SOME PROBLEMS IN QUANTUM MECHANICAL REPRESENTATION THEORY

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SOME PROBLEMS IN QUANTUM MECHANICAL REPRESENTATION THEORY:

I. RIGGED HILBERT SPACES FOR SOME IDEAL SYSTEM;

II. RAY AND CORAY REPRESENTATIONS OF SOME FINITE EXTENSIONS OF THE POINCARE GROUP

Thesis

Submitted by

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for the degree of

DOCTOR OF PHILOSOPHY

University of Edinburgh,

December, 1966.

ACKNOWLEDGEMENT

I wish to thank Professor N. Kemmer, F.R.S., for the hospitality of the Tait Institute, and Dr. P.W. Higgs for many useful discussions.

I also wish to extend my thanks to Professor A.O. Barut of the University of Colorado for suggesting the material in Section II, and to Dr. A. Böhm for suggesting the material in Section I of this thesis.

Thanks are also due to Mrs. R.W. Chester for her help in preparing and typing the thesis.

Finally I wish to acknowledge with thanks the receipt of a U.A.R. grant which made this work possible.

All the material presented in this dissertation is asserted to be original except where explicit references were cited.

The work described in Section II, Chapter 2 of this thesis has already been submitted to J.Math. Phys., for publication.

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INTRODUCTION

Recently there has been much interest in the group theoretical investigation of inherent dynamical symmetries of exactly solvable systems such as the non-relativistic Kepler problem, rigid rotator and the isotropic harmonic oscillator. One reason for such a study is that the structure of closely related problems in hadron physics may be studied, namely the use of higher symmetry of interactions in order to classify the spectrum of the system by the corresponding non-invariance dynamical groups. The concept of the dynamical groups originates from an idea by Barut[5] for obtaining the energy spectrum of a quantum mechanical system by a suitably chosen group. The use of such groups should not be confused with the symmetry groups in the usual sense, compact (e.g. SU(3) or non-compact (Lorentz group)). The non-compact (dynamical) groups contain in a precise way the complete information about the physical system, including degeneracy, mass or energy levels and so on. in Chapter 1(a) some of their properties are listed.

Given a quantum mechanical system one finds the appropriate dynamical group G which describes the system completely, i.e. one finds a one-to-one correspondence between the states of the physical system and one irreducible unitary representation of the dynamical group G. The representation space is then identified with the physical space ψ of the system, in which a positive definite hermitian function is defined by the ordinary scalar product. Completion with respect to the norm, defined by this

scalar product, gives the Hilbert space H. It is found that some of the operators, (generators of the dynamical group), which represent physical quantities, are not bounded operators in the Hilbert space H. Unbounded operators necessarily occur in unitary representation of non-compact Lie algebras[13]. Therefore it is not always permitted to carry out algebraic operations on these physical quantities. In order to be mathematically rigorous one needs always to be concerned with the domain of definition of an operator in the physical Hilbert space. The original formulation of quantum mechanics by von Neumann[12], namely that there is a one-to-one correspondence between observables and self-adjoint operators on the Hilbert space, is too restrictive. On the one hand, there are self-adjoint operators which are not observables[28], on the other hand not all unitary (or self-adjoint) operators have a complete set of eigenvectors lying in the physical Hilbert space, e.g, the translation operator in the Hilbert space L^2 of square integrable functions of one variable, has only the zero eigenvectors, while it has a complete set of eigenvectors $e^{-i\lambda x}$ lying outside it. A larger space is needed to give such functions an interpretation. Similarly in problems connected with analytic continuation into unphysical sheets, where one deals with functions which increase at infinity. In quantum field theory unbounded operators are the smeared field operators, they also occur in the canonical formulation, where one deals with representations of an infinite dimensional algebra. In the more recent development of current algebra similar problems occur, for example, a non-conserved generalized charge operator Q (being constructed out of fields), is not defined in the Hilbert space (considered as a physical state).*
In conclusion, the self-adjoint operators in the Hilbert space are

* Fabri & Picasso

not the simplest mathematical objects upon which the physical observables should be mapped. Therefore we look for another mathematical frame in which all algebraic operations on physical observables are permitted. It seems that the most suitable mathematical structure, upon which the structure of the physical objects of a physical system could be mapped ~~appears to be~~ is an algebra of bounded operators of the physical space Φ of the rigged Hilbert space $\Phi \subset H \subset \Phi^{\times}$ [10].

The choice of the rigged Hilbert space is preferred to the one used by Grossmann[17], in which he used a nested Hilbert space as a mathematical model. Given a nested Hilbert space, it will presumably be necessary to require at least one of the nesting mappings to be of the Hilbert-Schmidt type to ensure the existence of sufficiently generalized eigenvectors. Rigged Hilbert spaces involve only three spaces whereas a nested Hilbert space involves a collection of Hilbert spaces indexed by a partially ordered set. It was shown recently[27] that nested Hilbert spaces and rigged Hilbert spaces are closely related to each other. However such a discussion lies outside the scope of our presentation. The choice of the rigged Hilbert space as a physical space retains the advantage of Dirac's formulation of quantum mechanics, i.e. the existence of a complete set of eigenvectors for a complete set of commuting operators; also it is close enough to the usual quantum mechanical formulation. In Chapter 1(b) we list some of the well known properties of the rigged Hilbert space.

Rigged Hilbert space is introduced as the physical space for physical systems by the following postulate:

"The structure of a physical system can be mapped onto the mathematical structure of an algebra $\mathcal{A} \subset L(\Phi)$."

$L(\Phi)$ is the algebra of bounded operators, Φ is the physical space of the rigged Hilbert space $\Phi \subset H \subset \Phi^*$.

It should be remarked that statements such as are made in the postulate must necessarily remain vague. Notions like "physical system" cannot be mathematically defined up to now by their mathematical image and this kind of map.

The task now is to find this algebra $\mathcal{A}$. In Chapter 2 we give three examples to illustrate the above postulate and to indicate how one might proceed to find the algebra .

In practice, one starts with the ordinary Hilbert space H, and let $\mathcal{A}$ be the algebra of observables consisting of hermitian operators, not necessarily bounded operators. The domain of the algebra $\mathcal{A}$ is in general smaller than H. The rigged Hilbert space is constructed by making all the operators in $\mathcal{A}$ continuous. Operators with no eigenvectors in H might find their generalized eigenvectors in the space Φ^*.

CHAPTER I

(a) DYNAMICAL GROUPS

The dynamical groups for quantum mechanical systems are introduced by the following postulate[5-9,20].

"All states of a quantum mechanical system belong to a single irreducible unitary representation of a chosen non-compact group."

The energy is given as a simple function of the Casimir operators of the group of degeneracy of the energy G_0 (for definition see below). In the dynamical group G, the energy is an operator in the enveloping algebra[5], its spectrum is fixed once the Casimir operators are fixed.

Consider the set A_i of physical quantities (energy, and e.g. spin,...) which characterize completely a physical system. We define the group of degeneracy G_i with respect to the observable A_i as the group which transforms all the states of a given value of A_i into each other, its generators are those which commute with A_i. The dynamical group by definition is the smallest group which contains all G_i. Consider the group of degeneracy G_J of the spin J. For most quantum mechanical systems there are infinite numbers of states with the same spin J. Therefore G is non-compact for most quantum mechanical systems. Vice versa a physical system is identified by a complete set of its states defined as furnishing a faithful representation of a group. Since the most interesting systems (e.g. hydrogen atom, rigid rotator and the isotropic oscillator) have infinitely many linearly independent states, to be able to organize them into a single irreducible representation, the non-invariance group must be non-compact. The dynamical system is now defined by the non-invariance group and the dynamical variables are identified with the elements of the generalized enveloping algebra (e.g. the energy operator is an element of the enveloping algebra of $SO(3,1)$ for the rigid rotator). The non-compact non-invariance group thus becomes a dynamical model[32]. This is the essence of model making in atomic and nuclear physics[30]. The non-compact group G is semi-simple[9], the reason is that the invariant (i.e. an element of the enveloping algebra) operator of an invariant subgroup of a group is also an invariant operator of the group itself. The dynamical group by definition cannot have E as an invariant operator; it contains operators which allow us to go from one state with energy E_1, say, to another state with energy E_2. In other words, the shift operators (i.e. non-compact generators) take us from one representation of the symmetry group G_o to another. Therefore in general the complete unitary irreducible representation of the dynamical group G gives information about which and how many times each representation of any G_1 occurs.

Given a symmetry group G_o, $A_1, \ldots, A_n$ are its generators. Let $C_1, C_2, \ldots, C_\ell$ be the Casimir operators (ℓ is the rank of the group). The energy by definition is given by the functional equation:

$$E = f(C_1, \ldots\ldots, C_\ell) \qquad (1)$$

Equation (1) is a consequence of the fact that in physics particular representations of the groups rather than the groups themselves are used, therefore the Casimir operators must have the physical consequences(5) as they characterize the particular representation of the group G. They must be associated with the fundamental physical parameters which describe the physical system.

The dynamical group G contains besides the generators $A_1, \ldots\ldots, A_n$ the generators $B_1, \ldots\ldots, B_n$, say; these generators are chosen in such a way that the eigenstates of the energy operator E (e.g. Hamiltonian) constitute a basis for the unitary irreducible representation of G.

It may happen that the energy of the physical system (at hand) depends only on one quantum number; in such a case one expects that all Casimir operators C_i vanish except one, C_1 say. The functional form of the energy (1) reduces to

$$E = f(C_1) \qquad (2)$$

G_o becomes a maximal compact subgroup(25). The system is called a maximally degenerate or ideal system. Thus we define an ideal quantum mechanical system as a system in which the energy is given by the functional form (2); the group G_o is the maximal compact subgroup of G. The rigid rotator, hydrogen atom and the isotropic oscillator are ideal systems.

It is found that the representation of the dynamical group (see below) contains each and every representation of G_o only once. In the following paragraphs we adhere to the general method summarized above in order to derive the dynamical groups for the rigid rotator and the hydrogen atom. At the end of this section we make some general remarks.

I. The Hydrogen Atom (bound states problem)

(a) The Symmetry group.

It is well known that the treatment of a quantum system of two particles with Coulomb interaction reduces to that of a particle in the central potential $Z_1 Z_2 \frac{e^2}{r}$; for the hydrogen atom the two particles are subject to the attractive potential $\frac{-e^2}{r}$, if E is the energy of the electron proton system, then the problem is reduced to the solution of the Schrödinger equation,

$$(H - E)\,\psi(\bar{r}) = 0 \tag{1}$$

where

$$H = \frac{p^2}{2} - \frac{1}{r} \tag{2}$$

The units are chosen such that $m = e = \hbar = 1$.

It is possible to separate equation (1) using polar coordinates; one easily derives the energy levels for the hydrogen atom, given by the Balmer formula

$$E_n = -\frac{1}{2n^2} \qquad (3)$$

n takes integer values from 1 to ∞, each level is n^2-fold degenerate.

It is clear that equation (1) is invariant under spatial rotations, from which the angular momentum vector

$$\bar{L} = \bar{r} \times \bar{p} \qquad (4)$$

is a conserved quantity. Because the degeneracy exceeds the degeneracy due to rotational invariance, one looks for additional constants of motion to generate the higher symmetry[14,4,24]. For the Kepler problem the following vector

$$\bar{L} \times \bar{p} + \frac{\bar{r}}{r}$$

is also a conserved quantity. Pauli used the correspondence principle to investigate the commutation relations of the Hermitian part of it, i.e. the three vector:

$$\bar{F} = \tfrac{1}{2}(\bar{L} \times \bar{p} - \bar{p} \times \bar{L}) + \frac{\bar{r}}{r} \qquad (5)$$

The operators $\bar{L}$ and $\bar{F}$ are not independent, but the following relation holds:

$$\bar{L} \cdot \bar{F} = \bar{F} \cdot \bar{L} = 0 \qquad (6)$$

since $[H, \bar{L}] = [H, \bar{F}] = 0$ then $[H, [\bar{L}, \bar{F}]] = 0$, i.e. H is invariant under all transformations generated by $\bar{L}$, $\bar{F}$, i.e. the algebra:

$$[\bar{L}, \bar{L}] = i\bar{L} \qquad [\bar{L}, \bar{F}] = i\bar{F} \qquad [\bar{F}, \bar{F}] = -2iH\bar{L}$$

which is not a closed algebra. It is convenient to introduce the so-called Runge Lenz vector:

$$A = \frac{\bar{F}}{\sqrt{\pm 2H}} = \frac{\bar{F}}{\sqrt{\pm 2H}}$$

where $\pm$ sign is introduced in such a way that the vector $\bar{A}$ is well defined. For our case the minus sign is chosen since H is a negative c number (for the plus sign see[2]). Then H is invariant under the algebra:

$$[\bar{L}, \bar{L}] = i\bar{L} \qquad [\bar{L}, \bar{A}] = i\bar{A} \qquad [\bar{A}, \bar{A}] = i\bar{L} \qquad (7)$$

which is the algebra of the rotation group in 4-dimensional SO_4. The two vectors $\bar{K} = \frac{1}{2}(\bar{L} + \bar{A})$ and $\bar{K}' = \frac{1}{2}(\bar{L} - \bar{A})$ build up two commuting sets of operators, each one satisfying the commutation relations of the ordinary angular momentum. The two Casimir operators of the algebra (eq. (7)) are

$$F = \tfrac{1}{2}(L^2 + A^2) = K^2 + K'^2$$

$$G = \bar{L} . \bar{A} = \bar{A} . \bar{L} = K^2 - K'^2 = 0 \qquad (6)'$$

From equations (2) to (6) the Hamiltonian has the following form

$$\frac{1}{-2H} = (\bar{L} + \bar{A})^2 + 1 = (\bar{L} - \bar{A})^2 + 1 = L^2 + A^2 + 1$$

$$= 2F + 1 \qquad (8)$$

K^2, K'^2, K_3, K_3' forms a complete set of commuting operators, each state being represented by the ket:

$$\left| k\ k' \quad m\ m' \right\rangle \tag{9}$$

where, for example,

$$K_i^2 \left| k\ k' \quad m\ m' \right\rangle = k_i(k_i + 1) \left| k\ k' \quad m\ m' \right\rangle$$

$$k_i = k \quad \text{or} \quad k' \quad \text{as} \quad K_i = K \quad \text{or} \quad K' .$$

From equation (6) it follows that $k = k'$. It is not clear whether $k = k'$ has to be limited to integer values or can also take half integer values. From (8) and (9) it follows that

$$\frac{1}{-2E} = 4k(k + 1) + 1 = (2k + 1)^2 .$$

If we identify $(2k + 1)$ with the principal quantum number n we obtain equation (1), with n taking every integer value from 1 to ∞. We see that k is allowed to take the values $0, \frac{1}{2}, \ldots\ldots$. Since $\bar{L} = \bar{K} + \bar{K}'$, the well known addition theorem[24] for angular momentum shows that for a given $n = (2k + 1)$ the possible values of $\bar{L}$ are $0, 1, \ldots, 2k$. The degeneracy of the levels is equal to

$$\sum_{\ell=0}^{2k} (2\ell + 1) = (2k + 1)^2 = n^2 .$$

Thus one says that the hydrogen atom spectrum corresponds to an irreducible representation of the rotation group in which one of the Casimir operators (G) has zero eigenvalue, i.e. the states of the hydrogen atom fall into the n^2-dimensional representation of SO_4. There is, however, the further regularity that for each non-negative integral value of n there is only one multiplet of states furnishing the n^2-dimensional representation of $SO(4)$[4]. Can we incorporate this information into a group theoretic formulation? In other words what is the dynamical group G which describes the system completely?

(b) THE DYNAMICAL GROUP

It is a non-compact group, G, containing SO_4 as a maximal compact subgroup. Its unitary irreducible representation must contain all and every irreducible representation of SO_4 only once. It has been conjectured that (up to a choice[32]) the de Sitter group $SO(4,1)$ is adequate. The other generators are a vector $\bar{B}$ and a scalar S, with the following commutation relations:

$$[\bar{L}, \bar{L}] = i\bar{L} \quad [\bar{L}, \bar{A}] = i\bar{A} \quad [\bar{L}, \bar{B}] = i\bar{B} \quad [\bar{L}, S] = 0$$
$$[\bar{A}, \bar{A}] = i\bar{L} \quad [\bar{A}, \bar{B}] = iS \quad [S, \bar{A}] = i\bar{B} \quad [S, \bar{B}] = i\bar{A}$$
$$[\bar{B}, \bar{B}] = -i\bar{L} \tag{10}$$

The form of the operators $\bar{B}$ and S is given in terms of the dynamical variables $\bar{r}$, $\bar{p}$ classically by Bacry[3]. The expression of S and $\bar{B}$ are

$$S = \frac{1}{\sqrt{-2H}}\left[2Hu\phi - (1 + 2Hr)\frac{\partial\phi}{\partial u}\right]$$

$$\bar{B} = \left(r\frac{\partial\phi}{\partial u} - u\phi\right)\bar{p} + \frac{1}{r}\bar{r}$$

where

$$u = \bar{r}\cdot\bar{p}$$

$$\phi(H, u) = \sqrt{a - \frac{1}{2H}} \sin\left[\sqrt{-2H}\, u + \theta(H)\right]$$

where a is an arbitrary constant. $\theta(H)$ is an arbitrary function of classical energy. The quantum mechanical description is obtained by replacing the Poisson brackets by commutators, the c numbers $\overline{r}, \overline{p}$ by the corresponding operators. It is clear that there exists an infinite number of Hermitian operators corresponding to the function ϕ, therefore the solution is not unique. The non-uniqueness is due to the fact that ϕ satisfies a second order equation; its solution is given above. Thus S and $\overline{B}$ are essentially second-order differential operators in coordinate space. However, if one looks at the problem in the p-space, there we know that properly parametrized infinitesimal generators are linear differential operators. As a result of this, the solution is unique.

The p-space equation (1) reads

$$(p^2 + p_o^2)\, \psi(p) = \frac{1}{2\pi} \int d^2q \, \frac{\psi(q)}{|\overline{q} - \overline{p}|^2} \qquad (1)'$$

which shows no more than the ordinary rotational invariance in a three dimensional space. Let us project stereographically[2,15] the p-space on a four dimensional sphere of radius p_o^2 and let $\overline{\xi}$ be the point of this hypersphere corresponding to $\overline{p}$. The following relations hold

$$|\bar{p} - \bar{q}|^2 = \frac{(p^2 + p_o^2)(q^2 + p_o^2)}{2p_o^2} \, |\bar{\xi} - \bar{\eta}|^2 \qquad (2)'$$

$$\xi_k = \frac{2p_o \, p_k}{p_o^2 + p^2} \qquad\qquad k = 1,2,3 \qquad (3)'$$

$$\xi_o = \frac{p_o^2 - p^2}{p_o^2 + p^2}$$

We introduce the new wave-function

$$\Phi(\xi) = \frac{1}{4\, p_o^{5/2}} (p_o^2 + p^2)\, \psi(p) \qquad (4)'$$

With the aid of (2)' equation (1)' reads:

$$\Phi(\xi) = \frac{1}{2\pi p_o} \int \frac{\delta(\eta^2 - 1)\, \Phi(\eta)}{|\bar{\xi} - \bar{\eta}|^2} \, d^4\eta \qquad (5)'$$

which shows that the problem is rotationally invariant in a four dimensional space. The two functions $\psi(p)$ and $\phi(\xi)$ satisfy the same normalizability conditions.(2)(25) The generators of these rotations are given by

$$L_{\mu\nu} = -i\left(\xi_\mu \frac{\partial}{\partial \xi_\nu} - \xi_\nu \frac{\partial}{\partial \xi_\mu}\right)$$

$$\mu, \nu = 1,2,3,4$$

which generate the same algebra SO_4 with equations (7) to (8). To obtain the dynamical group we project the four-dimensional sphere on a five dimensional hyperboloid. The dynamical group now is the group of motions on the hyperboloid, their generators define a de Sitter group whose Lie algebra is given by (10). For completeness the generators have the form [25]

$$L_{ijk} = -i\left(p_j \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial p_j}\right) ; \quad (ijk) = (123)$$

$$A_i = i\left(\frac{p_i p_j}{p_o} + \delta_{ij} \frac{p_o^2 - p^2}{2p_o}\right) \frac{\partial}{\partial p_j} - 2\frac{p_i}{p_o}$$

$$S = i\left[\left(\frac{p_o^2 + 1}{2p_o}\right) p_i \frac{\partial}{\partial p_i} - \left(\frac{1 - p_o^2}{2p_o}\right)\left(\frac{p_o^2 - p^2}{p_o^2 + p^2}\right) p_o \frac{\partial}{\partial p_o}\right]$$

$$B_i = -i\left[\left(\frac{1+p_o^2}{2p_o^2}\right)\left\{\left(\frac{p_o^2+p^2}{2}\right)\delta_{ik} - p_i p_k\right\} \frac{\partial}{\partial p_k} + \left(\frac{1 - p_o^2}{p_o^2 + p^2}\right) p_o p_i \frac{\partial}{\partial p_o}\right]$$

The summation convention is used for repeated indices.

Now we want to find the irreducible unitary representation of the de Sitter group $SO_{4,1}$ which by definition describes the system completely.

The two Casimir operators of the algebra given by equations (10) are:

The quadratic operator:

$$Q = S^2 + B^2 - A^2 - L^2$$

and the quartic operator:

$$w = (s\bar{L} - \bar{A} \times \bar{B})^2 - \frac{1}{4}\left[L.(A + B) + (\bar{A} + \bar{B}).\bar{L}\right]^2$$

Since $\bar{L}.\bar{A} = 0$ we can show in the following that $\bar{L}.\bar{B} = 0$ for: multiply both sides (scalar multiplication) of the equation

$$\left[S, \bar{A}\right] = i\,\bar{B}$$

by $\bar{L}$,

$$\therefore \quad i\,\bar{L}.\bar{B} = \left[S, \bar{A}\right].\ \bar{L} = \left[S, \bar{A}.\bar{L}\right] - \bar{A}\ .\ \left[S, \bar{L}\right]$$

using the equation $\left[S, \bar{L}\right] = 0$. Hence obtain the result. The relation $S\bar{L} = \bar{A} \times \bar{B}$ holds. We have $w = 0$.

The last equality means that the energy depends only on one quantum number, as it should do. Therefore we look for a unitary irreducible representation for $SO(4,1)$ in which the quartic Casimir operator is zero.

set W, Q, K^2, K'^2, K_3, K_3' forms a complete set of commuting operators, every state being represented by the ket

$$\left| w = 0, \quad Q = a\ ;\ k\ k'\ k_3\ k_3' \right\rangle \qquad (11)$$

Such representations havebeen investigated by many people(12,23,33) from which one knows that there are two classes of unitary irreducible representations, one of which is chosen in such a way that it gives a complete description for the hydrogen atom system. (See Chapter 2).

(II) THE RIGID ROTATOR

The rigid rotator represents the motion of the nuclei in a diatomic molecule in the limit when the vibrational quanta are treated as infinitely large[19a]. The only degrees of freedom of the rotator are the angular momentum variables (θ, ϕ) fixing its spatial orientation. If $\bar{L}$ is the rotator's angular momentum vector, m the reduced mass, and r = mutual distance, the Hamiltonian of the system is given by:

$$H = \frac{L^2}{2I} \qquad (1)$$

where $I = mr^2$.

It is clear that the three vector $\bar{L}$ is a constant of motion, therefore the Hamiltonian is invariant under the transformation generated by the three vector $\bar{L}$, i.e. the algebra of the rotation group SO(3), i.e.

$$[\bar{L}, \bar{L}] = i \bar{L} \qquad (2)$$

The energy is given as a function of the Casimir operator, i.e. L^2 .

Representations of the Lie algebra (2) are well known, all unitary irreducible representations are finite dimensional. Every state is represented by the ket:

$$| j \,;\, j_3 \rangle \quad \text{or simply} \quad | j_3 \rangle \qquad (3)$$

where

$$L^2 \left| j_3 \right\rangle = j(j+1) \left| j_3 \right\rangle$$

$$L_3 \left| j_3 \right\rangle = j_3 \left| j_3 \right\rangle$$

and

$j_3 = -j, -j+1, \ldots, +j$, j takes all integral values; if E is the energy of the rotator, then:

$$E = \frac{j(j+1)}{2I} \tag{4}$$

which is the well known formula for the rotator's energy. The states of the rotator fall into (2j+1) dimensional representation of SO(3). For each integral value j (non negative), there is only one multiplet of states furnishing the (2j+1) dimensional representation of SO(3).

We want to identify the entire spectrum with a single irreducible representation of a larger group G. In other words one unitary irreducible representation of G is the sum of all irreducible representations of its maximal compact subgroup SO(3), each of them being contained only once.

It is well known that the entire spectrum is identified with a single irreducible representation of E(3) (in flat space) or the Lorentz group SO(3,1) (in curved space). In the case of E(3) the rotation generators are the dynamical variables of angular momentum, while the translation generators are the components of the radial vector on the unit sphere[32].

In the following we write down the explicit form of the generator of the non-invariance (i.e. dynamical) group in the classical case. The quantum version is obtained by replacing Poisson brackets by commutators and the c-numbers $\bar{r}, \bar{p}$ by their corresponding quantum mechanical operators.

The generators of the Lorentz group are the two three vectors $\bar{L}$ and $\bar{B}$ with the following commutation relations:

$$\{\bar{L}, \bar{L}\} = \bar{L} \tag{5.1}$$

$$\{\bar{L}, \bar{B}\} = \bar{B} \tag{5.2}$$

$$\{\bar{B}\ \bar{B}\} = -\bar{L} \tag{5.3}$$

where

$$\{f_1(r,p),\ f_2(r,p)\} = \sum_{i=1}^{3} \left(\frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial x_i}\right)$$

where f_1 and f_2 are two scalar functions.

Equation (5.1) is realized by the angular momentum vector

$$\bar{L} = \bar{r} \times \bar{p} \tag{6}$$

From equation (5.2), $\bar{B}$ is a vector under rotation, so it must have the form

$$\bar{B} = f\,\bar{p} + g\,\bar{r} \tag{7}$$

where f, g are scalar functions. In other words, they are functions of the three scalars r, p and $(\bar{r}.\bar{p})$. One possible solution is

$$\bar{B} = r\left(\bar{p} - \frac{(\bar{r}.\bar{p})}{r^2}\,\bar{r}\right) + \alpha\,\frac{\bar{r}}{r} \tag{8}$$

where α is a constant, to be determined (see below).

Equation (8) is determined up to an arbitrary function of L^2, which we choose to be one.

Vectors (6) and (7) are subject to the following conditions

$$\bar{L} \cdot \bar{B} = \bar{B} \cdot \bar{L} = 0 \tag{9}$$

$$B^2 - L^2 = \alpha^2 \tag{10}$$

Equations (9) and (10) are equivalent to saying that the rigid rotator admits the Lorentz group as a dynamical group in which one of the Casimir operators has zero eigenvalues. Equation (10) tells us more, namely that the Hamiltonian can have the form:

$$H = \frac{B^2 + \alpha^2}{2I} = \frac{P^2}{2I} \tag{11}$$

This form is preferred to the form given by (1) because the model admits generalizations which are more directly related to physics(7).

It is an easy matter to obtain the quantum version of the above formulation. Equations (5.1) - (5.3) become:

$$[\bar{L}, \bar{L}] = i\,\bar{L} \tag{5'.1}$$

$$[\bar{L}, \bar{B}] = i\,\bar{B} \tag{5'.2}$$

$$[\bar{B}, \bar{B}] = i\,\bar{L} \tag{5'.3}$$

It should be noted that only the Hermitian part of $\bar{B}$ is used in the above equations. Equation (9) remains unchanged. Equation (10) has the form

$$B^2 - L^2 = 1 + \alpha^2 \tag{10'}$$

Unitary irreducible representations of the algebra (5'.1) — (5'.3) are labelled by two numbers (j_o, a) with j_o = 0, ½, 1,..... and $-\infty < a < +\infty$ (the principal series) or $j_o = 0$ and $0 \leq ia \leq 1$ (the supplementary series). It is shown (Chapter 2) that the entire spectrum of the rigid rotator is given by the supplementary series (with $j_o = 0$).

We remark that the Euclidean group E(3) can be used as a dynamical group for the rigid rotator (i.e. one identifies the spectrum of the rigid rotator) with one irreducible unitary representation of the group E(3). Since E(3) is obtained from SO(3,1) by the method of group constructions,[18] therefore the constructed space is the desired representation space of E(3) (see Chapter 2).

(C) THE ISOTROPIC HARMONIC OSCILLATOR

(a) The Symmetry Group

It is well known[19] that the Hamiltonian of the n-dimensional oscillator is invariant under the SU(n) group.

To see this let us write the Hamiltonian H of the normalized (unit mass and coupling constant) harmonic oscillator in n dimensions as:

$$H = \tfrac{1}{2} \sum_{i=1}^{n} (p_i^{\,2} + q_i^{\,2}) \tag{1}$$

As usual, we define

$$a_j = \tfrac{1}{2}(p_j - iq_j) \tag{2}$$

$$a_j^* = \tfrac{1}{2}(p_j + iq_j) \tag{3}$$

In quantum mechanics one knows the p_i and q_i satisfy the following commutation relations:

$$[q_i, p_j] = i\,\delta_{ij} \tag{4}$$

from which it follows:

$$[a_i, a_j^*] = \delta_{ij} \tag{5}$$

Using (2), (3) and (5), the Hamiltonian H has the form

$$H = \sum_{i=1}^{n} (a_i^* a_i + \tfrac{1}{2}) \tag{6}$$

It is clear that the Hamiltonian is invariant under the algebra generated by the following operators

$$L_{ij} = \tfrac{1}{2}(a_i^* a_j + a_j^* a_i) \qquad i,j = 1,2,\ldots,n$$

The following commutation relations hold:

$$[L_{ij}, L_{\ell k}] = \delta_{ik} L_{\ell j} - \delta_{\ell j} L_{ik} \tag{7}$$

It is worth mentioning that the Hamiltonian (6) has the form

$$H = L_{ii} .$$

Thus we are left with $(n^2 - 1)$ generators. To show that the invariant algebra is the algebra of the compact $SU(n)$; we know that p and q are hermitian, therefore the generators (7) satisfy the following hermiticity conditions.

$$(L_{ij})^+ = L_{ji}, \qquad \text{hence the result.}$$

(b) <u>The Dynamical Group.</u>

Now we want to embed this algebra, into a larger algebra such that:

(i) This algebra is of finite dimension, and build a non-compact group G.

(ii) $SU(n)$ is a maximal compact subgroup of the non-compact group.

(iii) The eigenvalues of H constitute a basis for the unitary irreducible (infinite) representation of G.

The most immediate non-compact extension is obtained by adding to SU(n) algebra, the Hamiltonian and the n(n+1) generators,

$$\text{i.e.} \quad a_i^\dagger \; a_j^+ \; ; \quad a_i \; a_j \quad (i,j = 1,2,\ldots, n)$$

which give the algebra of the non-compact SP(2n).[(14)] However, it is shown that minimal compact algebra satisfying the above conditions is the non-compact SU(n,1)[(6)].

We remark that for $n = 1$, the problem is trivial; this is because $SU(1,1) \sim SP(2) \sim SO(2,1)$. For $n = 3$ see (8) for more details. Since our concern is the construction of the rigged Hilbert space of that finite system, it is irrelevant in this case ~~to tell~~ which is the adequate dynamical group (see Chapter 2).

I. REMARKS

If the Hamiltonian for a given quantum-mechanical system is invariant under a group of operations (which we call the symmetry group); then the eigenfunctions which correspond to each energy (or any other quantum number) eigenvalue form the basis of a representation of this symmetry group (or the degeneracy group). The degeneracy of an eigenstate is directly related to the dimensionality of the irreducible representations. One might consider the inverse problem, namely how to extract dynamical information, given the symmetry. An initial step along this line was taken[16], by Greenberg. It is shown, that given the symmetry group (e.g. SO_4 for the hydrogen atom, SU(2) for the two-dimensional oscillator) one can predict the form of the Hamiltonian. One might try to extend these ideas, to determine the dynamics of hadrons from the empirically observed symmetries among the strong interacting particles. It can be used also to test the various symmetry groups that have been considered so far (e.g. SU(3), SU(6)....); (see Section II).

II. REMARKS

It may happen that the states of a quantum system are correctly labelled by the irreducible representations of the group of degeneracy G_E (E is the energy), but the levels are not degenerate, then the function equation

$$E = f(C_i)$$

$$\text{or} \quad E = f(C_1) \quad \text{(for ideal systems)}$$

cannot hold where C_i $i = 1,2,\ldots$ are the Casimir operators of the symmetry group G_E.

In this case the actual group of degeneracy of E is a smaller group g_E. The group G_E can still be used either by saying that G_E is now the group of degeneracy of a new quantity E' given by:

$$E' = f(C_i)$$

or by using the representations of G_E with non-vanishing Casimir operators (for ideal systems all $C_i = 0$ except one). E is a function of a tensor operator in G_E:

$$E = f(C_i, D_i)$$

where D_i are the Casimir operators of the group g_E for ideal systems. E has the form: $E = f_1(C_i) + f_2(D_i)$.

Then the second term in the above equation gives the splitting of levels of the multiplets of the ideal system.

This is actually what happens in particle physics where we have only approximate symmetry. For example, high energy interacting particles are grouped in multiplets with respect to SU(3) representations; although they ~~happen to be~~ are not quite degenerate in mass.

III. REMARKS

Given a non-compact symmetry group (e.g. the dynamical group is a symmetry group in a general sense), one prefers to work with the corresponding Lie algebra and its representations by skew symmetric operators in a Hilbert space H (after completion with respect to a well defined norm). In this case the generators cannot be defined in the whole Hilbert space, but only on a dense domain D in the space H, therefore these operators (which are related to physical observables) must be specified by their domain of definition.

It may happen that some operators have a complete set of eigenvectors lying outside the Hilbert space.

In Section (2) we construct the rigged Hilbert space $\Phi \subset H \subset \Phi^*$. The determination of Φ solves the problem of the domain, the determination of Φ^* solves the problem of interpreting those eigenvectors which cannot be interpreted by postulating only the Hilbert space as the physical space.

IV. REMARKS

In quantum mechanical systems (which we are considering) we have the choice of starting with the dynamical system formulated in terms of dynamical variables (e.g. canonical variables p_i, q_i momenta and positions or related entities). Then we search for a non-invariance (dynamical) group, such that the entire spectrum of the system constitutes an irreducible representation of such a group. The generalized enveloping algebra of the dynamical group contains all the dynamical variables. In contrast, the dynamical structure of elementary particles is not well known. Here a new hypothesis should be made and be tested with the available experimental data.

One method of predicting the non-compact group G is to look for groups which have SU(6) as a ~~[illegible]~~ compact sub-group, for example, SU(6,6), U(12), SL(6,c)[11,26,31]. The difficulty with such groups is how one can identify so many invariants of such groups with well known physical quantities.

The restriction of using compact Lie groups as symmetry groups is not very clear. From Section 2 one learns that finite groups can be used as well. If this is so, then the dynamical group is the inhomogeneous Lorentz group extended by finite groups. It should be noted that it might be useful to use different finite groups to describe the internal structure of different physical systems, or different types of ray representations of a chosen finite group to describe various systems as well as their interactions. However, this is an open question needing to be answered.

CHAPTER I

SECTION B THE RIGGED HILBERT SPACE

A Hilbert space H is an infinite dimensional inner product space which is a complete metric space with respect to the metric generated by the inner product.

A function F, which relates to each element $f \in D$ (D is a subset of H) a definite complex number $F(f)$, is called a functional in the space H with domain D.

A function T, which relates to each element $f \in D$ (D is a subset of H) a particular element $Tf = g \in H$, is called an operator in the space H with domain D.

We are interested in linear operators and linear functionals. We introduce the following definitions:

Definition A functional F in H is said to be linear if

(i) Its domain of definition D is a linear manifold and

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$$ i.e. homogeneous and additive.

(ii) $$\sup_{f \in D,\ \|f\| \leq 1} |F(f)| < \infty$$ i.e. bounded.

where $f, g \in D$ and α, β are complex numbers.

If the functional F is continuous at one point and satisfies (i), then it satisfies (ii), i.e. bounded.

Definition An operator T is linear if its domain D is a linear manifold and if

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$$ i.e. homogeneous and additive

for all $f, g \in D$ and α, β are complex numbers.

In contrast to linear functionals linear operators need not be bounded. A linear operator is bounded if

$$\sup_{f \in D;\ \|f\| \leq 1} \|Tf\| < +\infty .$$

<u>Definition</u> A linear operator is completely continuous which maps a Hilbert space, H_1, into a Hilbert space H_2, if it carries any bounded set into a set which is compact.

<u>Theorem (1)</u> Every completely continuous operator is bounded.

<u>Proof</u>:

If not, there would exist a sequence $\{f_k\}_{k=1}^{\infty}$ for which

$\|f_k\| = 1$ and $\|Af_k\| \geqslant K$ $(k = 1,2,\ldots)$ which is impossible.

The following propositions hold(1)

1. If A is a completely continuous operator, then its adjoint A^* (i.e. A^*: $(Af,g)_2 = (f,A^*g)_1$) is a completely continuous operator.
2. If A is a continuous operator and B is a completely continuous operator then AB and BA are completely continuous operators.
3. Any finite linear combination of completely continuous operators is a completely continuous operator.

<u>Theorem (2)</u> A completely continuous operator A which maps H_1 into H_2 has the form $A = UT$(15) where T is a positive semi-definite (i.e. $(Tf,f) \geqslant 0$) completely continuous operator, and U is an isometric operator (i.e. $(Uf,\ Ug)_2 = (f,g)_1$ $f,\ g \in H$

the suffix 1 (2) means the scalar product is defined in H_1 (H_2)). Moreover a completely continuous operator has the form:

$$Af = \sum_{n=1}^{\infty} \lambda_n (f, e_n) h_n \qquad (1)$$

where $f \in H_1$; $\{e_n\}$, $\{h_n\}$ are orthonormal basis in H_1 and H_2 respectively:

$$\lambda_n \geqslant 0 \quad \text{and} \quad \lim_{n \to \infty} \lambda_n = 0 \quad .$$

Definition A completely continuous operator $A(= UT)$ is said to be of Hilbert-Schmidt type if it has the decomposition (1) with $\lambda_n \geqslant 0$ and $\sum_{n=1}^{\infty} \lambda_n^2$ converges.

Definition A completely continuous operator $A(= UT)$ is said to be nuclear if it has the decomposition (1) with λ_n 0 and $\sum_{n=1}^{\infty} \lambda_n$ converges.

In contrast to finite dimensional space, not every unitary ($\|f\| = \|Uf\| = \|U^{-1}f\|$) operator or self-adjoint operator $\left(A: (Af, g) = (f, A^* g) \text{ for all } f, g \in H\right)$ has a complete set of eigenvectors in the Hilbert space H. e.g. the translation operator U_a in the Hilbert space L^2 has only the zero eigenvector in L^2, while it has a complete set of eigenvectors (generalized eigenvectors) which lie outside the Hilbert space. Such eigenvectors cannot be interpreted unless one goes to a larger space. One usually obtains the Hilbert space as follows:

Consider a physical system which is described by a dynamical group, say. The states of this system span a linear space, Ψ, (representations of the dynamical groups) in which one defines a positive definite bilinear Hermitian functional, i.e. (ϕ, ψ); where $\phi, \psi \in \Psi$ such that:

(i) $(\phi_1 + \phi_2, \psi) = (\phi_1, \psi) + (\phi_2, \psi)$

(ii) $(a\phi, \psi) = a(\phi, \psi)$

(iii) $(\phi, \psi) = (\psi, \phi)$

(iv) (ϕ, ϕ) 0, and $(\phi, \phi) = 0$ only when $\phi = 0$.

Taking (ϕ, ψ) as a scalar product in Ψ and completing it with respect to the norm defined by this scalar product, one obtains a corresponding Hilbert space. If one is able to define another scalar product (which is possible) in the linear space Ψ, one obtains different Hilbert spaces corresponding to different scalar products.

In order to interpret eigenfunctions lying outside the Hilbert space, one has to consider with the Hilbert space a certain extension Φ' of it. Extension spaces are obtained by defining various topologies different from the one generated by the ordinary scalar product. For our purpose we shall make use of the concept of a rigged Hilbert space[10,15] $\Phi \subset H \subset \Phi^*$ defined below:

<u>Definition</u> A countable ~~normed~~ Hilbert space Φ is a complete linear topological space. The topology is given by a countable set of norms $\|\phi\|_n^2 = (\phi, \phi)_n$; $n = 1, 2, \ldots, \infty$. The norms are

compatible in the sense that if a sequence $\{\phi_k\}_n$ of elements of Φ converges to zero in the norm $\|\phi\|_n$ and is a fundamental sequence in the norm $\|\phi\|_m$ then it converges to zero in the norm $\|\phi\|_m$.

If we denote by Φ_n the space obtained by completing Φ with respect to the norm $\|\phi\|_n$, due to the completeness of Φ, then $\Phi = \bigcap_{n=1}^{\infty} \Phi_n$.

We could assume without loss of generality that the set of inequalities hold,

$\|\phi\|_1 \leq \|\phi\|_2 \ldots \leq \|\phi\|_n \leq \ldots$ which implies:

$$\Phi \subset \ldots \subset \Phi_n \ldots\ldots \ldots\ldots \subset \Phi_1 \qquad (5)$$

Along with the space we consider the adjoint space ,
i.e. the space of linear functionals on the space ϕ. If we denote by ϕ_{-k} the space ϕ_k^* the adjoint space of the Hilbert space ϕ_k. Then it is easily shown(15) that $\phi^* = \bigcup_{n=1}^{\infty} \phi_{-n}$. Since Φ_{-n} is a Hilbert space, then there is defined a norm $\|F\|_{-n} = \sqrt{(F, \phi)_{-n}}$ by the scalar product $(F, G)_{-n}$ in Φ_{-n}.

<u>Definition</u> A countable Hilbert space Φ is called nuclear if for every m there is an n such that the mapping T_m^n of the space Φ_n into the space Φ_m is nuclear.

It is clear that Φ^* is also a nuclear space, the mapping T_{-n}^{-m} (the adjoint operator of T_m^n for $n, m < -1$) is also nuclear.

It was mentioned in the introduction that among various spaces in which the mathematical structure of a physical system should be mapped, one is the rigged Hilbert space. In the following we give the definition of this space with some of its properties. Proofs are not presented (which can be found elsewhere)[(10,15)]).

<u>Definition</u> A rigged Hilbert space is a triple of spaces Φ, H, Φ^* such that:

a) Φ is a nuclear space, in which there is defined a non-degenerate scalar product (ϕ, ψ) ϕ , i.e.

$$(\phi, \psi_1 + \psi_2) = (\phi, \psi_1) + (\phi, \psi_2)$$

$$(\phi, \psi) = \overline{(\psi, \phi)}$$

$$(\phi, a\psi) = a(\phi, \psi) \; ; \; (\phi, \phi) \geq 0 ; (\phi, \phi) = 0 \text{ only when } \phi = 0$$

If $\lim_{n \to \infty} \psi_n = \psi$ then $\lim_{n \to \infty} (\phi, \psi_n) = (\phi, \psi)$.

b) H is the completion of Φ with respect to the norm defined by the scalar product (ϕ, ψ), where .

c) Φ^* is the adjoint space of Φ .

From the definition it follows that:[(15)]

i) Φ is a dense subset in H, i.e. there exists a continuous linear operator T which maps Φ onto an everywhere dense subset in H.

(ii) The adjoint operator T^* of T is antilinear and maps H on to an everywhere dense subset in H^* (the adjoint space of H).

It should be noted that if we define Φ^* as the space of antilinear functions on Φ , then T^* becomes a linear mapping.

The Realization of the Rigged Hilbert Space:

Let $\phi \subset H \subset \phi^*$ be a rigged Hilbert space. Consider the realization $h \quad h(x)$ of the space H as a space of functions with the scalar product:

$$(\phi, \psi) = \int_X \phi(x)\ \overline{\psi(x)}\ d\sigma(x)$$

where $\sigma(x)$ is a positive measure on some set X.

Then to each element $\phi \in \Phi$ there corresponds a function $\phi(x)$, associated by this realization with the element $T\phi \in H$. Thus we obtain a realization $\phi \to \phi(x)$ of the space Φ, induced by the realization $h \to h(x)$ of the space H. Φ^* is realized by the linear functionals on the space Φ. Moreover we could choose the realization of Φ, induced by the realization of H such that

$$\phi(x) = F_x(\phi) \text{ for every } x. \quad (15)$$

Definition A linear functional F on Φ is such that:

$$F(A\phi) = \lambda F(\phi) \quad \text{for every} \quad \phi \in \Phi$$ is called a

generalized eigenvector of the operator A, corresponding to the eigenvalue λ.

Theorem

(i) A unitary operator in a rigged Hilbert space has a complete system of generalized eigenvectors, corresponding to eigenvalues λ having modulus one.

(ii) A self-adjoint operator in a rigged Hilbert space has a complete system of generalized eigenvectors, corresponding to real eigenvalues.

Theorem [15]

If $\{T_k\}$, $1 < k < n$ (finite), is a system of commuting self-adjoint (unitary) op rators in a rigged Hilbert space, then the set of generalized eigenvectors of this system is complete.

In the following we shall illustrate our postulate given in the introduction, namely the use of the rigged Hilbert space as a physical space

Example 1: Let T_h be the operator of translation in the Hilbert space L^2 of functions on the line, having square integrable moduli. Let $f(x) \in L^2$ in such a way that:

$$T_h\, f(x) = f(x-h) = a\, f(x) \qquad (1)$$

Since $$F(\lambda) = \int f(x)\, e^{i\lambda x} dx \qquad (2)$$

is the Fourier transform of $f(x)$ and

$$f(x) = \frac{1}{2\pi} \int F(\lambda)\, e^{-i\lambda x}\, d\lambda \qquad (3)$$

It follows from (1) that:

$$e^{i\lambda h}\, F(\lambda) = a\, F(\lambda) \qquad (4)$$

$F(\lambda) \neq 0$ for points $a = e^{i\lambda h}$, otherwise $F(\lambda) = 0$, i.e. $F(\lambda)$ is different from zero only at a countable set of points. Therefore T_h has only the zero eigenvector in L^2.

Consider the complete set of functions $e^{-i\lambda x}$, then:

$$T_h\ e^{-i\lambda x} = e^{i\lambda h}\, e^{-i\lambda x}$$

i.e. $(e^{-i\lambda x})$ is an eigenfunction of the operator T_h, corresponding to the eigenvalue $e^{i\lambda h}$. The system of eigenfunctions $e^{-i\lambda x}$ is complete, in the sense that every $f(x) \in L^2$ has the form (3) and the following equality holds:

$$\int |f(x)|^2 \, dx = \frac{1}{2\pi} \int |F(\lambda)|^2 \, d\lambda \qquad (5)$$

Thus T_h has a complete set of eigenfunctions $e^{-i\lambda x}$ lying outside the Hilbert space.

Consider a rigged Hilbert space $\Phi \subset H \subset \Phi^*$. Consider a realization of H as L^2, the space of square integrable functions, then Φ is often realized by the space S of infinitely differentiable functions, rapidly decreasing on the real line together with their derivatives of any order and Φ^* is realised as the space S^* of tempered distributions(15). Now we say that the functions $e^{-i\lambda x}$ are generalised eigenvectors for the translation operator considered in the space S.
$F(\lambda) = (e^{-i\lambda x}, \phi(x))$ where $\phi(x) \in S = \Phi$ and $F(\lambda)$ is the Fourier transform of $F(\lambda)$. Equation (5) ensures that the set of generalised eigenvectors $e^{-i\lambda x}$ is complete, i.e. $F(\lambda) = 0$ implies $\phi(x) = 0$ (See Theorem 1).

CHAPTER 2

(I) CONSTRUCTION OF THE RIGGED HILBERT SPACE OF THE RIGID ROTATOR

(a) It is shown in Chapter 1 that the dynamical group of the rigid rotator is the Lorentz group, in which one of the two Casimir operators has zero eigenvalues.

The Lie algebra of G is generated by the two vectors $\overline{L}$ and $\overline{B}$ and the commutation relations are of the following form:

$$\begin{aligned} [\overline{L}, \overline{L}] &= i\,\overline{L} \\ [\overline{L}, \overline{B}] &= i\,\overline{B} \\ [\overline{B}, \overline{B}] &= -i\overline{L} \end{aligned} \tag{1}$$

with the two Casimir operators given by:

$$Q_1 = \overline{L}\,.\,\overline{B} \tag{2}$$

$$Q_2 = B^2 - L^2 \tag{3}$$

The construction of the representation space Ψ has been given in detail[21]. Starting with a vector $|0\,0\rangle$ (we use Dirac's notation) with the properties that:

(i) $J_1\,|0\,0\rangle = 0$

(ii) $\langle 0\,0, 0\,0\rangle = 1$

where $\langle\ ;\ \rangle$ is the scalar product. We obtain by successive application of the operators J_1 and B_1. the set of vectors:

$$|(j_0, a);\, j\, j_3\rangle = \psi_{j\, j_3} \tag{4}$$

where:

$$(j_0, a) \in X = \left[(j_0, a)\,\middle|\, j_0 = 0, \tfrac{1}{2}, 1, \ldots\,;\, -\infty < a < +\infty\right] \cup \left[(j_0, a)\,\middle|\, j_0 = 0,\ 0 \leq ia \leq 1\right]$$

$$j = j_o, j_o+1, \ldots \quad \ldots \quad \ldots , \text{ and}$$

$$j_3 = -j, -j+1, \ldots \quad \ldots \quad \ldots , +j$$

The pair (j_o, a) name the irreducible representation. The set of states (4) spans a linear space, the linear envelope of which we call Ψ. The two numbers (j_o, a) are related to the eigenvalues of the two Casimir operators,

$$\text{if} \quad Q_i \left|(j_o,a) ; j\, j_3\right\rangle = q_i \left|(j_o,a) ; j\, j_3\right\rangle \quad (i = 1,2..$$

Then:

$$q_1 = -j_o\, a \tag{4}$$

$$q_2 = (1 + a^2 - j_o^{\,2}) \tag{5}$$

We shall denote by $\left|j\, j_3\right\rangle$ the ket $\left|(j_o,a); j\, j_3\right\rangle$, writing

$$L_\pm = L_1 \pm i\, L_2$$

$$B_\pm = B_1 \pm i\, B_2$$

Then the operators $B_\pm$, $L_\pm$, B_3, L_3 have the form:

$$L_+ \left|j\, j_3\right\rangle = \sqrt{(j-j_3)(j+j_3+1)}\, \left|j \quad j_3+1\right\rangle \tag{6.1}$$

$$L_- \left|j\, j_3\right\rangle = \sqrt{(j+j_3)(j-j_3+1)}\, \left|j \quad j_3-1\right\rangle \tag{6.2}$$

$$L_3 \left|j\, j_3\right\rangle = j_3 \left|j \quad j_3\right\rangle \tag{6.3}$$

$$\begin{aligned} B_+ \left|j\, j_3\right\rangle = {} & \sqrt{(j-j_3)(j-j_3+1)}\; c_j \left|j-1, \; j_3+1\right\rangle \\ & - \sqrt{(j-j_3)(j+j_3+1)}\; A_j \left|j \quad j_3+1\right\rangle \\ & + \sqrt{(j+j_3+1)(j+j_3+2)}\; c_{j+1} \left|j+1, \; j_3+1\right\rangle \end{aligned} \tag{6.4}$$

$$B_- |j\ j_3\rangle = \sqrt{(j+j_3)(j+j_3+1)}\ c_j\ |j-1,\ j_3-1\rangle$$
$$- \sqrt{(j+j_3)(j+j_3+1)}\ A_j\ |j,\ j_3-1\rangle$$
$$- \sqrt{(j-j_3+1)(j-j_3+2)}\ c_{j+1}\ |j+1,\ j_3-1\rangle \qquad (6.5)$$

$$B_3 |j\ j_3\rangle = \sqrt{(j-j_3)(j+j_3)}\ c_j\ |j-1,\ j_3\rangle - j_3 A_j\ |j\ j_3\rangle$$
$$- \sqrt{(j+j_3+1)(j-j_3+1)}\ c_{j+1}\ |j+1,\ j_3\rangle \qquad (6.6)$$

where

$$A_j = \frac{i j_0 a}{j(j+1)} \quad \text{and} \quad c_j = \frac{i}{j}\sqrt{\frac{(j^2 - j_0^{\,2})(j^2 + a^2)}{4j^2 - 1}}$$

and

$$\langle j\ \ j_3\ ;\ j'\ \ j_3'\rangle = \delta_{jj'}\ \delta_{j_3 j_3'} \qquad (7)$$

For the rigid rotator, we have:

$$q_1 = 0$$

which implies that $j_0 = 0$

$$q_2 = 1 + a^2 \geqslant 0$$

If we take $q_2 = 0$, i.e. $ia = 1$, then the ground state (for $c_1 = 0$) is not included.

Therefore we have to consider the representation of the Lorentz group given by ~~principle~~ supplementary series.

The linear envelope of the set of states $|j\ \ j_3\rangle$ is the space $\Phi(j_0 = 0, a) = \Phi(a)$. Completing Φ with respect to the norm defined by the scalar product (7) one obtains the Hilbert space $H(a)$.

We introduce into Ψ the stronger topology given by the countable number of scalar products $\langle\ ,\ \rangle_p$ given by:

$$\langle \psi_{jj_3} , \psi_{j'j_3'} \rangle_p = \langle \psi_{jj_3} | (J^2 + \tfrac{1}{4})^p | \psi_{j'j_3'} \rangle$$

$$= (j + \tfrac{1}{2})^{2p} \ \delta_{jj'} \ \delta_{j_3 j_3'} \quad (8)$$

for all integer p.

Note that (7) is a special case of (8), i.e. with $p = 0$. The completion of the space Ψ with respect to the topology given by (8) is the linear topological space Φ. The norms: $\Phi \ni \phi = \sum_{jj_3} c_{jj_3} \psi_{jj_3}$ are given by:

$$\|\phi\|_p^2 = \sum_{j\, j_3} |c_{jj_3}|^2 \ (j+\tfrac{1}{2})^{2p} \quad (9)$$

From equation (9) it follows:

$$\cdots \|\phi\|_p \geq \cdots\cdots \geq \|\phi\|_1 \geq \|\phi\|_0 \overset{def}{=} (\phi , \phi) \quad (10)$$

so that the canonical sequence of P is:

$$\Phi \subset \cdots \subset \Phi_p \subset \Phi_{p-1} \subset \cdots \subset \Phi_1 \subset \Phi_0 \equiv H \quad (11)$$

The compatibility of the norms follows from the continuity of the operator $(J^2 + \frac{1}{4}I)$ which follows from the continuity (see below) of the operators $J_\pm, J_3$ and I, which is a completely continuous operator (it maps the space Φ into itself).

That is Φ is a countably normed Hilbert space. Now we identify the space Φ_0 with the Hilbert space H. The continuity of the scalar product in H is assured by equation (10).

Since Φ is complete, then every $\phi \in \Phi$ can be expressed as the limit of the sequence $\{\phi^{(n)}\}_n$ of elements of where

$$\phi_n = \sum_{j=0}^{n} \sum_{j_3=-j}^{+j} C^{(n)}_{jj_3} \psi_{jj_3} \qquad (12)$$

The limit is understood, with respect to the topology in the space Φ,

i.e.

$$\|\phi - \phi_{(n)}\|_p \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all } p.$$

from which we write:

$$\phi = \sum_{jj_3} C_{jj_3} \psi_{jj_3} \qquad (13)$$

Let ϕ^P denote the element $\phi \in \Phi$ considered as an element of the space Φ_p (i.e. the space obtained by completing Ψ with respect to the topology in Φ_p). ϕ^P is the limit in the norm of the sequence $\{\phi^P_{(n)}\}_n$ of elements in the space $\Phi_p{}^P$.

Because of the completeness of the space $_p$, we write:

$$\phi^P = \sum_{jj_3} C_{jj_3} \psi^P_{jj_3} \qquad (14)$$

From which we have

$$\langle \psi^P_{jj_3}, \phi \rangle_p = (j+\tfrac{1}{2})^{2p} C_{jj_3} \qquad (15)$$

where $\phi \in \Phi_p$, i.e. we have the form (14).

(Note that we wrote ϕ to mean ϕ^P).

Similarly we consider elements in the space Φ_{p-1}, if we denote such elements ϕ^{P-1}, then:

$$\phi^{P-1} = \sum_{jj_3} c_{jj_3} \psi^{P-1}_{jj_3} \tag{16}$$

We define the operator T^P_{P-1} by:

$$T^P_{P-1} \phi = \phi^{P-1} \tag{17}$$

where $\phi \in \Phi_P$ i.e. T^P_{P-1} maps the space Φ_P into a dense subset of the space Φ_{P-1}.

From equations (14) - (17) we have:

$$T^P_{P-1} \phi = \sum_{jj_3} \frac{1}{(j+\frac{1}{2})^{2P}} \langle \psi^P_{jj_3}, \phi \rangle_P \psi^{P-1}_{jj_3} \tag{18}$$

which we could write in the form

$$T^P_{P-1} = \sum_{j=0}^{\infty} \lambda_j \sum_{j_3=-j}^{+j} \langle \psi^P_{jj_3}, \phi \rangle_P \psi^{P-1}_{jj_3} \tag{18}$$

where $\phi \in \Phi_P$, $\{\psi^P_{jj_3}\}$ and $\{\psi^{P-1}_{jj_3}\}$ are orthonormal basis in the spaces Φ_P and Φ_{P-1} respectively. (Note that $\{\psi_{jj_3}\}$ is an orthonormal basis in the space H), $\lambda_j \geqslant 0$ and $\sum_{j=0}^{\infty} \lambda_j$ converges, for p integer. Therefore the operator T^P_{P-1} is a nuclear operator, hence Φ is a nuclear Hilbert space.

Now we consider the adjoint space $\Phi^\times$ of the space Φ. The elements of the space are given by the linear functionals $F(\phi) = \langle F, \phi \rangle$ where $\phi \in \Phi$, which can be expressed in the basis $\{\psi_{jj_3}\}$ by:

$$\Phi^* \ni F = \sum_{jj_3} f_{jj_3} \psi_{jj_3}$$

such that

$$\langle F, \phi \rangle = \sum_{jj_3} \bar{f}_{jj_3} C_{jj_3} \quad \text{exists.}$$

Now since for $\Phi \ni \phi = \sum_{jj_3} C_{jj_3} \psi_{jj_3}$, the sum

$$\|\phi\|_P^2 = \sum_{jj_3} |C_{jj_3}|^2 (j + \tfrac{1}{2})^{2P}$$ has to exist, and for

$$H \ni h = \sum_{jj_3} h_{jj_3} \psi_{jj_3},$$

the sum:

$$\sum_{jj_3} |h_{jj_3}|^2$$ exists, then $H \subset \Phi^*$.

This completes the construction of the rigged Hilbert space $\Phi \subset H \subset \Phi^*$.

(b) In the following we shall show that the algebra generated by the operators $L_\pm$, L_3, $B_\pm$, B_3 is contained in the algebra of bounded operators in the space Φ.

Since every completely continuous operator is bound, it is sufficient to show that the generators are ~~completely~~ continuous operators.

Consider the sequence $\{\phi_n\}$ of elements in the space Φ; the norms in Φ are given by:

$$\|\phi_n\|_P^2 = \sum_{j=0}^{n} \sum_{j_3=-j}^{+j} |C^{(n)}_{jj_3}|^2 (j + \tfrac{1}{2})^{2P} \tag{20}$$

Assume that

$$\|\phi_n\|_P^2 \rightarrow 0 \qquad \text{as} \quad n \rightarrow \infty \tag{21.1}$$

for all P.

i.e.

$$\sum_{j=0}^{n} \sum_{j_3=-j}^{+j} |c_{jj_3}^{(n)}|^2 (j + \tfrac{1}{2})^{2P} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

for all P (21.2)

Then:

$$\|J_3\,\phi_n\|_P^2 = (J_3\,\phi_n,\ J_3\,\phi_n)_P$$

$$= \sum_{jj_3}^{n} j_3^2\,|c_{jj_3}^{(n)}|^2\,(j + \tfrac{1}{2})^{2P}$$

$$\leqslant \sum_{jj_3} c_{jj_3}^{(n)\,2}\,(j + \tfrac{1}{2})^{2(P+1)}$$

$$= \|\phi_n\|_{P+1}^2$$

from which in view of (16):

$$\lim_{n\to\infty} \|J_3\,\phi_n\|_P = 0$$

.·. J_3 is a completely continuous operator, i.e. bounded.

Similarly,

$$\|J_\pm\,\phi_n\|_P^2 = \sum_{jj_3}^{n} |c_{jj_3}^{(n)}|^2\,(j\pm j_3+1)(j\mp j_3)(j+\tfrac{1}{2})^{2P}$$

$$\leqslant \sum_{jj_3}^{n} |c_{jj_3}^{(n)}|^2\,(2j+1)(2j)(j+\tfrac{1}{2})^{2P}$$

$$\leqslant 4\sum_{jj_3}^{n} |c_{jj_3}^{(n)}|^2\,(j+\tfrac{1}{2})^{2(P+1)}$$

$$= 4\,\|\phi_n\|_{P+1}^2$$

Then

$$\lim_{n\to\infty} \| J_{\pm} \phi_n \|_P = 0$$

i.e. $J_{\pm}$ are ~~completely~~ continuous operators, hence bounded. To show that B_3 is a ~~completely~~ continuous operator, for simplicity we fix j_3, moreover (without loss of generality we take $j_3 = 0$, then:

$$\| B_3 \phi_n \|_P^2 = \langle B_3 \phi_n | (\bar{J}^2 + \tfrac{1}{4})^P | B_3 \phi_n \rangle \qquad (22)$$

Denote $\psi_{j\, j_3=0}$ by ψ_j and $c^{(n)}_{j,j_3=0}$ by $c^{(n)}_j$, then:

$$B_3 \psi_j = j\, c_j\, \psi_{j-1} - (j+1)\, c_{j+1}\, \psi_{j+1} \qquad (23)$$

and $\phi_n = \sum_{j=0}^{n} c^{(n)}_j \psi_j$.

Instead of (21.2) we have

$$\sum_j |c^{(n)}_j|^2 (j+\tfrac{1}{2})^{2P} \longrightarrow 0 \qquad (24)$$

as $n \to \infty$ for all P.

Using equations (22) and (23), then equation (22) can be written in the following form

$$\begin{aligned} \| B_3 \phi_n \|_P^2 &= \sum_{j=0}^{n} |c^{(n)}_j|^2 \, |c_j|^2 \, j^2 \, (j-\tfrac{1}{2})^{2P} \\ &+ \sum_{j=0}^{n} |c^{(n)}_j|^2 \, |c_{j+1}|^2 \, (j+1)^2 \, (j+\tfrac{3}{2})^{2P} \\ &- 2 \,\mathrm{Re} \sum_{j=0}^{\infty} \bar{c}^{(n)}_{j+1} \, c^{(n)}_{j-1} \, \bar{c}_{j+1} \, c_j \, (j(j+1)(j+\tfrac{1}{2})^{2P} \\ &= I^{(n)}_1 + I^{(n)}_2 + I^{(n)}_3 \end{aligned} \qquad (25)$$

To show that B_3 is a ~~completely~~ continuous operator, it is sufficient to show that the right hand sum of (25) goes to zero as $n \to \infty$, because the norm is positive. It follows that

$$\| B_3 \psi_n \|_P^2 \to 0 \quad \text{as} \quad n \to \infty .$$

$$\because \quad e_j = \frac{1}{2} \sqrt{\frac{j^2 + a^2}{(j-\frac{1}{2})(j+\frac{1}{2})}} ,$$

~~substituteing~~ substitute the value of e_j, we get for example

$$I_1^{(n)} = \sum_{j=0}^{n} |c_j^{(n)}|^2 \; j^2 \; \frac{j^2 + a^2}{(j-\frac{1}{2})(j+\frac{1}{2})} \; (j-\tfrac{1}{2})^{2P}$$

$$\leq \frac{1}{4} \sum_{j=0}^{n} c_j^{(n)\,2} \quad \overline{(j+\tfrac{1}{2})^2 + a^2} \quad (j+\tfrac{1}{2})^{2P}$$

$$= \frac{1}{4} \sum_{j=0}^{n} |c_j^{(n)}|^2 \, (j+\tfrac{1}{2})^{2(P+1)} + \frac{a^2}{4} \sum_{j=0}^{n} |c_j^{(n)}|^2 \, (j+\tfrac{1}{2})^{2P}$$

Since the sum (24) goes to zero as $n \to \infty$ for all P

$$\therefore \quad \lim_{n \to \infty} I_1^{(n)} = 0 \qquad \text{for all } P$$

Similarly:

$$\lim_{n\to\infty} I_2^{(n)} = 0$$

and

$$\lim_{n\to\infty} I_3^{(n)} = 0 .$$

Hence B_3 is a ~~completely~~ continuous operator, i.e. bounded operator.

From the commutation relations:

$$[B_3, L_\pm] = B_\pm$$

and the facts that the multiplication of two ~~completely~~ continuous operators is again a ~~completely~~ continuous operator; also the finite linear combination of ~~completely~~ continuous operators is again a ~~completely~~ continuous operator.

This shows that $B_{\pm}$ are ~~completely~~ continuous operators, hence bounded.

This completes the proof of the second part of our postulate, i.e. $L_{\pm}$, L_3, $B_{\pm}$, B_3 are bounded operators in the space of the rigged Hilbert space $\Phi \subset H \subset \Phi^*$.

(c) <u>Remark</u>:

It is mentioned in Chapter I that we could have chosen the group $E(3)$ as the dynamical group for the rigid rotator. In other words, we can define a one to one correspondence between the entire spectrum of the rigid rotator and an irreducible representation of the group $E(3)$.

$E(3)$ is obtained from the Lorentz group by the method of group contraction[18]. The contracted space is the desired irreducible representation of $E(3)$. To show that the above con-traction of the rigged Hilbert space is still valid, we have to show that the contraction process is performed in a continuous manner.

For:

The irreducible representation space of the homogeneous Lorentz group is the direct sum of irreducible representation spaces of the rotation group:

$$H(a) = \sum_{j=0}^{\infty} \oplus \; m_j$$

where m_j is the representation space of SO_3 subgroup.

Now introduce the parameter λ such that

$$\lambda B_i = P_i \qquad i = 1,2,3 .$$

The commutation relations (1) become:

$$\begin{aligned} [\bar{L}, \bar{L}] &= i \bar{L} \\ [\bar{L}, \bar{P}] &= i \bar{P} \\ [\bar{P}, \bar{P}] &= -i\lambda^2 \bar{L} \end{aligned} \tag{26}$$

The second Casimir operator has the form:

$$\lambda Q^2 = \not{P}^2 - \lambda^2 L^2 \tag{27}$$

As $\lambda \to 0$, the commutation relation (26) goes to the commutation relation

$$\begin{aligned} [\bar{L}, \bar{L}] &= i \bar{L} \\ [\bar{L}, \bar{P}] &= i \bar{P} \\ [\bar{P}, \bar{P}] &= 0 \end{aligned}$$

which is the Lie algebra of the E(3) group.

We choose a sequence of irreducible representations in such a way that $\lambda^2 \beta^2 \to \varepsilon^2$ as $\lambda \to 0$ ($\beta^2 = 1 + a^2$ the eigen-value of Q_2; and ε^2 is arbitrary).

$$\therefore \qquad H(a) \longrightarrow H(\varepsilon)$$

i.e. ε characterize the representations of E(3). In the general case when $j_o \neq 0$, we carry out the same process, but fix j_o first. It is clear that the irreducible representation spaces m_j of the rotation group contained in $H(a)$ (or in general $H(a, j_o)$) are also contained in the contracted representations.

$$\therefore \quad \lambda^2\beta^2 = \lambda^2(1 + a^2 - j_o) = \lambda^2(1 + a^2) \rightarrow \varepsilon^2 \quad \text{as} \quad \lambda \rightarrow 0$$

$$(j_o = 0 \text{ for the rotator})$$

means that $\lambda^2 a^2 \rightarrow \varepsilon^2$ as $\lambda \rightarrow 0$.

Then the representation space $H(\varepsilon)$ is again a direct sum of irreducible representation spaces of the rotation group.

$$\text{i.e.} \quad H(\varepsilon) = \sum_{j=0}^{\infty} \oplus \, m_j \qquad \left(\text{or in general } H(\varepsilon, j_o) = \sum_{j=j_o}^{\infty} \oplus \, m_j \right)$$

Therefore the contraction is considered as a mapping from the set of real numbers $\{a\}$ to the set of real numbers $\{\varepsilon\}$ and as such can be performed continuously. This proves the continuity of group contraction.

CHAPTER 2

B. CONSTRUCTION OF THE RIGGED HILBERT SPACE FOR THE HYDROGEN ATOM

(a) The Representation Space Ψ

We look for a representation of the group $SO_{4,1}$ in which

(i) the quartic operator $w = 0$

(ii) the chain

$$SO_4(\bar{A}, \bar{L}) \subset SU_3(\bar{L}) \subset SO_2(L_3) \qquad (1)$$

of subgroups appears in a completely reduced form.

The Lie algebra of this group is given in Chapter I(b) by equations (10). $SO_4(\bar{A}, \bar{L})$ is the maximal compact subgroup of $SO_{4,1}$, all unitary irreducible representations of this subgroup are finite dimensional. Let (k,m) and (k',m') be the quantum numbers of the subgroups $SO_3(\frac{1}{2}(\bar{A} + \bar{L}))$ and $SO_3(\frac{1}{2}(\bar{A} - \bar{L}))$ respectively. Then every state is represented by ket:

$$|k \quad k' \quad m \quad m'\rangle \qquad (2)$$

The set of states (2) spans a linear space, which we denote by $V_{kk'}$. The dimension of this representation is $(2k+1)(2k'+1)$ where k, k' may be any pair of the sequence $0, \frac{1}{2}, 1, \ldots, \infty$.

Following Newton[23], one then assumes the existence of an irreducible unitary representation of $SO_{4,1}$ by unitary operators in a space Ψ. The bases in Ψ are chosen such that the restricted ~~induced~~ representations of the subgroups (1) are in a completely reduced form, then:

$$\psi = \sum_{k,k'}^{\infty} \oplus V_{kk'} \,, \qquad (3)$$

The infinite sum follows from the fact that $SO_{4,1}$ contains the

Lorentz group as a subgroup which has unitary infinite dimensional space.

Since $w(=0)$, Q, K^2, K'^2, K_3, K_3' form a complete set of commuting operators, where $\bar{K} = \frac{1}{2}(\bar{L} + \bar{A})$, $\bar{K}' = \frac{1}{2}(\bar{L} - \bar{A})$, then every state is represented by the ket

$$|\alpha, k, k', m\ m'\rangle \tag{3}$$

where $\alpha = \alpha_1, \alpha_2, \ldots,$ is a set of parameters which characterize completely the representation. We shall write $|kk'\ mm'\rangle$ instead of $|\alpha; k\ k'\ m\ m'\rangle$, except when necessary to display the representations.

The generators of $SO_{4,1}$ have the following form, e.g.

$$L_3\ |k\ k'\ m\ m'\rangle = (m + m')\ |k\ k'\ m\ m'\rangle$$

$$\frac{1}{i}B_+\ |k\ k'\ m\ m'\rangle = \sqrt{(k+m+1)(k'+m'+1)}\ A_{kk'}\ |k+\tfrac{1}{2}, k'+\tfrac{1}{2}\ m+\tfrac{1}{2}, m'+\tfrac{1}{2}\rangle$$
$$+ \sqrt{(k-m)(k'-m')}\ A_{k-\frac{1}{2},k'-\frac{1}{2}}|k-\tfrac{1}{2}\ k'-\tfrac{1}{2}\ m+\tfrac{1}{2}\ m'+\tfrac{1}{2}\rangle$$

$$\frac{1}{i}(-S+iB_3) = -\sqrt{(k+m+1)(k'-m'+1)}\ A_{kk'}\ |k+\tfrac{1}{2}, k'+\tfrac{1}{2}, m+\tfrac{1}{2}, m'-\tfrac{1}{2}\rangle$$
$$\ldots\quad \ldots + \sqrt{(k-m)(k'+m')}\ \ldots\ \ldots\ A_{k-\frac{1}{2},k'-\frac{1}{2}}\ |k-\tfrac{1}{2}, k'-\tfrac{1}{2}, m+\tfrac{1}{2}, m'-\tfrac{1}{2}\rangle$$

where

$$K^2\ |k\ k'\ m\ m'\rangle = k(k+1)\ |k\ k'\ m\ m'\rangle$$
$$K'^2\ |k\ k'\ m\ m'\rangle = k'(k'+1)\ |k\ k'\ m\ m'\rangle$$

$$W\ |k\ k'\ m\ m'\rangle = 0$$

$$Q\ |k\ k'\ m\ m'\rangle = \beta^2\ |k\ k'\ m\ m'\rangle \tag{4}$$

where $A_{kk'}$ are numerical coefficients depending on k, k' only. It is convenient to label the basis $|k\ k'\ m\ m'\rangle$ by $|j_o\ c, j\ j_3\rangle$

where j_o, c are related to k and k' by the following relations

$$(-j_o + c)^2 - 1 = 4k'(k' + 1)$$

$$(j_o + c)^2 - 1 = 4k(k + 1)$$

$$j_o = \left[\text{sign}\,(k - k')\right](k - k')$$

$$c = \left[\text{sign}\,(k - k')\right](k + k' + 1)$$

and (j, j_3) are the quantum numbers of $SO_3(\bar{L})$, where $j_3 = m + m'$. We write simply

$$\left| j_o\ c,\ j\ j_3 \right\rangle = \left| k\ k'\ j\ j_3 \right\rangle .$$

To find the form of generators on this basis, we multiply both sides of equations (4) by

$$I = \sum_{kk'jj_3} \left| j\ j_3\ k\ k' \right\rangle \left\langle k\ k'\ j\ j_3 \right|$$

and note that

$$\left\langle K\ K'\ j\ j_3 \,|\, k\ k'\ m\ m' \right\rangle = \delta_{Kk}\,\delta_{K'k'} \left\langle j\ j_3 \,|\, k\ k'\ m\ m' \right\rangle$$

where $\langle j\ j_3 \,|\, k\ k'\ m\ m' \rangle$ are the Clebsch-Gordon Coefficients[12,23].
Now we introduce the following notations

$$k,\ k' \qquad n = k + k' + 1 \qquad j_o = |k - k'| .$$

For the hydrogen atom, we have $k = k'$, then:

$$n = 2k + 1 \quad \text{and} \quad j_o = 0 .$$

Using the properties of the Clebsch-Gordon coefficients, we arrive at the following forms of the generators.

$$L_3 \; |n \; j \; j_3\rangle = j_3 \; |n \; j \; j_3\rangle \qquad (5.1)$$

$$L_+ \; |n \; j \; j_3\rangle = \sqrt{(j+j_3+1)(j-j_3)} \; |n \; j \; j_3+1\rangle \qquad (5.2)$$

$$L_- \; |n \; j \; j_3\rangle = \sqrt{(j-j_3)(j+j_3)} \; |n \; j \; j_3-1\rangle \qquad (5.3)$$

$$A_3 \; |n \; j \; j_3\rangle = C_j \sqrt{(j-j_3)(j+j_3)} \; |n \; j-1 \; j_3\rangle - C_{j+1} \sqrt{(j-j_3+1)(j+j_3+1)} \; |n \; j+1 \; j_3\rangle \qquad (5.4)$$

where

$$C_j = \sqrt{\frac{j^2 + n^2}{4j^2 - 1}} = \sqrt{\frac{n^2 - j^2}{4j^2 - 1}}$$

$$\frac{2}{i} S \; |n \; j \; j_3\rangle = A_n \sqrt{(n+j+1)(n-j)} \; |n+1 \; j \; j_3\rangle + A_{n-1}\sqrt{(n+j)(n-j+1)} \; |n-1 \; j \; j_3\rangle \qquad (5.5)$$

where A_n is a numerical coefficient depending on n only. Remembering

$$[S, A_\pm] = i B_\pm \qquad (5.6)$$

$$[S, A_3] = i B_3 \qquad (5.7)$$

$$[A_3, L_\pm] = \pm A_\pm \qquad (5.8)$$

We can easily calculate the form of the other generators B_3, $B_\pm$, $A_\pm$. There are two classes of unitary irreducible representations:

Class 1: The continuous series which is characterized by the Casimir operator:

$$Q = \beta^2 \geqslant 0 \; ; \quad \beta \text{ is a real number,}$$

for which the irreducible representation space $\psi(\beta, w = 0)$ reduces into the representation spaces of the subgroup in the following way

$$\psi(\beta,0) = \sum_{n=1}^{\infty} V_{n,0}$$

where $V_{n,0}$ is the representation space of SO_4 with dimension n^2 $(n = 2k + 1)$. In this case:

$$A_n = \sqrt{\frac{n(n+1) + \beta^2 - 2}{n(n+1)}}$$

Class 2: The discrete series, which is characterized by:

$$Q = -(N - 1)(N + 2) \leqslant 0 \quad , \quad N = 1, 2, \ldots\ldots \ldots$$

The irreducible representation space $\psi(N, 0)$ reduces into the representation spaces of the subgroups in the following way

$$\psi(N, 0) = \sum_{n=N+1}^{\infty} V_{n,0} \tag{6}$$

where $V_{n,0}$ is the representation space of SO_4 with dimensions n^2.

We know for the Hydrogen atom, that

$$k = \frac{n-1}{2} \qquad \text{where} \quad E_n = -\frac{1}{2n^2} \; .$$

In order to include all the states of the Hydrogen atom spectra, we must consider the continuous series. The discrete series does not include the ground state of the Hydrogen atom (i.e. $n = 2,3,\ldots$). β^2 must be greater than one, because if $\beta^2 = 0$ we see that $A_1 = 0$, again the ground state is not included. Therefore one says: "The Hydrogen atom spectra (bound states) is identified with one irreducible unitary representation of $SO_{4,1}$ characterized by $W = 0$ and given by the continuous series."

(b) <u>Construction of the Rigged Hilbert Space $\Phi \subset H \subset \Phi^*$</u>:

The set of states (3) constitutes an orthonormal basis for the linear space $\Phi(\beta, 0)$

$$\text{i.e.} \; \langle j_3 \; j \; n \,|\, n' \; j' \; j_3' \rangle = \delta_{nn'} \; \delta_{jj'} \; \delta_{j_3 j_3'} \tag{7}$$

The scalar product (7) defines a norm. Completion of the linear space Φ with respect to this norm gives the Hilbert space $H(\beta,0)$. We introduce into $\Phi(\beta,0)$ the stronger topology given by the countable number of norms defined by the scalar products:

$$\langle n \; j \; j_3 \,|\, n' \; j' \; j_3 \rangle_P = \langle n \; j \; j_3 \,|\, (L^2 + A^2 + 1)^P \,|\, n' \; j' \; j_3 \rangle \tag{8}$$

for all P (integer).

Completion with respect to the topology given by (8) ~~is~~ gives the space Φ . In the following we shall show that Φ is a countable Hilbert space.

For, let $\Phi \ni \phi = \sum_{njj_3} C^n_{j\,j_3} |n\,j\,j_3\rangle$,

the norms are given by:

$$\|\phi\|_p^2 = \sum_{njj_3} |C^n_{jj_3}|^2 \; n^{2p} \tag{9}$$

where equations (7) and (8) are used.

From equation (9) it is clear that,

$$\cdots\cdots \; \|\phi\|_p \geqslant \|\phi\|_{p-1} \; \cdots\cdots \geqslant \|\phi\|_1 \geqslant \|\phi\|_0 \tag{10}$$

That is to say the canonical sequence of P is the chain of spaces

$$\Phi \subset \cdots\cdots \; \Phi_p \cdots\cdots \; \cdots\cdots \; \Phi_1 \subset \Phi_0 \equiv H \tag{10'}$$

where we have identified the space Φ_0 with H.

The continuity of the scalar products in H is assumed by (10). The compatibility of the norms follow from the continuity of the operator $L^2 + A^2$ (the unit operator is a completely continuous operator which maps the space Φ onto itself).

Moreover, Φ is a nuclear Hilbert space.

Proof

To show that the space Φ is nuclear, it is sufficient to show that there is a map T^{P}_{P-1} which maps the space Φ_P into a dense subset of the space Φ_{P-1}, and it is nuclear.

Since Φ is complete, then any $\phi \in \Phi$ is the limit in the norms of a sequence $\{\phi_k\}$ of elements in the space Φ as $k \to \infty$, where

$$\phi_k = \sum_{n=1}^{k} \sum_{jj_3} C^{n}_{jj_3} \psi_{njj_3}$$

and

$$\psi_{njj_3} = |n\, jj_3\rangle .$$

i.e. $\|\phi - \phi_k\|_P \to 0$ as $k \to \infty$ for all P, which we shall write

$$\phi = \sum_{njj_3} C^{n}_{jj_3} \psi_{njj_3} \tag{11}$$

Let $\phi^P \in \Phi_P$ mean the element $\phi \in \Phi$ considered as an element of the space Φ_P. Again ϕ^P is the limit in the norm (defined for fixed P) of a sequence of elements $\{\phi^P_k\}$ of the space Φ_P, which we shall write

$$\phi^P = \sum_{njj_3} C^{n}_{jj_3} \psi^{P}_{njj_3} \tag{12}$$

where $\{\psi^{P}_{njj_3}\}$ is an orthonormal basis in the space Φ_P.

Therefore for $\phi \in \Phi_P$ (i.e. have the form (12)) we have

$$\langle \psi^{P}_{njj_3}, \phi \rangle_P = \sum_{n'j'j'_3} C^{n'}_{j'j'_3} \langle \psi^{P}_{njj_3} \; \psi^{P}_{n'j'j'_3} \rangle_P$$

$$= C^{n}_{jj_3} \; n^{2P}$$

$$\therefore \; C^{n}_{jj_3} = \frac{1}{n^{2P}} \langle \psi^{P}_{njj_3}, \phi \rangle_P \tag{13}$$

Similar to equation (12), we write $\phi^{P-1} \in \bar{\Phi}_{p-1}$ in the form

$$\phi^{P-1} = \sum_{njj_3} C^{n}_{jj_3} \; \psi^{P-1}_{njj_3} \tag{14}$$

Writing

$$T^{P}_{P-1} \; \phi = \phi^{P-1} \quad ; \qquad \text{where } \phi \in \bar{\Phi}_P .$$

Using equations (13) and (14) we obtain

$$T^{P}_{P-1} \; \phi = \sum_{njj_3} \frac{1}{n^{2P}} \langle \psi^{P}_{njj_3}, \phi \rangle_P \; \psi^{P-1}_{njj_3} \tag{15}$$

Writing:

$$\lambda_n = \frac{1}{n^{2P}}$$

it is clear that $\lambda_n > 0$ and $\sum_n \lambda_n$ converges. Therefore T^{P}_{P-1} is a nuclear map, hence $\bar{\Phi}$ is a nuclear space. The elements of the adjoint space Φ^{*} are given by the linear functionals:

$$F(\phi) = \langle F, \phi \rangle$$

(which can be chosen such that $\|F\| = \|\phi\|$)

every element $F \in \Phi^*$ is the limit of the sequence F_ℓ as $\ell \to \infty$ with respect to the topology given in Φ^*, which we shall write:

$$F = \sum_{njj_3} f^n_{jj_3} \psi_{njj_3}$$

such that

$$(F, \phi) = \sum_{njj_3} \bar{f}^n_{jj_3} \; c^{(n)}_{jj_3} \qquad \text{exists .}$$

Let $\Phi \ni \phi = \sum_{njj_3} c^n_{jj_3} \psi_{njj_3}$

the sum

$$\|\phi\|_P^2 = \sum_{njj_3} |c^n_{jj_3}|^2 \; n^{2P} \tag{16}$$

<u>exists for all P.</u>

and for $H \ni h = \sum_{njj_3} h^n_{jj_3} \psi_{njj_3}$

only the sum

$$\|h\|^2 = \sum_{njj_3} |h^{(n)}_{jj_3}|^2 \qquad \text{exists ,}$$

which implies that

$$H \subset \Phi^*$$

This completes the construction of the rigged Hilbert space

$\Phi \subset H \subset \Phi^*$.

In the following we shall show that the algebra generated by L_3, $L_{\pm}$, B_3, $B_{\pm}$, A_3, $A_{\pm}$, S is a subalgebra of the algebra of bounded operators, defined in the space Φ .

Since every ~~completely~~ continuous operator is a bounded operator, we shall show that the generators are ~~completely~~ continuous operators.

For, let $\phi \in \Phi$, it is the limit in the norm of the sequence $\{\phi_k\}$ of elements of Φ where

$$\|\phi_k\|_p^2 = \sum_{n=1}^{k} \sum_{jj_3} |C^n_{jj_3}|^2 \; n^{2P} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

$$\text{where} \quad j = 0,1,\ldots\ldots, \frac{n-1}{2} \quad \text{and} \quad j_3 = -j,\ldots\ldots, +j \qquad (17)$$

Then:

$$\|L_3 \; \phi_k\|_P^2 = \sum_{n=1}^{k} \sum_{jj_3} |C^{(n)}_{jj_3}|^2 \; j_3^2 \, n^{2P}$$

$$\leqslant \sum_{n=1}^{k} \sum_{jj_3} |C^n_{jj_3}|^2 \; n^{2(P+1)}$$

$$= \|\phi_k\|^2_{P+1}$$

in view of (17) which vanish for all p, then

$$\lim_{k \rightarrow \infty} \|L_3 \phi_k\|_P^2 = 0 .$$

i.e. L_3 is a ~~completely~~ continuous operator, hence bounded.

Similarly

$$\|L_{\pm}\phi_k\|_p^2 \leqslant 4\sum_{n=1}^{k}\sum_{jj_3}|c^n_{jj_3}|^2\, n^{2(P+1)}$$

$$= 4\,\|\phi_k\|^2_{P+1}$$

$$\therefore \lim_{k\to\infty}\|L_{\pm}\phi_k\|^2_P = 0$$

i.e. $L_{\pm}$ are ~~completely~~ continuous operators, hence bounded.

To show that A_3 is a ~~completely~~ continuous operator, for simplicity we fix j_3 and without loss of generality we set $j_3 = 0$.

Denote $\psi_{nj,j_3=0}$ by ψ_{nj} and $c^n_{j,j_3=0}$ by c^n_j .

Instead of equation (17) we have:

$$\|\phi_k\|_p^2 = \sum_{n=1}^{k}\sum_{j=0}^{\frac{n-1}{2}}|c^n_j|^2\, n^{2P} \to 0 \quad \text{as} \quad k\to\infty$$

for all p (integer) (17')

$\|A_3\,\phi_k\|_p^2$ can be written in the following form:

$$\|A_3\,\phi_k\|_P^2 = I_1^{(k)} + I_2^{(k)} - 2I_3^{(k)}$$

where

$$I_1^{(k)} = \sum_{n,j}^{k,\frac{n-1}{2}} j^2\,|c_j|^2\,|c_j^{(n)}|^2\, n^{2P}$$

$$I_2^{(k)} = \sum_{n,j}^{kj} (j+1)^2 \; |c_{j+1}|^2 \; |c_j^{(n)}|^2 \; n^{2P}$$

$$I_3^{(k)} = \mathrm{Re} \sum_{nj}^{kj} j(j+1) \; \bar{c}_{j+1} \; c_j \; \bar{c}_{j+1}^{(n)} \; c_{j-1}^{(n)} \; n^{2P}$$

Using the value of c_j and in view of (17), it is a straightforward calculation, to see that

$$\lim_{k \to \infty} I_i^{(k)} = 0 \quad \text{for } i = 1,2,3$$

i.e. A_3 is a ~~completely~~ continuous operator, hence bounded. From the commutation relations

$$[A_3, L_\pm] = \pm A_\pm$$

It follows that the operators $A_\pm$ are ~~completely~~ continuous operators, hence bounded.

S is a ~~completely~~ continuous operator, then from the commutation relations:

$$[S, A_3] = i B_3$$

$$[S, A_\pm] = i B_\pm$$

It follows that B_3, $B_\pm$ are ~~completely~~ continuous operators, hence bounded. Therefore it remains to show that S is a ~~completely~~ continuous operator. Since S leaves j and j_3 unchanged, then without loss of generality we set $j = j_3 = 0$. Denote $\psi_{n,j=0,j_3=0}$ by ψ_n and $c^{(n)}_{j=0,j_3=0}$ by $c^{(n)}$. Then instead of (17) we have

$$\sum_{n=1}^{k} |c(n)|^2 \quad n^{2P} \longrightarrow 0 \quad \text{as} \quad k \rightarrow \infty \qquad (17)''$$

for all P (integer).

$$\therefore \quad S\psi_n = \tfrac{1}{2} A_n \quad n(n+1) \quad \psi_{n+1} + \tfrac{1}{2} A_{n-1} \quad n(n+1) \quad \psi_{n-1}$$

where

$$A_n = \sqrt{\frac{n(n+1) + \beta^2 - 2}{n(n+1)}}$$

$$\therefore \quad \| S\psi_k \|_P^2 = I_1^{(k)} + I_2^{(k)} + I_3^{(k)}$$

where

$$I_1^{(k)} = \frac{1}{4} \sum_{n=1}^{k} |A_{n-1}|^2 \; |c(n)|^2 \quad n(n+1) \; (n-1)^{2P}$$

$$\leq \frac{1}{4} \; \|\phi_k\|^2_{P+1} + \frac{\beta^2}{4} \; \|\phi_k\|_P^2$$

in view of (17" we have:

$$\lim_{k \to \infty} I_1^{(k)} = 0$$

Similarly

$$I_2^{(k)} = \frac{1}{4} \sum_{n=1}^{k} |A_n|^2 \quad |c(n)|^2 \quad n(n+1) \; (n+1)^{2P}$$

which can be written as a finite sum of terms of the form

$\|\phi_k\|_r$ where $r = 0,1,\ldots, p$

each term goes to zero as $k \mapsto \infty$, hence:

$$\lim_{k \to \infty} I_3^{(k)} = 0$$

Similarly $\lim_{k \to \infty} I_3^{(k)} = 0$ (Note that equation (17) is valid also for $p = 0$).

i.e. S is a ~~completely~~ continuous operator, hence bounded. Therefore the algebra $\mathcal{A}((L_3, L_\pm, A_3, A_\pm, B_3, B_\pm, S)) \subset L(\Phi))$ where Φ is the physical space of the rigged Hilbert space $\Phi \subset H \subset \Phi^*$.

Remark I:

It is well known that the Euclidean group E(4) is also adequate for the Hydrogen atom, as a dynamical group, i.e. one can define a one-to-one correspondence between one unitary irreducible representation of E(4) and the entire spectrum of the Hydrogen atom.

E(4) is obtained from $SO_{4,1}$ by contracting the latter with respect to its SO_4 subgroup, the contracted space is the desired representation space of E(4). In order that our construction for the rigged Hilbert space is still valid in our case, i.e. all the generators of E(4) are bounded operators in the representation space of E(4) (i.e. the contracted space), we must show the contraction process can be performed in a continuous manner.

For, let us introduce the parameter λ, the three vector $\bar{P}$ and the scalar P_4, say, such that

$$\lambda B_i = P_i \qquad i = 1,2,3\ldots$$

$$\lambda S = P_4$$

The commutation relations (10) (Chapter 1(b)) become:

$$[\bar{L}, \bar{L}] = i\bar{L} \qquad [\bar{L}, \bar{A}] = i\bar{A} \qquad [\bar{A}, \bar{A}] = i\bar{L}$$

$$[\bar{P}, \bar{A}] = i\bar{P}_4 \qquad [\bar{P}, \bar{L}] = i\bar{P} \qquad [P_4, \bar{A}] = i\bar{P} \qquad [P_4, \bar{L}] = 0$$

$$[\bar{P}, \bar{P}] = -i\lambda^2\bar{L} \qquad [P_4, \bar{P}] = +i\lambda^2\bar{A} \qquad (1)$$

The two Casimir operators have the form (for the Hydrogen atom)

$$W = 0$$

$$\lambda^2 Q = P_4^{\,2} + P^2 - \lambda^2(A^2 + L^2) \qquad (2)$$

We know that the contents of the irreducible unitary representation (continuous series) of $SO_{4,1}$ in terms of unitary representations of the subgroup SO_4 are the V_n, where $n = 1,2,\ldots,\infty$. Each representation V_n (of SO_4) occurs only once

$$\text{i.e.} \quad H(p) = \sum_{n=1}^{\infty} V_n \qquad (3)$$

when $\lambda \longrightarrow 0$.

Then the Lie algebra (1) goes to the Lie algebra of the group E(4) from (2)

(III) CONSTRUCTION OF THE RIGGED HILBERT SPACE OF THE N-DIMENSIONAL OSCILLATOR

The quantum mechanical treatment of the N-dimensional oscillator is well known, so details shall be omitted.

The physical observables, momenta, positions and energy are to be mapped onto the elements p_i, q_i, E respectively, $(1 \leq i \leq N)$ of an algebra $\mathfrak{A}$ in which:

$$p_i q_i - q_i p_i + i\delta_{ij} I = 0 \qquad (1)$$

$$E - \sum_{i=1}^{N} (p_i^2 + q_i^2) = 0 \qquad (2)$$

It is more convenient to introduce the new operators

$$a_j = \tfrac{1}{2}(p_j - iq_j) \qquad 1 \leq j \leq N$$

$$a_j^* = \tfrac{1}{2}(p_j + iq_j) \qquad 1 \leq j \leq N$$

in terms of which equations (1) and (2) read:

$$a_i a^*_j - a^*_j a_i = \delta_{ij} I = \qquad (1)'$$

$$E - \sum_{i=1}^{N} (a_i^* a_i + \tfrac{1}{2}) = 0 \qquad (2)'$$

The algebra, generated by a_i and a_i^* $(i = 1,2,\ldots, N)$ coincides with the algebra generated by a_i, a_i^*, E.

Thus one says that the physical structure of the N-dimensional oscillator is to be mapped on the mathematical structure of an algebra generated by a_i, a_i^* modulo, the relations (1)' and

and (2)'.

From equation (2)' one sees immediately that our system admits SU(n) (special unitary group) as a symmetry group[19]. In the following we shall show that $\mathcal{A} \subset L(\Phi)$ where Φ is the physical space and p_i, q_i are extendable to self-adjoint operators in the Hilbert space H of the rigged Hilbert space $\Phi \subset H \subset \Phi^*$.

Following the quantum mechanical description of the N-dimensional oscillator, the set of states

$$\left|\nu_1, \nu_2, \ldots, \nu_n\right\rangle = \sum_{i=1}^{N} \left[(\nu_i)!\right]^{-1/2} \sum_{i=1}^{N} (a_i^+)^{\nu_i} \left|0\right\rangle \qquad (1)$$

~~spans~~ is an orthonormal basis for a linear space, the linear envelope of which we call Ψ,

where $|0\rangle$ stands for $|\overbrace{00 \ldots\ldots\ldots 0}^{\text{N-terms}}\rangle$

and $\nu_i = 0, 1, 2, \ldots, \infty$; for $i = 1, 2, \ldots, N$.

The orthogonality of the states (1) is given by

$$\langle \nu_1, \ldots, \nu_N \mid \nu_1', \ldots, \nu_N' \rangle = \delta_{\nu_1 \nu_1'}\, \delta_{\nu_2 \nu_2'} \ldots \delta_{\nu_N \nu_N'} \qquad (2)$$

where

$$\delta_{\nu_i \nu_i'} = 1 \quad \text{for} \quad \nu_i = \nu_i'$$
$$= 0 \quad \text{for} \quad \nu_i \neq \nu_i'$$

Completion with respect to the norm given by the scalar product (2) gives the Hilbert space H.

We introduce into Ψ the stronger topology given by means of the countable set of norms defined by the scalar products:

$$\langle \nu_1,\ldots,\nu_N \quad \nu'_1,\ldots,\nu'_N \rangle_P = (\nu_1,\ldots,\nu_N \,|\, (n+1)^P \,|\, \nu'_1,\ldots,\nu'_N \rangle \tag{3}$$

for all integer P

where

$$n = n_1 + n_2 + \ldots + n_N$$

and

$$n_i = a_i^* \; a_i \quad .$$

The operator n has the following properties

$$n \; |\nu_1\ldots\nu_N\rangle = \nu \quad |\nu_1\ldots \nu_N\rangle$$

where $\nu = \nu_1 + \nu_2 + \ldots + \nu_N$;

$$a_i(n+1)^P a_i^* = (n+2)^P(\nu_i+1) = \sum_{s=0}^{P} \binom{P}{s} (\nu+1)^s(\nu_i+1) \tag{4.2}$$

$$a_i^*(n+1)^P a_i = n^P n_i = \sum_{s=0}^{P} (-1)^{P-s}\binom{P}{s} (\nu+1)^s \nu_i \tag{4.3}$$

If we complete the space Φ with respect to the topology given by (3), we get the linear topological space Φ .

Introduce the following notation:

$$\psi_{[\nu]} = \psi_{\nu_1\ldots,\nu_N} = |\nu_1,\ldots,\nu_N\rangle$$

and

$$C_{[\nu]} = C_{\nu_1,\ldots,\nu_N} \; ,$$

Then $\Phi \ni \phi = \sum_{[\nu]=0}^{\infty} C_{[\nu]} \, \psi_{[\nu]}$

The norms in Φ are given by:

$$\|\phi\|_P^2 = \sum_{[\nu]=0}^{\infty} |C_{[\nu]}|^2 (\nu+1)^{2P} \qquad (5)$$

for all integer P. Note that for $P = 0$, (5) defines the ordinary scalar product in the Hilbert space H.

From (5) it follows that:

$$\|\phi\|_P \geq \cdots\cdots \geq \|\phi\|_1 \geq \|\phi\|_0 \qquad (6.1)$$

i.e. the canonical sequence of P is the chain of spaces

$$\Phi \subset \cdots\cdots \Phi_P \subset \Phi_{P-1} \subset \cdots \subset \Phi_1 \subset \Phi_0 = H \qquad (6.2)$$

Φ_P denotes the Hilbert space obtained by completing Φ with respect to the norm given in (5) for fixed P. The compatibility of the norms (5) follows from the continuity of the operator n (see below), that is to say Φ is a countably ~~normed~~ Hilbert space. Φ is a dense subset in the space H.

The continuity of the scalar product in H follows from (6), hence the linear map T which maps Φ into a dense subset of H, is continuous.

Since the space Φ is complete, then every $\phi \in \Phi$ is the limit in the norm of a sequence $\{\phi_{(n)}\}$ of elements of the space Φ, where

$$\phi_{[k]} = \sum_{[\nu]=0}^{k} C_{[\nu]} \phi_{[\nu]}$$

i.e.

$$\|\phi - \phi_{[k]}\|_P \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty .$$

We shall write

$$\phi = \sum_{[\nu]} C_{[\nu]} \psi_{[\nu]} \tag{7}$$

Let ϕ^P denote the element $\phi \in \Phi$ as an element of the space Φ_P; Φ is a dense subset in the space Φ_P. Due to the completeness of Φ_P, ϕ^P can be written as the limit in the norm of a sequence of elements of Φ_P, then:

$$\phi^P = \sum_{[\nu]} C_{[\nu]} \psi^P_{[\nu]} \tag{8}$$

where $\{\psi^P_{[\nu]}\}$ is an orthonormal basis in the space Φ_P. Similarly

$$\phi^{P-1} = \sum_{[\nu]} C_{[\nu]} \psi^{P-1}_{[\nu]} \tag{9}$$

for $\phi \in \Phi_P$ we have :

$$\langle \psi^P_{[\nu]}, \phi \rangle_P = C_{[\nu]} (\nu+1)^P$$

$$\therefore C_{[\nu]} = \frac{1}{(\nu+1)^P} \langle \psi^P_{[\nu]}, \phi \rangle_P \tag{10}$$

Define the linear map. T^P_{P-1} such that:

$$T^P_{P-1} \phi = \phi^{P-1} \qquad \phi^{P-1} \in \Phi_{P-1} \text{ and } \phi \in \Phi_P .$$

Using equations (7) - (10) we obtain

$$T^P_{P-1} \phi = \sum_{[\nu]} \frac{1}{(\nu+1)^{2P}} \langle \psi^P_{[\nu]}, \phi \rangle_P \psi^{P-1}_{[\nu]} \tag{11}$$

If we define $\lambda_\nu = \frac{1}{(\nu+1)^p}$, then

$$\lambda_\nu > 0$$

and $\sum^{\infty} \lambda_\nu$ converges for $p \neq 1$, i.e. T^p_{p-1} is a nuclear map, hence Φ is a nuclear space.

We have shown that

$$\Phi \subset H \tag{12}$$

by which there is defined a continuous map T which maps into a dense subset in H.

Let Φ^* be the adjoint space of the space Φ, i.e. the elements of Φ^* are given by the linear functional $F(\phi) = (F, \phi)$ when $\phi \in \Phi$. Every $\Phi^* \to F = \sum_{(\nu)=0}^{\infty} F_{[\nu]} \psi_{[\nu]}$ such that

$$(F, \phi) = \sum \bar{f}_{[\nu]} C_{[\nu]} \text{ exists.} \tag{13}$$

(13) comes from the fact that every linear functional on Φ is bounded.

Consider T^* (the adjoint operator of T); ~~and~~ it is defined by:

$T h'(\phi) = h'(T\phi)$ (note that $T\phi = h \in H$) since every linear functional $h' \in H$ (on the space H) can be written in the form

$$h'(h) = (h, h_1) \tag{14}$$

where $h, h_1 \in H$.

In particular for all

$$H \ni h = \sum_{[\nu]} h_{[\nu]} \phi_{[\nu]}$$

the sum

$$\|h\|^2 = \sum_{[\nu]} |h_{[\nu]}|^2 \quad \text{exists.}$$

Thus T^* can be considered as a mapping of H into Φ^*, hence $H \subset \Phi^*$.

This completes the construction of the rigged Hilbert space:

$$\Phi \subset H \subset \Phi^* .$$

It remains to show that the algebra $\mathcal{A}(a_i, a_i^*) \subset L(\Phi)$, i.e. the algebra of bounded operators.

For this purpose we shall show in the following that a_i, a_i^* for all i are completely continuous operators, hence they are bounded operators (Theorem).

Let $\{\phi^{(k)}\}$ be a Cauchy sequence of elements in the space Φ, since it is complete, then its limit in the norm (defined in the space Φ) exists as an element in Φ.

Choose a sequence such that

$$\lim_{k \to \infty} \|\phi^{(k)}\|_p = 0 \quad \text{for all integer } p$$

i.e. $$\lim_{(k) \to \infty} \sum_{(\nu)}^{(k)} |c_\nu|^2 (\nu+1)^p = 0 \qquad (15)$$

for all integer p.

Then

$$\|a_i^{*} \phi^{(k)}\|_P^2 = \sum_{(\nu)}^{(k)} \langle \nu_1, \ldots, \nu_N | a_i (n+1)^P a_i^{*} | \nu_1, \ldots, \nu_N \rangle$$

$$= \sum_{s=0}^{P} \binom{P}{s} \binom{(k)}{(\nu)} \langle \nu_1, \ldots, \nu_N | (n+1)^s (n_i+1) | \nu_1, \ldots, \nu_N \rangle$$

$$\leq \sum_{s=0}^{P} \binom{P}{s} \|\phi^{(k)}\|_{s+1}^2 \tag{16}$$

and

$$\|a_i \phi^{(k)}\|_P^2 = \sum_{(\nu)}^{(k)} \langle \nu_1, \ldots, \nu_N | a_i^{*} (n+1)^P a_i | \nu_1, \ldots, \nu_N \rangle$$

$$\leq \sum_{s=0}^{P+1} (-1)^{P-s+1} \binom{P+1}{s} \|\phi^{(k)}\|_s^2 \tag{17}$$

Because (15) vanish for all P, i.e.

$$\lim_{k\to\infty} \|\phi^{(k)}\|_0 = \ldots \lim \|\phi^{(k)}\|_P$$

$$= \lim_{k\to\infty} \|\phi^{(k)}\|_{P+1} = \ldots\ldots = 0 .$$

Then each term in the finite series (16) (or(17)) vanishes identically, hence

a_i and a_i^{*} $(i = 1, \ldots\ )$ are bounded operators:

from this it follows that the operator $n + \frac{N}{2} = \sum_{i=1}^{N} \left(a_i \ a_i + \frac{I}{2} \right)$ is bounded. This completes our construction for the rigged Hilbert space $\phi \subset H \subset \phi^{*}$ in which the algebra $\mathcal{A}$ generated by a_i, a_i^{*} (for all $i = 1,2,\ldots N$) is a subalgebra of the algebra of bounded operators in Φ.

The choice of the basis vectors is arbitrary, but they were chosen in such a way that the basis vectors are eigenstates of the energy operator $E = \left(n + \frac{N}{2}\right)$.

Remark

In quantum field theory, we deal with systems which have infinite (countable) degrees of freedom. The indices in (1) take the values from one to ∞. The construction of the Hilbert space for this algebra (i.e. the representation space is not unique) (eqn. (1) with $i = 1,2,\ldots,\infty$) admits various realizations which are inequivalent in contrast to the finite dimensional case, where all realizations of such algebra are unitarily equivalent.

It is of some value to extend the above construction of the rigged Hilbert space to include this more interesting case. It might be possible to construct the rigid Hilbert space $\Phi \subset H \subset \Phi^*$ such that, with various realizations of the Hilbert space H (as the representation space of this infinite dimensional algebra), one still gets the same spaces Φ and Φ^*.

SECTION II

Introduction

(a) The aspect of symmetry in Physics

The most powerful tool for studying symmetry properties of a physical system is given by the methods of group theory. It is clear that all symmetry properties of a given system can be described as groups of transformations which carry the system into itself, in other words, as transformations which leave the system invariant. This aspect of symmetry manifests itself in different ways[15]; we mention, for example (but not all of them):

(i) The solutions of the equations of motions of a physical system will be classified by their symmetry character. The solutions of Lorentz covariant field equations must span representations of the Lorentz group for example, all elementary particles can be classified by stating what particular representations the associated fields belong to (i.e. scalar, vector, etc.).

(ii) The invariance of the dynamical laws, under some symmetry group, give rise to conservation laws, from which certain group elements constants of motion can be constructed. The existence of such constants of motion then lead us to selection rules[19], accordingly certain processes become forbidden, e.g. the invariance under spatial reflections has the consequence that the operator of reflection commutes with the Hamiltonian, so that its eigenvalues, i.e. parity of the states is a conserved quantity. Processes connecting states with different parity are forbidden.

(b) <u>The law of transformation of symmetry operators in Quantum Theory</u>

In quantum theory physical states of a physical system are represented by rays in a complex Hilbert space H. If $\phi \in H$ is a vector corresponding to a physically realizable state, then ϕ and $a\phi$ (where a is a phase factor) represent one and the same state. It is useful to normalize the state vectors, i.e. to choose a vector from the ray of ϕ which is normalized,

$$\text{i.e.} \quad (\phi, \phi) = 1 \tag{1}$$

Even then, a phase factor (i.e. a factor of modulus one) remains free in ϕ. The invariance transformations are thus transformations in the complex Hilbert space H, which map rays into rays (i.e. take us from one possible state to another) such that the law of probability is conserved.

$$\text{i.e.} \quad |(\phi, \phi)|^2 = |(T\phi, T\psi)|^2 \quad \text{or} \quad |(\phi, \psi)| = |(T\phi , T\psi)| \tag{2}$$

where $\phi, \psi, \in H$ and T is the invariance transformation. Equation (2) remains valid if ϕ, ψ, $T\phi$ and $T\psi$ are multiplied by arbitrary (i.e. not necessarily equal) phase factors. As a consequence of equation (2)

(i) every symmetry transformation[20] can be replaced by unitary or antiunitarity transformations (i.e. $(T\phi, T\psi) = (\phi, \psi)$ if T is a unitary transformation and $(T\phi, T\psi) = (\psi, \phi)$ if T is an antiunitary transformation).

(ii) The symmetry transformations are determined up to a phase factor.

Now we consider a particular group of transformations (a set of transformations do not necessarily form a group; for example, the discrete transformations P, T, C form a group only when they are subject to a chosen normalization) $G = \{g_i\}$; to every transformation g_i we associate a unitary or antiunitary operator $D(g_i)$ (see (i)), in the Hilbert space of the physical system onto which we apply the symmetry group. Each $D(g_i)$ is determined up to a phase factor so that if

$$g_i\, g_j = g_k \qquad g_i, g_j, g_k \in G$$

then the corresponding operators will be:

$$D(g_i)\, D(g_j) = \omega(g_i, g_j)\, D(g_k) \tag{3}$$

if $D(g_i)$ is unitary

or

$$D(g_i)\, D^*(g_j) = \omega(g_i, g_j)\, D(g_k) \tag{4}$$

if $D(g_i)$ is antiunitary; all g_i, g_j, g_k G.

Equation (3) (or equation (3) and equation (4)) does (do) not define a true representation, but a representation up to a factor, which is usually called a quantum mechanical representation of ray (coray) representation. Therefore in quantum theory we speak about ray (coray) representations of the symmetry groups (see Chapter 1).

(c) Internal symmetry groups for elementary particles

In elementary particle physics one lacks a satisfactory dynamical theory, therefore it has become very important to exploit the symmetry properties which have been found empirically. An initial step was made by Wigner in his detailed study of irreducible representations of the inhomogeneous Lorentz group(10). The irreducible representations of this group are characterized by two numbers (m, s) conventionally identified with the generators corresponding to the mass and spin of the elementary system constituting the representation. This is the strictly kinematical group.

In the following we shall make no distinction as to whether the internal symmetry group originates from the kinematics or the dynamics of the physical system. The concept of isospin(9) (which is one of the foundation stones of modern physics) was introduced to express charge independence of nuclear forces (we remark that charge independence demands invariance under all rotations in the 3-dimensional iso-space).

With the number of different elementary particles rapidly growing, and with more available experimental information, new selection rules were found, reflecting the possible existence of additional symmetries given by larger groups, for example SU(3) and SU(6).

The ambitions of the group theoretical approach had grown considerably in the last few years in the course of extended studies of more complicated groups (i.e. larger than Poincaré group and SU(2) or SU(3) or SU(6)) to incorporate all internal symmetries found empirically with the space time group (Poincaré group). All attempts have this in common, namely that the internal symmetry properties are taken into account by introducing internal degrees of freedom which are independent of space and time; this means that the internal symmetry group and the Poincaré group appear as a direct product. Therefore a finite transformation belonging to SU(3), or SU(6), or even SU(2), transforms, in general, a permissible state vector into a linear combination of state vectors which belong to different Hilbert spaces (i.e. different irreducible representation spaces of the Poincaré group) and describe different physical systems which cannot be connected by any observable. In spite of using Lie groups of orders reaching several hundreds, it was not possible to satisfy all the imposed requirements. It has been shown recently that a symmetry scheme described by a Lie group which is of finite ~~order~~ dimension and which contains the Poincaré group as a subgroup,(13,14) is incompatible with the existence of a non-trivial mass formula, i.e. obe obtains a discrete mass formula by breaking such a symmetry scheme, and this is unsatisfactory. (It was not shown that the continuous case is actually realizable). Therefore a closer examination of symmetries is required. We shall limit ourselves to the basic question as to which kind of groups should be used in particular symmetry cases.

We start by making the following remarks:

(i) Given a number of particles (or elementary systems) we must distinguish between:

I: The symmetry in the grouping (multiplet structure) of a given set of particles and

II: The symmetry in the scattering of these particles, i.e. the symmetry of the S-matrix (whenever it is properly defined) describing the interaction of these particles.

In general, symmetry of type I is different from the symmetry of type II. It may happen that the two types of symmetry are the same; for example, in the SU(2) (isospin)-symmetry of the strong interactions in the absence of electromagnetic and weak interactions, the strongly interacting particles can be grouped into the singlet, doublet, triplet..... multiplets of SU(2) and at the same time, the strong interaction part of the S-matrix is invariant under $SU_2(1)$, but perhaps this is a special situation, for why is it that symmetries higher than the direct product of isospin and hypercharge have had some measure of success when dealing with masses and coupling constants, while the description they give of scattering amplitudes is much less transparent?[14].

(ii) It seems quite accidental that continuous parameter symmetry groups (i.e. Lie groups) have been preferred so far.

Physically realizable transformations belonging to the internal symmetry groups are discrete, therefore it is quite natural to consider finite symmetry groups rather than continuous symmetry groups when discussing internal symmetries. To make this point clear, we shall consider, only as an example, the isotopic spin symmetry group SU(2) and ask whether it is necessary to assume invariance under all rotations in the isospin space in charge independent theories? The answer is no! It was shown(5) that the consequences of the charge independent theory can be obtained from the requirements of permutation symmetry between two objects, and the law of charge conservation. This approach was taken by several people(5,12,22). We summarize their procedure as follows:

If the multiplets of a set of particles (or states of a set of elementary systems) are assigned to an irreducible representation of a finite group, under which the theory is invariant, then the invariance requirement for this group does not, in general, guarantee the conservation law of electric charge. In this case it is necessary, in addition, to postulate the law of charge conservation, thereby extending the original (finite) symmetry group to a higher symmetry with a corresponding enlargement of the irreducible representations.

(iii) It is very interesting to remark that finite groups contain only a limited number of operations, so that invariance requirements under certain finite groups are less restrictive than that under Lie groups.

From the above discussion, it appears that the use of finite groups as internal symmetry groups in particle physics, should be (at least) considered seriously. The use of finite groups retaining the advantage (in hope) of avoiding some of the difficulties of combining an internal symmetry group with the strict Poincare symmetry group, as well as obtaining interesting quantum numbers like baryon number etc., in a very natural manner. From the requirement of (b), we look for quantum representations of these symmetry groups.

In Chapter 1 we describe in some detail the ray representations of commutative finite groups. It is shown that the ray representations of these (simple) finite groups lead to such a richness of different classes (see appendix) and each class to different types of representations. Also one gets higher dimensional irreducible representations, that is, multiplet structure of states, much the same way, that one used to obtain from the representations of compact symmetry groups.

In Chapter II we obtain the quantum mechanical representations of n generalized parity operators. By a generalized parity operator we mean an operator whose square is the identity operator. The full inhomogeneous Lorentz group has two abstractly defined parity operators, a unitary parity operator corresponding to space reflections and the anti-unitary parity operator corresponding to time reflections. The product of two unitary parity[17] operators has been considered in the literature. The product of three parity operators was used[2] to classify leptons.

CHAPTER 1

INTRODUCTION TO QUANTUM MECHANICAL REPRESENTATIONS OF FINITE GROUPS

I. Mathematical Introduction

Definition (1)

A ray representation belonging to the factor system $w(g_i, g_j)$ of a group $G = \{g\}$ is a set of $\Gamma = \{D(g)\}$ of square non-singular, n-dimensional matrices $D(g)$ together with the correspondence

$$g \longrightarrow D(g) \qquad (1)$$

such that

$$D(g_i)\, D(g_j) = w(g_i, g_j)\; D(g_i\, g_j) \qquad (2)$$

for all $g_i\, g_j \in G$ and $|w(g_i, g_j)| = 1.$

Definition (2)

A ray representation $\Gamma = \{D(g)\}$ of the group $G = \{g\}$ is called a faithful representation if the correspondence given by (1) is one to one.

Definition (3)

Two ray representations $\{D^{(1)}(g)\}$ and $\{D^{(2)}(g)\}$ of the group $G = \{g\}$ are called ray equivalent if there exists a real function $\alpha(g)$ in G and a square non-singular matrix S such that:

$$D^{(2)}(g) = e^{i\alpha(g)}\; S\; D^{(1)}(g)\; S^{-1} \qquad (3)$$

Definition (4)

Two factor systems of a group G are equivalent if they belong to two equivalent ray representations, i.e. $S = 1$ (the unit matrix) then (3) has the form

$$D^{(2)}(g) = e^{i\alpha(g)} D^{(1)}(g) ,$$

where $\alpha(g)$ is a real function in G.
We denote $e^{i\alpha(g)} = w(g)$.

Then the two factor systems w and w' are related according to the equation:

$$w'(g_i, g_j) = \frac{w(g_i)\, w(g_j)}{w(g_i\, g_j)} w(g_i, g_j) \text{ for all } g_i, g_j \in G.$$

Definition (5)

Ray representations belonging to non-equivalent factor systems are said to be of different types.

It should be remarked that true representations (vector representation) of the group belong~~ing~~ only to one type, i.e. ~~mainly~~ the type for which $w(g_i, g_j) = 1$ for all g_i, g_j G.

If the factor systems of a group are classified according to ray equivalence, it can be shown that there is only a finite number of such classes. The class of factor systems form a finite abelian group. This group is called the multiplicator group(8).

Definition (6) #

A ray representation $D(g)$ of a group $G = \{g\}$ is called unitary if:

$$D^{-1}(g) = D^{+}(g) \quad \text{for all } g \in G.$$

In this Chapter we use + = hermitean conjugate, * = complex conjugate in some bas

As in the case of vector representations for finite groups (i.e. a representation defined by $g \rightarrow D(g)$ and $D(g_i)\, D(g_j) = D(g_i\, g_j)$) one easily proves the following propositions.[16]

<u>Proposition (1)</u>

The ray representations for a finite group can always be chosen to be unitary.

<u>Proposition (2)</u>

If $D^{(1)}(g)$ and $D^{(2)}(g)$ are two irreducible ray representations of a group G belonging to the same factor system with different dimensions, then if there exists a matrix such that

$$D^{(1)}(g)\, S = S\, D^{(2)}(g) \quad \text{for all} \quad g \in G \quad \text{then} \quad S = 0 \,.$$

<u>Proposition (3)</u>

If $D(g)$ be an irreducible representation of a group $G = \{g\}$ and if there exists a matrix S such that

$$D(g)\, S = S\, D(g) \quad \text{for all} \quad g \in G \quad \text{then} \quad S = \lambda I$$

where λ is constant and I is the unit matrix having the same dimensions of $D(g)$.

<u>Proposition (4)</u>

If G has the order g and its multiplicator B has order b, then the dimension of every irreducible ray representation of G is a divisor of bg, also the number of irreducible ray representations is bg (for proof see[10]).

<u>Definition (7)</u> Let G be a group, G' is another group which contains H' in its centre such that $G'/H' = G$. Then G' is called a covering group of G extended by the abelian group H'. The covering group G' of G is called a representative group (or the quantum mechanical representative group where H' is the multiplicator group of G).

<u>Proposition (5)</u>

Let $G = \{g\}$ be a group, $G' = (g'$ a covering group of G extended by the abelian group H' ; $D'(g') = \Gamma'$ is an irreducible representation of G', S' the distinct cosets of H'; $g(s')$ any subset of G' consisting of one element $g(s')$ from each coset S' of H' ; and $g \rightarrow g'(s')$ the isomorphism between G'/H' and G. Then $g \rightarrow D'(g')$ is an irreducible ray representation of G(10).

From the above theorem, it follows that one can find the ray representation of a group G by finding the vector representations of its covering group. However we shall avoid the construction of a covering group (i.e. the extension of G by the multiplicator group) by considering a much simpler group with different factor systems (see II).

When the symmetric group G, under consideration contains some elements with the time inversion operator with other operators (geometrical operators, e.g. space reflection operator or not, e.g. charge conjugation), the structure of the group becomes non-unitary, made up of unitary and anti-unitary operators. We shall call such groups <u>non-unitary groups</u>, their ray representations are called Coray representations.

It is convenient to express every anti-unitary operator as $A = UK$ where U is a unitary operator, and K is the complex conjugation operator. It is easily demonstrated that the product of two unitary or two anti-unitary ones is unitary, and the product of an anti-unitary operator and a unitary one is an anti-unitary operator, from which it follows that the non-unitary group contains an equal number of unitary and anti-unitary operators. The unitary operators form an invariant subgroup of index two; if we call this group N, then G is given in terms of N as:

$$G = N + KN$$

or, in other words, if G is generated by n generators, then $g_1, \ldots, g_{n-1}$ can be chosen to be unitary operators and g_n is/an anti-unitary operator.

<u>Definition</u>

A Coray representation belonging to a factor system $w(g_i, g_j)$ of a group $G = \{g\}$ is a set of matrices $D(g_i)$ such that

$$D(U_i)\, D(U_j) = w(U_i, U_j)\, D(U_i, U_j)$$
$$D(U_i)\, D(a_j) = w(U_i, a_j)\, D(U_i a_j)$$
$$D(a_i)\, D^*(u_j) = w(u_i, u_j)\, D(a_i, u_j)$$
$$D(a_i)\, D^*(a_j) = w(a_i, a_j)\, D(a_i, a_j)$$

The associative law restricts these factors to the following conditions

$$w(u, g)\, w(ug, g') = w(u, gg')\, w(g, g')$$

$$w(a, g)\, w^{*}(ag, g') = w(a, gg')\, w^{*}(g, g')$$

where g, u, a correspond to any general element, the unitary, and the anti-unitary element, respectively. Other restrictions depend on the choice of the normalizations of the unitary and anti-unitary operators respectively (see Chapter II(b)).

It is shown that representations (vector or true representations) of non-unitary groups are determined by the vector representations of its unitary subgroup[7,21]. In an analogous way Coray representations of non-unitary groups are obtained from the ray representations of its unitary subgroup. Similar definitions in this case are exactly the same as definitions (1) - (5) for any representation. The following propositions can easily be proved in exactly the same way as for unitary groups. Definition (6) is modified, for a Coray representation is unitary; $D^{-1}(g) = D^{+}(g)$ when g is unitary, and $D^{-1}(g) = D^{+}(g)$ with $D(g) = D^{*}(g)$ when g is anti-unitary, and all g G.

<u>Proposition</u>: Any Coray representation by non-singular matrices is equivalent to one by unitary matrices[11].

<u>Proposition</u>: If $\{D^{(1)}(g)\}$ and $\{D^{(2)}(g)\}$ are two Coray representations of a group G belonging to the same factor system with different dimensions, then if there exists a matrix S such that

$$D^{(1)}(u)\, S = S\, D^{(2)}(u)$$

$$D^{(1)}(a)\, S^{*} = S\, D^{(2)}(a)$$

where g, u, a correspond to any general element, the unitary and the anti-unitary element, respectively, then $S = 0$.

<u>Proposition:</u> If $D(g)$ be an irreducible Coray representation of a group $G = g$ and if there exist S such that

$$D(u)\, S = S\, D(u)\,, \quad u \text{ corresponds to any unitary element in } G,$$

$$D(a)\, S^{*} = S\, D(a)\,, \quad a \text{ corresponds to any anti-unitary element in } G,$$

for all $a, u \in G$ then $S = \lambda I$, where λ is a constant and I is the unit matrix having the same dimension as $D(g)$.

In Chapter II(b) we determine exactly in the same way as for unitary groups the corepresentations for non-unitary groups.

In this section, we are concerned only with finite groups. The unitarity or anti-unitarity of such groups is to be understood in the sense that the elements of the group correspond to unitary or antiunitary operators.

II. Ray Representations of Commutative Groups

Any finite group G can be generated by a (small) number of generating elements γ_i i.e. the elements $g \in G$ are obtained from all possible products of γ_i with a number of polynomial conditions of the form

$$f_\mu(\gamma_i) = \alpha_\mu \qquad \mu = 1, 2, \ldots, N, \tag{1}$$

depending on the group structure where f_μ are polynomials in γ_i of degree m_μ and α_μ are arbitrary numbers.

(e.g. The symmetric group S_n is generated by (i) $\gamma_1 = (12)$ and $\gamma_2 = (123\ldots n)$ where $\gamma_1{}^2 = 1$, $\gamma_2^n = 1, \ldots$ and $\gamma_1 \gamma_2 \neq \gamma_2 \gamma_1$ or (ii) by (n-1) partly commuting elements, namely, the (n-1) transpositions; $\gamma_1 = (12)$, $\gamma_2 = (123) \ldots \gamma_{n-1} = (n-1, n)$ where $\gamma_i{}^2 = 1$, $i = 1, 2, \ldots, n$ $(\gamma_i \gamma_{i+1})^3 = 1, \ldots$).

In the following we shall consider only groups whose generators commute, i.e. commutative groups only.

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the generators of a group G (of order m n) such that

$$\gamma_i \gamma_j = \gamma_j \gamma_i \tag{2}$$

In this case we could assume that there are no other numerical coefficients in f_μ given by (1) except α_μ.

The ray representations are defined by

$$D(\gamma_i)\, D(\gamma_j) = w_{ij}\, D(\gamma_i \gamma_j) \tag{3}$$

Interchanging i and j, using equation (2), we find that:

$$D(\gamma_i)\, D(\gamma_j) \;=\; C_{ij}\, D(\gamma_j)\, D(\gamma_i) \qquad (4)$$

where $C_{ij} = \dfrac{w_{ij}}{w_{ji}}$.

The restrictions (1) become

$$f_\mu\,(D\,(\gamma_i)) \;=\; \beta_\mu \qquad\qquad \mu = 1,2,\ldots,N \qquad (5)$$

where β_μ are arbitrary numbers, in general $\beta_\mu \neq \alpha_\mu$.

Replace $D(\gamma_i)$ by Γ_i and assume that this representation is unitary (see proposition (1)), then the above equations have the more compact form

$$\Gamma_i\,\Gamma_j = C_{ij}\;\Gamma_j\,\Gamma_i \qquad (3)'$$

$$f_\mu\,(\Gamma_i) \;=\; 1 \qquad (5)'$$

where we set $\beta_\mu = 1$ in (5), which is a consequence of the fact that two ray representations Γ_i and $\alpha_i\,\Gamma_i$ (α_i = 1) lead to equivalent factor systems. We shall refer to the phase factors C_{ij} as the "commutation factors". These factors C_{ij} cannot be absorbed into the Γ's because the multiplication of Γ_i by any factor a occurs on both sides of equation (3)' and hence the factor a cancels. But the commutation factors C_{ij} are restricted to a number of conditions similar to (5)'. In fact (3)' and (5)' determine all factors C_{ij}. Equations (3)' generate an algebra $\mathcal{A}$ with the basis elements

$$I,\;\; \Gamma_i\;,\;\; \Gamma_i\,\Gamma_j \quad (i \le j),\ldots\;;\;\; \Gamma_i\,\Gamma_j \ldots\ldots\; \Gamma_k \quad (i \le j \ldots \le k)\;\ldots \qquad (6)$$

which is just the basis of the representation of the group (Frobenius) algebra of the group G.

Note that the equality signs in $i \leq j$ etc., give us also all the terms of the form $\Gamma_1^{n_1}\Gamma_2^{n_2}\Gamma_3^{n_3}\ldots\ldots\ldots$.

Because of conditions (5)' the series (6) breaks after some power. We note that if we do not have (5)', the series (6) is infinite; in this case we have an infinite dimensional algebra. In our case the dimension of the algebra is equal to the order of the group G.

<u>Theorem:</u> The unitary ray representations of the group (3)' satisfy the generalized algebra $\mathcal{A}$ generated by (3)' and (5)' with the basis (6).

<u>Proof:</u> Since $\gamma_1, \ldots, \gamma_n$ generate the group G, then every element $g \in G$ is expressed in terms of the products of γ_i, for any $g_i, g_j \in G$ we have;

$$\therefore \quad D(g_i)\,D(g_j) = D(\gamma_{i_1}, \gamma_{i_2} \ldots \gamma_{i_N})\,D(\gamma_{j_1}\,\gamma_{j_2}, \ldots, \gamma_{j_M})$$

Using equation (4), then we arrive at the following equation

$$D(g_i)\,D(g_j) = \left(\prod_{\ell=1}^{M}\prod_{k=1}^{N} C_{i_k j_\ell}\right) D(g_j)\,D(g_i) \qquad (7)$$

Consider the corresponding elements in the algebra, i.e.

$\left[D(\gamma_{i_1})\ldots D(\gamma_{i_N})\right]$ and $\left[D(\gamma_{j_1})\,D(\gamma_{j_2})\ldots D(\gamma_{j_M})\right]$ (Note that $D(\gamma_j) = \Gamma_j$ and $D(\gamma_i) = \Gamma_i$). Using equation (4) (or (3)') we find that

$$\left(\Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_N}\right)\left(\Gamma_{j_1}\Gamma_{j_2}\cdots\Gamma_{j_M}\right) = \left(\prod_{\ell=1}^{M}\prod_{k=1}^{N} C_{i_k j_\ell}\right) \times \left(\Gamma_{j_1}\cdots\Gamma_{j_M}\right)\left(\Gamma_{i_1}\cdots\Gamma_{i_N}\right) \qquad (8)$$

From (7) and (8) we see that the commutation relations of the arbitrary elements of the group G are exactly the same as the commutation relations of two arbitrary (but corresponding) elements of the algebra (6). Hence the result.

Therefore, we can determine first the representation of the algebra and then pass to the representation of the group. Care should be taken, since the representations of the commutation relations (3)' with the restrictions (5)' differ from the ray representations of the group (see Chapter II(a)).

Our task now is to find irreducible representations of the algebra $\mathcal{A}$. For this purpose we interpret equation (3)' as the group composition law of a finite group G' consisting of the direct product of the set (6), the basis elements of $\mathcal{A}$, and the abelian group B generated by the coefficients C_{ij}. G' and B are finite groups; if the order of G is g (which is the dimension of $\mathcal{A}$) and b is the order of B then gb is the order of the group G'. Of course $G'/B = G$.

Note that G' is different for different possible C_{ij}). The representations of G (vector representations) are also representations of G', although they do not satisfy equation (3)'. Then the problem is reduced to the one of finding the additional representations of G' which satisfy (3)'.

Let k be the number of the desired additional representations of G' (which are the ray representations of G), and let $\ell_1, \ell_2, \ldots, \ell_k$ be their dimensions. Because the order of the group is equal to the sum of squares of the dimensions of inequivalent irreducible representations we have

$$gb = g + \ell_1^2 + \ell_2^2 + \dots + \ell_k^2$$

i.e.

$$\ell_1^2 + \ell_2^2 + \dots + \ell_k^2 = g(b-1) \qquad (8)$$

(Note that G is a commutative group, each element forms a separate class).

It remains to determine the integer k, then the solution of (8) is unique.

We make use of the fact that the total number of irreducible representations of G is equal to the number of conjugate classes. For a given C_{ij} we first determine the number of elements among the basis set (ℓ) of the algebra $\mathcal{A}$ which commute with all other basis elements. Let this number be s ($s \geq 1$, because the identity commutes with all other elements), then

$$k = s(b-1) \qquad (9)$$

(Note that every element of the group B form a separate class; the number of conjugate classes equal to $g + k = g + s(b-1)$) where

$$1 \leq s \leq g \qquad (10)$$

(The case $s = g$ arises when B is an invariant subgroup).

In the following we shall show the correspondence between the ray representations of finite groups and the fundamental representations of some Lie algebras.

The most general form of a commutative group is the direct product of n cyclic groups with orders m_i; $i = 1,2,\dots,n$;

In this case we have instead of (3)' and (5)', the following equations

$$\Gamma_i \ \Gamma_j = C_{ij} \ \Gamma_j \ \Gamma_i \tag{11}$$

and

$$\Gamma_i^{m_i} = 1 \qquad i = 1, \ldots, n \tag{12}$$

Multiplying both sides of equation (11) by $\Gamma_i^{m_i - 1}$ and $\Gamma_j^{m_j - 1}$ respectively, using equations (11) and (12), we immediately find that C_{ij} are subject to the following conditions

$$(C_{ij})^{m_i} = (C_{ij})^{m_j} = (C_{ij})^{d_{ij}} = 1 \tag{13}$$

where d_{ij} is the greatest common divisor of m_i and m_j. When all m_i are prime numbers, then $d_{ij} = 1$ hence $C_{ij} = 1$ for all $i, j, = 1, \ldots, n$. In this case all representations are equivalent to vector representations, and all irreducible representations are one dimensional. In general $d_{ij} \neq 1$.

Let us form the following commutators

$$\left[\Gamma_i^{\alpha_i}, \Gamma_j^{\alpha_j}\right] = \left(C_{ij}^{\alpha_i \alpha_j} - 1\right) \Gamma_j^{\alpha_j} \Gamma_i^{\alpha_i} \tag{14}$$

Set $L_{ij}^{\alpha_i \alpha_j} = \left(C_{ij}^{\alpha_i \alpha_j} - 1\right) \Gamma_j^{\alpha_j} \Gamma_i^{\alpha_i}$ and form the following commutators

$$\left[L_{ij}^{\alpha_i \alpha_j}, \Gamma_k^{\alpha_k}\right] = \left(C_{ik}^{\alpha_i \alpha_k} C_{jk}^{\alpha_j \alpha_k} - 1\right) \Gamma_k^{\alpha_k} \ L_{ij}^{\alpha_i \alpha_j} \tag{15}$$

and

$$\left[L_{ij}^{\alpha_i\alpha_j},\ L_{kl}^{\alpha_k\alpha_l}\right] = \left(C_{il}^{\alpha_i\alpha_l}\, C_{ik}^{\alpha_i\alpha_k}\, C_{jl}^{\alpha_j\alpha_l}\, C_{jk}^{\alpha_j\alpha_k} - 1\right) L_{kl}^{\alpha_k\alpha_l}\ L_{ij}^{\alpha_i\alpha_j} \qquad (16)$$

If these relations are not closed, we proceed, and set

$$L_{ijk}^{\alpha_i\alpha_j\alpha_k} = \left(C_{ik}^{\alpha_i\alpha_k}\, C_{jk}^{\alpha_j\alpha_k} - 1\right) \Gamma_k^{\alpha_k}\ L_{ij}^{\alpha_i\alpha_j}$$

and $$L_{ijkl}^{\alpha_i\alpha_j\alpha_k\alpha_l} = \left(C_{il}^{\alpha_i\alpha_l}\, C_{ik}^{\alpha_i\alpha_k}\, C_{jl}^{\alpha_j\alpha_l}\, C_{jk}^{\alpha_j\alpha_k} - 1\right) L_{kl}^{\alpha_k\alpha_l}\ L_{ij}^{\alpha_i\alpha_j}$$

and form the commutation relations as before between the quantities

$$L_i^{\alpha_i} = \Gamma_i^{\alpha_i},\ L_{ij}^{\alpha_i\alpha_j},\ L_{ijk}^{\alpha_i\alpha_j\alpha_k},\ \ldots\ \ldots\ \ldots$$

If we choose $\alpha_i = \frac{m_i}{2}$, it follows from equations (14), (15), and (16) that the $\frac{n(n+1)}{2}$ quantities $L_i^{m_i}$ and $L_{ij}^{m_i m_j}$ form a Lie algebra. This is the algebra SO_{n+1}. If we choose the integers $\alpha_i < \frac{m_i}{2}$ then the commutation relations a re not closed. We have to continue the process of taking the commutation relations P steps, say, such that $P\,\alpha_i = m_i$ for all $i = 1,2,\ldots, n$, whenever admissible. Thus we obtain a set of Lie/sub algebras whose fundamental irreducible representations are also irreducible representations of the subalgebras of the algebra $\mathcal{A}$. We consider in detail the case where all $m_i = 2$ in Chapter $\mathfrak{D}$(I).

We end this section with the following remarks:

(i) If the generating elements of the finite group G do not commute, then the above construction can possibly be generalized for any finite group.

(ii) It is very interesting to consider the algebra $\mathcal{A}$ with more complicated restrictions than those given by (12), and find all its subalgebras, whose irreducible representations are isomorphic to the fundamental representations of other Lie algebras. (e.g. Unitary Lie algebras,... ...). It is also interesting to relax the polynomial restrictions given by (5)'. In this case the series (6) is infinite and the algebra is of infinite dimensions. Such algebra can be related in one way or another to similar algebras in quantum field theory.

CHAPTER 2

INTRODUCTION

By a generalized parity operator we mean an operator whose square is the identity, i.e. an operator Γ such that $\Gamma^2 = 1$. This operator generates a two element symmetry group.

The generalized parity operators may be unitary or anti-unitary operators. The parity operator corresponding to the space reflection operator is unitary, while a generalized parity operator corresponding to the time reflection operator is anti-unitary. Other generalized parity operators are well defined, like charge conjugation, G-parity (isospin parity), etc. One reason for such consideration is that one hopes that the generalized parity operators would introduce more general additive quantities, which could be identified with the electron charge of baryon number and so on.(17) It should be remarked that in physical applications it is very useful to normalize the symmetry operators in such a way that they form a group. This is possible, since an arbitrary ~~normalization~~ choice of the phase factors cannot affect the physical contents of the theory.

In part I in this chapter we shall find the quantum mechanical representations of n-generalized parity (commuting) operators In part II we consider the more general case of both unitary and anti-unitary parity operators.

I. Unitary Ray Representations of the Generalized Parity Operators

Let G be the group generated by n elements Γ_1, $\Gamma_2, \ldots \Gamma_n$ such that

$$\Gamma_i^{\,2} = 1 \qquad (1)$$

$$\Gamma_i \; \Gamma_j = \Gamma_j \; \Gamma_i \qquad (2)$$

The group G consists of I, Γ_i and all possible distinct products of Γ_i's and is abelian. Its elements will be denoted collectively by g_i. Note that Γ's generate the group, i.e. any g_i is expressed as a product of Γ's. Thus we have

$$g_i^{\,2} = 1 \qquad (1)'$$

$$g_i \; g_j = g_j \; g_i \qquad (2)'$$

Let $U(g_i)$ denote the unitary ray representations. They satisfy

$$U(g_i)\, U(g_j) = w(i,j)\, U(g_i \; g_j)$$

$$U(I) = I \qquad (3)$$

and we have from equation (1)':

$$U^2(g_i) = \alpha_i \; ; \quad \alpha_i \text{ are constants.}$$

Two representations $U(g)$ and $\beta U(g)$ lead to equivalent factor systems. Hence because of (1)' the diagonal phases

~~which~~ can be chosen to be the identity so that

$$U^2(g_i) \quad = \quad I \tag{4}$$

In order to determine the remaining phases we pass from the group law (3) to the commutation relations

$$\begin{aligned} U(g_i)\,U(g_j) &= w(i,j)\,U(g_i,g_j) \\ &= w(i,j)\,U(g_j,g_i) = w(i,j)\,w^{-1}(j,i)\,U(g_j)\,U(g_i) \\ &= C_{ij}\,U(g_j)\,U(g_i) \end{aligned} \tag{5}$$

We shall refer to the phase factors C_{ij} as the "commutation factors". These factors C_{ij} cannot be absorbed into the U's because the multiplication of U by any factor α occurs on both sides of equation (5) and hence the factor α cancels. But the commutation factors C_{ij} must be restricted to ± 1. For, if we multiply equation (5) from the left by $U(g_j)\,U(g_i)$, we obtain

$$\begin{aligned} I &= C_{ij}\;U(g_j)\,U(g_i)\,U(g_j)\,U(g_i) \\ &= C_{ij}\,C_{ij}\,U(g_j)^2\;U(g_i)^{\,2} \quad = \quad C_{ij}^{\,2} \end{aligned}$$

$$\text{or} \quad C_{ij} \quad = \quad \pm 1 \tag{6}$$

In order to determine all independent C_{ij} it is sufficient to remember that the group is generated by $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, the commutation factors of these generating elements, i.e.

$$\begin{aligned} U(\Gamma_i)\,U(\Gamma_j) &= \mathring{C}_{ij}\,U(\Gamma_j)\,U(\Gamma_i) \\ U^2(\Gamma_i) &= 1 \end{aligned} \tag{7}$$

determine all other factors C_{ij}. In fact equations (7) generate a generalized Clifford algebra with 2n basis elements

$$U(\Gamma_i);\ U(\Gamma_i)\,U(\Gamma_j);\ \dots\dots;\ U(\Gamma_1)\,U(\Gamma_2)\dots\dots U(\Gamma_n) \qquad (8)$$
$$i < j$$

which is just the representation of the group (Frobenius) algebra of the group G. The ordinary Clifford algebra corresponds to all $C^{o}_{ij} = -1$ (4).

For any two group elements g_i, g_j expressed in terms of the products of Γ_i, we have

$$\begin{aligned} U(g_i)\,U(g_j) &= U(\Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_N})\;U(\Gamma_{j_1}\Gamma_{j_2}\cdots\cdots\Gamma_{j_M}) \\ &= \Big(\prod_{k=1}^{N}\prod_{\ell=1}^{M} C^{o}_{i_k j_\ell}\Big)\,U(g_j)\,U(g_i) \qquad (9) \\ &= C_{ij}\,U(g_j)\,U(g_i) \end{aligned}$$

which is exactly the same as the commutation relations of the two elements $[U(\Gamma_{i_1}),\dots U(\Gamma_{i_N})]$ and $[U(\Gamma_{j_1})\ \dots\ U(\Gamma_{j_M})]$ of the algebra (8). This shows that the unitary ray representations of the group (1)' and (2)' satisfy the generalized Clifford algebra generated by (7) with the basis (8).

We therefore determine first the representations of the algebra and then pass to the representation of the group. The representations of the commutation relations (7) differ from the ray representations of the group as follows: to a given representation of the Clifford algebra there corresponds as many ray representations of the group as there are phase factors w_{ij} satisfying

$$\frac{w_{ij}}{w_{ji}} = C_{ij} \quad .$$

This freedom in the choice of the w_{ij} is, however, restricted by the associative law of the group which we have not yet used. If one multiplies equation (3) from the left by $U(g_i\ g_j)$ and uses equations (4) and (2)', one gets:

$$U(g_i)\,U(g_j)\,U(g_i g_j) = w_{ij}$$

or

$$U(g_i)\,U(g_j)\,U(g_j g_i) = w_{ij}$$

or

$$w_{ji}^{-1}\,U(g_i)\,U(g_j)\,U(g_j)\,U(g_i) = w_{ij}$$

or

$$w_{ji}^{-1} = w_{ij} \; ; \quad C_{ij} = C_{ji}^{-1} \tag{10}$$

Consequently

$$C_{ij} = w_{ij}^2 = \pm 1$$

$$\text{i.e. } w_{ij} = \pm 1, \quad \pm i \tag{11}$$

It is now a relatively simple matter to pass from the representations of the commutation relations (i.e. Clifford algebra) to the ray representations of the group. Take a representation of the algebra with $\overset{o}{C}_{ij}$ fixed. From equation (11) the corresponding w_{ij} take only two possible values $\pm\sqrt{C_{ij}}$. (Note that C_{ij} is given in terms of the products of $\overset{o}{C}_{ij}$). Now quite generally, if $U(g)$ is a representation of the commutation relations, so is $a_i\,U(g_i)$; and if $U(g_i)$

corresponds to the phase systems w_{ij} of the ray representations of the group, then $a_i\, U(g_i)$ corresponds to the equivalent phase system

$$w_{ij} \quad \frac{a_i\, a_j}{a_{ij}} \quad .$$

In our case, because of (4), we have $a_i = \pm 1$. Note the difference between the equivalence of phase systems and the equivalence of representations. Two representations U and aU belong to equivalent phase systems, but they are in general not equivalent representations, that is, there exists no matrix S such that

$$SU(g)S^{-1} = a\, U(g) \qquad \text{for all} \qquad g \in G \, .$$

The concept of equivalent phase systems tells us simply that if we have found one representation, the other is trivially obtained by multiplication with a phase factor, like $U(g\ )$ and $-U(g)$. But the eigenvalues of the operator $U(g)$ in the two representations are of course the opposite of each other.

We now find all the representations of the commutation relations (7), or the representations of the algebra with the basis (8). Some special cases of this algebra are well known. If all $\overset{o}{C}_{ij} = 1$, hence all $C_{ij} = 1$. We have the trivial case of an abelian group and all irreducible representations are one dimensional.

If all $\overset{o}{C}_{ij} = -1$ ($C_{ij} = \pm 1$ according whether we have an even number of products of C_{ij} or odd number, respectively), the representation $U(g)$ forms a bona fide Clifford algebra whose representations are known. All other mixed cases take an intermediary position between these two extreme cases.

Because of equation (10) there are

$$\tfrac{1}{2}(n-1) = K \qquad \text{distinct factors} \quad C_{ij},$$

hence *a priori* there are 2^K different types of ray representations, depending on which c_{ij} are equal to "+1" and which are equal to "-1". If k of these factors are -1, they can be distributed in $\binom{K}{k}$ different ways among the K factors,

$$\binom{K}{1} + \binom{K}{2} + \binom{K}{k} + \ldots + \binom{K}{k} = 2^K$$

Consider now the class with k of the $\overset{\circ}{C}_{ij}$ being equal to -1. We distinguish two cases:

(i) The k minus signs are so distributed that we have a sub-Clifford algebra with r generating elements, $k = \tfrac{1}{2}(r-1)r$.

(ii) There is no sub-algebra which is a *bona fide* Clifford algebra. For example, if $k = 3$, then the case (i) corresponds to $\overset{\circ}{C}_{12} = \overset{\circ}{C}_{13} = \overset{\circ}{C}_{23} = -1$, then $\Gamma_1, \Gamma_2, \Gamma_3$ form a Clifford algebra. Case (ii) corresponds, say, to $\overset{\circ}{C}_{12} = \overset{\circ}{C}_{13} = \overset{\circ}{C}_{14} = -1$, that is Γ_1, Γ_2 anticommute but Γ_2 and Γ_3 commute. It is clear that we have case (i) only if k is of the form $k = \tfrac{1}{2} r(r-1)$, $r = 1,2,3,\ldots, n$. (Note we write simply Γ_i to mean $U(\Gamma_i)$).

Case (i): The representations are determined by those of the Clifford sub-algebra with r generating elements ($r = 1,2,\ldots, n$)

have been first determined by Jordan and Wigner[18] for even r. For even r there is a single irreducible representation (up to unitary equivalence) of the Clifford sub-algebra of degree $2^{r/2}$. Every other representation is completely reducible, faithful, having a dimension which is a multiple of $2^{r/2}$. For odd r a similar analysis exists. In this case there are two inequivalent, but non-faithful representations of ~~degree~~ dimension $2^{(r-\frac{1}{2})}$. The faithful representation is the direct sum of these two and, therefore, of ~~degree~~ dimension $2^{(r+\frac{1}{2})}$. Again every other representation is completely reducible and its ~~degree~~ dimension is a multiple of $2^{(r-\frac{1}{2})}$.

Case (ii): Except in the cases where, by relabelling the elements, we may obtain a Clifford sub-algebra as in the case (i), the generalized algebra with mixed signs $C^0_{ij} = \pm 1$ lead in general to new types of representations different from those of Clifford algebra. As a matter of fact, all of these algebras (including the Clifford algebra) are special instances of a much larger algebra with arbitrary C^0_{ij} (forming a group) whose irreducible representation have been determined in Chapter 1. In this case the prescription of determining the irreducible representations is as follows: Consider the group $\mathcal{E}$ consisting of the 2^n elements given in equation (8) and their negatives. The order of $\mathcal{E}$ is 2^{n+1}. The representations of the commutation relations (7) are also representations of $\mathcal{E}$. The factor group $\mathcal{E}/C_2$ is abelian and has 2^n one dimensional representations, where C_2 is the group of two elements (1, -1).

Following the procedure of Chapter 1, let k be the number of the additional representations and let $\ell_1, \ell_2, \ldots, \ell_k$

be their dimensions, then

$$\ell_1^2 + \ell_2^2 + \ell_k^2 + 2^n = 2^{n+1}$$

$$\therefore \quad \ell_1^2 + \ell_2^2 + \ell_k^2 = 2^n \qquad (13)$$

The number k is equal to the number of basis elements in the set (8) which commute with all 2^n elements. Thus, if all $C^0_{ij} = +1$ then $k = 2^n$, hence $\ell_1 = \ldots . \ell_k = 1$; and if all $C^0_{ij} = -1$, then $k = 1$ or 2, depending on whether n = even or odd, and $\ell = 2^{n/2}$ (n even), or $\ell_1 = \ell_2 = 2^{\frac{n-1}{2}}$ (n odd). Finally, to find the number of commuting elements in the set (8) we look at the table of C^0_{ij} and determine how many C^0_{ij}, $C^0_{ij}\, C^0_{ik}$, $C^0_{ij}\, C^0_{ik}\, C^0_{i\ell}, \ldots .$ are $+1$ for fixed i, all j (all j, k, ℓ). It follows from this that the dimensions of irreducible representations are determined by those of the sub-algebra containing - signs for C^0_{ij}. Thus all irreducible representations of dimensions 1, 2, 4, 8, ... $2^{(n/2)}$ occur where $(\frac{n}{2}) = \frac{n}{2}$ (even) or $\frac{n-1}{2}$ (odd n). This completes the enumeration of all irreducible representations. The case $n = 4$ is treated in detail and an explicit form of the matrices are given in an appendix.

II. Unitary Coray Representations of Generalized Parity Operators

We wish to represent n_1 generating elements by unitary operators $U(\Gamma_i)$, $i = 1,2,\ldots, n_1$, and n_2 generating elements by anti unitary operators $A(\Gamma_j)$ $j = 1,2,\ldots, n_2$. $n = n_1 + n_2$. Because the product of two unitary (or anti-unitary) operators is again a unitary operator, while the product of unitary and anti-unitary operators is again an anti-unitary operator, the whole algebra with 2^n basis elements splits into, say, m anti-unitary and 2^n-m unitary operators. In particular, the case where all generating elements are represented by anti-unitary operators corresponds to the case where $n_1 = 0$, in which

$$m = \binom{n}{1} + \binom{n}{3} + \ldots = 2^{n-1}$$

the number of unitary and anti-unitary operators is the same as those of the unitary representations of $(n-1)\Gamma$'s and the anti-unitary representations of one Γ. Thus we have:

<u>Theorem</u>: The ray representation of G generated by n elements (commuting) Γ_i in which any member of the Γ's are represented by anti-unitary operators can, by relabelling the elements and the proper adjustment of phases, be made to coincide with the ray representations in which only one Γ is represented by an anti-unitary operator.

The representation algebra is thus always generated by a direct product of the form:

$$(1,\ U(\Gamma_1) \ldots U(\Gamma_{n-1})) \otimes (I,\ A(\Gamma_n)) .$$

Among all discrete commuting symmetry transformations there can only be <u>one</u> anti-unitary generating operator (the others are the products).

We now discuss in general, the phases between anti-unitary and unitary operators.

As in Section I, we first normalize the unitary operators in such a way that the products of two unitary operators satisfy

$$U(g_i)^2 = 1 \qquad w_{ii} = 1$$

$$U(g_i)\, U(g_j) = C_{ij}\, U(g_i)\, U(g_j); \quad C_{ij} = \pm 1 \tag{14}$$

Next we look at the products of two anti-unitary operators. The determination of the phase factors here is slightly more complicated.

For the relation:[20]

$$A(g_i)\, A^*(g_i) = \varepsilon_{ii}\, U(g_i^2) = \varepsilon_{ii} \tag{15}$$

The diagonal phases ε_{ii} now cannot be normalized to unity by multiplying A with a phase factor. For in equation (15) we replace $A(g_i)$ by $w\, A(g_i) = A'(g_i)$ where $w = 1$, we see immediately that equation (15) holds for $A'(g_i)$ with the same ε_{ii}.

However, it follows from the associativity law that these phases are equal to ± 1, for the multiplication of the above equation by $A(g_i)$ from the left gives (see Chapter 1(a))

$$A(g_i)\, \left[A(g_i)\, A^*(g_i)\right]^* = A(g_i)\, \varepsilon_{ii}$$

or

$$A(g_i)\, A^*(g_i)\, A(g_i) = \varepsilon^*_{ii}\, A(g_i) \tag{16}$$

or $$\varepsilon_{ii} = \varepsilon^*_{ii} \tag{17}$$

i.e. ε_{ii} are real, hence the result.

To determine the off-diagonal phase factors in

$$A(g_i)\ A^*(g_j) = \varepsilon_{ij}\ U(g_i\ g_j) \tag{18}$$

and

$$A(g_j)\ A^*(g_i) = \varepsilon_{ji}\ U(g_j\ g_i) \tag{19}$$

multiply both sides of these two equations to obtain

$$A(g_j)\ A^*(g_i) \quad A(g_i)\ A^*(g_j) = \varepsilon_{ij}\ \varepsilon_{ji}$$

$$\varepsilon_{jj}\ \varepsilon^*_{ii} = \varepsilon_{ij}\ \varepsilon_{ji}$$

or

$$\varepsilon_{jj}\ \varepsilon_{ii} = \varepsilon_{ij}\ \varepsilon_{ji} \tag{20}$$

Because ε_{kk} are real and equal to ± 1, the product $\varepsilon_{ij}\ \varepsilon_{ji}$ is also equal to ± 1. From (18) and (19) we obtain

$$A(g_i)\ A^*(g_j) = d_{ij}\ A(g_j)\ A^*(g_i) \tag{21}$$

where $$d_{ij} = \varepsilon_{ij}/\varepsilon_{ji} \tag{22}$$

In contrast to the unitary case, the associativity law does not allow us to determine the ε_{ij}. To see this, we multiply equation (18) by $A(g_i)$ from the left:

$$A(g_i)\left[A\ (g_i)\ A^{+}(g_j)\right]^* = \varepsilon^*_{ij}\ \ A(g_i)U^*(g_i g_j)$$

and replace $U(g_i\, g_j)$ from equation (19) and use equations (16), (17) and (21) we obtain

$$\varepsilon_{ii}\, A(g_j) = \varepsilon^*_{ij}\, A(g_i)\, \varepsilon^{*-1}_{ji}\, A^*(g_j)\, A(g_i)$$

$$= \varepsilon^*_{ij}\, \varepsilon^{*-1}_{ji}\, d_{ij}\, \varepsilon^*_{ii}\, A(g_j)$$

$$\text{or} \quad d^*_{ij} = \varepsilon^*_{ij}/\varepsilon^*_{ji} = \frac{1}{d^*_{ij}} \qquad (22)'$$

i.e. the associativity law gives nothing new.

We can, however, pass to equivalent phase systems by multiplying in (18) $A(g_i)$ and $U(g_i\, g_j)$ by phase factors so that $\varepsilon_{ij} = 1$. But then ε_{ji} cannot be equal to +1 always, but from (20) it is equal to ± 1. Consequently, the representations are determined up to these arbitrary phase factors in the anti-unitary operators.

Finally we consider the product of one unitary and one anti-unitary operator. Here we have only the off-diagonal phase factors

$$U(g_i)\, A(g_j) = w_{ij}\, A(g_i\, g_j)$$

$$A(g_j)\, U^*(g_i) = w_{ji}\, A(g_j\, g_i) = w_{ji}\, A(g_i\, g_j) \qquad (24)$$

The commutation relations are given by

$$U(g_i)\, A(g_j) = f_{ij}\, A(g_j)\, U^*(g_i) \qquad (25)$$

where

$$f_{ij} = \frac{w_{ij}}{w_{ji}} = f_{ji}^{-1} \tag{26}$$

Again we make use of the associativity law as before and obtain

$$U(g_i)^2 A(g_i) = w_{ij} U(g_i) A(g_i\ g_j)$$

or

$$A(g_j) = w_{ij} U(g_i)\ w_{ji}^{-1}\ A(g_j)\ U^*(g_i)$$

$$= w_{ij}\ w_{ji}^{-1}\ f_{ij}\ A(g_j)\ U^*(g_i)^2$$

or $\quad f_{ij}{}^2 = 1 \qquad \therefore\ f_{ij} = \pm 1 \qquad (27)$

To determine w_{ij} we again pass, as in the previous case, to an equivalent phase system by multiplying the first equation of (24) by phase factors such that

$$w_{ij} = +1$$

but then

$$w_{ji} = \pm 1 \tag{23)'}$$

gives us two distinct types, as in equation (23). Both in (23) and (23)' we can choose $\varepsilon_{ij} = +1$ (or $f_{ij} = +1$) for $i \quad j$ then ε_{ji} (or f_{ji}) $= \pm 1$ holds for $i > j$.

Now we discuss the explicit representations of the group. We bring first G, by relabelling the group elements, to the form given in theorem (2). i.e. (n-1) unitary generators $U(\Gamma_i)$, $i = 1, \ldots, n-1$ and one anti-unitary generator $A(\Gamma_n)$. Because every anti-unitary operator is of the form

$A(\Gamma_n) = U(\Gamma_n) K$, where K is the complex conjugation, we have from (15)

$$U(\Gamma_n)\, U^*(\Gamma_n) = \varepsilon_{nn} = \pm 1 \tag{28}$$

In addition to the unitary condition

$$U(\Gamma_n)\, U^+(\Gamma_n) = U^+(\Gamma_n)\, U(\Gamma_n) = 1 \tag{29}$$

Thus the unitary matrix is symmetric or antisymmetric depending on whether $\varepsilon_{nn} = +1$ or -1 respectively. We have from (25) and (27)

$$U(\Gamma_1)\, U(\Gamma_n) = f_{1n}\, U(\Gamma_n)\, U^*(\Gamma_1) \tag{30}$$

Suppose we have a representation of $G(n-1)$, $U(\Gamma_1)$ as discussed in Section I, then equations (28) and (30) can be satisfied by real or pure imaginary matrices (for example,

$$\sigma_1 \sigma_2^* = +\sigma_2 \sigma_1 \quad \text{but} \quad \sigma_1 \sigma_3^* = -\sigma_3 \sigma_1)$$

We may take for $\varepsilon_{nn} = \mp 1$, $U^*(\Gamma_n) = \mp U(\Gamma_n)$; $\mp f_{1n} = c_{1n}$, then the representation of (30) is reduced again to the cases of Section I, but with the additional restriction $U^*(\Gamma_n) = \mp U(\Gamma_n)$.

APPENDIX

In this appendix, we shall write down exactly the form of the ray representations of a group G generated by four parity operators. The order of the group is 16 which is the dimension of the algebra $\mathcal{A}$. G has the form

$$G = (1, \Gamma_1) \otimes (1, \Gamma_2) \otimes (1, \Gamma_3) \otimes (1, \Gamma_4)$$

$$= 1, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_1 \Gamma_2, \ldots, \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$$

$$= 1, g_1, g_2, \ldots, g_{15} .$$

The multiplication law of the group is given by

$$\Gamma_i \Gamma_j = \Gamma_j \Gamma_i \qquad (1)$$

$$\Gamma_i^2 = 1 \qquad (2) \qquad i,j = 1,2,3,4$$

of course

$$g_\mu g_\nu = g_\nu g_\mu \qquad \mu,\nu = 0,1,\ldots, 15$$

where $g_o = 1$

and $g_\mu^2 = 1 \qquad \mu = 0,1,2,\ldots, 15$

The ray representation is given by:

$$U(\Gamma_i) U(\Gamma_j) = \varepsilon_{ij} U(\Gamma_j) U(\Gamma_i) \qquad (1)'$$

$$U^2(\Gamma_i) = 1 \qquad (2)'$$

where $\varepsilon_{ij} = \frac{w(i,j)}{w(j,i)} = w^2(i,j)$ (3)

(when $\varepsilon_{ij} = 1 \quad w(i,j) = 1$) (4)

where $\varepsilon_{ij}^2 = 1$ or $\varepsilon_{ij} = \pm 1$ (5)

The number of independent commutation factors is 6. Equation (1)' can be written in the following diagram.

$\epsilon_{ij} = \epsilon_{ji}^{-1}$	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	1	ε_{12}	ε_{13}	ε_{14}
Γ_2		1	ε_{23}	ε_{24}
Γ_3			1	ε_{34}
Γ_4				1

Figure 1

Since the representation of the group is determined by the representation of its algebra modulo equation (3), we shall determine only the representations of the algebra.

The number of different types of ray representations $= 2^6 = 64$. In physical problems these types should describe different physical systems. Since we are interested in writing the form of the representations, the interpretation of Γ's does not matter: types which have the same dimensionality define a class. In this case we have the following classes:

Class 1

All $\epsilon_{ij} = +1$, in this all the elements of the algebra commute, hence the elements of the group. We have only one dimensional irreducible representations. This class defines one type.

Class 2

$\epsilon_{12} = -1$ say, all others equal +1. This class defines 6 types. In this case we have the two dimensional representations:

$$U(\Gamma_1) = \sigma_1$$
$$U(\Gamma_2) = \sigma_2$$
$$U(\Gamma_1 \Gamma_2) = \sigma_3$$
$$U(\Gamma_3) = I$$
$$U(\Gamma_4) = I$$
$$\cdots\cdot \qquad \cdots\cdot$$

where I is the unit matrix of dimension 2, this representation is not faithful.

Class 2

$\epsilon_{12} = -1$, $\epsilon_{13} = -1$ others equal +1. This class defines 15 types.

In this case we cannot have the one dimensional representations, because some of the matrices anticommute. Because $U(\Gamma_2)$ and $U(\Gamma_3)$ commute, but both anticommute with $U(\Gamma_1)$, then we have only irreducible representations of dimension two.

For example:

$$U(\Gamma_1) = \sigma_1$$
$$U(\Gamma_2) = \sigma_2$$
$$U(\Gamma_3) = \sigma_2$$
$$U(\Gamma_1\Gamma_2) = \sigma_3$$
$$U(\Gamma_4) = I$$

We have $\ell_1^2 + \dots + \ell_k^2 = 2^4 = 16$ and we have $k = 2$. From the above formula we have $\ell_1 = \ell_2 = 2$, the other equivalent irreducible representations

$$U(\Gamma_1) = -\sigma_1$$
$$U(\Gamma_2) = -\sigma_2$$
$$U(\Gamma_3) = -\sigma_2$$
$$U(\Gamma_4) = I$$

~~Both~~ Neither representations ~~are not~~ is faithful.

If we repeat this process, we arrive at the following class:

all $\varepsilon_{ij} = -1$. This class defines only one type: the dimension of the irreducible representation is 4, the elements of the algebra have the form

$$U(\Gamma_1) = i\,\gamma_1$$

$$U(\Gamma_2) = i\,\gamma_2$$

$$U(\Gamma_3) = i\,\gamma_3$$

$$U(\Gamma_4) = i\,\gamma_4$$

$$U(\Gamma_i\,\Gamma_j) = i\,\gamma_i\,\gamma_j$$

$$U(\Gamma_i\,\Gamma_4) = \gamma_i\,\gamma_4$$

$$U(\Gamma_i\,\Gamma_j\,\Gamma_k) = \gamma_i\,\gamma_j\,\gamma_k$$

$$U(\Gamma_i\,\Gamma_j\,\Gamma_4) = i\,\gamma_i\,\gamma_j\,\gamma_4$$

$$U(\Gamma_1\,\Gamma_2\,\Gamma_3\,\Gamma_4) = i\,\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4$$

where $i, j, k = 1, 2, 3$. γ's are the Dirac matrices. The representation is faithful.

From the above analysis, we could write the representation matrices as a direct product of Pauli matrices and the 2×2 identity matrix I. Because the dimension of the irreducible representations are determined by the non-commutative sub-algebra. The representation with the highest dimensions, i.e. for all $\epsilon^{0}_{ij} = -1$ can be chosen in the following standard form:

(a) When n is <u>even</u>

$$U(\Gamma_1) = \sigma_1 \otimes I \otimes I \ldots\ldots \otimes I \quad (\tfrac{n}{2} \text{ factors})$$

$$U(\Gamma_2) = \sigma_2 \otimes I \otimes I \ldots\ldots \otimes I$$

$$U(\Gamma_3) = \sigma_3 \otimes \sigma_1 \otimes I \ldots\ldots \otimes I$$

$$U(\Gamma_4) = \sigma_3 \otimes \sigma_2 \otimes I \ldots\ldots \otimes I$$

$$U(\Gamma_5) = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes I \ldots\ldots \otimes I$$

$$U(\Gamma_6) = \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes I \ldots\ldots \otimes I$$

$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$

$$U(\Gamma_{n-1}) = \sigma_3 \otimes \sigma_3 \ldots\ldots \otimes \sigma_3 \otimes \sigma_1$$

$$U(\Gamma_n) = \sigma_3 \otimes \sigma_3 \ldots\ldots \otimes \sigma_3 \otimes \sigma_2$$

(b) n = odd.

There are two inequivalent irreducible representations

(i) $U(\Gamma_{2j} - 1) = U(\ _{2j-1})$ of (a)

$U(\Gamma_{2j}) = U(\Gamma_{2j})$ of (a) $\qquad j = 1, \ldots, \frac{n-1}{2}$

and

$$U(\Gamma_n) = \sigma_3 \otimes \sigma_3 \ldots\ldots \otimes \sigma_3 \qquad \frac{n-1}{2} \text{ factors} .$$

(ii) The negative of matrices in (i) .

REFERENCES - SECTION I

1. Akhiezer and Glazman, "Theory of linear operators in Hilbert space", Vol. I, (English translation by Unger, New York, 1962).

2. M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966); 38, 346 (1966).

3. H. Bacry, Nuovo Cimento A, 15, 22 (1966).

4. V. Bargmann, Z. Physik 99, 576 (1936).

5. A.O. Barut, Proceedings - Seminar on High Energy Physics and Elementary Particles, Trieste 1965 (I.A.E.A. Vienna), and references therein.

6. A.O. Barut, Trieste preprint IC/65/33.

7. A.O. Barut and A. Böhm, Trieste preprint IC/65/18.

8. A.O. Barut and S.R. Komy, under preparation.

9. A. Bohm, University of Marburg Report (August 1964).

10. A. Bohm, Lectures on "Rigged Hilbert Space", Vol. II, Proceedings of the Theoretical Physics Institute, University of Colorado, Summer 1966. See also Trieste preprint IC/64/9.

11. R. Dashen and M. Gell-Mann, Phys. Lett. 17, 142 (1965).

12. J. Dixmier, Bull. Soc. Math. France 89, 9 (1961).

13. H.D. Doebner and O. Melsheimer, Trieste preprint IC/66/97.

14. V. Fock, Z. Physik 98, 145 (1935).

15. J.M. Gelfand and N. Vilenkin, "Generalized Functions", Vol. 4 (English translation by A. Feinstein, Academic Press, 1964).

16. D.E. Greenberg, University of Chicago preprint (1966).

17. A. Grossman, J. Math. Phys. 5, 1025 (1964); 5, 54 (1965).

18. E. Inönü and E.P. Wigner, Proc. N.A.S. 39 (1953), 510; 40 (1954), 119.

19. H.J. Lipkin, "Lie groups for pedestrians", (Amsterdam).

20. N. Mukunda, L. O'Raifeartaigh and E.C.G. Sudarshan, Phys. Rev. Lett. 19, 1322 (1965).

21. M.A. Naimark, "Linear Representations of the Lorentz Group", (Pergamon Press, 1964.)

22. von Neumann, "Mathematical Foundations of Quantum Mechanics", (Princeton University Press (1955)).

23. T.D. Newton, Ann. of Math. 3, 730 (1950).

24. W. Pauli, Z. Physik 36, 336 (1926).

25. P. Budini, Nuovo Cimento A 44, 363 (1966), 44, 411 (1966).

26. P. Budini and C. Fronsdal, Phys. Rev. Lett. 14, 968 (1965).

27. J.E. Roberts, Comm. Math. Phys. 3, 98 (1966). Cambridge University preprint (July 1966).

28. P. Roman, "Advanced Quantum Theory", Chapter I. (Addison-Wesley Publishing Co., Inc. 1965).

29. M.E. Rose, "Elementary Theory of Angular Momentum", (Wiley, 1961, New York).

30. A. Salam, Trieste preprint IC/65/76.

31. A. Salam, R. Delbourgo and J. Strathdee, Proc. Roy. Soc. London.

32. E.C.G. Sudarshan, "Coral Gables Conference", Vol. III, (W.H. Freeman and Co., 1966).

33. L.H. Thomas, Ann. of Math. 42, 113 (1941).

REFERENCES - SECTION II

1. A.O. Barut, University of Colorado, Preprint U COL-65-8.
2. A.O. Barut, Phys. Rev <u>147</u> (1966), 978.
3. A.O. Barut and S.R. Komy, University of Colorado, Preprint U COL-66-4
4. H. Boerner, "Representations of Groups", (North Holland Pub. Co., Amsterdam, 1963).
5. S.K. Bo e and U. Singh, Delhi University, preprint (1966).
6. I.M. Case, R. Karplus and C.N. Yang, Phys. Rev. <u>101</u>, 874 (1956).
7. J.O. Dimmock, J. Math. Physics <u>4</u>, 1307 (1960).
8. M.H. Hamermesh, "Group Theory and its Applications", (Addison-Wesley Publishing Company, Inc., 1962).
9. W. Heisenberg, Z. f. Phys. <u>77</u>, 1 (1932).
10. J.S. Lemont, "Application of Finite Groups", (Academic Press, Inc., New York, 1959).
11. M.V. Murthy, J. Math. Phys. <u>7</u>, 853 (1966).
12. S. Okubo, J. Schechter and M. Veda, Annals of Phys. <u>32</u>, 424 (1965).
13. O'Raifeartaigh, Preprint NYO-3399-27-SU-23.
14. A. Pais, Rev. Mod. Phys. <u>38</u>, 215 (1966).
15. P. Roman, "Advanced Quantum Theory", Chapter 4. (Addison-Wesley Publishing Co., Inc., 1965).
16. P. Runda, J. Math. Physics <u>6</u>, 1273 (1965).
17. N. Tarimer, Phys. Rev. <u>140</u>, 477 (1965).
18. P. Jordan and E.P. Wigner, Z. phys. <u>47</u>, 631 (1928).
19. G.C. Wick, A.S. Wightman and E.P. Wigner, Phys. Rev. <u>88</u>, 101, (1952).
20. E.P. Wigner in "Group Theoretical Concepts and Methods in Elementary Particle Physics", Gordon & Breach, Science oublishers, Inc., New York, 1964.
21. E.P. Wigner, "Group Theory and Its Applications, Chap. 26. Academic Press Inc., New York, 1959).
22. Yoshio Yamaguchi, Phys. Lett. <u>9</u>, 281 (1964).