (y_1[n])^a(y_2[n])^b$$

$$\therefore y[n] \neq ay_1[n] + by_2[n]$$

So $T\{ax_1[n] + bx_2[n]\} \neq ay_1[n] + by_2[n]$

System is not linear.



Linearity Example 2 with details

$$x_1[n] \rightarrow \boxed{T} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{T} \rightarrow y_2[n]$$

$$x[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{T} \rightarrow y[n] = ay_1[n] + by_2[n]?$$

Checking Linearity of a System: Example 2

Linearity Example 2

$$x_1[n] \rightarrow \boxed{T} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{T} \rightarrow y_2[n]$$

$$x[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{T} \rightarrow y[n] = ay_1[n] + by_2[n]?$$

Checking Linearity of a System: Example 2

Consider a system with input $x[n]$ and output $y[n] = x[n] - x[n - 1]$:

Check if $T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n](?)$

$x_1[n], x_2[n]$ are two arbitrary input signals to a system.

Outputs are

$$y_1[n] = T\{x_1[n]\} = x_1[n] - x_1[n - 1]$$

$$y_2[n] = T\{x_2[n]\} = x_2[n] - x_2[n - 1].$$

Next consider an input $x[n] = ax_1[n] + bx_2[n]$.

Output $y[n] = x[n] - x[n - 1]$.

So $y[n] = ax_1[n] + bx_2[n] - ax_1[n - 1] - bx_2[n - 1]$.

$\therefore y[n] = a(x_1[n] - x_1[n - 1]) + b(x_2[n] - x_2[n - 1]).$

$$\therefore y[n] = ay_1[n] + by_2[n]$$

So $T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n]$

System is linear.

Linearity: Often easier to check with additivity and homogeneity (scaling)

- For a system to be linear, it must satisfy both the additivity and homogeneity properties:

- Additivity:

If $T\{x_1[n]\} = y_1[n]$ and $T\{x_2[n]\} = y_2[n]$

$$\rightarrow T\{x_1[n] + x_2[n]\} = y_1[n] + y_2[n]$$

means that the system satisfies the additivity property.

- Homogeneity or Scaling:

If $T\{x[n]\} = y[n] \rightarrow T\{ax[n]\} = ay[n]$

means that a system satisfies the homogeneity (scaling) property.

- If system is linear, then homogeneity with $a = 0$ implies:

If $T\{x[n]\} = y[n]$ then $T\{0 \cdot x[n]\} = 0 \cdot y[n]$

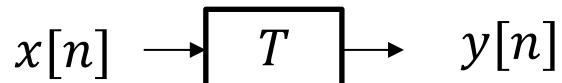
that is, an input sequence that is zero produces a zero output.

Linearity: Often easier to check with additivity and homogeneity (scaling)

- So one implication of homogeneity: If input to a linear system is $x[n] = 0$, the output should be $y[n] = 0$
- Consider the example $y[n] = e^{x[n]}$.
If $x[n] = 0$ for all n , then $y[n] = 1$ for all n .
So the system is not linear
- Consider the example $y[n] = x[n] + 3$.
If $x[n] = 0$ for all n , then $y[n] = 3$ for all n .
So the system is not linear



Linearity – Intuitive understanding



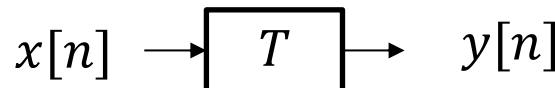
Consider some example transformations T

$y[n]$ can be expressed in terms of different functions of $x[n], x[n - 1], x[n - 2], \dots, x[n + 1], x[n + 2], \dots$. In addition, it may be expressed in terms of $y[n - 1], y[n - 2], \dots,$

- | | |
|---|--------|
| 1. $y[n] = ax[n]$ | 1. Yes |
| 2. $y[n] = \frac{1}{4}x[n - 3] + \frac{1}{2}x[n] + \frac{1}{4}x[n + 3]$ | 2. Yes |
| 3. $y[n] = 5x^3[n + 2]$ | 3. No |
| 4. $y[n] = e^{x[n]}$ | 4. No |
| 5. $y[n] = x[n] - x[n - 1]$ | 5. Yes |
| 6. $y[n] = n^2x[n]$ | 6. Yes |
| 7. $y[n] = x[n] + 3$ | 7. No |
| 8. $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n - k]$ | 8. Yes |

System Property – Shift-Invariance

Let $x[n]$ be an input to a system and $y[n]$ be the corresponding output: $y[n] = T\{x[n]\}$.



Next consider an input $x_1[n] = x[n - n_0]$ to the system and $y_1[n]$ be the corresponding output: $y_1[n] = T\{x_1[n]\}$. The system is shift-invariant if the following is true for an arbitrary integer n_0 and arbitrary input signals $x[n]$:

$$x_1[n] = x[n - n_0] \rightarrow \boxed{T} \rightarrow y_1[n] = y[n - n_0]$$

$y_1[n] = y[n - n_0]$, i.e. $T\{x[n - n_0]\} = y[n - n_0]$.

Shift Invariance: Example



$$x_1[n] = x[n - n_0] \rightarrow \boxed{T} \rightarrow y_1[n] = y[n - n_0]$$

Shift Invariance: Example



$$x_1[n] = x[n - n_0] \rightarrow \boxed{T} \rightarrow y_1[n] = y[n - n_0]$$

Checking Shift Invariance of a System: Example

Consider a system with input $x[n]$ and output $y[n] = nx[n]$:

Check if $T\{x[n - n_0]\} = y[n - n_0]$. (?)

Let $x[n]$ be an arbitrary input to a system.

The corresponding output is $y[n] = T\{x[n]\} = nx[n]$.

Now consider an input $x_1[n] = x[n - n_0]$ to the system.

The corresponding output is $y_1[n] = nx_1[n] = nx[n - n_0]$.

Question: Is $y_1[n] = y[n - n_0]$?

Since $y[n] = nx[n]$, $y[n - n_0] = [n - n_0]x[n - n_0]$.

So $y_1[n] \neq y[n - n_0]$

\Rightarrow System is not shift-invariant (It is shift-varying).



Shift Invariance: Example (Cont'd)

Shift Invariance: Example (Cont'd)

Checking Shift Invariance of a System: Example (continued)

Again consider a system with input $x[n]$ and output $y[n] = nx[n]$:

Observation: Output depends on a time-varying coefficient n in $y[n] = nx[n]$.

Sometimes a simple counter-example may be used.

Consider an input $x[n] = \delta[n]$.

Then $y[n] = T(\delta[n]) = n\delta[n] = 0$.

Next consider an input $x_1[n] = \delta[n - 1]$

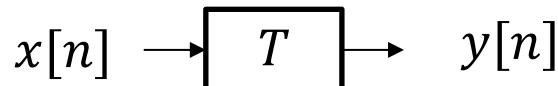
Now the output is $y_1[n] = T(\delta[n - 1]) = n\delta[n - 1] = \delta[n - 1]$

So $y_1[n] \neq y[n - n_0]$

⇒ System is not shift-invariant (It is shift-varying).



Shift Invariance – Intuitive understanding (system parameters change with time)



Consider some example transformations T

$y[n]$ can be expressed in terms of different functions of $x[n], x[n - 1], x[n - 2], \dots, x[n + 1], x[n + 2], \dots$. In addition, it may be expressed in terms of $y[n - 1], y[n - 2], \dots$,

- | | |
|---|--------|
| 1. $y[n] = ax[n]$, a is a constant | 1. Yes |
| 2. $y[n] = nx[n]$ | 2. No |
| 3. $y[n] = \frac{1}{4}x[n - 3] + \frac{1}{2}x[n] + \frac{1}{4}x[n + 3]$ | 3. Yes |
| 4. $y[n] = 5x^3[n]$ | 4. Yes |
| 5. $y[n] = e^{x[n]}$ | 5. Yes |
| 6. $y[n] = e^{nx[n]}$ | 6. No |
| 7. $y[n] = x[n] + 3$ | 7. Yes |
| 8. $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n - k]$ | 8. Yes |

System Property – Memoryless and Causal systems

System Property – Memoryless and Causal systems

Memoryless system:

A system is memoryless if the output $y[n_0]$ at “instant” n_0 depends only on input sample $x[n_0]$.

- $y[n] = x^2[n] + (n + 1)^2$. Is the system memoryless?
Yes.
- $y[n] = \frac{1}{2}(x[n - 1] + x[n + 1])$. Is the system memoryless?
No.

Causality:

Output $y[n_0]$ depends on input samples $\{x[n_0], x[n_0 - 1], x[n_0 - 2], \dots\}$ and not on “future” samples $\{x[n_0 + 1], x[n_0 + 2], \dots\}$

- $y[n] = \frac{1}{2}(x[n - 1] + x[n + 1])$. Is the system causal?
No. $y[n_0]$ depends on future sample $x[n_0 + 1]$
- $y[n] = \cos(n + 1)x[n]$. Is the system causal?
Yes.

System Property: Bounded-Input Bounded-Output (BIBO) Stability

- What is a bounded sequence (or DT signal)?
- $x[n]$ is said to be bounded if there exists a positive number (bound) B such that $|x[n]| \leq B$ for all integers n
- Question: Is $x[n] = \cos 0.25\pi n$ bounded?
- Question: Is $x[n] = n$ bounded?

- Notation:
 - B_I for bound on input signal
 - B_O for bound on output signal



System Property – BIBO Stability and example



System Property – BIBO Stability and example

■ Bounded-Input Bounded-Output (BIBO)

(BIBO) Stability:

A system is BIBO stable if the following is true:

If the system input $x[n]$ is bounded for all $n \in \mathbb{Z}$, i.e | $x[n]$ | $\leq B_I < \infty$, $\forall n \in \mathbb{Z}$, for a given real positive B_I
then the output $y[n]$ is bounded for all $n \in \mathbb{Z}$, i.e. | $y[n]$ | $\leq B_O < \infty$, $\forall n \in \mathbb{Z}$, for some real positive B_O

Example 1 (stability)

- $y[n] = e^{x[n]}$, $x[n]$ real. Is the system stable?

Let | $x[n]$ | $\leq B_I < \infty$, $\forall n \in \mathbb{Z}$.

Now | $y[n]$ | = | $e^{x[n]}$ | $\leq e^{|x[n]|} \leq e^{B_I} \triangleq B_O$.

We found a B_O bounding the output \Rightarrow System is stable.

BIBO stability example 2

BIBO stability example 2

(BIBO) Stability:

Example 2 (stability)

- $y[n] = n2^{x[n]}$. Is the system stable?

Let $|x[n]| \leq B_I < \infty, \forall n \in \mathbb{Z}$.

Now $|y[n]| = |n2^{x[n]}| = |n||2^{x[n]}|$.

Let $x[n] = B_I$ for all $n \in \mathbb{Z}$. Then $|y[n]| = |n||2^{B_I}|$.

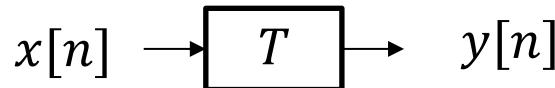
Suppose we are given any positive real B_O , however large.

We can find an N large enough so that $|N||2^{B_I}| > B_O$.

For all $|n| \geq N$, $|y[n]| = |n||2^{B_I}| \geq |N||2^{B_I}| > B_O$.

System is not stable.

DT systems: Implication of Linearity and Shift-Invariance



Linearity implies:

$$\text{If } x_1[n] \rightarrow \boxed{T} \rightarrow y_1[n] \quad \text{and } x_2[n] \rightarrow \boxed{T} \rightarrow y_2[n]$$
$$\text{Then } x[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{T} \rightarrow y[n] = \boxed{T}(ax_1[n] + bx_2[n]) \\ = aT(x_1[n]) + bT(x_2[n]) \\ = ay_1[n] + by_2[n]$$

Generalization to more sum of than two sequences applied to a linear system:

If the input is $x[n] = \sum_{k=-\infty}^{\infty} a_k x_k[n]$, and $T(x_k[n]) = y_k[n]$, then the output is

$$y[n] = T\left(\sum_{k=-\infty}^{\infty} a_k x_k[n]\right) = \sum_{k=-\infty}^{\infty} a_k T(x_k[n]) = \sum_{k=-\infty}^{\infty} a_k y_k[n]$$

Shift Invariance implies:

$$\text{If } x_k[n] \rightarrow \boxed{T} \rightarrow y_k[n] \quad \text{then } x_k[n - n_0] \rightarrow \boxed{T} \rightarrow y_k[n - n_0]$$



Recall: Signal representation as a sum of scaled and shifted unit impulse signals

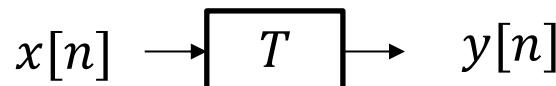
Any signal $x[n]$ can be represented as a sum of scaled and shifted unit impulse signals

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$$

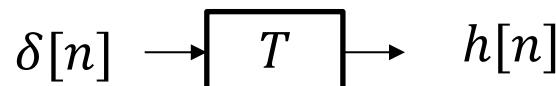
$x[k]$ is a scalar, not a signal/sequence

$\delta[n - k]$ is a signal – an impulse signal with a shift by an integer k

$x[n]$ is applied to a linear shift-invariant system:



In the special case where the input is an impulse signal:



we will denote the output as $h[n]$, and call $h[n]$ the system impulse response.

Linear Shift-Invariant (LSI) System

Linear Shift-Invariant (LSI) System

We will deal primarily with LSI systems in this course.

That is, the system is both linear and shift-invariant.

Consider a LSI system T characterized by impulse response $h[n]$. That is, $T(\delta[n]) = h[n]$.

Now consider an arbitrary input $x[n]$ to the system:

$$x[n] \rightarrow [T] \rightarrow y[n] = T(x[n])$$

Now $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ where $x[k]$ are scalars and $\delta[n-k]$ are shifted impulse sequences.

So $y[n] = T(x[n]) = T(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]).$

Since T is linear, $y[n] = \sum_{k=-\infty}^{\infty} x[k]T(\delta[n-k]).$

Since T is shift-invariant $T(\delta[n-k]) = h[n-k].$

Therefore $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$

Linear Shift-Invariant (LSI) System

LSI system is characterized by its impulse response $h[n]$

Linear Shift-Invariant (LSI) System

LSI system is characterized by its impulse response $h[n]$

- Output of LSI system $y[n]$ is the convolution sum of input $x[n]$ and $h[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$$

Once you know the impulse response $h[n]$, you can determine the output $y[n]$ for any input $x[n]$.

Stability and causality of an LSI system can be inferred from $h[n]$.

- Stable LSI system $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Later, we will see why these two conditions are valid.

- Causal LSI system $\Leftrightarrow h[n] = 0, n < 0$

- Impulse response duration: FIR (finite-duration impulse response), IIR (infinite-duration impulse response)