Philips Chair for Medical Information Technology (MedIT)
Helmholtz-Institute for Biomedical Engineering
RWTH Aachen University

Pauwelsstr. 20

52074-Aachen, Germany

**Master Course in Electrical Engineering**

# Assignment description: Quadruple Tank

Berno J.E. Misgeld

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# 1 Introduction

The quadruple tank is a relatively simple multivariable-input multivariable-output (MIMO) system. By changing the ratios of control valves, we can change the system behavior to minimum or non-minimum phase. The system can be controlled either by multiple single-input single-output (decentralised controller) (SISO) loops or only by one MIMO controller. The aims of this assignment are as follow:

• To analyse the nature of this system

• To interpret the meaning of relative gain array (RGA)

• To evaluate the concept of uncertainty modeling and $H_\infty$ loop-shaping design

• To gain an insight of controlling a MIMO system with minimum and non-minimum phases response

## 1.1 System description [3]

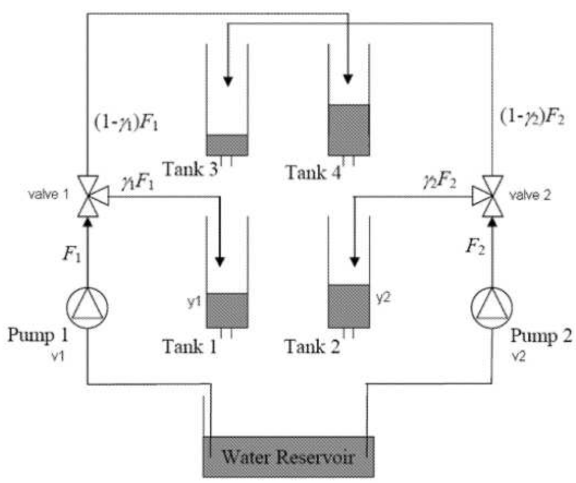
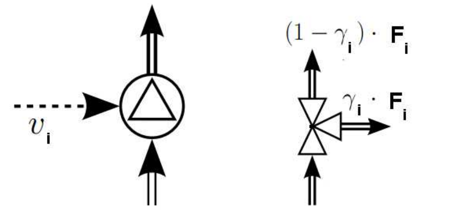



Figure 1: A schematic diagram of quadruple tank system.

The system of quadruple tanks is given in Figure 1. The objective is to control the level in the lower two tanks with two pumps. The voltage inputs to the pumps are $v_1$ and $v_2$ leading to the corresponding flow $F_1 = k_1 v_1$ and $F_2 = k_2 v_2$ and the outputs corresponding to water levels are $h_1$ and $h_2$ , where $A_i$ is the cross sectional of Tank i, $a_i$ is the cross sectional of outlet hole from Tank i. The parameters $\gamma_1$ and $\gamma_2 \in [0,1]$ determine how the valves are set. The flow to Tank 1 is $\gamma_1 k_1 v_1$ and the flow to Tank 4 is $(1-\gamma_1)k_1 v_1$. Similarly, the flow to Tank 2 is $\gamma_2 k_2 v_2$ and the flow to Tank 3 is $(1-\gamma_2)k_2 v_2$. Note that ideal pumps with no dynamics are assumed for a simplication in our model.




(a) Symbol of a pump with input voltage $v_i$ generating flow

(b) Symbol of valve splitting up pump generated flow ($\gamma_i \in (0,1)$)

Figure 2: Symbols used in the system configuration.

The flow of water into each tank are controlled by valves. By changing the flow ratios of the valve sections, it makes the system minimum or non-minimum phase.

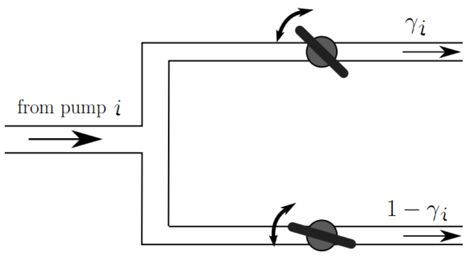



Figure 3: An anatomy of valve used in this system.

## 1.2 Organization of assignment handout

In Section 2, mathematical modelling of quadruple tank system will be given in the first section based upon mass balance and Bernoulli's law. The theory for trimming and linearisation of the quadruple tank system will be discussed in the second section. In Section 3, the exercises of the assignment are provided. Finally, Section 4 ends with a listing of the files and models provided with an instruction for compiling the report for final submission.

# 2 Mathematical modelling

## 2.1 Mass balance [3]

The principle of mass balance is governed in every tank $i$, which is based on the fact that the amount of water in each tank is equalled to the difference between the amount of water flow into the tank and the amount of water leaving out that particular tank.

$$\dot{V}_i = q_{in,i} - q_{out,i} = A_i \cdot \dot{h}_i \tag{1}$$

where $V_i$ denotes volume of water in Tank $i$,

$q_{in,i}$ is inflow into Tank $i$,

$q_{out,i}$ is outflow from Tank $i$,

$A_i$ symbolizes the cross-sectional area of Tank $i$

$\dot{h}_i$ represents a rate change of water level in Tank $i$.

## 2.2 Bernoulli's law [3]

Bernoulli's law can be derived from the principle of conservation of energy for incompressible liquids moving at low Mach numbers. In a steady flow, the sum of kinetic energy and potential energy remain constant. It can be expressed in the following equation.

$$p + \frac{1}{2}\rho v_w^2 + \rho g h = const \tag{2}$$

where $p$ denotes the pressure at the chosen point, $\rho$ is the density of the fluid at all points, $v_w$ symbolizes the fluid flow speed at the particular point and $g$ denotes gravity.

By considering on the water surface ($v_w$=0) and the bottom of each tank (h = 0), the outflow from Tank $i$ can be estimated ($a_i$ is the cross-sectional area of outlet in Tank $i$):

$$q_{out,i} = a_i\sqrt{2gh_i}. \tag{3}$$

## 2.3 Quadruple tank system

Based on mass balance and Bernoulli's laws, a mathematical model representing the dynamic of quadruple tank is derived as follow:

$$\frac{dh_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}v_1 \tag{4}$$

$$\frac{dh_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}v_2 \tag{5}$$

$$\frac{dh_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}v_2 \tag{6}$$

$$\frac{dh_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}v_1 \tag{7}$$

### 2.4 Trimming and linearisation of the system

For the successful design of linear controller, we have to linearise the nonlinear quadruple tank model. A general nonlinear time-variant system can be represented by the following form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \end{aligned} \tag{8}$$

with system dimensions $\mathbf{x} \in \mathbb{R}^{n\times 1}$, $\mathbf{u} \in \mathbb{R}^{m\times 1}$, $\mathbf{y} \in \mathbb{R}^{p\times 1}$ and the resulting linear system matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ of appropriate dimensions. We first have to find the equilibrium point, that is the point where all state-derivatives evaluate to zero.

$$\mathbf{f}(\mathbf{x}_{eq}, \mathbf{u}_{eq}, t_e q) = \mathbf{0}. \tag{9}$$

Note that a general nonlinear system may have one, multiple or no equilibrium points at pre-defined state conditions. For the linearization of the quadruple tank dynamics at a certain operating condition, one Simulink model and one M-script file are provided. The **incomplete** script file initialises all parameters and defines operating conditions for which the nonlinear model has to be linearised. Using the Simulink models, the quadruple tank dynamics is trimmed and linearised. For the linearisation of the quadruple tank dynamics, the Taylor-series approximation of first order is assumed:

$$\begin{aligned} \frac{d\Delta\mathbf{x}}{dt} &\approx \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{eq}, \mathbf{u}=\mathbf{u}_{eq}} \Delta\mathbf{x} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}_{eq}, \mathbf{u}=\mathbf{u}_{eq}} \Delta\mathbf{u} \\ \Delta\mathbf{y} &\approx \left.\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{eq}, \mathbf{u}=\mathbf{u}_{eq}} \Delta\mathbf{x} + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}_{eq}, \mathbf{u}=\mathbf{u}_{eq}} \Delta\mathbf{u} \end{aligned}, \tag{10}$$

for which the linear model resolves in

$$\begin{aligned} \frac{d\Delta\mathbf{x}(t)}{dt} &= \mathbf{A}\Delta\mathbf{x}(t) + \mathbf{B}\Delta\mathbf{u}(t) \\ \Delta\mathbf{y}(t) &= \mathbf{C}\Delta\mathbf{x}(t) + \mathbf{D}\Delta\mathbf{u}(t) \end{aligned}. \tag{11}$$

The state-space matrices are given by the partial differentiation

$$\begin{aligned} \mathbf{A} &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_{eq}\\ \mathbf{u}=\mathbf{u}_{eq}}} \in \mathbb{R}^{n\times n} \\ \mathbf{B} &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_{eq}\\ \mathbf{u}=\mathbf{u}_{eq}}} \in \mathbb{R}^{n\times m} \\ \mathbf{C} &= \left.\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_{eq}\\ \mathbf{u}=\mathbf{u}_{eq}}} \in \mathbb{R}^{p\times n} \\ \mathbf{D} &= \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_{eq}\\ \mathbf{u}=\mathbf{u}_{eq}}} \in \mathbb{R}^{p\times m} \end{aligned}, \tag{12}$$

which results in the water level dynamic derivatives for the quadruple tank system.

Linearising the nonlinear equations yields the following state-space form around the defined equilibrium point.

$$\begin{aligned} \begin{bmatrix} \dot{h}_1^* \\ \dot{h}_2^* \\ \dot{h}_3^* \\ \dot{h}_4^* \end{bmatrix} &= \begin{bmatrix} -\frac{a_1}{A_1}\sqrt{\frac{g}{2h_{10}}} & 0 & \frac{a_3}{A_1}\sqrt{\frac{g}{2h_{30}}} & 0 \\ 0 & -\frac{a_2}{A_2}\sqrt{\frac{g}{2h_{20}}} & 0 & \frac{a_4}{A_2}\sqrt{\frac{g}{2h_{40}}} \\ 0 & 0 & -\frac{a_3}{A_3}\sqrt{\frac{g}{2h_{30}}} & 0 \\ 0 & 0 & 0 & -\frac{a_4}{A_4}\sqrt{\frac{g}{2h_{40}}} \end{bmatrix} \cdot \begin{bmatrix} h_1^* \\ h_2^* \\ h_3^* \\ h_4^* \end{bmatrix} \\ &+ \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \end{aligned} \tag{13}$$

$$\begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} h_1^* \\ h_2^* \\ h_3^* \\ h_4^* \end{bmatrix} \tag{14}$$

## 2.5 System parameters and equilibrium points

The following table denotes a reference of system constants [2]:

Table 1: The parameter values of system.

| Constant | Value | Unit |
|---|---|---|
| $A_1, A_3$ | 30 | $cm^2$ |
| $A_2, A_4$ | 35 | $cm^2$ |
| $a_1, a_3$ | 0.071 | $cm^2$ |
| $a_2, a_4$ | 0.057 | $cm^2$ |
| $g$ | 981 | $cm/s^2$ |

**Note that** the model and control of the quadruple tank are evaluated at two different operating points $P_{mp}$ for minimum phase characteristics and $P_{np}$ for non-minimum phase characteristic. The parameter values are defined below:

Table 2: The parameter values of quadruple tank system at two equilibrium points.

| Parameters | $P_{mp}$ | $P_{np}$ | Unit |
|---|---|---|---|
| $(h_1^0, h_2^0)$ | (12.1, 12.6) | (6.79, 8.78) | $cm$ |
| $(h_3^0, h_4^0)$ | (2.50, 2.49) | (2.97, 4.17) | $cm$ |
| $(v_1^0, v_2^0)$ | (2.99, 2.97) | (2.53, 2.35) | $V$ |
| $(k_1, k_2)$ | (3.33, 3.35) | (3.14, 3.29) | $cm^3/Vs$ |
| $(\gamma_1, \gamma_2)$ | (0.60, 0.50) | (0.35, 0.30) | - |

# 3 Description of assignment tasks

## 3.1 Task 1: System analysis

### 3.1.1 Question 1: Trimming the model

Trim the model at the following (two different) operating points with the following conditions, namely two cases for $\gamma_1$ and $\gamma_2$ :

- $\gamma_1 = 0.60$ and $\gamma_2 = 0.50$ (minimum phase condition)
- $\gamma_1 = 0.35$ and $\gamma_2 = 0.30$ (non-minimum phase condition)

Compare the results with the symbolic linearisation.

**Note: All subsequent questions in this section and in other sections are based on both minimum phase and non-minimum phase conditions and sometimes called two different operating points. The answers for both cases have to be reported and organized creatively whenever it is necessary.**

### 3.1.2 Question 2: System representation and analysis

Represent the system in transfer function matrix form and calculate poles and zeros. Compare numerically using MATLAB. Determine associated pole and zero directions.

### 3.1.3 Question 3: Relative Gain Array (RGA)

Show that the RGA at $G(0)$ can be represented by $\begin{bmatrix} \lambda & (1-\lambda) \\ (1-\lambda) & \lambda \end{bmatrix}$, where $\lambda = \dfrac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1}$. Determine the RGA for different cases of $\gamma$ and discuss the results for the system. Additionally, determine the dynamic RGA and discuss implications for control.

### 3.1.4 Question 4: Parametric uncertainty for nonlinearities

Compare the entries of the state-space model linearised at the two different operating points. Propose ranges of parametric uncertainty if we want to model the nonlinearities as uncertainty.

### 3.1.5 Question 5: Singular values

Plot the singular value of the plant at different operating points and discuss the results in terms of multivariable control system properties (conditioning, bandwidth, etc.). Provide multiple results in a minimal number of plots.

## 3.2 Task 2: Decentralised control and uncertainty modeling

Two decentralised PI controllers (ignoring the coupling) are given for the two $\gamma_1$ and $\gamma_2$ configuration.

$$K_1 = \begin{bmatrix} K_1(1+\frac{1}{T_1 s}) & 0 \\ 0 & K_2(1+\frac{1}{T_2 s}) \end{bmatrix}$$

Use the RGA to analyse if the given input-output pairing is a good choice for the two $\gamma_1$ and $\gamma_2$ configurations. Based on your results, is it possible to find a fixed structure decentralised controller that can successfully stabilise minimum and non-minimum phase cases? (Hint: You might use the Niederlinski Index (NI) for the proof. The NI can be used to analyse the stability of control-loop pairings with previously evaluated steady-state RGA results

$$NI = \frac{\det(\mathbf{G}_0)}{\prod_{i=1}^{n} g_{ii,0}}, \tag{15}$$

with $\mathbf{G}_0$ the steady-state gain matrix and $g_{ii,0}$ the corresponding entries of the steady-state gain matrix pairings. A positive NI indicates stability in the control-loop pairings, while a negative NI corresponds to instability.)

### 3.2.1 Question 1: Decentralised controller based on linear and nonlinear models

Compute $K_1$, $K_2$, $T_1$ and $T_2$ in order to satisfy setting time $t_s < 45$ sec and overshoot $M_p <$ 10% and test the controller in the linearized model at different operating points (plot them into one figure). Compare with results with nonlinear simulations.

**Hint:** Based on a structure of second-order system $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$. The settling time and overshoot can be computed by $t_s \approx \frac{4.6}{\zeta\omega_n}$ and $M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$.

### 3.2.2 Question 2: Model uncertainty and weighting function

Assume uncertain parameters of the model $\gamma_1$ , $\gamma_2$ , $k_1$ , $k_2$ are ± 10% of nominal value and $a_1$ , $a_2$ , $a_3$ , $a_4$ are ± 5% of nominal value. Model the uncertainty via dynamic uncertainty.

Use an approximate weighting function. Describe the results and use operating point $x_0$ as the nominal model.

### 3.2.3 Question 3: Robust stability with dynamic uncertainty

Determine robust stability for the two decentralised controllers with the modeled dynamic uncertainty.

### 3.2.4 Question 4: Robust stability with parametric uncertainty

Determine robust stability for the two decentralised controllers with parametric uncertainties and describe your findings compared to robust stability with dynamic weighting function.

## 3.3 Task 3: Robust control design using mixed sensitivity approach

Specification for robust control design:

- Nominal stability
- Robust stability and model uncertainties
- If started with empty tanks, the output errors are less than 2 cm after 10 mins
- Steady state error is zero for unit step reference

### 3.3.1 Question 1: $H_\infty$ loop-shaping

Specify your $H_\infty$ loop-shaping weighting functions for a multivariable controller design for performance and uncertainty.

### 3.3.2 Question 2: $H_\infty$ mixed sensitivity design prerequisites

Design robust $H_\infty$ - controller via the mixed sensitivity procedure and determine control response with the multivariable plant for linear and nonlinear model.

### 3.3.3 Question 3: Nonlinear model and a $H_\infty$ controller

Consider the nonlinear model including sensor noise. Is the $H_\infty$ - controller response appropriate? If it is inappropriate, redesign $H_\infty$ loop-shaping controller using an additional weighting for $\mathbf{K}(s) \cdot \mathbf{S}(s) = \mathbf{R}(s)$

Note: Actuator limitations exist and should be regarded with 20 % of tolerance bound.

### 3.3.4 Question 4: Robust performance

Determine robust performance with respect to performance weighting and compare with decentralised PI-controllers (Use performance bound from 3.3.3 in this section).

## 3.4 Task 4: Loop/Plant modification

Modelling of a leckage in Tanks 3 and 4. A leckage can occur with diameter $A_{d30} = 0.08 \quad cm^2$ and $A_{d40} = 0.08 \quad cm^2$ at the bottom of the tank and the leckage outflows are out of the system, leading to the loss of volume in Tanks 3 and 4 only.

### 3.4.1 Question 1: Nonlinear model for leckage

Set up the nonlinear model for leckage.

### 3.4.2 Question 2: Linearisation for leckage

Linearise model for leckage and scale the whole system.

### 3.4.3 Question 3: Controller design for leckage

Redesign the $H_\infty$ loop-shaping controller with respect to disturbance rejection.

### 3.4.4 Question 4: Evaluation of disturbance rejection for leckage

Evaluate Robust Stability/Robust Performance and time response for the controller with regard to disturbance rejection.

### 3.4.5 Question 5: Evaluation of controller based on leckage

Use representative disturbance rejection simulated to evaluate controller response.

#### 3.4.6 Question 6: $\mu$ -synthesis [A bonus question!!!]

Controller design using $\mu$ -synthesis including the previous defined uncertainty and additional parameter variations provided by a number of linearised model. Describe the resulting performance and stability bounds with the structured singular value. Reduce the uncertainty to that of Question 3.2.2. What is the result? What about bounds for the uncertainty model?

## 4 List of provided files and information for report

### 4.1 List of provided files

Included to the uploaded zip file are the following files:

List of model files with description

- QUADTANK.M: Script file containing all nominal parameters necessary for the quadruple tank system.
- NLQUADTANK.MDL: Simulink model of the nonlinear quadruple tank system.

### 4.2 Information for writing the report

The deadline for submission of the report is announced via the online system of the course. Concerning guidelines of how to write the report, please regard the following information.

#### 4.2.1 Software used

Please hand-in the Matlab code for your assignment on a CD-ROM, in addition to your hard-copy report. Do not hand in temporary files of for example early stages of work. Software files provided, should be adjusted to the latest version of your work.
If you use software from third sources that solve or help to solve the given problems, provide code or references. You do not need to mention standard Matlab functions or toolbox code (like e.g. functions contained in the Robust Control Toolbox). If you develop your own code in addition to existing Matlab functions, please provide that on the CD-ROM. **All submitted code must be compatible with Matlab 2020b or lower versions**.

#### 4.2.2 Report writing

The report for this coursework can be written in English or German. The submitted report should be well organised and contain about 15-25 pages including introduction and bibliography and it must not exceed 30 pages. Identify formulas by reference numbers and refer to them accordingly. Use standard lecture nomenclature for formulas. If necessary provide a discussion and conclusion at the end of each particular question. In case you are not used to mathematical writing, refer to the literature, for example [1].

#### 4.2.3 Plagiarism

**If you copy from third sources (Figures, code or text), provide references. You are responsible for your report, so do not give it away. Even though you can solve the problems in a team you should write your own report, providing your individual interpretation of the achieved results. Do not copy discussions from your teammates. Of course, there is a penalty on this.**

#### 4.2.4 Figures

Figures in the report should be provided with appropriate axes description and units. If no units apply, denote this in the axis, e.g. y [-]. Provide figures that are easily readable, including legends to indentify various lines. Provide only most important results using figures. Limit the number of figures in your report and discuss the results presented in each figure sufficiently.

## References

[1] Paul R Halmos. How to write mathematics. *Enseign. Math*, 16(2):123–152, 1970.

[2] Karl Henrik Johansson. The quadruple-tank process: A multivariable laboratory process with an adjustable zero. *IEEE Transactions on Control Systems Technology*, 8(3):456–465, 2000.

[3] S. RICHTER. Quad tanks. *IfA Fachpraktikum , Automatic Control Laboratory, ETH Zürich*, Experiment(3.4):1–25, 2010.