WEIGHTING FUNCTION SELECTION IN H^∞ DESIGN

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Abstract : We examine an H^∞ mixed sensitivity design problem formulated to achieve good disturbance rejection and tracking properties as well as robust stability. A two degree-of-freedom controller is used, then, as is well known, the tracking objectives can be considered separately. A major concern in H^∞ design is the selection of weighting functions to satisfy the design specifications. In this paper we give conditions on the weights which guarantee the optimization problem is well-defined; describe how a *constrained* optimization can be posed so that improper and unstable weights can be used; describe the effect the weights have on various design objectives; show how an appropriate selection of weights can reduce a 2-block optimization problem (requiring an iterative solution) to a 1-block problem (with a closed-form solution); give an illustrative example. In particular we show how the pole-zero cancellation phenomenon in H^∞ design can be exploited or prevented by a suitable choice of weight, and how an optimal controller with a given frequency roll-off can be obtained.

Keywords : H^∞ optimal control, multivariable control systems, robustness, pole placement, disturbance rejection, tracking.

INTRODUCTION

We consider the linear multivariable feedback control system shown in Figure 1, where the plant G is to be controlled by compensator K. The configuration is quite general and uses what is called a two-parameter compensator or controller (Youla, 1985). Disturbances to the system are represented by an equivalent disturbance d at the plant output, w represents measurement noise, and r is a reference signal to be followed by the controlled output y. The function of K is to stabilize the plant, to reduce the effects of disturbances and measurement noise, and to achieve desirable response to the reference signal, all in the presence of plant model uncertainty.

A realization of the two-parameter controller is shown in Figure 2, where K_1 is a feedforward controller from r to the plant input u and K_2 is a feedback controller from the measured outputs to u. It is an example of a 2 degree-of-freedom (DOF) controller. The function of K_2 is primarily to reject disturbance d and to robustly stabilize G, and K_1 is to improve tracking performance.

From Figure 2, the controlled output y and the plant input u are given by

$$y = GS'K_1r + Sd + GK_2Sw \tag{1}$$

and

$$u = S'K_1r + K_2S(d+w) \tag{2}$$

where $S' = (I-K_2G)^{-1}$ and $S = (I-GK_2)^{-1}$ are the input and output sensitivity matrices respectively.

We will assume that G is an $m \times p$ *strictly* proper real-rational matrix with $p \geq m$. For a given G, let $G = NM^{-1}$ $(= \tilde{M}^{-1}\tilde{N})$ be a right (left) coprime factorization defined in $\Re\mathbf{H}^\infty$ space and let $(N, M, \tilde{N}, \tilde{M}, X, Y, \tilde{X}, \tilde{Y})$ be $\Re\mathbf{H}^\infty$ matrices such that

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I \,. \tag{3}$$

Then all stabilizing feedback controllers of G, in terms of a free $Q_d \in \Re\mathbf{H}^\infty$ (Youla, 1976), are given by

$$K_2 = (Y - MQ_d)(X - NQ_d)^{-1} = (\tilde{X} - Q_d\tilde{N})^{-1}(\tilde{Y} - Q_d\tilde{M}) \,. \tag{4}$$

DISTURBANCE REJECTION

To consider disturbance rejection alone, we can assume $r = w = 0$. Then we see from (1) and (2) that the properties of the feedback system can be characterized by the functions S and K_2S. That is, the sensitivity matrix determines how much effect the disturbance d has on the controlled output y, and the power matrix K_2S determines how much effect the disturbance d (and the measurement w) has on the plant input u. Consequently, good disturbance attenuation, robustness, limited bandwidth and compensator roll-off can be obtained by minimizing the weighted cost function (Kwakernaak, 1986)

$$\|Z_d\|_\infty := \left\| \begin{matrix} W_1SW_d \\ W_2K_2SW_d \end{matrix} \right\|_\infty \tag{5}$$

over the set of all stabilizing feedback compensators K_2 where W_1, W_2 and W_d are chosen to tailor the solution to meet the design specifications. For example, W_d can be regarded as a generator which characterizes all relevant disturbances. The selection of weighting functions is considered in more detail later.

Problem (5) is illustrated in Figure 3 where the transfer function from v_d to $[e_1^T \; e_2^T]^T$ is given by

$$Z_d = \begin{bmatrix} W_1W_d \\ 0 \end{bmatrix} + \begin{bmatrix} W_1G \\ W_2 \end{bmatrix} (I-K_2G)^{-1}K_2W_d \,. \tag{6}$$

By direct computation we obtain from (5) the affine optimization:

$$\min_{Q_d \in \Re\mathbf{H}^\infty} \|T_{11} - T_{12}Q_dT_{21}\|_\infty \; (=: \gamma_o) \tag{7}$$

where

$$T_{11} = \begin{bmatrix} W_1X\tilde{M}W_d \\ W_2Y\tilde{M}W_d \end{bmatrix} \tag{8}$$

$$T_{12} = \begin{bmatrix} W_1N \\ W_2M \end{bmatrix} \tag{9}$$

$$T_{21} = \tilde{M}W_d \,. \tag{10}$$

It can be seen explicitly from (8)-(10) that the solvability of (7) depends on the controlled plant and the weighting functions as defined by the following conditions (e.g. Francis (1987))

- T_{11}, T_{12} and T_{21} are stable
- $T_{12}(s)$ and $T_{21}(s)$ are full column and row rank, respectively, for all s in $j\Re$ (the imaginary axis).

Doyle (1984) has argued that a realistic design problem will always satisfy these conditions. Any H^∞ design problem which satisfies the above requirements will be called *well-defined*.

For a given G it is clearly important to be able to choose weighting functions such that the design problem (5) is well-defined. The following lemma gives conditions on the weighting functions W_1, W_2, W_d which ensure that (7) is well-defined.

Lemma 1 *Let $\tilde{M}^{-1}\tilde{N}$ be any left coprime factorization of G and let the weighting functions in (5) be chosen as follows:*

- *W_1 is proper, stable and minimum phase.*
- *W_2 is a unit in $\Re\mathbf{H}^\infty$.*
- *$W_d = \tilde{M}^{-1}W_3$ where W_3 is a unit in $\Re\mathbf{H}^\infty$.*

Then optimization problem (5) is well-defined.

In this lemma, the weighting functions are restricted to being proper so that the whole solution procedure can be carried out using state-space models. However, we may, in general, allow some improper weights in problem (5). To do this and still use state-space representations in the solution, we can formulate (5) into a constrained H^∞ design problem as described later. Without loss of generality, we also restrict the stable weights W_1, W_2 and W_3 to be minimum phase. Note that any nonminimum phase $W_1, W_2, W_3 \in \Re\mathbf{H}^\infty$ can be factorized into

$$W_1 = W_{1i}W_{1o}, \quad W_2 = W_{2i}W_{2o} \quad \text{and} \quad W_3 = W_{3o}W_{3i}$$

where W_{1i}, W_{2i} and W_{3i} are inner, and therefore do not affect the cost of (5).

Remark 1 The characterization of W_d $(= \tilde{M}^{-1}W_3)$ is used to prevent the rank deficiency of T_{21} when G has poles in $j\Re$. This provides a simple way of dealing with the $j\Re$ poles. The issue of a well-defined H^∞ design problem is also considered in Safonov (1986) and O'Young *et al.* (1989). It was claimed in O'Young *et al.* (1989) that by imposing saturation constraints and accounting for additive instrument noise in the sensor and actuator signals, the possibility of rank deficiency in T_{21} and/or T_{12} originating from the plant poles and/or zeros in $j\Re$ is completely eliminated. It is obvious, however, that by this approach the problem becomes more complex and more computations are needed. Another method (Safonov, 1986) is to use a conformal transformation to perturb the $j\omega$-axis thereby shifting the offending poles. With the transformation, however, the physical meaning of frequency response is not easily recovered. Also in Safonov (1986) weighting functions are used to cancel $j\Re$ poles as proposed here. It is worth noting that if the controlled plant has $j\Re$ zeros and the design objectives are characterized to minimize $\|[\,I - S^T \quad S^T\,]^T\|_\infty$ as studied for example in Safonov and Chiang (1986), then this problem is not well-defined and is difficult to make well-defined using *proper* weights.

TRACKING

The feedforward compensator K_1 can be designed to improve the tracking performance of the system. To consider a tracking problem, we assume $d = w = 0$ and define a tracking error signal as $r - y$, where the desired reference signal r is generated by W_t. To make the design problem practical, we need to prevent saturation of the control signal u. Analogous to the disturbance minimization criterion for finding K_2, we therefore define the tracking performance criterion as:

$$\|Z_t\|_\alpha := \left\| \begin{matrix} W_1(I - GS'K_1)W_t \\ W_2S'K_1W_t \end{matrix} \right\|_\alpha \qquad (11)$$

where α represents either the L^2 or L^∞-norm, W_1, W_2 are given as in (5), and W_t is a unit in $\Re\mathbf{H}^\infty$ (when $\alpha = \infty$) or an $L^\infty[0\,\infty)$ function (when $\alpha = 2$). This problem is illustrated in Figure 4.

Remark 2 In contrast to the disturbance minimization problem (5), we also consider L^2 optimization since for some situations the reference signal r may be known *a priori*. In order to have good tracking performance with respect to a known reference signal, for example a step, it is better to use L^2-norm optimization Stein and Athans (1987).

It is well known that K can be represented in terms of two independent free parameters in $\Re\mathbf{H}^\infty$; a structure which provides two explicit and independent design freedoms.

Let the two-parameter controller $K = [K_1 \quad K_2]$ be implemented as in Figure 5:

$$K_2 = D_c^{-1}N_{k2} \quad \text{and} \quad K_1 = D_c^{-1}N_{k1} \qquad (12)$$

where $D_c = \tilde{X} - Q_d\tilde{N}$ and $N_{k2} = \tilde{Y} - Q_d\tilde{M}$. The advantage of this implementation is that K stabilizes G if K_2 stabilizes G and N_{k1} is stable. Therefore, unlike in Figure 2, K_1 need not be stable.

Let $N_{k1} = Q_t$ where $Q_t \in \Re\mathbf{H}^\infty$. Then all stabilizing controllers are given by

$$K = (\tilde{X} - Q_d\tilde{N})^{-1}\,[Q_t \quad \tilde{Y} - Q_d\tilde{M}] \qquad (13)$$

where K contains two free parameters Q_d and Q_t in $\Re\mathbf{H}^\infty$.

With the 2DOF controller structure shown in Figure 5, we obtain

$$\begin{aligned} Z_t &= \begin{bmatrix} W_1W_t \\ 0 \end{bmatrix} + \begin{bmatrix} -W_1G \\ W_2M \end{bmatrix} S'K_1W_t \\ &= \begin{bmatrix} W_1W_t \\ 0 \end{bmatrix} - \begin{bmatrix} W_1N \\ -W_2M \end{bmatrix} Q_tW_t \end{aligned}$$

and hence the affine optimization problem :

$$\min_{Q_t \in \Re\mathbf{H}^\infty} \|\bar{T}_{11} - \bar{T}_{12}Q_t\bar{T}_{21}\|_\alpha \qquad (14)$$

where

$$\bar{T}_{11} = \begin{bmatrix} W_1W_t \\ 0 \end{bmatrix}, \quad \bar{T}_{12} = \begin{bmatrix} W_1N \\ -W_2M \end{bmatrix}, \quad \bar{T}_{21} = W_t\,.$$

Remark 3 From the formulation, we see that Z_t has only one dependent variable Q_t. That is, the closed-loop transfer function from r to u (and, hence to y) is independent of Q_d. This implies that K provides an extra design freedom Q_t which can be used to improve tracking performance. Hence the disturbance minimization and tracking problems can be solved independently. Moreover, since T_{12} in (9) is only different in sign from $\bar{T}_{12}$ above, there is the potential to save on computations and to reduce the complexity of the final controller.

Lemma 2 *Suppose that the disturbance problem with cost function Z_d is well-defined. Then Z_t is a well-defined cost function, if W_t is a unit in $\Re\mathbf{H}^\infty$ and L^∞-norm optimization is considered. For L^2-norm optimization, W_t is only required to be an $L^\infty[0,\infty)$ function such as a step function.*

CONSTRAINED OPTIMIZATION

When the weights satisfy the conditions of Lemmas 1 and 2, we obtain, in general, an H^∞ optimal controller K with a nonzero D $(= K(\infty))$ matrix. Hence, if one of our design objectives is to require the feedback compensator to have a high frequency roll-off, then we could never achieve this goal from the above problem formulations. This practical requirement is often desirable for controller implementation and robustness with respect to high frequency plant uncertainties and measurement noise . Therefore there is a need to be able to formulate design problems from (5) and (11) which satisfy this roll-off demand and at the same time permit a solution using only state-space models. The latter is required for numerical reliability of the computations.

First, we explain the idea of constrained optimization. Let $\mathbf{K}$ denote the set of all stabilizing controllers for plant G and let W_c

be any proper function. Define an augmented plant G_a as

$$G_a = GW_c \tag{15}$$

and assume that G_a has no hidden unstable modes. Then let the set of all stabilizing controllers for G_a be denoted by $\hat{\mathbf{K}}$. From the properness of stabilizing controllers and the absence of hidden unstable modes, we have

$$W_c\hat{\mathbf{K}} \subseteq \mathbf{K} . \tag{16}$$

These two sets will be equal when W_c is a unit in $\Re\mathbf{H}^\infty$. In other words, if W_c is either strictly proper or unstable, then the set $W_c\hat{\mathbf{K}}$ is strictly smaller than $\mathbf{K}$.

Thus let controller K stabilizing G be given by

$$K = W_c\hat{K}, \quad \hat{K} \in \hat{\mathbf{K}} . \tag{17}$$

Then the final K has the desired form since $\hat{K}$ is proper, $W_c\hat{K}$ has no hidden unstable modes and W_c can be chosen to reflect the required roll-off in K.

If W_c is strictly proper, then the optimal solution procedure for $\hat{K}$ can be performed completely in state-space and the resulting K is strictly proper since the optimal $\hat{K}$ is at most biproper. From a design point of view, W_c can include integrator and unstable modes in order to reject constant and unstable disturbances, *i.e.*, W_c can be chosen to include the disturbance model. More on this later.

With respect to the original optimization problem, we are looking for an optimal stabilizing K from the subset $W_c\hat{\mathbf{K}}$ of $\mathbf{K}$. We hence call the problem a 'constrained' optimization. In the following, we define constrained design problems corresponding to (5) and (11).

Given W_c such that G_a (defined by GW_c) has no hidden unstable modes, the *constrained disturbance minimization* problem is to find a stabilizing controller $\hat{K}_2$ which minimizes

$$\|\hat{Z}_d\|_\infty := \left\| \begin{matrix} W_1 S_a W_d \\ W_2 \hat{K}_2 S_a W_d \end{matrix} \right\|_\infty \tag{18}$$

where $S_a = (I - G_a\hat{K}_2)^{-1}$, and the feedback-loop controller K_2 is given by $W_c\hat{K}_2$. Similarly, the *constrained tracking* problem is to find a $\hat{K}_1$ which minimizes

$$\|\hat{Z}_t\|_\alpha := \left\| \begin{matrix} W_1(I - G_a S_a' \hat{K}_1)W_t \\ W_2 S_a' \hat{K}_1 W_t \end{matrix} \right\|_\alpha \tag{19}$$

where $S_a' = (I - \hat{K}_2 G_a)^{-1}$ and the feedforward controller K_1 is given by $W_c\hat{K}_1$.

Since $S_a = S$ and $G_a S_a' \hat{K}_1 = GS'K_1$, we know that W_c does not change the closed-loop transfer functions from d to y and from r to y.

Remark 4 Here we have shown that in a constrained problem, the controller output u due to d and r is constrained by W_c. In fact, W_c acts functionally as a part of W_2. We explain this fact below.

Suppose that $W_c \in \Re\mathbf{H}^\infty$ is minimum phase. The unstable poles of G and G_a are the same, and therefore it makes sense to put the same W_d in both (5) and (18). Then we have from (18)

$$\hat{Z}_d = \begin{bmatrix} W_1 W_d \\ 0 \end{bmatrix} + \begin{bmatrix} W_1 G \\ W_2 W_c^{-1} \end{bmatrix} K_2(I - GK_2)^{-1} W_d . \tag{20}$$

Comparing Z_d of (6) and $\hat{Z}_d$ above, we see that the only difference is that W_2 in (6) is now replaced by $W_2W_c^{-1}$. This suggests that we can choose W_2 to be an improper function in problem (5), but solve problem (18) instead. To see this, assume that W_2 in (5) is an improper function and W_2 is expressed as $\hat{W}_2\hat{W}_c^{-1}$ where $\hat{W}_2$ is a unit in $\Re\mathbf{H}^\infty$ and $\hat{W}_c$ is strictly proper and minimum phase. As shown above, there exists an equivalent constrained problem $\hat{Z}_d$ with $W_c = \hat{W}_c$ and $W_2 = \hat{W}_2$. Hence when we solve the equivalent $\hat{Z}_d$, we find the solution for the problem Z_d, which includes an improper weight W_2. Note that this effect also appears in the tracking problem.

CHOICE OF WEIGHTS

The selection of weights in H^∞ design is problem specific and therefore it is difficult to provide a definitive set of rules for building and modifying weights. The following are general guidelines:

- Use only stable and diagonal weights.
- Restrict the diagonal elements to be minimum phase, real-rational functions.
- Scale the weights with respect to a (normalized) targeted cost.

For the disturbance minimization problem (18), however, we can be more specific about the weights. Note that for notational convenience, the coprime factorizations of G_a are denoted by $NM^{-1} = \tilde{M}^{-1}\tilde{N}$ in the following.

Choice of W_c

We first consider the weight W_c used in the constrained optimization problem.

- *Protection against Measurement Noise:*

The effect of measurement noise on the control system output and the plant input is characterized by the relations $y = GK_2Sw$ and $u = K_2Sw$. (Note that $S = S_a = (I - GW_c\hat{K}_2)^{-1}$ and $K_2 = W_c\hat{K}_2$.) Hence the effect of measurement noise is limited if $K_2(j\omega)S(j\omega)$ is adequately bounded over the noise frequencies. Therefore to mitigate the effects of high-frequency noise, W_c should be chosen to be a low-pass filter with a desired frequency roll-off rate.

- *Robust Disturbance Rejection:*

Suppose that the disturbance is an unstable signal injected at the plant output. In this case, for the plant output to be insensitive to the disturbance, W_c should include the disturbance model.

Choice of W_d

W_d was introduced as the disturbance generator and in Lemma 1 was assumed to be of the form $\tilde{M}^{-1}W_3$. One interpretation of this is that the disturbance generated by W_3 is assumed to be injected inside the (augmented) plant between the coprime factors. However, W_d also has the following important functions:

- *Pole Placement:*

W_d can be regarded as a mechanism for partial pole placement (Kwakernaak, 1986). This can be seen as follows where the solution is assumed to be an equalizing solution (or a super-optimal solution). First the closed-loop poles of the optimal feedback system include the stable (transmission) zeros of $W_3W_3^\sim$. Therefore if $W_d = I$ (equivalently $W_3 = \tilde{M}$) the closed-loop poles of the optimal system include the stable zeros of $\tilde{M}\tilde{M}^\sim$. Thus if $W_d = I$, the stable poles of $\tilde{M}^{-1}$ (*i.e.*, the stable open-loop poles) reappear as closed-loop poles, and the unstable poles of $\tilde{M}^{-1}$ (*i.e.*, the unstable open-loop poles) reappear as closed-loop poles but at their mirror-image positions with respect to $j\Re$. If $W_d \neq I$, it can be used to modify the 'apparent' open-loop poles thereby effecting partial pole placement; recall $W_3 = \tilde{M}W_d$.

- *Stopband Shaping:*

The other function of W_d is stopband shaping. To see this effect, we assume that W_2 is small so that the upper term in Z_d is dominant. Then for an equalizing solution, we have $Z_d^\sim Z_d = \gamma_o^2 I$ so that $W_1SW_d \approx \gamma_o E$, where E is inner. If $W_1 = I$, then $S \approx \gamma_o E W_d^{-1}$, and we see that S is shaped proportionally to W_d^{-1}.

Choice of W_1 and W_2

By the argument of the preceding paragraph, W_1 also has a readily recognizable effect on the shape of S. For stopband shape adjustment, W_1 can be chosen to be a high-gain low-pass filter. In contrast to W_1, W_2 is normally chosen as a high-pass filter in order to achieve robust stability with respect to high-frequency model uncertainties. Moreover, as a standard criterion of optimization, W_1 and W_2 provide a trade-off between control effort and performance. However, it is clear that there is some overlap between the functional effects of W_1, W_2, W_c and W_d. For practical design, it is desirable to reduce this overlap as much as possible.

To investigate this situation, we will partly transform weights W_1, W_2 inside the feedback loop. For simplicity, we here assume $W_c = I$ and both W_1 and W_2 are units in $\Re\mathbf{H}^\infty$ with the form of $W_1(s) = W_{11}\phi(s)$ and $W_2(s) = W_{21}W_{22}(s)$ where W_{11}, W_{21} are real positive definite constant matrices and $\phi(s)$ is a scalar function in $\Re\mathbf{H}^\infty$. Define

$$G_a = \phi G W_{22}^{-1} \quad \text{and} \quad K_a = W_{22} K_2 \phi^{-1} .$$

Then we obtain

$$W_1 S = W_1 (I - GK_2)^{-1} = W_{11}(I - G_a K_a)^{-1}\phi \qquad (21)$$

and

$$W_2 K_2 S = W_{21} K_a (I - G_a K_a)^{-1}\phi . \qquad (22)$$

Note that here we can see that the effect of ϕ and W_{22} can be achieved partially by setting W_c as ϕW_{22}^{-1}.

Let Φ_l and Φ_r be spectral factors satisfying

$$\begin{aligned} \Phi_l^\sim \Phi_l &= N^\sim W_1^\sim W_1 N + M^\sim W_2^\sim W_2 M & (23)\\ \Phi_r \Phi_r^\sim &= \tilde{N} W_2^{-1}(W_2^{-1})^\sim \tilde{N}^\sim + \tilde{M} W_1^{-1}(W_1^{-1})^\sim \tilde{M}^\sim . & (24) \end{aligned}$$

Then the pairs $(\phi N \Phi_l^{-1},\ W_{22} M \Phi_l^{-1})$ and $(\Phi_r^{-1}\tilde{N} W_{22}^{-1},\ \Phi_r^{-1}\tilde{M}\phi^{-1})$ denoted by $(N_a,\ M_a)$ and $(\tilde{N}_a,\ \tilde{M}_a)$ respectively, are coprime factorizations of G_a. Thus we obtain from (23) and (24) that

$$\begin{aligned} \begin{bmatrix} N_w \\ M_w \end{bmatrix} &:= \begin{bmatrix} W_{11} N_a \\ W_{21} M_a \end{bmatrix} & (25)\\ [\tilde{N}_w \quad \tilde{M}_w] &:= [\tilde{N}_a W_{21}^{-1} \quad \tilde{M}_a W_{11}^{-1}] & (26) \end{aligned}$$

are inner and co-inner respectively. From the Bezout identity of (3), we also obtain

$$\begin{bmatrix} \tilde{X}_w & -\tilde{Y}_w \\ -\tilde{N}_w & \tilde{M}_w \end{bmatrix} \begin{bmatrix} M_w & Y_w \\ N_w & X_w \end{bmatrix} = I \qquad (27)$$

where

$$\begin{aligned} \begin{bmatrix} Y_w \\ X_w \end{bmatrix} &= \begin{bmatrix} W_2 Y \Phi_r \\ W_1 X \Phi_r \end{bmatrix} \left(=: \begin{bmatrix} W_{21} Y_a \\ W_{11} X_a \end{bmatrix} \right) & (28)\\ [\tilde{X}_w \quad -\tilde{Y}_w] &= [\Phi_l \tilde{X} W_2^{-1} \quad -\Phi_l \tilde{Y} W_1^{-1}] . & (29) \end{aligned}$$

Let Q_w denote $\Phi_l Q_d \Phi_r$. Then, from (21) and (22), $\hat{Z}_d$ in (18) is given by

$$\hat{Z}_d = \begin{bmatrix} X_w - N_w Q_w \\ Y_w - M_w Q_w \end{bmatrix} \Phi_r^{-1} \tilde{M} W_d . \qquad (30)$$

Thus by Lemma 1 (*i.e.*, $W_d = \tilde{M}^{-1} W_3$)

$$\begin{aligned} \hat{Z}_d &= \begin{bmatrix} X_w - N_w Q_w \\ Y_w - M_w Q_w \end{bmatrix} \Phi_r^{-1} W_3 \\ &= \begin{bmatrix} W_{11} & 0 \\ 0 & W_{21} \end{bmatrix} \begin{bmatrix} X_a - N_a Q_w \\ Y_a - M_a Q_w \end{bmatrix} \Phi_r^{-1} W_3 . \end{aligned}$$

From this we can see that if W_1 and W_2 are units, their dynamic parts (ϕ and W_{22}) can be included in W_c and W_3. Note that Φ_r is a function of W_1 and W_2.

Remark 5 For combining both disturbance minimization and tracking problems, we suggest that W_1 and W_2 are chosen as constant matrices to reduce the size of controller K. Their function will be regarded as the trade-off between y (or $r - y$) and u when solving optimal solutions.

Moreover, if W_3 is chosen as Φ_r, then the corresponding Nehari problem will be reduced to a simple one as shown below.

By an all-pass transformation we simplify $\|\hat{Z}_d\|_\infty$ as follows:

$$\begin{aligned} \|\hat{Z}_d\|_\infty &= \left\| \begin{matrix} X_w - N_w Q_w \\ Y_w - M_w Q_w \end{matrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} N_w & \tilde{M}_w^\sim \\ M_w & -\tilde{N}_w^\sim \end{bmatrix}^\sim \begin{bmatrix} X_w - N_w Q_w \\ Y_w - M_w Q_w \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{matrix} \tilde{M}_w^\sim Y_w + \tilde{N}_w^\sim X_w - Q_w \\ I \end{matrix} \right\|_\infty . \end{aligned}$$

This type of problem actually belongs to the class of 1-block Model-Matching Problems (MMP) (McFarlane, 1988) and hence is considerably easier to solve numerically.

Solution Analysis

Based on Remark 5, we will assume that W_1 and W_2 are positive definite symmetric constant matrices in the following.

By Lemma 1 and (7), the MMP of (18) is given by

$$\min_{\hat{K}_2 \text{ stabilizing}} \|\hat{Z}_d\|_\infty = \min_{Q_d \in \Re\mathbf{H}^\infty} \|T_{11} - T_{12} Q_d T_{21}\|_\infty \qquad (31)$$

where

$$T_{11} = \begin{bmatrix} W_1 X W_3 \\ W_2 Y W_3 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} W_1 N \\ W_2 M \end{bmatrix}, \quad T_{21} = W_3 . \qquad (32)$$

To form a Nehari problem from (31), we require an inner T_{12}. This can be done below by the construction of a coprime factorization of G_a (recall $G_a = NM^{-1} = \tilde{M}^{-1}\tilde{N}$).

Let $(A, B, C, 0)$ be a minimal realization of G_a (G and W_c are strictly proper so is G_a). From Nett *et al.* (1984), right coprime factorizations of G_a in terms of state-space realizations are given by

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A + BF & BD_r \\ \hline F & D_r \\ C & 0 \end{array} \right] \qquad (33)$$

where F is a state-feedback matrix and D_r is a nonsingular constant matrix. To make T_{12} inner, F and D_r are given by

$$\begin{aligned} D_r &= W_2^{-1} & (34)\\ F &= -(W_2^T W_2)^{-1} B^T P_o & (35) \end{aligned}$$

where

$$A^T P_o + P_o A - P_o B (W_2^T W_2)^{-1} B^T P_o + C^T W_1^T W_1 C = 0. \quad (36)$$

Similarly, we need a left coprime factorization of G_a such that $[-\tilde{N} W_2^{-1} \quad \tilde{M} W_1^{-1}]$ is co-inner. This is given by

$$[\tilde{N} \quad \tilde{M}] = \left[\begin{array}{c|cc} A + HC & B & H \\ \hline D_l C & 0 & D_l \end{array} \right] \qquad (37)$$

where

$$\begin{aligned} D_l &= W_1 & (38)\\ H &= -P_c C^T W_1^T W_1 & (39) \end{aligned}$$

and

$$A P_c + P_c A^T - P_c C^T W_1^T W_1 C P_c + B (W_2^T W_2)^{-1} B^T = 0. \quad (40)$$

Note that in fact we have (Bucy, 1972)

$$A + HC = (I + P_c P_o)(A + BF)(I + P_c P_o)^{-1} .$$

Realizations of the corresponding Bezout identity which only involve the eigenvalues of $A + BF$ are given by

$$\begin{bmatrix} M & Y \\ N & X \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & BW_2^{-1} & -HW_1^{-1} \\ \hline F & W_2^{-1} & 0 \\ C & 0 & W_1^{-1} \end{array} \right] \qquad (41)$$

and

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + HC & -B & H \\ \hline W_2 F & W_2 & 0 \\ W_1 C & 0 & W_1 \end{array} \right] . \qquad (42)$$

By the all-pass transformation matrix $\begin{bmatrix} W_1 N & W_1^{-1}\tilde{M}^\sim \\ W_2 M & -W_2^{-1}\tilde{N}^\sim \end{bmatrix}$ the Nehari problem of (31) is given by

$$\min_{\hat{K}_2 \text{ stabilizing}} \|\hat{Z}_d\|_\infty = \min_{Q \in \Re\mathbf{H}^\infty} \left\| \begin{matrix} R_1 - Q \\ R_2 \end{matrix} \right\|_\infty \qquad (43)$$

where

$$\begin{aligned} R_1 &= (M^\sim W_2^T W_2 Y + N^\sim W_1^T W_1 X) W_3 \\ R_2 &= W_3 \\ Q &= Q_d W_3 . \end{aligned}$$

Remark 6 The above formulation is explicitly in terms of the augmented plant G_a and weights W_1, W_2, W_3. The Nehari problem has been reached by solving just two AREs. From a numerical point view, this approach should be more efficient than that followed by Doye (1984).

Choice of W_3

It is clear that the complexity of the solution of (43) relies on the weight W_3. For the simple case $W_3 = I$ (*i.e*, $W_d = \tilde{M}^{-1}$), problem (43) is a 1-block problem and the optimal cost is $(1+\lambda_1(P_oP_c))^{\frac{1}{2}}$. For more practical designs, $W_3 \neq I$, and W_3 provides a mechanism for assigning desired closed-loop poles as discussed earlier.

Remark 7 (Paper in preparation). When $W_3W_3^{\sim} = \tilde{M}\tilde{M}^{\sim}$ (i.e. $W_d = I$), the *stable* poles of G_a will be cancelled by the optimal feedback controller $\hat{K}_2$ (resulting from an equalizing solution Q_d). However, for the case $W_3 = I$ (i.e. $W_d = \tilde{M}^{-1}$), we can prove that the optimal feedback system (with an equalizing solution) has no pole-zero cancellations between the plant and controller. In general, we can show that if the zeros of W_3 are different from the zeros of both $\tilde{M}$ and $\tilde{N}$, then the optimal feedback system will have no pole-zero cancellations between the plant and controller.

In the following, we derive a state-space formula for constructing W_3 in terms of two constant matrices. From (42), we have

$$\tilde{M} = \left[\begin{array}{c|c} A+HC & H \\ \hline W_1C & W_1 \end{array}\right] \quad \left(\tilde{M}^{-1} = \left[\begin{array}{c|c} A & -HW_1^{-1} \\ \hline C & W_1^{-1} \end{array}\right]\right) .$$

Using the pole placement method (Kautsky *et al.*, 1985), we can find a matrix H_d such that $A + H_dW_1C$ has specified eigenvalues. For more generality, we can also introduce a matrix D_d which results in $A + H_dW_1D_dC$ having the same eigenvalues as $A + H_dW_1C$. Hence we define

$$W_d = \left[\begin{array}{c|c} A & -H_dW_1 \\ \hline D_dC & I \end{array}\right]$$

and then $W_3 = \tilde{M}W_d$ will have its zeros equal to the eigenvalues of $A + H_dW_1C$ and its poles as the eigenvalues of $A + HC$ (*i.e.* the poles of $\tilde{M}$). For the case $D_d = I$,

$$W_3 = \left[\begin{array}{c|c} A+HC & H - H_dW_1 \\ \hline W_1C & W_1 \end{array}\right] .$$

If the realization is in an *observable canonical form*, the effect of D_d will be clear. D_d in the example shown below is chosen to make W_d diagonal.

Remark 8 The above provides a simple way of choosing weight W_3 (equivalently W_d) in terms of two constant matrices H_d and D_d. Using this construction, it is reasonably straightforward to keep as closed-loop poles those open-loop poles which are in desired locations and to shift the rest. Also, with this construction there exists an optimal $\hat{K}_2$ with McMillan degree ($deg(\hat{K}_2)$) strictly less than $deg(G_a)$, and for the final controller $K_2(= W_c\hat{K}_2)$ we have that $deg(K_2) < deg(\hat{K}_2) + deg(W_c)$ if there is at least one zero of W_3 which is also a pole of W_c (paper in preparation).

For more details on the solution to the tracking problem and the choice of W_t see (Tsai, 1989).

A DESIGN EXAMPLE

Consider the design example in Kwakernaak (1986) where the plant transfer function is

$$G(s) = \begin{bmatrix} \frac{1}{s^2} & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

and the design objectives are :

(a) Disturbance attenuation on both channels up to 1 *rad/sec*.

(b) Rejection of constant disturbances.

(c) Compensator roll-off of 20 dB/decade starting at the lowest frequency without affecting (a).

To achieve specifications (b) and (c), W_c is chosen as

$$W_c = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s} \end{bmatrix} .$$

and hence the augmented plant is

$$G_a = GW_c = \begin{bmatrix} \frac{1}{s^2(s+1)} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s(s+2)} \end{bmatrix} .$$

To combine disturbance minimization with tracking, (from Remark 5) we let $W_1 = I$ and $W_2 = 0.1\ I$ to trade-off the control input and output tracking errors. Moreover, to satisfy the design goal (a), we choose W_d (equivalently, W_3) such that the undesired poles of G_a at the origin are moved to $\frac{1}{\sqrt{2}}(-1 \pm i)$ and -1 respectively. Based on an observable canonical realization of G_a, we thus choose

$$D_d = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad H_d = \begin{bmatrix} -1 & 0 \\ -(1+\sqrt{2}) & 0 \\ -\sqrt{2} & 0 \\ 0 & -2 \\ 0 & -1 \end{bmatrix}$$

and then obtain

$$W_d = \begin{bmatrix} \frac{s^2+\sqrt{2}s+1}{s^2} & 0 \\ 0 & \frac{s+1}{s} \end{bmatrix} . \qquad (44)$$

In the tracking part, we consider a step reference signal and hence L^2 optimization is treated. Thus we choose $W_t = \frac{1}{s}I$. By solving two AREs for P_o and P_c, we obtain the eigenvalues of both $A + BF$ and $A + HC$ as

$$-1.0302 \pm 1.5708i, \ -2.0126, -2.8251 \pm 2.4699i \ .$$

The L^2 optimal feedforward Q_t is given by $N^{\sim}(0)$ (Tsai ,1989). It can be seen that there will be no steady-state tracking errors to steps since $M(0) = 0$ and $I - N(0)Q_t = 0$.

In the disturbance minimization, we obtain the optimal cost $\gamma_o = 1.8068$ using γ-iteration (Chu, 1985) and an optimal (equalizing) solution Q_d by formulae in Glover (1984) . Finally, the optimal 2DOF controller is given by $K = W_c\hat{K}$ where $\hat{K} = [\hat{K}_1,\ \hat{K}_2]$ can be found from Q_t and Q_d using (13)

We find that $deg(\hat{K}_2) = 4$ (*i.e.* $deg(G_a) - 1$) and $deg(K_2) = 5$ since the pole of W_c at -1 is cancelled by the controller as expected. The poles of $S = (I - GK_2)^{-1}$ are given by

$$-4.6112,\ -1.9885 \pm 1.7500i,\ -1.0599, -2,\ -1,\ -1,\ \frac{-1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$$

Note that the last 5 poles are also zeros of W_3. Because of this choice of W_3, S has poles at -1 and -2 which are also open-loop poles. Note also that the eigenvalues of $A + BF$ (i.e. poles of M, N, X, and Y) do not appear as closed-loop poles (as might be expected) because of the construction of W_3.

Finally, the singular values of the loop gain (G_aK) are shown in Figure 6. Note the high gains at low frequencies (less than 1 *rad/sec*) for disturbance attenuation and the low gains at high frequencies for robustness, (see Tsai (1989) for details).

CONCLUSIONS

The selection of weights is crucial in H^{∞} design. For example, an H^{∞} controller will often cancel open-loop stable poles and if these are in undesirable positions (e.g. lightly damped) an unsatisfactory design will result. In this paper, we have considered the selection of weights for a two-block mixed sensitivity problem and looked in detail at the effect the weights have on the various design objectives and the solution process. A constrained optimization is introduced (by augmenting the plant as in McFarlane (1988)) so that improper and unstable weights can be used and a controller with desired frequency roll-off obtained . It is also shown how the pole-zero cancellation phenomenon can be exploited or prevented by a suitable choice of weight.

REFERENCES

Bucy, R. (1972). The Riccati equation and its bounds. *J. Comput. Systems.*, **6**, 343.

Chu, C.C. (1985). H^∞ optimization and robust multivariable control. *Ph.D. thesis*, Dept. of Elec. Eng., University of Minnesota.

Doyle, J.C. (1984). Lecture notes in advances in multivariable control. *ONR/Honeywell Workshop*, Minneapolis, MN.

Francis, B.A. (1987). *A course in H_∞ control theory.* Springer.

Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their L^∞ error bounds. *Int. J. Control*, **39**, 1115.

Kautsky, J., N.K. Nichols and P. Van Dooren (1985). Robust pole assignment in linear state feedback. *Int. J. Control*, **41**, 1129.

Kwakernaak, H. (1986). A polynomial approach to minimax frequency domain optimization of multivariable feedback systems. *Int. J. Control*, **44**, 117.

McFarlane, D. (1988). Robust controller design using normalized coprime factor plant descriptions. *D.Phil Thesis*, University of Cambridge.

Nett, C.N., C.A. Jacobson, and M.J. Balas, (1984). A connection between state-space and doubly coprime fractional representations. *IEEE Trans. Auto. Control*, **AC-29**, 831.

O'Young, S.D., I. Postlethwaite and D.-W. Gu (1989). A treatment of $j\omega$-axis model-matching transformation zeros in the optimal H^2 and H^∞ control designs. *IEEE Trans. Auto. Control*, **AC-34**, 551.

Safonov, M.G. and R.Y. Chiang (1986). CACSD using the state-space L^∞ theory – a design example. *Proc. IEEE Conf. on CACSD.* Washington, DC.

Safonov, M.G. (1986). Imaginary-axis zeros in multivariable H^∞-optimal control. *Proc. of the NATO Advanced Research workshop on Modelling, Robustness and Sensitivity reduction in control systems*, University of Gronigen, Netherlands.

Stein, G. and M. Athans (1987). The LQG/LTR procedure for multivariable feedback control design. *IEEE Trans. Auto. Control*, **AC-32**, 105.

Tsai, M.C. (1989). Super-optimal control system design for multivariable plants. *D. Phil. Thesis, Oxford University.*

Youla, D.C., H. Jabr and J.J. Bongiorno (1976). Modern Wiener-Hopf design of optimal controllers-Part 2: the multivariable case. *IEEE Trans. Auto. Control*, **AC-21**, 319.

Youla, D.C. and J.J. Bongiorno (1985). A feedback theory of two degree-of-freedom optimal Weiner-Hopf design. *IEEE Trans. Auto. Control*, **AC-30**, 652.

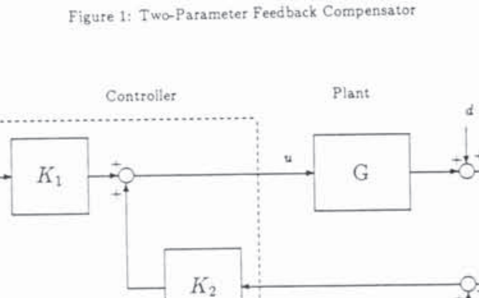

Figure 1: Two-Parameter Feedback Compensator

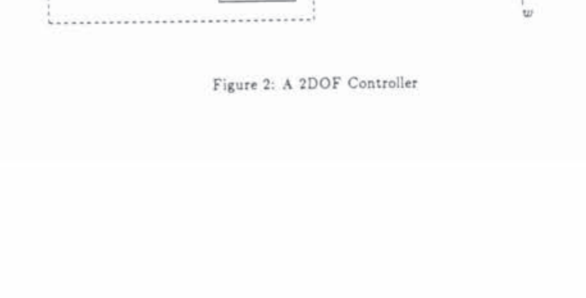

Figure 2: A 2DOF Controller

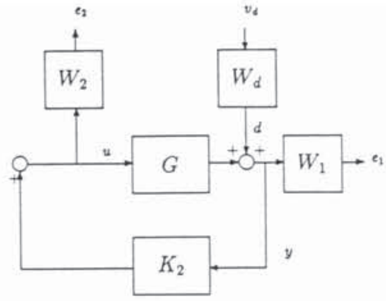

Figure 3: Disturbance Rejection Problem

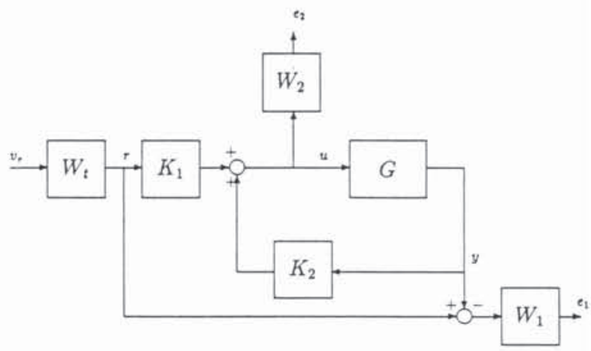

Figure 4: Tracking Problem

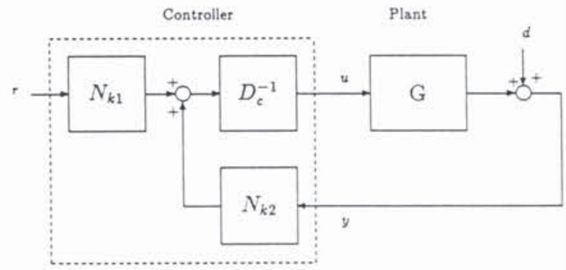

Figure 5: Implementation of a General 2DOF Controller

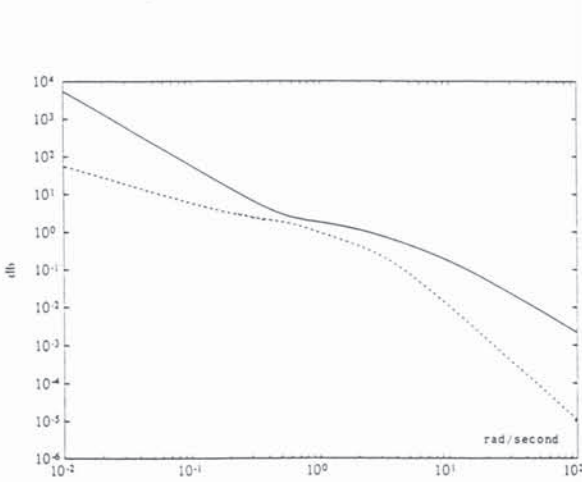

Figure 6: Singular values of the loop gain : G K