1 Previous Formulations

1.1 MILP

$$
\begin{aligned}
\min & \sum_{s \in \mathcal{V}} \sum_{t \in \mathrm{N}_G^+(s) \setminus \{d^{\mathrm{e}}\}} x_{s,t} c^{\mathrm{v}} \\
& + \sum_{t \in \mathcal{T}} \sum_{s \in \mathrm{N}_G^-(t)} \left[x_{s,t} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_{t} \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t} \right) \right] && \text{(MILP)} \\
\text{s.t.} & \sum_{t \in \mathrm{N}_G^-(s)} x_{t,s} = \sum_{t \in \mathrm{N}_G^+(s)} x_{s,t} && \text{for all } s \in V \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.15) \\
& \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = 1 && \text{for all } t \in \mathcal{V} && (3.16) \\
& \sum_{t \in C^{-1}(c)} \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = 1 && \text{for all } c \in \mathcal{C} && (3.17) \\
& \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \leq x_{s,t} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.18) \\
& e_s \leq f^0_s && \text{for all } s \in \mathcal{V} && (3.19) \\
& 0 \leq e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.12) \\
& e_t \leq 1 - f^{\mathrm{t}}_t - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.13) \\
& e_t \leq e_s - x_{s,t} \left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t \right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t} \right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.14) \\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t) \in A && (3.20) \\
& z_{s,r,t} \in \{0,1\} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t), r \in \mathcal{R}_{s,t} && (3.21) \\
& e_s \in [0,1] && \text{for all } s \in V \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.22)
\end{aligned}
$$

1.2 AMILP

$$
\begin{aligned}
\min \sum_{s \in \mathcal{V}} & \sum_{t \in \mathrm{N}_G^+(s) \setminus \{d^{\mathrm{e}}\}} x_{s,t} c^{\mathrm{v}} \\
& + \sum_{t \in \mathcal{T}} \sum_{s \in \mathrm{N}_G^-(t)} \left[x_{s,t} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_{t} \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t} \right) \right] && \text{(AMILP)} \\
\text{s.t.} & \sum_{t \in \mathrm{N}_G^-(s)} x_{t,s} = \sum_{t \in \mathrm{N}_G^+(s)} x_{s,t} && \text{for all } s \in V \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.15) \\
& \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = 1 && \text{for all } t \in \mathcal{T} \cup \mathcal{V} && (6.2) \\
& \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \leq x_{s,t} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.18) \\
& e_s \leq f^0_s && \text{for all } s \in \mathcal{V} && (3.19) \\
& 0 \leq e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.12) \\
& e_t \leq 1 - f^{\mathrm{t}}_t - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.13) \\
& e_t \leq e_s - x_{s,t} \left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t \right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t} \right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.14) \\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t) \in A && (3.20) \\
& z_{s,r,t} \in \{0,1\} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t), r \in \mathcal{R}_{s,t} && (3.21) \\
& e_s \in [0,1] && \text{for all } s \in V \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.22)
\end{aligned}
$$

2 Ideas

2.1 LP Approach

We consider the possibility of multimodal transport. The following holds:

- Each costumer has a set of alternative multimodal routes; exactly one of them must be fulfilled.
- A multimodal route consists of a sequence of trips; each of them can be a car trip or a public transport trip.

The routes are given in advance. The car trips are adjusted in such a way, that they fit to the public transport routes (in location and time). It is not possible to model each route as a trip, because then the car availabilities are not considered.
We model a flow of the cars. The public transport trips are only constraints for this flow. From the fulfilled routes, we can easily determine the corresponding public transport trips and maybe add their costs to the objective function.

MMILP

Let $\mathcal{T}$ be the set of car trips, $\mathcal{M}$ the set of multimodal routes, $\mathcal{C}$ the set of costumers. Then, $C : \mathcal{M} \to \mathcal{C}$ maps the routes to costumers and $M : \mathcal{T} \to \mathcal{M}$ maps the trips to routes.
We build on the (MILP). The constraint (3.17) is modified to

$$\sum_{t \in (M \circ C)^{-1}(c)} \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} \geq 1 \qquad \text{for all } c \in \mathcal{C}. \tag{1}$$

We have ≥ 1 instead of $= 1$ because the route can have more than one trip. Due to minimization, not more than one route will be selected. To guarantee, that each trip of a selected route is fulfilled, we insert

$$\sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} \geq \sum_{s \in \mathrm{N}_G^-(t')} x_{s,t'} \qquad \text{for all } t \in \mathcal{T}, t' \in M^{-1}(t) \setminus \{t\}. \tag{2}$$

Considerations

1. Is it sufficient to only allow more than one leg for multimodal transport?
2. Are there more effects on the model? No refueling at public transport

3. What are the costs for public transport? No vehicle costs, only trip costs
4. How are the data for public transport created?
5. Is there a connection between the heuristic (Knoll) and the optimal approach (Kaiser)? Branch and Bound with result of the heuristic as a starting point
6. Does the costumer prefer a route? Less total time, less costs, fewer changes

2.2 Heuristic

This is an extension to the Successive Heuristic (Knoll, cap. 10). This heuristic further allows alternative routes for costumers. Each costumer has a set of alternative multi-modal routes, consisting of a sequence of trips. For each costumer, exactly one route has to be fulfilled; if a route is fulfilled, every trip of this route is fulfilled.

First Approach

In contrast to the previous heuristic, not the trips but the costumer requests are split. This means, all trips of a route and all routes of a costumer are in the same splitting. How the splitting is performed (e.g., first start time, last end time) is not specified here. For each splitting, we apply (MMILP) to find an optimal solution and connect the solutions as before.
Our goal is the model equivalence between the heuristic and the (MMILP). For this, we consider the following example:

Example 1. Let t_1, t_2, t_3 with $t_1 \rightarrow t_2 \rightarrow t_3$ be trips with the following properties:

Trip	Start	End	Route	Costumer
t_1	8:00	8:15	M_1	C_1
t_2	8:30	8:45	M_2	C_2
t_3	9:00	9:15	M_1	C_1

Table 1: Trips

In this case, costumer C_1 uses public transport between 8:15 and 9:00. The duty (t_1, t_2, t_3) is a feasible result of the (MMILP).

We distinguish between 3 cases:

1. t_1, t_2, t_3 are in the same splitting
   $\Rightarrow (t_1, t_2, t_3)$ is feasible since (MMILP)
2. t_1, t_3 are in the same splitting, t_2 is before or after this splitting
   $\Rightarrow (t_1, t_3)$ and (t_2) are not connectable

3. t_1, t_3 are not in the same splitting
   $\Rightarrow$ this is not possible according to the heuristic

Question 1. Can we guarantee that case 2 cannot happen? Can we make restrictions to the splitting length, route extensions, trips lengths or choice of splitting, such that case 2 cannot happen?
$\Rightarrow$ No. The splitting point can be arbitrarily bad.

Improved Approach

With this example we have seen, that it has to be possible to split the trips of a costumer. If these trips are in different splittings, we are not able to maintain feasibility of the whole problem. Therefore, we need a new approach:
We create the splittings as in the Successive Heuristic. Each trip is assigned to a splitting according to its start time. If one trip is chosen (in the earlier processed splitting), then the other trips of this route are fixed in their (later processed) splittings. This means, these trips have to be fulfilled in the later splittings.

Example 2. We have only one costumer C_1 with two alternative multimodal routes M_1, M_2. For this holds:

$$M_1 = (t_1, t_2) \quad M_2 = (t_3) \qquad\qquad \mathcal{T}_1 = \{t_1\} \quad \mathcal{T}_2 = \{t_2, t_3\}$$

We assume that $\mathcal{T}_1$ is processed first. If we set the costumer constraints (1) in $\mathcal{T}_1$, then the heuristic is forced to use M_1. Otherwise, the heuristic does not choose t_1 in $\mathcal{T}_1$ due to optimality and therefore is forced to use M_2.

To avoid this problem, we insert dummy trips when this is necessary. This means, we insert

$$t^{\mathrm{d}}_{i,m} \qquad\qquad \text{for all } c \in \mathcal{C}, m \in C^{-1}(c), i \in [n]$$

if there exist $s \in (M \circ C)^{-1}(c)$ with $s \in \mathcal{T}_i$ and there is no $s' \in M^{-1}(m)$ with $s' \in \mathcal{T}_i$. The dummy trips do not affect the model except for the costumer constraint (1) and in the objective function. We define $\mathcal{T}^{\mathrm{d}}_{i,c}$ as the set of dummy nodes for splitting $i \in [n]$ and costumer $c \in \mathcal{C}$. Then, the modified costumer constraints in splitting i are

$$\sum_{t \in (M \circ C)^{-1}(c)} \sum_{s \in \mathrm{N}^-_G(t)} x_{s,t} + \sum_{t \in \mathcal{T}^{\mathrm{d}}_{i,c}} y_t \geq 1 \qquad\qquad \text{for all } c \in \mathcal{C} \tag{3}$$

Using this approach, we have to decide already in splitting 1 which multimodal route is chosen for costumer C_1. This is difficult because we do not know which costs arise with using M_1 or M_2 in advance.

Question 2. How can we estimate the costs for the multimodal routes?

We can introduce some multimodal route costs, by which we decide at the beginning of a splitting, which route is chosen for the costumer. There we have to consider not only the sum of the trip costs but also factors like the fuel states or the location of the cars (to reduce the deadhead costs). The route costs can be introduced similar to w^{heur} (Kaiser, Knoll, cap. 4).

Feasibility

The heuristic has to provide only feasible solutions for the main problem. Furthermore, each feasible solution of the main problem should be a feasible result in the heuristic.
Except for the choice of trips, the subproblems are solved and connected in a feasible way with the Successive Heuristic. The dummy nodes only occur in (3) and in the objective, the trip choice only in (2) and (3). These constraints are maintained as follows: The route is chosen for the costumer definitively in the first splitting where the costumer occurs (either by a trip or a dummy trip). Therefore, each costumer has a route (3). Then, all other trips of this costumer are fixed in the following splittings (2). Hence, the heuristic provides a feasible solution.
Consider a feasible overall solution. The trips that have to be used in this solution are already chosen in a feasible way. Then, the heuristic is the same as the Successive Heuristic (Knoll, cap. 10).

Considerations

1. How long are the routes? Important for the choice of the splitting lengths
2. Are all routes of a costumer in a similar time window? Restrict number of possible splittings for costumers

3 Problem Formulation

3.1 Notation and Model

This formulation models the problem of optimal integration of autonomous vehicles in car sharing, considering multimodal transport.

Notation

We are given a set of vehicles $\mathcal{V}$ and a set of costumers $\mathcal{C}$. For public transport, we have a set of available stations $\mathcal{S}$ and a set of public transport rides $\mathcal{P}$. A ride $p \in \mathcal{P}$ is a sequence of stops at time points $p = ((s_1, z_1), \ldots, (s_k, z_k))$ with $s_i \in \mathcal{S}$ and z_i a time point for $i \in [k]$.
We are further given a set of trips $\mathcal{T}$; each trip $t \in \mathcal{T}$ is either a car trip or a public transport trip and has a start and end location $p_t^{\text{start}}, p_t^{\text{end}}$ and a start and end time $z_t^{\text{start}}, z_t^{\text{end}}$. Accordingly, we define $\mathcal{T} = \mathcal{T}_{\text{car}} \mathbin{\dot\cup} \mathcal{T}_{\text{public}}$. A public transport trip $t \in \mathcal{T}$ is a subsequence of a public transport ride $p \in \mathcal{P}$ and it holds

$$p_t^{\text{start}} = s_i^p \qquad p_t^{\text{end}} = s_j^p \qquad z_t^{\text{start}} = z_i^p \qquad z_t^{\text{end}} = z_j^p$$

for some $i < j$.
The start position and the starting time of a vehicle $v \in \mathcal{V}$ is p_v and z_v. The time, a vehicle needs from its start location to a trip or from one trip to another is $t_{s,t}$ for $s \in \mathcal{V} \cup \mathcal{T}, t \in \mathcal{T}$.
We define a partial order $\preceq$ on $\mathcal{V} \times \mathcal{T}$. We say

$$s \preceq t \qquad \text{if } z_s^{\text{end}} + t_{s,t} \leq z_t^{\text{end}} \qquad \text{for } s \in \mathcal{V} \times \mathcal{T}, t \in \mathcal{T}$$

The expression $s \preceq t$ means, that one car is able to fulfill both trips, first s and then t. If one of the trips is a public transport trip, one costumer can use both of these trips.
We are given a set of multimodal routes $\mathcal{M}$. A route $m = (t_1, \ldots, t_k)$ is a sequence of trips with to following properties:

$$p_{t_i}^{\text{end}} = p_{t_{i+1}}^{\text{start}} \qquad t_i \preceq t_{i+t} \qquad t_i \in \mathcal{T}_{\text{car}} \Rightarrow t_{i+1} \in \mathcal{T}_{public} \qquad \text{for all } i \in [k-1]$$

We define the route start and end locations and times

$$p_m^{\text{start}} = p_{t_1}^{\text{start}} \qquad p_m^{\text{end}} = p_{t_k}^{\text{end}} \qquad z_m^{\text{start}} = z_{t_1}^{\text{start}} \qquad z_m^{\text{end}} = z_{t_k}^{\text{end}}.$$

Each costumer $c \in \mathcal{C}$ has a set of alternative routes. $C : \mathcal{M} \to \mathcal{C}$ maps the routes to costumers and $M : \mathcal{T} \to \mathcal{M}$ maps the trips to routes. For each route of the same costumer $m \in C^{-1}(c)$, the start and end positions are the same, the start and end times may differ.
Additionally, we have a set of refuel stations $\mathcal{R}$. A refuel station $r \in \mathcal{R}$ has the location p_r. In this model, a car is allowed to refuel at most once between two trips. We define $f^{\mathrm{d}}_{s,t}$ for $s \in \mathcal{V} \cup \mathcal{T} \cup \mathcal{R}, t \in \mathcal{T} \cup \mathcal{R}$ as the amount, the fuel level decreases along the deadhead trip. f^{t}_t for $t \in \mathcal{T} \cup \mathcal{R}$ is the amount of fuel, the car needs for a trip. For $r \in \mathcal{R}$ holds $f^{\mathrm{t}}_r \leq 0$.

Problem Description

A feasible solution is a schedule of trips for every vehicle including refueling stops and a sequence of trips for every costumer. These trips are fulfilled by the scheduled car or by public transport according to its timetable. For this, we have the following constraints:

- Each car is able to serve its scheduled trips, considering time and location.
- The fuel state of each car is always in a feasible range.
- Each costumer is able to complete his trip, considering time and location.
- For each costumer, exactly one trip is chosen.

The goal is to find a cost-minimal feasible schedule considering all these constraints.

Costs

We have the following types of costs:

- Vehicles costs c^{v}: unit costs for each used car
- Deadhead costs $c^{\mathrm{d}}_{s,t}$ for $s \in \mathcal{V} \cup \mathcal{T} \cup \mathcal{R}, t \in \mathcal{T} \cup \mathcal{R}$: costs, if a car drives to a trip or a refuel station without a costumer using it
- Trip costs c^{t}_t for $t \in \mathcal{T}_{\mathrm{car}}$: costs for fulfilling a trip

For public transport, we define either trip costs for each public transport trip or fixed costs for each costumer using public transport. Finally, we define costs to consider the costumer preferences. These costs can be the total time or the number of changes.

- Trip costs c^{t}_t for $t \in \mathcal{T}_{\mathrm{public}}$: costs for using public transport
- Route-dependent costs c^{r}_m for $m \in \mathcal{M}$: costs for costumer preferences and unit costs for using public transport

3.2 LP Considerations

Creation of Routes

In reality, we are not given a set of multimodal routes. We have only $\mathcal{C}, \mathcal{S}, \mathcal{P}, \mathcal{V}$. For each costumer $c \in \mathcal{C}$, we have a start and end location $p_c^{\text{start}}, p_c^{\text{end}}$ and a time interval $\left[z_c^{\text{start}}, z_c^{\text{end}}\right]$, in which all routes are located.
How we determine the routes, we have not yet considered. Since a car can drive to every station, where the public transport trip starts, the number of alternative routes can be very large. Therefore, we will have to develop a preprocessing in order to reduce the number of alternatives.

LP Constraints

We build on the (MILP) formulation (Kaiser, Knoll, cap. 3). For our problem, we make the following adaptions: The variables $x_{s,t} \in \{0,1\}, z_{s,r,t} \in \{0,1\}, e_s \in [0,1]$ are the same as in the (MILP). For the trips in this formulation, only the car trips $t \in \mathcal{T}_{\text{car}}$ are considered.
We initialize new variables $u_t \in \{0,1\}$ for $t \in \mathcal{T}_{\text{public}}$ and $v_m \in \{0,1\}$ for $m \in \mathcal{M}$. u_t indicate, whether a public transport trip is fulfilled or not; v_m indicate, whether a multimodal route is fulfilled.
We replace the constraint (3.17) by

$$\sum_{m \in C^{-1}(c)} v_m = 1 \qquad \text{for all } c \in \mathcal{C} \tag{4}$$

$$\sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} \geq v_m \qquad \text{for all } m \in \mathcal{M}, t \in M^{-1}(m) \cap \mathcal{T}_{\text{car}} \tag{5}$$

$$u_t \geq v_m \qquad \text{for all } m \in \mathcal{M}, t \in M^{-1}(m) \cap \mathcal{T}_{\text{public}} \tag{6}$$

This formulation is equivalent to (1) and (2), but is more suitable to formulate the objective function.

Objective Function

This objective function considers unit vehicle costs, unit public transport costs, trip costs for vehicles and public transport, deadhead costs for vehicles and user preferences.

$$\min \sum_{s\in\mathcal{V}} \sum_{t\in \mathrm{N}_G^+(s)\setminus\{d^{\mathrm{e}}\}} x_{s,t}c^{\mathrm{v}} + \sum_{t\in\mathcal{T}_{\mathrm{public}}} u_t c_t^{\mathrm{t}} + \sum_{m\in\mathcal{M}} v_m c_m^{\mathrm{r}}$$
$$+ \sum_{t\in\mathcal{T}_{\mathrm{car}}} \sum_{s\in \mathrm{N}_G^-(t)} \left[x_{s,t}\left(c_{s,t}^{\mathrm{d}} + c_t^{\mathrm{t}}\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c_{s,r}^{\mathrm{d}} + c_{r,t}^{\mathrm{d}} - c_{s,t}^{\mathrm{d}}\right)\right] \tag{MILP'}$$

4 Iterative Heuristic

This is an iterative heuristic. We find an initial solution or the problem. We choose the routes for the costumers and then solve the resulting problem (equivalent to the problem without costumer choice). Considering this solution, we find costumers, for which the route choices were bad. For these costumers, we find more suitable routes. With this, we improve the solution step by step.

Initial solution

Let $[n]$ be the set of partial instances, let $\sigma \in S_n$ with $\sigma(n) = 1$ be the order, in which the partial instances are processed. This means, partial instance $\sigma(i)$ is solved at the i-th position, the first partial instance is solved at last.
We consider partial instance $\sigma(i) \in [n]$ and costumer $c \in \mathcal{C}$, let i be the first solved instance, where multimodal routes of this costumer occur. This means, in this partial instance we are choosing definitively, which route is taken by this costumer. For the costumer, we have some trips in this splitting. For all routes of this costumer, we may have trips in subsequent splittings, too. For all routes, that are not represented in this splitting, we have dummy trips $t^{\mathrm{d}}_{i,m}$.
Since we do not have any knowledge about the subsequent partial instances, we want to estimate the route costs as exactly as possible. The total costs are

$$C^* = C^{\mathrm{v}} + C^{\mathrm{r}} + C^{\mathrm{t}} + C^{\mathrm{d}}. \tag{7}$$

We can only determine the trip costs and the route costs in advance. Thus, we define

$$C_1(m) := c^{\mathrm{r}}_m + \sum_{t \in m} c^{\mathrm{t}}_t \qquad \text{for all } m \in \mathcal{M}. \tag{8}$$

These costs are used to determine which route is chosen. For this, we replace the route costs c^{r} in (MILP') as follows:

$$\hat{c}^{\mathrm{r}}_m := c^{\mathrm{r}} + \sum_{t \in m \setminus \mathcal{T}_i} c^{\mathrm{t}}_t \qquad \text{for all } m \in \mathcal{M} \tag{9}$$

In $\hat{c}^{\mathrm{r}}$, we consider additionally the trip costs of the trips, that are not in this splitting. It is beneficial, if there are many trips of this costumer in the considered splitting. In this case, we can consider structure of the problem in more detail, according to the (MILP'). The other case is, that there are only few trips of this costumer in the splitting, the majority in other splittings. Then, we cannot consider many structural properties.

With solving this partial instance, we have chosen implicitly the route of this costumer. Therefore, in the subsequent partial instances, this costumer uses, are already certain. This trips are then fixed in the respective constraints.

Subproblem

Given a solution of the problem, the subproblem is to find a costumer with a bad route choice. This means, for this costumer there is another route, such that the total costs are lower if this route is chosen. Then, we can exchange these routes and compute a new solution considering the new route.

Question 3. How can we find a bad route?
We have no efficient method to find a route, for which we can guarantee better total costs.

An initial idea is to compute the costs, one route in the solution contributes to the entire solution. Then, we can compare this to the cost, with which we estimated the route costs before. If the actual costs are considerably higher than the estimate costs, this costumer is a candidate for exchanging routes.
Since we cannot determine the contributing costs exactly, we try to estimate these costs. Let $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u}, \bar{v})$ be a solution of the (MILP'). To determine the contributing costs for route $m \in \mathcal{M}$, we first define the following auxiliary costs for every trip $t \in \mathcal{T}$ of the solution:

Definition 1. Vehicle costs $c^{\mathrm{v}}_{S,t}$: Let $v \in \mathcal{V}$ be the vehicle covering t and k_v the number of trips covered by v:

$$c^{\mathrm{v}}_{S,t} := \frac{c^{\mathrm{v}}}{k_v} \tag{10}$$

Refueling costs $c^{\mathrm{refuel}}_{S,t}$: Let $r \in \mathcal{R}$ be the next refuel station used after t and $\mathcal{T}_r$ all trips covered since the last station, let $\bar{z}_{s,r,s'} = 1$:

$$c^{\mathrm{refuel}}_{S,t} := \frac{f^{\mathrm{t}}_t}{\sum_{t' \in \mathcal{T}_r} f^{\mathrm{t}}_t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,s'} - c^{\mathrm{d}} s, s' \right) \tag{11}$$

If the vehicle is not refueled after t, then $c^{\mathrm{refuel}}_{S,t} := 0$.
Deadhead costs $c^{\mathrm{d}}_{S,t}$: Let $s \in \mathcal{V} \cup \mathcal{T}_{\mathrm{car}}, s' \in \mathcal{T}_{\mathrm{car}}$ be the trips covered directly before and after t by vehicle v, i.e. $\bar{x}_{s,t} = \bar{x}_{t,s'} = 1$:

$$c^{\mathrm{d}}_{S,t} := \frac{1}{2} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{d}}_{t,s'} \right) \tag{12}$$

If t is the last trip of the duty, i.e. $\bar{x}_{t,d^{\mathrm{e}}} = 1$, then $c^{\mathrm{d}}_{S,t} := \frac{1}{2} c^{\mathrm{d}}_{t,s}$.