Optimal Integration of Autonomous Vehicles in Car Sharing

Development of a Heuristic considering Multimodal Transport and Integration in an Optimal Framework

Martin Sperr

October 23, 2016

Contents

1 Introduction **1**

2 Problem Description and Classification **3**
2.1 Situation and Issue 3
2.2 Problem Description and Notation 4
2.3 Classification 8
2.4 Complexity 10

3 Mathematical Models **13**
3.1 Task Graph 13
3.2 Arc Flow Formulation 15

4 Successive Heuristics **19**
4.1 Splitting the Problem 20
4.1.1 Splitting Trip and Vehicle Set 20
4.1.2 Identifying the Subproblems 23
4.2 General Setting 23
4.3 Costumer-dependent Splitting 26
4.4 Time-dependent Splitting 31
4.4.1 Basic Idea 31
4.4.2 Iterative Approach 35
4.4.3 Restricted Approach 38
4.4.4 Improvements 40

5 Optimal Approach **41**
5.1 Dantzig-Wolfe-Decomposition 41
5.2 Master Problem without Route Choice Restriction 43
5.3 Solving the Relaxed Master Problem 45
5.4 Master Problem with Route Choice Restriction 46
5.5 Solving the Relaxed Master Problem 48

6 Instance Creation **51**
6.1 Route Creation 51

Previous Formulations **55**
1 (MILP) 55
2 (AMILP) 56
3 (LMILP) 57

Chapter 1

Introduction

In this thesis, a heuristic for the routing of autonomous vehicles is refined. It is based on two master's theses [Kai16] and [Kno16] which were created earlier on the same faculty. While [Kno16] develops heuristic solution methods for a simplified model, [Kai16] examines a decomposition approach for solving the original model exactly. In contrast to the previous theses, an extended model is regarded here, where the possibility of multimodal transport is considered. First, the previously developed heuristic is extended to the possibility of multimodal transport. Then, this heuristic is integrated in the already existing optimal approach.

In commercial car sharing, a costumer rents a car a limited period of time. In the classic version, the costumer gets the car on a fixed location and returns it to this location after usage. In contrast to this, free-floating car sharing allows the costumer to pick any available car and to put it down somewhere in the operation area. The costumer usually books the car beforehand, typically via a smartphone application. He pays a certain amount per minute of car usage. This method is obviously more costumer-friendly since the costumer has no effort in getting to and from the renting location. But this means significantly more effort for the car sharing supplier. He has to provide a comprehensive offer of available cars, such that there is always a car where the costumer needs it. Further he is responsible for refueling and servicing the cars, wherever they are. Costumers may park their car where it suits them and simultaneously only rent a car if it is within a small walking distance to their current position. Therefore, the distribution of the cars heavily depends on the costumer behavior. This might lead to an imbalance of supply and demand.

A possible solution for this is the usage of autonomous vehicles. Although they are not available on present day, this topic is highly researched. Autonomous vehicles may be available within the next ten to twenty years (cf. [Hau15]). The obvious advantage of autonomous cars is that they do not need a driver. An autonomous car is able change the position after satisfying a costumer on its own. The car can drive to a refuel station or to a position where it is needed next. For the costumer, this behavior is similar to calling a taxi. The car picks him up on his present location and takes him to his destination. The supplier profits since he does not need employees for refueling or relocating the cars.

Besides cars, it is often advantageous for the costumer to use public transport. For suitable trips, the usage of public transport is often faster and particularly cheaper. It is further more efficient in a city with many cars. But with public transport, there also some bad facts arise. The next station may be too far away for walking or unfavorable changing times increase the total travel time. In these cases it is often a good idea to combine car sharing and public transport in one trip. The costumer uses the car for driving to a station with a good connection and continues the trip by using public transport. The combination of different types of transport is called "multiple-leg". While in the previous master's theses the number of legs was restricted to one, the here developed solution methods can cope with multiple legs.
The introduction of autonomous cars is a huge change in the service provided to the costumer. Therefore, it is hard to predict the changing of the costumer behavior. This estimation is not part of this thesis, the focus lies rather in the potential of autonomous cars. Therefore, actual renting data from present day are used. In order to model realistic alternatives for the costumer the data are slightly modified, for example by splitting existing trips into a car trip and a public transport trip. The goal is to find an optimal fleet size for the car sharing provider. For finding the optimal fleet size, the number of needed cars has to be traded off against the total driven distance. And these costs have to be considered while providing a good service for the costumer. The determined fleet size can be compared to the actual fleet size in order to estimate the cost saving with autonomous cars.
The setting for the underlying theses was the following: Each costumer has a set of alternative trips, where one trip each has to be fulfilled in the solution. Each trip is a single-leg trip, i. e. the trip can either be a car or a public transport trip. [Kai16] provides an approach to solve this problem exactly via column generation. [Kno16] provides a fast heuristic via a time-dependent splitting of the trips. This heuristic requires the further restriction that each costumer has only one alternative, i. e. each trip has to be fulfilled. In contrast, this thesis provides solution methods without both of these restrictions. The heuristic developed in [Kno16] is extended in order to cope with multiple-leg multimodal transport. The goal of this heuristic is to determine a good solution fast. Afterwards, the extended heuristic is integrated in the optimal framework developed in [Kai16], in order to find an optimal solution for the problem.

Chapter 2

Problem Description and Classification

In this chapter, the problem is stated in detail and the notation is introduced. The problem is classified by relating it to known problems in literature and its complexity is determined. Finally, known approaches to similar problems are regarded. Most of the following considerations are already part of the underlying theses [KK16], except for the fact that multiple leg is allowed. For better clarity to the reader, all crucial results are repeated here.

2.1 Situation and Issue

We regard the situation of free-floating car sharing as it exists today in combination with autonomous vehicles. Free-floating car sharing means that a costumer can rent an available car wherever and whenever there is one and use it as long as he needs it. After usage, he parks the car somewhere in the operation area. We assume the existence of autonomous vehicles. An autonomous vehicle behaves the same as if a human being drives it, but without that a human being is necessarily present. Instead of looking for a car, a costumer books a car via a smartphone application and gets picked up by the car at the desired start location and at the desired start time. For the costumer, this seems similar to a taxi service.

The car sharing issue is combined with public transport as it is known today. There is a fixed schedule, according to which the bus or train visits public transport stations in a row at certain time points. A possible route for a costumer may look as follows: The costumer is picked up at his start position by a car and is brought to a station where he gets on a train. After finishing the train trip, he is picked up again by another car and is brought to his destination. It is also possible to change trains during this public transport trip. This behavior is very advantageous for the costumer. While an in-between train trip is cheaper than a pure car trip, the combination of car and train is faster than a pure public transport trip since does not have walking and changing time.

2.2 Problem Description and Notation

In order to realize the previously described problem, we introduce a formal notation for the problem. We have a set of costumers with known travel requests. Each of those can be realized by a number of precomputed multimodal routes. Each route consists of a sequence of trips. A trip is either a car trip or a public transport trip and has a fixed start and end position, as well as a fixed start and end time. Fulfilling a route means that the costumer takes all the trips of this route in a row, i. e. he starts at the start point of the first trip and is finished at the end point of the last trip. The transition between two subsequent trips is the changing from a car to a train or the other way round. Each costumer has to be satisfied, i. e. he has to be able to fulfill one of his alternative routes.

Although we formally define public transport trips here, we regard only the car trips in detail. Public transport is maintained by an own schedule and is not of interest for this issue. The availability of suitable public transport trips has to be respected while creating the routes (Chapter 6). There, the car trips are created in such a way that they are suitable to the existing public transport trips. Here, we assume the existence of feasible routes and therefore only control the behavior of the cars.

For fulfilling the car trips, we have a set of vehicles. For each vehicle, we have a position where it starts and a time from when it is available. A vehicle can drive from its start point to a trip's start point, execute this trip, and then drive from the trip's end point to the next trip's start point. After fulfilling its last trip, the car stays at the end point of the last trip. The sequence, in which the vehicle executes the trips, is called the duty of the vehicle.

A further restriction to the problem is that the vehicles have a maximal range that they can drive without refueling. In the model, refueling points are included.

The objective is to create a schedule for the vehicles, such that for each costumer exactly one route is fulfilled, i. e. every trip of this route is fulfilled. Further, each vehicle has a feasible fuel state all the time and visits a refuel station when necessary.

Costumers, Trips and Vehicles

We are given a set of car trips $\mathcal{T}$. Each trip $t \in \mathcal{T}$ has a start and end location $p_t^{\text{start}}, p_t^{\text{end}}$ and a start and end time $z_t^{\text{start}}, z_t^{\text{end}}$. For the sake of completeness, we define the set of public transport trips $\mathcal{T}_{\text{public}}$ with the same properties for $t \in \mathcal{T}_{\text{public}}$.

The set of vehicles is denoted by $\mathcal{V}$. The start position and the start time of a vehicle $v \in \mathcal{V}$ is p_v and z_v.

We have the set of costumers $\mathcal{C}$ and the set of multimodal routes $\mathcal{M}$. A route $m = (t_1, \ldots, t_{k_m})$ is a finite sequence of trips with the following properties:

$$p_{t_i}^{\text{end}} = p_{t_{i+1}}^{\text{start}} \qquad z_{t_i}^{\text{end}} \leq z_{t_{i+1}}^{\text{start}} \qquad \text{for all } i \in [k-1].$$

We define the route start and end locations and times for $m \in \mathcal{M}$ as

$$p_m^{\text{start}} := p_{t_1}^{\text{start}} \qquad p_m^{\text{end}} := p_{t_k}^{\text{end}} \qquad z_m^{\text{start}} := z_{t_1}^{\text{start}} \qquad z_m^{\text{end}} := z_{t_k}^{\text{end}}.$$

The mapping $M : \mathcal{T} \to \mathcal{M}$ indicates to which route a trip belongs. Each costumer $c \in \mathcal{C}$ has a finite set of alternative routes. The mapping $C : \mathcal{M} \to \mathcal{C}$ shows which route belongs to which costumer.
We use the notation

$$\begin{aligned}
C^{-1}(c) &:= C^{-1}(\{c\}) = \{m \in \mathcal{M} \mid C(m) = c\} && \text{for } c \in \mathcal{C}\\
M^{-1}(m) &:= M^{-1}(\{m\}) = \{t \in \mathcal{T} \mid M(t) = m\} && \text{for } m \in \mathcal{M}\\
(M \circ C)^{-1}(c) &:= M^{-1}\left(C^{-1}(c)\right) = \{t \in \mathcal{T} \mid C(M(t)) = c\} && \text{for } c \in \mathcal{C}
\end{aligned}$$

for all routes of a costumer, all trips of a route and all trips of a costumer, respectively. For each route of the same costumer $m \in C^{-1}(c)$, the start and end positions are the same, the start and end times may differ. We define the costumer start and end times for $c \in \mathcal{C}$

$$z_c^{\text{start}} := \min_{m \in C^{-1}(c)} z_m^{\text{start}} \qquad\qquad z_c^{\text{end}} := \max_{m \in C^{-1}(c)} z_m^{\text{end}}.$$

Fuel and Refueling

We have to consider fuel restrictions. Fuel can be any form of energy the considered vehicle is powered with. For each vehicle, the fuel level is in the interval $[0, 1]$, where 1 means full capacity and 0 is empty. We call a trip without a costumer, i. e. a trip between two trips, a deadhead trip. A car may visit a refuel station only during a deadhead trip. For simplicity of the model, each car is allowed to refuel at most once between two trips. On a refuel station, there are no capacity constraints, i. e. two or more vehicles may refuel at the same time on the same station.
For refueling, we have a set of refuel stations $\mathcal{R}$. A refuel station $r \in \mathcal{R}$ has a location p_r. We define $f_{s,t}^{\text{d}}$ for $s \in \mathcal{V} \cup \mathcal{T} \cup \mathcal{R}$, $t \in \mathcal{T} \cup \mathcal{R}$ as the amount, the fuel level decreases along a deadhead trip. f_t^{t} for $t \in \mathcal{T} \cup \mathcal{R}$ is the amount of fuel, a vehicle needs for a trip. For $r \in \mathcal{R}$ holds $f_r^{\text{t}} \leq 0$. f_v^0 for $v \in \mathcal{V}$ is the initial fuel state of a car.

Ordering of the Trips

We define the time, a vehicle needs to get from position p_1 to p_2 as t_{p_1,p_2}. We define

$$t_{s,t} = \begin{cases} t_{p_s^{\text{end}}, p_t^{\text{start}}} & \text{if } s, t \in \mathcal{T} \\ t_{p_s, p_t^{\text{start}}} & \text{if } s \in \mathcal{V} \cup \mathcal{R}, t \in \mathcal{T} \\ t_{p_s^{\text{end}}, p_t} & \text{if } s \in \mathcal{T}, t \in \mathcal{R} \\ t_{p_s, p_t} & \text{if } s \in \mathcal{V}, t \in \mathcal{R} \end{cases}$$

as the time a vehicle needs from one trip to another.
In order to decide whether a vehicle is able to fulfill two trips in a row, we define a partial ordering on the set of vehicles and trips. The set of public transport trips is left out in this definition.

Definition 1 (Order of trips)**.** The binary relation $\prec$ on $\mathcal{V} \cup \mathcal{T}$ is defined as follows:

$$
\begin{aligned}
s \prec t \quad &:\Leftrightarrow \quad \left(z_s^{\text{end}} + t_{s,t} \leq z_t^{\text{start}}\right) \wedge ((M \circ C)(s) \neq (M \circ C)(t) \vee M(s) = M(t)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } s \in \mathcal{V} \cup \mathcal{T}, t \in \mathcal{T} \\
s \nprec t \quad &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } s \in \mathcal{V} \cup \mathcal{T}, t \in \mathcal{V}
\end{aligned}
$$

The binary relation $\preceq$ on $\mathcal{V} \cup \mathcal{T}$ is defined as:

$$
s \preceq t \qquad :\Leftrightarrow \qquad s = t \wedge s \prec t \qquad \text{for all } s, t \in \mathcal{V} \cup \mathcal{T}
$$

The expression $s \prec t$ means, that one car is able to fulfill both trips, first s and then t. A car must not cover two trips of the same costumer, except they belong to the same route. This results from the problem setting, that for each costumer exactly one route is fulfilled.

Remark 1. Note, that $\preceq$ is not a partial order on $\mathcal{V} \cup \mathcal{T}$ since the transitivity is missing. Let $t_1, t_2, t_3 \in \mathcal{T}$ with

$$
z_{t_1}^{\text{end}} + t_{t_1,t_2} \leq z_{t_2}^{\text{start}} \qquad\qquad z_{t_2}^{\text{end}} + t_{t_2,t_3} \leq z_{t_3}^{\text{start}}
$$

$$
(M \circ C)(t_1) = (M \circ C)(t_3) \qquad (M \circ C)(t_1) \neq (M \circ C)(t_2) \qquad M(t_1) \neq M(t_3)
$$

Then,

$$
t_1 \preceq t_2 \preceq t_3 \qquad\qquad t_1 \npreceq t_3
$$

Problem Description

We define the considered problem as follows: Find a schedule of trips for every vehicle including refueling stops and a sequence of trips for every costumer. Therefore, the car trips are fulfilled by the scheduled car and the public transport trips by public transport according to its timetable. For this, we have the following conditions:

- Each car is able to serve its scheduled trips, considering time and location.
- The fuel state of each car is always in a feasible range.
- Each costumer is able to complete his route, considering time and location.
- For each costumer, exactly one route is chosen.

The goal is to find a cost-minimal feasible schedule considering all these constraints.

Costs

We introduce the following types of costs:

- Vehicles costs c^{v}: unit costs for each used vehicle
- Deadhead costs $c^{\mathrm{d}}_{s,t}$ for $s \in \mathcal{V} \cup \mathcal{T} \cup \mathcal{R}, t \in \mathcal{T} \cup \mathcal{R}$: costs, if a vehicle drives to a trip or a refuel station without a costumer using it
- Trip costs c^{t}_t for $t \in \mathcal{T}$: costs for fulfilling a trip

For public transport, we define either trip costs for each public transport trip or fixed costs for each costumer using public transport. Finally, we define costs to consider the costumer preferences.

- Trip costs c^{t}_t for $t \in \mathcal{T}_{\text{public}}$: costs for using public transport
- Route-dependent costs $\bar{c}^{\mathrm{r}}_m$ for $m \in \mathcal{M}$: costs for costumer preferences and unit costs for using public transport

Since the trip costs for public transport are connected with the choice of the route, we easily add these costs to the trip costs.

$$c^{\mathrm{r}}_m := \bar{c}^{\mathrm{r}}_m + \sum_{t \in m \cap \mathcal{T}_{\text{public}}} c^{\mathrm{t}}_t \qquad \text{for } m \in \mathcal{M}$$

The route costs additionally include costumer preferences. Each costumer has a set of alternative routes. He does not choose the route by himself, but this is decided by the problem whatever fits best into the system. The user preference costs work as penalty terms for an inconvenient route choice. This means, a route that is disadvantageous for the costumer is penalized. Then, either the less favorable route is chosen (if it fits better into the system) and this is penalized. Or the more favorable route is chosen although it is not so good for the system. With this, a realistic costumer behavior is modeled.
The costumer preferences are e. g. the total travel time, the number of changes or the costs for the costumer. Typically, a pure car trip is faster but more expensive. Further, a late departure time or an early arrival time can be criteria for this cost function.

Additional Assumptions

In the following, we summarize all the assumptions we made on the input data.
All costs are non-negative.

$$c^{\mathrm{v}} \geq 0 \qquad c^{\mathrm{d}}_{s,t} \geq 0 \qquad c^{\mathrm{t}}_t \geq 0 \qquad \bar{c}^{\mathrm{r}}_m \geq 0 \qquad \text{for all } s,t \in \mathcal{T}, m \in \mathcal{M}. \tag{2.1}$$

The fuel consumption is non-negative, except for refueling.

$$f_t^{\mathrm{t}} \geq 0 \qquad f_r^{\mathrm{t}} \leq 0 \qquad \text{for all } t \in \mathcal{T}, r \in \mathcal{R} \tag{2.2}$$

$$f_{s_1,s_2}^{\mathrm{d}} \geq 0 \qquad \text{for all } s_1 \in \mathcal{V} \cup \mathcal{T} \cup \mathcal{R}, s_2 \in \mathcal{T} \cup \mathcal{R} \tag{2.3}$$

There are no zero-time rentals.

$$z_t^{\mathrm{start}} < z_t^{\mathrm{end}} \qquad \text{for all } t \in \mathcal{T} \tag{2.4}$$

We assume the Triangle Inequalities for $s \in \mathcal{V} \cup \mathcal{T} \cup \mathcal{R}$ and $r, t \in \mathcal{T} \cup \mathcal{R}$:

$$t_{s,t} \leq t_{s,r} + t_{r,t} \qquad c_{s,t}^{\mathrm{d}} \leq c_{s,r}^{\mathrm{d}} + c_{r,t}^{\mathrm{d}} \qquad f_{s,t}^{\mathrm{d}} \leq f_{s,r}^{\mathrm{d}} + f_{r,t}^{\mathrm{d}} \tag{2.5}$$

From (2.1) and (2.5) we get:

$$c_{s,t}^{\mathrm{d}} \leq c_{s,r}^{\mathrm{d}} + c_r^{\mathrm{t}} + c_{r,t}^{\mathrm{d}} \qquad \text{for all } s, r, t \in \mathcal{T} \tag{2.6}$$

2.3 Classification

We want to classify our problem in relation to other known problems in literature and state the difficulty of these problems.

Vehicle Scheduling Problems

According to the structure of the problem stated in Section 2.2, we regard the field of vehicles scheduling problems (VSP). [BK09] define the (VSP) as follows: "Given a set of timetabled trips with fixed travel (departure and arrival) times and start and end locations as well as traveling times between all pairs of end stations, the objective is to find an assignment of trips to vehicles such that each trip is covered exactly once, each vehicle performs a feasible sequence of trips and the overall costs are minimized." The complexity of some different variants of the (VSP) is regarded by [LK81].
A similar problem formulation is the dial-a-ride problem (DARP). [CL07] discuss the differences of the (DARP) to other vehicle routing problems and write: "What makes the DARP different from most such routing problems is the human perspective. When transporting passengers, reducing user inconvenience must be balanced against minimizing operating costs." The basic formulations of (VSP) and (DARP) are the same, therefore we use the formulation (VSP).

Depot Variants

In [BK09], there are two main variants for the (VSP) with respect to the fact where vehicles start and return. In the single depot case (SD-VSP), there is one depot from where all vehicles start. After usage, all vehicles return to this depot. The multiple depot case (MD-VSP) means that there is more than one depot and from each depot starts a certain number of vehicles. After usage, each vehicle returns to the depot from where it has started.

In order to make our problem more realistic, we have more than one depot. We have more than one vehicle and each vehicle starts from its start position. The vehicles do not have a certain point, where there have to return after usage, i. e. they can stay, wherever the last trip of their duty ends. [DP95] claim that "if the vehicle[s] are allowed to return to a depot different from its origin depot, [...] the problem can be solved as a single depot instance." We see, that our problem is in the single depot case (SD-VSP) concerning the depot variant.

In [DF54] is proven that (SD-VSP) can be solved in polynomial time, i. e. (SD-VSP) is in $\mathcal{P}$. In contrast, the multidepot case (MD-VSP) is $\mathcal{NP}$-hard as shown in [BCG87].

Additional Constraints

In the basic (SD-VSP), all existing trips have to be fulfilled. For more generality, there were additional cover constraints introduced in the underlying master theses. There are "costumers with sets of alternative trips out of which exactly one trip shall be fulfilled, respectively." ([KK16, p.10]) In our problem, even more general cover constraints are required. We have costumers with sets of alternative routes, consisting of trips; for each costumer, exactly one route has to be fulfilled, i. e. each of its trips is fulfilled. We call these constraints "multi-leg" cover constraints and write the problem (VSP-MC). This is a generalization to the previous cover constraints, as can be seen easily by rewriting them: There are costumers with sets of alternative routes, where each route consists of exactly one trip. According to this reformulation, we call the primary constraints "single-leg" cover constraints and write the problem (VSP-SC). We see in Section 2.4 that (VSP-SC) is already $\mathcal{NP}$-hard.

We further have to respect the fuel constraints. In literature, there are named general resources like time, mileage or fuel, summarized in the general term "route constraints" (cf. [BK09, p. 16], [Raf83]). The respective problems with route constraints are called (SD-VSP-RC) and (MD-VSP-RC). [FP95] describe the (VSP) with time constraints, [Raf83] presents the (VSP) with path constraints, what is a more general formulation. In these models, a vehicle returns to the depot after the respective resource is exhausted, while the vehicle has the possibility to refuel in our model. The problem with not refilling the resource is a special case of the problem with the possibility to refill the resource. We see in Section 2.4 that (SD-VSP-RC) is already $\mathcal{NP}$-hard.

In summary, we have a problem with two types of constraints where each individually makes the problem $\mathcal{NP}$-hard. The first one is the multi-leg cover constraint and the second one is the fuel constraint. The only difference to [KK16] is that the single-leg cover constraint is replaced by the multi-leg cover constraint. Their solution methods are extended to the requirements in this thesis. To the knowledge of the author, a combination of these constraints, as well as these constraints on their own, are not treated in literature so far.

2.4 Complexity

We regard the complexity of our problem. As we have seen in Section 2.3, the problem can be modeled as a single depot vehicle scheduling problem (SD-VSP) with resource constraints, the possibility of refueling and multi-leg cover constraints. The (SD-VSP) itself can be solved in polynomial time, which is proven by [DF54]. It can be formulated as minimum-cost flow problem. If we extend the basic formulation with one of the additional constraints, it gets $\mathcal{NP}$-hard.

Theorem 1 (VSP with resource constraints)**.** *The vehicle scheduling problem with resource constraints and the special objective of minimizing the number of used vehicles is $\mathcal{NP}$-hard.*

This theorem is proven in [KK16, p. 11] by a polynomial reduction of the bin packing problem. This theorem holds for the case of resource constraints without the possibility of refilling the resources. Since this is a special case of resource constraints with refueling, this problem is also $\mathcal{NP}$-hard.

Theorem 2 (VSP with cover constraints)**.** *The vehicle scheduling problem with cover constraints and the special objective of minimizing the number of used vehicles is $\mathcal{NP}$-hard.*

This theorem is proven in [KK16, p. 12] by a polynomial reduction of the set cover problem. This theorem treats the case of single-leg cover constraints.

Theorem 3 (VSP with multi-leg cover constraints)**.** *The vehicle scheduling problem with multi-leg cover constraints and the special objective of minimizing the number of used vehicles is $\mathcal{NP}$-hard.*

Proof. We prove this statement by a polynomial reduction of the (VSP-SC). Consider a (VSP-SC) with vehicles $\hat{\mathcal{V}}$, costumers $\hat{\mathcal{C}}$, trips $\hat{\mathcal{T}}$ and the function $\hat{C} : \hat{\mathcal{T}} \to \hat{\mathcal{C}}$ that maps trips to their respective costumers. The problem is defined in detail in [KK16, pp. 5-8]. We create the corresponding (VSP-MC) as follows: $\mathcal{V} := \hat{\mathcal{V}}$, $\mathcal{C} := \hat{\mathcal{C}}$, $\mathcal{T} := \hat{\mathcal{T}}$ stay the same. We define the multimodal routes as one-element sequence for each trip

$$\mathcal{M} := \left\{ (t) \mid t \in \hat{\mathcal{T}} \right\}$$

and the mappings

$$M : \mathcal{T} \to \mathcal{M},\ M(t) = (t), \qquad C : \mathcal{M} \to \mathcal{C},\ C(m) = \hat{C}(t) \qquad \text{for } t \in m \text{ unique.}$$

The solution of (VSP-SC) can easily be mapped to a solution of the corresponding (VSP-MC) and vice versa. If a trip is chosen in (VSP-SC), the respective route is chosen in (VSP-MC). Then, every trip of this route is fulfilled and the costumer is satisfied. The other way round, if a route is chosen in (VSP-MC), the trip contained in this route is chosen in (VSP-SC) and the costumer is satisfied. The feasibility of the vehicle duties is not affected by this procedure.
This is a polynomial reduction, (VSP-SC) is is $\mathcal{NP}$-hard by Theorem 2 and hence (VSP-MC) is $\mathcal{NP}$-hard.

□

With the Theorems 1 and 3 we can see, that our problem gets $\mathcal{NP}$-hard only with cover constraints or with resource constraints and considering the number of vehicles. The problem gets even harder, as we do not only consider the number of vehicles but include the operational cost and penalty terms for costumer preferences. Further, we want to have the possibility to refuel during the process, i. e. we have negative resource cost. Finally, we want to apply both of these constraints simultaneously.
We will model our problem as a mixed-integer linear program in a size polynomial in the input size. Therefore, it is possible to verify a solution in polynomial time. This means, our problem is $\mathcal{NP}$-complete.

Chapter 3

Mathematical Models

We introduce the mathematical model, with which we want to solve the previously described problem. First we define the underlying task graph and afterwards we develop an arc flow formulation on this graph. The main idea is that we model a flow of the vehicles to the trips with additional requirements in order to fulfill the cover constraints and the fuel constraints. As mentioned before, we totally neglect public transport trips. They have only an indirect effect on the model by contribution to the trip creation and the route cost.

3.1 Task Graph

We introduce the task graph, on which the model is based. This is a directed graph corresponding to the relation $\prec$ which we defined in Section 2.2. The graph is basically the same as used in [KK16] with the only difference that the costumer and route considerations are adapted here.

Definition 2 (Task Graph)**.** Let $d^{\mathrm{s}}, d^{\mathrm{e}}$ be special vertices describing the source and sink of the vehicle flow. We define the task graph as $\hat{G} = (\hat{V}, \hat{A})$, where

$$\hat{V} := \{d^{\mathrm{s}}, d^{\mathrm{e}}\} \cup \mathcal{V} \cup \mathcal{T}$$

is the vertex set consisting of the source, the sink, the vehicle set $\mathcal{V}$ and the trip set $\mathcal{T}$. The arc set is

$$\hat{A} := (\{d^{\mathrm{s}}\} \times \mathcal{V}) \cup \left\{(s,t) \in (\mathcal{V} \cup \mathcal{T})^2 \,|\, s \prec t\right\} \cup ((\mathcal{V} \cup \mathcal{T}) \times \{d^{\mathrm{e}}\}) \,.$$

A vertex $s \in \mathcal{V}$ represents the initial state of a vehicle s where it becomes available for the first time. Each d^{s}-d^{e}-path in $\hat{G}$ is the duty of one vehicle, i.e. this vehicle fulfills the trips in the order given by the path. Hence, two trips are connected only if it is possible that one car fulfills both trips, i.e. the relation $\prec$ holds.

Lemma 1. *$\hat{G}$ is a directed acyclic graph.*

Proof. Assume there is a cycle in $\hat{G}$. The source d^{s} and sink d^{e} have only ingoing, respectively outgoing arcs and are therefore not part of the cycle. For $v \in \mathcal{V}$, all ingoing arcs come from d^{s}, hence $v \in \mathcal{V}$ are not part of the cycle, too. This means, a cycle consists only of trips.
Consider an arbitrary cycle of trips $t_1, \dots, t_k \in \mathcal{T}$, $k \geq 2$. These trips form a cycle, i. e. $t_1 \prec \dots \prec t_k$ and $t_k \prec t_1$. With Definition 1 and the assumptions (2.4) and (2.5) holds:

$$z_i^{\text{start}} < z_i^{\text{end}} \leq z_i^{\text{end}} + t_{t_i,t_{i+1}} \leq z_{i+1}^{\text{start}} < z_{i+1}^{\text{end}} \qquad \text{for all } i \in [k-1]$$

$$\Rightarrow \qquad z_1^{\text{start}} < z_k^{\text{end}} \qquad \Rightarrow \qquad z_k^{\text{end}} + t_{t_1,t_k} > z_1^{\text{start}} \qquad \Rightarrow \qquad t_k \not\prec t_1$$

This is a contradiction to the existence of a cycle.

□

In order to consider refueling and refuel stations, we introduce an extended task graph.

Definition 3 (Extended Task Graph)**.** For every $s, t \in \mathcal{V} \cup \mathcal{T}$ with $s \prec t$ we create a copy of $\left\{r \in \mathcal{R} | z_s^{\text{end}} + t_{s,r} + t_{r,t} \leq z_t^{\text{start}}\right\}$ denoted by $\mathcal{R}_{s,t}$. This means, various copied sets are pairwise disjoint. The expression $r \in \mathcal{R}_{s,t}$ means that a vehicle is able to finish trip s, then drive to refuel station r and then start trip t in time.
We define the extended task graph $G = (V, A)$ with vertex set

$$V := \hat{V} \mathbin{\dot\cup} \bigcup_{\substack{s,t \in \mathcal{V} \mathbin{\dot\cup} \mathcal{T} \\ s \prec t}} \mathcal{R}_{s,t}$$

and arc set

$$A := \hat{A} \mathbin{\dot\cup} \{(s,r) | s,t \in \mathcal{V} \mathbin{\dot\cup} \mathcal{T}, s \prec t, r \in \mathcal{R}_{s,t}\} \mathbin{\dot\cup} \{(r,t) | s,t \in \mathcal{V} \mathbin{\dot\cup} \mathcal{T}, s \prec t, r \in \mathcal{R}_{s,t}\}.$$

It is possible that there is a copy of each refuel station for each feasible pair of trips. This leads to an enormous size of the task graph and thus a bad solution behavior is expected. [KK16, pp. 24-30] describe a method to reduce the size of $\mathcal{R}_{s,t}$ without cutting the optimal solution. This method only considers Pareto optimal refuel station w. r. t. a suitable function. From now on, we will use $G = (V, A)$ with restricted $\mathcal{R}_{s,t}$.

Lemma 2. *$\hat{G}$ is a directed acyclic graph.*

Proof. Assume there is a cycle in G. In comparison to $\hat{G}$, only arcs (s, r) and (r, t) for $r \in \mathcal{R}_{s,t}, s \prec t$ were added. Assume there is a cycle containing $r \in \mathcal{R}_{s,t}$. r has only one ingoing arc (s, r) and one outgoing arc (r, t) and only if the arc (s, t) exists. There is no cycle on the vertices $\{s, r, t\}$. Every other cycle containing r is also a cycle using the arc (s, t). This is a contradiction to the fact that $\hat{G}$ is cycle-free as proven in Lemma 1.

□

In the extended task graph G, a d^{s}-d^{e}-path further represents the duty of a vehicle. The additional arcs $(s, r), (r, t)$ for $r \in \mathcal{R}_{s,t}$ describe a possible detour between the trips s and t in order to refuel at refuel station r.
We introduce the following frequently used notation:

$$\mathrm{N}_G^-(t) := \{s \in V \mid (s,t) \in A\} \qquad \mathrm{N}_G^+(t) := \{t \in V \mid (s,t) \in A\}$$

$\mathrm{N}_G^-(t)$ is the set of in-neighbors of $t \in V$, $\mathrm{N}_G^+(s)$ is the set of out-neighbors of $s \in V$.

3.2 Arc Flow Formulation

In the following, we model the problem via a flow of the vehicles on the extended task graph. The trips and multimodal routes are given in advance. The fact, whether two trips can be fulfilled subsequently in one duty, is already given by the underlying task graph. We additionally have to model the cover constraints and the fuel constraints. Since the duties of various vehicles are disjoint w. r. t. $\mathcal{T}$, we are able to use one common set of variables for the flow of one vehicle. From the flow, we can easily extract the individual d^{s}-d^{e}-paths in order to identify the duties of the respective vehicles.

Basic Model

We model the arc flow as a mixed-integer linear program. The formulation is basically built on the (MILP) formulation as described in [KK16, p. 34]. We use the following decision variables:

- $x_{s,t} \in \{0, 1\}$ for $(s,t) \in A$: indicates, whether trip $t \in \mathcal{T}$ is fulfilled after $s \in \mathcal{V} \cup \mathcal{T}$
- $z_{s,r,t} \in \{0, 1\}$ for $t \in \mathcal{T}, s \in \mathrm{N}_G^-(t), r \in \mathcal{R}_{s,t}$: indicates, whether refuel station $r \in \mathcal{R}$ is visited between s and t
- $e_s \in [0, 1]$ for $s \in V \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\}$: states the fuel of the respective vehicle after fulfilling trip $s \in \mathcal{T}$

If $s \in \mathcal{V}$, then $x_{s,t}$ determines whether trip t is the first trip fulfilled by s and e_s is the initial fuel state f_s^0 of vehicle s.
Additionally to (MILP), we introduce decision variables in order to ensure the cover constraints:

- $u_m \in \{0, 1\}$ for $m \in \mathcal{M}$: indicates whether multimodal route m is fulfilled

The basic constraints are developed in [KK16, pp. 21-34] and not explained in detail here. The basic constraints are the flow conservation constraint, the constraint for considering every car and the constraints ensuring feasible fuel states all the time.

Costumer and Route Constraints

In (MILP), each costumer has a set of alternative trips and from this set, exactly one trip has to be fulfilled. This is modeled as follows:

$$\sum_{t \in C^{-1}(c)} \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = 1 \qquad \text{for all } c \in \mathcal{C} \tag{3}$$

In contrast to (MILP), here each costumer has a set of alternative routes consisting of trips and from this set, exactly one route has to be fulfilled. Therefore, we replace (3) by the following formulation:

$$\sum_{m \in C^{-1}(c)} u_m = 1 \qquad \text{for all } c \in \mathcal{C} \tag{3.1}$$

$$\sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = u_m \qquad \text{for all } m \in \mathcal{M}, t \in m \tag{3.2}$$

The constraint (3.1) says, that for every costumer exactly one route is fulfilled. The constraint (3.2) says, if a route is fulfilled then every trip of this route must be fulfilled.

Objective Function

The objective function in (MILP) is given by

$$\sum_{s \in \mathcal{V}} \sum_{t \in \mathrm{N}_G^+(s) \setminus \{d^{\mathrm{e}}\}} x_{s,t} c^{\mathrm{v}} + \sum_{t \in \mathcal{T}} \sum_{s \in \mathrm{N}_G^-(t)} \left[x_{s,t} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_t \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t} \right) \right]$$

considering the vehicle costs c^{v}, the trip costs c^{t}_t for $t \in \mathcal{T}$ and the deadhead costs c^{d}. What is missing, are the route-dependent costs c^{r}. Thus, we add the term

$$\sum_{m \in \mathcal{M}} u_m c^{\mathrm{r}}_m$$

to the objective function.

LP Formulation

Putting all this together, we get the following formulation, called (MMILP):

$$
\begin{aligned}
\min \quad & \sum_{s\in\mathcal{V}} \sum_{t\in \mathrm{N}^+_G(s)\setminus\{d^{\mathrm{e}}\}} x_{s,t}c^{\mathrm{v}} + \sum_{m\in\mathcal{M}} u_m c^{\mathrm{r}}_m \\
& + \sum_{t\in\mathcal{T}} \sum_{s\in \mathrm{N}^-_G(t)} \left[x_{s,t}\left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_t\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t}\right)\right] && \text{(MMILP)} \\
\text{s.t.} \quad & \sum_{s\in \mathrm{N}^-_G(t)} x_{s,t} = \sum_{s\in \mathrm{N}^+_G(t)} x_{t,s} && \text{for all } t\in V\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} \quad (3.3) \\
& \sum_{s\in \mathrm{N}^-_G(t)} x_{s,t} = 1 && \text{for all } t\in\mathcal{V} \quad (3.4) \\
& \sum_{m\in C^{-1}(c)} u_m = 1 && \text{for all } c\in\mathcal{C} \quad (3.1) \\
& \sum_{s\in \mathrm{N}^-_G(t)} x_{s,t} = u_m && \text{for all } m\in\mathcal{M}, t\in m \quad (3.2) \\
& \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t) \quad (3.5) \\
& e_s \le f^0_s && \text{for all } s\in\mathcal{V} \quad (3.6) \\
& 0 \le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t) \quad (3.7) \\
& e_t \le 1 - f^{\mathrm{t}}_t - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t) \quad (3.8) \\
& e_t \le e_s - x_{s,t}\left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t}\right) + (1 - x_{s,t}) \\
& && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t) \quad (3.9) \\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t)\in A \quad (3.10) \\
& z_{s,r,t} \in \{0,1\} && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t), r\in\mathcal{R}_{s,t} \quad (3.11) \\
& e_s \in [0,1] && \text{for all } s\in V\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} \quad (3.12) \\
& u_m \in \{0,1\} && \text{for all } m\in\mathcal{M} \quad (3.13)
\end{aligned}
$$

Chapter 4

Successive Heuristics

In this chapter, successive heuristics are introduced in order to solve our problem. As seen in Section 2.4, the problem is $\mathcal{NP}$-hard even if we apply one of the restrictions, the cover constraints or the fuel constraints, individually. Our goal is to develop a heuristic that can cope with both multi-leg cover constraints and fuel constraints. We build our heuristic on a heuristic for a simpler version of the problem, developed in the underlying theses. [Kno16] present heuristical solution methods for the problem only with fuel constraints. The problem settings assumes that there is a set of trips where each of these trips shall be fulfilled. They already claim, that solving a complete instance of 24 hours to optimality is not possible with their respective computing capacity. Therefore it is a plausible assumption that an optimal solution for our problem cannot be expected in reasonable time.

Their solution methods are based on the idea of splitting the complete instance according to several time intervals. For each interval, only the trips starting in the respective interval are considered. From this formulation emerge several separate partial instances that are still loosely connected to each other. Each of these partial instances is solved separately and then the partial solutions are connected to a complete feasible solution. Two different approaches are presented in order to solve the problem: The constraints connecting the partial instances are relaxed by using Lagrange Relaxation. With suitable computation of Lagrange multipliers, the partial instances are solved in parallel. In the other one, the partial instances are solved successively, where the respective connecting constraints are fixed beginning from the end.

An adaption of the cover constraints to the heuristic using Lagrange Relaxation seems not practicable. This heuristics heavily exploits the loosely connection of the partial instances. The cover constraints strongly influence the complete instance by selecting the fulfilled trips, the multi-leg cover constraints even require an additional set of variables, belonging to none of the partial instances. Therefore, an additional relaxing of these cover constraints is not a promising approach. Instead, we focus on the second approach of Successive Heuristics.

The crucial difficulty for this procedure is to ensure the costumer satisfaction. In particular, if trips of a costumer are wide apart in terms of time, these trips will lie in different splittings. This makes it hard to keep control over the trip selection in

separately solved partial instances.
We first define the splitting of the instance and the arising adaptions of task graph and model. Then, we describe the heuristic in general. Finally, we introduce different splitting methods, one according to the costumers and one according to time.

4.1 Splitting the Problem

4.1.1 Splitting Trip and Vehicle Set

In order to create the partial instances, we define splittings of $\mathcal{V}$ and $\mathcal{T}$. In contrast to [Kno16], we define the splittings in a general way.

Definition 4 (Splitting)**.** Let $n \in \mathbb{N}$ and let

$$\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i \qquad\qquad \mathcal{V} = \bigcup_{i=1}^{n} \mathcal{V}_i$$

be partitions of the set of trips, respectively vehicles. Then we call $\{\mathcal{T}_i \mid i \in [n]\}$ and $\{\mathcal{V}_i \mid i \in [n]\}$ splitting of $\mathcal{T}$ and $\mathcal{V}$ and $\mathcal{T}_i$ and $\mathcal{V}_i$ partial trip respectively vehicle set.

Adaption of the Task Graph

We transform our task graph such that it contains the splittings as defined in Definition 4. For this, we introduce so called split points connecting the partial sets. Arcs that connected two partial sets before, take a detour over the respective split point in the transformed graph.

Definition 5 (Transformed Task Graph)**.** Let $\{\mathcal{T}_1, \dots \mathcal{T}_n\}$ be a splitting of $\mathcal{T}$ according to Definition 4. Then we define:

1. Split Point: Let $s \in \mathcal{T}_i$ for $i \in [n]\backslash\{1\}$. For $j \in [i-1]$, we define the split point $\mathrm{SP}_j(s)$ with $p^{\text{start}}_{\mathrm{SP}_j(s)} = p^{\text{end}}_{\mathrm{SP}_j(s)} =: p^{\text{start}}_s$, $z^{\text{start}}_{\mathrm{SP}_j(s)} = z^{\text{end}}_{\mathrm{SP}_j(s)} =: z^{\text{start}}_s$ and $f^{\text{t}}_{\mathrm{SP}_j(s)} =: 0$.

2. For $i \in [n]\backslash\{1\}$ and $j \in [i-1]$, we define $\mathcal{P}_{j,i} := \{\mathrm{SP}_j(s) \mid s \in \mathcal{T}_i\}$.

3. Partial Split Point Set: For $j \in [n-1]$, we define the partial split point set $\mathcal{P}_j := \bigcup_{i=j+1}^{n} \mathcal{P}_{j,i}$.

4. Split Point Set: We define the split point set $\mathcal{P} := \bigcup_{j=1}^{n-1} \mathcal{P}_j$.

Let $G = (V, A)$ be the task graph, $\{\mathcal{V}_1, \dots, \mathcal{V}_n\}$ a splitting of $\mathcal{V}$.

5. Transformed Task Graph: We define the transformed task graph $\overline{G} = \left(\overline{V}, \overline{A}\right)$ with vertex set

$$\overline{V} := V \cup \mathcal{P} = V \cup \{\mathrm{SP}_i(s) \mid i \in [n-1], j \in [n+1] \backslash [i], s \in \mathcal{T}_j\}$$

and arc set

$$\begin{aligned}\overline{A} := &(d^{\mathrm{s}} \times \mathcal{V}) \cup \bigcup_{i=1}^{n} \{(s,t) \in (\mathcal{V}_i \cup \mathcal{T}_i) \times (\mathcal{T}_i \cup \mathcal{P}_i) \mid s \prec t\} \\ &\cup \bigcup_{i=1}^{n} \left\{(s,t) \in \left(\left(\bigcup_{j=1}^{i-1} \mathcal{P}_{j,i}\right) \times \mathcal{T}_i\right) \mid s = \mathrm{SP}_i(t)\right\} \cup ((\mathcal{V} \cup \mathcal{T}) \times \{d^{\mathrm{e}}\})\end{aligned}$$

Adaption of the Model

In order to adapt (MMILP) to the transformed task graph, we make the following considerations:
For all split points we define the costs and fuel states as

$$c_s^{\mathrm{t}} := 0 \qquad c_{s,t}^{\mathrm{d}} := 0 \qquad f_s^{\mathrm{t}} := 0 \qquad f_{s,t}^{\mathrm{d}} := 0 \qquad \text{for } s \in \mathcal{P}, t \in \mathrm{N}_{\overline{G}}^{+}(s)$$

since $p_s^{\mathrm{end}} = p_t^{\mathrm{start}}$ and $z_s^{\mathrm{end}} = z_t^{\mathrm{start}}$. Furthermore, refueling is not possible between s and t.
In the transformed task graph, the arcs between two trips of different splittings are replaced by the detour over the splitting point. Therefore, the trip costs of trip directly after a split point are not considered in the objective function any more. In order to compensate this, we add the following term to the objective function:

$$\sum_{s \in \mathcal{P}} \sum_{t \in \mathrm{N}_{\overline{G}}^{+}(s)} x_{s,t} c_t^{\mathrm{t}}$$

We want to ensure the flow conservation also in the new nodes $\mathcal{P}$, thus we add the inequality

$$\sum_{t \in \mathrm{N}_{\overline{G}}^{-}(s)} x_{t,s} = \sum_{t \in \mathrm{N}_{\overline{G}}^{+}(s)} x_{s,t} \qquad \text{for all } s \in \mathcal{P} \tag{4.1}$$

The equations (3.3) and (4.1) are contracted to

$$\sum_{t \in \mathrm{N}_{\overline{G}}^{-}(s)} x_{t,s} = \sum_{t \in \mathrm{N}_{\overline{G}}^{+}(s)} x_{s,t} \qquad \text{for all } s \in \overline{V} \backslash \{d^{\mathrm{s}}, d^{\mathrm{e}}\} \tag{4.2}$$

$$
\begin{aligned}
\min \quad & \sum_{s\in\mathcal{V}} \sum_{t\in \mathrm{N}^+_{\overline{G}}(s)\setminus\{d^{\mathrm{e}}\}} x_{s,t}c^{\mathrm{v}} + \sum_{s\in\mathcal{P}} \sum_{t\in \mathrm{N}^+_{\overline{G}}(s)} x_{s,t}c^{\mathrm{t}}_t + \sum_{m\in\mathcal{M}} u_m c^{\mathrm{r}}_m \\
& + \sum_{t\in\mathcal{T}\cup\mathcal{P}} \sum_{s\in \mathrm{N}^-_{\overline{G}}(t)\setminus\mathcal{P}} \left[x_{s,t}\left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_t\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t}\right)\right] && \text{(SMILP)}\\
\text{s.t.} \quad & \sum_{t\in \mathrm{N}^-_{\overline{G}}(s)} x_{t,s} = \sum_{t\in \mathrm{N}^+_{\overline{G}}(s)} x_{s,t} \qquad \text{for all } s\in\overline{V}\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} && (4.2)\\
& \sum_{s\in \mathrm{N}^-_{\overline{G}}(t)} x_{s,t} = 1 \qquad \text{for all } t\in\mathcal{V} && (3.4)\\
& \sum_{m\in C^{-1}(c)} u_m = 1 \qquad \text{for all } c\in\mathcal{C} && (3.1)\\
& \sum_{s\in \mathrm{N}^-_{\overline{G}}(t)} x_{s,t} = u_m \qquad \text{for all } m\in\mathcal{M}, t\in m && (3.2)\\
& \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} \qquad \text{for all } t\in\mathcal{T}\cup\mathcal{P}, s\in \mathrm{N}^-_{\overline{G}}(t)\setminus\mathcal{P} && (4.3)\\
& e_s \le f^0_s \qquad \text{for all } s\in\mathcal{V} && (3.6)\\
& 0 \le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}f^{\mathrm{d}}_{s,r} \qquad \text{for all } t\in\mathcal{T}\cup\mathcal{P}, s\in \mathrm{N}^-_{\overline{G}}(t)\setminus\mathcal{P} && (4.4)\\
& e_t \le 1 - f^{\mathrm{t}}_t - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}f^{\mathrm{d}}_{r,t} \qquad \text{for all } t\in\mathcal{T}\cup\mathcal{P}, s\in \mathrm{N}^-_{\overline{G}}(t)\setminus\mathcal{P} && (4.5)\\
& e_t \le e_s - x_{s,t}\left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t}\right) + (1 - x_{s,t})\\
& \qquad \text{for all } t\in\mathcal{T}\cup\mathcal{P}, s\in \mathrm{N}^-_{\overline{G}}(t)\setminus\mathcal{P} && (4.6)\\
& e_t \le e_s - x_{s,t}f^{\mathrm{t}}_t + (1 - x_{s,t}) \qquad \text{for all } s\in\mathcal{P}, t\in \mathrm{N}^+_{\overline{G}}(s) && (4.10)\\
& x_{s,t}\in\{0,1\} \qquad \text{for all } (s,t)\in\overline{A} && (4.7)\\
& z_{s,r,t}\in\{0,1\} \qquad \text{for all } t\in\mathcal{T}\cup\mathcal{P}, s\in \mathrm{N}^-_{\overline{G}}(t)\setminus\mathcal{P}, r\in\mathcal{R}_{s,t} && (4.8)\\
& e_s\in[0,1] \qquad \text{for all } s\in\overline{V}\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} && (4.9)\\
& u_m\in\{0,1\} \qquad \text{for all } m\in\mathcal{M} && (3.13)
\end{aligned}
$$

The fuel constraints are adapted in the following way: (3.5), (3.7), (3.8) and (3.9) hold also on the arcs leading to $\mathcal{P}$ and are therefore replaced by (4.3), (4.4), (4.5) and (4.6). Further the arcs leading from a split points to its respective trips have to be considered. Since refueling is not possible there, we have only to adapt (3.9). Since $f^{\mathrm{d}}_{s,t} = 0$ and refueling is not possible between s and t, the constraint reads as follows:

$$e_t \leq e_s - x_{s,t} f_t^{\mathrm{t}} + (1 - x_{s,t}) \qquad \text{for all } s \in \mathcal{P}, t \in \mathrm{N}^{+}_{\overline{G}}(s) \tag{4.10}$$

The costumer constraints (3.1) are not affected by transforming the graph. The decision whether a trip $t \in \mathcal{T}$ is fulfilled is still given by $\sum_{s \in \mathrm{N}^{-}_{\overline{G}}(t)} x_{s,t}$, no matter if the ingoing arc is a split point or not. Thus, the route constraints (3.2) do not change either.
Putting all together, we have the formulation (SMILP).

4.1.2 Identifying the Subproblems

Given a splitting of $\mathcal{T}$ and $\mathcal{V}$, we describe how the subproblems of (SMILP) are created. For each partial trip and vehicle set $\mathcal{T}_i$, $\mathcal{V}_i$, we solve a partial instance $\mathcal{I}_i$. We call the solution of a partial instance I_i partial solution S_i.

Partial Instances

First we define the task graph with which we can solve the partial instances. The transformed task graph $\overline{G}$ covers the complete instance, but contains the partial split sets from the splittings of $\mathcal{V}$ and $\mathcal{T}$. This graph only contains the respective partial trip and vehicle set. It additionally contains a start point set $\hat{\mathcal{V}}_i$ and an end point set $\hat{\mathcal{P}}_i$. How these sets are defined is explained later.

Definition 6 (Partial Transformed Task Graph)**.** Let $i \in [n]$. For a set of start points $\hat{\mathcal{V}}_i$, a set of end points $\hat{\mathcal{P}}_i$ and the partial trip set $\mathcal{T}_i$, the partial transformed task graph is the directed graph $\overline{G}_i = \left(\overline{V}_i, \overline{A}_i\right)$ with vertex set

$$\overline{V}_i := \{d^{\mathrm{s}}, d^{\mathrm{e}}\} \mathbin{\dot\cup} \hat{\mathcal{V}}_i \mathbin{\dot\cup} \mathcal{T}_i \mathbin{\dot\cup} \hat{\mathcal{P}}_i$$

and arc set

$$\begin{aligned}\overline{A}_i := &\left(\{d^{\mathrm{s}}\} \times \left(\hat{\mathcal{V}}_i \mathbin{\dot\cup} \mathcal{T}_i \mathbin{\dot\cup} \hat{\mathcal{P}}_i\right)\right) \mathbin{\dot\cup} \left\{(s,t) \in \left(\hat{\mathcal{V}}_i \mathbin{\dot\cup} \mathcal{T}\right) \times \left(\mathcal{T}_i \mathbin{\dot\cup} \hat{\mathcal{P}}_i\right) \mid s \preceq t\right\} \\ &\mathbin{\dot\cup} \left(\left(\hat{\mathcal{V}}_i \mathbin{\dot\cup} \mathcal{T}_i \mathbin{\dot\cup} \hat{\mathcal{P}}_i\right) \times \{d^{\mathrm{e}}\}\right)\end{aligned}$$

4.2 General Setting

In this section, we describe the general setting of the Successive Heuristics. First we describe how the partial task graph is created, given a splitting of $\mathcal{V}$ and $\mathcal{T}$. It is based on $\overline{G}$ and contains start and end points, which are created in the partial instances solved before this. Then we treat the order in which the partial instances are solved. The first partial instance is a special instance since there the vehicles come into play. Therefore, this instance is solved last. We explain how start and end points are are created out of a partial solution. Finally, we describe the feasible connection of the partial instances to an overall solution.

Order of Solving the Partial Instances

Consider a splitting $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ and $\{\mathcal{V}_1, \dots, \mathcal{V}_n\}$ for $\mathcal{T}$ and $\mathcal{V}$ respectively. Let $\sigma \in S_n$ be a permutation of $[n]$ with $\sigma(n) = 1$. σ indicates in which order the partial instances are solved. This means, partial instance $\sigma(i) \in [n]$ is solved at the i-th position, the first partial instance is solved at last. The actual specification of σ follows in the description of the respective heuristic.

Determination of Start and End Points

The sets of start and end points $\hat{\mathcal{V}}_i, \hat{\mathcal{P}}_i$ are initially empty for all $i \in [n]$. Assume, we have solved the partial instance $\sigma(i)$ just now. Based on the received partial solution, we update the start point set of the next partial instance after $\sigma(i)$ and we update the end point set of the next partial instance before $\sigma(i)$ which is not yet solved. This means, we update $\hat{\mathcal{V}}_{\sigma(j)}$ and $\hat{\mathcal{P}}_{\sigma(k)}$ for $j = \arg\min_{j>i} \{\sigma(j) \mid \sigma(j) > \sigma(i)\}$ and $k = \arg\min_{k>i} \{\sigma(k) \mid \sigma(k) < \sigma(i)\}$.
For each duty of the partial solution that does either visit no node out of $\hat{\mathcal{V}}_{\sigma(i)}$ or no node out of $\hat{\mathcal{P}}_{\sigma(i)}$, we create a start point and/or an end point. If a duty starts with a trip or an end point s, we create an end point out of it. The end point t has the following properties

$$p_t^{\text{start}} = p_t^{\text{end}} := p_s^{\text{start}} \qquad z_t^{\text{start}} = z_t^{\text{end}} := z_s^{\text{start}} \qquad f_t^0 := e_s + f_s^{\text{t}}$$

where e_s is the respective value of decision variable e in the partial solution. We add t to the end point set $\hat{\mathcal{P}}_{\sigma(k)}$. If a duty ends with a trip or a start point s, we create a start point out of it. The start point t has the following properties

$$p_t^{\text{start}} = p_t^{\text{end}} := p_s^{\text{end}} \qquad z_t^{\text{start}} = z_t^{\text{end}} := z_s^{\text{end}} \qquad f_t^0 := e_s$$

where e_s is the respective value of decision variable e in the partial solution. We add t to the start point set $\hat{\mathcal{V}}_{\sigma(j)}$.

Remark 2. Since $\sigma(n) = 1$, the set $\{k > i \mid \sigma(k) < \sigma(i)\}$ is never empty for $i \in [n-1]$. Therefore, it is always possible to create an end point. If there is no later partial instance left, which is not yet solved, i. e. $\{j > i \mid \sigma(j) > \sigma(i)\} = \emptyset$, then we create no start points out of $\sigma(i)$.

If a duty consists of exactly one trip, then for this trip we create both a start and an end point.

Solving Partial Instance $\sigma(n) = 1$

Feasible Connection of Partial Solutions

Algorithm 1: Successive Heuristic (general setting)

Input: splitting $\mathcal{T} = \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$, $\mathcal{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_n\}$, $\sigma \in S_n$ with $\sigma(n) = 1$
Output: overall solution S

foreach $i \in [n]$ **do**
  $\hat{\mathcal{V}}_{\sigma(i)} \leftarrow \emptyset$;
  $\hat{\mathcal{P}}_{\sigma(i)} \leftarrow \emptyset$;
end
foreach $i \in [n-1]$ **do**
  solve partial instance $\sigma(i)$, receive partial solution $S_{\sigma(i)}$ with duty set $D_{\sigma(i)}$;
  $j = \arg\min_{j>i} \{\sigma(j) \mid \sigma(j) > \sigma(i)\}, k = \arg\min_{k>i} \{\sigma(k) \mid \sigma(k) < \sigma(i)\}$;
  foreach $D_{\sigma(i)} \ni d = (s_1, \ldots, s_l)$ **do**
    if $s_1 \in \mathcal{T}_{\sigma(i)} \cup \hat{\mathcal{P}}_{\sigma(i)}$ **then**
      create end point t;
      $p_t^{\text{start}} \leftarrow p_{s_1}^{\text{start}}, p_t^{\text{end}} \leftarrow p_{s_1}^{\text{start}}, z_t^{\text{start}} \leftarrow z_{s_1}^{\text{start}}, z_t^{\text{end}} \leftarrow z_{s_1}^{\text{start}}, f_t^0 \leftarrow e_{s_1} + f_{s_1}^{\text{t}}$;
      $\hat{\mathcal{P}}_{\sigma(k)} \leftarrow \hat{\mathcal{P}}_{\sigma(k)} \cup \{t\}$;
    end
    if $s_l \in \hat{\mathcal{V}}_{\sigma(i)} \cup \mathcal{T}_{\sigma(i)}$ **then**
      create start point t;
      $p_t^{\text{start}} = p_t^{\text{end}} \leftarrow p_{s_l}^{\text{end}}, z_t^{\text{start}} = z_t^{\text{end}} \leftarrow z_{s_l}^{\text{end}}, f_t^0 := e_{s_l}$;
      $\hat{\mathcal{V}}_{\sigma(j)} \leftarrow \hat{\mathcal{V}}_{\sigma(j)} \cup \{t\}$;
    end
  end
end
$\hat{\mathcal{V}}_1 \leftarrow \mathcal{V}$;
solve partial instance 1, receive partial solution 1 with duty set D_1;
feasibly connect $\{S_1, \ldots, S_n\}$ to S;
return S

4.3 Costumer-dependent Splitting

In contrast to the splitting performed in (Knoll, cap. 8), the trips are not split according to their start times but according to their costumers' start times. This means, that all trips of a route and all routes of a costumer are in the same splitting. For each spitting, we apply (CMILP$_i$) to receive an optimal partial solution and connect the partial solutions to a feasible overall solution.

Splitting

Definition 7 (Costumer-dependent Splitting)**.** Given points in time c_i for $i \in [n-1]$ with $c_i < c_{i+1}$ for $i \in [n-2]$. We first define a splitting of the costumers $\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i$ as

$$\mathcal{C}_i := \begin{cases} \{c \in \mathcal{C} \mid z_c^{\text{start}} \leq c_1\} & \text{for } i = 1 \\ \{c \in \mathcal{C} \mid c_{i-1} < z_c^{\text{start}} \leq c_i\} & \text{for } i \in [n-1] \backslash \{1\} \\ \{c \in \mathcal{C} \mid c_{n-1} < z_c^{\text{start}}\} & \text{for } i = n. \end{cases}$$

Based on the costumer splitting, we define the splittings of $\mathcal{T}$ and $\mathcal{V}$ as

$$\mathcal{T}_i := \{t \in \mathcal{T} \mid (M \circ C)(t) \in \mathcal{C}_i\} \qquad \text{for } i \in [n]$$

and

$$\mathcal{V}_i := \begin{cases} \{v \in \mathcal{V} \mid z_v \leq c_1\} & \text{for } i = 1 \\ \{v \in \mathcal{V} \mid c_{i-1} < z_v \leq c_i\} & \text{for } i \in [n-1] \backslash \{1\} \\ \{v \in \mathcal{V} \mid c_{n-1} < z_v\} & \text{for } i = n. \end{cases}$$

We denote the formulation (SMILP) with a splitting according to Definition 7 as (CMILP).

Solving of the Partial Instances

Since for costumer $c \in \mathcal{C}_i$ all his trips are in splitting $\mathcal{T}_i$, costumer c has to be satisfied only in the partial instance i. For solving the partial instances, we modify (SMILP) as follows:
Instead of (3.1) and (3.2) we have the constraints

$$\sum_{m \in C^{-1}(c)} u_m = 1 \qquad \text{for all } c \in \mathcal{C}_i \tag{4.11}$$

$$\sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = u_m \qquad \text{for all } m \in C^{-1}(\mathcal{C}_i), t \in m \tag{4.12}$$

We do not have vehicles in the partial split set any more. Instead we have to ensure that each start and end point is visited. Therefore we modify (3.4) to

$$\sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = 1 \qquad \text{for all } t \in \hat{\mathcal{V}}_i \cup \hat{\mathcal{P}}_i \tag{4.13}$$

To guarantee the fuel level f^0 in $\hat{\mathcal{P}}_i$ we introduce the constraint

$$f^0_s \leq e_s \qquad \text{for all } s \in \hat{\mathcal{P}}_i \tag{4.14}$$

We introduce two additional constraints. The first one guarantees that the fuel level at the beginning of a duty, if it starts with $s \in \mathcal{T}_i$ is at most $f^{\max}_s$. The second one guarantees that the fuel level at the end of a duty, if it ends with $s \in \mathcal{T}_i$ is at least $f^{\max}_s$.

$$e_s + f^{\mathrm{t}}_s \leq f^{\max}_s + (1 - x_{d^{\mathrm{s}},s}) \cdot \left(1 + f^{\mathrm{t}}_s\right) \qquad \text{for all } s \in \mathcal{T}_i \tag{4.15}$$

$$f^{\min}_s \leq e_s + (1 - x_{s,d^{\mathrm{e}}}) \qquad \text{for all } s \in \mathcal{T}_i \tag{4.16}$$

Restricting all other constraints to vertices of the partial task graph, we have the following formulation:

$$
\begin{aligned}
\min \quad & \left(\sum_{s \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i} x_{d^{\mathrm{s}},s} - \sum_{s \in \hat{\mathcal{P}}_i} x_{s,d^{\mathrm{e}}} \right) c^{\mathrm{v}} + \sum_{m \in C^{-1}(\mathcal{C}_i)} u_m c^{\mathrm{r}}_m \\
& \sum_{t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i} \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \{d^{\mathrm{s}}\}} \left[x_{s,t} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_t \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t} \right) \right] && (\mathrm{CMILP}_i) \\
\text{s.t.} \quad & \sum_{t \in \mathrm{N}^-_{\overline{G}_i}(s)} x_{t,s} = \sum_{t \in \mathrm{N}^+_{\overline{G}_i}(s)} x_{s,t} \qquad \text{for all } s \in \overline{V}_i \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (4.17) \\
& \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = 1 \qquad \text{for all } t \in \hat{\mathcal{V}}_i \cup \hat{\mathcal{P}}_i && (4.13) \\
& \sum_{m \in C^{-1}(c)} u_m = 1 \qquad \text{for all } c \in \mathcal{C}_i && (4.11) \\
& \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = u_m \qquad \text{for all } m \in C^{-1}(\mathcal{C}_i), t \in m && (4.12) \\
& \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} \qquad \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (4.18) \\
& e_s \le f^0_s \qquad \text{for all } s \in \hat{\mathcal{V}}_i && (4.19) \\
& f^0_s \le e_s \qquad \text{for all } s \in \hat{\mathcal{P}}_i && (4.14) \\
& 0 \le e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} \qquad \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (4.20) \\
& e_t \le 1 - f^{\mathrm{t}}_t - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} \qquad \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (4.21) \\
& e_t \le e_s - x_{s,t} \left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t \right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t} \right) + (1 - x_{s,t}) \\
& \qquad \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (4.22) \\
& e_t \le e_s - x_{s,t} f^{\mathrm{t}}_t + (1 - x_{s,t}) \qquad \text{for all } s \in \hat{\mathcal{P}}_i, t \in \mathrm{N}^+_{\overline{G}_i}(s) && (4.23) \\
& e_s + f^{\mathrm{t}}_s \le f^{\max}_s + (1 - x_{d^{\mathrm{s}},s}) \cdot \left(1 + f^{\mathrm{t}}_s \right) \qquad \text{for all } s \in \mathcal{T}_i && (4.15) \\
& f^{\min}_s \le e_s + (1 - x_{s,d^{\mathrm{e}}}) \qquad \text{for all } s \in \mathcal{T}_i && (4.16) \\
& x_{s,t} \in \{0,1\} \qquad \text{for all } (s,t) \in \overline{A}_i && (4.24) \\
& z_{s,r,t} \in \{0,1\} \qquad \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i, r \in \mathcal{R}_{s,t} && (4.25) \\
& e_s \in [0,1] \qquad \text{for all } s \in \overline{V}_i \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (4.26) \\
& u_m \in \{0,1\} \qquad \text{for all } m \in C^{-1}(\mathcal{C}_i) && (3.13)
\end{aligned}
$$

Model Equivalence

This heuristic formulation is not equivalent to the original formulation (MMILP). This is shown by the following example.

Example 1. Let t_1, t_2, t_3 with $t_1 \prec t_2 \prec t_3$ be trips with the properties shown in Table 4.1

Trip	Start	End	Route	Costumer
t_1	8:00	8:15	m_1	C_1
t_2	8:30	8:45	m_2	C_2
t_3	9:00	9:15	m_1	C_1

Table 4.1: Trips

In this case, costumer C_1 uses public transport between 8:15 and 9:00. The duty (t_1, t_2, t_3) is a feasible result of the (MMILP).
If there is a split point at 8:15 then the splittings are $\mathcal{T}_1 = \{t_1, t_3\}, \mathcal{T}_2 = \{t_2\}$. Hence, there is one split point $\mathrm{SP}_1(t_2)$ with $z^{\text{start}}_{\mathrm{SP}_1(t_2)} = 8{:}30$. The partial solution of instance 1 is (t_1, t_3) and $t_3 \not\prec \mathrm{SP}_1(t_2)$. Thus, the partial solutions cannot be feasibly connected to the solution (t_1, t_2, t_3).

With this example we have seen, that the formulations (CMILP) and (MMILP) are not equivalent. It is even possible, that an optimal solution of (MMILP) is not feasible in (CMILP).
Although the formulations are not equivalent, we can give an estimation on the objective value when we make some restrictions.

Definition 8. For $n \geq 3$, consider a costumer set $\mathcal{C}$ and split points c_i for $i \in [n-1]$ with $c_i < c_{i+1}$ for all $i \in [n-2]$. We define the following values:

- Costumer Extension for $c \in \mathcal{C}$: $L_{\mathrm{C}}(c) := \max\limits_{t \in (M \circ C)^{-1}(c)} z^{\text{start}}_t - \min\limits_{t \in (M \circ C)^{-1}(c)} z^{\text{start}}_t$
- Costumer Extension: $L_{\mathrm{C}} := \max\limits_{c \in \mathcal{C}} L_{\mathrm{C}}(c)$
- Splitting Length: $L_{\mathrm{S}} := \min\limits_{i \in [n-1]} c_{i+1} - c_i$

Theorem 4. *For $n \geq 3$, consider the problem with costumer set $\mathcal{C}$ and split points c_i for $i \in [n-1]$ with $c_i < c_{i+1}$ for all $i \in [n-2]$. Let*

$$L_{\mathrm{C}} \leq L_{\mathrm{S}} \tag{4.27}$$

Let $d := (t_1, \ldots, t_k)$ be the duty of a vehicle of a feasible solution of the (MMILP). *Then, there are duties $d_1 \cup d_2 = d$, where d_1, d_2 are part of a feasible solution of*

(CMILP). *Moreover, there holds*

$$\operatorname{cost}(d_1) + \operatorname{cost}(d_2) \leq 2 \cdot \operatorname{cost}(d)\,. \tag{4.28}$$

Proof. We consider the vehicle duty $d = (t_1, \dots t_k)$. We write $s \prec t$ according to Definition 1, i.e. (s,t) is feasible in (MMILP). We write $s \to t$ iff (s,t) is feasible in (CMILP).
Consider $s \prec t$ with $s \not\to t$ and costumers $C_s := (M \circ C)(s)$ and $C_t := (M \circ C)(t)$. Then s is in a later splitting than t. There are split points c_{l-1}, c_l, c_{l+1} for $l \in [n]$ with

$$z_s^{\text{start}} < z_t^{\text{start}} \qquad z_{C_t}^{\text{start}} \leq c_l < z_{C_s}^{\text{start}} \qquad c_l + L_{\text{S}} \leq c_{l+1} \qquad z_{C_s}^{\text{start}} \leq z_s^{\text{start}} \leq z_{C_s}^{\text{start}} + L_{\text{C}}$$

Since (4.27), holds

$$\begin{aligned} z_{C_s}^{\text{start}} \leq z_s^{\text{start}} < z_t^{\text{start}} \leq z_{C_t}^{\text{start}} + L_{\text{C}} \leq c_l + L_{\text{C}} \leq c_l + L_{\text{S}} \leq c_{l+1} \\ z_{C_t}^{\text{start}} \geq z_t^{\text{start}} - L_{\text{C}} > z_s^{\text{start}} - L_{\text{C}} \geq z_{C_s}^{\text{start}} - L_{\text{C}} > c_l - L_{\text{C}} \geq c_l - L_{\text{S}} \geq c_{l-1} \end{aligned}$$

and therefore $t \in \mathcal{T}_l, s \in \mathcal{T}_{l+1}$. Here, we use $c_0 := -\infty, c_{n+1} := +\infty$.

Feasibility For arbitrary $i \in [k-2]$ holds: $t_i \prec t_{i+1} \prec t_{i+2}$, therefore also $t_i \prec t_{i+2}$. We prove that t_{i+2} can be appended after t_i or t_{i+1}. We differentiate between the following cases:

1. $t_{i+1} \to t_{i+2}$: Clear.
2. $t_{i+1} \not\to t_{i+2}$: Then holds $t_{i+2} \in \mathcal{T}_l$ and $t_{i+1} \in \mathcal{T}_{l+1}$ for some $l \in [k]$. From $t_i \prec t_{i+2}$ follows $t_i \in \bigcup_{j=1}^{l+1} \mathcal{T}_j$. Therefore $t_i \to t_{i+1}$ or $t_i \to t_{i+2}$.
    - $t_i \to t_{i+2}$: Clear.
    - $t_i \not\to t_{i+2}$: Then holds $t_{i+2} \in \mathcal{T}_l$ and $t_i, t_{i+1} \in \mathcal{T}_{l+1}$ and therefore $t_i \to t_{i+1}$. For $i' \geq i$ holds $t_{i'} \in \bigcup_{j=l}^{n} \mathcal{T}_j$ and therefore $t_{i+1} \to t_{i'}$ or $t_{i+2} \to t_{i'}$. Thus, every later trip can be appended after on of these duties.

We have seen that two duties d_1, d_2 can fulfill the trips of duty d, such that d_1 and d_2 are feasible in (CMILP). Each trip can be appended to d_1 or to d_2.

Costs The costs of duty d are

$$\operatorname{cost}(d) = c^{\text{v}} + c^{\text{d}}_{v,t_1} + c^{\text{t}}_{t_1} + \sum_{i=2}^{k} \left(c^{\text{d}}_{t_{i-1},t_i} + c^{\text{t}}_{t_i} \right).$$

Each duty d_1, d_2 has cost $c^{\text{d}}_{t,t'} + c^{\text{t}}_{t'} + c^{\text{d}}_{t',t''}$ if trip t' is covered and cost $c^{\text{d}}_{t,t''}$ if not. According to (2.6), the costs for not covering the trip do not exceed the costs for covering. Therefore we have

$$\operatorname{cost}(d_1) + \operatorname{cost}(d_2) \leq 2 \cdot \operatorname{cost}(d)\,.$$

□

Corollary 1. *Consider the problem with* $L_{\mathrm{C}} \leq L_{\mathrm{S}}$. *Let* S_1 *be a feasible solution of* (MMILP)*. Then there exists a solution* S_2 *feasible also in* (CMILP)*.*

$$\operatorname{val}(S_2) \leq 2 \cdot \operatorname{val}(S_1)$$

4.4 Time-dependent Splitting

The developed formulation (CMILP) based on a costumer-dependent splitting is not equivalent to the original formulation (MMILP). The goal now is to develop a splitting that is equivalent and create a heuristic based on this splitting. Therefore, it is necessary that trips of the same costumer may be in different splittings. This leads to the following problem: When the partial instances are solved successively, we need a possibility to still guarantee the costumer satisfaction for the entire problem. This has to be applied already in the partial instance, although we do not have any knowledge about the trips of the same costumer in the later solved partial instances.

4.4.1 Basic Idea

Splitting

We split the sets $\mathcal{T}$ and $\mathcal{V}$ according to their start times.

Definition 9 (Time-dependent Splitting)**.** Given points in time c_i for $i \in [n-1]$ with $c_i < c_{i+1}$ for $i \in [n-2]$. We define the splitting of $\mathcal{T}$ and $\mathcal{V}$ as follows:

$$\mathcal{T}_i := \begin{cases} \{t \in \mathcal{T} \mid z_t^{\text{start}} \leq c_1\} & \text{for } i = 1 \\ \{t \in \mathcal{T} \mid c_{i-1} < z_t^{\text{start}} \leq c_i\} & \text{for } i \in [n-1] \backslash \{1\} \\ \{t \in \mathcal{T} \mid c_{n-1} < z_t^{\text{start}}\} & \text{for } i = n \end{cases}$$

and

$$\mathcal{V}_i := \begin{cases} \{v \in \mathcal{V} \mid z_v \leq c_1\} & \text{for } i = 1 \\ \{v \in \mathcal{V} \mid c_{i-1} < z_v \leq c_i\} & \text{for } i \in [n-1] \backslash \{1\} \\ \{v \in \mathcal{V} \mid c_{n-1} < z_v\} & \text{for } i = n \end{cases}$$

We denote the formulation (SMILP) with a splitting according to Definition 9 as (TMILP).

Solving of the Partial Instances

Since the trips of the same costumer may be in different splittings, we cannot easily guarantee the costumer satisfaction only in just one partial instance. We have to put great effort in this issue. For this, we first define the earliest partial instance in which a trip of a costumer arises as follows: Let $\sigma \in S_n$ with $\sigma(1) = n$ be the order in which the partial instances are solved.

$$\gamma : \mathcal{C} \to [n] \qquad \gamma(c) := \underset{i \in [n]}{\arg\min} \left\{ \sigma(i) \in [n] \mid \left((M \circ C)^{-1}(c) \cap \mathcal{T}_i \right) \neq \emptyset \right\}$$

Depending on γ and $\{\mathcal{T}_1, \dots \mathcal{T}_n\}$ we define a partition $\mathcal{C} = \{\mathcal{C}_1, \dots \mathcal{C}_n\}$ as

$$\mathcal{C}_i := \{c \in \mathcal{C} \mid \gamma(c) = i\} \qquad \text{for } i \in [n]$$

Consider costumer $c \in \mathcal{C}$ and the partial instance $\gamma(c) \in [n]$. In this partial instance, a multimodal route $m \in C^{-1}(c)$ for the costumer is chosen and this choice is definite. This means, in all subsequent processed partial instances, all trips $t \in m$ are fixed to be chosen before solving and all trips $t \in ((M \circ C)^{-1}(c) \backslash m)$ are fixed to be neglected. In partial instance $\gamma(c)$ we have at least one trip of this costumer. But there are also trips that are in other splittings. There are even multimodal routes with no trip in this splitting at all. These routes must not be neglected. Therefore, we need a method to choose the routes where all routes $m \in C^{-1}(c)$ are considered. Therefore, we try to estimate the costs of the routes in advance.

The solving of the partial instances is again based on (SMILP). In comparison the (CMILP$_i$) there are only few changes.

The costumer constraint (4.11) is basically the same. Notice, that the definition of $\mathcal{C}_i$ has changed. The route constraint (4.12) is restricted to the trips that are actually in this splitting. So the new constraints are

$$\sum_{m \in C^{-1}(c)} u_m = 1 \qquad \text{for all } c \in \mathcal{C}_i \tag{4.29}$$

$$\sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = u_m \qquad \text{for all } m \in C^{-1}(\mathcal{C}_i), t \in m \cap \mathcal{T}_i \tag{4.30}$$

For the costumer constraint it is irrelevant if there are trips of the considered routes in this splitting.

After solving the partial instance, all determined u_m are fixed for the later processed partial instances. The fixed route decisions from the previous partial instances have an impact on the instance, too.

Let $\bar{u}_m \in [0, 1]$ be the fixed route choices from the previous instances. Define

$$\overline{\mathcal{C}}_i := \{c \in \mathcal{C} \mid \gamma(c) < \sigma(i)\} \tag{4.31}$$

as the costumers that are already treated. Then, we introduce the constraint

$$\sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = \bar{u}_m \qquad \text{for all } m \in C^{-1}\left(\overline{\mathcal{C}}_i\right), t \in m \cap \mathcal{T}_i \tag{4.32}$$

which ensures that the previous route choices are considered.

Cost Estimation

In order to choose a route in a partial instance, we have to estimate the costs for these routes in advance in all subsequent instances. The entire cost for the problem consists of vehicle costs c^{v}, trip costs c^{t}, deadhead costs c^{d} and route costs c^{r}. While we can determine the trip costs and route costs easily for a route, the vehicle costs and trip costs strongly depend on the environment of the route and cannot be determined. We therefore focus on the trip and route costs and define the estimated route costs as follows:

$$C_1(m) := c^{\mathrm{r}}_m + \sum_{t \in m} c^{\mathrm{t}}_t \qquad \text{for } m \in \mathcal{M}$$

We use these costs in the (TMILP$_i$) to define the modified route costs

$$\hat{c}^{\mathrm{r}}_m := c^{\mathrm{r}}_m + \sum_{t \in m \setminus \mathcal{T}_i} c^{\mathrm{t}}_t \qquad \text{for } m \in \mathcal{M}$$

and add

$$\sum_{m \in C^{-1}(\mathcal{C}_i)} u_m \hat{c}^{\mathrm{r}}_m$$

to the objective function.

Remark 3. The trips in the same splitting $t \in (m \cap \mathcal{T}_i)$ are not considered in $\hat{c}^{\mathrm{r}}_m$ since they are already part of the objective function. The other trips $t \in (m \setminus \mathcal{T}_i)$ are added to $\hat{c}^{\mathrm{r}}_m$, such that they have an impact on the choice of the routes.
Consider a trip t that is decided before this partial instance, i.e. $t \in (M \circ C)\left(\overline{\mathcal{C}}_i\right)$. Its trip costs c^{t}_t arise twice in the objective functions. Once in the partial instance $\gamma\left((M \circ C)(t)\right)$ as part of $\hat{c}^{\mathrm{r}}_{M(t)}$ and once in partial instance i as c^{t}_t. But since in partial instance i, the trip has fulfilled anyway, these costs are only an additional factor that does not influence the solution.

LP Formulation

The entire formulation (TMILP$_i$) reads as follows:

$$
\begin{aligned}
\min \quad & \left(\sum_{s \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i} x_{d^{\mathrm{s}},s} - \sum_{s \in \hat{\mathcal{P}}_i} x_{s,d^{\mathrm{e}}} \right) c^{\mathrm{v}} + \sum_{m \in C^{-1}(\mathcal{C}_i)} u_m \hat{c}^{\mathrm{r}}_m \\
& \sum_{t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i} \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \{d^{\mathrm{s}}\}} \left[x_{s,t} \left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_t \right) + \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t} \right) \right] && (\mathrm{TMILP}_i) \\
\text{s.t.} \quad & \sum_{t \in \mathrm{N}^-_{\overline{G}_i}(s)} x_{t,s} = \sum_{t \in \mathrm{N}^+_{\overline{G}_i}(s)} x_{s,t} && \text{for all } s \in \overline{V}_i \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (4.17) \\
& \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = 1 && \text{for all } t \in \hat{\mathcal{V}}_i \cup \hat{P}_i && (4.13) \\
& \sum_{m \in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C}_i && (4.29) \\
& \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = u_m && \text{for all } m \in C^{-1}(\mathcal{C}_i), t \in m \cap \mathcal{T}_i && (4.30) \\
& \sum_{s \in \mathrm{N}^-_{\overline{G}_i}(t)} x_{s,t} = \bar{u}_m && \text{for all } m \in C^{-1}\left(\overline{\mathcal{C}}_i\right), t \in m \cap \mathcal{T}_i && (4.32) \\
& \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \leq x_{s,t} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (4.18) \\
& e_s \leq f^0_s && \text{for all } s \in \hat{\mathcal{V}}_i && (4.19) \\
& f^0_s \leq e_s && \text{for all } s \in \hat{\mathcal{P}}_i && (4.14) \\
& 0 \leq e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (4.20) \\
& e_t \leq 1 - f^{\mathrm{t}}_t - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (4.21) \\
& e_t \leq e_s - x_{s,t} \left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t \right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t} \right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i && (4.22) \\
& e_t \leq e_s - x_{s,t} f^{\mathrm{t}}_t + (1 - x_{s,t}) && \text{for all } s \in \hat{\mathcal{P}}_i, t \in \mathrm{N}^+_{\overline{G}_i}(s) && (4.23) \\
& e_s + f^{\mathrm{t}}_s \leq f^{\max}_s + (1 - x_{d^{\mathrm{s}},s}) \cdot \left(1 + f^{\mathrm{t}}_s\right) && \text{for all } s \in \mathcal{T}_i && (4.15) \\
& f^{\min}_s \leq e_s + (1 - x_{s,d^{\mathrm{e}}}) && \text{for all } s \in \mathcal{T}_i && (4.16) \\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t) \in \overline{A}_i && (4.24) \\
& z_{s,r,t} \in \{0,1\} && \text{for all } t \in \mathcal{T}_i \cup \hat{\mathcal{P}}_i, s \in \mathrm{N}^-_{\overline{G}_i}(t) \setminus \hat{\mathcal{P}}_i, r \in \mathcal{R}_{s,t} && (4.25) \\
& e_s \in [0,1] && \text{for all } s \in \overline{V}_i \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (4.26) \\
& u_m \in \{0,1\} && \text{for all } m \in C^{-1}(\mathcal{C}_i) && (3.13)
\end{aligned}
$$

4.4.2 Iterative Approach

We use the previously developed heuristic for an iterative approach. We compute an initial solution while we choose the routes with cost function C_1. Then we determine the actual cost of this route in the entire solution and compare the estimated cost with the actual cost.

Initial Solution

We determine a solution with the heuristic developed in Section 4.4.1. Given a solution $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u})$ of the (TMILP), we determine

$$C_1(c) := C_1(m) \qquad \text{for } c \in \mathcal{C}, m \in C^{-1}(c) \text{ with } \bar{u}_m = 1$$

Finding Bad Route Choice

Given a solution of the problem, the subproblem is to find a costumer with a bad route choice. This means, for this costumer there is another route, such that the total costs are lower if this route is chosen. Then, we can exchange these routes and compute a new solution considering the new route.
An initial idea is to compute the costs, one route in the solution contributes to the entire solution. Then, we can compare this to the cost, with which we estimated the route costs before. If the actual costs are considerably higher than the estimated costs, this costumer is a candidate for exchanging routes.
Since we cannot determine the contributing costs exactly, we try to estimate these costs.
Let $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u})$ be a solution of (MMILP). To determine the contributing costs for route $m \in \mathcal{M}$, we define the following auxiliary costs for every trip $t \in \mathcal{T}$ of the solution:
Vehicle costs $c_t^{\mathrm{v}}(S)$: Let $v \in \mathcal{V}$ be the vehicle covering t and k_v the number of trips covered by v:

$$c_t^{\mathrm{v}}(S) := \frac{c^{\mathrm{v}}}{k_v}$$

Refueling costs $c_t^{\mathrm{refuel}}(S)$: Let $r \in \mathcal{R}$ be the next refuel station used after t and T_r all trips covered since the last station, let $\bar{z}_{s,r,s'} = 1$:

$$c_t^{\mathrm{refuel}}(S) := \frac{f_t^{\mathrm{t}}}{\sum_{t' \in T_r} f_t^{\mathrm{t}}} \left(c_{s,r}^{\mathrm{d}} + c_{r,s'}^{\mathrm{d}} - c_{s,s'}^{\mathrm{d}} \right)$$

If the vehicle is not refueled after t, then $c_t^{\mathrm{refuel}}(S) := 0$.

Deadhead costs $c^{\mathrm{d}}_t(S)$: Let $s \in \mathcal{V} \cup \mathcal{T}, s' \in \mathcal{T}$ be the trips covered directly before and after t by vehicle v, i.e. $\bar{x}_{s,t} = \bar{x}_{t,s'} = 1$:

$$c^{\mathrm{d}}_t(S) := \frac{1}{2}\left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{d}}_{t,s'}\right)$$

If t is the last trip of the duty, i.e. $\bar{x}_{s,t} = \bar{x}_{t,d^{\mathrm{e}}} = 1$, then $c^{\mathrm{d}}_t(S) := \frac{1}{2}c^{\mathrm{d}}_{s,t}$.
With these auxiliary costs we can define new route costs which describe the contribution of a multimodal route to the entire solution better:

Definition 10 (Improved Cost Estimation)**.** Let $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u})$ be a solution of the (MMILP). With the auxiliary costs described before, we define the improved cost estimation for all multimodal routes $m \in \{m \in \mathcal{M} \mid \bar{u}_m = 1\}$:

$$C_2(S, m) := C_1(m) + \sum_{t \in m}\left(c^{\mathrm{v}}_t(S) + c^{\mathrm{refuel}}_t(S) + c^{\mathrm{d}}_t(S)\right)$$

We further define

$$C_2(S, c) := C_2(S, m) \qquad\qquad \text{for } c \in \mathcal{C}, m \in C^{-1}(c) \text{ with } \bar{u}_m = 1$$

Now we can evaluate our previous estimation for the route contribution. If $C_2(S, c)$ is significantly higher than $C_1(S, c)$ then the probability is high that we made a bad route choice for costumer $c \in \mathcal{C}$.
We therefore determine

$$c^* := \arg\max_{c \in \mathcal{C}} \frac{C_2(S, c)}{C_1(S, c)}$$

The probability is high that we made a bad route choice for costumer c^*. Thus, we look at the route choice for c^* again.

Remark 4. For simplicity of notation, we assume that S is a solution of (MMILP). This is possible since the formulations (TMILP) and (MMILP) are equivalent.

Subproblem

Let $S = (\bar{x}, \bar{z}, \bar{e}, \bar{u})$ be a solution of (MMILP) and $c \in \mathcal{C}$ a candidate for a bad route choice. We define the following subproblem (HSP$_c$): Assume the schedule according to S for the entire time without $[z^{\mathrm{start}}_c, z^{\mathrm{end}}_c]$ and all route choices for costumers except c as fix. Determine an optimal schedule within these restrictions.
Considering only the trips of c and the trips chosen in S, we define

$$\mathcal{T}^c := \left\{ t \in \mathcal{T} \mid (M \circ C)(t) = c \vee \sum_{s \in \mathrm{N}^-_G(t)} \bar{x}_{s,t} = 1 \right\}$$

and the splittings $\mathcal{T}^c = \{\mathcal{T}_1^c, \mathcal{T}_2^c, \mathcal{T}_3^c\}$ and $\mathcal{V} = \{\mathcal{V}_1^c, \mathcal{V}_2^c, \mathcal{V}_3^c\}$ by

$$\mathcal{T}_i^c := \begin{cases} \{t \in \mathcal{T}^c \mid z_t^{\text{start}} < z_c^{\text{start}}\} & \text{if } i = 1 \\ \left\{t \in \mathcal{T}^c \mid z_c^{\text{start}} \leq z_t^{\text{start}} \leq z_c^{\text{end}}\right\} & \text{if } i = 2 \\ \left\{t \in \mathcal{T}^c \mid z_c^{\text{end}} < z_t^{\text{start}}\right\} & \text{if } i = 3 \end{cases}$$

and

$$\mathcal{V}_i^c := \begin{cases} \{v \in \mathcal{V} \mid z_v < z_c^{\text{start}}\} & \text{if } i = 1 \\ \left\{v \in \mathcal{V} \mid z_c^{\text{start}} \leq z_v \leq z_c^{\text{end}}\right\} & \text{if } i = 2 \\ \left\{v \in \mathcal{V} \mid z_v < z_c^{\text{end}}\right\} & \text{if } i = 3 \end{cases}$$

We then define the start point set $\hat{\mathcal{V}}_2$ and the end point set $\hat{\mathcal{P}}_2$

$$\hat{\mathcal{V}}_2 := \{s \in \mathcal{T}_1^c \mid \bar{x}_{s,t} = 1 \text{ for } t \in (\mathcal{T}_2^c \cup \mathcal{T}_3^c \cup \{d^{\text{e}}\})\} \cup \mathcal{V}_1 \cup \mathcal{V}_2$$
$$\hat{\mathcal{P}}_2 := \{t \in \mathcal{T}_3^c \mid \bar{x}_{s,t} = 1 \text{ for } t \in (\{d^{\text{s}}\} \cup \mathcal{T}_1^c \cup \mathcal{T}_2^c)\}$$

With these definitions, we can adapt the formulation (TMILP$_i$) for $i = 2$ to (HSP$_c$). The only modified constraints are the costumer and route constraints (4.29), (4.30), (4.32) and (3.13). They are replaced by

$$\sum_{m \in C^{-1}(c)} u_m = 1 \tag{4.33}$$

$$\sum_{s \in \mathrm{N}_{G_2}^-(t)} x_{s,t} = u_m \qquad \text{for all } m \in C^{-1}(c), t \in m \tag{4.34}$$

$$\sum_{s \in \mathrm{N}_{G_2}^-(t)} x_{s,t} = \bar{u}_{M(t)} \qquad \text{for all } t \in \mathcal{T}_2 \backslash (M \circ C)^{-1}(c) \tag{4.35}$$

$$u_m \in \{0, 1\} \qquad \text{for all } m \in M^{-1}(c) \tag{4.36}$$

The objective function is the same as in (TMILP$_i$).
We can easily receive a new entire solution: Transform the original solution S into three partial solutions $\{S_1^c, S_2^c, S_3^c\}$ according to the splitting $\mathcal{T}^c = \{\mathcal{T}_1^c, \mathcal{T}_2^c, \mathcal{T}_3^c\}$ and $\mathcal{V}^c = \{\mathcal{V}_1^c, \mathcal{V}_2^c, \mathcal{V}_3^c\}$. Let $\hat{S}_2^c$ be an optimal solution of (HSP$_c$). Then feasibly connect the partial solutions $\left\{S_1^c, \hat{S}_2^c, S_3^c\right\}$ to a new solution $\hat{S}$.
The original partial solution S_2^c is a feasible solution of (HSP$_c$). Therefore, with this method we cannot get a worse entire solution than before.
After completing this step, we can apply this procedure to the costumer with the second-highest ratio of $\frac{C_2(S,c)}{C_1(c)}$.

Remark 5. The costumer extension L_{C} is not bounded explicitly like (4.27). But also here a small costumer extension is beneficial due to the size of the (HSP$_c$).

4.4.3 Restricted Approach

We consider again the special case that the costumer extension is smaller than the splitting length.

$$L_\mathrm{C} \leq L_\mathrm{S} \tag{4.27}$$

For costumer $c \in \mathcal{C}$ there are $i \in [n-1]$ with time points $c_{i-1} \leq z_c^{\mathrm{start}} < c_i$ and therefore

$$z_t^{\mathrm{start}} \leq z_c^{\mathrm{start}} + L_\mathrm{C} < c_i + L_\mathrm{C} \leq c_i + L_\mathrm{S} \leq c_{i+1} \qquad \text{for all } t \in (M \circ C)^{-1}(c)$$

We again state $c_0 := -\infty$ and $c_n := +\infty$. Thus we have

$$t \in (\mathcal{T}_i \cup \mathcal{T}_{i+1}) \qquad \text{for all } t \in (M \circ C)^{-1}(c)$$

This means, for every costumer $c \in \mathcal{C}$ at most two splittings are affected. We can exploit this property for solution methods.

Each costumer is represented in at most two subsequent splittings. Costumers, that are only represented in one splitting, are neglected here since the costumer satisfaction is already ensured in (TMILP$_i$).

For each costumer, we basically distinguish between two cases: There are more trips of the costumer in the splitting whose partial instance is processed first (Case 1). Or there are more trips in the splitting whose partial instance is processed later (Case 2). In Case 1, the cost estimation for the routes is easy since most of the structure is already contained in the first processed partial instance. In Case 2, there is not much structure in the first processed partial instance, so the cost prediction will be imprecise.

To prevent an imprecise cost estimation in Case 2, we inspect the possibility to reverse a previous route choice in the later processed partial instance if we find a better alternative there. For this, we have to think about the cost saving for a belated trip deletion.

Consider partial instance $i \in [n]$ and the costumer set

$$\mathcal{C}_i^\mathrm{R} := \left\{ c \in \mathcal{C} \mid \gamma(c) \in \{i-1, i+1\} \wedge \left((M \circ C)^{-1}(c) \cap \mathcal{T}_i \right) \neq \emptyset \right\}$$

and the route set

$$\mathcal{M}_i^\mathrm{R} := \left\{ m \in \mathcal{M} \mid C(m) \in \mathcal{C}_i^\mathrm{R} \wedge m \subset \mathcal{T}_i \right\}$$

$\mathcal{C}_i^\mathrm{R}$ are all costumers represented in $\mathcal{T}_i$ but initially treated in another partial instance, $\mathcal{M}_i^\mathrm{R}$ are all routes of these costumers where all trips are in $\mathcal{T}_i$.

We regard the possibility to revise a previous route choice if we find a better alternative in partial instance i. For this, we think about the cost saving for subsequent trip deletion. As in Section 4.4.2, the cost function $C_1(m)$ is used for cost estimation.

Costs for Trip Replacement

Consider partial instance i where partial instances $i-1$ or $i+1$ are processed before, i.e. $\sigma(i) < \sigma(i-1)$ or $\sigma(i) < \sigma(i+1)$. If both are processed later, we have $\mathcal{C}_i^{\mathrm{R}} = \emptyset$ and this procedure is not considered at all.

Let $\bar{m}(c) \in C^{-1}(c)$ be the unique route with $\bar{u}_m = 1$ for all $c \in \mathcal{C}_i^{\mathrm{R}}$. Let $s_1(t) \in \left(\left\{d^{\mathrm{s}}_{\gamma(c)}\right\} \mathbin{\dot\cup} \hat{\mathcal{V}}_{\gamma(c)} \mathbin{\dot\cup} \mathcal{T}_{\gamma(c)}\right)$, $s_2(t) \in \left(\mathcal{T}_{\gamma(c)} \mathbin{\dot\cup} \hat{\mathcal{P}}_{\gamma(c)} \mathbin{\dot\cup} \left\{d^{\mathrm{e}}_{\gamma(c)}\right\}\right)$ be the unique trips with $\bar{x}_{s_1,t} = \bar{x}_{t,s_2} = 1$ for all $t \in (\bar{m}(c) \backslash \mathcal{T}_i)$.

If we delete trip t in partial instance $\gamma(c)$, the saved costs are

$$c^{\mathrm{d}}_{s_1(t),t} + c^{\mathrm{t}}_t + c^{\mathrm{d}}_{t,s_2(t)} - c^{\mathrm{d}}_{s_1(t),s_2(t)}$$

We state the deadhead costs $c^{\mathrm{d}}_{d^{\mathrm{s}},t} = c^{\mathrm{d}}_{t,d^{\mathrm{e}}} =: 0$ for all $t \in \mathcal{T}_{\gamma(i)}$.

We modify the formulation (TMILP$_i$) such that a belated route exchange is possible in the restricted case with (4.27). The formulation is called (RTMILP$_i$). The underlying partial task graph is not modified.

We introduce new decision variables u^c for $c \in \mathcal{C}_i^{\mathrm{R}}$. They indicate whether the route choice for costumer c is confirmed or not. If a deletion of the route determined in the partial instances $i-1$ or $i+1$ is possible, it is necessary to add routes of the same costumer. Since a belated insertion of trips is difficult, we restrict ourselves to the routes that have only trips in $\mathcal{T}_i$. We therefore introduce decision variables u_m for all $m \in \mathcal{M}_i^{\mathrm{R}}$.

We add the following term to the objective function:

$$\sum_{m \in \mathcal{M}_i^{\mathrm{R}}} u_m c_m - \sum_{c \in \mathcal{C}_i^{\mathrm{R}}} (1 - u^c) \left[\sum_{t \in (\bar{m}_c \backslash \mathcal{T}_i)} \left(c^{\mathrm{d}}_{s_1(t),t} + c^{\mathrm{t}}_t + c^{\mathrm{d}}_{t,s_2(t)} - c^{\mathrm{d}}_{s_1(t),s_2(t)} \right) \right]$$

For every $c \in \mathcal{C}_i^{\mathrm{R}}$, either the previous choice must be confirmed or a new route is chosen. We therefore add the constraint

$$u^c + \sum_{\substack{m \in \mathcal{M}_i^{\mathrm{R}} \\ C(m)=c}} u_m = 1 \qquad \text{for all } c \in \mathcal{C}_i^{\mathrm{R}} \tag{4.37}$$

We have $\overline{\mathcal{C}}_i = \mathcal{C}_i^{\mathrm{R}}$ since (4.27). The constraint (4.32) ensures the route decisions of the previous partial instances. It is replaced by

$$\sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = u^c \qquad \text{for all } c \in \mathcal{C}_i^{\mathrm{R}}, t \in \bar{m}(c) \cap \mathcal{T}_i \tag{4.38}$$

$$\sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = u_m \qquad \text{for all } t \in M^{-1}\left(\mathcal{M}_i^{\mathrm{R}}\right) \tag{4.39}$$

$$\sum_{s \in \mathrm{N}^{-}_{\overline{G}_i}(t)} x_{s,t} = 0 \qquad \text{for all } c \in \mathcal{C}_i^{\mathrm{R}}, t \in M^{-1}\left(C^{-1}(c) \setminus \left(\mathcal{M}_i^{\mathrm{R}} \cap \{\bar{m}_c\}\right)\right) \cap \mathcal{T}_i \tag{4.40}$$

The route satisfaction for the previously decided trips is ensured by (4.38) for the decided tour, by (4.39) for all routes in $\mathcal{T}_i$ and by (4.40) for all other trips in $\mathcal{T}_i$.
Note that $\mathcal{C}_i \cap \mathcal{C}_i^{\mathrm{R}} = \emptyset$. Hence, (4.29) and (4.30) are not influenced by them.
Finally, we replace (3.13) by

$$u_m \in \{0,1\} \qquad \text{for all } m \in C^{-1}\left(\mathcal{C}_i\right) \mathbin{\dot\cup} \mathcal{M}_i^{\mathrm{R}} \tag{4.41}$$

$$u^c \in \{0,1\} \qquad \text{for all } c \in \mathcal{C}_i^{\mathrm{R}} \tag{4.42}$$

4.4.4 Improvements

- If (4.27) does not hold: Find a possibility to arrange splittings such that at most 2 splittings are affected.
- If a route is chosen where no trip is in this splitting: Choice of routes among routes with no trip in this splitting in the next processed partial instance.
- (RTMILP) does not work if two subsequent trips, fulfilled by the same car, are deleted.

Chapter 5

Optimal Approach

5.1 Dantzig-Wolfe-Decomposition

We adapt the path flow formulation from (Kaiser, Knoll, cap. 3.3) by applying Dantzig-Wolfe-Decomposition. This can be used, if many of the constraints have only impact on a small number of variables and these variables can be grouped. The structure of the problem looks like:

$$
\left(\begin{array}{cccccccccccc}
\star & \cdots & \star & \star & \cdots & \star & & & & \star & \cdots & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \cdots & & \vdots & \ddots & \vdots \\
\star & \cdots & \star & \star & \cdots & \star & & & & \star & \cdots & \star \\
\hline
\star & \cdots & \star & 0 & \cdots & 0 & & & & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \cdots & & \vdots & \ddots & \vdots \\
\star & \cdots & \star & 0 & \cdots & 0 & & & & 0 & \cdots & 0 \\
0 & \cdots & 0 & \star & \cdots & \star & & & & & & \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \ddots & & & \vdots & \\
0 & \cdots & 0 & \star & \cdots & \star & & & & & & \\
 & & & & & & & & & 0 & & 0 \\
 & \vdots & & & \ddots & & & \ddots & & \vdots & \ddots & \vdots \\
 & & & & & & & & & 0 & & 0 \\
0 & \cdots & 0 & & & & 0 & \cdots & 0 & \star & \cdots & \star \\
\vdots & \ddots & \vdots & & \cdots & & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & & & & 0 & \cdots & 0 & \star & \cdots & \star
\end{array}\right)
$$

The constraints concerning more variables are called linking constraints.

Identification of the Subproblems

Consider the (MMILP). A natural choice for the subproblem is the duty of each vehicle $v \in \mathcal{V}$. We define (x^v, z^v, e^v) for $v \in \mathcal{V}$ as the specific variables for this vehicle. We can therefore define the set of feasible configurations for vehicle $v \in \mathcal{V}$ as follows:

$$
\begin{aligned}
X_v := \Big\{ & (x, z, e) \in \{0,1\}^A \times \{0,1\}^{A \cap (\mathcal{V} \cup \mathcal{T}_{\mathrm{car}})^2 \times \mathcal{R}} \times [0,1]^{\mathcal{V} \cup \mathcal{T}_{\mathrm{car}}} \mid \\
& \sum_{t \in \mathrm{N}_G^-(s)} x_{t,s} = \sum_{t \in \mathrm{N}_G^+(s)} x_{s,t} && \text{for all } s \in V \setminus \{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (3.15) \\
& \sum_{s \in \mathrm{N}_G^-(t)} x_{s,v} = 1 && && (5.1) \\
& \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} = 0 && \text{for all } t \in \mathcal{V} \setminus \{v\} && (5.2) \\
& \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.18) \\
& e_s \le f_s^0 && \text{for all } s \in \mathcal{V} && (3.19) \\
& 0 \le e_s - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f_{s,r}^{\mathrm{d}} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.12) \\
& e_t \le 1 - f_t^{\mathrm{t}} - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t} f_{r,t}^{\mathrm{d}} && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.13) \\
& e_t \le e_s - x_{s,t}\left(f_{s,t}^{\mathrm{d}} + f_t^{\mathrm{t}}\right) - \sum_{r \in \mathcal{R}_{s,t}} z_{s,r,t}\left(f_{s,r}^{\mathrm{d}} + f_r^{\mathrm{t}} + f_{r,t}^{\mathrm{d}} - f_{s,t}^{\mathrm{d}}\right) + (1 - x_{s,t}) \\
& && \text{for all } t \in \mathcal{T}, s \in \mathrm{N}_G^-(t) && (3.14) \\
& \Big\}
\end{aligned}
$$

We denote the set of feasible duties for any vehicle by $X := \bigcup_{v \in \mathcal{V}} X_v$. We write the cost for configuration (x^v, z^v, e^v) as $g(x^v, z^v, e^v)$. Putting all together, we can rewrite (MMILP) as

$$
\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} g(x^v, z^v, e^v) + \sum_{m \in \mathcal{M}} u_m c_m^{\mathrm{r}} \\
\text{s.t.} \quad & \sum_{m \in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C} && (3.1) \\
& \sum_{v \in \mathcal{V}} \sum_{s \in \mathrm{N}_G^-(t)} x_{s,t}^v = u_m && \text{for all } m \in \mathcal{M}, t \in m && (5.3) \\
& (x^v, z^v, e^v) \in X_v && \text{for all } v \in \mathcal{V} \\
& u_m \in \{0,1\}^{\mathcal{M}}
\end{aligned}
$$

Reduction of the Master Problem

We define the linear mapping

$$\psi : X \to \{0,1\}^{\mathcal{T}_{\text{car}}} \qquad (x, z, e) \mapsto \left(\sum_{s \in \mathrm{N}_G^-(t)} x_{s,t} \right)_{t \in \mathcal{T}_{\text{car}}} \tag{5.4}$$

and rewrite (MMILP) by using $y^v := \psi\left(x^v, z^v, e^v\right)$:

$$\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} \min g\left(\psi^{-1}\left(y^v\right) \cap X_v\right) + \sum_{m \in \mathcal{M}} u_m c_m^{\mathrm{r}} \\
\text{s.t.} \quad & \sum_{m \in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C} \\
& \sum_{v \in \mathcal{V}} y_t^v = u_m && \text{for all } m \in \mathcal{M}, t \in m \\
& y^v \in \psi\left(X_v\right) && \text{for all } v \in \mathcal{V} \\
& u_m \in \{0,1\} && \text{for all } m \in \mathcal{M}
\end{aligned} \tag{3.1}$$

We can see that (3.1) depends only on u_m. Therefore, we create two different versions of this formulation: One with restriction of the route choice and one without restriction.

5.2 Master Problem without Route Choice Restriction

Column Generation

For every $v \in \mathcal{V}$, let $\mathcal{I}_v$ be an index set for the finitely many points in $\psi\left(X_v\right)$ and let the columns of $Y^v \in \mathbb{R}^{\mathcal{T}_{\text{car}} \times \mathcal{I}_v}$ be exactly those points. Let $G^v \in \mathbb{R}^{1 \times \mathcal{I}}$ be the respective values of $\min g\left(\psi^{-1}(\cdot) \cap X_v\right)$. Then we can reformulate the master problem as

$$
\begin{aligned}
\min \quad & \sum_{v\in\mathcal{V}} G^v\lambda^v + \sum_{m\in\mathcal{M}} u_m c^{\mathrm{r}}_m && && \text{(IMP)}\\
\text{s.t.} \quad & \sum_{v\in\mathcal{V}} Y^v_{t,\cdot}\lambda^v = u_m && \text{for all } m\in\mathcal{M}, t\in m\\
& \sum_{m\in C^{-1}(c)} u_m = 1 && \text{for all } c\in\mathcal{C}\\
& \sum_{i\in\mathcal{I}_v} \lambda^v_i = 1 && \text{for all } v\in\mathcal{V}\\
& \lambda^v \in \{0,1\}^{\mathcal{I}_v} && \text{for all } v\in\mathcal{V}\\
& u_m \in \{0,1\} && \text{for all } m\in\mathcal{M}
\end{aligned}
$$

We regard the LP-relaxation by dropping the integrality constraints:

$$
\begin{aligned}
\min \quad & \sum_{v\in\mathcal{V}} G^v\lambda^v + \sum_{m\in\mathcal{M}} u_m c^{\mathrm{r}}_m && && \text{(LMP)}\\
\text{s.t.} \quad & \sum_{v\in\mathcal{V}} Y^v_{t,\cdot}\lambda^v = u_m && \text{for all } m\in\mathcal{M}, t\in m\\
& \sum_{m\in C^{-1}(c)} u_m = 1 && \text{for all } c\in\mathcal{C}\\
& \sum_{i\in\mathcal{I}_v} \lambda^v_i = 1 && \text{for all } v\in\mathcal{V}\\
& \lambda^v \in \mathbb{R}^{\mathcal{I}_v}_{\geq 0} && \text{for all } v\in\mathcal{V}\\
& u_m \geq 0 && \text{for all } m\in\mathcal{M}
\end{aligned}
$$

We reduce the size by considering only subsets $\mathcal{J}_v \subset \mathcal{I}_v$ and formulate the relaxed restricted master problem:

$$
\begin{aligned}
\min \quad & \sum_{v\in\mathcal{V}} G^v_{\mathcal{J}_v}\lambda^v + \sum_{m\in\mathcal{M}} u_m c^{\mathrm{r}}_m && && \text{(LRMP)}\\
\text{s.t.} \quad & \sum_{v\in\mathcal{V}} Y^v_{t,\mathcal{J}_v}\lambda^v = u_m && \text{for all } m\in\mathcal{M}, t\in m\\
& \sum_{m\in C^{-1}(c)} u_m = 1 && \text{for all } c\in\mathcal{C}\\
& \sum_{i\in\mathcal{J}_v} \lambda^v_i = 1 && \text{for all } v\in\mathcal{V}\\
& \lambda^v \in \mathbb{R}^{\mathcal{J}_v}_{\geq 0} && \text{for all } v\in\mathcal{V}\\
& u_m \geq 0 && \text{for all } m\in\mathcal{M}
\end{aligned}
$$

For the dual relaxed restricted master problem, we introduce dual variables $\gamma \in \mathbb{R}^{\mathcal{T}_{\text{car}}}$, $\mu \in \mathbb{R}^{\mathcal{V}}$ and $\eta \in \mathbb{R}^{\mathcal{C}}$. The dual problem is:

$$
\begin{aligned}
\max \quad & \sum_{c \in \mathcal{C}} \eta_c + \sum_{v \in \mathcal{V}} \mu_v && \text{(DLRMP)} \\
\text{s.t.} \quad & \sum_{t \in \mathcal{T}_{\text{car}}} Y^v_{t,i} \gamma_t + \mu_v \leq G^v_i && \text{for all } v \in \mathcal{V}, i \in \mathcal{J}_v \\
& \eta_{C(m)} - \sum_{t \in m} \gamma_t \leq c^{\text{r}}_m && \text{for all } m \in \mathcal{M} \\
& \gamma \in \mathbb{R}^{\mathcal{T}_{\text{car}}} \\
& \mu \in \mathbb{R}^{\mathcal{V}} \\
& \eta \in \mathbb{R}^{\mathcal{C}}
\end{aligned}
$$

5.3 Solving the Relaxed Master Problem

Let $(\gamma^*, \mu^*, \eta^*)$ be a solution of (DLRMP) with $\mathcal{J}_v \subset \mathcal{I}_v$ for all $v \in \mathcal{V}$. We want to find out whether $(\gamma^*, \mu^*, \eta^*)$ corresponds to an optimal solution of (LMP). This is the case if it is feasible for the dual relaxed master problem, i.e. the following constraints hold for the entire set $\mathcal{I}_v$. This means, the following equation holds for $(\gamma^*, \mu^*, \eta^*)$:

$$
\sum_{t \in \mathcal{T}_{\text{car}}} Y^v_{t,i} \gamma^*_t + \mu^*_v \leq G^v_i \qquad \text{for all } v \in \mathcal{V}, i \in \mathcal{I}_v \tag{5.5}
$$

In order to find an optimal solution of (LMP) we have to find indices $i \in \mathcal{I}_v$ where the previous constraints are violated. This leads to the following subproblem:

$$
\text{Find } i \in \mathcal{I}_v \backslash \mathcal{J}_v \text{ s.t.} \qquad \sum_{t \in \mathcal{T}_{\text{car}}} Y^v_{t,i} \gamma^*_t + \mu^*_v > G^v_i \qquad \text{for } v \in \mathcal{V}
$$

We call this problem vehicle subproblem (SP^{v}_v).

Vehicle Subproblem

The vehicle subproblem for finding violated constraints (5.5) reads for $v \in \mathcal{V}$ as follows:

$$
\begin{aligned}
\min \quad & g\left(x^v, z^v, e^v\right) - \sum_{t \in \mathcal{T}_{\text{car}}} \sum_{s \in \mathrm{N}^-_G(t)} x^v_{s,t} \gamma^*_t && (\text{SP}^{\text{v}}_v) \\
\text{s.t.} \quad & (x^v, z^v, e^v) \in X_v
\end{aligned}
$$

The constraints (5.1) and (5.2) ensure, that exactly vehicle v is used and the others are not used. The subproblem is equivalent to the Shortest Path Problem with Resource Constraints (SPPRC). In (Kaiser, cap. 7) there is provided a way to solve the (SPPRC) efficiently.
Let $(\bar{x}^v, \bar{z}^v, \bar{e}^v)$ be an optimal solution of $(\mathrm{SP}_v^{\mathrm{v}})$. If $\mathrm{val}\,(\bar{x}^v, \bar{z}^v, \bar{e}^v) < \mu_v^*$ then add this to $\mathcal{J}_v$ and repeat the master problem.

5.4 Master Problem with Route Choice Restriction

Remember the formulation from before:

$$\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} \min g\left(\psi^{-1}\left(y^v\right) \cap X_v\right) + \sum_{m \in \mathcal{M}} u_m c_m^{\mathrm{r}} & & \\
\text{s.t.} \quad & \sum_{m \in C^{-1}(c)} u_m = 1 & & \text{for all } c \in \mathcal{C} \qquad (3.1) \\
& \sum_{v \in \mathcal{V}} y_t^v = u_m & & \text{for all } m \in \mathcal{M}, t \in m \\
& y^v \in \psi\left(X_v\right) & & \text{for all } v \in \mathcal{V} \\
& u_m \in \{0,1\} & & \text{for all } m \in \mathcal{M}
\end{aligned}$$

The constraints (3.1) depend only on u_m. Therefore, we create another subproblem for the choice of routes. We define the set of feasible route choices as follows:

$$\hat{X} := \left\{ \{0,1\}^{\mathcal{M}} | \sum_{m \in C^{-1}(c)} u_m = 1 \text{ for all } c \in \mathcal{C} \right\}$$

We introduce variable $\hat{u}$ and the respective route cost function $\hat{g}$ and rewrite (MMILP) again:

$$\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} \min g\left(\psi^{-1}\left(y^v\right) \cap X_v\right) + \hat{g}\left(\hat{u}\right) & & \\
\text{s.t.} \quad & \sum_{v \in \mathcal{V}} y_t^v = u_m & & \text{for all } m \in \mathcal{M}, t \in m \\
& y^v \in \psi\left(X_v\right) & & \text{for all } v \in \mathcal{V} \\
& \hat{u} \in \hat{X} & &
\end{aligned}$$

Column Generation

For every $v \in \mathcal{V}$, let $\mathcal{I}_v$ be an index set for the finitely many points in $\psi(X_v)$ and let the columns of $Y^v \in \mathbb{R}^{\mathcal{T}_{\text{car}} \times \mathcal{I}_v}$ be exactly those points. Let $\hat{\mathcal{I}}$ be an index set for the finitely many points in $\hat{X}$ and let the columns of $\hat{Y} \in \mathbb{R}^{\mathcal{M} \times \hat{\mathcal{I}}}$ be exactly those points. Let $G^v \in \mathbb{R}^{1 \times \mathcal{I}}$ be the respective values of $\min g\left(\psi^{-1}(\cdot) \cap X_v\right)$ and $\hat{G} \in \mathbb{R}^{1 \times \hat{\mathcal{I}}}$ be the respective route costs. Then we can reformulate the master problem as

$$
\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} G^v \lambda^v + \hat{G}\hat{\lambda} && && \text{(IMMP)}\\
\text{s.t.} \quad & \sum_{v \in \mathcal{V}} Y^v_{t,\cdot} \lambda^v = \hat{Y}_{m,\cdot} \hat{\lambda} && \text{for all } m \in \mathcal{M}, t \in m\\
& \sum_{i \in \mathcal{I}_v} \lambda^v_i = 1 && \text{for all } v \in \mathcal{V}\\
& \sum_{i \in \hat{\mathcal{I}}} \hat{\lambda}_i = 1\\
& \lambda^v \in \{0,1\}^{\mathcal{I}_v} && \text{for all } v \in \mathcal{V}\\
& \hat{\lambda} \in \{0,1\}^{\hat{\mathcal{I}}}
\end{aligned}
$$

We regard the LP-relaxation by dropping the integrality constraints:

$$
\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}} G^v \lambda^v + \hat{G}\hat{\lambda} && && \text{(LMMP)}\\
\text{s.t.} \quad & \sum_{v \in \mathcal{V}} Y^v_{t,\cdot} \lambda^v = \hat{Y}_{m,\cdot} \hat{\lambda} && \text{for all } m \in \mathcal{M}, t \in m\\
& \sum_{i \in \mathcal{I}_v} \lambda^v_i = 1 && \text{for all } v \in \mathcal{V}\\
& \sum_{i \in \hat{\mathcal{I}}} \hat{\lambda}_i = 1\\
& \lambda^v \in \mathbb{R}^{\mathcal{I}_v}_{\geq 0} && \text{for all } v \in \mathcal{V}\\
& \hat{\lambda} \in \mathbb{R}^{\hat{\mathcal{I}}}_{\geq 0}
\end{aligned}
$$

We reduce the size by considering only subsets $\mathcal{J}_v \subset \mathcal{I}_v$ and $\hat{\mathcal{J}} \subset \hat{\mathcal{I}}$ and formulate the relaxed restricted master problem:

$$
\begin{aligned}
\min \quad & \sum_{v\in\mathcal{V}} G^v_{\mathcal{J}_v}\lambda^v + \hat{G}_{\hat{\mathcal{J}}}\hat{\lambda} && && \text{(LRMMP)}\\
\text{s.t.} \quad & \sum_{v\in\mathcal{V}} Y^v_{t,\mathcal{J}_v}\lambda^v = \hat{Y}_{m,\hat{\mathcal{J}}}\hat{\lambda} && \text{for all } m\in\mathcal{M}, t\in m\\
& \sum_{i\in\mathcal{J}_v} \lambda^v_i = 1 && \text{for all } v\in\mathcal{V}\\
& \sum_{i\in\hat{\mathcal{J}}} \hat{\lambda}_i = 1\\
& \lambda^v \in \mathbb{R}^{\mathcal{J}_v}_{\geq 0} && \text{for all } v\in\mathcal{V}\\
& \hat{\lambda} \in \mathbb{R}^{\hat{\mathcal{J}}}_{\geq 0}
\end{aligned}
$$

For the dual relaxed restricted master problem, we introduce dual variables $\gamma \in \mathbb{R}^{\mathcal{T}_{\mathrm{car}}}$, $\mu \in \mathbb{R}^{\mathcal{V}}$ and $\alpha \in \mathbb{R}$. The dual problem is:

$$
\begin{aligned}
\max \quad & \sum_{v\in\mathcal{V}} \mu_v + \alpha && && \text{(DLRMMP)}\\
\text{s.t.} \quad & \sum_{t\in\mathcal{T}_{\mathrm{car}}} Y^v_{t,i}\gamma_t + \mu_v \leq G^v_i && \text{for all } v\in\mathcal{V}, i\in\mathcal{J}_v\\
& \alpha - \sum_{m\in\mathcal{M}}\sum_{t\in m} \hat{Y}_{m,i}\gamma_t \leq \hat{G}_i && \text{for all } i\in\hat{\mathcal{J}}\\
& \gamma \in \mathbb{R}^{\mathcal{T}_{\mathrm{car}}}\\
& \mu \in \mathbb{R}^{\mathcal{V}}\\
& \alpha \in \mathbb{R}
\end{aligned}
$$

5.5 Solving the Relaxed Master Problem

Let $(\gamma^*, \mu^*, \alpha^*)$ be a solution of (DLRMMP) with $\mathcal{J}_v \subset \mathcal{I}_v$ for all $v \in \mathcal{V}$ and $\hat{\mathcal{J}} \subset \hat{\mathcal{I}}$. We want to find out whether $(\gamma^*, \mu^*, \alpha^*)$ corresponds to an optimal solution of (LMMP). This is the case if it is feasible for the dual relaxed master problem, i.e. the following constraints hold for the entire sets $\mathcal{I}_v$ and $\hat{\mathcal{I}}$. This means, the following equations hold for $(\gamma^*, \mu^*, \alpha^*)$:

$$
\sum_{t\in\mathcal{T}_{\mathrm{car}}} Y^v_{t,i}\gamma^* + \mu^*_v \leq G^v_i \qquad \text{for all } v\in\mathcal{V}, i\in\mathcal{I}_v \tag{5.6}
$$

$$
\alpha^* - \sum_{m\in\mathcal{M}}\sum_{t\in m} \hat{Y}_{m,i}\gamma^*_t \leq \hat{G}_i \qquad \text{for all } i\in\hat{\mathcal{I}} \tag{5.7}
$$

In order to find an optimal solution of (LMMP) we have to find indices $i \in \mathcal{I}_v$ or $j \in \hat{\mathcal{I}}$ where the previous constraints are violated. This leads to the following subproblems:

$$\begin{aligned}
&\text{Find } i \in \mathcal{I}_v \backslash \mathcal{J}_v \text{ s.t.} && \sum_{t \in \mathcal{T}_{\text{car}}} Y^v_{t,i} \gamma^*_t + \mu^*_v > G^v_i && \text{for } v \in \mathcal{V} \\
&\text{Find } i \in \hat{\mathcal{I}} \backslash \hat{\mathcal{J}} \text{ s.t.} && \alpha^* - \sum_{m \in \mathcal{M}} \sum_{t \in m} \hat{Y}_{m,i} \gamma^*_t > \hat{G}_i &&
\end{aligned}$$

The vehicle subproblem ($\mathrm{SP}^{\mathrm{v}}_v$) was already considered before. For the route choice, an additional subproblem arises.

Route Subproblem

The route subproblem for finding violated constraints (5.7) reads as follows:

$$\begin{aligned}
\min \quad & \sum_{m \in \mathcal{M}} u_m \left(c^{\mathrm{r}}_m + \sum_{t \in m} \gamma^*_t \right) && && (\mathrm{SP}^{\mathrm{m}}) \\
\text{s.t.} \quad & \sum_{m \in C^{-1}(c)} u_m = 1 && \text{for all } c \in \mathcal{C} \\
& u_m \in \{0, 1\} && \text{for all } m \in \mathcal{M}
\end{aligned}$$

This problem is easy to solve: For every $c \in \mathcal{C}$ choose the multimodal route $m \in C^{-1}(c)$ with the smallest cost $c^{\mathrm{r}}_m + \sum_{t \in m} \gamma^*_t$.
Let $\bar{u}$ be an optimal solution of (SP^{m}). If $\operatorname{val}(\bar{u}) < \alpha^*$ then add this to $\hat{\mathcal{J}}$ and continue the master problem.

Chapter 6

Instance Creation

6.1 Route Creation

We are not given the set of routes $\mathcal{M}$ in advance. For each costumer $c \in \mathcal{C}$, we have start and end location $p_c^{\text{start}}, p_c^{\text{end}}$ and a start and end time $\hat{z}_c^{\text{start}}, \hat{z}_c^{\text{end}}$. All the trips of the costumer lie in this interval, i.e.

$$\hat{z}_c^{\text{start}} \leq z_m^{\text{start}} \qquad z_m^{\text{end}} \leq \hat{z}_c^{\text{end}} \qquad \text{for all } m \in C^{-1}(c).$$

Basic Restrictions

To simplify the creation of the routes, we make some assumptions. For every route $m \in \mathcal{M}$ holds:

- There are not two car trips in a row.
- There is no car trip between two public transport trips.
- The number of public transport trips is restricted. Usually, one can reach every station with at most two changes.
- We define a walking distance d^{walk}. If the distance between the start position and the first station or between the last station and the end position, there is no car trip necessary.

We assume that we have some oracle that provides the set of feasible public transport routes for costumer $c \in \mathcal{C}$:

$$\begin{aligned} M_c = \quad & \Big\{ (s_1, z_1, s_2, z_2) \,|\, s_1, s_2 \in \mathcal{S}, \hat{z}_c^{\text{start}} \leq t_1 < t_2 \leq \hat{z}_c^{\text{end}}, \text{ there is a public} \\ & \text{transport route from } s_1 \text{ to } s_2 \text{ with start time } z_1 \text{ and end time } z_2 \Big\} \end{aligned}$$

The fact, whether the costumer changes during his usage of public transport, has no effect on the model. Thus, we can consider each element in M_c as a public transport trip.

Route Creation

We create the set of multimodal routes $\mathcal{M}$. For this, we set a car trip before and after each public transport trip in order to bring the costumer from his start to his destination, except when it is possible to walk the distance. We also have to consider the given time restrictions. Further, we create the pure car trips. How the set $\mathcal{M}$ is created in detail, is described in algorithm 2.
Until now, we do not consider any changing times between a car trip and a public transport trip.
Further, we assume that the given costumer start and and times are feasible, i.e. $\hat{z}_c^{\text{start}} + t_{p_c^{\text{start}}, p_c^{\text{end}}} \leq \hat{z}_c^{\text{end}}$ for all $c \in \mathcal{C}$.

Further Restrictions

If the routes are created as described in algorithm 2, there are routes using every available station as long as it is feasible. Most of these routes are obviously bad for the costumer since they cause a big detour. What is more, a large number of routes enlarge the problem size and leads to a bad performance for solving it. Therefore, we try to restrict the set of alternatives to a reasonable size.

Example 2. Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ with a single public transport ride serving all stations. Let $\mathcal{C} = \{c_1, c_2\}$ with $p_{c_1}^{\text{end}} = s_n$ and $p_{c_2}^{\text{start}} = s_k$ for a certain $k \in [n-1]$. The alternative routes are

$$\mathcal{M} = \underbrace{\left\{\left(\left(p_{c_1}^{\text{start}}, s_i\right), (s_i, s_n)\right) | i \in [n-1]\right\}}_{\text{for } c_1} \cup \underbrace{\left\{\left(s_k, p_{c_2}^{\text{end}}\right)\right\}}_{\text{for } c_2}$$

with $(p_{c_1}^{\text{start}}, s_k) \prec \left(s_k, p_{c_2}^{\text{end}}\right)$ and $(p_{c_1}^{\text{start}}, s_i) \not\prec \left(s_k, p_{c_2}^{\text{end}}\right)$ for all $i \in [n] \backslash \{k\}$.
We get the only solution, where only one car is needed, when c_1 drives to s_k, wherever the station s_k is. Every route of c_1 can be the optimal route, considering the other costumers. Therefore, an exact reduction of $\mathcal{M}$ is not possible without the the risk of cutting off the optimal solution.

It is not practicable to consider all possible multimodal routes due to computation reasons. But it is also not possible to reduce the number of routes without risking to lose the optimal solution. Hence, we try to make reasonable restrictions which keep the problem size small.

Pareto Optimality

Algorithm 2: Creation of the routes

Input: costumer set $\mathcal{C}$; $p_c^{\text{start}}, p_c^{\text{end}}, \hat{z}_c^{\text{start}}, \hat{z}_c^{\text{end}}, M_c$ for all $c \in \mathcal{C}$
Output: set of routes $\mathcal{M}$, set of trips $\mathcal{T}_{\text{car}}, \mathcal{T}_{\text{public}}$

$\mathcal{T}_{\text{car}} \leftarrow \emptyset$;
$\mathcal{T}_{\text{public}} \leftarrow \emptyset$;
$\mathcal{M} \leftarrow \emptyset$;
foreach $c \in \mathcal{C}$ **do**
    foreach $(s_1, z_1, s_2, z_2) \in M_c$ **do**
        create public transport trip t;
        $p_t^{\text{start}} \leftarrow s_1, p_t^{\text{end}} \leftarrow s_2, z_t^{\text{start}} \leftarrow z_1, z_t^{\text{end}} \leftarrow z_2$;
        create car trips t_1, t_2;
        $p_{t_1}^{\text{start}} \leftarrow p_c^{\text{start}}, p_{t_1}^{\text{end}} \leftarrow s_1, z_{t_1}^{\text{start}} \leftarrow z_1 - t_{p_c^{\text{start}}, s_1}, z_{t_1}^{\text{end}} \leftarrow z_1$;
        $p_{t_2}^{\text{start}} \leftarrow s_2, p_{t_2}^{\text{end}} \leftarrow p_c^{\text{end}}, z_{t_2}^{\text{start}} \leftarrow z_2, z_{t_2}^{\text{end}} \leftarrow z_2 + t_{s_2, p_c^{\text{end}}}$;
        if $\hat{z}_c^{\text{start}} \leq z_{t_1}^{\text{start}} \wedge z_{t_2}^{\text{end}} \leq \hat{z}_c^{\text{end}}$ **then**
            create multimodal route m;
            $\mathcal{T}_{\text{public}} \leftarrow \mathcal{T}_{\text{public}} \cup \{t\}$;
            if $d_{p_c^{\text{start}}, s_1} \geq d^{\text{walk}}$ **then** $m \leftarrow (t_1, t)$; $\mathcal{T}_{\text{car}} \leftarrow \mathcal{T}_{\text{car}} \cup \{t_1\}$;
            else $m \leftarrow (t)$;
            if $d_{s_2, p_c^{\text{end}}} \geq d^{\text{walk}}$ **then** append t_2 to m; $\mathcal{T}_{\text{car}} \leftarrow \mathcal{T}_{\text{car}} \cup \{t_2\}$;
            $C(m) \leftarrow c$;
            $\mathcal{M} \leftarrow \mathcal{M} \cup \{m\}$;
        end
    end
    create car trips t_1, t_2;
    $p_{t_1}^{\text{start}} \leftarrow p_c^{\text{start}}, p_{t_1}^{\text{end}} \leftarrow p_c^{\text{end}}, z_{t_1}^{\text{start}} \leftarrow \hat{z}_c^{\text{start}}, z_{t_1}^{\text{end}} \leftarrow \hat{z}_c^{\text{start}} + t_{p_c^{\text{start}}, p_c^{\text{end}}}$;
    $p_{t_2}^{\text{start}} \leftarrow p_c^{\text{start}}, p_{t_2}^{\text{end}} \leftarrow p_c^{\text{end}}, z_{t_2}^{\text{start}} \leftarrow \hat{z}_c^{\text{end}} - t_{p_c^{\text{start}}, p_c^{\text{end}}}, z_{t_2}^{\text{end}} \leftarrow \hat{z}_c^{\text{end}}$;
    create multimodal routes m_1, m_2;
    $m_1 \leftarrow (t_1), m_2 \leftarrow (t_2)$;
    $\mathcal{T}_{\text{car}} \leftarrow \mathcal{T}_{\text{car}} \cup \{t_1, t_2\}, \mathcal{M} \leftarrow \mathcal{M} \cup \{m_1, m_2\}$;
end
return $\mathcal{M}, \mathcal{T}_{\text{car}}, \mathcal{T}_{\text{public}}$

The idea is to choose only Pareto optimal multimodal routes (cf. Kaiser/Knoll, cap. 3.2.2) in order to determine good routes.

Definition 11 (Pareto optimality)**.** Let $V \subset \mathbb{R}^n$.

1. The partial order $\leq$ on $\mathbb{R}^n$ is given by

$$v \leq w \qquad :\Leftrightarrow \qquad v_i \leq w_i \qquad \forall i \in [n] \qquad \text{for all } v, w \in \mathbb{R}^n$$

2. An element $w \in V$ is Pareto optimal in V if it is minimal with respect to $\leq$ in V, i.e.

$$v \leq w \qquad \Rightarrow \qquad v = w \qquad \text{for all } v \in V$$

3. The Pareto frontier of V with respect to $\leq$ is the set of Pareto optimal elements in V, i.e.

$$\min_{\leq} V := \{w \in V | \forall v \in V : v \leq w \Rightarrow v = w\}$$

Let $m \in \mathcal{M}$ be a multimodal route. We define

$$\varphi : \mathcal{M} \to \mathbb{R}^5 \qquad m \mapsto \begin{pmatrix} c^{\mathrm{r}} + \sum_{t \in m \cap \mathcal{T}_{\mathrm{car}}} c_t^{\mathrm{t}} \\ c^{\mathrm{r}} \\ |\mathcal{T}_{\mathrm{car}} \cap \{t \in m\}| \\ \sum_{t \in m \cap \mathcal{T}_{\mathrm{car}}} z_t^{\mathrm{end}} - z_t^{\mathrm{start}} \\ \sum_{t \in m \cap \mathcal{T}_{\mathrm{car}}} f_t^{\mathrm{t}} \end{pmatrix}$$

The function φ grades a route to their costs, their route costs, the number of cars needed, the time of a car needed and the fuel consumption.
From now on, we will use the Pareto frontier of $\varphi(\mathcal{M})$ as a restricted route set:

$$\hat{\mathcal{M}} := min_{\leq} \varphi(\mathcal{M}) \tag{6.1}$$

Previous Formulations

1 (MILP)

$$
\begin{aligned}
\min \sum_{s\in\mathcal{V}} & \sum_{t\in \mathrm{N}^+_G(s)\setminus\{d^{\mathrm{e}}\}} x_{s,t} c^{\mathrm{v}} \\
&+ \sum_{t\in\mathcal{T}} \sum_{s\in \mathrm{N}^-_G(t)} \left[x_{s,t}\left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_{t}\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t}\right)\right] && \text{(MILP)} \\
\text{s.t.} & \sum_{t\in \mathrm{N}^-_G(s)} x_{t,s} = \sum_{t\in \mathrm{N}^+_G(s)} x_{s,t} && \text{for all } s \in V\setminus\{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (1)\\
& \sum_{s\in \mathrm{N}^-_G(t)} x_{s,t} = 1 && \text{for all } t\in\mathcal{V} && (2)\\
& \sum_{t\in C^{-1}(c)} \sum_{s\in \mathrm{N}^-_G(t)} x_{s,t} = 1 && \text{for all } c\in\mathcal{C} && (3)\\
& \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t) && (4)\\
& e_s \le f^0_s && \text{for all } s\in\mathcal{V} && (5)\\
& 0 \le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t) && (6)\\
& e_t \le 1 - f^{\mathrm{t}}_t - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t) && (7)\\
& e_t \le e_s - x_{s,t}\left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_t\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_r + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t}\right) + (1 - x_{s,t}) \\
& && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t) && (8)\\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t)\in A && (9)\\
& z_{s,r,t} \in \{0,1\} && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}^-_G(t), r\in\mathcal{R}_{s,t} && (10)\\
& e_s \in [0,1] && \text{for all } s\in V\setminus\{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (11)
\end{aligned}
$$

2 (AMILP)

$$\begin{aligned}
\min \sum_{s\in\mathcal{V}} \sum_{t\in \mathrm{N}_G^+(s)\setminus\{d^{\mathrm{e}}\}} & x_{s,t}c^{\mathrm{v}} \\
+\sum_{t\in\mathcal{T}} \sum_{s\in \mathrm{N}_G^-(t)} & \left[x_{s,t}\left(c^{\mathrm{d}}_{s,t}+c^{\mathrm{t}}_t\right)+\sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c^{\mathrm{d}}_{s,r}+c^{\mathrm{d}}_{r,t}-c^{\mathrm{d}}_{s,t}\right)\right] && \text{(AMILP)}\\
\text{s.t.} \sum_{t\in \mathrm{N}_G^-(s)} x_{t,s} = \sum_{t\in \mathrm{N}_G^+(s)} x_{s,t} & && \text{for all } s\in V\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} && (1)\\
\sum_{s\in \mathrm{N}_G^-(t)} x_{s,t} = 1 & && \text{for all } t\in\mathcal{T}\cup\mathcal{V} && (12)\\
\sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} & && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t) && (4)\\
e_s \le f^0_s & && \text{for all } s\in\mathcal{V} && (5)\\
0 \le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}f^{\mathrm{d}}_{s,r} & && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t) && (6)\\
e_t \le 1 - f^{\mathrm{t}}_t - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}f^{\mathrm{d}}_{r,t} & && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t) && (7)\\
e_t \le e_s - x_{s,t}\left(f^{\mathrm{d}}_{s,t}+f^{\mathrm{t}}_t\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} & \left(f^{\mathrm{d}}_{s,r}+f^{\mathrm{t}}_r+f^{\mathrm{d}}_{r,t}-f^{\mathrm{d}}_{s,t}\right)+(1-x_{s,t})\\
& && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t) && (8)\\
x_{s,t}\in\{0,1\} & && \text{for all } (s,t)\in A && (9)\\
z_{s,r,t}\in\{0,1\} & && \text{for all } t\in\mathcal{T}, s\in \mathrm{N}_G^-(t), r\in\mathcal{R}_{s,t} && (10)\\
e_s\in[0,1] & && \text{for all } s\in V\setminus\{d^{\mathrm{s}},d^{\mathrm{e}}\} && (11)
\end{aligned}$$

3 (LMILP)

$$
\begin{aligned}
\min & \sum_{s\in\mathcal{V}} \sum_{t\in \mathrm{N}^{+}_{\overline{G}}(s)\setminus\{d^{\mathrm{e}}\}} x_{s,t} c^{\mathrm{v}} + \sum_{s\in P} \sum_{t\in \mathrm{N}^{+}_{\overline{G}}(s)} x_{s,t} c^{\mathrm{t}}_{t} \\
& + \sum_{t\in\mathcal{T}\cup P} \sum_{s\in \mathrm{N}^{-}_{\overline{G}}(t)} \left[x_{s,t}\left(c^{\mathrm{d}}_{s,t} + c^{\mathrm{t}}_{t}\right) + \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(c^{\mathrm{d}}_{s,r} + c^{\mathrm{d}}_{r,t} - c^{\mathrm{d}}_{s,t}\right)\right] && \text{(LMILP)}\\
\text{s.t.} & \sum_{t\in \mathrm{N}^{-}_{\overline{G}}(s)} x_{t,s} = \sum_{t\in \mathrm{N}^{+}_{\overline{G}}(s)} x_{s,t} && \text{for all } s\in\overline{V}\setminus\{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (13)\\
& \sum_{s\in \mathrm{N}^{-}_{\overline{G}}(t)} x_{s,t} = 1 && \text{for all } t\in\mathcal{T}\cup\mathcal{V} && (14)\\
& \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} \le x_{s,t} && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P && (15)\\
& e_s \le f^{0}_{s} && \text{for all } s\in\mathcal{V} && (16)\\
& 0 \le e_s - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{s,r} && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P && (17)\\
& e_t \le 1 - f^{\mathrm{t}}_{t} - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t} f^{\mathrm{d}}_{r,t} && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P && (18)\\
& e_t \le e_s - x_{s,t}\left(f^{\mathrm{d}}_{s,t} + f^{\mathrm{t}}_{t}\right) - \sum_{r\in\mathcal{R}_{s,t}} z_{s,r,t}\left(f^{\mathrm{d}}_{s,r} + f^{\mathrm{t}}_{r} + f^{\mathrm{d}}_{r,t} - f^{\mathrm{d}}_{s,t}\right) + (1 - x_{s,t}) \\
& && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P && (19)\\
& e_t \le e_s - x_{s,t} f^{\mathrm{t}}_{t} + (1 - x_{s,t}) && \text{for all } s\in P, t\in \mathrm{N}^{+}_{\overline{G}}(s) && (20)\\
& x_{s,t} \in \{0,1\} && \text{for all } (s,t)\in\overline{A} && (21)\\
& z_{s,r,t} \in \{0,1\} && \text{for all } t\in\mathcal{T}\cup P, s\in \mathrm{N}^{-}_{\overline{G}}(t)\setminus P, r\in\mathcal{R}_{s,t} && (22)\\
& e_s \in [0,1] && \text{for all } s\in\overline{V}\setminus\{d^{\mathrm{s}}, d^{\mathrm{e}}\} && (23)
\end{aligned}
$$

Bibliography

[BCG87] A. A. Bertossi, P. Carraresi, and G. Gallo. On some matching problems arising in vehicle scheduling models. *Networks*, 17.3:271–281, 1987.

[BK09] Stefan Bunte and Natalie Kliewer. An overview on vehicle scheduling models. *Public Transport*, 1.4:299–317, 2009.

[CL07] J.-F. Cordeau and G. Laporte. The dial-a-ride problem: Models and algorithms. *Annals of Operations Research*, 153.1:29–46, 2007.

[DF54] G. B. Dantzig and D. R. Fulkerson. Minimizing the number of tankers to meet a fixed schedule. *Naval Research Logistics Quarterly*, 1.3:217–222, 1954.

[DP95] J. Daduna and J. Pinto Paixão. Vehicle scheduling for public mass transit - an overview. *Computer-Aided Transit Scheduling*, 430:76–90, 1995.

[FP95] R. Freling and J. Pinto Paixão. Vehicle scheduling with time constraint. *Computer-Aided Transit Scheduling*, 430:130–144, 1995.

[Hau15] Jan Hauser. Amerika schaltet auf autopilot. *Frankfurter Allgemeine Zeitung*, 2015.

[Kai16] Marcus Kaiser. Optimal integration of autonomous vehicles in car sharing: A decomposition approach in consideration of multimodal transport. Master's thesis, Technische Universität München, 2016.

[KK16] Marcus Kaiser and Martin Knoll. Optimal integration of autonomous vehicles in car sharing: Overview and basic models. Master's thesis, Technische Universität München, 2016. part 1.

[Kno16] Martin Knoll. Optimal integration of autonomous vehicles in car sharing: A decomposition approach and fastening heuristics. Master's thesis, Technische Universität München, 2016.

[LK81] J. K. Lenstra and A. H. G. Rinnooy Kan. Complexity of vehicle routing and scheduling problems. *Networks*, 11.2:221–227, 1981.

[Raf83] S. Raff. Routing and scheduling of vehicles and crews: The state of the art. *Computers and Operations Research*, 10.2:63–211, 1983.