ON THE FIXED CHARGE TRANSPORTATION PROBLEM

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Abstract

The paper primarily investigates the application of mixed-variable techniques (due to J. F. Benders) to the Fixed Charge Transportation Problem. On account of the special structure of the problem several parts of the general algorithm can be simplified. The mathematical programming part of the general iteration becomes essentially a zero-one integer programming problem. The linear programming part reduces to a transportation problem with certain prohibited routes, which is discussed in detail. An example for the iterative procedure is given.

A. INTRODUCTION

In the first part of these notes we define the symbols and formulate the Fixed Charge Transportation Problem, following in part the developments of M. L. Balinsky[1]. The reader may find a description of this problem also in standard texts, such as in reference [2]. While the statement of the problem is simple, its practical solution is known to offer great difficulties.

In the main body of the paper we consider the Fixed Charge Problem as a special case of a mixed-variables programming problem as treated by J. F. Benders[3]. The special structure of the problem at hand allows the construction of an iterative algorithm for the solution of the Fixed Charge Transportation Problem which consists of alternating executions of Integer Programming and Transportation Problems without surcharges. The difficulty of possible infeasible problems is dealt with in a manner considerably simpler than that necessitated by the general case of alternating mathematical programming and linear programming problems.

The application of the iterative procedure is finally illustrated by means of a simple test problem. Our results seem to indicate that it should be possible to write machine programs which could solve problems of moderate size, say of about 30 sources and destinations, on currently available computers. The success of a machine program will depend to a great extent on the efficiency of current integer programming techniques, most of which are based on the work of R. E. Gomory (e.g. [4]). Again the special nature of the problem discussed here should allow the use of simplifications which are not applicable in general.

B. TERMINOLOGY, SYMBOLS

In the following we shall use standard transportation problem or vector terminology, whichever is more convenient at a particular stage. In this section we briefly define some of the symbols. The variables can be considered to be integers.

$$\left.\begin{array}{l} \text{Source availabilities: } a_1, a_2, \dots, a_m;\ a_i > 0 \\ \text{Destination requirements: } b_1, b_2, \dots, b_n;\ b_j > 0 \\ \text{Shipments: } x_{ij} \geq 0,\ i = 1, 2, \dots, m;\ j = 1, 2, \dots, n \\ \text{Costs: } c_{ij} \geq 0,\ i = 1, 2, \dots, m;\ j = 1, 2, \dots, n \\ \text{Surcharges: } q_{ij} \geqslant 0,\ i = 1, 2, \dots, m;\ j = 1, 2, \dots, n \end{array}\right\} \tag{2.1}$$

Objective functions (to be maximized):

$$\left.\begin{array}{l} \text{Normal Transportation Problem:} \\ \quad g(x) = -\ \Sigma\, c_{ij}\, x_{ij} \\ \text{Fixed Charge Transportation Problem:} \\ \quad g(x, y) = -\ \Sigma(c_{ij} x_{ij} + q_{ij} y_{ij}) \\ \text{where } \ y_{ij} = 1 \longleftrightarrow x_{ij} \neq 0 \\ \qquad\quad\ y_{ij} = 0 \longleftrightarrow x_{ij} = 0 \end{array}\right\} \tag{2.2}$$

Constraints:

$$\left.\begin{array}{ll} f_i(x) \equiv a_i - \sum_{j=1}^{n} x_{ij} \geqslant 0 & i = 1, 2, \dots, m \\ f_{m+j}(x) \equiv -b_j + \sum_{i=1}^{m} x_{ij} \geqslant 0, & j = 1, 2, \dots, n \end{array}\right\} \tag{2.3}$$

It is convenient to introduce the following vectors and matrices: (' stands for transposition, e.g. the change of a column into a row vector)

$$x' \equiv (x_1, x_2, \dots, x_\nu) \equiv (x_{11}, x_{12}, \dots, x_{mn}),\quad \nu \equiv m \cdot n$$

$$y' \equiv (y_1, y_2, \dots, y_\nu) \equiv (y_{11}, y_{12}, \dots, y_{mn})$$

$$c' \equiv (c_1, c_2, \dots, c_\nu) \equiv (c_{11}, c_{12}, \dots, c_{mn})$$

$$q' \equiv (q_1, q_2, \ldots, q_\nu) \equiv (q_{11}, q_{12}, \ldots, q_{mn})$$

$$\delta' \equiv (\delta_1, \delta_2, \ldots, \delta_\mu) \equiv (a_1, a_2, \ldots, a_m; -b_1, -b_2, \ldots, -b_n),\ \mu \equiv m+n$$

$$u' \equiv (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_\mu) \equiv (u_1, u_2, \ldots, u_m; v_1, v_2, \ldots, v_n) \tag{2.4}$$

The "dual variables" introduced in the last definition are constrained by:

$$u \geqslant 0 \quad \text{or} \quad u_i \geqslant 0 \quad , \quad v_j \geqslant 0 \quad . \tag{2.5}$$

Introducing the vector:

$$F(x) \equiv (f_1(x), f_2(x), \ldots, f_\mu(x)) \tag{2.6}$$

and the $\mu \times \nu$ matrix (characteristic for the Transportation Problem):

$$\underset{(\mu,\nu)}{{}^{T}A} = \begin{pmatrix} 11\ldots1 & & & & \\ & 11\ldots1 & & \ddots & \\ & & & & 11\ldots1 \\ -1 & & -1 & & -1 \\ \ \ -1 & & \ \ -1 & & \ \ -1 \\ \ \ \ \ \ddots & & \ \ \ \ \ddots & & \ \ \ \ \ddots \\ \ \ \ \ \ \ -1 & & \ \ \ \ \ \ -1 & & \ \ \ \ \ \ -1 \end{pmatrix} \tag{2.7}$$

we can represent the constraints (2.3) alternatively by:

$$F(x) \geqslant 0 \quad \text{or} \quad {}^{T}A\,x \leqslant \delta \quad . \tag{2.8}$$

The normal transportation problem can be stated in the two forms:

Primal

$$\text{Min}\ \{c'\,x \mid {}^{T}A\,x \leqslant \delta\ ,\ x \geqslant 0\},$$

Dual

$$\text{Max}\ \{-u'\,\delta \mid {}^{T}A'\,u \geqslant -c,\ u \geqslant 0\} \quad . \tag{2.9}$$

The minimum of the cost function $c'x$ can be shown to be equal to the maximum of the dual form:

$$d(u_i, v_j) \equiv -u'\delta = \sum_{j=1}^{n} b_j v_j - \sum_{i=1}^{m} a_i u_i \quad . \tag{2.10}$$

Even though the symbol $d(u_1, u_2, \ldots, u_m; v_1, v_2, \ldots, v_n)$ would be more natural, we have chosen $d(u_i, v_j)$ for brevity.

C. THE FIXED CHARGE TRANSPORTATION PROBLEM

The Fixed Charge Problem can be formulated as follows (for part of the formulation compare M. L. Balinsky[1]).

Maximize

$$g(x, y) \equiv -(c'\,x + q'\,y) \tag{3.1}$$

Subject to the constraints:

$$F(x) \equiv \delta - {}^{T}A\,x \geqslant 0 \quad , \quad x \geqslant 0 \tag{3.2}$$

$$H(x, y) \equiv My - x \geqslant 0 \tag{3.3}$$

$$\underset{(\nu,\nu)}{M} \equiv \begin{pmatrix} M_{11} & & 0 \\ & M_{12} & \\ 0 & & \ddots \\ & & & M_{mn} \end{pmatrix} \quad , \quad M_{ij} \equiv \min(a_i, b_j) \tag{3.4}$$

$$\underset{(\nu,1)}{K}(y) \equiv Yy - y \geqslant 0 \quad , \quad y \geqslant 0 \tag{3.5}$$

$$Y \equiv \begin{pmatrix} y_{11} & & 0 \\ 0 & y_{12} & \\ & & \ddots \\ & & & y_{mn} \end{pmatrix} \tag{3.6}$$

$$-K(y) = y - Yy \geqslant 0 \quad , \quad y \geqslant 0 \quad . \tag{3.7}$$

Relation (3.2) represents the constraints of a normal Transportation Problem. Condition (3.3) explicitly restricts the shipments x_{ij} to be less than M_{ij}. In conjunction with the nonlinear constraint $K(y) = 0$, which has been written in terms of inequalities for purely formal reasons, it also ensures that:

$$y_{ij} = 0 \quad \text{implies} \quad x_{ij} = 0 \tag{3.8}$$

$$\text{and} \quad x_{ij} > 0 \quad \text{implies} \quad y_{ij} = 1 \quad . \tag{3.9}$$

At a maximum of $g(x, y)$ the possible combination $y_{ij} = 1$ and $x_{ij} = 0$ will evidently not occur, so that all of the conditions of (2.2) will be satisfied.

In the case of a zero surcharge q_{ij}, however, we may find it expedient, at various stages in the algorithm, to set $y_{ij} = 1$ regardless of x_{ij}.

D. A METHOD FOR SOLVING MIXED VARIABLES PROBLEMS DUE TO BENDERS

1. The Benders Algorithm[3]

In Chapter E we shall consider the Fixed Charge Problem as a special case of the Mixed Variables Problem (assumed to be feasible and to possess an optimum solution)

$$\max\ \{\gamma'\,x + f(y) \mid Ax + E(y) \leqslant b,\ x \geqslant 0,\ y \in S\} \tag{4.1}$$

$$\text{or}\ \max\ \{x_0 \mid x_0 \leqslant \gamma'\,x + f(y)\ ,\ Ax + E(y) \leqslant b,\ x \geqslant 0,\ y \in S\} \tag{4.2}$$

investigated by J. F. Benders in Ref. 3. In (4.1), (4.2) we consider S to be any closed and bounded set of vectors in ν-dimensional Euclidean space. We shall later restrict the vectors to have 0 or 1 components only. Furthermore, we consider any other constraints on the y - components, such as may be known to us a priori, to merely form part of the definition of S.

The main theme of Benders' paper may for our purposes perhaps best, even if incompletely and somewhat inaccurately, be summarized as follows. Associate with (4.1) the two auxiliary problems

$$\max \left\{ x_o \;\middle|\; \begin{pmatrix} x_o \\ y \end{pmatrix} \in G \right\} \tag{4.3}$$

$$G \equiv \bigcap_{(u_o,u)' \in C} \left\{ \begin{pmatrix} x_o \\ y \end{pmatrix} \;\middle|\; u_o x_o + u' E(y) - u_o f(y) \leqslant u'b,\; y \in s \right\} \tag{4.4}$$

and $\max \{\gamma' x \mid A x \leqslant b - E(\bar{y}), x \geqslant 0\}$ (4.5)

Here u_o is a scalar, u a vector of suitable dimensions (not to be confused with the u of (2.4)) and $\bar{y}$ a solution vector of a problem of type (4.3) The set C is defined by

$$C \equiv \left\{ \begin{pmatrix} u_o \\ u \end{pmatrix} \;\middle|\; A' u - \gamma u_o \geqslant 0,\; u \geqslant 0,\; u_o \geqslant 0 \right\} \tag{4.6}$$

For feasible problems (4-1) - (4.3) Benders shows that

ⓐ. If $(\bar{x}_o, \bar{x}, \bar{y})$ is an optimal solution of (4.2), or equivalently $(\bar{x}, \bar{y})$ is an optimal solution of (4.1) and $\bar{x}_o = \gamma' \bar{x} + f(\bar{y})$, then $(\bar{x}_o, \bar{y})$ and $\bar{x}$ are optimal solutions of (4.3) and (4.5) respectively.

ⓑ. If $(\bar{x}_o, \bar{y})$ is an optimal solution of (4.3) then (4.5) is feasible and has the optimal solution $\bar{x}$ with $\gamma' \bar{x} = \bar{x}_o - f(\bar{y})$. Furthermore $(\bar{x}, \bar{y})$ is an optimum solution of (4.1) with objective function $\bar{x}_o$.

Benders then shows that it is possible to approximate G iteratively by a sequence of sets $G(Q^k)$, $k = 0, 1, \ldots$, where

$$G(Q^k) = \bigcap_{(u_o,u)' \in Q^k \subset C} \left\{ \begin{pmatrix} x_o \\ y \end{pmatrix} \;\middle|\; u_o x_o + u' E(y) - u_o f(y) \leqslant u'b,\; y \in S \right\} \quad . \tag{4.7}$$

The iterative procedure starts with some subset, Q^o, of C and consists of alternative solution of problems structured after (4.3) and (4.5). In our special case we have an explicit choice of Q^o, with $u_o^o = 1$, in mind.

2. First Part of Iterative Step

During the k^{th} iterative step solve the mathematical programming problem

$$\max \left\{ x_o \;\middle|\; \begin{pmatrix} x_o \\ y \end{pmatrix} \in G(Q^k) \right\} \tag{4.8}$$

This problem is feasible and has an optimum solution if the initially posed problem has these properties. Let

$$\begin{pmatrix} x_o^k \\ y^k \end{pmatrix}$$

be the optimum solution of (4.8) in the k^{th} iterative step.

3. Second Part of Iterative Step

Solve the linear programming problem

$$\max \{\gamma' x \mid Ax \leqslant b - E(y^k), x \geqslant 0\} \tag{4.9}$$

or the corresponding dual problem

$$\min \{ (b - E(y^k))' \cdot u \mid A' u \geqslant \gamma, u \geqslant 0\} \tag{4.10}$$

The principal output of either problem is to be a vector u^k (or two vectors $\hat{u}^k$, $\tilde{u}^k$) which can be used to generate one (or two) additional constraints in (4.7). At the termination of the algorithm the solution x^k of (4.9) must also be at hand.

Our assumptions for problems (4.1), (4.2) imply that (4.10) is always feasible.

Regarding the problems (4.9) and (4.10) one, therefore, has to consider two possibilities.

ⓐ. Problem (4.9) is feasible. This is known to imply that (4.9) and (4.10) have finite optimum solutions, say x^k and u^k respectively. In that case we consider the function

$$F^k \equiv F(u^k, x_o^k, y^k) \equiv x_o^k - f(y^k) - (b - E(y^k))' u^k \tag{4.11}$$

which can be shown (see next section) to be equal to or larger than zero.

Equality implies that the iterative procedure has terminated

$$F^k = 0 \rightarrow x_o^k = \bar{x}_o,\; x^k = \bar{x},\; y^k = \bar{y} \quad . \tag{4.12}$$

On inequality one forms the set

$$Q^{k+1} = Q^k \cup \left\{ \begin{pmatrix} 1 \\ u^k \end{pmatrix} \right\} \tag{4.13}$$

and executes the next, the $(k + 1)^{th}$, iterative step.

ⓑ. Problem (4.9) is not feasible. Equivalently we can say that (4.10), while feasible, has no finite optimum. Therefore, there must exist a halfline

$$\{u \mid u = \hat{u}^k + \lambda \tilde{u}^k,\; \lambda \geqslant 0\} \tag{4.14}$$

such that the objective function of (4.10) can be made arbitrarily large in magnitude by the selection of a sufficiently large λ. According to reference 3 the simplex method leads to $\hat{u}^k$ as a vertex of P and to $\tilde{u}^k$ as a direction vector of an extreme halfline of C_o, where

$$P \equiv \{u \mid A'u \geqslant \gamma,\ u \geqslant 0\},$$
$$C_o \equiv \{u \mid A'\ u \geqslant 0,\ u \geqslant 0\} \qquad (4.15)$$

Having obtained $\hat{u}^k$ and $\tilde{u}^k$, one executes the $(k+1)^{th}$ iterative step with the set

$$Q^{k+1} = Q^k \cup \left\{\begin{pmatrix} o \\ \tilde{u}^k \end{pmatrix}\right\} \text{ if } F(\hat{u}^k, x_o^k, y^k) \leqslant 0 \qquad (4.16)$$

or the set

$$Q^{k+1} = Q^k \cup \left\{\begin{pmatrix} o \\ \tilde{u}^k \end{pmatrix}, \begin{pmatrix} 1 \\ \hat{u}^k \end{pmatrix}\right\} \text{ if}$$
$$F(\hat{u}^k, x_o^k, y^k) > 0 \qquad (4.17)$$

4. Termination Rules

The rules given above can be justified as follows.

Comparing the sets (4.4) and (4.7), we evidently have $G \subset G(Q^k)$. Accordingly the optimal solution $\bar{x}_o$ of (4.3), which by (b) is also the optimal solution of (4.1), is less than the maximum value, x_o^k, of x_o in the k^{th} iterative step.

$$\bar{x}_o \leqslant x_o^k \quad,\quad k = 1, 2, \ldots \qquad (4.18)$$

On the other hand, the optimal solution of (4.2) is given by

$$\bar{x}_o = [\gamma'\ x + f(y)]_{x=\bar{x},\ y=\bar{y}} \qquad (4.19)$$

and the substitution of any other feasible solution (x,y) of (4.2) must lead to a smaller value for $[\gamma'\ x + f(y)]$. Comparing the constraints of problems (4.8) and (4.9) with those of (4.2), we note that (x^k, y^k), if it exists, is feasible for (4.1) and (4.2). Hence we have, using also the equality of optimal primal and dual objective functions of (4.9) and (4.10), that

$$\bar{x}_o \geqslant \gamma'\ x^k + f(y^k) = (b - E(y^k))'\ u^k + f(y^k) \qquad (4.20)$$

Combining (4.18) and (4.20) we see that the function F^k of (4.11) is indeed, in the case of a feasible linear programming problem (4.9), non-negative and that $F^k = 0$ implies attainment of the optimal solution $\bar{x}_o$. The inequalities (4.18) and (4.20) also ensure that $(1, u^k)'$ does not yet belong to Q^k (see the constraints of (4.7)), so that the construction (4.13) indeed enlarges the set Q and improves the representation of G by G(Q).

In case of infeasibility of (4.9), it is known (as we shall have occasion to show in Chapter E) that

$$(b - E(y^k))'\ \tilde{u}^k < 0 \qquad (4.21)$$

It is evident, then, that $(0, \tilde{u}^k)'$ does not belong to Q^k. $(1, u^k)'$ may or may not belong to Q^k, so that we obtain the constructions (4.16) and (4.17).

Benders shows, considering the sets G and G(Q) more carefully, that the number of steps (4.13) or (4.16), (4.17) must be finite, so that the process indeed terminates with a solution of problem (4.1).

In the case of the Fixed Charge Transportation problem the finiteness of the process will be established directly in section 4 of Chapter E.

E. THE FIXED CHARGE TRANSPORTATION PROBLEM AS A SPECIAL MIXED VARIABLES PROBLEM

1. Formulation

We now make a number of identifications which exhibit the Fixed Charge Problem as a special case of (4.1). (Compare with 3.1 - 3.4).

$$\gamma = -c \quad,\quad f(y) = -q'\ y \qquad (5.1)$$

$$\underset{(\mu+\nu,\nu)}{A} = \begin{pmatrix} {}^T\!A \\ I_\nu \end{pmatrix} \quad,\quad \underset{(\nu,\nu)}{I_\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad (5.2)$$

$$\underset{(1,\mu+\nu)}{E(y)'} = (\underbrace{0, 0, \ldots, 0}_{\mu};\ \underbrace{-M_{11}y_{11}, -M_{12}y_{12}, \ldots, -M_{mn}y_{mn}}_{\nu}) \qquad (5.3)$$

$$\underset{(1,\mu+\nu)}{b'} = (\underbrace{\delta'}_{\mu};\ \underbrace{0, 0, \ldots, 0}_{\nu}) \qquad (5.4)$$

The constraints (3.5) and (3.7) are taken into account by a restriction of the domain of y, S, to vectors with 0-1 components.

The vector u can for convenience, i.e. in the interest of conformity with standard Transportation Problem terminology, be structured as follows:

$$\underset{(1,\mu+\nu)}{u'} = \underbrace{u_1, u_2, \ldots, u_m; v_1, v_2, \ldots, v_n}_{\mu};\ \underbrace{w_{11}, w_{12}, \ldots, w_{mn}}_{\nu}) \qquad (5.5)$$

The sets defined in (4.6) and (4.7) can now be expressed in the form

$$C = \left\{\begin{pmatrix} u_o \\ u_i \\ v_j \\ w_{ij} \end{pmatrix} \middle|\ u_o c_{ij} + u_i - v_j + w_{ij} \geqslant 0;\right.$$
$$\left. u_o \geqslant 0,\ u_i \geqslant 0,\ v_j \geqslant 0,\ w_{ij} \geqslant 0\right\} \qquad (5.6)$$

$$G(Q^k) = \bigcap_{(u_o, u_i, v_j, w_{ij}) \in Q^k} \left\{ \begin{pmatrix} x_o \\ y \end{pmatrix} \middle| u_o x_o \leq -d(u_i, v_j) + \sum_{i,j} y_{ij} \, (M_{ij} \, w_{ij} - q_{ij} \, u_o) \right\} \quad (5.7)$$

($d(u_i, v_j)$ being defined by (2.10), and the indices i, j always ranging from 1 to m and 1 to n respectively).

The iterative procedure starts from an initial subset Q^o of C, which we may take as consisting of only one vector, say V^o.

$$Q^o = \{V^o\}, \; V^{o\prime} \equiv (1, u_i^o, v_j^o, w_{ij}^o) \quad . \quad (5.8)$$

Many choices of V^o are possible. Since it is desired to make x_o as small as possible, we assign a large value to $d(u_i^o, v_j^o)$ by determining it as the optimal dual objective function of a Transportation Problem. That is we choose u_i^o, v_j^o as the optimal dual variables of the problem

$$\max \{-c' x \mid {}^T\!A x \leq \delta, \; x \geq 0\} \; . \quad (5.9)$$

The solution (u_i^o, v_j^o) satisfies the constraints of the dual problem, so that (compare [5, pp 97, 98])

$$c_{ij} + u_i^o - v_j^o \begin{cases} \geq 0 & \text{for } x_{ij}^o = 0 \\ = 0 & \text{for } x_{ij}^o > 0 \end{cases} \quad (5.10)$$

$$\sum c_{ij} \, x_{ij}^o = -\sum_{i=1}^{m} a_i \, u_i^o + \sum_{j=1}^{n} b_j v_j^o = d(u_i^o, v_j^o) \; . \quad (5.11)$$

Hence, we have that $V^o \in C$, no matter how the w_{ij}^o are chosen. We shall set

$$w_{ij}^o = 0 \quad \text{for all } i, j \quad (5.12)$$

a choice which is motivated by the form which the minimization problem (4.10) takes in the u, v, w variables:

$$A: \; \min \{-d(u_i, v_j) + \sum_{i,j} y_{ij}^k M_{ij} \, w_{ij} \mid c_{ij} + u_i - v_j + w_{ij} \geq 0, \; (u_i, v_j, w_{ij}) \geq 0\} \quad (5.13)$$

2. First Part of Iterative Step

We have to maximize (by a suitable choice of y) the scalar x_o subject to

$$y_{ij} = 0 \text{ or } 1 \quad (5.14)$$

$$u_o^\alpha x_o \leq -d(u_i^\alpha, v_j^\alpha) + \Sigma y_{ij} (M_{ij} w_{ij}^\alpha - q_{ij} u_o^\alpha) \quad (5.15)$$

$$\alpha = 0, 1, \ldots \quad .$$

The system of inequalities is augmented by one or two inequalities during the second part of the iterative step. The component u_o^α is 1 for $\alpha = 0$ and 1 or 0 for $\alpha > 0$. Since it is possible to choose $(u_i^\alpha, v_j^\alpha, w_{ij}^\alpha)$ as integers throughout the iterative procedure, the maximal value of x_o is integral. Hence we can solve (5.14), (5.15) as an integer programming problem in y and x_o.

It is at this stage desirable to impose any constraints on the y_{ij} which may be known a priori to be necessarily satisfied by the final $\overline{y}_{ij}$. As we shall see in the discussion of the second part of the iterative step, the y_{ij}^k determine the distribution of prohibited ($y_{ij}^k = 0$) routes of a Transportation Problem. Since the final optimal solution must solve such a Transportation Problem, we may impose the obviously necessary conditions (the second condition presupposes: $\Sigma a_i = \Sigma b_j$):

$$\Sigma a_i y_{ij} \geq b_j \quad , \; \Sigma b_j y_{ij} \geq a_i \quad (5.16)$$

for every maximization problem.

We can clearly also set $y_{ij} = 1$ whenever $q_{ij} = 0$, without altering the objective function and the solution to the problem.

As already mentioned before, the above conditions can be interpreted as additional permissible specifications of the domain of y, S.

3. Second Part of Iterative Step

In its primal formulation, the linear programming problem (4.9), (4.10), (5.13) can be interpreted as a Transportation Problem with routes (i,j) prohibited whenever $y_{ij}^k = 0$. It need not be feasible.

$$A: \; \max\{-c' x \mid {}^T\!A x \leq \delta, \, x_{ij} \leq M_{ij} \, y_{ij}^k, \, x_{ij} \geq 0\} \; (5.17)$$

We associate with (5.17) a modified, normal Transportation Problem (identified in the sequel by a letter B) in which the prohibited routes of (5.17) are represented by routes with prohibitively large shipping costs $c_{ij} + L_{ij}$.

$$B: \; \max\{-(c + L^k)' x \equiv -\overline{c}' x \mid {}^T\!A x \leq \delta, \, x \geq 0\} \quad (5.18)$$

$$L_{ij}^k = L \text{ (large integer) for } y_{ij}^k = 0 \qquad (5.19)$$
$$= 0 \qquad \text{for } y_{ij}^k = 1$$

In a machine program one must take L to be one of the largest numbers that can be represented in the machine. One then tacitly assumes that, for problems which are at all tractable, no other variable comes anywhere near L. In some of the following arguments we shall talk of quantities (rather loosely) as being 0(L), or of order L, if they are of the form a + b L, a and b being small integers (b > 0). We do not anticipate having to compute such quantities.

At this stage one might discover that the original problem already had some prohibited routes, themselves represented by large costs. We have to assume, therefore, not only that $\Sigma a_i \geqslant \Sigma b_j$ but also that the original problem has a solution which does not require the use of its prohibited routes. For a machine program we would indeed have to use two levels of "largeness", several orders of magnitude apart, the higher level being reserved for the originally prohibited routes.

We can now distinguish two types of solutions of (5.18)(B) according to whether or not the solution involves routes which were prohibited in (5.17) (A).

a. <u>The optimal solution of B does not involve L.</u> In this case we have $\min \bar{c}'x|_B = \min c'x|_A$, and the optimal primal variables of B, $\overline{X}^B$, evidently also solve the more restrictive problem A: $\overline{X}^B = \overline{X}^A$.

Most algorithms which solve the Transportation Problem (5.18) will also arrive at the final optimal solution $(\overline{u}_i^B, \overline{v}_j^B)$ of the problem dual to (5.18):

$$\text{B: } \min \{u'\delta = -d(u_i, v_j) \,|\, {}^T A'u \geqslant -\bar{c},\, u \geqslant 0\}. \quad (5.20)$$

The equality at the optimum of primal and dual objective functions for both problem A and problem B yields:

$$-d(\overline{u}_i^B, \overline{v}_j^B) = -\bar{c}'\,\overline{X}^B = -c'\,\overline{X}^A =$$
$$= -d(\overline{u}_i^A, \overline{v}_j^A) + \Sigma y_{ij}^k M_{ij} \overline{w}_{ij}^A \quad (5.21)$$

The constraints of (5.20) ensure that

$$c_{ij} + L_{ij}^k + \overline{u}_i^B - \overline{v}_j^B \geqslant 0,\ \overline{u}_i^B \geqslant 0,$$
$$\overline{v}_j^B \geqslant 0. \quad (5.22)$$

A set of optimal dual variables for problem A need only satisfy (5.21) and the feasibility constraints of problem A (see 5.13).

$$c_{ij} + \overline{w}_{ij}^A + \overline{u}_i^A - \overline{v}_j^A \geqslant 0,\ \overline{u}_i^A \geqslant 0,\ \overline{v}_j^A \geqslant 0,$$
$$\overline{w}_{ij}^A \geqslant 0 \quad (5.23)$$

We may, therefore, set

$$u_i^k = \overline{u}_i^A = \overline{u}_i^B \quad,\quad v_j^k = \overline{v}_j^A = \overline{v}_j^B$$
$$w_{ij}^k = \overline{w}_{ij}^A = 0 \text{ for } y_{ij}^k = 1 \text{ (i.e., for } L_{ij}^k = 0)$$
$$w_{ij}^k = \overline{w}_{ij}^A \geqslant 0 \text{ for } y_{ij}^k = 0 \text{ and}$$
$$c_{ij} + u_i^k - v_j^k \geqslant 0$$
$$w_{ij}^k = \overline{w}_{ij}^A \geqslant |\, c_{ij} + u_i^k - v_j^k \,| \text{ for } y_{ij}^k = 0 \text{ and}$$
$$c_{ij} + u_i^k - v_j^k < 0. \quad (5.24)$$

The inequality signs in (5.24) permit a degree of arbitrariness in the choice of the w_{ij}^k. Since one wishes, however, to make the inequalities (5.15) as restrictive as possible, one will prefer to take the smallest permissible values for the w_{ij}^k.

With the final vector $u^k = (u_i^k, v_j^k, w_{ij}^k)$ we can now evaluate the function defined by (4.11).

$$F^k \equiv F\,(u^k, x_o^k, y^k) = x_o^k + d\,(u_i^k, v_j^k) +$$
$$+ \Sigma\, y_{ij}^k\, (q_{ij} - M_{ij}\, w_{ij}^k) \quad (5.25)$$

and perform the termination test (4.12) and the construction (4.13) which leads to the next, the $(k+1)^{th}$, iterative step.

b. <u>The optimal objective function of the modified problem (5.18) depends on L.</u>

The original problem, (5.17), is then evidently not feasible. In the sequel we assume that the Transportation Problem (5.18) is approached by means of the Ford-Fulkerson Algorithm described in reference [5], pp. 95 etc., and implemented in the machine program [6]. In this algorithm the dual variables are initialized by

$$u_i^! = 0 \quad,\quad v_j^! = \min_i\, (\bar{c}_{ij} = c_{ij} + L) \quad (5.26)$$

and changed in accordance with the scheme

$$u_j' = \begin{cases} u_i^o & \text{when } i \in I \\ u_i^o + \Delta & \text{when } i \in \bar{I} \end{cases}$$
$$v_j' = \begin{cases} v_j^o & \text{when } j \in J \\ v_j^o + \Delta & \text{when } j \in \bar{J} \end{cases}, \quad (5.27)$$

whenever a preceding "labeling procedure" (executed at a level identified by a superscript °) which results in a partitioning of the set of all i (and j) into $I, \bar{I}$ (and $J, \bar{J}$), fails to lead to a feasible new shipment. In this section the superscript ' identifies a level after a dual variable change (in case (5.26) the initialization). As a consequence of (5.27) the dual objective function $d(u_i^o, v_j^o)$ can be shown to increase (to $d\,(u'_i, u'_j)$) strictly monotonically by an additive term $k\Delta$, where

$$\Delta = \min_{I,\bar{J}}\ (\bar{c}_{ij} + u_i^o - v_j^o) > 0,$$
$$k = \sum_{\bar{J}} b_j - \sum_{\bar{I}} a_i > 0. \quad (5.28)$$

If the optimal dual objective function of (5.18) is to depend on L, the algorithm must at some point lead to a variable change (5.27), (5.28) with $\Delta = O(L)$ and $d\,(u'_i, v_j^!) = O(L)$, or the initialization (5.26) must already give rise to a v'_j of order L. At that instant we are in a position to construct a halfline $u^k = \hat{u}^k = \lambda\, \tilde{u}^k$, $\lambda \geqslant 0$, along which the dual objective function of (5.13), i.e. of problem A, goes to (minus) infinity with increasing λ.

We first consider the vectors $u^o = (u_i^o, v_j^o, w_{ij}^o)$

and $u' = (u'_i, v'_j, w'_{ij})$ which are, as far as the u_i, v_j are concerned, composed of the dual variables before and after the dual variable change (5.27) (or the initialization (5.26); here the u_i^o, v_j^o are taken as zero). With an appropriate choice of the w_{ij}^o and w'_{ij}, the halfline $u^* = u^o + \lambda u'$ can be shown to have the properties required of $u^k = \hat{u}^k + \lambda \tilde{u}^k$.

First we notice that the feasibility requirement of problem A for u^*, namely

$$c_{ij} + u_i^* - v_j^* + w_{ij}^* \geq 0 ,$$
$$u_i^* \geq 0, v_j^* \geq 0, w_{ij}^* \geq 0 , \tag{5.29}$$

can be satisfied for all λ if u^o is constructed so as to be feasible:

$$c_{ij} + u_i^o - v_j^o + w_{ij}^o \geq 0,$$
$$u_i^o \geq 0, v_j^o \geq 0, w_{ij}^o \geq 0 \tag{5.30}$$

and if u' satisfies

$$u'_i - v'_j + w'_{ij} \geq 0, u'_i \geq 0,$$
$$v'_j \geq 0, w'_{ij} \geq 0. \tag{5.31}$$

If the initialization already gives rise to $v'_j = O(L)$, we can satisfy (5.30) by taking u^oas the Nullvector. Otherwise, we know that the u_i^o, v_j^o satisfy (5.22). Hence, we satisfy (5.30) by setting, as in (5.24):

$$w_{ij}^o = 0 \quad \text{for} \quad y_{ij}^k = 1$$
$$w_{ij}^o \geq 0 \quad \text{for} \quad y_{ij}^k = 0 \text{ and } c_{ij} + u_i^o - v_j^o \geq 0$$
$$w_{ij}^o \geq | c_{ij} + u_i^o - v_j^o| \text{ for } y_{ij}^k = 0 \text{ and}$$
$$c_{ij} + u_i^o - v_j^o < 0. \tag{5.32}$$

Very similarly we simply choose the w'_{ij} so as to satisfy (5.31), i.e. we set

$$\begin{aligned} w'_{ij} &\geq 0 && \text{if} \quad u'_i - v'_j \geq 0 \\ w'_{ij} &\geq | u'_i - v'_j | && \text{if} \quad u'_i - v'_j < 0 . \end{aligned} \tag{5.33}$$

Again one will usually take the minimal values for w_{ij}^o and w'_{ij}.

We now turn to the dual objective function of problem A, evaluated for u^*.

$$-d(u_i^*, v_j^*) + \Sigma M_{ij} w_{ij}^* y_{ij}^k = -d(u_i^o, v_j^o) + \overbrace{\Sigma M_{ij} w_{ij}^o y_{ij}^k}^{0} + \lambda [-d(u'_i, v'_j) + \Sigma M_{ij} w'_{ij} y_{ij}^k] \tag{5.34}$$

We show for each of two cases that this function goes to $-\infty$ with increasing λ. It will be sufficient to demonstrate that the quantity in brackets is negative, e.g. by showing that

$$d(u'_i, v'_j) = O(L) \text{ and } \Sigma M_{ij} w'_{ij} y_{ij}^k \neq O(L),$$

Case 1 One of the v'_j is $O(L)$ by virtue of the initialization (5.26). Clearly $d(u'_i, v'_j)$ is larger than zero and $O(L)$. However, according to (5.26) and (5.33) the w'_{ij} are $O(L)$ only for columns for which all $\overline{c_{ij}} = O(L)$, that is for which all y_{ij}^k vanish. It follows that $\Sigma M_{ij} w'_{ij} y_{ij}^k \neq O(L)$.

Case 2 After a dual variable change in which $\Delta = O(L)$, we have (by 5.27, 5.28) that $d(u'_i, v'_j) = O(L)$; $u'_i = O(L)$ for $i \in \overline{I}$, $v'_j = O(L)$ for $j \in \overline{J}$. Regarding (5.33) (with minimal w'_{ij}) in the light of (5.27) we observe that $w'_{ij} = O(L) \rightarrow i \in I$ and $j \in \overline{J}$. But the fact that Δ (of order L) has been chosen as $\min_{I, \overline{J}} (\overline{c}_{ij} + u_i^o - v_j^o)$ implies that all $\overline{c_{ij}}$ with $i \in I, j \in \overline{J}$ are O(L), so that the corresponding y_{ij}^k are 0. Thus all w'_{ij} of order L in $\Sigma w'_{ij} M_{ij} y_{ij}^k$ have zero coefficients and the sum cannot be of order L.

It is clear now that the halfline $u^k = \hat{u}^k + \lambda \tilde{u}^k$ can be identified with $(u^k)_1 = u^* = u^o + \lambda u'$. In particular we may set $\hat{u}^k$ equal to u^o and $\tilde{u}^k$ equal to u'. The only disadvantage of this is that some of the components of $\tilde{u}^k$ depend on the large number L. As a consequence the integer programming problem (5.14), (5.15) involves L. This need not necessarily cause difficulties. In Section F we shall solve a numerical example in this manner.

The essential relationships to be obeyed by $\tilde{u}$ (we drop the superscript k for convenience) are (5.31) and (5.33) (with ' replaced by ~) and the strict inequality

$$d(\tilde{u}_i, \tilde{v}_j) > \Sigma M_{ij} \tilde{w}_{ij} y_{ij}^k . \tag{5.35}$$

The constraints (5.31) ensure that $\binom{0}{\tilde{u}} \in C$ (see (5.6)) and (5.35) guarantees that $\binom{0}{\tilde{u}} \notin Q^k$ (see (5.7), (5.15)).

It is clear that instead of the halfline $(u^k)_1 = u^o + \lambda u'$ one could also have taken $(u^k)_2 = (0,0, \ldots,) + \lambda u'$. As far as the iterative procedure is concerned the choice of a zero-vector for $\hat{u}^k$ means that no vector of the form $\binom{1}{u^k}$ is added to Q^k.

We now turn our attention to the construction of yet other vectors $\tilde{u}$ which are given numerically rather than in terms of L. A convenient way of doing this is to replace L by ℓ wherever it appears, explicitly or implicitly in the relations (5.26) to (5.39). This is to have the effect, formally, of changing Δ to δ and (u'_i, v'_j, w'_{ij}) to $(\tilde{u}_i, \tilde{v}_j, \tilde{w}_{ij})$. The existence of a positive ℓ which satisfies (5.35) follows from the (assumed) existence of L.

In case 1 above, governed by the relations (5.26) and (5.33) (with ' replaced by ~), we can determine an initial estimate (not necessarily integral) of ℓ, say ℓ_1, from (5.35), as the solution of:

$$f(\ell) = \Sigma b_j \tilde{v}_j - \Sigma M_{ij} \tilde{w}_{ij} y_{ij}^k = 0 \qquad (5.36)$$

Let J_1 ($\overline{J}_1$) denote the set of indices j for which $\tilde{v}_j$, and therefore $\tilde{w}_{ij}$, depends (does not depend) on ℓ. We note again that $\Sigma M_{ij} \tilde{w}_{ij} y_{ij}^k$ does not depend on ℓ, since $j \in J_1 \rightarrow y_{ij}^k = 0$ for all i.

The linear function $f(\ell)$ is therefore of the form:

$$f(\ell) = c_1 + \ell \sum_{j \in J_1} b_j \,. \qquad (5.37)$$

The constant c_1 can be evaluated numerically from (5.36). Let ℓ_1 be the solution of (5.37). We then set

$$\begin{aligned} \ell &= 1 && \text{if} \quad \ell_1 \leqslant 0 \\ \ell &= [\ell_1 + 1] && \text{if} \quad \ell_1 > 0 \end{aligned} \qquad (5.38)$$

The bracket [ ] stands for "the largest integer contained in."

In case 2 above, it is easy to obtain a numerical value for ℓ which satisfies all constraints. However, it is not completely clear as to what is, in some sense, an optimal choice of ℓ. The following approach seems most reasonable.

The knowledge of the existence of an ℓ large enough for the validity of (5.35) is based on the partitioning I, J of the "labeling procedure" mentioned after (5.27). This partitioning can be maintained by the "non-admissibility" requirements (see [5, p. 98]):

$$c_{ij} + \ell + u_i^o - v_j^o > 0 \text{ for all } (i,j)\text{: } y_{ij}^k = 0, \qquad (5.39)$$

in the sense that a replacement of L by ℓ in (5.19) during the labeling procedure preceding the dual variable change (5.27) leaves the sets I, J intact.

Regarding the choice of the $\tilde{w}_{ij}$ we observe the following relations (using (5.27), (5.28), (5.33) in terms of δ and ℓ):

$$\begin{aligned} &(i \in I,\ j \in J),\ (i \in \overline{I},\ j \in \overline{J}) \rightarrow \tilde{u}_i - \tilde{v}_j = u_i^o - v_j^o \\ &(i \in \overline{I},\ j \in J) \rightarrow \tilde{u}_i - \tilde{v}_j = u_i^o - v_j^o + \delta \\ &(i \in I,\ j \in \overline{J}) \rightarrow \tilde{u}_i - \tilde{v}_j = u_i^o - v_j^o - \delta, \text{ also: } y_{ij}^k = 0. \end{aligned} \qquad (5.40)$$

The constraints added to the integer equation subproblem (5.14), (5.15) by vectors of the form $\binom{o}{\tilde{u}}$ essentially serve the purpose of forcing the introduction of non-vanishing y_{ij}^k into the solution vector y^k. To this end it is desirable to choose the $\tilde{w}_{ij}$ as small as possible, but to make $d(\tilde{u}_i, \tilde{v}_j) = d(u_i^o, v_j^o) + k \cdot \delta$ large. In order to obtain a maximum number of vanishing $\tilde{w}_{ij}$ we demand that

$$\delta \equiv m + \ell \geqslant v_j^o - u_i^o \text{ for } [(i \in \overline{I},\ j \in J) \text{ and } u_i^o < v_j^o]$$

$$m \equiv \min_{I, \overline{J}} (c_{ij} + u_i^o - v_j^o) \qquad (5.41)$$

In conjunction with (5.40) and (5.33) this ensures that $\tilde{w}_{ij} = 0$ for $(i, j) \in (\overline{I}, J)$. The requirements (5.40) and (5.41) now allow us to state the inequality (5.35) in the simple form:

$$d(u_i^o, v_j^o) + k(\ell + m) > \sum_{\Omega} (v_j^o - u_i^o) M_{ij} y_{ij}^k \qquad (5.42)$$

$$\Omega \equiv \{(i, j) \mid [(i \in I, j \in J) \text{ and } v_j^o > u_i^o] \text{ or } [(i \in \overline{I},\ j \in \overline{J}) \text{ and } v_j^o > u_i^o]\}.$$

The pairs $(i, j) \notin \Omega$ need not be considered in $\Sigma \tilde{w}_{ij} M_{ij} y_{ij}^k$, because either $\tilde{w}_{ij}$ or y_{ij}^k vanishes.

If ℓ_1, ℓ_2 and ℓ_3 are the smallest positive integers which satisfy (5.39), (5.41) and (5.42) respectively, then we set

$$\ell = \max(\ell_1, \ell_2, \ell_3). \qquad (5.43)$$

Taking the smallest permissible values for ℓ_1, ℓ_2, ℓ_3 has the advantage of possibly generating vanishing $\tilde{w}_{ij}$ for $(i \in I, j \in \overline{J})$. Increasing ℓ has no discernible overall advantage. It causes a desirable increase in $d(\tilde{u}_i, \tilde{v}_j)$, but an undesirable increase as well in $\tilde{w}_{ij}$ for $(i \in I, j \in \overline{J})$. We do not know, at this point, what the corresponding impact on the integer programming problem is.

After the construction of $\hat{u}^k$ and $\tilde{u}^k$ we can proceed in accordance with (4.16) - (4.17). In a computer program, of course, one of the major difficulties may be the large dimension of the arrays w_{ij}^k needed for every maximization problem (4.8). It should be noted that the rules for the computation of the w_{ij}^k are simple and involve only the (m + n) quantities u_i^k, v_j^k. One will therefore store the latter variables and compute the w_{ij}^k as needed.

4. Termination Rules

According to the inequalities (4.18) and (4.20), the final optimum of the objective function, $-\overline{x}_o$, is bounded above and below by a sequence of "trial solutions."

$$-x_o^k \leqslant -\overline{x}_o \leqslant +d(u_i^k, v_j^k) + \Sigma y_{ij}^k (q_{ij} - M_{ij} w_{ij}^k) \qquad (5.44)$$

The termination rules test for equality between the left and right side of (5.44). On account of $G(Q^{k+1}) \subset G(Q^k)$, the sequence of integers $\{-x_o^k\}$ is nondecreasing and terminates with the value $-\overline{x}_o$.

We can show that the number of iterative steps is finite. To this end we notice that the set of iterative steps can be partitioned into two sets, characterized by feasibility or non-feasibility of the transportation sub-problem.

When the transportation sub-problem is not feasible, the set Q^k is augmented to Q^{k+1} by addition of a vector $\binom{0_k}{u}$ which satisfies (5.35), namely

$$0 > -d(u_i^k, v_j^k) + \Sigma y_{ij}^k M_{ij} w_{ij}^k \quad . \qquad (5.45)$$

But $y^{k+1}, y^{k+2}, \ldots$ must satisfy, for all $\binom{0^\alpha}{u} \in Q^{k+1}$, $Q^{k+2}, \ldots$, the inequality

$$0 \leq -d(u_i^\alpha, v_j^\alpha) + \Sigma y_{ij} M_{ij} w_{ij}^\alpha \quad . \qquad (5.46)$$

y^k does not satisfy (5.46) for $\alpha = k$ and can, therefore, never again solve the maximum problem. Since there exist only 2^ν different vectors y^k, the set of iterations with non-feasible problems is finite.

When the transportation problem is feasible, the set Q^k is augmented to Q^{k+1} by addition of a vector $\binom{1_k}{u}$ which satisfies (5.44) in the strict sense:

$$x_o^k > -d(u_i^k, v_j^k) + \Sigma y_{ij}^k (M_{ij} w_{ij}^k - q_{ij}) \cdot \quad (5.47)$$

If there is to be an infinite sequence of iterative steps, there must also be an infinite sequence of successive steps in which the value of x_o does not change. Let us assume that the k^{th} iteration is in such an infinite sequence. Then, for all $\binom{1_\beta}{u} \in Q^{k+1}, Q^{k+2}, \ldots$, the vector $y^{k+1}, y^{k+2}, \ldots$ must satisfy the inequality

$$x_o^k \leq -d(u_i^\beta, v_j^\beta) + \Sigma y_{ij} (M_{ij} w_{ij}^\beta - q_{ij}) . (5.48)$$

Again we see that y^k does not satisfy (5.47) for $\beta = k$ and will, therefore, as long as x_o remains equal to x_o^k, not recur as a solution vector during iterations k+1, k+2, ... Since there are only 2^ν different vectors y^k, the hypothesis of an infinite sequence of iterative steps cannot be maintained.

AN ILLUSTRATIVE EXAMPLE

We consider the following simple Fixed Charge Transportation Problem:

a_i \ b_j	3	4	1
2	2	4	1
1	1	3	2
5	1	1	2

c_{ij}

1	1	0
0	1	1
10	1	1

q_{ij} (surcharges)

The array of $M_{ij} = \min(a_i, b_j)$ is given by:

2	2	1
1	1	1
3	4	1

In view of the large surcharge $q_{31} = 10$, the solution to our problem is easily seen to be:

$$\bar{x} = (2,0,0,1,0,0,0,4,1)$$

with a total cost of

$$\Sigma c_{ij} \bar{x}_{ij} + \Sigma q_{ij} \bar{y}_{ij} = 11 + 3 = 14$$

Initialization: The simple transportation problem without surcharges has the solution

$$x^o = (1,0,1,1,0,0,1,4,0)$$

$$u^o = (0,1,1,2,2,1,0,0,0,0,0,0,0,0,0) \quad .$$

This can be verified by means of the equality between $\Sigma c_{ij} x_{ij}^o$ (= 9) and $d(u_i^o, v_j^o) = -\Sigma a_i u_i^o + \Sigma b_j v_j^o$ (= − 6 + 15).

FIRST ITERATIVE STEP

1. Maximize x_o subject to $y_{ij} = 0$ or 1 and

$$\Sigma a_i y_{ij} \geq b_j, \ \Sigma b_j y_{ij} \geq a_i \qquad (6.1)$$

$$x_o \leq -d(u_i^o, v_j^o) + \Sigma y_{ij} (M_{ij} w_{ij}^o - q_{ij}) \ .$$

Instead of the full set of 6 a priori inequalities (6.1) we shall, for simplicity, only use 2. This cannot change the final result. Our system of inequalities is now

$$3 \leq 2y_{11} + y_{21} + 5y_{31}$$

$$2 \leq 3y_{11} + 4y_{12} + y_{13}$$

$$x_o \leq -9 - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 10 & 1 & 1 \end{pmatrix} * y$$

The symbol * stands for scalar multiplication, here between the vectors q and y.

It is easy to see that $x_o|_{max} = -10$. We note that the components y_{13} and y_{21} are arbitrary. In order to arrive at feasible transportation problems as soon as possible, we adopt the convention of assigning values of 1 to arbitrary components.

$$x_o^1 = -10 \quad , \quad y^1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Modified Transportation Problem

$$\begin{array}{c|ccc|} & 3 & 4 & 1 \\ \hline 2 & 2 & 4+L & 1 \\ 1 & 1 & 3+L & 2+L \\ 5 & 1+L & 1+L & 2+L \\ \hline \end{array}$$

The initialization (5.26) leads to

$u' = (0,0,0,1,\ 1+L,\ 1,\ \ldots)$: Problem not feasible.

We can take $\hat{u}^1 = u^o$ as the Nullvector.

L: The vector $u^1 = \tilde{u}^1$ is constructed in accordance with (5.26), (5.33) if L is to be retained.

$$\begin{array}{c|ccc} \tilde{u}_i \backslash \tilde{v}_j & 1 & 1+L & 1 \\ \hline 0 & 1 & 1+L & 1 \\ 0 & 1 & 1+L & 1 \\ 0 & 1 & 1+L & 1 \end{array} \qquad \tilde{w}_{ij}$$

$d(\tilde{u}_i,\ \tilde{v}_j) = 3 + 4\,(1+L) + 1$

$u^1 = (\tilde{u}_i,\ \tilde{v}_j,\ \tilde{w}_{ij}) = (0,0,0,1,1+L,1;\ 1,1+L,1,1,1+L,1,1,1+L,1)$

The vector $\begin{pmatrix} 0 \\ u^1 \end{pmatrix}$ is added to Q^o, resulting in the new inequality

$$d(\tilde{u}_i,\ \tilde{v}_j) = 8 + 4L \leq \Sigma\, y_{ij}\, M_{ij}\, \tilde{w}_{ij}$$

$$8 + 4L \leq \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1+L & 1 \\ 1 & 1+L & 1 \\ 1 & 1+L & 1 \end{pmatrix} * y$$

The symbol $\times$ stands for a product in which the entries in the resultant matrix are the products of corresponding entries in the factor matrices. $\times$ has priority over $*$.

ℓ: To evaluate ℓ we use equations (5.36), (5.37).

$J_1 = \{2\}$

$$f(\ell) = b_1 + b_2(1+\ell) + b_3 - \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1+\ell & 1 \\ 1 & 1+\ell & 1 \\ 1 & 1+\ell & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 8 - (2 + 1 + 1) + 4\,\ell = 4 + 4\,\ell$$

$$f(\ell) = 0 \longrightarrow \ell_1 = -1 \longrightarrow \ell = 1$$

$$\begin{array}{c|ccc} \tilde{u}_i \backslash \tilde{v}_j & 1 & 2 & 1 \\ \hline 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array} \qquad d(\tilde{u}_i,\ \tilde{v}_j) = 3 + 8 + 1 = 12$$

The new inequality is: $12 \leq \begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 1 \\ 3 & 8 & 1 \end{pmatrix} * y.$

SECOND ITERATIVE STEP

1. $\text{Max } x_o \mid y_{ij} = 0 \text{ or } 1$

$3 \leq 2\, y_{11} + y_{21} + 5\, y_{31}$

$2 \leq 3\, y_{11} + 4\, y_{12} + y_{13}$

$$8 + 4L \leq \begin{pmatrix} 2 & 2+2L & 1 \\ 1 & 1+L & 1 \\ 3 & 4+4L & 1 \end{pmatrix} * y \text{ or}$$

$$12 \leq \begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 1 \\ 3 & 8 & 1 \end{pmatrix} * y$$

$$x_o \leq -9 - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 10 & 1 & 1 \end{pmatrix} * y$$

The solution of this problem is evidently not affected by the choice of the inequality above.

$$y^2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad x_o^2 = -11$$

2. Modified Transportation Problem

$$\begin{array}{c|ccc} & 3 & 4 & 1 \\ \hline 2 & 2 & 4+L & 1 \\ 1 & 1 & 3+L & 2+L \\ 5 & 1+L & 1 & 2+L \end{array} \qquad \begin{array}{c|ccc} u_i \backslash v_j & 1 & 1 & 1 \\ \hline 0 & 1 & 3+L & 0 \\ 0 & 0 & 2+L & 1+L \\ 0 & L & 0 & 1+L \end{array}$$

$\bar{c}_{ij}$ $\qquad$ $e_{ij} = \bar{c}_{ij} + u_i - v_j$

The transportation problem is solved by a sequence of labeling, breakthrough and dual variable change steps (see reference [5]).

Label, and Breakthrough: Label:

$$
\begin{array}{c|ccc|c}
 & 2,1 & 3,5 & 1,2 & \\
\hline
- & & & 1 & \leftarrow 2 \\
- & 1 & & & \leftarrow 1 \\
- & & 4 & & \leftarrow 5 \\
\hline
 & 3 & 4 & 1 &
\end{array}
\qquad
\begin{array}{c|ccc|c}
 & & 3,1 & 1,1 & \\
\hline
- & & & 1 & 1 \\
- & 1 & & & 0 \\
- & & 4 & & 1 \\
\hline
 & 2 & 0 & 0 &
\end{array}
$$

Dual variable change, $I = \{1,3\}$; $\bar{I} = \{2\}$; $J = \{2,3\}$; $\bar{J} = \{1\}$

$\delta = \min_{I\bar{J}} e_{ij} = \min(1, L) = 1$

$$
\begin{array}{c|ccc}
u_i \backslash v_j & 2 & 1 & 1 \\
\hline
0 & 0 & 3+L & 0 \\
1 & 0 & 3+L & 2+L \\
0 & L-1 & 0 & 1+L
\end{array}
\quad e_{ij}
$$

Label:

$$
\begin{array}{c|ccc|c}
 & 1,1 & 3,1 & 1,1 & \\
\hline
- & 1 & & 1 & \leftarrow 1 \\
 & 1 & & & 0 \\
- & & 4 & & 1 \\
\hline
 & 2 & 0 & 0 &
\end{array}
$$

Label: 3,1

$$
\begin{array}{ccc|c}
1 & & 1 & 0 \\
1 & & & 0 \\
 & 4 & & 1 \\
\hline
1 & 0 & 0 &
\end{array}
$$

$I = \{3\}$, $\bar{I} = \{1,2\}$

$J = \{2\}$, $\bar{J} = \{1,3\}$

Dual variable change: $\delta = \min_{I,\bar{J}} e_{ij} = \min(L-1, L+1) = L-1$; Problem not feasible

u^o:

$$
\begin{array}{c|ccc}
u_i^o \backslash v_j^o & 2 & 1 & 1 \\
\hline
0 & 0 & 3 & 0 \\
1 & 0 & 3 & 2 \\
0 & -1 & 0 & 1
\end{array}
\quad c_{ij} + u_i^o - v_j^o
$$

$\hat{u} = u^o = (0,1,0,2,1,1 \;;\; 0,0,0,0,0,0,1,0,0)$ by (5.32)

Test whether to add $\binom{1_2}{\hat{u}^2}$ to Q^1 (see 5.25):

$$F(\hat{u}^2) = x_o^2 + d(\hat{u}_i^2, \hat{v}_j^2) + \Sigma y_{ij}^2 \; (q_{ji} - M_{ij} \hat{w}_{ij}^2)$$

$$= -11 + (-1 + 6 + 4 + 1) +$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 7 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= -11 + 10 + 2 = 1 > 0$$

$\therefore$ add the inequality: $x_o \leq -10 - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 7 & 1 & 1 \end{pmatrix} * y$

$\tilde{u}$:

L)

$$
\begin{array}{c|ccc}
\tilde{u}_i \backslash \tilde{v}_j & L+1 & 1 & L \\
\hline
L-1 & 2 & 0 & 1 \\
L & 1 & 0 & 0 \\
0 & L+1 & 1 & L
\end{array}
$$

$\tilde{u}_i$, $\tilde{v}_j$ by (5.27)

$\tilde{w}_{ij}$ by (5.33)

$$d(\tilde{u}_i, \tilde{v}_j) = -2(L-1) - L + 3(L+1) + 4 + L = L + 9$$

New inequality: $d(\tilde{u}_i, \tilde{v}_j) \leq \Sigma M_{ij} \tilde{w}_{ij} y_{ij}$

$$L + 9 \leq \begin{pmatrix} 4 & 0 & 1 \\ 1 & 0 & 0 \\ 3L+3 & 4 & L \end{pmatrix} * y$$

ℓ) (5.39): $\ell_1 = 2$

(5.41): $m = -1$

$\ell - 1 \geq 1 \rightarrow \ell_2 = 2$

(5.42): $k = \sum_{\bar{J}} b_j - \sum_{\bar{I}} a_i = 3+1 - (2+1) = 1$

$$10 + (\ell - 1) > \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix} *$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$9 + \ell > 10$

$\ell_3 = 2$

$\ell = \max(\ell_1, \ell_2, \ell_3) = 2$

New inequality: $11 \leq \begin{pmatrix} 4 & 0 & 1 \\ 1 & 0 & 0 \\ 9 & 4 & 2 \end{pmatrix} * y$

THIRD ITERATIVE STEP

1. Max $x_o \mid y_{ij} = 0$ or 1

$3 \leq 2\, y_{11} + y_{21} + 5\, y_{31}$

$2 \leq 3\, y_{11} + 4\, y_{12} + y_{13}$

$$x_o \leq -9 - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 10 & 1 & 1 \end{pmatrix} * y$$

$$x_o \leq -10 - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 7 & 1 & 1 \end{pmatrix} * y$$

$$8 + 4L \leq \begin{pmatrix} 2 & 2+2L & 1 \\ 1 & 1+L & 1 \\ 3 & 4+4L & 1 \end{pmatrix} * y \quad \text{or}$$

$$12 \leq \begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 1 \\ 3 & 8 & 1 \end{pmatrix} * y$$

$$9 + L \leq \begin{pmatrix} 4 & 0 & 1 \\ 1 & 0 & 0 \\ 3+3L & 4 & L \end{pmatrix} * y \quad \text{or}$$

$$11 \leq \begin{pmatrix} 4 & 0 & 1 \\ 1 & 0 & 0 \\ 9 & 4 & 2 \end{pmatrix} * y$$

Again it is easily seen for this system that the solution does not depend on whether the inequalities with L or the numerical inequalities are used.

$$x_o^3 = -13 \;, \; y^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The second inequality involving x_o does not influence the distribution of the y_{ij} but causes a smaller value of x_o^3.

2. Modified Transportation Problem

$$\begin{array}{c|ccc|} & 3 & 4 & 1 \\ \hline 2 & 2 & 4+L & 1 \\ 1 & 1 & 3+L & 2+L \\ 5 & 1+L & 1 & 2 \\ \hline \end{array}$$

Proceeds as in second iteration, until

$$\begin{array}{c|ccc} u_i \backslash v_j & 2 & 1 & 1 \\ \hline 0 & 0 & 3+L & 0 \\ 1 & 0 & 3+L & 2+L \\ 0 & L-1 & 0 & 1 \end{array} \qquad e_{ij}$$

Label:

$$\begin{array}{cccc|c} & & 3,1 & & \\ \hline & 1 & & 1 & 0 \\ & 1 & & & 0 \\ - & & 4 & & 1 \\ \hline & 1 & 0 & 0 & \end{array}$$

Dual variable change: $\delta = \min (L-1, \; 1) = 1$

Label:

$$\begin{array}{c|ccc|c} u_i \backslash v_j & 3 & 1 & 2 & \\ \hline 1 & 0 & 4+L & 0 & 3,1 \\ 2 & 0 & 4+L & 2+L & \\ 0 & L-2 & 0 & 0 & - \\ \hline \end{array} \qquad \begin{array}{ccc|c} 1,1 & 3,1 & 3,1 & \\ \hline 1 & & 1 & 0 \\ 1 & & & 0 \\ & 4 & \bigcirc & 1 \\ \hline 1 & 0 & 0 & \end{array}$$

Breakthrough!

Solution:

$$x^3 = \begin{array}{|ccc|} \hline 2 & & 0 \\ 1 & & \\ & 4 & 1 \\ \hline \end{array}$$

While this solution is the final solution, the termination test is not yet positive and one further iteration is required.

We construct u^3 in accordance with (5.24).

$$\begin{array}{c|ccc} u_i^3 \backslash v_j^3 & 3 & 1 & 2 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array} \qquad d(u_i^3, v_j^3) = -4 + 15 = 11$$

w_{ij}^3

Termination test (see (4.11), (5.25))

$$F^3 = x_o^3 + d(u_i^3, v_j^3) + \Sigma y_{ij}^3 \, (q_{ij} - M_{ij} \, w_{ij}^3) =$$
$$= -13 + 11 + 3 > 0$$

New inequality (5.15) with $u_o^3 = 1$):

$$x_o \leq -11 + \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 10 & 1 & 1 \end{pmatrix} \right] * y$$

FOURTH ITERATION

1. We have to solve the same system as in the third iterative step, with

$$x_o \leq -11 - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix} * y \quad \text{added.}$$

Clearly the y part of the solution remains the same.

$$y^4 = y^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \ x_o^4 = -14$$

2. The modified transporation problem remains unchanged,

$x^4 = x^3$, $u^4 = u^3$,

and the problem terminates:

$F^4 = -14 + 11 + 3 = 0.$

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