Investigação Operacional
Operational Research

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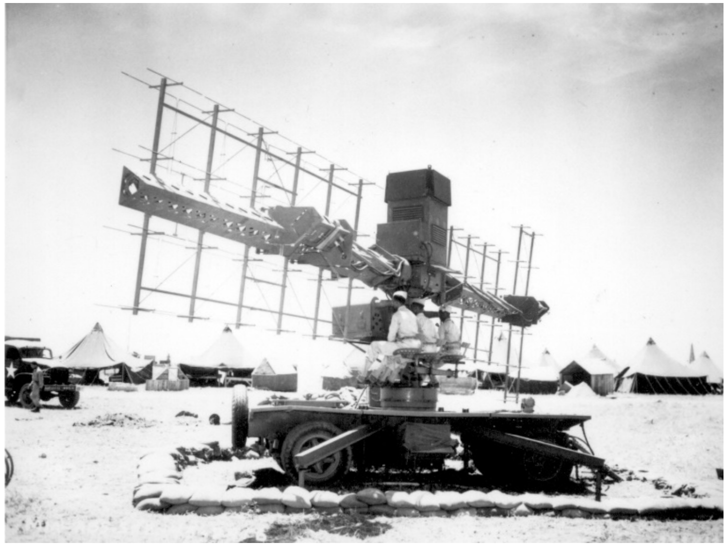
The origins of Operational Research

During 2nd World War, some military leaders asked for help to reputed scientists and mathematicians in order to analyse and solve several **military operational** problems:

- radar installation
- railways and submarine operations
- planning guard teams
- mines and bombs placing.



This field of study was initially called *Military Operations Research (!)*

Later, since these methodologies and techniques could be applied to several different areas, the name was adapted to *Operations Research* ☺

OR backgorund

1947	Project Scoop (Scientific Computation of Optimum Programs) with George Dantzig and others. It was developed Simplex Method for linear problems.
1950's	Considerable activity, with several mathematical developments, particularly in queuing theory and mathematical programming.
1960's	More and more activity, developments and ideas.
1970's	Some disappointment and less activity. Discovery of NP-complete problems. More realistic expectations.
1980's	Personal computers arising. Easier access to huge volumes of information. Managers more willing to use mathematical models.
1990's	Increase in using and developing decision support systems based on OR models. New technologies for optimization simulation and modeling languages. Connection between OR and AI techniques.

OR in the XXI century

Lots of opportunities for work and research on OR

- Data, data, data...

– Lots of business data

– Increasing need of support for decision making

- Increasing need of coordination for an efficient use of the available resources

- Transports, Mobility, Environment

- Manufacturing, Finance, Health,

- Bioinformatics: Human Genome project and all its variants and applications.

Some successful examples of application of OR

Organization	Year	Savings (per year) (US$)
• South Africa Defense Force	1997	1,1 billion

– Redesign and optimize the size and format of defense forces and their army systems

• Proctor and Gamble	1997	200 millions

– Redesign the production and distributions system in USA in order to reduce costs and increase the speed to market

• IBM	2000	750 millions

– Reengineering the global supply chain in order to attend clients more quickly, keeping the stock at the minimum.

• Continental Airlines	2003	40 millions

– Re-optimizing the crews assignment when unexpected deviations in the airline schedules occur.

Methodology of OR

One possible definition of OR:

“*OR is the scientific method applied to the decision problems context*”

A decision problem exists when:

- there is at least one decision agent (someone able to make decisions);
- There are at least two alternative lines of action to follow;
- There is at least one objective, once the decision agent chooses one line of action;
- The lines of action do not attain the same level of satisfaction for the objective(s).

Methodology of OR

The scientific method consists on the application of the following phases that overlap and interact with each other.

1. Problem definition and data gathering
2. Modeling the problem through a mathematical formulation
3. Model validation
4. Obtaining one (or more) solutions for the proposed model
5. Implementing the obtained solution or system

1. Problem definition and data gathering

- Need to:
  - study the organization and the system for which the problem appears
  - Identify the decision agents
  - Identify the main objectives of the organization (strategic, tactical, operational)
  - Select the objectives suited for the problem
  - Identify the minimal, reasonable, and ideal levels for objective satisfaction
- Need of multidisciplinary teams
- Gathering and selecting relevant information:
  - already available (databases, other systems)
  - to collect (e.g: a new database, surveys)

The outcome: a report containing a short and clear description of the problem, presenting guidelines and recommendations for its resolution. This document will evolve throughout the project being updated whenever new information is collected.

2. Modeling the problem through a mathematical formulation

A model: an idealized representation of reality

Mathematical model: a set of mathematical expressions representing the behavior of a complex system.

Choosing the most adequate model is a complex task:

- When the model is too simple, probably it will not consider some important aspects of the problem.
- When the model is too complex, it may no not be computationally tractable

A problem may be modeled in different ways, so choosing the appropriate model could be a success decision factor for the project.

It is very important to consider the availability and precision of the model input data.

3. Model validation

Model validation usually implies the implementation and execution (in a computer) of the chosen algorithm in order to guarantee that:

- ✓ input data and parameters do not contain errors
- ✓ the algorithm does not have logical errors
- ✓ The software does not have errors
- ✓ The algorithm represents correctly the model
- ✓ The results seem reasonable: sometimes, the algorithm is executed with historical data (if available) and the algorithm results are compared with the real past results.

4. Obtaining one (or more) solutions for the proposed model

We can use generic software (Excel, Lingo, CPLEX) or develop a particular algorithm for the specific problem.

In practice, the proposal of a solution involves the analysis of several solutions obtained under different conditions in order to acquire some sensibility to the data variability.

> For example, if the input data was different would the solution be affected? Why or why not? This type of questions is called <u>what-if analysis</u>.

Attention: the optimal solutions are obtained for a particular <u>model</u> !!

- they should correspond to "satisfying" solutions for the problem.
- the ideal solution may not be attainable.

It's better

to solve approximately the exact problem

than

to solve exactly the approximate problem

5. Implementing the obtained solution or system

This phase involves the implementation of the results of the study or the implementation of the algorithm as an operational tool or a software application (such as a Decision Support System).

Many OR projects successfully cross the previous phases and fail in the implementation …lots of work that will not have any effect in the organization…

In order to avoid this we have to:

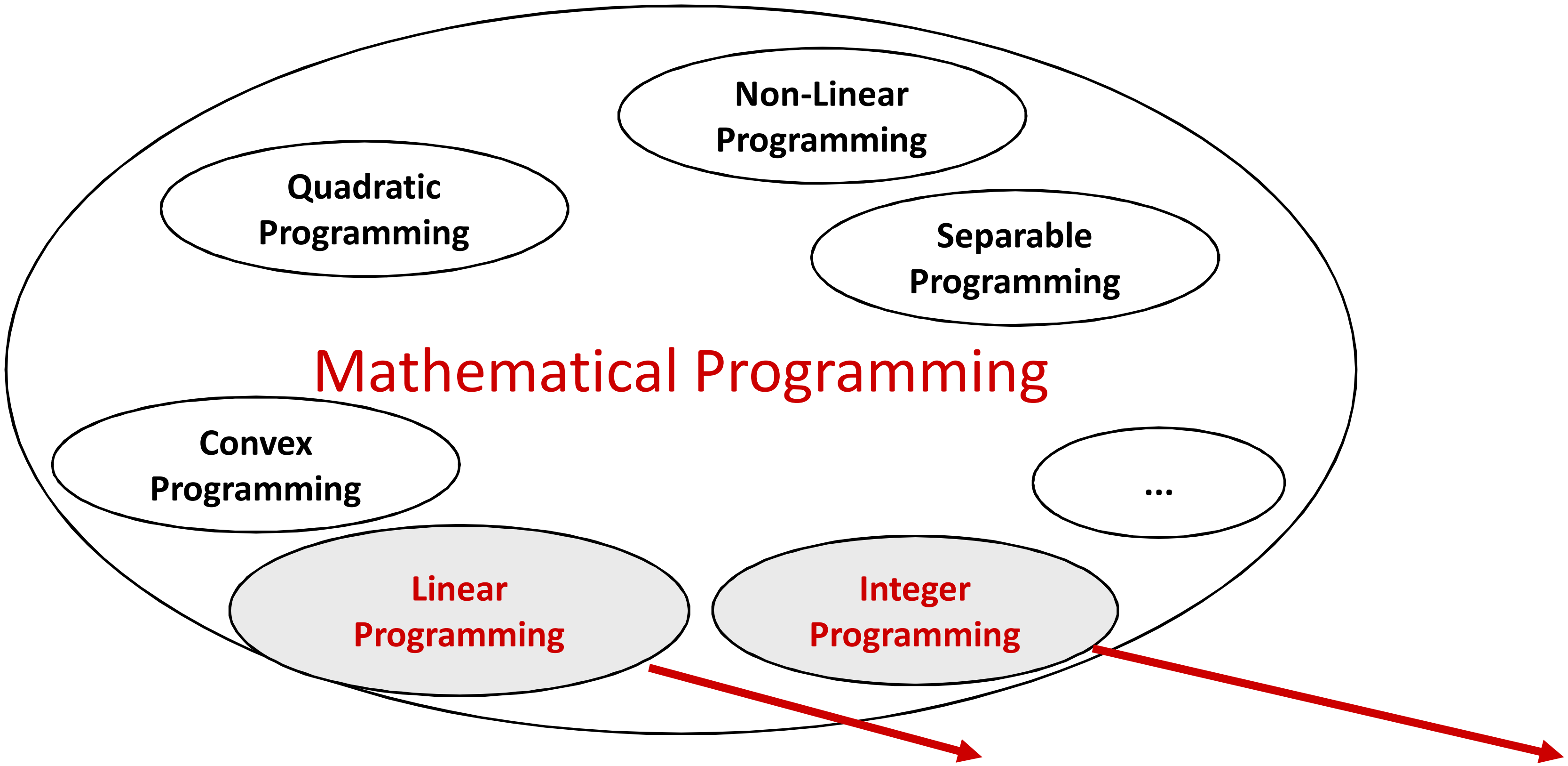
- Timely plan the implementation phase;
- Involve the client since the beginning;
- Provide adequate formation to the users;
- Provide user manuals and project documentation;
- Keep on testing and validating the proposed solutions, correcting deviations that can still occur.

10 guidelines for a good problem formulation

1. Do not create a complex model when a simple one is enough.
2. Do not fit the problem to a particular resolution technique that we want to use.
3. Solve accurately the chosen model. Only then you are able to know if some inconsistencies of the solutions provided by the model are related to the model…
4. Validate the model before implementing it.
5. The model is not the reality.
6. The model is not forced to do and it cannot be criticized by not doing what it was not meant to do.
7. Do not overestimate the models.
8. One of the main advantages of modeling is the modeling process.
9. A model is not better than the information that we used to build it.
10. Models never replace decision agents.

Some models we are going to learn in this course

This Linear Programming Problem belongs to a group of problems known as Mathematical Programming Problems, which are characterized by having a single objective and are subject to a set of constraints (which features are different for each class of problems).



In this course we will only address Linear Programming and Integer Programming

Linear Programming

- First stated in this form by **George B. Dantzig**, it is an amazing fact that literally thousands of decision (programming) problems from business, industry, government and the military can be stated (or approximated) as linear programming problems.
- Although there were some precursor attempts at stating such problems in mathematical terms, notably by the Russian mathematician **Leonid V. Kantorovich** in 1939, Dantzig's general formulation, combined with his method of solution, the simplex method, revolutionized decision making.
- The name "linear programming" was suggested to Dantzig by the economist **Tjalling C. Koopmans**.

Both Kantorovich and Koopmans were awarded the 1975 Nobel prize in economics for their contributions to the theory of optimum allocation of resources.

The untold story

- Most people familiar with the origins and development of linear programming were amazed and disappointed that **Dantzig did not receive the Nobel prize** along with Koopmans and Kantorovich (a Nobel prize can be shared by up to three recipients).
- Shortly after the award, Koopmans talked about his displeasure with the Nobel selection and told he had earlier written to Kantorovich suggesting that they both <u>refuse the prize</u>, certainly a most difficult decision for both, but especially so for Kantorovich who was not recognized in URSS….

Kantorovich said :

"In the spring of 1939 I gave some more reports – at the Polytechnic Institute and the House of Scientists, but several times met with the objection that the work used mathematical methods, and in the West the mathematical school in economics was an anti-Marxist school and mathematics in economics was a means for apologists of capitalism."

Linear Programming and the Simplex method were explained by George Dantzig in 1948 at a meeting held at the University of Wisconsin.

In the discussion after his lecture, someone from the audience said:

"Yes, but... we all know the world is <u>nonlinear</u>..."

John von Neumann, who was also there, stood up and said:

"Mr. Chairman, Mr. Chairman,

if the speaker does not mind, I would like to reply for him.

The speaker titled his talk 'linear programming' and carefully stated his axioms.

If you have an application that satisfies the axioms, well use it.

If it does not, then don't."

***John von Neumann** (1903-1957) was a Hungarian-American mathematician, physicist, inventor, computer scientist. He was a pioneer of quantum mechanics and of concepts of cellular automata, the universal constructor and the digital computer.*

After this episode, Dantzig's colleagues decided to hang this cartoon outside his office…

Top Ten Algorithms of the XXth Century

Computing in Science& Engineering, a joint publication of the American Institute of Physics and the IEEE Computer Society January/February 2000

1946 - Metropolis Algorithm for Monte Carlo

1947 - Simplex Method for Linear Programming

1950 - Krylov Subspace Iteration Methods

1951 - The Decompositional Approach to Matrix Computations

1957 - The Fortran Optimizing Compiler

1959 - QR Algorithm for Computing Eigenvalues

1962 - Quicksort Algorithm for Sorting

1965 - Fast Fourier Transform

1977 - Integer Relation Detection

1987 - Fast Multipole Method

Cereals, Ltd

Cereals, Ltd is a company specialized in preparing and packing wheat and corn for several retailing stores. There are no limits concerning the supply of these cereals and demand can be considered unlimited (in other words, the whole production is sold).
The production is processed in three phases: Pre-Processing (I), Processing (II) and Packing (III)

	I Pre-Processing	II Processing	III Packing	
Weekly production capacity (h)	120	100	150	
Time (h) needed to prepare 1 ton				Profit(€/ton)
Wheat	6 h	1 h	5 h	4
Corn	2 h	4 h	5 h	3

Currently, the company produces 18 tons of wheat and 6 tons of corn per week.
Is this the best option?

Cereals, Ltd - Formulation

Decision variables

x = tons of wheat to produce weekly

y = tons of corn to produce weekly

Constraints

$$6x + 2y \leq 120$$

$$x + 4y \leq 100$$

$$5x + 5y \leq 150$$

$$x \geq 0, y \geq 0$$

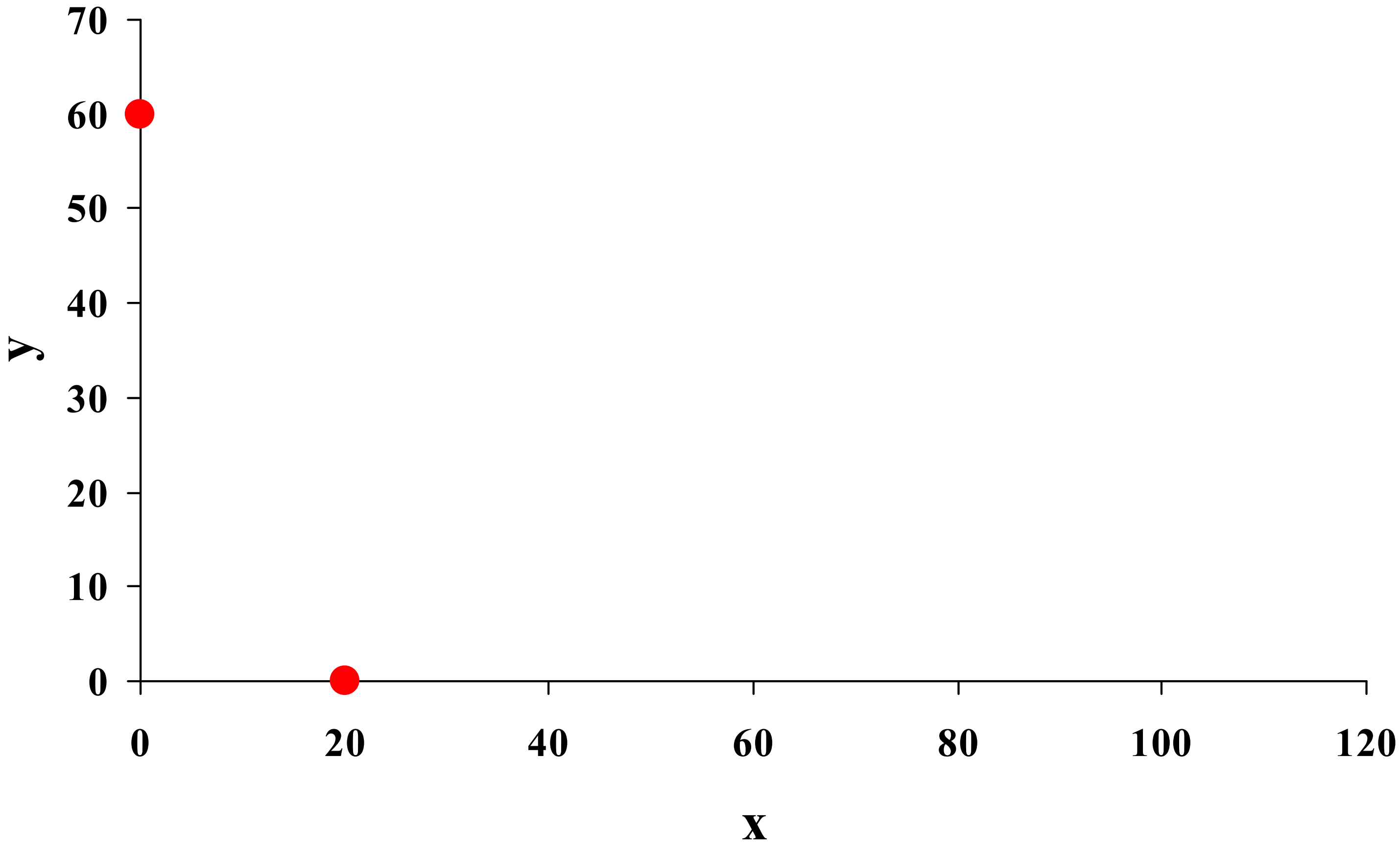
Objective function: to maximize the profit

$$\max 4x + 3y$$

$$\max\ 4x + 3y$$

s.a
$$6x + 2y \le 120$$
$$x + 4y \le 100$$
$$5x + 5y \le 150$$
$$x \ge 0,\ y \ge 0$$

Graphical method



Each constraint can be represented graphically by a plane region.
For the inequality

$$6x + 2y \le 120$$

consider the straight line corresponding to this constraint:

$$6x + 2y = 120$$

And find any two points in the line:

$$x = 0 \Rightarrow y = 60$$
$$y = 0 \Rightarrow x = 20$$

The points (0,60) and (20,0) belong to the line

$$\max\ 4x + 3y$$

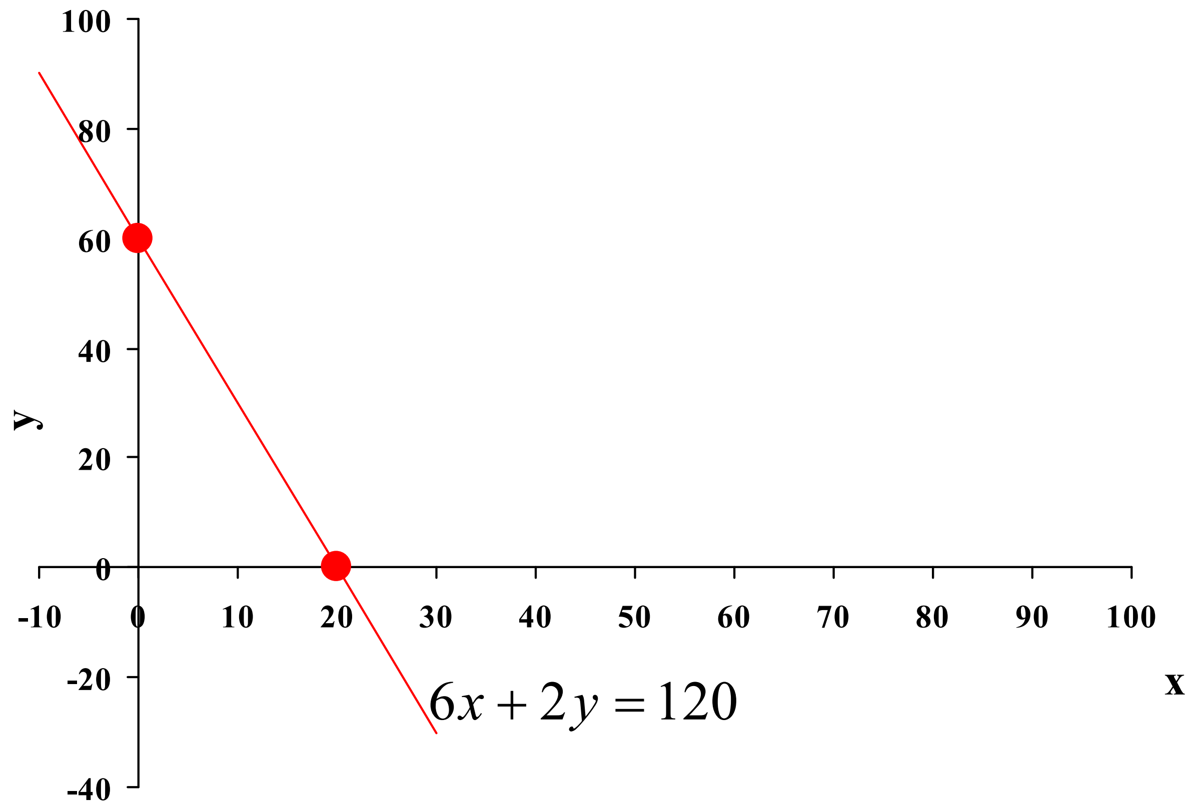
s.a $6x + 2y \le 120$

$x + 4y \le 100$

$5x + 5y \le 150$

$x \ge 0, y \ge 0$

Graphical method



The straight line $6x + 2y = 120$ divides the plane in two half-planes

Which one of them satisfies the inequality $6x + 2y \le 120$?

Consider, for example, the point (x,y)=(0,0)

Replacing it in the inequality we have $6 \times 0 + 2 \times 0 = 0 \le 120$

$$\max 4x + 3y$$

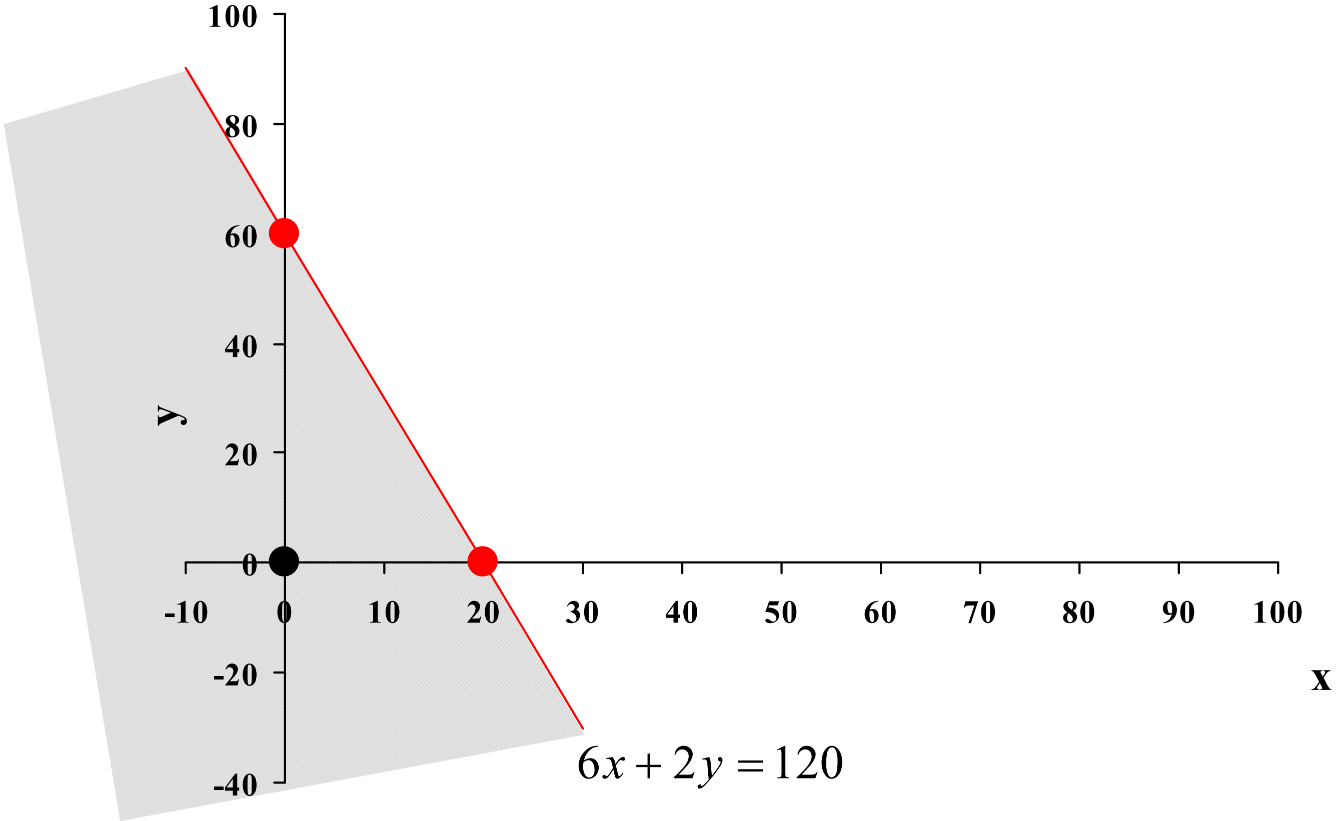
s.a $6x + 2y \leq 120$

$x + 4y \leq 100$

$5x + 5y \leq 150$

$x \geq 0, y \geq 0$

Graphical method

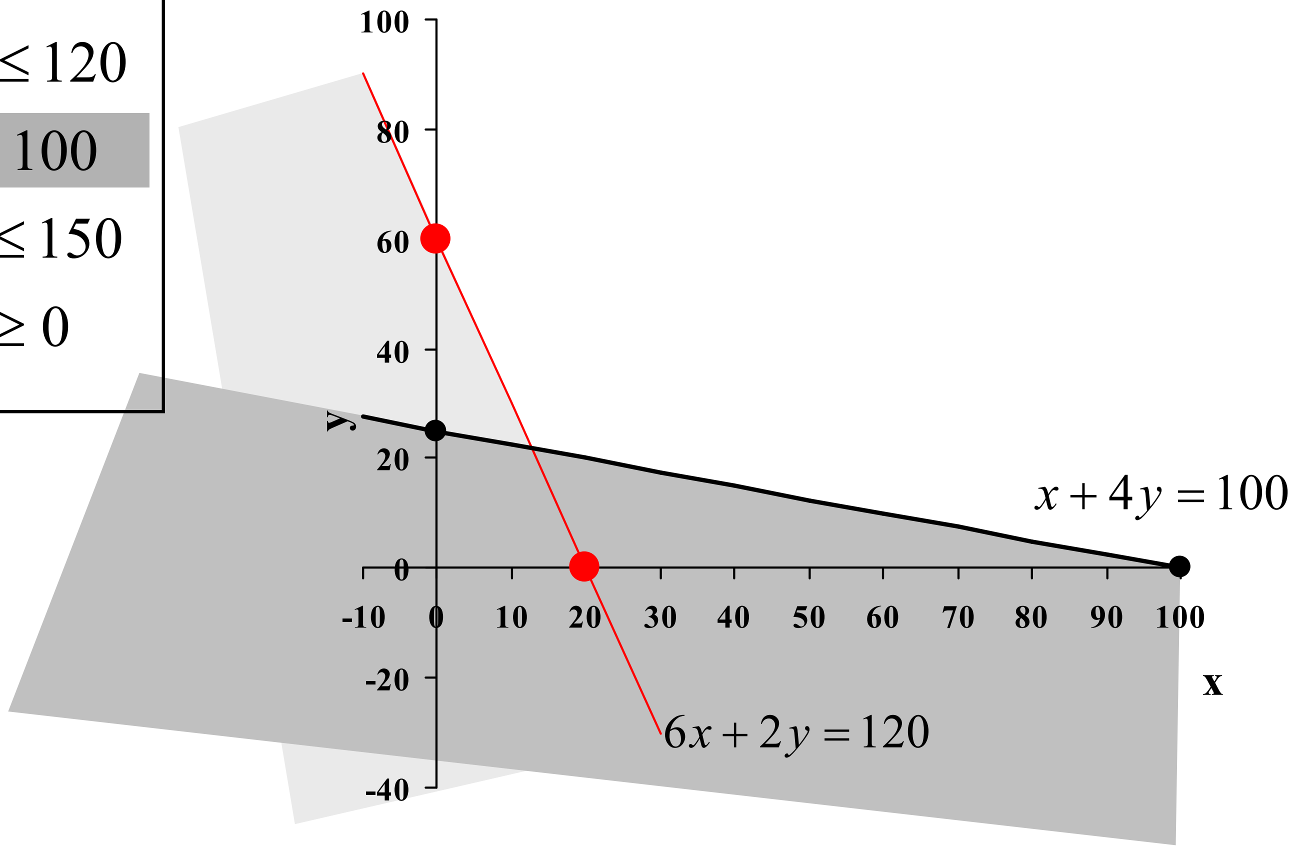


The inequality $6x + 2y \leq 120$ is satisfied by all points in the shaded zone

$$\max 4x + 3y$$

s.a
$$6x + 2y \leq 120$$
$$x + 4y \leq 100$$
$$5x + 5y \leq 150$$
$$x \geq 0, y \geq 0$$

Graphical method



For the constraint
$$x + 4y \leq 100$$

Consider the straight line $x + 4y = 100$

$x = 0 \Rightarrow y = 25$

$y = 0 \Rightarrow x = 100$

Points (0,25) and (100,0) belong to the line

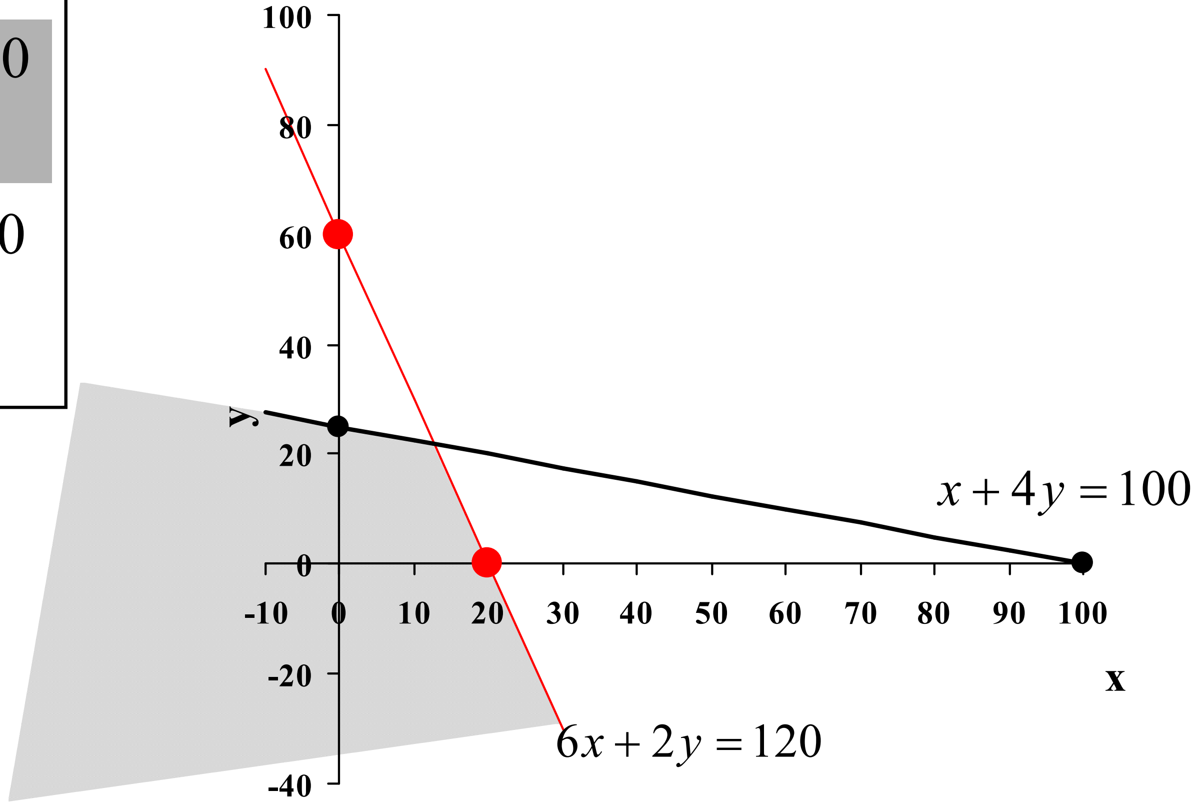
Point (0,0) satisfies the inequality $x + 4y \leq 100$

$$\max 4x + 3y$$

s.a

$$6x + 2y \le 120$$
$$x + 4y \le 100$$
$$5x + 5y \le 150$$
$$x \ge 0, y \ge 0$$

Graphical method

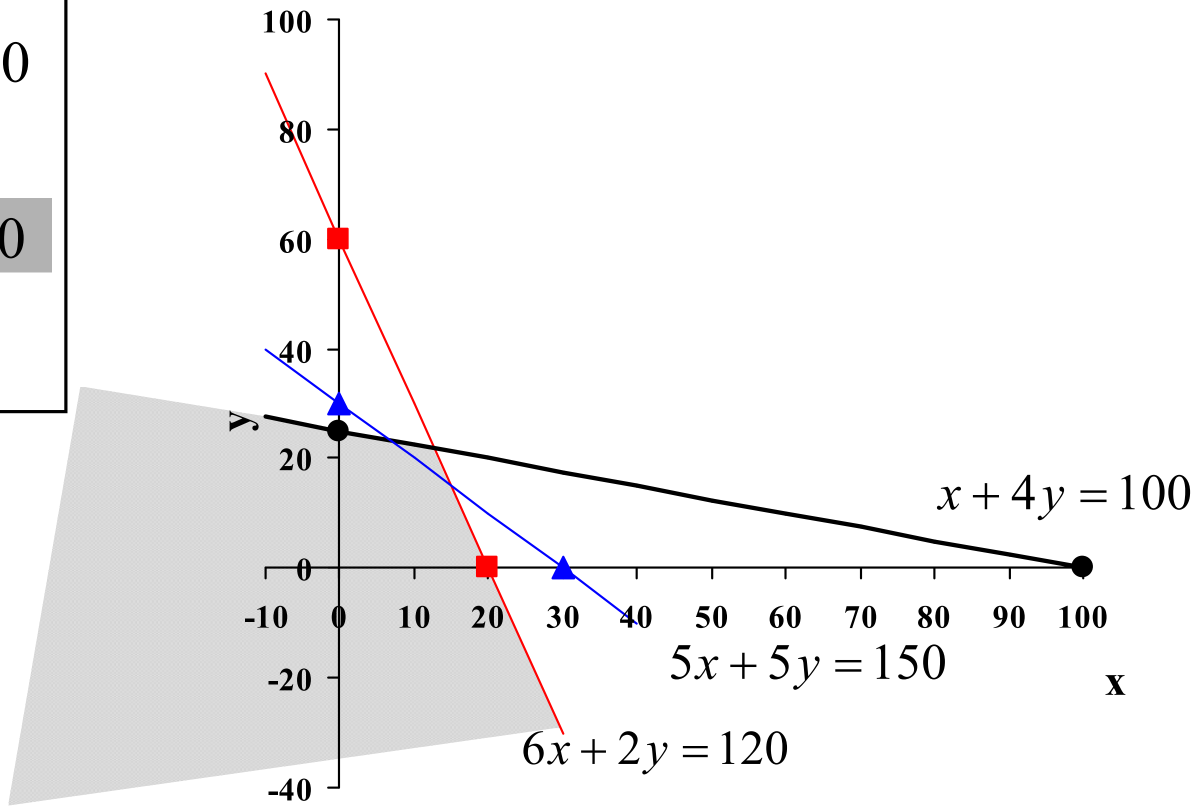


The two inequalities considered together are represented by the intersection of the half-planes, represented by the shaded area .

$$\max\ 4x+3y$$

s.a
$$6x+2y\le 120$$
$$x+4y\le 100$$
$$5x+5y\le 150$$
$$x\ge 0,\ y\ge 0$$

Graphical method



For the constraint
$$5x+5y\le 150$$

Consider the straight line $5x+5y=150$

$$x=0\Rightarrow y=30$$
$$y=0\Rightarrow x=30$$

Points (0,30) and (30,0) belong to the straight line

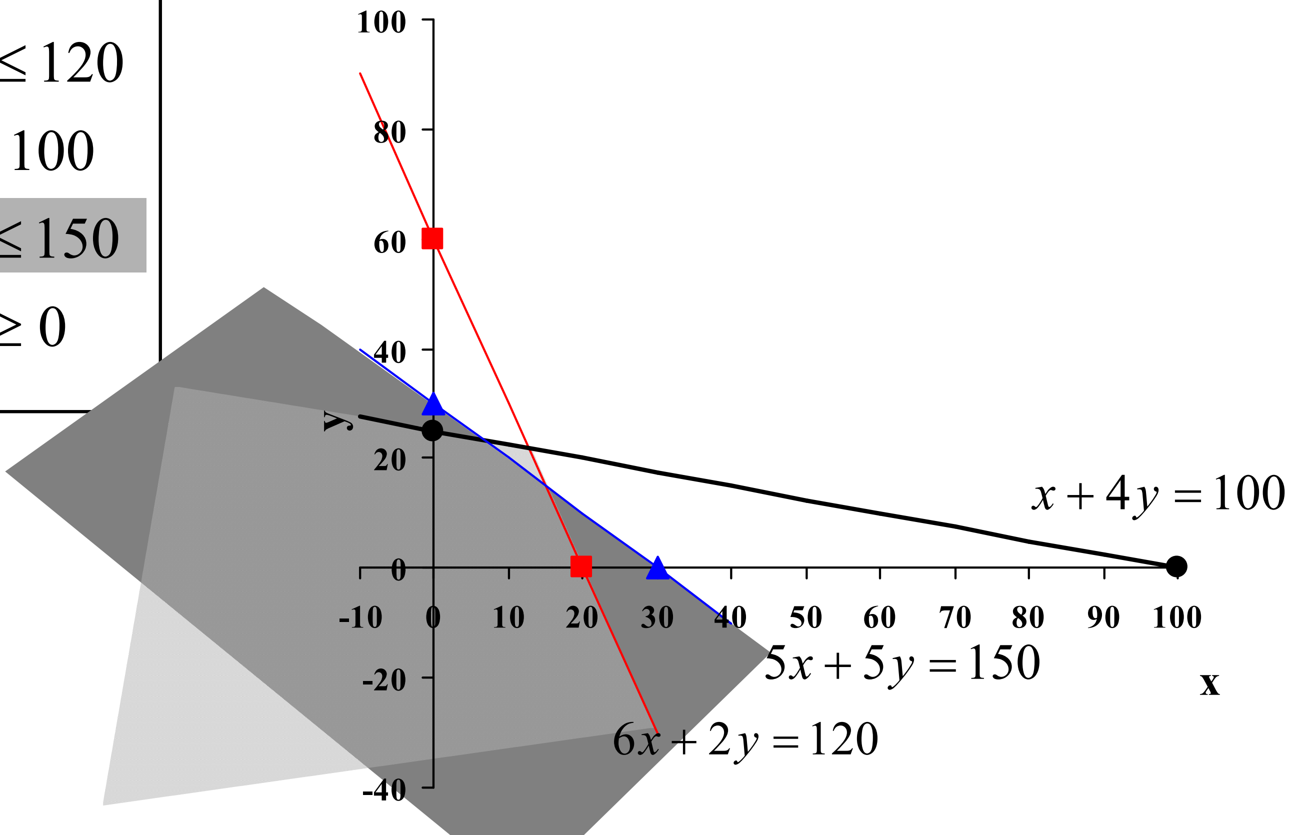
Point (0,0) satisfies the inequality $5x+5y\le 150$

Graphical method

$$\max\ 4x + 3y$$

s.a

$$6x + 2y \le 120$$
$$x + 4y \le 100$$
$$5x + 5y \le 150$$
$$x \ge 0,\ y \ge 0$$



For the constraint

$$5x + 5y \le 150$$

Consider the straight line $5x + 5y = 150$

$$x = 0 \Rightarrow y = 30$$
$$y = 0 \Rightarrow x = 30$$

Points (0,30) and (30,0) belong to the straight line

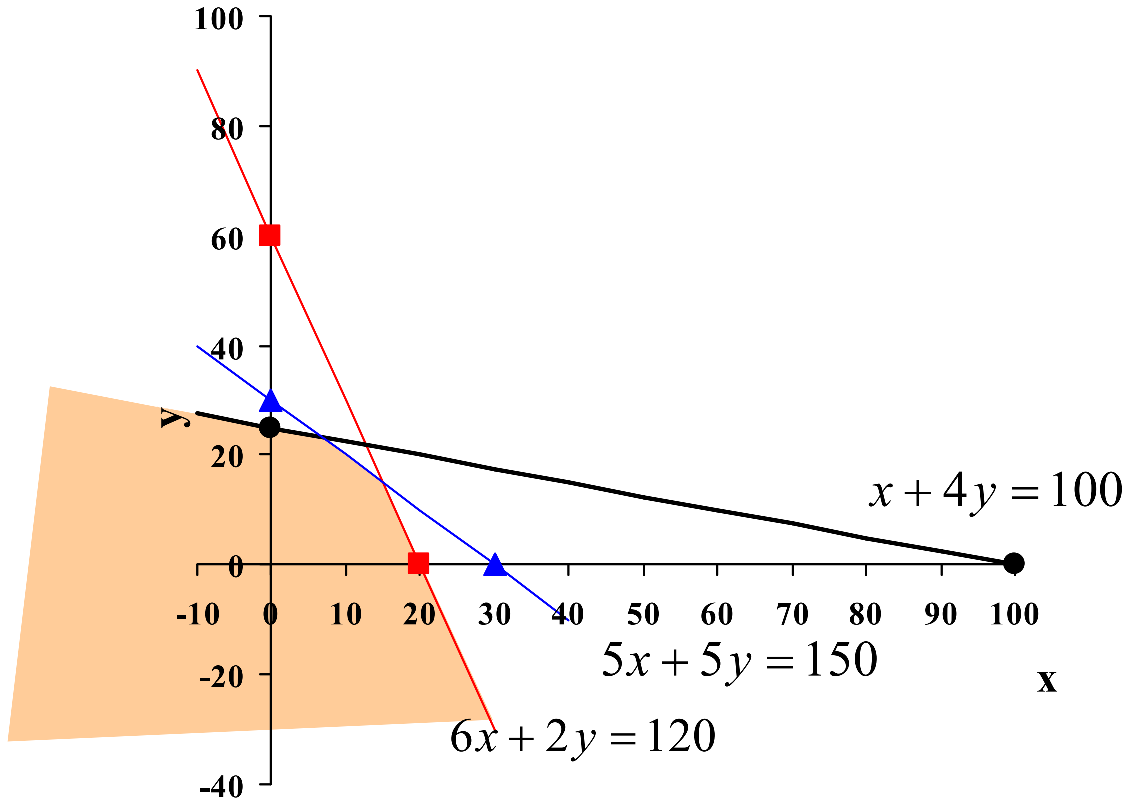
Point (0,0) satisfies the inequality $5x + 5y \le 150$

$$\max\ 4x + 3y$$

s.a

$$6x + 2y \leq 120$$
$$x + 4y \leq 100$$
$$5x + 5y \leq 150$$
$$x \geq 0, y \geq 0$$

Graphical method



The three inequalities are represented by the shaded area.

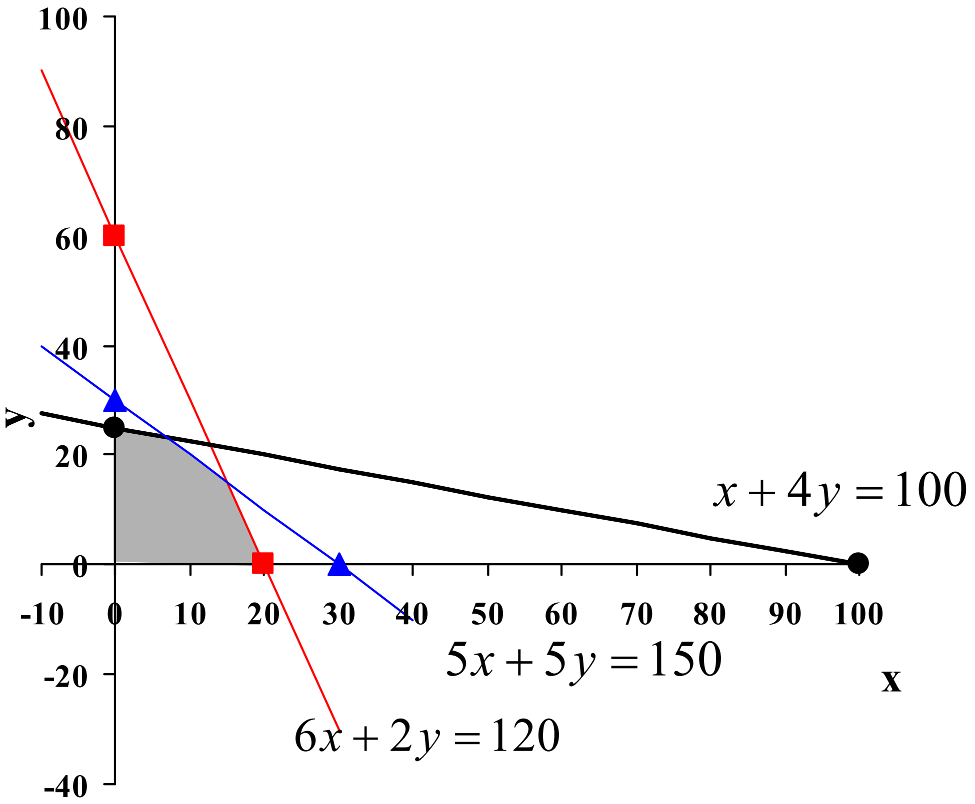
It only remains to consider the non-negativity constraints $x \geq 0, y \geq 0$

$$\max 4x + 3y$$

s.a

$$6x + 2y \le 120$$
$$x + 4y \le 100$$
$$5x + 5y \le 150$$
$$x \ge 0, y \ge 0$$

Graphical method



The shaded area represents the feasible solutions region

$$\max\ 4x + 3y$$

s.a
$$6x + 2y \leq 120$$
$$x + 4y \leq 100$$
$$5x + 5y \leq 150$$
$$x \geq 0, y \geq 0$$

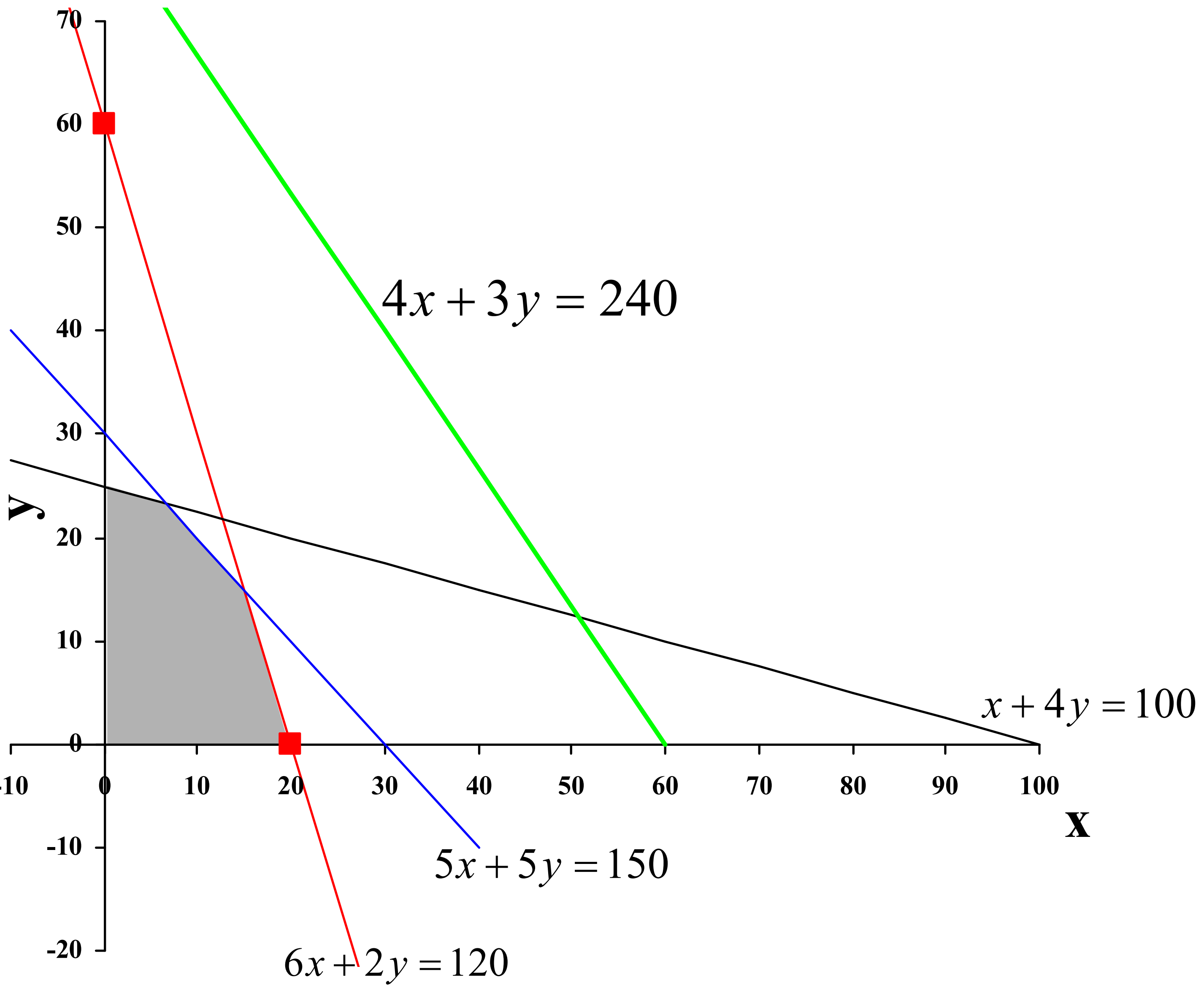
How can we represent the objective function?

Assign an arbitrary value to the objective function.

For example, to obtain a profit of 240 €:

$$4x + 3y = 240$$

Graphical method



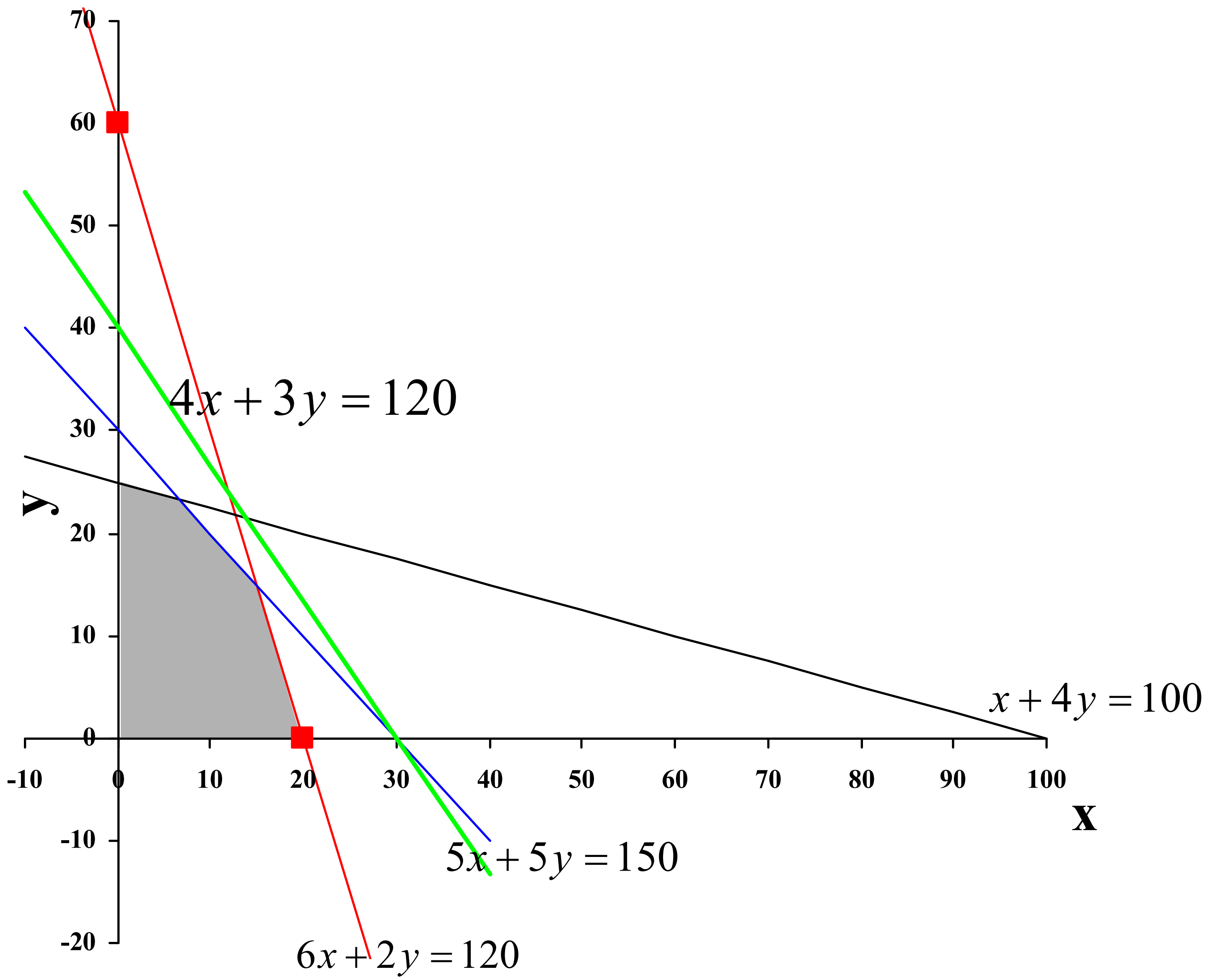
$$\max\ 4x + 3y$$

s.a
$$6x + 2y \leq 120$$
$$x + 4y \leq 100$$
$$5x + 5y \leq 150$$
$$x \geq 0,\ y \geq 0$$

And for a profit of 120 €:

$$4x + 3y = 120$$

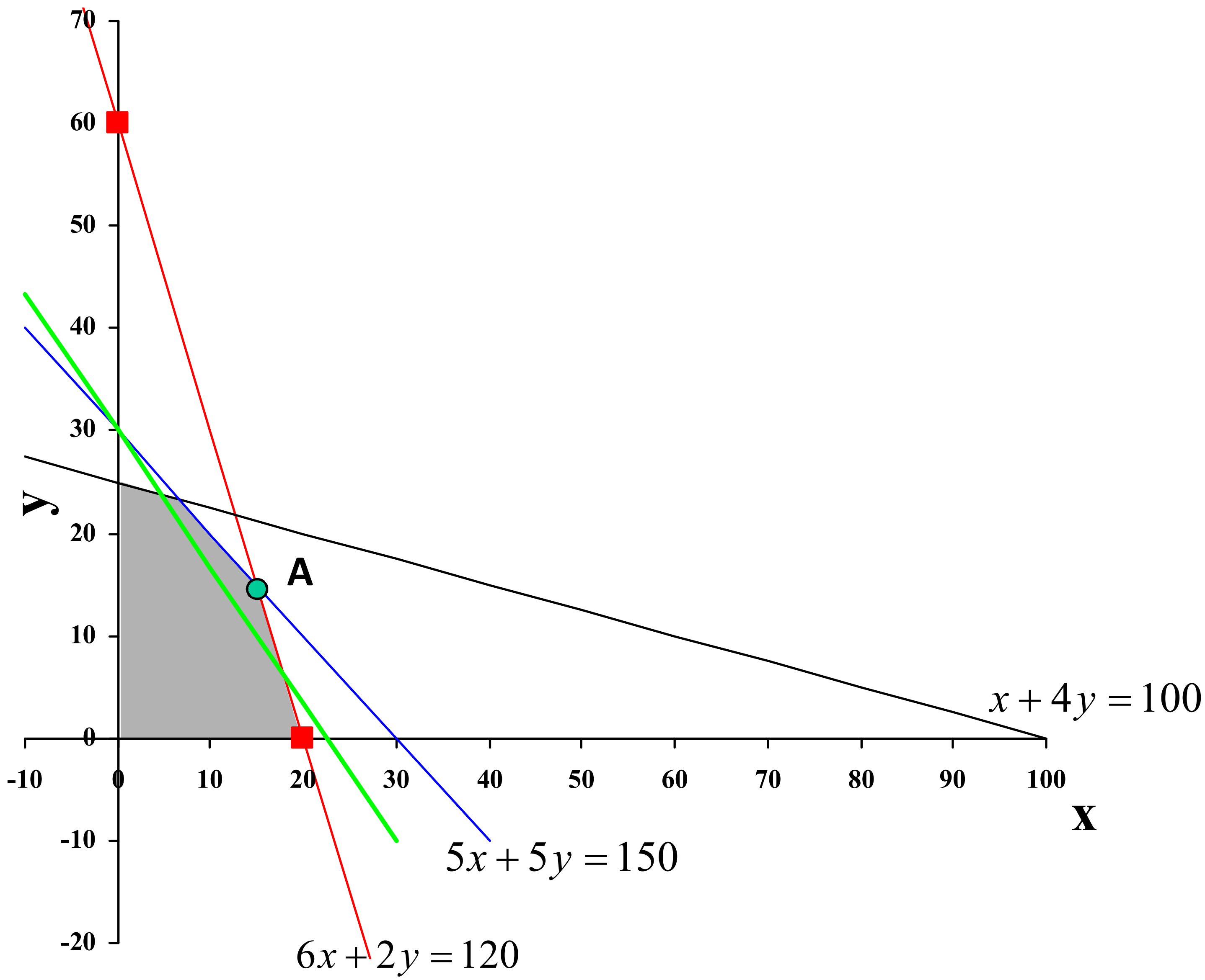
Graphical method



$$\max\ 4x+3y$$

s.a
$$6x+2y\le 120$$
$$x+4y\le 100$$
$$5x+5y\le 150$$
$$x\ge 0,\ y\ge 0$$

Graphical method



Currently, the company produces 18 tons of wheat and 6 tons of corn each week.
Hence the week profit is 90 €

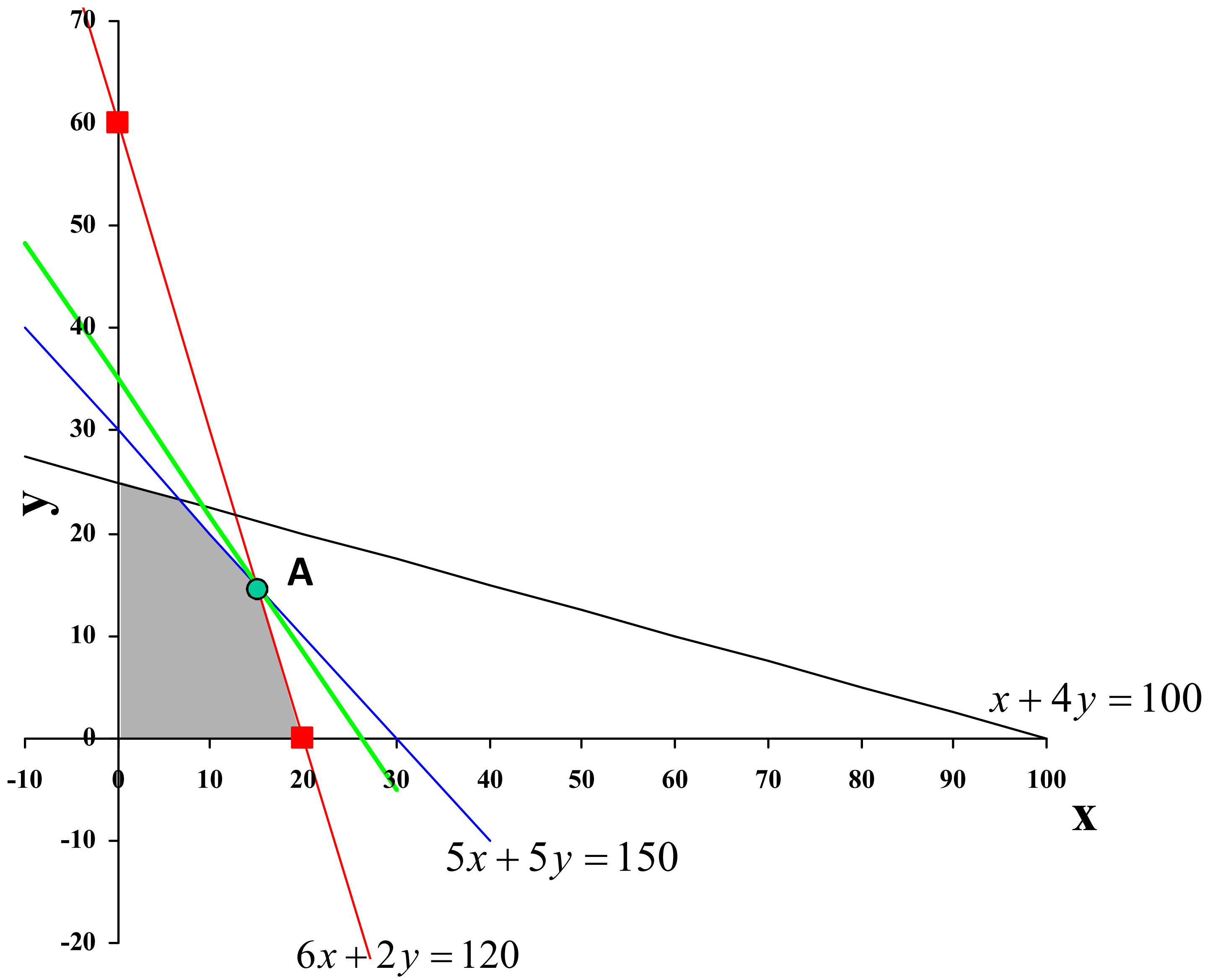
$$4x+3y=90$$

Is the current solution the optimal solution?

$$\max\ 4x + 3y$$

s.a
$$6x + 2y \le 120$$
$$x + 4y \le 100$$
$$5x + 5y \le 150$$
$$x \ge 0, y \ge 0$$

Graphical method



In order to obtain the optimal solution we have to calculate the intersection point (A) of the straight lines:

$$6x + 2y = 120$$
$$5x + 5y = 150$$

A = (15,15)

This production plan yields a profit of 105 €

$$4x + 3y = 105$$

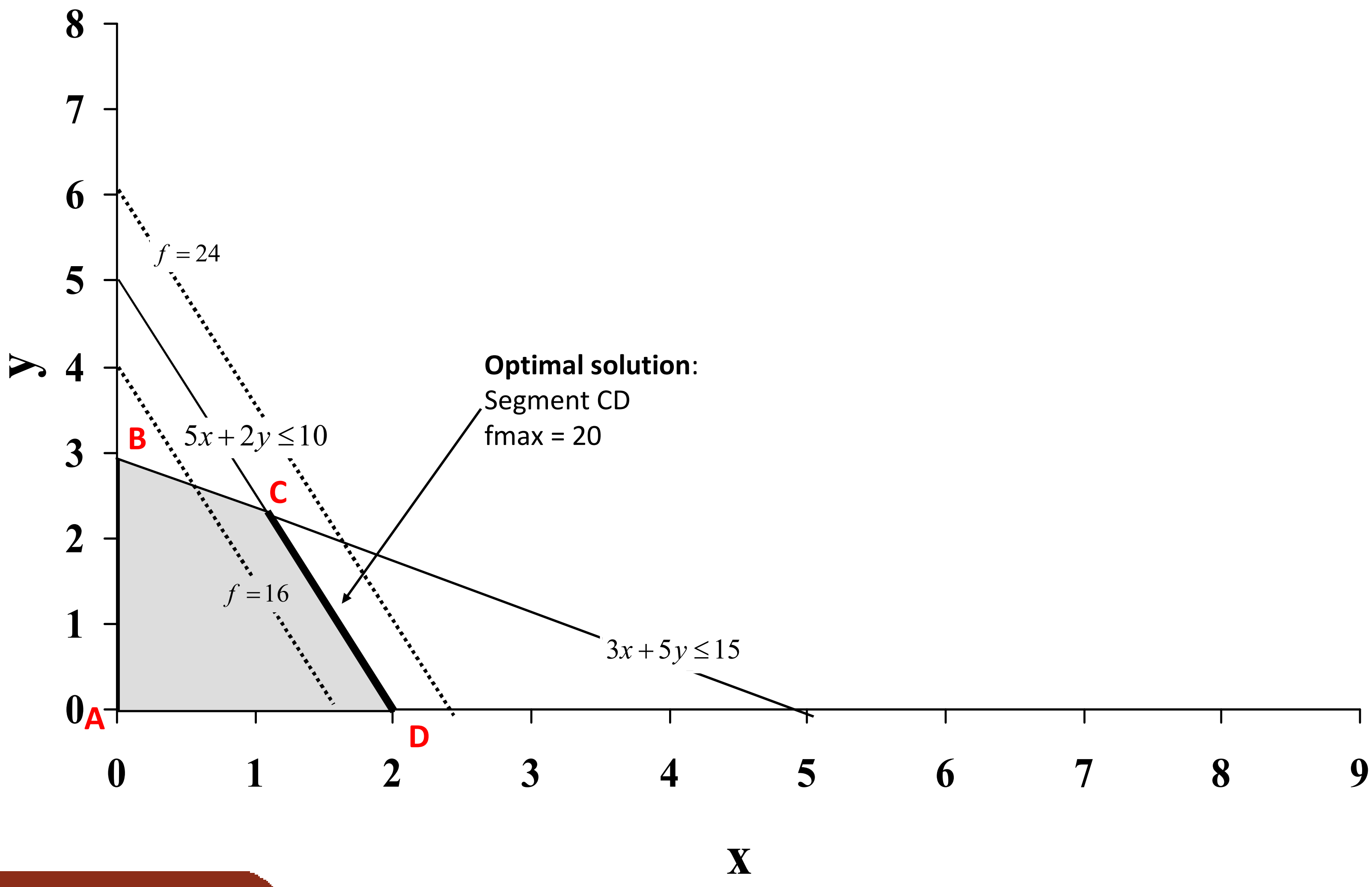
Particular cases of Linear Programming

$$\max 10x + 4y$$

s.a

$$3x + 5y \le 15$$
$$5x + 2y \le 10$$
$$x \ge 0, y \ge 0$$

Infinite optimal solutions

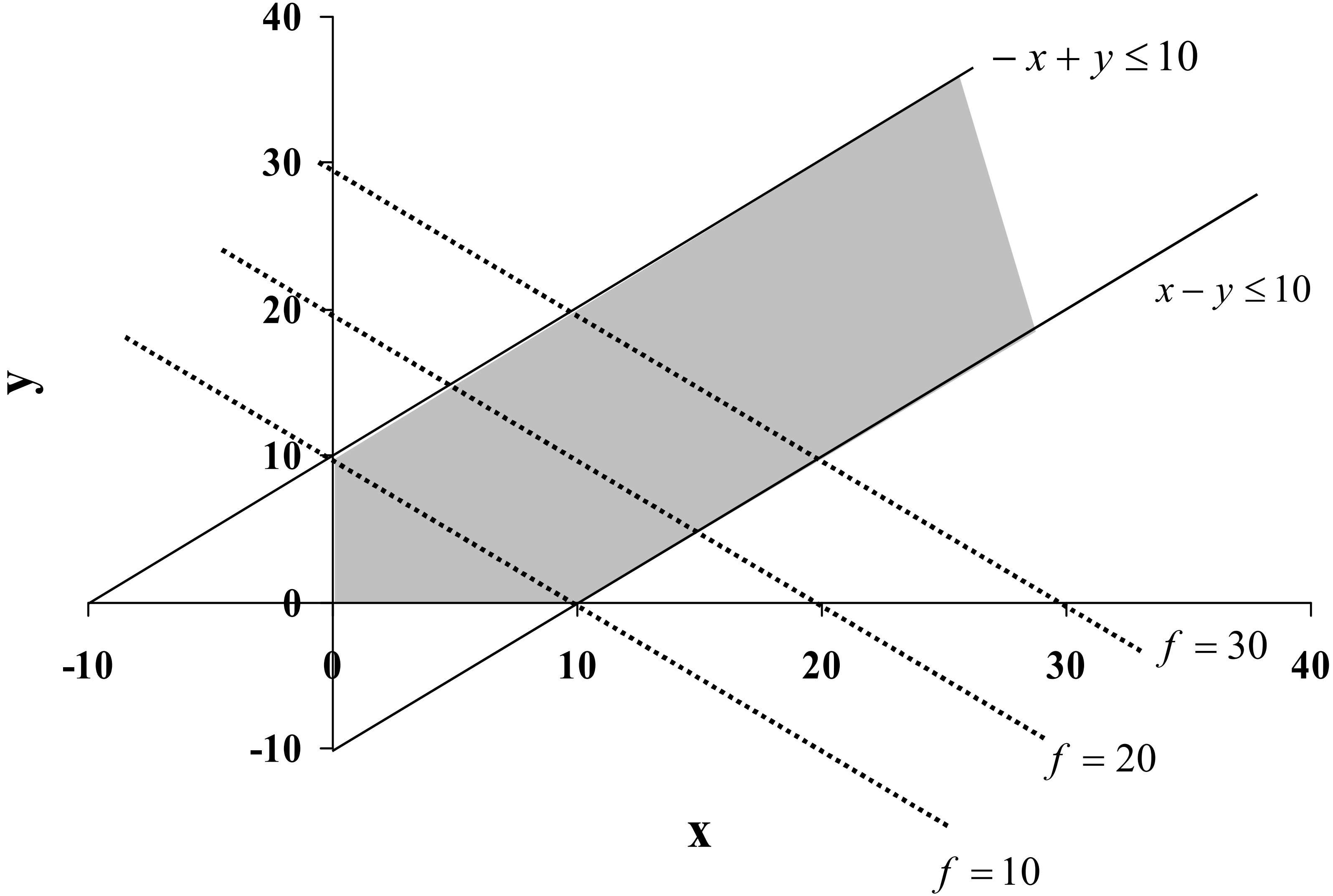


Unlimited optimal solution

$$\max x + y$$

s.a

$$x - y \leq 10$$
$$-x + y \leq 10$$
$$x \geq 0, y \geq 0$$

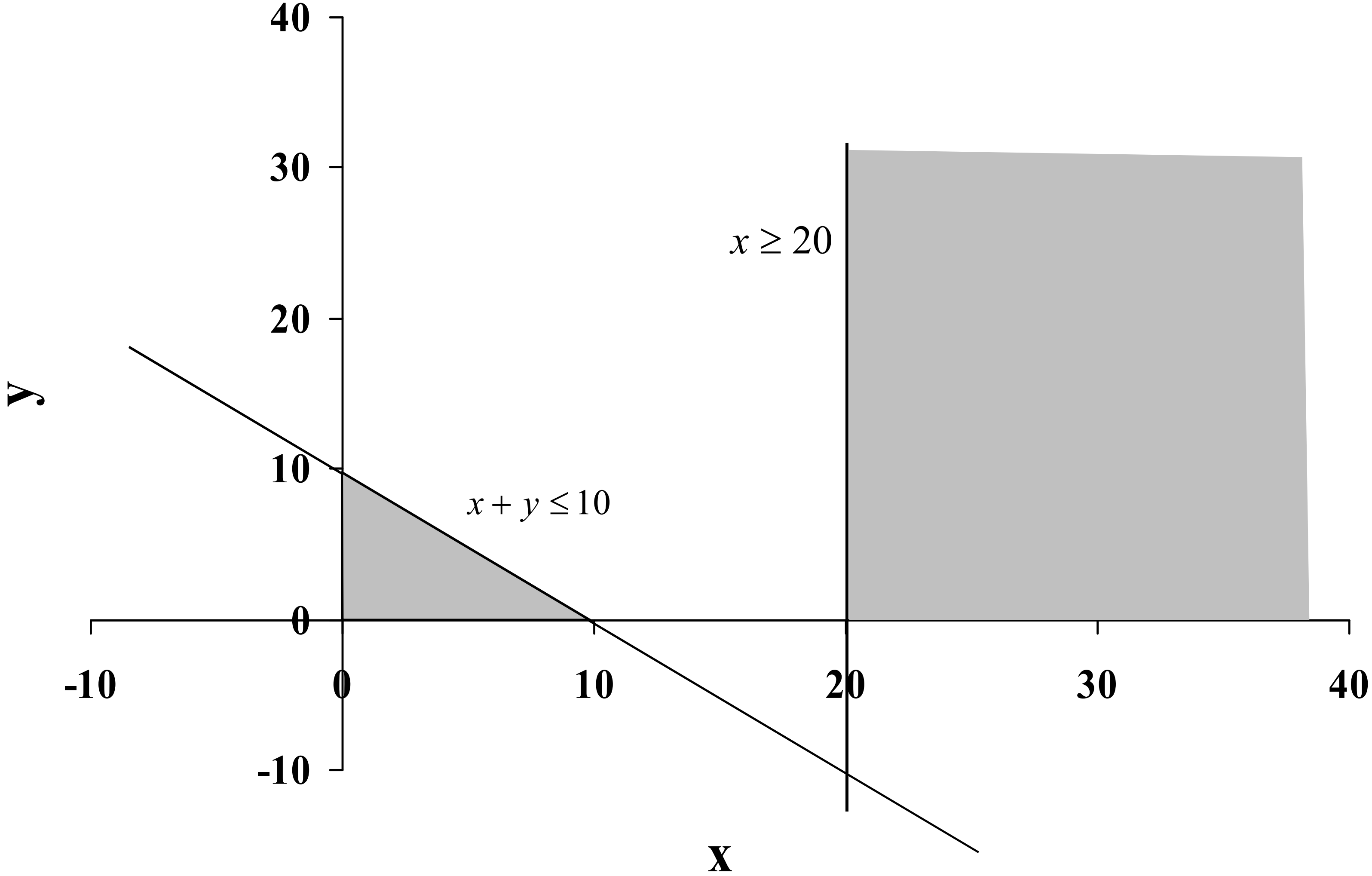


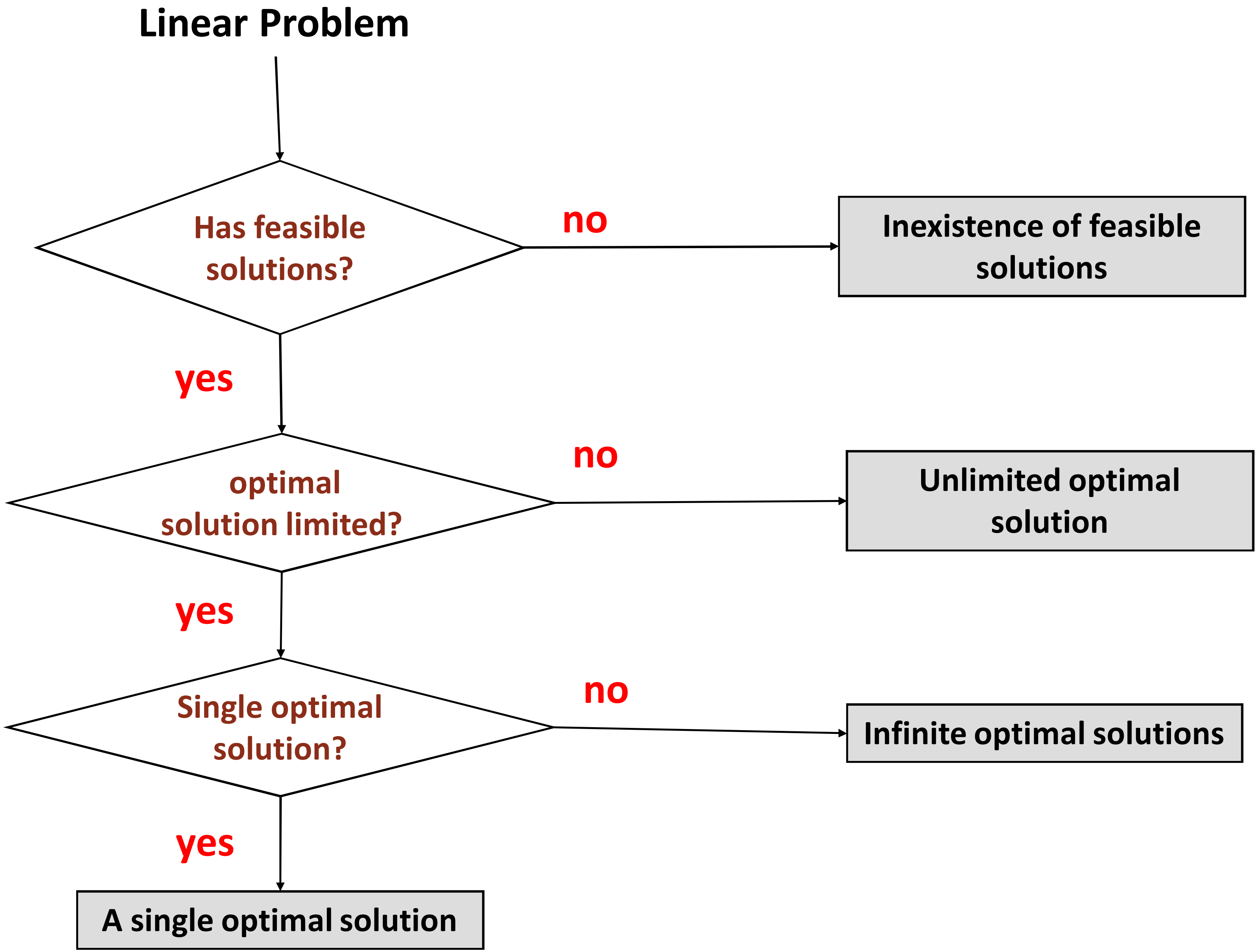
Inexistence of a feasible solution

$$\max\ x + 2y$$

s.a

$$x + y \leq 10$$
$$x \geq 20$$
$$x \geq 0, y \geq 0$$



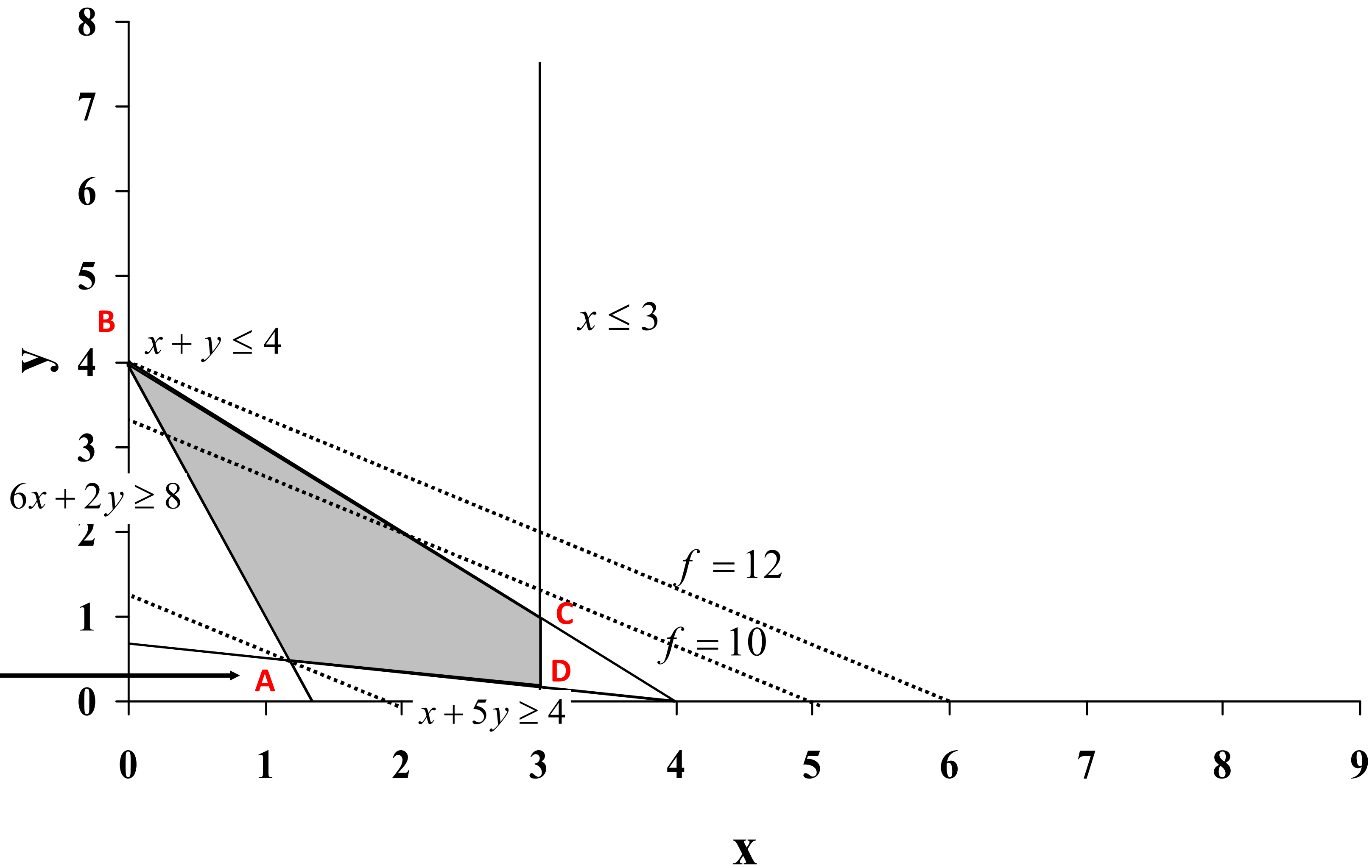

Linear Problem
Has feasible solutions?
no
Inexistence of feasible solutions
yes
optimal solution limited?
no
Unlimited optimal solution
yes
Single optimal solution?
no
Infinite optimal solutions
yes
A single optimal solution

Exercise

$$\min 2x + 3y$$

s.a

$$x + y \le 4$$
$$6x + 2y \ge 8$$
$$x + 5y \ge 4$$
$$x \le 3$$
$$x \ge 0, y \ge 0$$



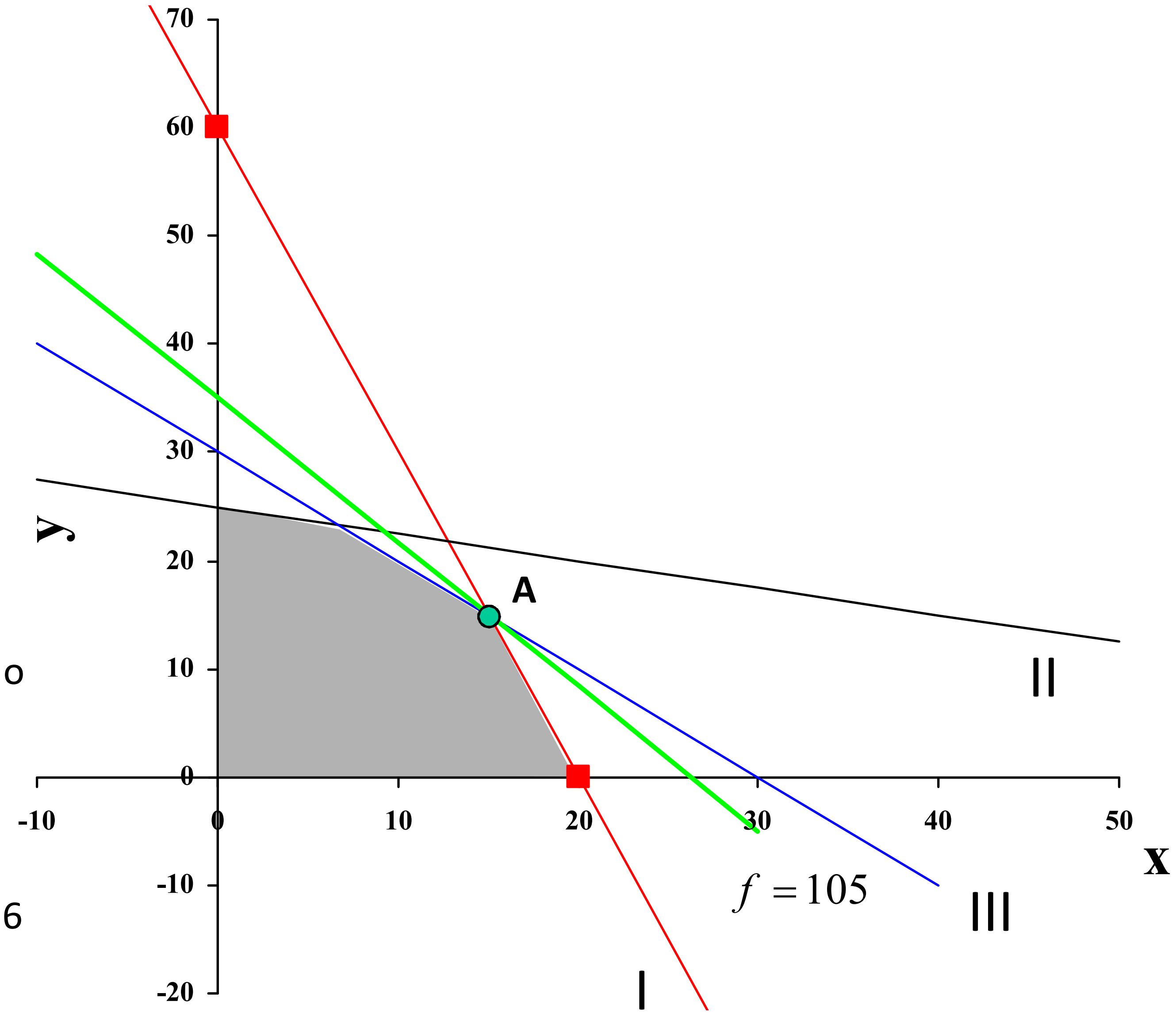
Optimal solution:
Point A
fmin = 4
X = 1,14, y = 0,57

$$\max\ 4x + 3y$$

s.a

$$6x + 2y \le 120$$
$$x + 4y \le 100$$
$$5x + 5y \le 150$$
$$x \ge 0, y \ge 0$$

Consider again the example of Cereals, Ltd



Solução óptima A = (15,15)

Lucro: $4x + 3y = 105$

O que acontece se for possível aumentar o lucro de cada tonelada de trigo para 4,35 €?

E se a capacidade de produção na secção III (empacotamento) for reduzida para 126 h/semana?