Master Thesis

Optimal Integration of Autonomous Vehicles in Car Sharing: Development of a Heuristic considering Multimodal Transport and Integration in an Optimal Framework

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1 Problem Formulation

1.1 Problem Description and Notation

This formulation models the problem of optimal integration of autonomous vehicles in car sharing, considering multimodal transport.

Notation

We are given a set of vehicles $\mathcal{V}$ and a set of costumers $\mathcal{C}$. For public transport, we have a set of available stations $\mathcal{S}$ and a set of public transport rides $\mathcal{P}$. A ride $p \in \mathcal{P}$ is a finite sequence of stations at time points $p = ((s_1, z_1), \ldots, (s_k, z_k))$ with $s_i \in \mathcal{S}$ and z_i a time point for $i \in [k]$.
We are further given a set of trips $\mathcal{T}$; each trip $t \in \mathcal{T}$ is either a car trip or a public transport trip and has a start and end location $p_t^{\text{start}}, p_t^{\text{end}}$ and a start and end time $z_t^{\text{start}}, z_t^{\text{end}}$. Accordingly, we define $\mathcal{T} = \mathcal{T}_{\text{car}} \mathbin{\dot\cup} \mathcal{T}_{\text{public}}$. A public transport trip $t \in \mathcal{T}_{\text{public}}$ is a connected subsequence of a public transport ride $p \in \mathcal{P}$ and it holds

$$p_t^{\text{start}} = s_i^p \qquad p_t^{\text{end}} = s_j^p \qquad z_t^{\text{start}} = z_i^p \qquad z_t^{\text{end}} = z_j^p$$

for some $i < j$.
The start position and the starting time of a vehicle $v \in \mathcal{V}$ is p_v and z_v.
Additionally, we have a set of refuel stations $\mathcal{R}$. A refuel station $r \in \mathcal{R}$ has a location p_r. In this model, a car is allowed to refuel at most once between two trips. We define $f_{s,t}^{\text{d}}$ for $s \in \mathcal{V} \cup \mathcal{T} \cup \mathcal{R}, t \in \mathcal{T} \cup \mathcal{R}$ as the amount, the fuel level decreases along the deadhead trip. f_t^{t} for $t \in \mathcal{T} \cup \mathcal{R}$ is the amount of fuel, the car needs for a trip. For $r \in \mathcal{R}$ holds $f_r^{\text{t}} \leq 0$. f_v^0 for $v \in \mathcal{V}$ is the initial fuel state of a car. The fuel of a car is in the interval $[0, 1]$ describing the relative fuel state.
We define the time, a car needs to get from position p_1 to p_2, as t_{p_1,p_2}. We define

$$t_{s,t} = \begin{cases} t_{p_s^{\text{end}}, p_t^{\text{start}}} & \text{if } s, t \in \mathcal{T}_{\text{car}} \\ t_{p_s, p_t^{\text{start}}} & \text{if } s \in \mathcal{V} \cup \mathcal{R}, t \in \mathcal{T}_{\text{car}} \\ t_{p_s^{\text{end}}, p_t} & \text{if } s \in \mathcal{T}_{\text{car}}, t \in \mathcal{R} \\ t_{p_s, p_t} & \text{if } s \in \mathcal{V}, t \in \mathcal{R} \end{cases}$$

as the time a car needs from one trip to another.
We are given a set of multimodal routes $\mathcal{M}$. A route $m = (t_1, \ldots, t_k)$ is a sequence of trips with the following properties:

$$p_{t_i}^{\text{end}} = p_{t_{i+1}}^{\text{start}} \qquad z_{t_1}^{\text{end}} \leq z_{t_{i+1}}^{\text{start}} \qquad t_i \in \mathcal{T}_{\text{car}} \Rightarrow t_{i+1} \in \mathcal{T}_{public} \qquad \text{for all } i \in [k-1].$$

The mapping $M : \mathcal{T} \to \mathcal{M}$ shows to which route a trip belongs. We define the route start and end locations and times for $m \in \mathcal{M}$

$$p_m^{\text{start}} := p_{t_1}^{\text{start}} \qquad p_m^{\text{end}} := p_{t_k}^{\text{end}} \qquad z_m^{\text{start}} := z_{t_1}^{\text{start}} \qquad z_m^{\text{end}} := z_{t_k}^{\text{end}}.$$

Each costumer $c \in \mathcal{C}$ has a finite set of alternative routes. The mapping $C : \mathcal{M} \to \mathcal{C}$ shows which route belongs to which costumer. For each route of the same costumer $m \in C^{-1}(c)$, the start and end positions are the same, the start and end times may differ. We define the costumer start and end times for $c \in \mathcal{C}$

$$z_c^{\text{start}} := \min_{m \in C^{-1}(c)} z_m^{\text{start}} \qquad z_c^{\text{end}} := \max_{m \in C^{-1}(c)} z_m^{\text{end}}. \tag{1}$$

Problem Description

The problem is the following: Find a schedule of trips for every vehicle including refueling stops and a sequence of trips for every costumer. Therefore, the car trips are fulfilled by the scheduled car and the public transport trips by public transport according to its timetable. For this, we have the following conditions:

- Each car is able to serve its scheduled trips, considering time and location.
- The fuel state of each car is always in a feasible range.
- Each costumer is able to complete his route, considering time and location.
- For each costumer, exactly one route is chosen.

The goal is to find a cost-minimal feasible schedule considering all these constraints.

Costs

We have the following types of costs:

- Vehicles costs c^{v}: unit costs for each used car
- Deadhead costs $c^{\text{d}}_{s,t}$ for $s \in \mathcal{V} \cup \mathcal{T} \cup \mathcal{R}, t \in \mathcal{T} \cup \mathcal{R}$: costs, if a car drives to a trip or a refuel station without a costumer using it
- Trip costs c^{t}_t for $t \in \mathcal{T}_{\text{car}}$: costs for fulfilling a trip

For public transport, we define either trip costs for each public transport trip or fixed costs for each costumer using public transport. Finally, we define costs to consider the costumer preferences.

- Trip costs c_t^{t} for $t \in \mathcal{T}_{\mathrm{public}}$: costs for using public transport
- Route-dependent costs $\bar{c}_m^{\mathrm{r}}$ for $m \in \mathcal{M}$: costs for costumer preferences and unit costs for using public transport

Since the trip costs for public transport are connected with the choice of the route, we easily add these costs to the trip costs.

$$c_m^{\mathrm{r}} := \bar{c}_m^{\mathrm{r}} + \sum_{t \in m \cap \mathcal{T}_{\mathrm{public}}} c_t^{\mathrm{t}} \qquad \text{for } m \in \mathcal{M}$$

The route costs additionally include costumer preferences. This can be the total travel time, the number of changes or the costs for the costumers. Typically, a pure car trip is faster but more expensive. Further, a late departure time or an early arrival time can be criteria for this cost function.

Partial Order of the Trips

In order to decide whether a car can fulfill two trips in a row, we define a partial ordering of the car set and the set of car trips. The set of public transport trips is left out in this definition.

Definition 1 (Partial order of trips)**.** The binary relation $\prec$ on $\mathcal{V} \times \mathcal{T}_{\mathrm{car}}$ is defined as follows:

$$\begin{aligned} s \prec t \qquad &:\Leftrightarrow \qquad \left(z_s^{\mathrm{end}} + t_{s,t} \leq z_t^{\mathrm{start}}\right) \wedge \left(C(s) \neq C(t) \vee M(s) = M(t)\right) \\ & \qquad\qquad \text{for all } s \in \mathcal{V} \cup \mathcal{T}_{\mathrm{car}}, t \in \mathcal{T}_{\mathrm{car}} \\ s \not\prec t \qquad & \qquad\qquad \text{for all } s \in \mathcal{V} \cup \mathcal{T}_{\mathrm{car}}, t \in \mathcal{V} \end{aligned}$$

The binary relation $\preceq$ on $\mathcal{V} \times \mathcal{T}_{\mathrm{car}}$ we define as:

$$s \preceq t \qquad :\Leftrightarrow \qquad s = t \wedge s \prec t \qquad \text{for all } s, t \in \mathcal{V} \cup \mathcal{T}_{\mathrm{car}}$$

The expression $s \prec t$ means, that one car is able to fulfill both trips, first s and then t. A car must not cover two trips of the same costumer, except they belong to the same route. This results from the problem setting, that for each costumer exactly one route is fulfilled.

Assumptions

We make the following assumptions to our model: All costs are non-negative, i.e.

$$c^{\mathrm{v}} \geq 0 \qquad c^{\mathrm{d}}_{s,t} \geq 0 \qquad c^{\mathrm{t}}_{t} \geq 0 \qquad \bar{c}^{\mathrm{r}}_{m} \geq 0 \qquad \text{for all } s,t \in \mathcal{T}, m \in \mathcal{M}. \tag{2}$$

Further we assume the Triangle Inequality:

$$c^{\mathrm{d}}_{t_1,t_3} \leq c^{\mathrm{d}}_{t_1,t_2} + c^{\mathrm{d}}_{t_2,t_3} \qquad \text{for all } t_1,t_2,t_3 \in \mathcal{T}_{\mathrm{car}} \tag{3}$$

From (2) and (3) we get:

$$c^{\mathrm{d}}_{t_1,t_3} \leq c^{\mathrm{d}}_{t_1,t_2} + c^{\mathrm{t}}_{t_2} + c^{\mathrm{d}}_{t_2,t_3} \qquad \text{for all } t_1,t_2,t_3 \in \mathcal{T}_{\mathrm{car}} \tag{4}$$

1.2 Route Creation

We are not given the set of routes $\mathcal{M}$ in advance. For each costumer $c \in \mathcal{C}$, we have start and end location $p_c^{\mathrm{start}}, p_c^{\mathrm{end}}$ and a start and end time $\hat{z}_c^{\mathrm{start}}, \hat{z}_c^{\mathrm{end}}$. All the trips of the costumer lie in this interval, i.e.

$$\hat{z}_c^{\mathrm{start}} \leq z_m^{\mathrm{start}} \qquad z_m^{\mathrm{end}} \leq \hat{z}_c^{\mathrm{end}} \qquad \text{for all } m \in C^{-1}(c).$$

Basic Restrictions

To simplify the creation of the routes, we make some assumptions. For every route $m \in \mathcal{M}$ holds:

- There are not two car trips in a row.
- There is no car trip between two public transport trips.
- The number of public transport trips is restricted. Usually, one can reach every station with at most two changes.
- We define a walking distance d^{walk}. If the distance between the start position and the first station or between the last station and the end position, there is no car trip necessary.

We assume that we have some oracle that provides the set of feasible public transport routes for costumer $c \in \mathcal{C}$:

$$M_c = \Big\{ (s_1, z_1, s_2, z_2) \,|\, s_1, s_2 \in \mathcal{S}, \hat{z}_c^{\text{start}} \leq t_1 < t_2 \leq \hat{z}_c^{\text{end}}, \text{ there is a public transport route from } s_1 \text{ to } s_2 \text{ with start time } z_1 \text{ and end time } z_2 \Big\}$$

The fact, whether the costumer changes during his usage of public transport, has no effect on the model. Thus, we can consider each element in M_c as a public transport trip.

Route Creation

We create the set of multimodal routes $\mathcal{M}$. For this, we set a car trip before and after each public transport trip in order to bring the costumer from his start to his destination, except when it is possible to walk the distance. We also have to consider the given time restrictions. Further, we create the pure car trips. How the set $\mathcal{M}$ is created in detail, is described in algorithm 1.

Until now, we do not consider any changing times between a car trip and a public transport trip.

Further, we assume that the given costumer start and and times are feasible, i.e. $\hat{z}_c^{\text{start}} + t_{p_c^{\text{start}}, p_c^{\text{end}}} \leq \hat{z}_c^{\text{end}}$ for all $c \in \mathcal{C}$.

Further Restrictions

If the routes are created as described in algorithm 1, there are routes using every available station as long as it is feasible. Most of these routes are obviously bad for the costumer since they cause a big detour. What is more, a large number of routes enlarge the problem size and leads to a bad performance for solving it. Therefore, we try to restrict the set of alternatives to a reasonable size.

Example 1. Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ with a single public transport ride serving all stations. Let $\mathcal{C} = \{c_1, c_2\}$ with $p_{c_1}^{\text{end}} = s_n$ and $p_{c_2}^{\text{start}} = s_k$ for a certain $k \in [n-1]$. The alternative routes are

$$\mathcal{M} = \underbrace{\Big\{ \Big(\Big(p_{c_1}^{\text{start}}, s_i \Big), (s_i, s_n) \Big) \,|\, i \in [n-1] \Big\}}_{\text{for } c_1} \cup \underbrace{\Big\{ \Big(s_k, p_{c_2}^{\text{end}} \Big) \Big\}}_{\text{for } c_2}$$

with $(p_{c_1}^{\text{start}}, s_k) \prec \Big(s_k, p_{c_2}^{\text{end}} \Big)$ and $(p_{c_1}^{\text{start}}, s_i) \not\prec \Big(s_k, p_{c_2}^{\text{end}} \Big)$ for all $i \in [n] \setminus \{k\}$.

Algorithm 1: Creation of the routes

Input: costumer set $\mathcal{C}$; $p_c^{\text{start}}, p_c^{\text{end}}, \hat{z}_c^{\text{start}}, \hat{z}_c^{\text{end}}, M_c$ for all $c \in \mathcal{C}$
Output: set of routes $\mathcal{M}$, set of trips $\mathcal{T}_{\text{car}}, \mathcal{T}_{\text{public}}$

$\mathcal{T}_{\text{car}} \leftarrow \emptyset$;
$\mathcal{T}_{\text{public}} \leftarrow \emptyset$;
$\mathcal{M} \leftarrow \emptyset$;
foreach $c \in \mathcal{C}$ **do**
    foreach $(s_1, z_1, s_2, z_2) \in M_c$ **do**
        create public transport trip t;
        $p_t^{\text{start}} \leftarrow s_1, p_t^{\text{end}} \leftarrow s_2, z_t^{\text{start}} \leftarrow z_1, z_t^{\text{end}} \leftarrow z_2$;
        create car trips t_1, t_2;
        $p_{t_1}^{\text{start}} \leftarrow p_c^{\text{start}}, p_{t_1}^{\text{end}} \leftarrow s_1, z_{t_1}^{\text{start}} \leftarrow z_1 - t_{p_c^{\text{start}}, s_1}, z_{t_1}^{\text{end}} \leftarrow z_1$;
        $p_{t_2}^{\text{start}} \leftarrow s_2, p_{t_2}^{\text{end}} \leftarrow p_c^{\text{end}}, z_{t_2}^{\text{start}} \leftarrow z_2, z_{t_2}^{\text{end}} \leftarrow z_2 + t_{s_2, p_c^{\text{end}}}$;
        if $\hat{z}_c^{\text{start}} \leq z_{t_1}^{\text{start}} \wedge z_{t_2}^{\text{end}} \leq \hat{z}_c^{\text{end}}$ **then**
            create multimodal route m;
            $\mathcal{T}_{\text{public}} \leftarrow \mathcal{T}_{\text{public}} \cup \{t\}$;
            if $d_{p_c^{\text{start}}, s_1} \geq d^{\text{walk}}$ **then** $m \leftarrow (t_1, t)$; $\mathcal{T}_{\text{car}} \leftarrow \mathcal{T}_{\text{car}} \cup \{t_1\}$;
            else $m \leftarrow (t)$;
            if $d_{s_2, p_c^{\text{end}}} \geq d^{\text{walk}}$ **then** append t_2 to m; $\mathcal{T}_{\text{car}} \leftarrow \mathcal{T}_{\text{car}} \cup \{t_2\}$;
            $C(m) \leftarrow c$;
            $\mathcal{M} \leftarrow \mathcal{M} \cup \{m\}$;
        end
    end
    create car trips t_1, t_2;
    $p_{t_1}^{\text{start}} \leftarrow p_c^{\text{start}}, p_{t_1}^{\text{end}} \leftarrow p_c^{\text{end}}, z_{t_1}^{\text{start}} \leftarrow \hat{z}_c^{\text{start}}, z_{t_1}^{\text{end}} \leftarrow \hat{z}_c^{\text{start}} + t_{p_c^{\text{start}}, p_c^{\text{end}}}$;
    $p_{t_2}^{\text{start}} \leftarrow p_c^{\text{start}}, p_{t_2}^{\text{end}} \leftarrow p_c^{\text{end}}, z_{t_2}^{\text{start}} \leftarrow \hat{z}_c^{\text{end}} - t_{p_c^{\text{start}}, p_c^{\text{end}}}, z_{t_2}^{\text{end}} \leftarrow \hat{z}_c^{\text{end}}$;
    create multimodal routes m_1, m_2;
    $m_1 \leftarrow (t_1), m_2 \leftarrow (t_2)$;
    $\mathcal{T}_{\text{car}} \leftarrow \mathcal{T}_{\text{car}} \cup \{t_1, t_2\}, \mathcal{M} \leftarrow \mathcal{M} \cup \{m_1, m_2\}$;
end
return $\mathcal{M}, \mathcal{T}_{\text{car}}, \mathcal{T}_{\text{public}}$

We get the only solution, where only one car is needed, when c_1 drives to s_k, wherever the station s_k is. Every route of c_1 can be the optimal route, considering the other costumers. Therefore, an exact reduction of $\mathcal{M}$ is not possible without the the risk of cutting off the optimal solution.

It is not practicable to consider all possible multimodal routes due to computation reasons. But it is also not possible to reduce the number of routes without risking to lose the optimal solution. Hence, we try to make reasonable restrictions which keep the problem size small.

Pareto Optimality

The idea is to choose only Pareto optimal multimodal routes (cf. Kaiser/Knoll, cap. 3.2.2) in order to determine good routes.

Definition 2 (Pareto optimality)**.** Let $V \subset \mathbb{R}^n$.

1. The partial order $\leq$ on $\mathbb{R}^n$ is given by
$$v \leq w \qquad :\Leftrightarrow \qquad v_i \leq w_i \qquad \forall i \in [n] \qquad \text{for all } v, w \in \mathbb{R}^n$$
2. An element $w \in V$ is Pareto optimal in V if it is minimal with respect to $\leq$ in V, i.e.
$$v \leq w \qquad \Rightarrow \qquad v = w \qquad \text{for all } v \in V$$
3. The Pareto frontier of V with respect to $\leq$ is the set of Pareto optimal elements in V, i.e.
$$\min_{\leq} V := \{w \in V | \forall v \in V : v \leq w \Rightarrow v = w\}$$

Let $m \in \mathcal{M}$ be a multimodal route. We define

$$\varphi : \mathcal{M} \to \mathbb{R}^5 \qquad m \mapsto \begin{pmatrix} c^{\mathrm{r}} + \sum_{t \in m \cap \mathcal{T}_{\mathrm{car}}} c_t^{\mathrm{t}} \\ c^{\mathrm{r}} \\ |\mathcal{T}_{\mathrm{car}} \cap \{t \in m\}| \\ \sum_{t \in m \cap \mathcal{T}_{\mathrm{car}}} z_t^{\mathrm{end}} - z_t^{\mathrm{start}} \\ \sum_{t \in m \cap \mathcal{T}_{\mathrm{car}}} f_t^{\mathrm{t}} \end{pmatrix}$$

The function φ grades a route to their costs, their route costs, the number of cars needed, the time of a car needed and the fuel consumption.
From now on, we will use the Pareto frontier of $\varphi(\mathcal{M})$ as a restricted route set:

$$\hat{\mathcal{M}} := min_{\leq} \varphi(\mathcal{M}) \tag{5}$$

2 Mathematical Models

We introduce the mathematical model, with which the previously described problem is solved.

2.1 Task Graph

For tackling the problem, we introduce a task graph, on which the model is based. The graph is basically the same as in (Kaiser, Knoll, cap. 3.1) with the restriction, that only car trips $t \in \mathcal{T}_{\text{car}}$ are considered.

Definition 3 (Task Graph)**.** Let $d^{\text{s}}, d^{\text{e}}$ be special vertices describing the source and sink of the vehicle flow. We define the task graph as $\hat{G} = (\hat{V}, \hat{A})$, where

$$\hat{V} := \{d^{\text{s}}, d^{\text{e}}\} \mathbin{\dot\cup} \mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\text{car}}$$

is the vertex set consisting of the source, the sink, the vehicle set $\mathcal{V}$ and the set of car trips $\mathcal{T}_{\text{car}}$. The arc set is

$$\hat{A} := (\{d^{\text{s}}\} \times \mathcal{V}) \mathbin{\dot\cup} \left\{(s,t) \in (\mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\text{car}})^2 \,|s \prec t\right\} \mathbin{\dot\cup} ((\mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\text{car}}) \times \{d^{\text{e}}\})\,.$$

A vertex $s \in \mathcal{V}$ represents the initial state of a vehicle s where it becomes available for the first time. Each $d^{\text{s}} - d^{\text{e}}$-path in $\hat{G}$ is the duty of one vehicle, i.e. this vehicle fulfills the trips in the order given by the path. Hence, two trips are connected only if it is possible that one car fulfills both trips, i.e. the relation $\prec$ holds.
To consider the refuel stations, we introduce an extended task graph.

Definition 4 (Extended Task Graph)**.** For every $s, t \in \mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\text{car}}$ with $s \prec t$ we create a copy of $\left\{r \in \mathcal{R} | z_s^{\text{end}} + t_{s,r} + t_{r,t} \leq z_t^{\text{start}}\right\}$ denoted by $\mathcal{R}_{s,t}$. This means, various copied sets are pairwise disjoint. We define the extended task graph $G = (V, A)$ with vertex set

$$V := \hat{V} \mathbin{\dot\cup} \dot{\bigcup_{\substack{s,t \in \mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\text{car}} \\ s \prec t}}} \mathcal{R}_{s,t}$$

and arc set

$$A := \hat{A} \mathbin{\dot\cup} \{(s,r) | s,t \in \mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\text{car}}, s \prec t, r \in \mathcal{R}_{s,t}\} \mathbin{\dot\cup} \{(r,t) | s,t \in \mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\text{car}}, s \prec t, r \in \mathcal{R}_{s,t}\}\,.$$

In this graph, each feasible refuel station is considered for each feasible pair of trips. There is a method to reduce the size of $\mathcal{R}_{s,t}$ significantly without losing the optimal solution. This method is described in (Kaiser, Knoll, cap. 3.2.2). From now on, we will use $G = (V, A)$ with restricted $\mathcal{R}_{s,t}$.

2.2 Arc Flow Formulation

We develop a model for solving the problem via a flow of the cars. The multimodal routes are given in advance. The car trips are adjusted in such a way, that they fit to the public transport routes (in location and time). It is not possible to model each route as a trip because then the car availabilities are not considered.
We model a flow of the cars. The public transport trips work only as constraints for this flow. Constraints are the fulfilling of one multimodal route per costumer and the fuel constraints.

Basic Model

We model the arc flow as an ILP. The formulation is basically built on (MILP), described in (Kaiser, Knoll, cap. 3.2). We use the following decision variables:

- $x_{s,t} \in \{0,1\}$ for $(s,t) \in A$: indicates, whether trip $t \in \mathcal{T}_{\text{car}}$ is fulfilled after $s \in \mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\text{car}}$
- $z_{s,r,t} \in \{0,1\}$ for $t \in \mathcal{T}_{\text{car}}, s \in \mathrm{N}_G^-(t), r \in \mathcal{R}_{s,t}$: indicates, whether refuel station $r \in \mathcal{R}$ is visited between s and t
- $e_s \in [0,1]$ for $s \in V \setminus \{d^{\text{s}}, d^{\text{e}}\}$: states the fuel of the respective car after fulfilling trip $s \in \mathcal{T}_{\text{car}}$

If $s \in \mathcal{V}$, then $x_{s,t}$ determines, whether trip t is the first trip fulfilled by s and e_s is the initial fuel state f_s^0 of vehicle s.
Additionally to (MILP) we introduce decision variables to determine the fulfilling of routes:

- $u_m \in \{0,1\}$ for $m \in \mathcal{M}$: indicates whether multimodal route m is fulfilled

The basic constraints are developed in detail in (Kaiser, Knoll, cap. 3.2) and not shown in detail here. The basic constraints are the flow conservation constraint, the constraint for considering every car and the constraints guaranteeing feasible fuel states all the time.

Costumer and Route Constraints

In (MILP), each costumer has a set of alternative trips and from this set, exactly one trips has to be fulfilled. This is modeled as follows:

$$\sum_{t \in C^{-1}(c)} \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = 1 \qquad \text{for all } c \in \mathcal{C} \tag{3.17}$$

In contrast to (MILP), here each costumer has a set of alternative routes consisting of trips and from this set, exactly one route has to be fulfilled. Therefore, we replace (3.17) by the following formulation:

$$\sum_{m \in C^{-1}(c)} u_m = 1 \qquad \text{for all } c \in \mathcal{C} \tag{6}$$

$$\sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = u_m \qquad \text{for all } m \in \mathcal{M}, t \in m \tag{7}$$

The constraint (6) says, that for every costumer exactly one route is fulfilled. The constraint (7) says, if a route is fulfilled then every trip of this route must be fulfilled.

Objective Function

The objective function in (MILP) is given by

$$\sum_{s \in \mathcal{V}} \sum_{t \in \mathrm{N}_G^+(s) \setminus \{d^{\mathrm{e}}\}} x_{s,t} c^{\mathrm{v}} + \sum_{t \in \mathcal{T}} \sum_{s \in \mathrm{N}_G^-(t)} \left[x_{s,t} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_t \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t} \right) \right]$$

considering the vehicle costs c^{v}, the trip costs c^{t}_t for $t \in \mathcal{T}_{\mathrm{car}}$ and the deadhead costs c^{d}. What is missing, are the route-dependent costs c^{r}. Thus, we add

$$\sum_{m \in \mathcal{M}} u_m c^{\mathrm{r}}_m$$

to the objective function.

LP Formulation

Putting all this together, we get the following formulation, called (MMILP):

$$
\begin{aligned}
\min \quad & \sum_{s\in\mathcal{V}} \sum_{t\in \mathrm{N}_G^+(s)\setminus\{d^{\mathrm{e}}\}} x_{s,t}c^{\mathrm{v}} + \sum_{m\in\mathcal{M}} u_m c_m^{\mathrm{r}} \\
& + \sum_{t\in\mathcal{T}} \sum_{s\in \mathrm{N}_G^-(t)} \left[x_{s,t}\left(c_{s,t}^{\mathrm{d}} + c_t^{\mathrm{t}}\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c_{s,r}^{\mathrm{d}} + c_{r,t}^{\mathrm{d}} - c_{s,t}^{\mathrm{d}}\right)\right] && \text{(MMILP)} \\
\text{s.t.} \quad & \sum_{s\in \mathrm{N}_G^-(t)} x_{s,t} = \sum_{s\in \mathrm{N}_G^+(t)} x_{t,s} && \text{for all } t \in V\setminus\{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.15) \\
& \sum_{s\in \mathrm{N}_G^-(t)} x_{s,t} = 1 && \text{for all } t \in \mathcal{V} && (3.16) \\
& \sum_{m\in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C} && (6) \\
& \sum_{s\in \mathrm{N}_G^-(t)} x_{s,t} = u_m && \text{for all } m \in \mathcal{M}, t \in m && (7) \\
& \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.18) \\
& e_s \le f_s^0 && \text{for all } s \in \mathcal{V} && (3.19) \\
& 0 \le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f_{s,r}^{\mathrm{d}} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.12) \\
& e_t \le 1 - f_t^{\mathrm{t}} - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f_{r,t}^{\mathrm{d}} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.13) \\
& e_t \le e_s - x_{s,t}\left(f_{s,t}^{\mathrm{d}} + f_t^{\mathrm{t}}\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(f_{s,r}^{\mathrm{d}} + f_r^{\mathrm{t}} + f_{r,t}^{\mathrm{d}} - f_{s,t}^{\mathrm{d}}\right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.14) \\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t) \in A && (3.20) \\
& z_{s,r,t} \in \{0,1\} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t), r \in \mathcal{R}_{s,t} && (3.21) \\
& e_s \in [0,1] && \text{for all } s \in V\setminus\{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.22) \\
& u_m \in \{0,1\} && \text{for all } m \in \mathcal{M} && (8)
\end{aligned}
$$

3 Heuristics

There is already a heuristic for solving an easier version of the problem (Knoll, cap. 10). This heuristic only handles the case without costumers. This means, there is a trip set $\mathcal{T}$ and each of these trips has to be fulfilled. This is even a simplification to (MILP). We try to extend this heuristic such that it can tackle the problem considering multimodal transport.

3.1 Splitting the Problem

We define the splitting of the task graph similarly to (Knoll, cap. 8) with the difference, that the splittings can be defined generally here.

Definition 5 (Splitting)**.** Let $n \in \mathbb{N}$ and let

$$\mathcal{T}_{\text{car}} = \bigcup_{i=1}^{n} \mathcal{T}_i \qquad\qquad \mathcal{V} = \bigcup_{i=1}^{n} \mathcal{V}_i$$

be partitions of the set of car trips, respectively vehicles. Then we call $\{\mathcal{T}_i \mid i \in [n]\}$ and $\{\mathcal{V}_i \mid i \in [n]\}$ splitting of $\mathcal{T}$ and $\mathcal{V}$ and $\mathcal{T}_i$ and $\mathcal{V}_i$ partial trip respectively vehicle set.

Definition 6 (Transformed Task Graph)**.** Let $\{\mathcal{T}_1, \dots \mathcal{T}_n\}$ be a splitting of $\mathcal{T}_{\text{car}}$ according to Definition 5. Then we define:

1. Split Point: Let $s \in \mathcal{T}_i$ for $i \in [n]\backslash\{1\}$. For $j \in [i-1]$, we define the split point $\text{SP}_j(s)$ with $p^{\text{start}}_{\text{SP}_j(s)} = p^{\text{end}}_{\text{SP}_j(s)} =: p^{\text{start}}_s, z^{\text{start}}_{\text{SP}_j(s)} = z^{\text{end}}_{\text{SP}_j(s)} =: z^{\text{start}}_s$ and $f^{\text{t}}_{\text{SP}_j(s)} =: 0$.
2. For $i \in [n]\backslash\{1\}$ and $j \in [i-1]$, we define $P_{j,i} := \{\text{SP}_j(s) \mid s \in \mathcal{T}_i\}$.
3. Partial Split Point Set: For $j \in [n-1]$, we define the partial split point set $P_j := \bigcup_{i=j+1}^{n} P_{j,i}$.
4. Split Point Set: We define the split point set $P := \bigcup_{j=1}^{n-1} P_j$.

Let $G = (V, A)$ the task graph, $\{\mathcal{V}_1, \dots, \mathcal{V}_n\}$ be a splitting of $\mathcal{V}$.

5. Transformed Task Graph: We define the transformed task graph $\overline{G} = \left(\overline{V}, \overline{A}\right)$ with vertex set

$$\overline{V} := V \cup P = V \cup \{\text{SP}_i(s) \mid i \in [n-1], j \in [n+1]\backslash[i], s \in \mathcal{T}_j\}$$

and arc set

$$\begin{aligned}\overline{A} := (d^{\mathrm{s}} \times \mathcal{V}) \mathbin{\dot\cup} \bigcup_{i=1}^{n} \{(s,t) \in (\mathcal{V}_i \mathbin{\dot\cup} \mathcal{T}_i) \times (\mathcal{T}_i \mathbin{\dot\cup} P_i) \mid s \prec t\} \\ \mathbin{\dot\cup} \bigcup_{i=1}^{n} \left\{ (s,t) \in \left(\left(\bigcup_{j=1}^{i-1} P_{j,i} \right) \times \mathcal{T}_i \right) \mid s = \mathrm{SP}_i(t) \right\} \mathbin{\dot\cup} ((\mathcal{V} \mathbin{\dot\cup} \mathcal{T}_{\mathrm{car}}) \times \{d^{\mathrm{e}}\})\end{aligned}$$

Since later we want to solve the problem for each splitting separately, we define an own graph for each splitting.

Adaption of the Model

We adapt the model (MMILP) to the transformed task graph and therefore get the formulation called (SMILP). For this, we make the following considerations:
For all split points we define the costs and fuel states as

$$c^{\mathrm{t}}_s := 0 \qquad c^{\mathrm{d}}_{s,t} := 0 \qquad f^{\mathrm{t}}_s := 0 \qquad f^{\mathrm{d}}_{s,t} := 0 \qquad \text{for } s \in P, t \in \mathrm{N}^{+}_{\overline{G}}(s)$$

since $p^{\mathrm{end}}_s = p^{\mathrm{start}}_t$ and $z^{\mathrm{end}}_s = z^{\mathrm{start}}_t$. Furthermore, refueling is not possible between s and t.
Since the arcs leading to a trip after a split point are not considered in the objective function any more, the following term including these costs is added to the objective function.

$$\sum_{s \in P} \sum_{t \in \mathrm{N}^{+}_{\overline{G}}(s)} x_{s,t} c^{\mathrm{t}}_t \tag{9}$$

The flow conservation should also be included in P, therefore we modify (3.15) to

$$\sum_{t \in \mathrm{N}^{-}_{\overline{G}}(s)} x_{t,s} = \sum_{t \in \mathrm{N}^{+}_{\overline{G}}(s)} x_{s,t} \qquad \text{for all } s \in \overline{V} \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} \tag{8.1}$$

The fuel constraints (3.18), (3.12), (3.13) and (3.14) should hold also on the arcs leading to P and are therefore replaced with (8.3), (8.5), (8.6) and (8.7).
Further the arcs between the split points and their respective trips have to be considered. Since refueling is not possible there, we have only to adapt (3.14). Since $f^{\mathrm{d}}_{s,t} = 0$ and refueling is not possible between s and t, the constraint reads as follows:

$$e_t \leq e_s - x_{s,t} f_t^{\mathrm{t}} + (1 - x_{s,t}) \qquad \text{for all } s \in P, t \in \mathrm{N}_{\overline{G}}^{+}(s) \tag{8.8}$$

Putting all together, we have the following formulation:

$$\begin{aligned}
\min \quad & \sum_{s \in \mathcal{V}} \sum_{t \in \mathrm{N}_{\overline{G}}^{+}(s) \setminus \{d^{\mathrm{e}}\}} x_{s,t} c^{\mathrm{v}} + \sum_{s \in P} \sum_{t \in \mathrm{N}_{\overline{G}}^{+}(s)} x_{s,t} c_t^{\mathrm{t}} + \sum_{m \in \mathcal{M}} u_m c_m^{\mathrm{r}} \\
& + \sum_{t \in \mathcal{T} \cup P} \sum_{s \in \mathrm{N}_{\overline{G}}^{-}(t) \setminus P} \left[x_{s,t} \left(c_{s,t}^{\mathrm{d}} + c_t^{\mathrm{t}} \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c_{s,r}^{\mathrm{d}} + c_{r,t}^{\mathrm{d}} - c_{s,t}^{\mathrm{d}} \right) \right] && \text{(SMILP)} \\
\text{s.t.} \quad & \sum_{t \in \mathrm{N}_{\overline{G}}^{-}(s)} x_{t,s} = \sum_{t \in \mathrm{N}_{\overline{G}}^{+}(s)} x_{s,t} && \text{for all } s \in \overline{V} \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} \quad (8.1) \\
& \sum_{s \in \mathrm{N}_{\overline{G}}^{-}(t)} x_{s,t} = 1 && \text{for all } t \in \mathcal{V} \quad (3.16) \\
& \sum_{m \in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C} \quad (6) \\
& \sum_{s \in \mathrm{N}_{\overline{G}}^{-}(t)} x_{s,t} = u_m && \text{for all } m \in \mathcal{M}, t \in m \quad (7) \\
& \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \leq x_{s,t} && \text{for all } t \in \mathcal{T} \cup P, s \in \mathrm{N}_{\overline{G}}^{-}(t) \setminus P \quad (8.3) \\
& e_s \leq f_s^0 && \text{for all } s \in \mathcal{V} \quad (3.19) \\
& 0 \leq e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f_{s,r}^{\mathrm{d}} && \text{for all } t \in \mathcal{T} \cup P, s \in \mathrm{N}_{\overline{G}}^{-}(t) \setminus P \quad (8.5) \\
& e_t \leq 1 - f_t^{\mathrm{t}} - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f_{r,t}^{\mathrm{d}} && \text{for all } t \in \mathcal{T} \cup P, s \in \mathrm{N}_{\overline{G}}^{-}(t) \setminus P \quad (8.6) \\
& e_t \leq e_s - x_{s,t} \left(f_{s,t}^{\mathrm{d}} + f_t^{\mathrm{t}} \right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(f_{s,r}^{\mathrm{d}} + f_r^{\mathrm{t}} + f_{r,t}^{\mathrm{d}} - f_{s,t}^{\mathrm{d}} \right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T} \cup P, s \in \mathrm{N}_{\overline{G}}^{-}(t) \setminus P \quad (8.7) \\
& e_t \leq e_s - x_{s,t} f_t^{\mathrm{t}} + (1 - x_{s,t}) && \text{for all } s \in P, t \in \mathrm{N}_{\overline{G}}^{+}(s) \quad (8.8) \\
& x_{s,t} \in \{0, 1\} && \text{for all } (s,t) \in \overline{A} \quad (8.9) \\
& z_{s,r,t} \in \{0, 1\} && \text{for all } t \in \mathcal{T} \cup P, s \in \mathrm{N}_{\overline{G}}^{-}(t) \setminus P, r \in \mathcal{R}_{s,t} \quad (8.10) \\
& e_s \in [0, 1] && \text{for all } s \in \overline{V} \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} \quad (8.11) \\
& u_m \in \{0, 1\} && \text{for all } m \in \mathcal{M} \quad (8)
\end{aligned}$$

The costumer constraints (6) are not affected by transforming the graph. The decision whether a trip $t \in \mathcal{T}_{\text{car}}$ is fulfilled is still given by $\sum_{s \in \mathrm{N}^{-}_{\overline{G}}(t)} x_{s,t}$, no matter if this is after a split point or not. So, the route constraints (7) do not change either.

Theorem 1 (Equivalence of (MMILP) and (SMILP))**.** *Let* $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ *and* $\{\mathcal{V}_1, \dots, \mathcal{V}_n\}$ *be splittings of* $\mathcal{T}_{\text{car}}$ *and* $\mathcal{V}$*, respectively. Let* S *be a feasible solution of* (MMILP) *and* $\hat{S}$ *be a feasible solution of* (SMILP)*.*
We can convert S *into a feasible solution of* (SMILP) *with the same objective function value and we can convert* $\hat{S}$ *into a feasible solution of* (MMILP) *with the same objective function value.*

3.2 Successive Heuristics

Consider a splitting $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ and $\{\mathcal{V}_1, \dots, \mathcal{V}_n\}$ for $\mathcal{T}_{\text{car}}$ and $\mathcal{V}$ respectively. Define $\sigma \in S_n$ be a permutation of $[n]$ with $\sigma(n) = 1$. σ indicates in which order the partial instances are solved. This means, partial instance $\sigma(i) \in [n]$ is solved at the i-th position, the first partial instance is solved at last. The actual description of σ and the splittings follows later.
For every partial instance $i \in [n]$ we have a set of start points $\hat{\mathcal{V}}_i$ and end points $\hat{\mathcal{P}}_i$. How these sets are determined is explained later.

Partial Instances

The (SMILP) is provided in order to split (MMILP) into partial instances. For solving the partial instances, we have to make further definitions.

Definition 7 (Partial Transformed Task Graph)**.** Let $i \in [n]$. For a set of start points $\hat{\mathcal{V}}_i$, a set of end points $\hat{\mathcal{P}}_i$ and the partial trip set $\mathcal{T}_i$, the partial transformed task graph is the directed graph $\overline{G}_i = \left(\overline{A}_i, \overline{V}_i\right)$ with vertex set

$$\overline{V}_i := \{d^{\text{s}}, d^{\text{e}}\} \mathbin{\dot\cup} \hat{\mathcal{V}}_i \mathbin{\dot\cup} \mathcal{T}_i \mathbin{\dot\cup} \hat{\mathcal{P}}_i$$

and arc set

$$\begin{aligned}\overline{A}_i := &\left(\{d^{\text{s}}\} \times \left(\hat{\mathcal{V}}_i \mathbin{\dot\cup} \mathcal{T}_i \mathbin{\dot\cup} \hat{\mathcal{P}}_i\right)\right) \mathbin{\dot\cup} \left\{(s,t) \in \left(\hat{\mathcal{V}}_i \mathbin{\dot\cup} \mathcal{T}_{\text{car}}\right) \times \left(\mathcal{T}_i \mathbin{\dot\cup} \hat{\mathcal{P}}_i\right) \mid s \preceq t\right\} \\ &\mathbin{\dot\cup} \left(\left(\hat{\mathcal{V}}_i \mathbin{\dot\cup} \mathcal{T}_i \mathbin{\dot\cup} \hat{\mathcal{P}}_i\right) \times \{d^{\text{e}}\}\right)\end{aligned}$$

3.3 Costumer-dependent Splitting

In contrast to the splitting performed in (Knoll, cap. 8), the trips are not split according to their start times but according to their costumers' start times. This means, that all trips of a route and all routes of a costumer are in the same splitting. For each spitting, we apply (CMILP_i) to receive an optimal partial solution and connect the partial solutions to a feasible overall solution.

Splitting

Definition 8 (Costumer-dependent Splitting)**.** Given points in time c_i for $i \in [n-1]$ with $c_i < c_{i+1}$ for $i \in [n-2]$. We first define a splitting of the costumers $\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i$ as

$$\mathcal{C}_i := \begin{cases} \{c \in \mathcal{C} \mid z_c^{\text{start}} \leq c_1\} & \text{for } i = 1 \\ \{c \in \mathcal{C} \mid c_{i-1} < z_c^{\text{start}} \leq c_i\} & \text{for } i \in [n-1]\backslash\{1\} \\ \{c \in \mathcal{C} \mid c_{n-1} < z_c^{\text{start}}\} & \text{for } i = n. \end{cases}$$

Based on the costumer splitting, we define the splittings of $\mathcal{T}_{\text{car}}$ and $\mathcal{V}$ as

$$\mathcal{T}_i := \{t \in \mathcal{T}_{\text{car}} \mid (M \circ C)(t) \in \mathcal{C}_i\} \qquad \text{for } i \in [n]$$

and

$$\mathcal{V}_i := \begin{cases} \{v \in \mathcal{V} \mid z_v \leq c_1\} & \text{for } i = 1 \\ \{v \in \mathcal{V} \mid c_{i-1} < z_v \leq c_i\} & \text{for } i \in [n-1]\backslash\{1\} \\ \{v \in \mathcal{V} \mid c_{n-1} < z_v\} & \text{for } i = n. \end{cases}$$

We denote the formulation (SMILP) with a splitting according to Definition 8 as (CMILP).

Solving of the Partial Instances

Since for costumer $c \in \mathcal{C}_i$ all his trips are in splitting $\mathcal{T}_i$, costumer c has to be satisfied only in the partial instance i. For solving the partial instances, we modify (SMILP) as follows:
Instead of (6) and (7) we have the constraints

$$\sum_{m \in C^{-1}(c)} u_m = 1 \qquad \text{for all } c \in \mathcal{C}_i \tag{10}$$

$$\sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = u_m \qquad \text{for all } m \in C^{-1}\left(\mathcal{C}_i\right), t \in m \tag{11}$$

We do not have vehicles in the partial split set any more. Instead we have to ensure that each start and end point is visited. Therefore we modify (3.16) to

$$\sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = 1 \qquad \text{for all } t \in \hat{\mathcal{V}}_i \cup \hat{\mathcal{P}}_i \tag{12}$$

To guarantee the fuel level f^0 in $\hat{\mathcal{P}}_i$ we introduce the constraint

$$f^0_s \leq e_s \qquad \text{for all } s \in \hat{\mathcal{P}}_i \tag{13}$$

We introduce two additional constraints. The first one guarantees that the fuel level at the beginning of a duty, if it starts with $s \in \mathcal{T}_i$ is at most $f^{\max}_s$. The second one guarantees that the fuel level at the end of a duty, if it ends with $s \in \mathcal{T}_i$ is at least $f^{\max}_s$.

$$e_s + f^{\mathrm{t}}_s \leq f^{\max}_s + (1 - x_{d^{\mathrm{s}},s}) \cdot \left(1 + f^{\mathrm{t}}_s\right) \qquad \text{for all } s \in \mathcal{T}_i \tag{14}$$

$$f^{\min}_s \leq e_s + (1 - x_{s,d^{\mathrm{e}}}) \qquad \text{for all } s \in \mathcal{T}_i \tag{15}$$

Restricting all other constraints to vertices of the partial task graph, we have the following formulation:

$$
\begin{aligned}
\min \quad & \left(\sum_{s \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i} x_{d^{\mathrm{s}},s} - \sum_{s \in \hat{\mathcal{P}}_i} x_{s,d^{\mathrm{e}}} \right) c^{\mathrm{v}} + \sum_{m \in C^{-1}(\mathcal{C}_i)} u_m c^{\mathrm{r}}_m \\
& \sum_{t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i} \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} \left[x_{s,t} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_t \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t} \right) \right] && (\mathrm{CMILP}_i) \\
\text{s.t.} \quad & \sum_{t \in \mathrm{N}^-_{\overline{G}_i}(s)} x_{t,s} = \sum_{t \in \mathrm{N}^+_{\overline{G}_i}(s)} x_{s,t} && \text{for all } s \in \overline{V}_i \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} \quad (16) \\
& \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = 1 && \text{for all } t \in \hat{\mathcal{V}}_i \cup \hat{\mathcal{P}}_i \quad (12) \\
& \sum_{m \in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C}_i \quad (10) \\
& \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = u_m && \text{for all } m \in C^{-1}(\mathcal{C}_i), t \in m \quad (11) \\
& \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i \quad (17) \\
& e_s \le f^0_s && \text{for all } s \in \hat{\mathcal{V}}_i \quad (18) \\
& f^0_s \le e_s && \text{for all } s \in \hat{\mathcal{P}}_i \quad (13) \\
& 0 \le e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i \quad (19) \\
& e_t \le 1 - f^{\mathrm{t}}_t - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i \quad (20) \\
& e_t \le e_s - x_{s,t} \left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t \right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t} \right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i \quad (21) \\
& e_t \le e_s - x_{s,t} f^{\mathrm{t}}_t + (1 - x_{s,t}) && \text{for all } s \in \hat{\mathcal{P}}_i, t \in \mathrm{N}^+_{\overline{G}_i}(s) \quad (22) \\
& e_s + f^{\mathrm{t}}_s \le f^{\max}_s + (1 - x_{d^{\mathrm{s}},s}) \cdot \left(1 + f^{\mathrm{t}}_s \right) && \text{for all } s \in \mathcal{T}_i \quad (14) \\
& f^{\min}_s \le e_s + (1 - x_{s,d^{\mathrm{e}}}) && \text{for all } s \in \mathcal{T}_i \quad (15) \\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t) \in \overline{A}_i \quad (23) \\
& z_{s,r,t} \in \{0,1\} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i, r \in \mathcal{R}_{s,t} \quad (24) \\
& e_s \in [0,1] && \text{for all } s \in \overline{V}_i \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} \quad (25) \\
& u_m \in \{0,1\} && \text{for all } m \in C^{-1}(\mathcal{C}_i) \quad (8)
\end{aligned}
$$

Model Equivalence

This heuristic formulation is not equivalent to the original formulation (MMILP). This is shown by the following example.

Example 2. Let t_1, t_2, t_3 with $t_1 \prec t_2 \prec t_3$ be trips with the properties shown in Table 1

Trip	Start	End	Route	Costumer
t_1	8:00	8:15	m_1	C_1
t_2	8:30	8:45	m_2	C_2
t_3	9:00	9:15	m_1	C_1

Table 1: Trips

In this case, costumer C_1 uses public transport between 8:15 and 9:00. The duty (t_1, t_2, t_3) is a feasible result of the (MMILP).
If there is a split point at 8:15 then the splittings are $\mathcal{T}_1 = \{t_1, t_3\}, \mathcal{T}_2 = \{t_2\}$. Hence, there is one split point $\mathrm{SP}_1(t_2)$ with $z^{\text{start}}_{\mathrm{SP}_1(t_2)} = 8{:}30$. The partial solution of instance 1 is (t_1, t_3) and $t_3 \not\prec \mathrm{SP}_1(t_2)$. Thus, the partial solutions cannot be feasibly connected to the solution (t_1, t_2, t_3).

With this example we have seen, that the formulations $(CMILP)$ and (MMILP) are not equivalent. It is even possible, that an optimal solution of (MMILP) is not feasible in (SMILP).
Although the formulations are not equivalent, we can give an estimation on the objective value when we make some restrictions.

Definition 9. For $n \geq 3$, consider a costumer set $\mathcal{C}$ and split points c_i for $i \in [n-1]$ with $c_i < c_{i+1}$ for all $i \in [n-2]$. We define the following values:

- Costumer Extension for $c \in \mathcal{C}$: $L_{\mathrm{C}}(c) := \max\limits_{t \in (M \circ C)^{-1}(c)} z^{\text{start}}_t - \min\limits_{t \in (M \circ C)^{-1}(c)} z^{\text{start}}_t$
- Costumer Extension: $L_{\mathrm{C}} := \max\limits_{c \in \mathcal{C}} L_{\mathrm{C}}(c)$
- Splitting Length: $L_{\mathrm{S}} := \min\limits_{i \in [n-1]} c_{i+1} - c_i$

Theorem 2. *For $n \geq 3$, consider the problem with costumer set $\mathcal{C}$ and split points c_i for $i \in [n-1]$ with $c_i < c_{i+1}$ for all $i \in [n-2]$. Let*

$$L_{\mathrm{C}} \leq L_{\mathrm{S}} \tag{26}$$

Let $d := (t_1, \dots, t_k)$ be the duty of a vehicle of a feasible solution of the (MMILP)*. Then, there are duties $d_1 \uplus d_2 = d$, where d_1, d_2 are part of a feasible solution of (CMILP). Moreover, there holds*

$$\operatorname{cost}(d_1) + \operatorname{cost}(d_2) \leq 2 \cdot \operatorname{cost}(d). \tag{27}$$

Proof. We consider the vehicle duty $d = (t_1, \dots t_k)$. We write $s \prec t$ according to Definition 1, i.e. (s,t) is feasible in (MMILP). We write $s \to t$ iff (s,t) is feasible in (CMILP).
Consider $s \prec t$ with $s \not\to t$ and costumers $C_s := (M \circ C)(s)$ and $C_t := (M \circ C)(t)$. Then s is in a later splitting than t. There are split points c_{l-1}, c_l, c_{l+1} for $l \in [n]$ with

$$z_s^{\text{start}} < z_t^{\text{start}} \qquad z_{C_t}^{\text{start}} \leq c_l < z_{C_s}^{\text{start}} \qquad c_l + L_{\mathrm{S}} \leq c_{l+1} \qquad z_{C_s}^{\text{start}} \leq z_s^{\text{start}} \leq z_{C_s}^{\text{start}} + L_{\mathrm{C}}$$

Since (26), holds

$$\begin{aligned} z_{C_s}^{\text{start}} \leq z_s^{\text{start}} < z_t^{\text{start}} \leq z_{C_t}^{\text{start}} + L_{\mathrm{C}} \leq c_l + L_{\mathrm{C}} \leq c_l + L_{\mathrm{S}} \leq c_{l+1} \\ z_{C_t}^{\text{start}} \geq z_t^{\text{start}} - L_{\mathrm{C}} > z_s^{\text{start}} - L_{\mathrm{C}} \geq z_{C_s}^{\text{start}} - L_{\mathrm{C}} > c_l - L_{\mathrm{C}} \geq c_l - L_{\mathrm{S}} \geq c_{l-1} \end{aligned}$$

and therefore $t \in \mathcal{T}_l, s \in \mathcal{T}_{l+1}$. Here, we use $c_0 := -\infty, c_{n+1} := +\infty$.

Feasibility For arbitrary $i \in [k-2]$ holds: $t_i \prec t_{i+1} \prec t_{i+2}$, therefore also $t_i \prec t_{i+2}$. We prove that t_{i+2} can be appended after t_i or t_{i+1}. We differentiate between the following cases:

1. $t_{i+1} \to t_{i+2}$: Clear.
2. $t_{i+1} \not\to t_{i+2}$: Then holds $t_{i+2} \in \mathcal{T}_l$ and $t_{i+1} \in \mathcal{T}_{l+1}$ for some $l \in [k]$. From $t_i \prec t_{i+2}$ follows $t_i \in \bigcup_{j=1}^{l+1} \mathcal{T}_j$. Therefore $t_i \to t_{i+1}$ or $t_i \to t_{i+2}$.
   - $t_i \to t_{i+2}$: Clear.
   - $t_i \not\to t_{i+2}$: Then holds $t_{i+2} \in \mathcal{T}_l$ and $t_i, t_{i+1} \in \mathcal{T}_{l+1}$ and therefore $t_i \to t_{i+1}$. For $i' \geq i$ holds $t_{i'} \in \bigcup_{j=l}^{n} \mathcal{T}_j$ and therefore $t_{i+1} \to t_{i'}$ or $t_{i+2} \to t_{i'}$. Thus, every later trip can be appended after on of these duties.

We have seen that two duties d_1, d_2 can fulfill the trips of duty d, such that d_1 and d_2 are feasible in (CMILP). Each trip can be appended to d_1 or to d_2.

Costs The costs of duty d are

$$\operatorname{cost}(d) = c^{\mathrm{v}} + c^{\mathrm{d}}_{v,t_1} + c^{\mathrm{t}}_{t_1} + \sum_{i=2}^{k} \left(c^{\mathrm{d}}_{t_{i-1},t_i} + c^{\mathrm{t}}_{t_i} \right).$$

Each duty d_1, d_2 has cost $c^{\mathrm{d}}_{t,t'} + c^{\mathrm{t}}_{t'} + c^{\mathrm{d}}_{t',t''}$ if trip t' is covered and cost $c^{\mathrm{d}}_{t,t''}$ if not. According to (4), the costs for not covering the trip do not exceed the costs for covering. Therefore we have

$$\operatorname{cost}(d_1) + \operatorname{cost}(d_2) \leq 2 \cdot \operatorname{cost}(d).$$

□

Corollary 1. *Consider the problem with $L_{\mathrm{C}} \leq L_{\mathrm{S}}$. Let S_1 be a feasible solution of* (MMILP)*. Then there exists a solution S_2 feasible also in (CMILP).*

$$\operatorname{val}(S_2) \leq 2 \cdot \operatorname{val}(S_1)$$

3.4 Time-dependent Splitting

The developed formulation (CMILP) based on a costumer-dependent splitting is not equivalent to the original formulation (MMILP). The goal now is to develop a splitting that is equivalent and create a heuristic based on this splitting. Therefore, it is necessary that a splitting of trips of the same costumer is possible. This leads to the following problem: When the partial instances are solved successively, we need a possibility to still guarantee the costumer satisfaction for the entire problem. This has to be applied already in the partial instance, although we do not have any knowledge about the trips of the same costumer in the later solved partial instances.

3.4.1 Basic Idea

Splitting

Definition 10 (Time-dependent Splitting)**.** Given points in time c_i for $i \in [n]$ with $c_i < c_{i+1}$ for $i \in [n-1]$. We define the splitting of $\mathcal{T}_{\mathrm{car}}$ and $\mathcal{V}$ as follows:

$$\mathcal{T}_i := \begin{cases} \{t \in \mathcal{T}_{\mathrm{car}} \mid z^{\mathrm{start}}_t \leq c_1\} & \text{for } i = 1 \\ \{t \in \mathcal{T}_{\mathrm{car}} \mid c_{i-1} < z^{\mathrm{start}}_t \leq c_i\} & \text{for } i \in [n]\backslash\{1\} \\ \{t \in \mathcal{T}_{\mathrm{car}} \mid c_n < z^{\mathrm{start}}_t\} & \text{for } i = n+1. \end{cases}$$

and

$$\mathcal{V}_i := \begin{cases} \{v \in \mathcal{V} \mid z_v \leq c_1\} & \text{for } i = 1 \\ \{v \in \mathcal{V} \mid c_{i-1} < z_v \leq c_i\} & \text{for } i \in [n] \backslash \{1\} \\ \{v \in \mathcal{V} \mid c_n < z_v\} & \text{for } i = n+1. \end{cases}$$

We denote the formulation (SMILP) with a splitting according to Definition 10 as (TMILP).

Solving of the Partial Instances

Since the trips of the same costumer may be in different splittings, we cannot easily guarantee the costumer satisfaction only in just one partial instance. We have to put great effort in this issue. For this, we first define the earliest partial instance in which a trip of a costumer arises as follows: Let $\sigma \in S_n$ with $\sigma(1) = n$ be the order in which the partial instances are solved.

$$\gamma : \mathcal{C} \to [n] \qquad \gamma(c) := \arg\min_{i \in [n]} \left\{ \sigma(i) \in [n] \mid \left((M \circ C)^{-1}(c) \cap \mathcal{T}_i \right) \neq \emptyset \right\}$$

Depending on γ and $\{\mathcal{T}_1, \dots \mathcal{T}_n\}$ we define a partition $\mathcal{C} = \{\mathcal{C}_1, \dots \mathcal{C}_n\}$ as

$$\mathcal{C}_i := \{c \in \mathcal{C} \mid \gamma(c) = i\} \qquad \text{for } i \in [n]$$

Consider costumer $c \in \mathcal{C}$ and the partial instance $\gamma(c) \in [n]$. In this partial instance, a multimodal route $m \in C^{-1}(c)$ for the costumer is chosen and this choice is definite. This means, in all subsequent partial instances, all trips $t \in m$ are fixed to be chosen before solving and all trips $t \in (M \circ C)^{-1}(c) \backslash m$ are fixed to be neglected.

In partial instance $\gamma(c)$ we have at least one trip of this costumer. But there are also trips that are in other splittings. There are even multimodal routes with no trip in this splitting at all. These routes must not be neglected. Therefore, we need a method to choose the routes where all routes $m \in C^{-1}(c)$ are considered. Therefore, we try to estimate the costs of the routes in advance.

The solving of the partial instances is again based on (SMILP). In comparison the (CMILP$_i$) there are only few changes.

The costumer constraint (10) is basically the same. Notice, that the definition of $\mathcal{C}_i$ has changed. The route constraint (11) is restricted to the trips that are actually in this splitting. So the new constraints are

$$\sum_{m\in C^{-1}(c)} u_m = 1 \qquad \text{for all } c \in \mathcal{C}_i \tag{28}$$

$$\sum_{s\in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = u_m \qquad \text{for all } m \in C^{-1}\left(\mathcal{C}_i\right), t \in m \cap \mathcal{T}_i \tag{29}$$

For the costumer constraint it is irrelevant if there are trips of the considered routes in this splitting.
After solving the partial instance, all determined u_m are fixed for the later processed partial instances. The fixed route decisions from the previous partial instances have an impact on the instance, too.
Let $\bar{u}_m \in [0,1]$ be the fixed route choices from the previous instances. Define

$$\overline{\mathcal{C}}_i := \{c \in \mathcal{C} \mid \gamma(c) < \sigma(i)\} \tag{30}$$

as the costumers that are already treated. Then, we introduce the constraint

$$\sum_{s\in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = \bar{u}_m \qquad \text{for all } m \in C^{-1}\left(\overline{\mathcal{C}}_i\right), t \in m \cap \mathcal{T}_i \tag{31}$$

which ensures that the previous route choices are considered.

Cost Estimation

In order to choose a route in a partial instance, we have to estimate the costs for these routes in advance in all subsequent instances. The entire cost for the problem consists of vehicle costs c^{v}, trip costs c^{t}, deadhead costs c^{d} and route costs c^{r}. While we can determine the trip costs and route costs easily for a route, the vehicle costs and trip costs strongly depend on the environment of the route and cannot be determined. We therefore focus on the trip and route costs and define the estimated route costs as follows:

$$C_1(m) := c^{\mathrm{r}}_m + \sum_{t\in m} c^{\mathrm{t}}_t \qquad \text{for } m \in \mathcal{M}$$

We use these costs in the (TMILP$_i$) to define the route costs

$$\hat{c}^{\mathrm{r}}_m := \hat{c}^{\mathrm{r}}_m + \sum_{t\in m\setminus\mathcal{T}_i} c^{\mathrm{t}}_t \qquad \text{for } m \in \mathcal{M}$$

and add

$$\sum_{m \in C^{-1}(\mathcal{C}_i)} u_m \hat{c}^{\mathrm{r}}_m$$

to the objective function.
In $\hat{c}^{\mathrm{r}}_m$, the trips in the same splitting are not considered since they are already part of the objective function. The trips that are not in the same splitting are added to $\hat{c}^{\mathrm{r}}_m$, such that they have an impact on the choice of the routes.
Consider a trip t that is decided before this partial instance, i.e. $t \in (M \circ C)\left(\overline{\mathcal{C}}_i\right)$. Its trip costs c^{t}_t arise twice in the objective functions. Once in the partial instance $\gamma\left((M \circ C)(t)\right)$ as part of $\hat{c}^{\mathrm{r}}_{M(t)}$ and once in partial instance i as c^{t}_t. But since in partial instance i, the trip has fulfilled anyway, these costs are only an additional factor that does not influence the solution.

$$
\begin{aligned}
\min \quad & \left(\sum_{s \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i} x_{d^{\mathrm{s}},s} - \sum_{s \in \hat{\mathcal{P}}_i} x_{s,d^{\mathrm{e}}} \right) c^{\mathrm{v}} + \sum_{m \in C^{-1}(\mathcal{C}_i)} u_m \hat{c}^{\mathrm{r}}_m \\
& \sum_{t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i} \sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t) \setminus \{d^{\mathrm{e}}\}} \left[x_{s,t} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_{t} \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t} \right) \right] && (\mathrm{TMILP}_i) \\
\text{s.t.} \quad & \sum_{t \in \mathrm{N}^{-}_{\overline{G}_i}(s)} x_{t,s} = \sum_{t \in \mathrm{N}^{+}_{\overline{G}_i}(s)} x_{s,t} && \text{for all } s \in \overline{V}_i \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (16) \\
& \sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = 1 && \text{for all } t \in \hat{\mathcal{V}}_i \cup \hat{P}_i && (12) \\
& \sum_{m \in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C}_i && (28) \\
& \sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = u_m && \text{for all } m \in C^{-1}(\mathcal{C}_i), t \in m \cap \mathcal{T}_i && (29) \\
& \sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = \bar{u}_m && \text{for all } m \in C^{-1}\left(\overline{\mathcal{C}}_i\right), t \in m \cap \mathcal{T}_i && (31) \\
& \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^{-}_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (17) \\
& e_s \le f^0_s && \text{for all } s \in \hat{\mathcal{V}}_i && (18) \\
& f^0_s \le e_s && \text{for all } s \in \hat{\mathcal{P}}_i && (13) \\
& 0 \le e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^{-}_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (19) \\
& e_t \le 1 - f^{\mathrm{t}}_t - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^{-}_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (20) \\
& e_t \le e_s - x_{s,t} \left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t \right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t} \right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^{-}_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (21) \\
& e_t \le e_s - x_{s,t} f^{\mathrm{t}}_t + (1 - x_{s,t}) && \text{for all } s \in \hat{\mathcal{P}}_i, t \in \mathrm{N}^{+}_{\overline{G}_i}(s) && (22) \\
& e_s + f^{\mathrm{t}}_s \le f^{\max}_s + (1 - x_{d^{\mathrm{s}},s}) \cdot \left(1 + f^{\mathrm{t}}_s \right) && \text{for all } s \in \mathcal{T}_i && (14) \\
& f^{\min}_s \le e_s + (1 - x_{s,d^{\mathrm{e}}}) && \text{for all } s \in \mathcal{T}_i && (15) \\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t) \in \overline{A}_i && (23) \\
& z_{s,r,t} \in \{0,1\} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^{-}_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i, r \in \mathcal{R}_{s,t} && (24) \\
& e_s \in [0,1] && \text{for all } s \in \overline{V}_i \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (25) \\
& u_m \in \{0,1\} && \text{for all } m \in C^{-1}(\mathcal{C}_i) && (8)
\end{aligned}
$$

3.4.2 Iterative Approach

We use the previously developed heuristic for an iterative approach. We compute an initial solution while we choose the routes with cost function C_1. Then we determine the actual cost of this route in the entire solution and compare the estimated cost with the actual cost.

Initial Solution

We determine a solution with the heuristic developed in Section 3.4.1. Given a solution $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u})$ of the (TMILP), we determine

$$C_1(c) := C_1(m) \qquad \text{for } c \in \mathcal{C}, m \in C^{-1}(c) \text{ with } \bar{u}_m = 1$$

Finding Bad Route Choice

Given a solution of the problem, the subproblem is to find a costumer with a bad route choice. This means, for this costumer there is another route, such that the total costs are lower if this route is chosen. Then, we can exchange these routes and compute a new solution considering the new route.
An initial idea is to compute the costs, one route in the solution contributes to the entire solution. Then, we can compare this to the cost, with which we estimated the route costs before. If the actual costs are considerably higher than the estimate costs, this costumer is a candidate for exchanging routes.
Since we cannot determine the contributing costs exactly, we try to estimate these costs. Let $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u})$ be a solution of the (TMILP). To determine the contributing costs for route $m \in \mathcal{M}$, we define the following auxiliary costs for every trip $t \in \mathcal{T}$ of the solution:
Vehicle costs $c_t^{\mathrm{v}}(S)$: Let $v \in \mathcal{V}$ be the vehicle covering t and k_v the number of trips covered by v:

$$c_t^{\mathrm{v}}(S) := \frac{c^{\mathrm{v}}}{k_v}$$

Refueling costs $c_t^{\mathrm{refuel}}(S)$: Let $r \in \mathcal{R}$ be the next refuel station used after t and T_r all trips covered since the last station, let $\bar{z}_{s,r,s'} = 1$:

$$c_t^{\mathrm{refuel}}(S) := \frac{f_t^{\mathrm{t}}}{\sum_{t' \in T_r} f_t^{\mathrm{t}}} \left(c_{s,r}^{\mathrm{d}} + c_{r,s'}^{\mathrm{d}} - c_{s,s'}^{\mathrm{d}} \right)$$

If the vehicle is not refueled after t, then $c_t^{\mathrm{refuel}}(S) := 0$.

Deadhead costs $c_t^{\mathrm{d}}(S)$: Let $s \in \mathcal{V} \cup \mathcal{T}_{\mathrm{car}}, s' \in \mathcal{T}_{\mathrm{car}}$ be the trips covered directly before and after t by vehicle v, i.e. $\bar{x}_{s,t} = \bar{x}_{t,s'} = 1$:

$$c_t^{\mathrm{d}}(S) := \frac{1}{2}\left(c_{s,t}^{\mathrm{d}} + c_{t,s'}^{\mathrm{d}}\right)$$

If t is the last trip of the duty, i.e. $\bar{x}_{s,t} = \bar{x}_{t,d^{\mathrm{e}}} = 1$, then $c_t^{\mathrm{d}}(S) := \frac{1}{2}c_{s,t}^{\mathrm{d}}$.
With these auxiliary costs we can define new route costs which describe the contribution of a multimodal route to the entire solution better:

Definition 11 (Improved Cost Estimation)**.** Let $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u})$ be a solution of the (TMILP). With the auxiliary costs described before, we define the improved cost estimation for all multimodal routes $\{m \in \mathcal{M} \mid \bar{u}_m = 1\}$:

$$C_2(S, m) := C_1(m) + \sum_{t \in m}\left(c_t^{\mathrm{v}}(S) + c_t^{\mathrm{refuel}}(S) + c_t^{\mathrm{d}}(S)\right)$$

We further define

$$C_2(S, c) := C_2(S, m) \qquad \text{for } c \in \mathcal{C}, m \in C^{-1}(c) \text{ with } \bar{u}_m = 1$$

Now we can evaluate our previous estimation for the route contribution. If $C_2(S, c)$ is significantly higher than $C_1(S, c)$ then the probability is high, the we made a bad route choice for costumer $c \in \mathcal{C}$.
We therefore determine

$$c^* := \arg\max_{c \in \mathcal{C}} \frac{C_2(S, c)}{C_1(S, c)}$$

The probability is high that we made a bad route choice for costumer c^*. Thus, we look at the route choice for c^* again.

Subproblem

Let $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u})$ be a solution of (TMILP) and $c \in \mathcal{C}$ a candidate for a bad route choice. We define the following subproblem (HSP$_c$): Assume the schedule according to S for the entire time without $[z_c^{\mathrm{start}}, z_c^{\mathrm{end}}]$ and all route choices for costumers except c as fix. Determine an optimal schedule within these restrictions.
Considering only the trips of c and the trips chosen in S, we define

$$\mathcal{T}^c := \left\{ t \in \mathcal{T}_{\text{car}} \mid (M \circ C)(t) = c \vee \sum_{s \in \mathrm{N}^-_{\overline{G}}(t)} x_{s,t} = 1 \right\}$$

and the splittings $\mathcal{T}^c = \{\mathcal{T}_1^c, \mathcal{T}_2^c, \mathcal{T}_3^c\}$ and $\mathcal{V}^c = \{\mathcal{V}_1^c, \mathcal{V}_2^c, \mathcal{V}_3^c\}$ by

$$\mathcal{T}_i^c := \begin{cases} \{t \in \mathcal{T}^c \mid z_t^{\text{start}} < z_c^{\text{start}}\} & \text{if } i = 1 \\ \left\{t \in \mathcal{T}^c \mid z_c^{\text{start}} \le z_t^{\text{start}} \le z_c^{\text{end}}\right\} & \text{if } i = 2 \\ \left\{t \in \mathcal{T}^c \mid z_c^{\text{end}} < z_t^{\text{start}}\right\} & \text{if } i = 3 \end{cases}$$

and

$$\mathcal{V}_i^c := \begin{cases} \{v \in \mathcal{V} \mid z_v < z_c^{\text{start}}\} & \text{if } i = 1 \\ \left\{v \in \mathcal{V} \mid z_c^{\text{start}} \le z_v \le z_c^{\text{end}}\right\} & \text{if } i = 2 \\ \left\{v \in \mathcal{V} \mid z_v < z_c^{\text{end}}\right\} & \text{if } i = 3 \end{cases}$$

We then define the start point set $\hat{\mathcal{V}}_2$ and the end point set $\hat{\mathcal{P}}_2$

$$\begin{aligned} \hat{\mathcal{V}}_2 &:= \{s \in \mathcal{T}_1^c \mid \bar{x}_{s,t} = 1 \text{ for } t \in (\mathcal{T}_2^c \mathbin{\dot\cup} \mathcal{T}_3^c \mathbin{\dot\cup} \{d^{\text{e}}\})\} \mathbin{\dot\cup} \mathcal{V}_1 \mathbin{\dot\cup} \mathcal{V}_2 \\ \hat{\mathcal{P}}_2 &:= \{t \in \mathcal{T}_3^c \mid \bar{x}_{s,t} = 1 \text{ for } t \in (\{d^{\text{s}}\} \mathbin{\dot\cup} \mathcal{T}_1^c \mathbin{\dot\cup} \mathcal{T}_2^c)\} \end{aligned}$$

With these definitions, we can adapt the formulation (TMILP$_i$) for $i = 2$ to (HSP$_c$). The only constraints we have to change are the costumer and route constraints (28), (29), (31) and (8). These are replaced with

$$\sum_{m \in C^{-1}(c)} u_m = 1 \tag{32}$$

$$\sum_{s \in \mathrm{N}^-_{\overline{G}}(t)} x_{s,t} = u_m \qquad \text{for all } m \in C^{-1}(c), t \in m \tag{33}$$

$$\sum_{s \in \mathrm{N}^-_{\overline{G}}(t)} x_{s,t} = \bar{u}_{M(t)} \qquad \text{for all } t \in \mathcal{T}_2 \setminus (M \circ C)^{-1}(c) \tag{34}$$

$$u_m \in \{0, 1\} \qquad \text{for all } m \in M^{-1}(c) \tag{35}$$

The objective function can be applied as (TMILP$_i$).

We can easily receive a new total solution: Transform the original solution S into three partial solutions $\{S_1^c, S_2^c, S_3^c\}$ according to the splitting $\mathcal{T}^c = \{\mathcal{T}_1^c, \mathcal{T}_2^c, \mathcal{T}_3^c\}$ and $\mathcal{V}^c = \{\mathcal{V}_1^c, \mathcal{V}_2^c, \mathcal{V}_3^c\}$. Let $\hat{S}_2^c$ be an optimal solution of (HSP$_c$). Then feasibly connect the partial solutions $\left\{S_1^c, \hat{S}_2^c, S_3^c\right\}$ to a new solution $\hat{S}$.
The original partial solution S_2 is a feasible solution of (HSP$_c$). Therefore, with this method we cannot get a worse entire solution than before.
After completing this step, we can apply this procedure to the costumer with the second-highest ratio of $\frac{C_2(S,c)}{C_1(c)}$.

3.4.3 Restricted Approach

3.4.4 Improvements

4 Optimal Approach

4.1 Dantzig-Wolfe-Decomposition

We adapt the path flow formulation from (Kaiser, Knoll, cap. 3.3) by applying Dantzig-Wolfe-Decomposition. This can be used, if many of the constraints have only impact on a small number of variables and these variables can be grouped. The structure of the problem looks like:

$$
\left(\begin{array}{cccccccccccc}
\star & \cdots & \star & \star & \cdots & \star & & & & \star & \cdots & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \cdots & & \vdots & \ddots & \vdots \\
\star & \cdots & \star & \star & \cdots & \star & & & & \star & \cdots & \star \\
\hline
\star & \cdots & \star & 0 & \cdots & 0 & & & & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \cdots & & \vdots & \ddots & \vdots \\
\star & \cdots & \star & 0 & \cdots & 0 & & & & 0 & \cdots & 0 \\
0 & \cdots & 0 & \star & \cdots & \star & & & & & & \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \ddots & & & \vdots & \\
0 & \cdots & 0 & \star & \cdots & \star & & & & & & \\
& & & & & & & & & 0 & & 0 \\
& \vdots & & & \ddots & & & \ddots & & \vdots & \ddots & \vdots \\
& & & & & & & & & 0 & & 0 \\
0 & \cdots & 0 & & & & 0 & \cdots & 0 & \star & \cdots & \star \\
\vdots & \ddots & \vdots & & \cdots & & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & & & & 0 & \cdots & 0 & \star & \cdots & \star
\end{array}\right)
$$

The constraints concerning more variables are called linking constraints.

Identification of the Subproblems

Consider the (MMILP). A natural choice for the subproblem is the duty of each vehicle $v \in \mathcal{V}$. We define (x^v, z^v, e^v) for $v \in \mathcal{V}$ as the specific variables for this vehicle. We can therefore define the set of feasible configurations for vehicle $v \in \mathcal{V}$ as follows:

$$
\begin{aligned}
X_v := \Big\{ &(x,z,e) \in \{0,1\}^A \times \{0,1\}^{A \cap (\mathcal{V} \cup \mathcal{T}_{\text{car}})^2 \times \mathcal{R}} \times [0,1]^{\mathcal{V} \cup \mathcal{T}_{\text{car}}} \mid \\
&\sum_{t \in \mathrm{N}_G^-(s)} x_{t,s} = \sum_{t \in \mathrm{N}_G^+(s)} x_{s,t} && \text{for all } s \in V \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.15)\\
&\sum_{s \in \mathrm{N}_G^-(t)} x_{s,v} = 1 && && (36)\\
&\sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = 0 && \text{for all } t \in \mathcal{V} \setminus \{v\} && (37)\\
&\sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \leq x_{s,t} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.18)\\
&e_s \leq f_s^0 && \text{for all } s \in \mathcal{V} && (3.19)\\
&0 \leq e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f_{s,r}^{\mathrm{d}} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.12)\\
&e_t \leq 1 - f_t^{\mathrm{t}} - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f_{r,t}^{\mathrm{d}} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.13)\\
&e_t \leq e_s - x_{s,t}\left(f_{s,t}^{\mathrm{d}} + f_t^{\mathrm{t}}\right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t}\left(f_{s,r}^{\mathrm{d}} + f_r^{\mathrm{t}} + f_{r,t}^{\mathrm{d}} - f_{s,t}^{\mathrm{d}}\right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.14)\\
&\Big\}
\end{aligned}
$$

We denote the set of feasible duties for any vehicle by $X := \bigcup_{v \in \mathcal{V}} X_v$. We write the cost for configuration (x^v, z^v, e^v) as $g\,(x^v, z^v, e^v)$. Putting all together, we can rewrite (MMILP) as

$$
\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} g\,(x^v, z^v, e^v) + \sum_{m \in \mathcal{M}} u_m c_m^{\mathrm{r}} \\
\text{s.t.} \quad & \sum_{m \in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C} && (6)\\
& \sum_{v \in \mathcal{V}} \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t}^v - u_m = 0 && \text{for all } m \in \mathcal{M}, t \in m && (38)\\
& (x^v, z^v, e^v) \in X_v && \text{for all } v \in \mathcal{V} \\
& u_m \in \{0,1\}^{\mathcal{M}}
\end{aligned}
$$

We can see that (6) depends only on u_m. Therefore, we create another subproblem for the choice of routes. We define the set of feasible route choices as follows:

$$\hat{X} := \left\{ \{0,1\}^{\mathcal{M}} | \sum_{m \in C^{-1}(c)} u_m = 1 \text{ for all } c \in \mathcal{C} \right\}$$

We introduce variable $\hat{u}$ and the route cost function $\hat{g}$ and rewrite (MMILP) again:

$$\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} g\left(x^v, z^v, e^v\right) + \hat{g}\left(\hat{u}\right) & & \\
\text{s.t.} \quad & \sum_{v \in \mathcal{V}} \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t}^v - u_m = 0 & & \text{for all } m \in \mathcal{M}, t \in m \qquad (38) \\
& \left(x^v, z^v, e^v\right) \in X_v & & \text{for all } v \in \mathcal{V} \\
& \hat{u} \in \hat{X} & &
\end{aligned}$$

The only linking constraints are (38).

Reduction of the Master Problem

We define the linear mapping

$$\psi : X \to \{0,1\}^{\mathcal{T}_{\text{car}}} \qquad (x, z, e) \mapsto \left(\sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} \right)_{t \in \mathcal{T}_{\text{car}}}$$

and rewrite (MMILP) by using $y^v := \psi\left(x^v, z^v, e^v\right)$:

$$\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} \min g\left(\psi^{-1}\left(y^v\right) \cap X_v\right) + \hat{g}\left(\hat{u}\right) & & \\
\text{s.t.} \quad & \sum_{v \in \mathcal{V}} y_t^v - u_m = 0 & & \text{for all } m \in \mathcal{M}, t \in m \\
& y^v \in \psi\left(X_v\right) & & \text{for all } v \in \mathcal{V} \\
& \hat{u} \in \hat{X} & &
\end{aligned}$$

Column Generation

For every $v \in \mathcal{V}$, let $\mathcal{I}_v$ be an index set for the finitely many points in $\psi\left(X_v\right)$ and let the columns of $Y^v \in \mathbb{R}^{\mathcal{T}_{\text{car}} \times \mathcal{I}_v}$ be exactly those points. Let $\hat{\mathcal{I}}$ be an index set for the finitely many points in $\hat{X}$ and let the columns of $\hat{Y} \in \mathbb{R}^{\mathcal{M}\times\hat{\mathcal{I}}}$ be exactly those points. Let $G^v \in \mathbb{R}^{1\times\mathcal{I}}$ be the respective values of $\min g\left(\psi^{-1}(\cdot) \cap X_v\right)$ and $\hat{g} \in \mathbb{R}^{1\times\hat{\mathcal{I}}}$ be the respective route costs. Then we can reformulate the master problem as

$$\begin{aligned}
\min \quad & \sum_{v\in\mathcal{V}} G^v\lambda^v + \hat{g}\hat{\lambda} && \text{(IMMP)}\\
\text{s.t.} \quad & \sum_{v\in\mathcal{V}} Y^v_{t,\cdot}\lambda^v = \hat{Y}_{m,\cdot}\hat{\lambda} && \text{for all } m \in \mathcal{M}, t \in m\\
& \sum_{i\in\mathcal{I}_v} \lambda^v_i = 1 && \text{for all } v \in \mathcal{V}\\
& \sum_{i\in\hat{\mathcal{I}}} \hat{\lambda}_i = 1 &&\\
& \lambda^v \in \{0,1\}^{\mathcal{I}_v} && \text{for all } v \in \mathcal{V}\\
& \hat{\lambda} \in \{0,1\}^{\hat{\mathcal{I}}} &&
\end{aligned}$$

We regard the LP-relaxation by dropping the integrality constraints:

$$\begin{aligned}
\min \quad & \sum_{v\in\mathcal{V}} G^v\lambda^v + \hat{g}\hat{\lambda} && \text{(LMMP)}\\
\text{s.t.} \quad & \sum_{v\in\mathcal{V}} Y^v_{t,\cdot}\lambda^v = \hat{Y}_{m,\cdot}\hat{\lambda} && \text{for all } m \in \mathcal{M}, t \in m\\
& \sum_{i\in\mathcal{I}_v} \lambda^v_i = 1 && \text{for all } v \in \mathcal{V}\\
& \sum_{i\in\hat{\mathcal{I}}} \hat{\lambda}_i = 1 &&\\
& \lambda^v \in \mathbb{R}^{\mathcal{I}_v}_{\geq 0} && \text{for all } v \in \mathcal{V}\\
& \hat{\lambda} \in \mathbb{R}^{\hat{\mathcal{I}}}_{\geq 0} &&
\end{aligned}$$

We reduce the size by considering only subsets $\mathcal{J}_v \subset \mathcal{I}_v$ and $\hat{\mathcal{J}} \subset \hat{\mathcal{I}}$ and formulate the relaxed restricted master problem:

$$
\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} G^v_{\mathcal{J}_v} \lambda^v + \hat{g}_{\hat{\mathcal{J}}} \hat{\lambda} && && \text{(LRMMP)} \\
\text{s.t.} \quad & \sum_{v \in \mathcal{V}} Y^v_{t,\mathcal{J}_v} \lambda^v = \hat{Y}_{m,\hat{\mathcal{J}}} \hat{\lambda} && \text{for all } m \in \mathcal{M}, t \in m \\
& \sum_{i \in \mathcal{J}_v} \lambda^v_i = 1 && \text{for all } v \in \mathcal{V} \\
& \sum_{i \in \hat{\mathcal{J}}} \hat{\lambda}_i = 1 \\
& \lambda^v \in \mathbb{R}^{\mathcal{J}_v}_{\geq 0} && \text{for all } v \in \mathcal{V} \\
& \hat{\lambda} \in \mathbb{R}^{\hat{\mathcal{J}}}_{\geq 0}
\end{aligned}
$$

For the dual relaxed restricted master problem, we introduce dual variables $\gamma \in \mathbb{R}^{\mathcal{T}_{\text{car}}}$, $\mu \in \mathbb{R}^{\mathcal{V}}$ and $\alpha \in \mathbb{R}$. The dual problem is:

$$
\begin{aligned}
\max \quad & \sum_{v \in \mathcal{V}} \mu_v + \alpha && && \text{(DLRMMP)} \\
\text{s.t.} \quad & \sum_{t \in \mathcal{T}_{\text{car}}} Y^v_{t,i} \gamma_t + \mu_v \leq G^v_i && \text{for all } v \in \mathcal{V}, i \in \mathcal{J}_v \\
& \alpha - \sum_{m \in \mathcal{M}} \sum_{t \in m} \hat{Y}_{m,i} \gamma_t \leq \hat{g}_i && \text{for all } i \in \hat{\mathcal{J}} \\
& \gamma \in \mathbb{R}^{\mathcal{T}_{\text{car}}} \\
& \mu \in \mathbb{R}^{\mathcal{V}} \\
& \alpha \in \mathbb{R}
\end{aligned}
$$

4.2 Solving the Relaxed Master Problem

Let $(\gamma^*, \mu^*, \alpha^*)$ be a solution of (DLRMMP) with $\mathcal{J}_v \subset \mathcal{I}_v$ for all $v \in \mathcal{V}$ and $\hat{\mathcal{J}} \subset \hat{\mathcal{I}}$. We want to find out whether $(\gamma^*, \mu^*, \alpha^*)$ corresponds to an optimal solution of the (LMMP). This is the case if it is feasible for the dual relaxed master problem, i.e. the following constraints hold for the entire sets $\mathcal{I}_v$ and $\hat{\mathcal{I}}$. This means, the following equations hold for $(\gamma^*, \mu^*, \alpha^*)$:

$$
\sum_{t \in \mathcal{T}_{\text{car}}} Y^v_{t,i} \gamma^* + \mu^*_v \leq G^v_i \qquad \text{for all } v \in \mathcal{V}, i \in \mathcal{I}_v \tag{39}
$$

$$
\alpha^* - \sum_{m \in \mathcal{M}} \sum_{t \in m} \hat{Y}_{m,i} \gamma^*_t \leq \hat{g}_i \qquad \text{for all } i \in \hat{\mathcal{I}} \tag{40}
$$

In order to find an optimal solution of (LMMP) we have to find indices $i \in \mathcal{I}_v$ or $j \in \hat{\mathcal{I}}$ where the previous constraints are violated. This leads to the following subproblems:

1. $\mathrm{SP}_v^{\mathrm{v}}$ for $v \in \mathcal{V}$: Find $i \in \mathcal{I}_v \backslash \mathcal{J}_v$ s.t.
$$\sum_{t \in \mathcal{T}_{\mathrm{car}}} Y_{t,i}^v \gamma_t^* + \mu_v^* > G_i^v$$

2. SP^{m}: Find $i \in \hat{\mathcal{I}} \backslash \hat{\mathcal{J}}$ s.t.
$$\alpha^* - \sum_{m \in \mathcal{M}} \sum_{t \in m} \hat{Y}_{m,i} \gamma_t^* > \hat{g}_i$$

Vehicle Subproblem

The vehicle subproblem for finding violated constraints (39) reads for $v \in \mathcal{V}$ as follows:

$$\begin{aligned} \min \quad & g\left(x^v, z^v, e^v\right) - \sum_{t \in \mathcal{T}_{\mathrm{car}}} \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t}^v \gamma_t^* \\ \text{s.t.} \quad & \left(x^v, z^v, e^v\right) \in X_v \end{aligned} \tag{$\mathrm{SP}_v^{\mathrm{v}}$}$$

The constraints (36) and (37) ensure, that exactly vehicle v is used and the others are not used. The subproblem is equivalent to the Shortest Path Problem with Resource Constraints (SPPRC). In (Kaiser, cap. 7) there is provided a way to solve the (SPPRC) efficiently.
Let $(\bar{x}^v, \bar{z}^v, \bar{e}^v)$ be an optimal solution of $(\mathrm{SP}_v^{\mathrm{v}})$. If $\operatorname{value}(\bar{x}^v, \bar{z}^v, \bar{e}^v) < \mu_v^*$ then add this to $\mathcal{J}_v$ and continue the master problem.

Route Subproblem

The route subproblem for finding violated constraints (40) reads as follows:

$$\begin{aligned} \min \quad & \sum_{m \in \mathcal{M}} u_m \left(c_m^{\mathrm{r}} + \sum_{t \in m} \gamma_t^* \right) & & \\ \text{s.t.} \quad & \sum_{m \in C^{-1}(c)} u_m = 1 & & \text{for all } c \in \mathcal{C} \\ & u_m \in \{0,1\} & & \text{for all } m \in \mathcal{M} \end{aligned} \tag{SP^{m}}$$

This problem is easy to solve: For every $c \in \mathcal{C}$ choose the multimodal route $m \in C^{-1}(c)$ with the smallest cost $c_m^{\mathrm{r}} + \sum_{t \in m} \gamma_t^*$.
Let $\bar{u}$ be an optimal solution of $(\mathrm{SP}^{\mathrm{m}})$. If $\operatorname{value}(\bar{u}) < \alpha^*$ then add this to $\hat{\mathcal{J}}$ and continue the master problem.

Previous Formulations

(MILP)

$$
\begin{aligned}
\min \sum_{s\in\mathcal{V}} & \sum_{t\in \mathrm{N}_G^+(s)\setminus\{d^{\mathrm{e}}\}} x_{s,t} c^{\mathrm{v}} \\
& + \sum_{t\in\mathcal{T}} \sum_{s\in\mathrm{N}_G^-(t)} \left[x_{s,t}\left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_t\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t}\right)\right] && \text{(MILP)}\\
\text{s.t.} & \sum_{t\in\mathrm{N}_G^-(s)} x_{t,s} = \sum_{t\in\mathrm{N}_G^+(s)} x_{s,t} && \text{for all } s\in V\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} && (3.15)\\
& \sum_{s\in\mathrm{N}_G^-(t)} x_{s,t} = 1 && \text{for all } t\in\mathcal{V} && (3.16)\\
& \sum_{t\in C^{-1}(c)} \sum_{s\in\mathrm{N}_G^-(t)} x_{s,t} = 1 && \text{for all } c\in\mathcal{C} && (3.17)\\
& \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t\in\mathcal{T}, s\in\mathrm{N}_G^-(t) && (3.18)\\
& e_s \le f^0_s && \text{for all } s\in\mathcal{V} && (3.19)\\
& 0 \le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t\in\mathcal{T}, s\in\mathrm{N}_G^-(t) && (3.12)\\
& e_t \le 1 - f^{\mathrm{t}}_t - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t\in\mathcal{T}, s\in\mathrm{N}_G^-(t) && (3.13)\\
& e_t \le e_s - x_{s,t}\left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t}\right) + (1 - x_{s,t})\\
& && \text{for all } t\in\mathcal{T}, s\in\mathrm{N}_G^-(t) && (3.14)\\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t)\in A && (3.20)\\
& z_{s,r,t} \in \{0,1\} && \text{for all } t\in\mathcal{T}, s\in\mathrm{N}_G^-(t), r\in\mathcal{R}_{s,t} && (3.21)\\
& e_s \in [0,1] && \text{for all } s\in V\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} && (3.22)
\end{aligned}
$$

(AMILP)

$$
\begin{aligned}
\min \sum_{s\in\mathcal{V}} \sum_{t\in \mathrm{N}_G^+(s)\setminus\{d^{\mathrm{e}}\}} & x_{s,t} c^{\mathrm{v}} \\
+ \sum_{t\in\mathcal{T}} \sum_{s\in \mathrm{N}_G^-(t)} & \left[x_{s,t}\left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_{t}\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t}\right)\right] && \text{(AMILP)}\\
\text{s.t.} \sum_{t\in \mathrm{N}_G^-(s)} x_{t,s} &= \sum_{t\in \mathrm{N}_G^+(s)} x_{s,t} \qquad \text{for all } s\in V\setminus\{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.15)\\
\sum_{s\in \mathrm{N}_G^-(t)} x_{s,t} &= 1 \qquad \text{for all } t\in\mathcal{T}\cup\mathcal{V} && (6.2)\\
\sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} &\le x_{s,t} \qquad \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t) && (3.18)\\
e_s &\le f^0_s \qquad \text{for all } s\in\mathcal{V} && (3.19)\\
0 &\le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} \qquad \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t) && (3.12)\\
e_t &\le 1 - f^{\mathrm{t}}_t - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} \qquad \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t) && (3.13)\\
e_t &\le e_s - x_{s,t}\left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t}\right) + (1 - x_{s,t})\\
&\qquad \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t) && (3.14)\\
x_{s,t} &\in \{0,1\} \qquad \text{for all } (s,t)\in A && (3.20)\\
z_{s,r,t} &\in \{0,1\} \qquad \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t), r\in\mathcal{R}_{s,t} && (3.21)\\
e_s &\in [0,1] \qquad \text{for all } s\in V\setminus\{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.22)
\end{aligned}
$$

(LMILP)

$$
\begin{aligned}
\min & \sum_{s\in\mathcal{V}} \sum_{t\in \mathrm{N}^{+}_{\overline{G}}(s)\setminus\{d^{\mathrm{e}}\}} x_{s,t}c^{\mathrm{v}} + \sum_{s\in P}\sum_{t\in \mathrm{N}^{+}_{\overline{G}}(s)} x_{s,t}c^{\mathrm{t}}_{t} \\
& + \sum_{t\in\mathcal{T}\cup P}\sum_{s\in \mathrm{N}^{-}_{\overline{G}}(t)} \left[x_{s,t}\left(c^{\mathrm{d}}_{s,t}+c^{\mathrm{t}}_{t}\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c^{\mathrm{d}}_{s,r}+c^{\mathrm{d}}_{r,t}-c^{\mathrm{d}}_{s,t}\right)\right] && \text{(LMILP)}\\
\text{s.t.} & \sum_{t\in \mathrm{N}^{-}_{\overline{G}}(s)} x_{t,s} = \sum_{t\in \mathrm{N}^{+}_{\overline{G}}(s)} x_{s,t} && \text{for all } s\in\overline{V}\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} && (8.1)\\
& \sum_{s\in \mathrm{N}^{-}_{\overline{G}}(t)} x_{s,t} = 1 && \text{for all } t\in\mathcal{T}\cup\mathcal{V} && (8.2)\\
& \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P && (8.3)\\
& e_s \le f^{0}_{s} && \text{for all } s\in\mathcal{V} && (8.4)\\
& 0 \le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}f^{\mathrm{d}}_{s,r} && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P && (8.5)\\
& e_t \le 1 - f^{\mathrm{t}}_{t} - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}f^{\mathrm{d}}_{r,t} && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P && (8.6)\\
& e_t \le e_s - x_{s,t}\left(f^{\mathrm{d}}_{s,t}+f^{\mathrm{t}}_{t}\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(f^{\mathrm{d}}_{s,r}+f^{\mathrm{t}}_{r}+f^{\mathrm{d}}_{r,t}-f^{\mathrm{d}}_{s,t}\right) + (1-x_{s,t}) \\
& && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P && (8.7)\\
& e_t \le e_s - x_{s,t}f^{\mathrm{t}}_{t} + (1-x_{s,t}) && \text{for all } s\in P, t\in \mathrm{N}^{+}_{\overline{G}}(s) && (8.8)\\
& x_{s,t}\in\{0,1\} && \text{for all } (s,t)\in\overline{A} && (8.9)\\
& z_{s,r,t}\in\{0,1\} && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P, r\in\mathcal{R}_{s,t} && (8.10)\\
& e_s\in[0,1] && \text{for all } s\in\overline{V}\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} && (8.11)
\end{aligned}
$$