The Two-Phase Simplex Method

Motivation

The Two-Phase method is based on the following simple observation:

- Suppose that we have a linear programming problem in canonical form and we wish to generate a feasible solution (not necessarily optimal) such that a given variable, say x_3, is equal to zero.
- Then, all we have to do is solve the linear programming problem obtained from the original problem by replacing the original objective function by min x_3.
- If more than one variable is required to be equal to zero, then we replace the original objective function by the sum of all the variables we want to set to zero.
- Observe that because of the non-negativity constraint, the sum of any collection of variables cannot be negative. Hence the smallest possible feasible value of such a sum is zero. If the smallest feasible sum is strictly positive, then the implication is that it is impossible to set all the designated variables to zero.

The Two-phase Simplex Method

Cereals Ltd revisited:

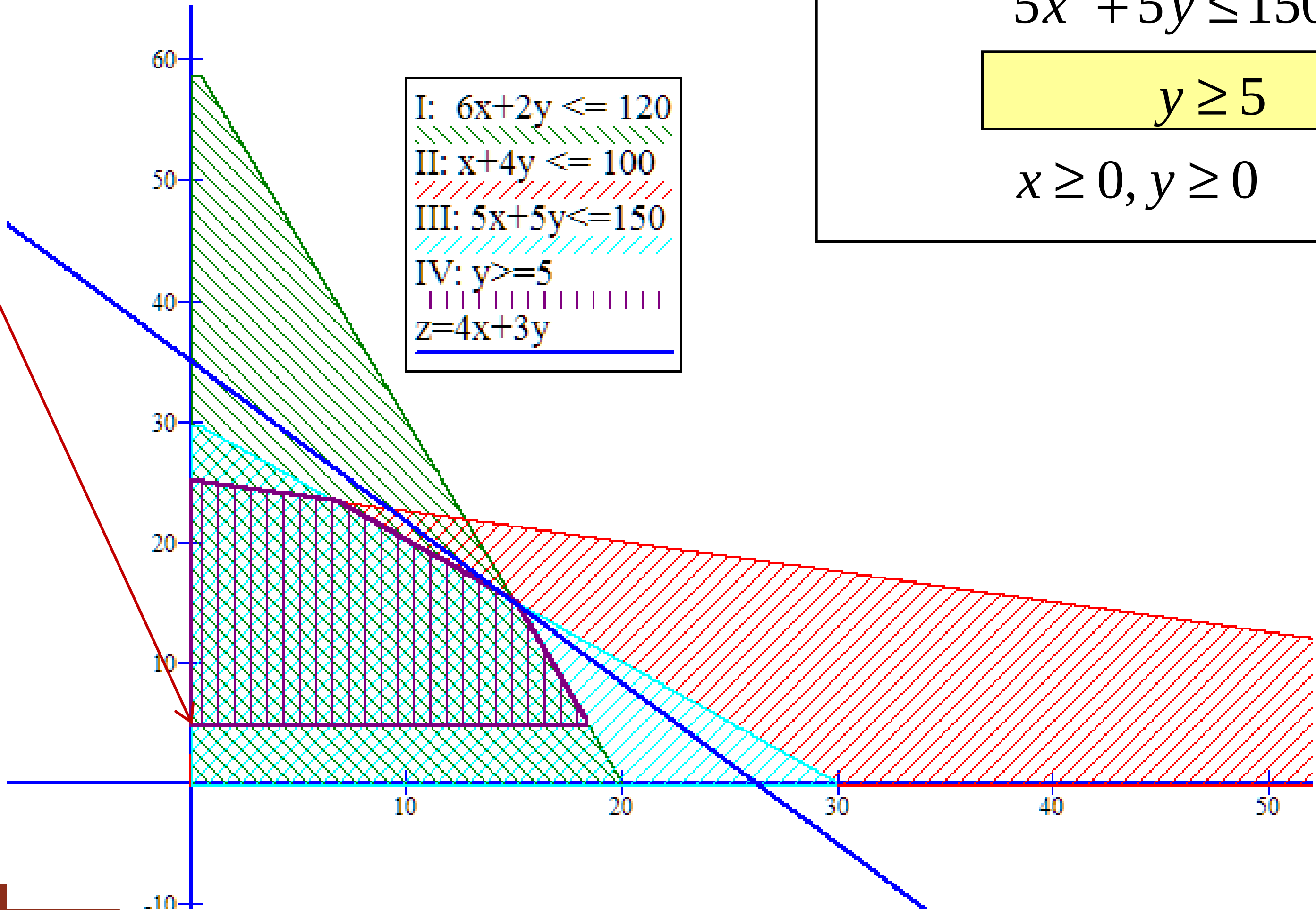
$$\max f = 4x + 3y$$

$$\text{s.to} \quad \begin{aligned} 6x + 2y &\leq 120 \\ x + 4y &\leq 100 \\ 5x + 5y &\leq 150 \\ \boxed{y \geq 5} \\ x \geq 0, y &\geq 0 \end{aligned}$$

Consider that a <u>new constraint is added</u> to the original problem, imposing that the minimum amount of corn (y) to produce is 5 tons.

Graphically, we can see that the optimal solution does not change.

However, in order to apply the Simplex Method we need to find a feasible basic solution, since the point (0,0) does no longer belong to the feasible region.



The Two-phase Simplex Method

Canonical form

$$\begin{aligned} \max \mathrm{f} &= 4x_1 + 3x_2 \\ \text{s.a} \quad 6x_1 + 2x_2 &\le 120 \\ x_1 + 4x_2 &\le 100 \\ 5x_1 + 5x_2 &\le 150 \\ x_2 &\ge 5 \\ x_1 \ge 0, x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned} &\max \mathrm{f} = 4x_1 + 3x_2 \\ &6x_1 + 2x_2 + s_1 = 120 \\ &x_1 + 4x_2 + s_2 = 100 \\ &5x_1 + 5x_2 + s_3 = 150 \\ &x_2 - s_4 = 5 \\ &x_1, x_2, s_1, s_2, s_3, s_4 \ge 0 \end{aligned}$$

The point (0,0) is not a feasible solution, hence (0,0,120,100,150,-5) is not a feasible basic solution. Remember that all the variables must be non-negative and in this solution S_4 = -5 !!!

How can we find a feasible basic solution?

The Two-phase Simplex Method

Original problem in canonical form

$$\max f = 4x_1 + 3x_2$$

$$\begin{aligned}
6x_1 + 2x_2 + s_1 &= 120 \\
x_1 + 4x_2 + s_2 &= 100 \\
5x_1 + 5x_2 + s_3 &= 150 \\
x_2 - s_4 &= 5 \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}$$

Auxiliary problem in canonical form

$$\boxed{\min F = a_1 = 5 - x2 + s4}$$

$$\begin{aligned}
6x_1 + 2x_2 + s_1 &= 120 \\
x_1 + 4x_2 + s_2 &= 100 \\
5x_1 + 5x_2 + s_3 &= 150 \\
x_2 - s_4 \boxed{+ a_1} &= 5 \\
x_1, x_2, s_1, s_2, s_3, s_4, a_1 &\geq 0
\end{aligned}$$

artificial variable

Remember the motivation idea:

Applying this simple idea to **artificial variables** we obtain the following recipe:

To set all the artificial variables to zero, solve a linear programming problem derived from the canonical form of the original problem by replacing the original objective function by the minimization of the sum of all the artificial variables.
If the optimal value of the modified objective function is not equal to zero, then the problem (system of constraints) is not feasible.

The Two-phase Simplex Method

1st phase: Find a feasible basic solution for the original problem

For each ‘>=‘ or ‘=‘ constraint, add an artificial variable.

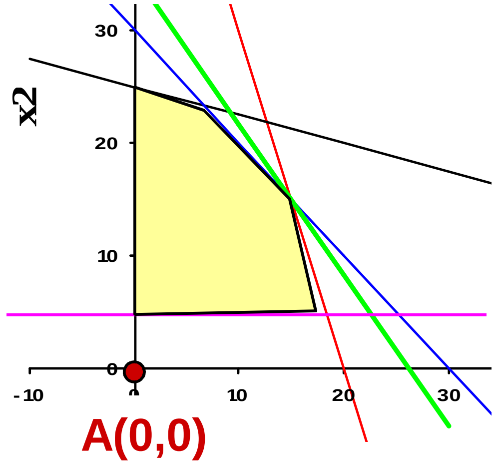
The artificial variables allow us to have a basic solution for an auxiliary problem for which the objective function is the minimization of the sum of the artificial variables.

The Simplex Method is then applied to this auxiliary problem.

The optimal solution of the auxiliary problem will be (if exists) a feasible basic solution for the original problem.

2nd phase: Once obtained a feasible basic solution for the original problem, the Simplex method is applied to solve it.

1st phase: Find a feasible basic solution for the original problem



Solve the auxiliary problem:

$$\min \mathrm{F} = a_1 = 5 - x2 + s4$$

$$6x_1 + 2x_2 + s_1 = 120$$

$$x_1 + 4x_2 + s_2 = 100$$

$$5x_1 + 5x_2 + s_3 = 150$$

$$x_2 - s_4 + a_1 = 5$$

$$x_1, x_2, s_1, s_2, s_3, s_4, a_1 \geq 0$$

Point A is not feasible for the original problem

Iteration 0 : Point A (0, 0)

	basis	X1	X2	S1	S2	S3	S4	a1	value		
L1	S1	6	2	1	0	0	0	0	120	120/2 =	60
L2	S2	1	4	0	1	0	0	0	100	100/4 =	25
L3	S3	5	5	0	0	1	0	0	150	150/5 =	30
L4	a1	0	1	0	0	0	-1	1	5	**5/1 =**	**5**
max	f	4	3	0	0	0	0	0	0		
min	F	0	-1	0	0	0	1	0	-5		

We are minimizing the auxiliary problem F. Hence we choose to enter the basis the variable with the most negative coefficient in the objective function F.

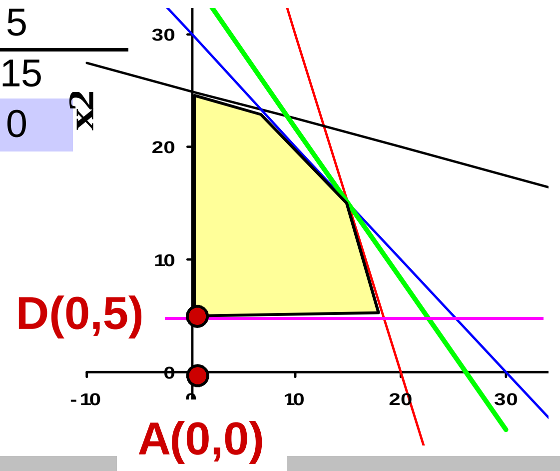
1st phase: Find a feasible basic solution for the original problem

Iteration 0 : Point A (0, 0)

	basis	X1	X2	S1	S2	S3	S4	a1	value		
L1	S1	6	2	1	0	0	0	0	120	120/2 =	60
L2	S2	1	4	0	1	0	0	0	100	100/4 =	25
L3	S3	5	5	0	0	1	0	0	150	150/5 =	30
L4	a1	0	1	0	0	0	-1	1	5	**5/1 =**	**5**
	f	4	3	0	0	0	0	0	0		
	F	0	-1	0	0	0	1	0	-5		

Iteration 1 : Point D (0, 5)

	basis	X1	X2	S1	S2	S3	S4	a1	value		
L'1 = L1-2L'4	S1	6	0	1	0	0	2	-2	110	110/6 =	18,3
L'2 = L2-4L'4	S2	1	0	0	1	0	4	-4	80	80/1 =	80
L'3 =L3-5L'4	S3	5	0	0	0	1	5	-5	125	125/5=	25
L'4 = L4	X2	0	1	0	0	0	-1	1	5		
f' = f-3L'4		4	0	0	0	0	3	-3	-15		
F' = F+L'4		0	0	0	0	0	0	1	0		



End of 1st phase: we found a feasible basic solution for the original problem: **point D**

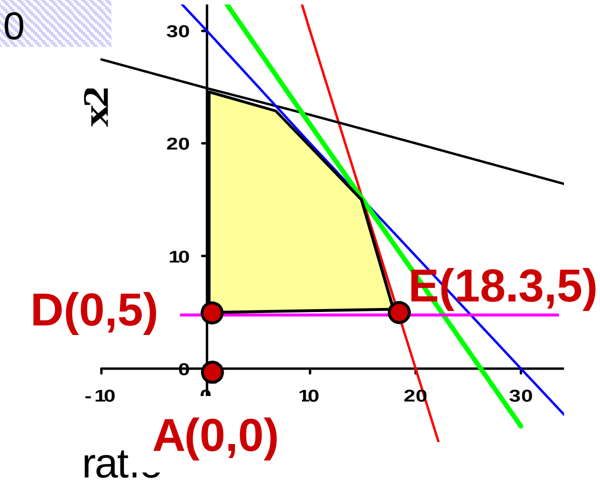
2nd phase: apply the Simplex method to the original problem

Iteration 1 : Point D (0, 5)

	basis	X1	X2	S1	S2	S3	S4	a1	value		
L'1 = L1-2L'4	S1	6	0	1	0	0	2	-2	110	110/6 =	18,3
L'2 = L2-4L'4	S2	1	0	0	1	0	4	-4	80	80/1 =	80
L'3 =L3-5L'4	S3	5	0	0	0	1	5	-5	125	125/5=	25
L'4 = L4	X2	0	1	0	0	0	-1	1	5		
Lf' = f-3L'4	f	4	0	0	0	0	3	-3	-15		
LF' = LF+L'4	F	0	0	0	0	0	0	1	0		

Remove the auxiliary problem column (F) and the artificial variable row (a1) from the tableaux.

Solve the original problem (f): the variable that enters the basis is x1



	basis	X1	X2	S1	S2	S3	S4	value	rat...	
L"1 = L'1/6	X1	1	0	0,1667	0	0	0,3333	18,333	18,3/0,33 =	55
L"2 = L'2-L"1	S2	0	0	-0,167	1	0	3,6667	61,667	6,67/2,67 =	17
L"3 = L'3-5L"1	S3	0	0	-0,833	0	1	3,3333	33,333	33,3/3,3 =	10
L"4 = L'4	X2	0	1	0	0	0	-1	5		
f" = f-4L"1		0	0	-0,667	0	0	1,6667	-88,33		

Iteration 2 : Point E (18.3, 5)

2nd phase: apply the Simplex method to the original problem.

Iteration 2 : Point E (18.3, 5)

	basis	X1	X2	S1	S2	S3	S4	value	ratio	
L"1 = L'1/6	X1	1	0	0,1667	0	0	0,3333	18,333	18,3/0,33 =	55
L"2 = L'2-L"1	S2	0	0	-0,167	1	0	3,6667	61,667	6,67/2,67 =	17
L"3 = L'3-5L"1	S3	0	0	-0,833	0	1	3,3333	33,333	33,3/3,3 =	10
L"4 = L'4	X2	0	1	0	0	0	-1	5		
f" = f'-4L"1		0	0	-0,667	0	0	1,6667	-88,33		

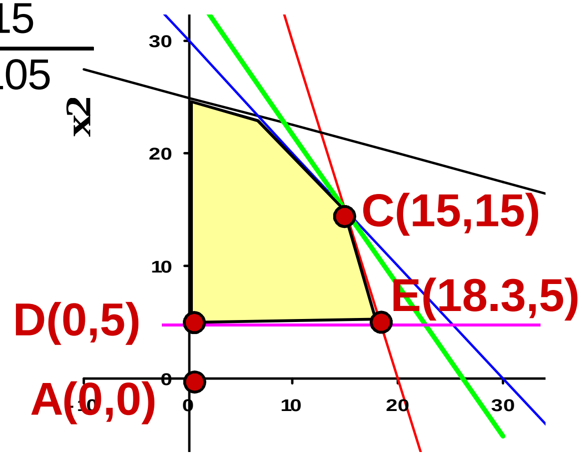
Iteration 3 : Point C (15,15)

	basis	X1	X2	S1	S2	S3	S4	value
L"'1=L"1-0,33L"'3	X1	1	0	0,25	0	-0,099	0	15
L"'2=L"2-3,67L"'3	S2	0	0	0,75	1	-1,1	0	25
L"'3=L"3/3,33	S4	0	0	-0,25	0	0,3	1	10
L"'4=L"4+L"'3	X2	0	1	-0,25	0	0,3	0	15
f"=f'-1,667L"'3		0	0	-0,25	0	-0,5	0	-105

End fo 2nd phase

Optimal solution: Point C : X1 = 15
X2 = 15

fmax = 105



The Two-phase Simplex Method: final remarks

At the end of the Phase 1 we can distinguish between two cases, namely:

Case 1: $F_{min} > 0$: The optimal value of F is greater than zero.

- we conclude that the LP problem under consideration has no feasible solutions.

Case 2: $F_{min} = 0$: The optimal value of F is equal to zero.

- implies that the constraints are feasible, hence the problem under consideration has a feasible solution.

It should be noted that although Case 2 implies that all the artificial variables are equal to zero, this does not mean that they are all out of the basis.

So it is necessary to consider Case 2 in more detail, namely:

Case 2.1: All the artificial variables are non-basic.

- we proceed to Phase 2 of the 2-Phase method replacing the objective function of Phase 1 with the original objective function.

Case 2.2: some of the artificial variables are in the basis.

- This case represents a degenerate basis, namely a situation where one or more of the basic variables are equal to zero. To take a degenerate artificial variable out of the basis we pivot on any non-artificial variable whose coefficient in the row of the artificial is not equal to zero and enter it into the basis.
- If the coefficients of all the non-artificial variables in that row are zeros, then the conclusion is that the constraint is redundant and thus can be ignored.

Special cases in Simplex Method

1. Infeasibility of the problem (no feasible solutions)
2. Unbound optimal value
3. Multiple optimal solutions
5. Degeneracy

1. Infeasibility of the problem

$$\begin{array}{lrcrcl} \max f = x_1 + x_2 & & & & & \\ \text{s.to} & & & & & \\ & x_1 & + x_2 & \leq & 1 \\ & 2x_1 & + 3x_2 & \geq & 6 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

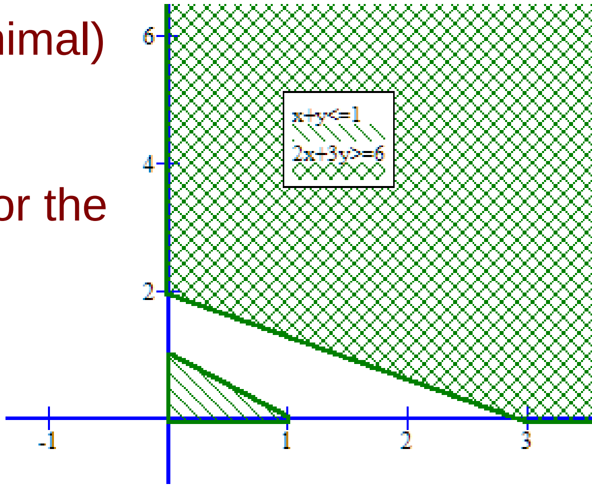
Auxiliary Problem (phase 1)

$$\begin{array}{llllll} \min F = a_1 & & & & & \\ \text{s.to} & & & & & \\ x_1 & + x_2 & + s_1 & & & = 1 \\ 2x_1 & + 3x_2 & & - s_2 & + a_1 & = 6 \\ x_1, & x_2, & s_1, & s_2, & a_1 & \geq 0 \end{array}$$

Basis	x1	x2	s1	s2	a1	Value
← s1	1	1	1	0	0	1
a1	2	3	0	-1	1	6
F	-2	-3	0	1	0	-6
f	1	1	0	0	0	0
		↑				

Basis	x1	x2	s1	s2	a1	Value
x2	1	1	1	0	0	1
a1	-1	0	-3	-1	1	3
F	1	0	3	1	0	-3
f	0	0	-1	0	0	-1

At the end of phase 1, F is optimal (minimal) and positive while a1 is still basic (and positive).
Hence there are no feasible solutions for the original problem (see slide 11).



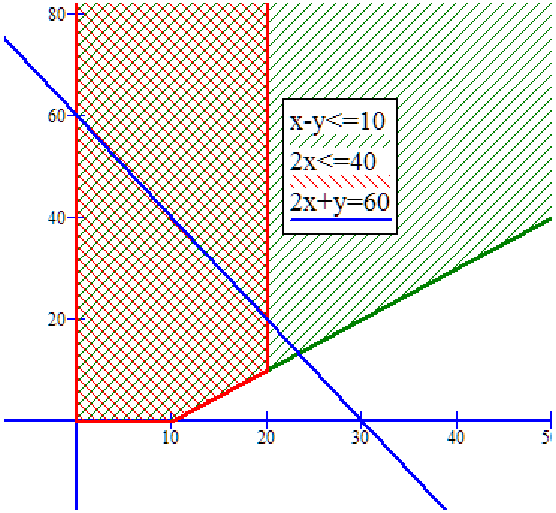
2. Unbound optimal value

$$\max f = 2x_1 + x_2$$
s.a
$$\begin{aligned} x_1 \quad - x_2 &\le 10 \\ 2x_1 \qquad &\le 40 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Base	x1	x2	s1	s2	Valor
← s1	1	-1	1	0	10
s2	2	0	0	1	40
f	2	1	0	0	0
	↑				

Base	x1	x2	s1	s2	Valor
x1	1	-1	1	0	10
← s2	0	2	-2	1	20
f	0	3	-2	0	-20
		↑			

Base	x1	x2	s1	s2	Valor
x1	1	0	0	1/2	20
x2	0	1	-1	1/2	10
f	0	0	1	-3/2	-20



- At some iteration of the simplex method, a non-basic variable with positive coefficient can enter the basis without a bound on its value (maximization)

- This means we can bring that variable in the basis and increase the *z*-value to +∞ (since the variable can be increased to +∞).

3. Multiple optimal solutions

$$\max \ f = 4x_1 + 2x_2$$

s.a

$$2x_1 + x_2 \leq 2$$
$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

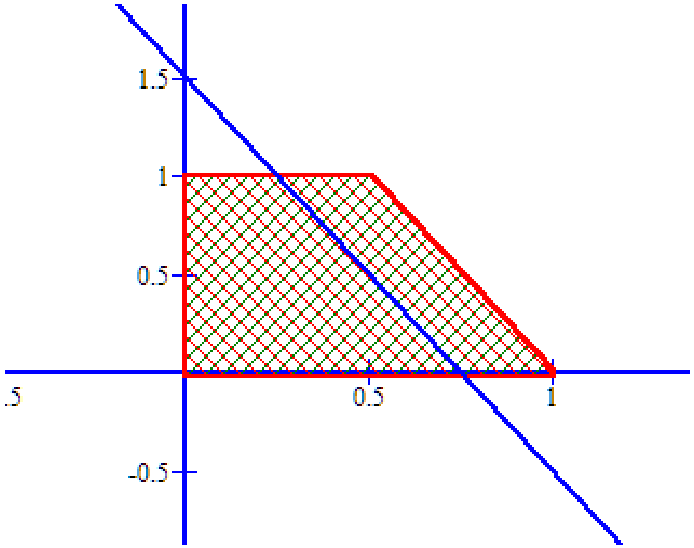
Basis	x1	x2	s1	s2	Value
← s1	2	1	1	0	2
s2	0	1	0	1	1
f	4	2	0	0	0

↑

Basis	x1	x2	s1	s2	Value
x1	1	0,5	0,5	0	1
s2	0	1	0	1	1
f	0	0	-2	0	-4

– There is a non-basic variable (x2) with 0 coefficient in the optimal table.
– This means we can bring that variable in the basis without changing the f-value.

The new solution would also be optimal.



5. Degeneracy

<u>Degeneracy</u>: occurs when a basic feasible solution has one or more <u>basic</u> variables with value 0

In a <u>n-dimensional space</u>, we have degeneracy when at least <u>n+1 constraints</u> intersect at the same vertex of the feasible region.

For example, when n=2, we have a degeneracy if 3 or more constraints intersect in the same vertex of the feasible region.

A degeneracy can occur:

(i) at the initial basic solution, or

(ii) at any iteration, when a tie occurs in the candidate variables to leave the basis (we can choose arbitrarily, although it is common practice to choose the lowest index variable).

Degeneracy and infinite loops

- Theoretically, on degenerated soluitons, the algorithm of the Simplex method can get into an infinite loop without progressing to the optimum solution.
- However, the resolution of real problems has shown that this issue rarely occurs.
- Software implementations of the Simplex algorithm ensure that it does not get into a loop.

Degeneracy at the initial basic solution

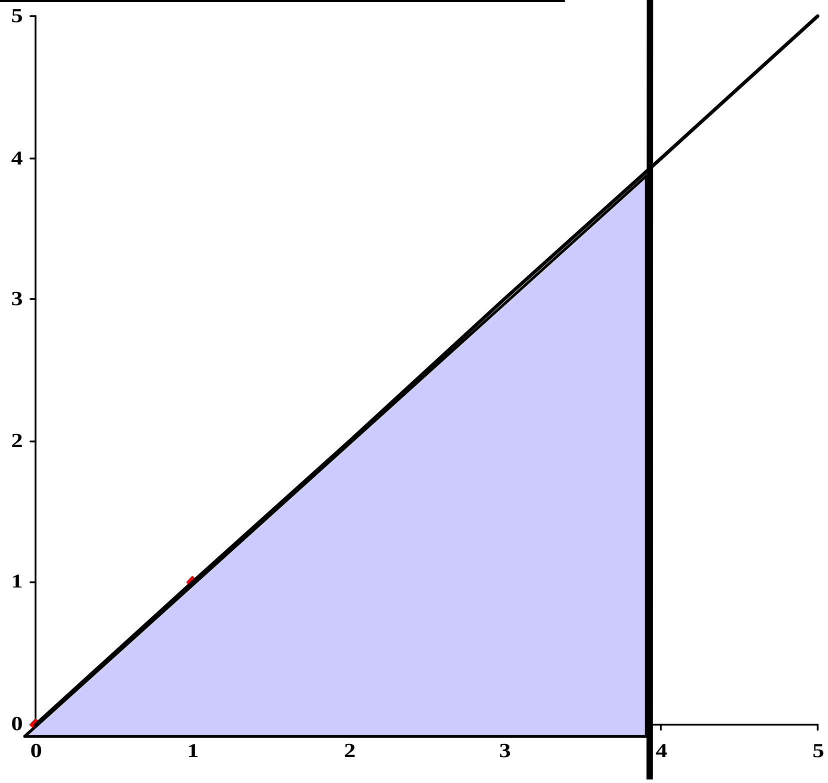
$$\begin{aligned} \max \quad & f = x_1 + 2x_2 \\ \text{s.a} \quad & \\ & x_1 \le 4 \\ & -x_1 + x_2 \le 0 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & f = x_1 + 2x_2 \\ \text{s.a} \quad & \\ & x_1 + s_1 = 4 \\ & -x_1 + x_2 + s_2 = 0 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$

Base	x1	x2	s1	s2	Valor
s1	1	0	1	0	4
s2	-1	1	0	1	0
f	1	2	0	0	0

Base	x1	x2	s1	s2	Valor
s1	1	0	1	0	4
x2	-1	1	0	1	0
f	3	0	0	-2	0

Base	x1	x2	s1	s2	Valor
x1	1	0	1	0	4
x2	0	1	1	1	4
f	0	0	-3	-2	-12



Degeneracy at any iteration of Simplex

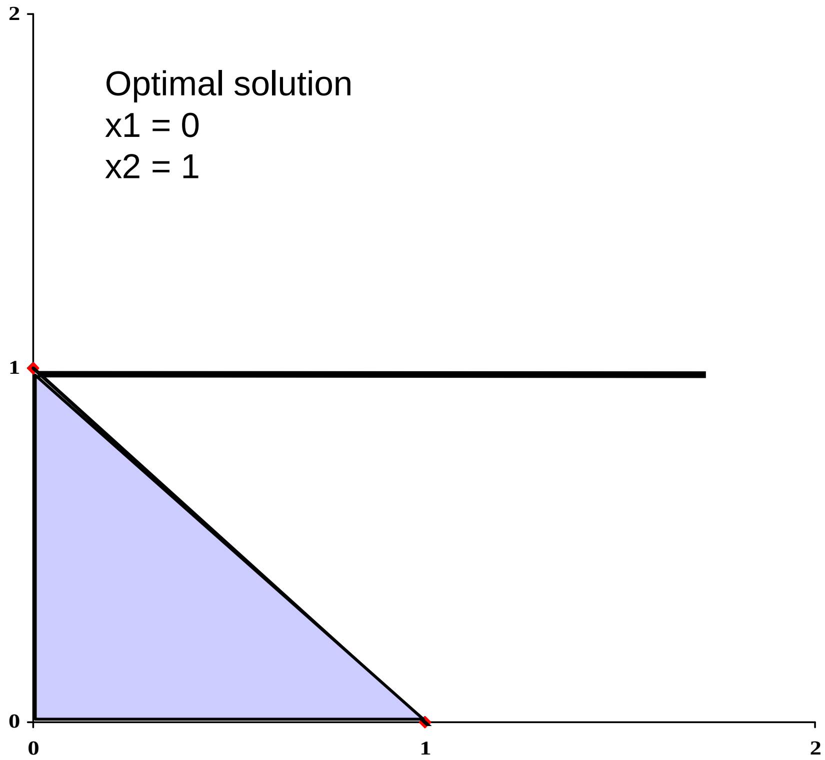
$$\max f = x_1 + 2x_2$$
s.a
$$\begin{aligned} x_2 &\leq 1 \\ x_1 + x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Base	x1	x2	s1	s2	Valor
s1	0	1	1	0	1
s2	1	1	0	1	1
f	1	2	0	0	0

Base	x1	x2	s1	s2	Valor
x2	0	1	1	0	1
s2	1	0	-1	1	0
f	1	0	-2	0	-2

Base	x1	x2	s1	s2	Valor
x2	0	1	1	0	1
x1	1	0	-1	1	0
f	0	0	-1	-1	-2

$$\max f = x_1 + 2x_2$$
s.a
$$\begin{aligned} x_2 + s_1 &= 1 \\ x_1 + x_2 + s_2 &= 1 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$



Optimal solution
x1 = 0
x2 = 1

Exercise

The following Simplex tableaux represents the final solution of a maximization linear problem. Which are the values that constants ***a***, ***b***, ***c*** e ***d*** can take in the following situations:

i. The solution is optimal and unique;
ii. There are several alternative optimal solutions;
iii. The optimal solution is unbounded;
iv. The optimal solution is degenerated.

Base	x_1	x_2	x_3	x_4	x_5	Valor
x_4	-4	$\boldsymbol{b}$	0	1	0	1
x_3	2	-1	1	0	0	4
x_5	3	$\boldsymbol{c}$	0	0	1	$\boldsymbol{a}$
f	-2	$\boldsymbol{d}$	0	0	0	-10

Solution

(i) Para a solução ser óptima $\Rightarrow d \le 0$. Dado que tem que ser única $\Rightarrow d \neq 0$;
Dada a restrição de não negatividade das variáveis: $a \ge 0$;
Assim: $d < 0 \wedge a \ge 0$.

ii) $d = 0$ por forma a que, se a variável não básica $x2$ entrasse para a base, o valor da função objectivo não se alterasse;
Dada a restrição de não negatividade das variáveis: $a \ge 0$;
Para garantir que, ao entrar para a base a variável não básica $x2$, saia desta uma outra variável, é necessário que $\left(\frac{1}{b} \ge 0 \vee \frac{a}{c} \ge 0\right)$, que é equivalente a $(b > 0 \vee c > 0)$

Assim: $d = 0 \wedge a \ge 0 \wedge (b > 0 \vee c > 0)$

iii) Para a solução ser óptima e não limitada é necessário que ao entrar a variável não básica $x2$ para a base, não saia de lá nenhuma variável básica, ou seja, $x2$ pode entrar para a base com um valor qualquer ilimitado.

Assim, $d > 0 \wedge a \ge 0 \wedge c < 0 \wedge b < 0$

iv) $a = 0 \wedge d < 0$ (b e c quaisquer) .

Fundamental concepts on Duality

Duality

Duality is one of the most important findings in Linear Programming. It shows that each linear programming problem is associated to a second linear problem, known as the **Dual Problem**.

Applications of Duality concepts:

- The relationships between the primal and the dual problems provide important information to sensitivity analysis.
- Resolution of Linear Programming problems:

  Sometimes it is easier to solve the Dual problem instead of solving the original problem (Primal problem). Since the optimal value of both objective functions is the same, and there is a correspondence between the optimal solutions of each problem, it may be easier to solve the dual problem first and deduce the optimal solution of the primal problem next.

- Resolution of Transportation Problems

The standard form and the Dual problem

A **maximization** problem is in the standard form if all the variables are non-negative and all the constraints are of **<=** type.

A **minimization** problem is in the standard form if all the variables are non-negative and all the constraints are of **>=** type.

Example

Primal

$$
\begin{array}{l}
\max f = x_1 + 2x_2 \\
\text{s.to} \\
\begin{array}{cccc}
x_1 & + x_2 & \leq & 4 \\
 & x_2 & \leq & 3 \\
 & x_1, x_2 & \geq & 0
\end{array}
\end{array}
$$

$\Rightarrow$

Dual

$$
\begin{array}{l}
\min g = 4y_1 + 3y_2 \\
\text{s.to} \\
\begin{array}{cccc}
y_1 & & \geq & 1 \\
y_1 & + y_2 & \geq & 2 \\
 & y_1, y_2 & \geq & 0
\end{array}
\end{array}
$$

Generalization

Primal Problem

$$\begin{aligned} &\max \quad c^T.x \\ &\text{s.to} \quad A\,x \le b \\ &\qquad\quad x_j \ge 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} &\min \quad b^T.\,y \\ &\text{s.to} \quad A^T\,y \ge c \\ &\qquad\quad y_i \ge 0 \end{aligned}$$

Correspondence between Primal and Dual

max f(x_j)	**min g(y_i)**
Constraint i (<=)	**Variable y_i (y_i >= 0)**
Variable x_j (x_j >= 0)	**Constraint j (>=)**
f coefficients	**Righthand side c**
Righthand side b	**g coefficients**

Example

Primal

$$\begin{aligned}
\max f &= x_1 + 2x_2 \\
\text{s.a} & \\
x_1 + x_2 &\le 4 \\
x_2 &\le 2 \\
x_1, x_2 &\ge 0
\end{aligned}$$

Dual

$$\begin{aligned}
\min g &= 4y_1 + 2y_2 \\
\text{s.a} & \\
y_1 &\ge 1 \\
y_1 + y_2 &\ge 2 \\
y_1, y_2 &\ge 0
\end{aligned}$$

Canonical Form (Phase 2)

$$\begin{aligned}
\max f &= x_1 + 2x_2 \\
\text{s.a} & \\
x_1 + x_2 + s_1 &= 4 \\
x_2 + s_2 &= 2 \\
x_1, x_2, s_1, s_2 &\ge 0
\end{aligned}$$

Canonical Form (Phase 1)

$$\begin{aligned}
\min G &= a_1 + a_2 = 3 - 2y_1 - y_2 + t_1 + t_2 \\
\text{s.a} & \\
y_1 - t_1 + a_1 &= 1 \\
y_1 + y_2 - t_2 + a_2 &= 2 \\
y_1, y_2, t_1, t_2, a_1, a_2 &\ge 0
\end{aligned}$$

So far, we have analyzed a maximization primal LP problem and the corresponding dual. What happens if the primal problem is not in the standard form?

Converting to the standard form (maximization)

standard form

$$\boxed{\begin{aligned} &\max \ c^T.x \\ &\text{s.a} \quad A\,x \le b \\ &\qquad x_i \ge 0 \end{aligned}}$$

Non-standard form	Standard form
Min f	Max (-f)
$\sum_{j=1}^{n} a_{ij}x_j \ge b_i$	$-\sum_{j=1}^{n} a_{ij}x_j \le -b_i$
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\sum_{j=1}^{n} a_{ij}x_j \le b_i \wedge -\sum_{j=1}^{n} a_{ij}x_j \le -b_i$
xj unrestricted in sign	$x_j^+ - x_j^-, x_j^+ \ge 0, x_j^- \ge 0$

In order to find the dual of any LP problem, we must express it (the primal) in one of the standard forms:

maximize constraints <=	or	minimize constraints >=

Primal

$$\begin{array}{l} \min f = 2x_1 + 4x_2 \\ \text{s.a} \\ \begin{array}{cccc} 6x_1 & +10x_2 & \leq & 60 \\ 2x_1 & +x_2 & \geq & 10 \\ 2x_1 & +3x_2 & = & 18 \\ & x_1, x_2 & \geq & 0 \end{array} \end{array}$$

Primal

$$\begin{array}{l} \min f = 2x_1 + 4x_2 \\ \text{s.a} \\ \begin{array}{cccc} -6x_1 & -10x_2 & \geq & -60 \\ 2x_1 & +x_2 & \geq & 10 \\ 2x_1 & +3x_2 & \geq & 18 \\ -2x_1 & -3x_2 & \geq & -18 \\ & x_1, x_2 & \geq & 0 \end{array} \end{array}$$

Dual

$$\begin{array}{l} \max f = -60y_1 + 10y_2 + 18(y_3 - y_4) \\ \text{s.a} \\ \begin{array}{ccccc} -6y_1 & +2y_2 & +2(y_3 - y_4) & \leq & 2 \\ -10y_1 & +y_2 & +3(y_3 - y_4) & \leq & 4 \\ & y_1, y_2 & y_3, y_4 & \geq & 0 \end{array} \end{array}$$

If we have a variable $x \in \Re$, we replace it by two new variables such as $x = x^+ - x^-$, com $x^+, x^- >= 0$.

To each equality constraint in the primal problem there is a dual variable unrestricted in sign.

...coming back to our example...

Primal problem resolution

1st tableaux
(2nd phase)

Basis	x1	x2	s1	s2	Value
s1	1	1	1	0	4
s2	0	1	0	1	2 →
max f	1	2 ↑	0	0	0

2nd tableaux

Basis	x1	x2	s1	s2	Value
s1	1	0	1	-1	2 →
x2	0	1	0	1	2
f	1 ↑	0	0	-2	-4

3rd tableaux

Optimal solution

Basis	x1	x2	s1	s2	Value
x1	1	0	1	-1	2
x2	0	1	0	1	2
f	0	0	-1	-1	-6

Dual problem resolution

1st tableaux
(1st phase)

Basis	y1	y2	t1	t2	a1	a2	Value
a1	1	0	-1	0	1	0	1 →
a2	1	1	0	-1	0	1	2
min G	-2	-1	1	1	0	0	-3
g	4	2	0	0	0	0	0

2nd tableaux

Basis	y1	y2	t1	t2	a1	a2	Value
y1	1	0	-1	0	1	0	1
a2	0	1	1	-1	-1	1	1 →
G	0	-1	-1	1	2	0	-1
g	0	2	4	0	-4	0	-4

3rd tableaux
optimal solution 1st phase and optimal solution of the original problem

Basis	y1	y2	t1	t2	a1	a2	Value
y1	1	0	-1	0	1	0	1
y2	0	1	1	-1	-1	1	1
G	0	0	0	0	1	1	0
g	0	0	2	2	-2	-2	-6

Some Fundamental Duality Properties

Weak Duality Theorem:

if x is a feasible solution for the primal minimization problem and y is a feasible solution for the dual maximization problem, then weak duality implies $g(y) <= f(x)$ where f and g are the objective functions for the primal and dual problems respectively

Strong Duality Theorem

If the primal (dual) problem has a finite optimal solution, then so does the dual (primal) problem and these two values are equal. That is, $g^* = f^*$

These properties describe the key relationships between any pair of primal and dual solutions.
One useful application is for evaluating a proposed solution for the primal problem.

For example, suppose that x is a primal feasible solution proposed for implementation and that a feasible dual solution y was found, such that $cx = yb$. Hence, we can deduce that x must be optimal without solving the problem by the Simplex method.

Even if $cx<yb$, then yb still provides an upper bound on f, so if $yb-cx$ is very small, we can, sometimes, use x as a good approximation of the optimal solution.

Some Fundamental Duality Properties

Complementary Slackeness Property

If, in an optimal solution of a linear program, the value of the dual variable associated with a constraint is nonzero, then that constraint must be satisfied with equality. Further, if a constraint is satisfied with strict inequality, then its corresponding dual variable must be zero.

The values of the optimal dual solution are the shadow prices in the primal problem

Symmetry Property

For any primal and corresponding dual problems, all the relations are symmetrical because the dual of the dual problem is the primal.

Duality Theorem

The following are the only possible relationships between the primal and dual problems:

1. If one of the problems (primal or dual) has feasible solutions and a bounded objective function (and so has an optimal solution), then so does the other problem.
2. If one of the problems has feasible solutions and an unbounded objective function (and so no optimal solutions), then the other problem has no feasible solutions.
3. If one of the problems has no feasible solutions, then the other problem has either no feasible solutions or an unbounded objective function.

Economic interpretation

Usual economic interpretation of a primal problem is the standard form: :

Quantity	Interpretation
x_j	Level of activity j
c_j	Unit profit from activity j
f	Total profit from all activities
b_i	Amount of resource i available
a_{ij}	Amount of resource i consumed per unit of activity j

Economic interpretation of the dual problem

The economic interpretation of the dual problem is directly based on the corresponding interpretation of the primal in its standard form.
Since the dual optimal value is equal to the optimal primal one (profit), we can say that:

$$\begin{array}{ll} \min & g = b^T . y \\ \text{s.a} & A^T y \geq c \\ & y_i \geq 0 \end{array}$$

yi is the value of resource i
yi >= 0 says that the contribution to profit per unit of resource *i* (*i* 1, 2, . . . , *m*) must be nonnegative: otherwise, it would be better not to use this resource at all.

In other words, y_i are the shadow prices.

$$g = \sum_{i=1}^{m} b_i y_i$$

corresponds to minimizing the total implicit value of the resources consumed by the activities.

Economic interpretation of the dual problem

$$\begin{aligned} \min\ & g = b^T . y \\ \text{s.a}\ & A^T y \geq c \\ & y_i \geq 0 \end{aligned}$$

Since each unit of activity j in the primal problem uses aij units of resource i,

$$\sum_{i=1}^{m} a_{ij} y_i$$

is interpreted as the current contribution to profit of the mix of resources that would be consumed if 1 unit of activity *j* were used (*j* = 1, 2, . . . , *n*)..

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

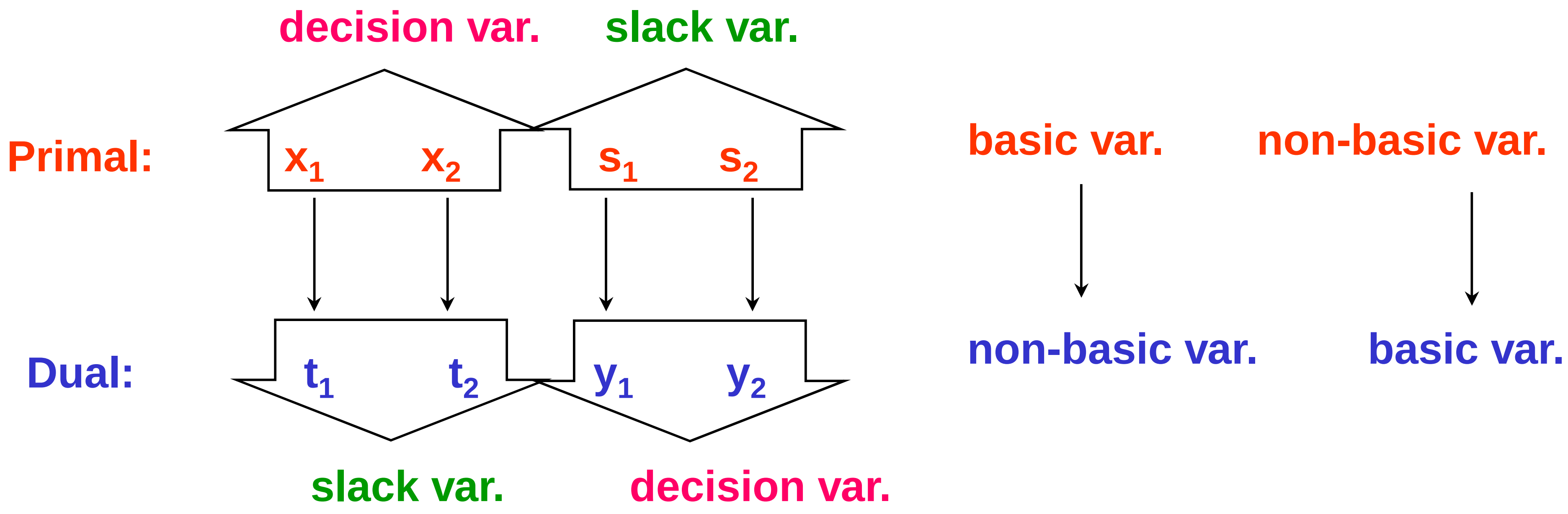
says that the actual contribution to profit of the above mix of resources must be at least as much as if they were used by 1 unit of activity *j;* otherwise, we would not be making the best possible use of these resources.

Primal-dual relationships

- The optimal solutions in both problems are equal:

  max f = min g

- there is a correspondence between primal and dual variables



Comparison between the final Primal and Dual simplex tableaux

Final Primal Tableau

Basis	x1	x2	s1	s2	Value
x1	1	0	1	-1	2
x2	0	1	0	1	2
max f	0	0	-1	-1	-6

Final Dual tableau

Basis	y1	y2	t1	t2	Value
y1	1	0	-1	0	1
y2	0	1	1	-1	1
min g	0	0	2	2	-6

Primal-dual relationships

- the value of a primal (dual) variable is the absolute value of the o,f. coefficient of the corresponding dual (primal) variable.

$x_1 = 2 \quad = |\text{ coeff. of } t_1 \text{ in } g = 2| = 2$

$x_2 = 2 \quad = |\text{ coeff. of } t_2 \text{ in } g = 2| = 2$

$y_1 = 1 \quad = |\text{ coeff. of } s_1 \text{ in } f = -1| = 1$

$y_2 = 1 \quad = |\text{ coeff. of } s_2 \text{ in } f = -1| = 1$

- the coefficient that in the primal tableau is at the intersection of a row (basic var.) and a non-basic column is the symmetrical of the coefficient that, in the dual tableau is at the intersection of the corresponding column and row dual variables.

Primal – Intersection of x_1 (basic) and s_2 (non-basic) = -1

Dual – Intersection of t_1 (non-basic) and y_2 (basic) = 1

Empresa de madeira

A company producing two products A and B intends to optimize its week production plans in order mo maximize the profit. The unit profits of A and B are of €8 and €5.

The production of A and B implies the consumption of other resources, namely, timber and labour hours.

The following table shows the amount of resources used per unit of A and B produced.

Product	Timber(m)	Labour hours
A	30	5
B	20	10

The company has 300 m of timber/wek and 110 hours of week hous available.

(a) Which is the optimal production plan?

(b) Market studies indicate that the profit of B will increase of € 1/6 per week, while the profit of A will remain constant. Taking this tendency into account, for how many weeks will the previous optimal solution remain optimal?

(c) At what price will the company pay for extra labour hours?

$$\max L = 8X_1 + 5X_2$$

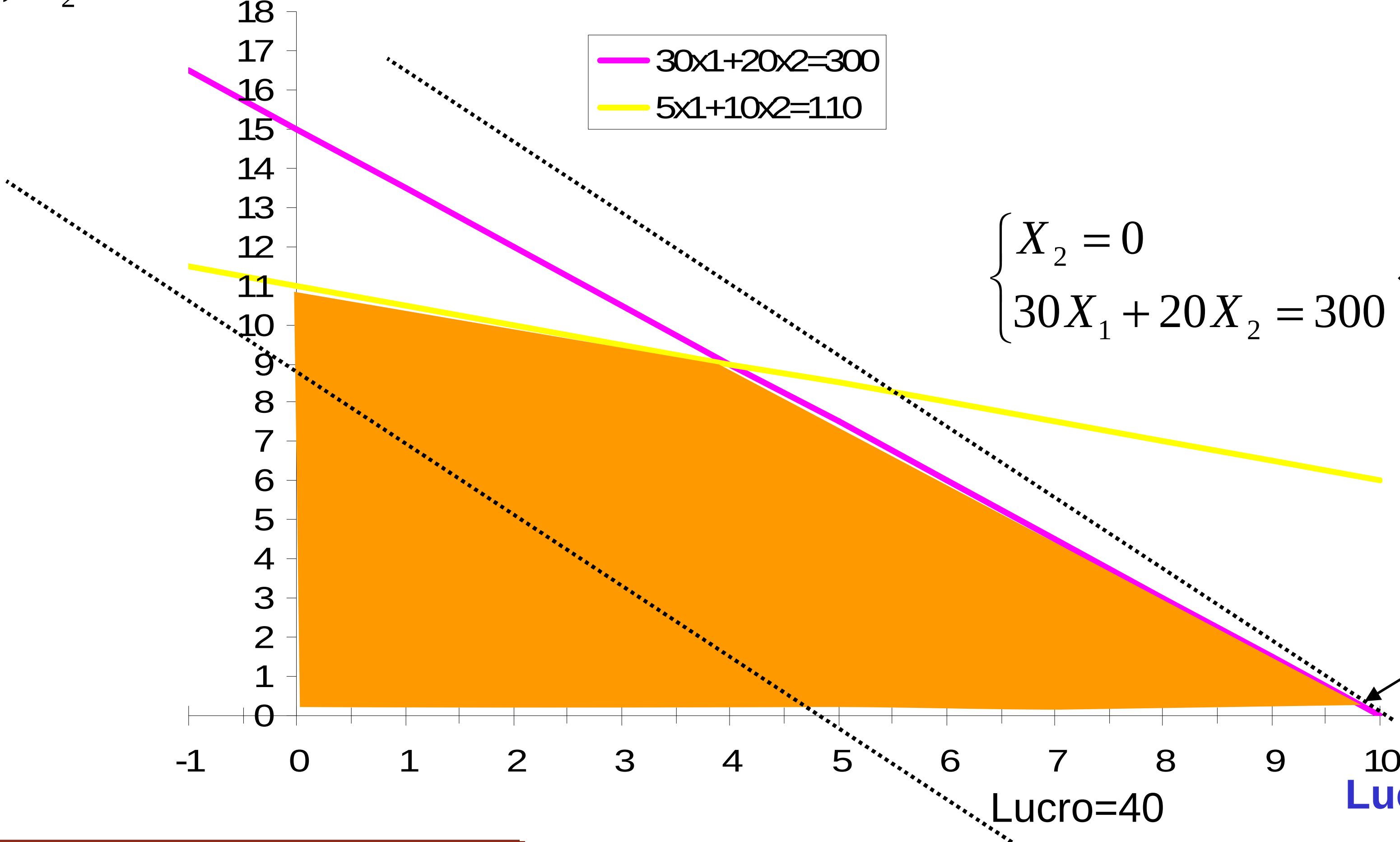
$$30X_1 + 20X_2 \leq 300$$

$$5X_1 + 10X_2 \leq 110$$

$$X_1, X_2 \geq 0$$

Notes:

- **Reduced cost of X2:** if the profit of X2 increases to 5.33, it becomes profitable to produce X2 (X2 >0)
- **Shadow price of timber constraint:** if we one exta unit of timber the profit increases from €80 to €80,2667



$$\begin{cases} X_2 = 0 \\ 30X_1 + 20X_2 = 300 \end{cases} \Leftrightarrow \begin{cases} X_2 = 0 \\ X_1 = 10 \end{cases}$$

Solução óptima

Lucro=80

Primal

Variable -->	X1	X2	Direction	R. H. S.
Maximize	8	5		
C1	30	20	<=	300
C2	5	10	<=	110
LowerBound	0	0		
UpperBound	M	M		
VariableType	Continuous	Continuous		

Iter. 1

		X1	X2	Slack_C1	Slack_C2		
Basis	C(j)	8,0000	5,0000	0	0	R. H. S.	Ratio
Slack_C1	0	30,0000	20,0000	1,0000	0	300,0000	10,0000
Slack_C2	0	5,0000	10,0000	0	1,0000	110,0000	22,0000
	C(j)-Z(j)	8,0000	5,0000	0	0	0	

Iter. 2

		X1	X2	Slack_C1	Slack_C2		
Basis	C(j)	8,0000	5,0000	0	0	R. H. S.	Ratio
X1	8,0000	1,0000	0,6667	0,0333	0	10,0000	
Slack_C2	0	0	6,6667	-0,1667	1,0000	60,0000	
	C(j)-Z(j)	0	-0,3333	-0,2667	0	80,0000	

Dual

Variable -->	C1	C2	Direction	R. H. S.
Minimize	300	110		
X1	30	5	>=	8
X2	20	10	>=	5
LowerBound	0	0		
UpperBound	M	M		
VariableType	Continuous	Continuous		

Iter. 1

		C1	C2	Surplus_X1	Surplus_X2	Artificial_X1	Artificial_X2		
Basis	C(j)	300,0000	110,0000	0	0	0	0	R. H. S.	Ratio
Artificial_X1	M	30,0000	5,0000	-1,0000	0	1,0000	0	8,0000	0,2667
Artificial_X2	M	20,0000	10,0000	0	-1,0000	0	1,0000	5,0000	0,2500
	C(j)-Z(j)	300,0000	110,0000	0	0	0	0	0	
	* Big M	-50,0000	-15,0000	1,0000	1,0000	0	0	0	

Iter. 2

		C1	C2	Surplus_X1	Surplus_X2	Artificial_X1	Artificial_X2		
Basis	C(j)	300,0000	110,0000	0	0	0	0	R. H. S.	Ratio
Artificial_X1	M	0	-10,0000	-1,0000	1,5000	1,0000	-1,5000	0,5000	0,3333
C1	300,0000	1,0000	0,5000	0	-0,0500	0	0,0500	0,2500	M
	C(j)-Z(j)	0	-40,0000	0	15,0000	0	-15,0000	75,0000	
	* Big M	0	10,0000	1,0000	-1,5000	0	2,5000	0	

Iter. 3

		C1	C2	Surplus_X1	Surplus_X2	Artificial_X1	Artificial_X2		
Basis	C(j)	300,0000	110,0000	0	0	0	0	R. H. S.	Ratio
Surplus_X2	0	0	-6,6667	-0,6667	1,0000	0,6667	-1,0000	0,3333	
C1	300,0000	1,0000	0,1667	-0,0333	0	0,0333	0	0,2667	
	C(j)-Z(j)	0	60,0000	10,0000	0	-10,0000	0	80,0000	
	* Big M	0	0	0	0	1,0000	1,0000	0	

Final Primal tableau

		X1	X2	Slack_C1	Slack_C2		
Basis	C(j)	8,0000	5,0000	0	0	R. H. S.	Ratio
X1	8,0000	1,0000	0,6667	0,0333	0	10,0000	
Slack_C2	0	0	6,6667	-0,1667	1,0000	60,0000	
	C(j)-Z(j)	0	-0,3333	-0,2667	0	80,0000	

Reduced cost of X2: 0,33
Shadow price of Slack_C1: 0,2667

	basic	non-basic	non-basic	basic
Primal	**X1**	**X2**	**Slack_C1**	**Slack_C2**
Dual	**Surplus_X1**	**Surplus_X2**	**C1**	**C2**
	non-basic	basic	basic	non-basic

Final Dual tableau

		C1	C2	Surplus_X1	Surplus_X2	Artificial_X1	Artificial_X2		
Basis	C(j)	300,0000	110,0000	0	0	0	0	R. H. S.	Ratio
Surplus_X2	0	0	-6,6667	-0,6667	1,0000	0,6667	-1,0000	0,3333	
C1	300,0000	1,0000	0,1667	-0,0333	0	0,0333	0	0,2667	
	C(j)-Z(j)	0	60,0000	10,0000	0	-10,0000	0	80,0000	
	* Big M	0	0	0	0	1,0000	1,0000	0	

Exercise A

a) Define the dual problem corresponding to the following linear program

$$\begin{aligned}
\max \quad & f = 2x_1 + 3x_2 \\
\text{s.a} \quad & \\
& 0.25x_1 + 0.50x_2 \leq 40 \\
& 0.40x_1 + 0.20x_2 \leq 40 \\
& 0.80x_2 \leq 40 \\
& x_1, x_2 \geq 0
\end{aligned}$$

b) Knowing that the following tableau corresponds to the optimal primal solution, write the final dual tableau.

Basis	x1	x2	s1	s2	s3	Value
x1	1	0	-1,33	3,33	0	80
s3	0	0	-2,13	1,33	1	8
x2	0	1	2,67	-1,67	0	40
max f	0	0	-5,33	-1,67	0	-280

Resolution of Exercise A

a) Dual problem

$$\min \; g = 40y_1 + 40y_2 + 40y_3$$

s.a

$$\begin{array}{lllcl} 0.25y_1 & + 0.40y_2 & & \geq & 2 \\ 0.50y_1 & + 0.20y_2 & + 0.8y_3 & \geq & 3 \\ & y_1, y_2 & & \geq & 0 \end{array}$$

	b	**b**	nb	nb	**b**
Primal	**x1**	**x2**	s1	s2	**s3**
Dual	s4	s5	**y1**	**y2**	y3
	nb	nb	**b**	**b**	nb

b) Final dual Simplex tableau.

Primal

Base	x1	x2	s1	s2	s3	Valor
x1	1	0	-1,33	3,33	0	80
s3	0	0	-2,13	1,33	1	8
x2	0	1	2,67	-1,67	0	40
max f	0	0	-5,33	-1,67	-1	-280

Dual

Base	y1	y2	y3	s4	s5	Valor
y1	1	0	2,13	1,33	-2,67	5,33
y2	0	1	-1,33	-3,33	1,67	1,67
min g	0	0	8	80	40	-280