Basics of non-stabilizerness

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This paper reviews some basic knowledge related the **non-stabilizerness**, or **magic**, which is an important object investigated in the area of quantum information. This paper aims to provide an intuitive understanding of this interesting concept rather than simply stacking some dull maths. This note would be updated every so often, depending on my schedule, so do not be surprised if it suddenly ends somewhere :)

I. INTRODUCTION

AhaAha.

II. NOTATIONS

This section list some notations we used through out the paper.

First of all, letter n always refers to the number of qubits, i.e. the dimension of a n-qubit quantum state is 2^n.

For a single qubit or a spin-1/2 system, we identify

$$|0\rangle \equiv |\uparrow\rangle,\ |1\rangle \equiv |\downarrow\rangle \tag{1}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ denote the state of spin up and down, respectively. In the Pauli-z basis, we have

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2}$$

Attention that if without further explanation, the matrix form of any operator occuring in this paper is written in the Pauli-z basis.

We denoete the three Pauli operators by X, Y, Z, and their matrix forms are

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3}$$

For an n-qubit state, we label the qubits from 1 to n, from left to right. For example, the quantum state $|\phi\rangle = |01\rangle$ includes two qubits, and the first qubit is in the state of $|0\rangle$, and the second qubit is in the state of $|1\rangle$.

For unitary operation, we use the subscript to signify which qubit the operator is acting on. For example, X_1 means we act the Pauli-x operator on the first qubit, and apparently, $X_1|\phi\rangle = X_1|01\rangle = |11\rangle$.

III. STABILIZER FORMALISM

A. Stabilizer

Given a quantum state $|\psi\rangle$ and a unitary operation U, we say $|\psi\rangle$ is *stabilized by* U if and only if

$$U|\psi\rangle = |\psi\rangle \tag{4}$$

The notion that $|\psi\rangle$ is stabilized means it is unchanged after the operation of U. In other words, the state is stable. Accordingly, we call U the **stabilizer** of $|\psi\rangle$. Of course, $|\psi\rangle$ is just an eigenstate of U with eigenvalue exactly 1, but as we will see later, the introduction of stabilizer is useful, since it can give us some very interesting result about the computational complexity of simulating a quantum system classically.

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As an example, we consider an EPR pair

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{5}$$

which is stabilized by both X_1X_2 and Z_1Z_2 (easy to check). Therefore, a state can have more than one stabilizer.

In addition, a stabilizer can stabilize more than one state. For example, both $|00\rangle$ and $|11\rangle$ are stabilized by Z_1Z_2.

It is easy to prove that the the multiplication of different stabilizers is still a stabilizer. Diffferent stabilizers then can constitute a group, making it natural to introduce the language of group theory.

B. Stabilizer group and Pauli group

Now we can generalize the concept of stabilizer to the case of a set of unitaries.

Given a unitary group $\mathscr{U}$, and some n-qubit state (vector) space V_s, we say V_s is *stabilized by* $\mathscr{U}$ if and only if

$$\forall\, U \in \mathscr{U},\ |\psi\rangle \in V_s \Rightarrow U|\psi\rangle = |\psi\rangle \tag{6}$$

Similarly, we call the group of unitaries $\mathscr{U}$ a **stabilizer group** (also abbriviated as **stabilizer**) of the state space V_s.

Equivalently (easy to check), given a unitary group $\mathscr{U}$, we can define V_s as

$$V_s := \bigcap_{U\in\mathscr{U}} \left\{ |\psi\rangle \;\middle|\; U|\psi\rangle = |\psi\rangle \right\} \tag{7}$$