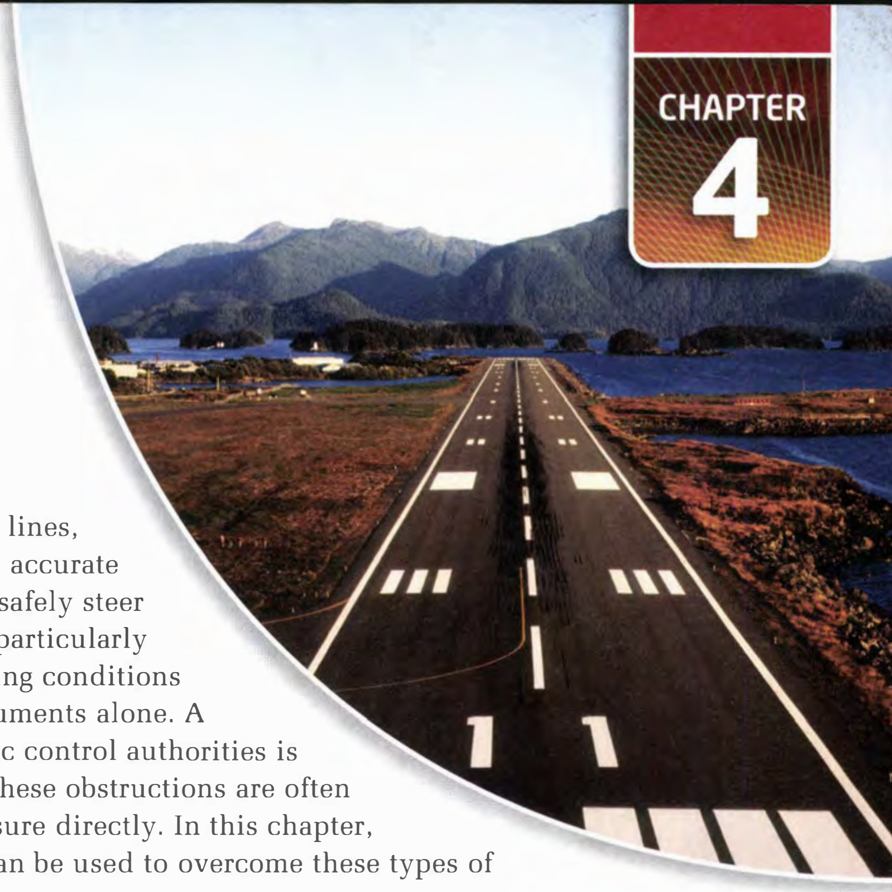
CHAPTER 4

Trigonometry



Airports are often surrounded by mountains, trees, buildings, power lines, and other obstructions. Pilots need accurate information about these objects to safely steer clear of them. This information is particularly important when weather and lighting conditions force pilots to navigate using instruments alone. A challenge for aviation and air traffic control authorities is that the distances and heights for these obstructions are often difficult, if not impossible, to measure directly. In this chapter, you will learn how trigonometry can be used to overcome these types of challenges.

By the end of this chapter, you will

- determine the exact values of the sine, cosine, and tangent of the special angles: 0°, 30°, 45°, 60°, and 90°
- determine the values of the sine, cosine, and tangent of any angle from 0° to 360°, through investigation using a variety of tools and strategies
- determine the measures of two angles from 0° to 360° for which the value of a given trigonometric ratio is the same
- define the secant, cosecant, and cotangent ratios for angles in a right triangle in terms of the sides of the triangle, and relate these ratios to the cosine, sine, and tangent ratios
- prove simple trigonometric identities, using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$; the quotient identity $\tan x = \frac{\sin x}{\cos x}$; and the reciprocal identities $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$, and $\cot x = \frac{1}{\tan x}$
- pose and solve problems involving right triangles and oblique triangles in two-dimensional settings, using the primary trigonometric ratios, the cosine law, and the sine law (including the ambiguous case)
- pose and solve problems involving right triangles and oblique triangles in three-dimensional settings, using the primary trigonometric ratios, the cosine law, and the sine law

Prerequisite Skills

Refer to the Prerequisite Skills Appendix on pages 478 to 495 for examples of the topics and further practice.

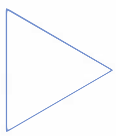
Classify Triangles

1. Measure the sides and angles of each triangle. Classify each as completely as possible, using terms such as equilateral, isosceles, scalene, and right.

a)

b)

c)



d)

Angle Sum of a Triangle

2. Sketch each triangle using pencil and paper or geometry software. Then, determine the indicated values by measuring.

a) In $\triangle ABC$, $\angle A = 50°$ and $\angle B = 70°$. Determine the value of $\angle C$.

b) $\triangle DEF$ is isosceles such that DE = DF and $\angle D = 40°$. Determine the measures of the other two angles.

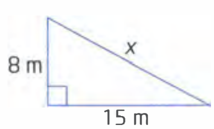
c) $\triangle GHI$ is equilateral. Find the measures of all angles.

d) $\triangle JKL$ is isosceles and $\angle K = 90°$. Find the measures of the other two angles.

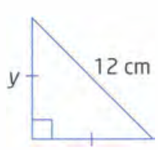
Use the Pythagorean Theorem

3. Determine the measure of the unknown side in each triangle. Round your answer to one decimal place, if necessary.

a)



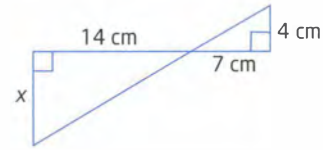
b)



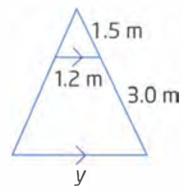
Use Similar Triangles

4. Use similar triangles to determine the unknown value in each figure.

a)



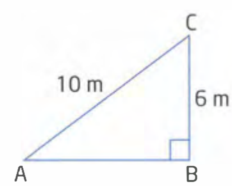
b)



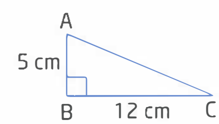
Find Primary Trigonometric Ratios

5. In each triangle, determine the primary trigonometric ratios for $\angle A$ and $\angle C$.

a)



b)



6. Use a calculator to find the approximate primary trigonometric ratios for each angle. Round your answers to four decimal places, if necessary. Be sure to set your calculator to degrees, rather than radians.

a) 30° **b)** 45° **c)** 60°

Determine an Angle Given a Trigonometric Ratio

7. Find each angle measure, to the nearest degree.

a) $\angle A$ given that $\sin A = 0.5299$

b) $\angle B$ given that $\cos B = \frac{3}{4}$

c) $\angle C$ given that $\tan C = \frac{1}{\sqrt{3}}$

Technology Tip

To access the functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ on your calculator, you may need to press another key, such as 2nd, SHIFT, or INV.

Apply Trigonometric Ratios to Problems

8. In $\triangle ABC$, $\angle A = 50°$, $\angle B = 90°$, and $a = 8$ cm.

a) Sketch the triangle, labelling all given measurements. Which trigonometric ratio would you use to determine the value of c? Justify your choice.

b) Determine c, to the nearest centimetre.

c) Determine $\angle C$.

9. A ladder of length 6.0 m is leaning against a building to reach a window 5.8 m above the ground. Safety instructions for the ladder indicate that a safe angle to erect the ladder is between 70° and 80° with the ground.

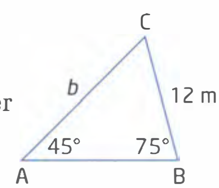
a) Use a protractor and a ruler to draw an accurate scale diagram of this situation.

b) Predict whether this setup is safe. Justify your prediction.

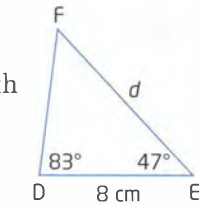
c) Select and use a trigonometric ratio to check your prediction.

Apply the Sine Law and the Cosine Law

10. Determine the length of the unknown side b in $\triangle ABC$. Round your answer to one decimal place, if necessary.



11. Determine the length of the unknown side d in $\triangle DEF$, to the nearest tenth of a centimetre.



12. In $\triangle PQR$, $q = 18$ m, $r = 14$ m, and $\angle P = 48°$.

a) Select the appropriate trigonometric tool to find the measure of p. Justify your selection.

b) Determine the measure of p, to the nearest metre.

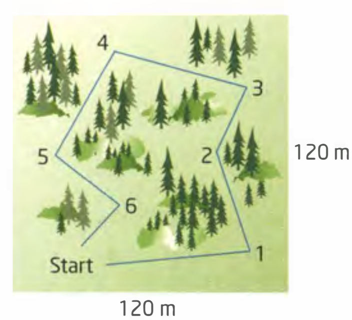
c) Find, to the nearest degree, the measure of $\angle Q$.

Connections

A trigonometric tool is an equation used to solve for an unknown. Examples of trigonometric tools are the primary trigonometric ratios, the sine law, and the cosine law.

13. You are given $\triangle ABC$ such that $a = 15$ cm, $\angle A = 35°$, and $\angle C = 55°$. You want to find the measure of c. Select the most appropriate trigonometric tool to use. Justify your selection.

Chantal is a member of an orienteering club. Orienteering is an activity in which participants make their way around a course, stopping at checkpoints along the way. At the starting point, racers are given a map, a compass to measure direction, and a pedometer to measure distance. They use these tools to find their way through the course.



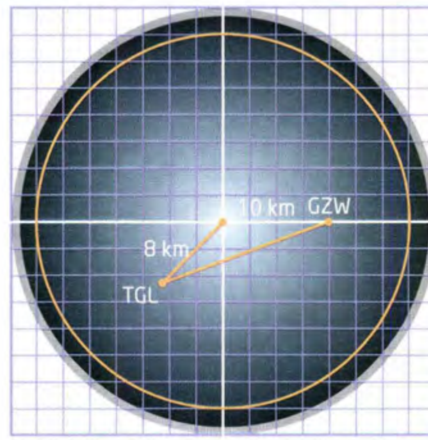
For a mathematics project, Chantal created an orienteering course. Her directions require participants to use trigonometry to determine directions and distances between checkpoints. Only paper and pencil are allowed for the first three legs, but participants are given a calculator at checkpoint #3. As a participant in Chantal's race, you will use the trigonometry you learn throughout this chapter to make your way around the course.

4.1

Special Angles

Aircraft pilots often cannot see other nearby planes because of clouds, fog, or visual obstructions. Air Traffic Control uses software to track the location of aircraft to ensure that they are kept a safe distance from one another. The software uses trigonometry to make these calculations. The radar screen here shows an aircraft, identified by GZW, 10 km east of the control tower, and another, identified as TGL, 8 km southwest of the tower. To find the distance between the two aircraft, the software can use the cosine law, but it needs the cosine of an obtuse angle. Are there trigonometric ratios for angles greater than 90°? If so, how are they calculated?

In this section, you will learn how to find the primary trigonometric ratios for any angle from 0° to 360°.



Tools

- computer with *The Geometer's Sketchpad®*

or

- grid paper

unit circle

- a circle with centre at the origin and a radius of 1 unit

Technology Tip

In *The Geometer's Sketchpad®*, you can drag the unit point on the *x*-axis to make the circle a convenient size. When measuring distances using the **Measure** menu, be sure to select **Coordinate Distance** to apply the proper scale factor.

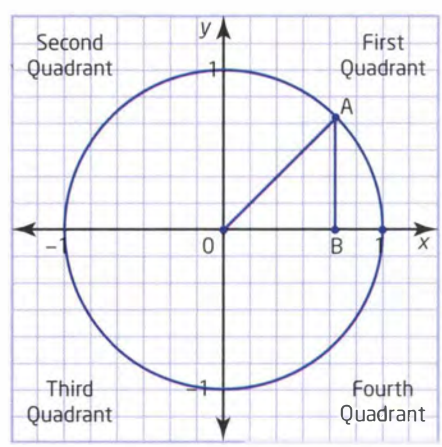
Investigate A

How can you find exact trigonometric ratios?

Some triangles contain known angles and sides. You can use these triangles to find exact trigonometric ratios for special angles.

1. Draw a set of axes. Using the origin as the centre, draw a circle. The radius of this circle will represent 1 unit. This circle is known as a **unit circle**.

   If you are using grid paper for this investigation, create a unit circle of workable size to help with accuracy.



2. a) Draw a 45° angle in the first quadrant by placing the **initial arm** on the x-axis. Extend the **terminal arm** of the angle until it meets the circle at point A, such that OA is a radius of the circle. This representation is known as an **angle in standard position**.

   b) Draw a vertical line from point A to the x-axis, and label the intersection point B.

   c) Draw a line from the origin to point B to form $\triangle$OAB.

3. **Reflect** Classify $\triangle$OAB. Be as specific as you can.

4. Use the Pythagorean theorem to find the side lengths of $\triangle$OAB. Leave your answers in radical form. Do not convert to a decimal.

5. a) Use the side lengths to find exact expressions for sin 45°, cos 45°, and tan 45°.

   b) Use a calculator to evaluate these expressions to 4 decimal places.

6. Use a calculator to determine the values of sin 45°, cos 45°, and tan 45°.

7. a) **Reflect** How do the trigonometric ratios obtained from the triangle compare to those obtained from the calculator?

   b) What is the relationship between cos 45° and the measure of side OB of $\triangle$OAB?

   c) What is the relationship between sin 45° and the measure of side AB of $\triangle$OAB?

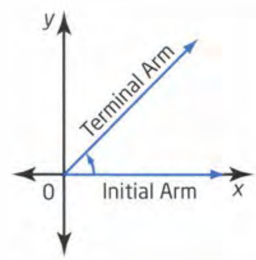
   d) Find the coordinates of point A. How do the two relationships in parts b) and c) relate to the coordinates of point A?

initial arm

- first arm, or ray, of an angle drawn on a Cartesian plane that meets the other (terminal) arm of the angle at origin

terminal arm

- the arm of an angle that meets the initial arm at the origin and rotates around the origin counterclockwise to form a positive angle or clockwise to form a negative angle



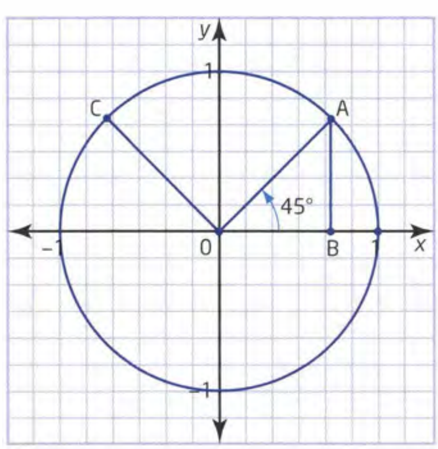
angle in standard position

- the position of an angle when its initial arm is on the positive x-axis and its vertex is at the origin

Investigate B

How can you find trigonometric ratios for angles greater than 90°?

1. Line segment OA from Investigate A forms an angle of 45° with the x-axis. Reflect point A in the y-axis to obtain point C, and join point C to the origin. What is the measure of the angle between OC and the negative x-axis? This angle is referred to as a **reference angle**. The significance of knowing the reference angle is that the values of the six trigonometric functions for any angle greater than 90° are the same as the corresponding values for its reference angle—with a possible change in sign.



Technology Tip

In *The Geometer's Sketchpad®*, you can use the **Transform** menu to rotate the terminal arm about the origin to form an exact angle. For help with *The Geometer's Sketchpad®*, refer to the Technology Appendix on pages 496 to 516.

reference angle

- the acute angle between the terminal arm and the x-axis of an angle in standard position

2. What are the coordinates of point C?

3. Look back at the relationships found in step 7d) of Investigate A. Use the coordinates of point C to determine cos 135° and sin 135°.

4. **Reflect** How can you use the coordinates of point C to represent tan 135°?

5. **a)** Find tan 135°.

   b) Find the slope of the terminal arm OC. How does this slope relate to tan 135°?

6. Use a calculator to compare the trigonometric ratios that you found in steps 3 and 5 with the calculator values of sin 135°, cos 135°, and tan 135°.

Connections

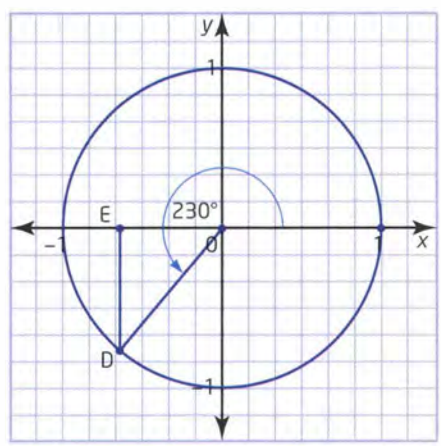
How does a calculator find trigonometric ratios for angles? The sine of an angle x can be expressed as a series of values involving powers of x and various coefficients. The calculator adds the terms of the series to find the sine, using enough terms to obtain the desired accuracy. Similar series exist for cosine and tangent. These are known as Taylor series. Visit the McGraw-Hill Ryerson Web site and follow the links to learn more about Taylor series.

Investigate C

How can you use a unit circle to find the trigonometric ratios for any angle?

1. Starting from the x-axis, plot a point D such that the line OD forms an angle of 230° measured counterclockwise from the positive x-axis.

2. To find the coordinates of point D, draw a vertical line to meet the x-axis at point E. The angle EOD is the reference angle for the angle in standard position.



3. Measure OE and ED. Use the measurements to determine the coordinates of point D. Record the coordinates of point D.

4. Find sin 230°, cos 230°, and tan 230°.

5. **Reflect** Explain why sin 230° and cos 230° are negative, but tan 230° is positive.

Example 1

Determine the Primary Trigonometric Ratios for 30° and 60°

Another triangle whose side lengths and angles are known is an equilateral triangle.

a) Draw an equilateral triangle with side length 2 units such that the base is horizontal. From the top vertex, draw a vertical line to form two congruent right triangles.

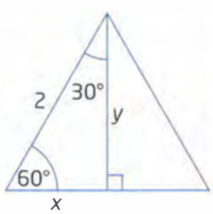
b) What are the measures of the angles in these triangles?

c) Find the side lengths of the base and height of one of these triangles. Leave answers in radical form where appropriate.

d) Use the side lengths and angle measures to find exact values of the trigonometric ratios for 30° and 60°.

Solution

a)



b) The altitude of the triangle bisects the top angle into two 30° angles. The angles in each triangle are 30°, 60°, and 90°.

c) Let x represent the base of one of the triangles. x is half the length of one side, or 1 unit.

Let y represent the height of the triangle. Since the triangle is a right triangle, the Pythagorean theorem applies.

$$x^2 + y^2 = 2^2$$
$$1^2 + y^2 = 4$$
$$y^2 = 3$$
$$y = \sqrt{3}$$

Since lengths are positive, discard the negative value for y.

d) Since the adjacent side to the 60° angle measures 1 unit, the opposite side measures $\sqrt{3}$ units, and the hypotenuse measures 2 units,

$$\sin 60° = \frac{\sqrt{3}}{2}, \cos 60° = \frac{1}{2}, \text{ and}$$

$$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Similarly,

$$\sin 30° = \frac{1}{2}, \cos 30° = \frac{\sqrt{3}}{2}, \text{ and } \tan 30° = \frac{1}{\sqrt{3}}$$

Technology Tip

A computer algebra system (CAS) can display either exact or approximate values of the trigonometric ratios for special angles. You will learn more about how to use a CAS in the Use Technology section following Section 4.2.

Example 2

Trigonometric Ratios for 0°, 90°, 180°, and 270°

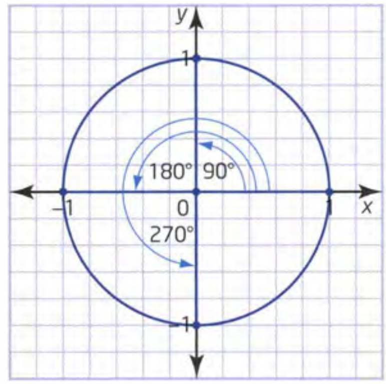
Use a unit circle to find exact values of the trigonometric ratios for 0°, 90°, 180°, and 270°.

Connections

θ is the lowercase form of the Greek letter theta. Greek letters are often used to represent variable quantities in science and mathematics. Other letters often used for angles are α, β, and ϕ (alpha, beta, and phi).

Solution

Choose a point on the terminal arm of each angle in the unit circle.



For an angle of 0°, the required point is on the x-axis at (1, 0).

$\sin \theta = y \qquad \cos \theta = x \qquad \tan \theta = \dfrac{y}{x}$

$\sin 0° = 0 \qquad \cos 0° = 1 \qquad \tan 0° = \dfrac{0}{1}$

$\qquad\qquad\qquad\qquad\qquad\qquad = 0$

For an angle of 90°, the required point is on the y-axis at (0, 1).

$\sin \theta = y \qquad \cos \theta = x \qquad \tan \theta = \dfrac{y}{x}$

$\sin 90° = 1 \qquad \cos 90° = 0 \qquad \tan 90° = \dfrac{1}{0}$ Division by 0 is undefined.

tan 90° is undefined.

For an angle of 180°, the required point is on the x-axis at (−1, 0).

$\sin \theta = y \qquad \cos \theta = x \qquad \tan \theta = \dfrac{y}{x}$

$\sin 180° = 0 \qquad \cos 180° = -1 \qquad \tan 180° = \dfrac{0}{-1}$

$\tan 180° = 0$

For an angle of 270°, the required point is on the y-axis at (0, −1).

$\sin \theta = y \qquad \cos \theta = x \qquad \tan \theta = \dfrac{y}{x}$

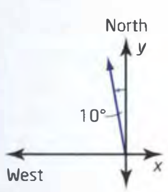
$\sin 270° = -1 \qquad \cos 270° = 0 \qquad \tan 270° = \dfrac{-1}{0}$

tan 270° is undefined.

Example 3

Connections

To represent 10° west of north, start from north and turn 10° toward the west.



Apply Trigonometric Ratios

An air traffic controller observes that a ValuAir flight is 20 km due east of the control tower, while a First Class Air flight is 25 km in a direction 10° west of north from the control tower.

a) What is the angle of separation of the two aircraft as seen from the tower?

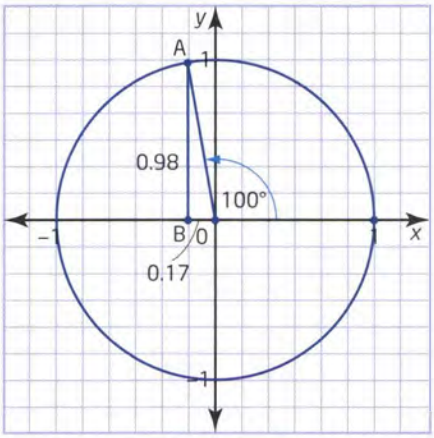
b) Construct a unit circle to determine the cosine of the angle in part a).

Solution

a) From east to north is an angle of 90°. A further angle of 10° results in an angle of separation of 100°.

b) Use geometry software or grid paper to construct a unit circle, and plot the point A required for an angle of 100°. Draw a vertical line to meet the x-axis at point B to complete the triangle. Measure the sides of the triangle to determine the coordinates of point A.

Reminder: If you are using grid paper, apply an appropriate scale factor.



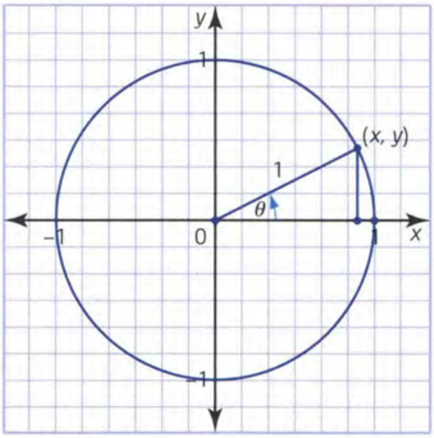
The coordinates of point A are approximately $(-0.17, 0.98)$. Therefore, $\cos 100° \doteq -0.17$.

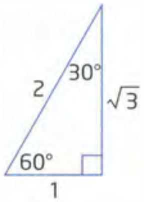
Technology Tip

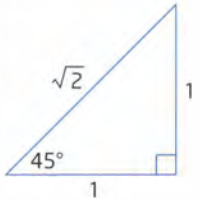
You can obtain accuracy to about one decimal place when using grid paper. Using geometry software generally allows more accuracy. With *The Geometer's Sketchpad®*, for example, you can set the **Preferences** under the **Edit** menu to measure up to five decimal places.

Key Concepts

- Using a unit circle is one way to find the trigonometric ratios for angles greater than 90°.
- Any point on a unit circle can be joined to the origin to form the terminal arm of an angle. The angle θ is measured starting from the initial arm along the positive x-axis, proceeding counterclockwise to the terminal arm.
- The coordinates of the point (x, y) on a unit circle are related to θ such that $x = \cos \theta$ and $y = \sin \theta$.
- $\tan \theta = \frac{y}{x}$
- Exact trigonometric ratios for special angles can be determined using special triangles.







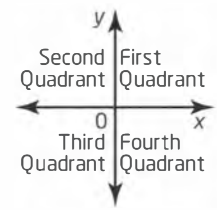
- The exact trigonometric ratios for 45° are $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, and $\tan 45° = 1$.

Communicate Your Understanding

C1 As the terminal arm moves counterclockwise from the positive x-axis around the circle, trace what happens to the sign of $\cos \theta$ as you move from 0° to 360°. Explain why this happens in terms of coordinates. Then, do the same trace for $\sin \theta$. Finally, trace what happens for $\tan \theta$.

C2 Some trigonometric ratios for certain angles are undefined. Give two examples. Explain why they are undefined.

C3 Which trigonometric ratios are positive in the fourth quadrant? Which are negative? Explain why.



A Practise

For help with questions 1 to 4, see Investigate A and Examples 1 and 2.

1. Compare the exact values of the trigonometric ratios for 30° and 60° to the trigonometric ratios calculated by a calculator.

2. Compare the exact values of the trigonometric ratios from Example 2 to the trigonometric ratios calculated by a calculator.

3. a) Use a unit circle to represent an angle of 30°. Draw a triangle and use it to write the three primary trigonometric ratios in exact form for 30°.

b) Use a unit circle to represent an angle of 60°. Draw a triangle and use it to determine the exact primary trigonometric ratios for 60°.

4. In a table, summarize the exact trigonometric ratios for the angles 0°, 30°, 45°, 60°, and 90°. Add and complete a column for the ratios as given by a calculator, correct to 4 decimal places.

For help with questions 5 and 6, see Investigate B.

5. a) When using a unit circle to find trigonometric ratios for 135°, a reference angle of 45° is used. What reference angle should you use to find the trigonometric ratios for 120°?

b) Construct a unit circle to find the exact values of the three primary trigonometric ratios for 120°.

6. Construct a unit circle to find the exact values of the three primary trigonometric ratios for 315°.

For help with questions 7 and 8, see Investigate C and Example 3.

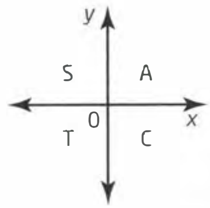
7. Use a unit circle to find the approximate primary trigonometric ratios for 40°. Measure any side lengths needed. Compare your answers to those generated by calculator, correct to 4 decimal places.

8. Use a unit circle to find the approximate primary trigonometric ratios for 310°. Measure any side lengths needed. Compare your answers to those generated by calculator.

9. Create a table to summarize the exact values of the primary trigonometric ratios for 0°, 90°, 180°, 270°, and 360°.

10. **a)** Which trigonometric ratios are positive for angles in the first quadrant? second quadrant? third quadrant? fourth quadrant?

b) One way to remember the signs of trigonometric ratios is called the CAST rule, as shown (the letters spell CAST, moving counterclockwise, beginning in the fourth quadrant. What do the letters in each quadrant stand for?



B Connect and Apply

11. A pine tree that is 10 m tall is damaged in a windstorm such that it leans sideways to make an angle of 60° with the ground.

a) Represent this situation with a diagram.

b) Find an exact expression for the length of the shadow of the tree when the sun is directly overhead.

12. A sailboat is 12 km north of a lighthouse. A motor cruiser is 12 km east of the same lighthouse.

a) Use trigonometry to find an exact expression for the distance between the two boats.

b) Check your answer using another method.

13. Tall structures are sometimes stabilized with ropes or cables attached to the ground. These stabilizers are known as guy wires. A flagpole is stabilized by two guy wires attached to the top of the pole. On one side, a 25-m-long wire makes an angle of 60° with the ground. The sine of the angle formed by the second wire and the ground equals the cosine of the angle of the first guy wire.

a) Represent this situation with a diagram.

b) Determine the length, to the nearest tenth of a metre, of the second guy wire without calculating any angles.

c) Why is it not necessary to find the angle that the second guy wire makes with the ground to solve the problem?

d) Determine the angle made by the second guy wire with the ground.

14. **Use Technology** Use a calculator for this question.

a) Copy and complete the table.

θ	$\sin\theta$	Quadrant	Sign
30°			
150°			
210°			
330°			

b) Relate the sign of $\sin\theta$ with the quadrant. Are the signs as you expected?

c) Now, work backward. Find the angle that satisfies

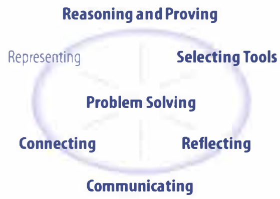
i) $\sin\theta = 0.5$ **ii)** $\sin\theta = -0.5$

The calculator will give you only one answer for each, despite the fact that there are two angles between 0° and 360° that have each value. Note also that the calculator expressed the second angle as −30° and not 330°. For angles between 180° and 360°, the calculator starts at the positive x-axis and proceeds in a clockwise direction. Angles measured in this direction are defined as negative.

d) Construct a similar table for $\cos\theta$, using the angles 60°, 120°, 240°, and 300°. Then, start with the cosine, and find the angle for both positive and negative values. Note the answers provided by the calculator.

e) Select suitable angles to test $\tan\theta$. Note how the calculator presents the angles when you work backward.

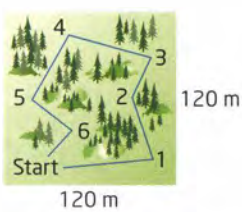
15. a) Pose a real-world problem that can be solved using trigonometric ratios for special angles without using a calculator. Solve your problem to ensure that there is a solution.



b) Trade your problem with a classmate. Solve each other's problem.

c) Trade solutions. Judge the mathematical correctness of the solution. Look for proper form and careful use of mathematical symbols.

16. Chapter Problem You are about to begin Chantal's trigonometric orienteering course. Prepare a set of axes on grid paper to make a map of your progress, labelling the Start position at the origin. All coordinates for the course will be positive. You may not use a calculator to help you until you reach checkpoint #3. Use the instructions below to calculate the direction and distance to checkpoint #1. Draw this leg on your map and label the angle and distance. Choose and record a suitable scale.



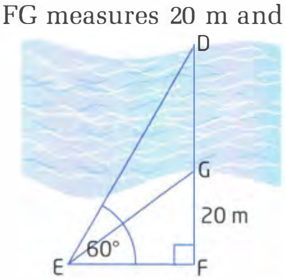
Checkpoint #1

Direction: North of east

Use the angle in the first quadrant that has a sine of $\frac{1}{2}$.

Distance: The result of evaluating $-40(\cos 150° \times \tan 135° \times \sin 300°)$

17. Stefan has set up a right $\triangle EFG$ on one side of a river such that FG measures 20 m and $\angle DEF$ measures 60°. EG bisects $\angle DEF$. Without using a calculator, determine the width, DG, of the river.



✓ Achievement Check

18. In $\triangle PQR$, $\angle Q = 90°$, $\angle P = 60°$, and $\angle R = 30°$. PR = 1 unit. Extend side QR to T such that PR = RT. Join PT.

a) Draw a diagram to represent this situation.

b) Calculate the exact measure of $\angle T$. Justify your answer.

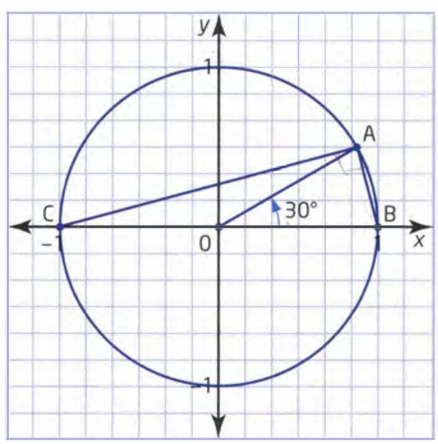
c) What lengths do you need to know to find tan T? Explain.

d) Determine the exact value of the unknown lengths in part c). Do not use a calculator. Justify your reasoning.

e) Find the exact value for tan T.

C Extend

19. Consider an angle of 30° in standard position on a unit circle. Join A to B and to C as shown. Show that the lengths of the sides of $\triangle ABC$ satisfy the Pythagorean theorem and that $\angle CAB = 90°$.



20. Refer to question 19. Let $\angle AOB$ be any angle in the first quadrant, and let the coordinates of A be (x, y). Show that the sides of $\triangle ABC$ satisfy the Pythagorean theorem and that $\angle CAB = 90°$.

21. The town of Dainfleet is planning to build a municipal swimming pool in the shape of a regular hexagon. The projected cost of the pool depends on its area. Without using a calculator, show that the side length, ℓ, of the pool is related to its area, A, by the formula

$$\ell = \sqrt{\frac{2A}{3\sqrt{3}}}$$

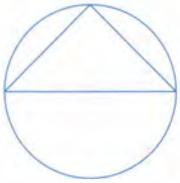
22. **Math Contest** An equilateral triangle has a height of $3\sqrt{3}$ cm. Its perimeter is

A 12 cm **B** 18 cm
C 6 cm **D** $9\sqrt{3}$ cm
E $18 + 3\sqrt{3}$ cm

23. **Math Contest** The parallel sides of a trapezoid have lengths of 7 cm and 15 cm. The two lower base angles are 30° and 60°. The area of the trapezoid is

A $22\sqrt{3}$ cm^2 **B** $14\sqrt{3}$ cm^2
C 22 cm^2 **D** $30\sqrt{3}$ cm^2
E $8\sqrt{3}$ cm^2

24. **Math Contest** A circle has an inscribed isosceles triangle with one side as the diameter. What is the ratio of the area of the triangle to the area of the circle?



A π:2 **B** 1:2π
C 1:π **D** 2:π
E 1:4

Career Connection

Building codes exist to ensure that structures are properly built with suitable building materials. As a building inspector, Christopher visits sites during all phases of the construction to make sure regulations are being followed. Initially, a site must be able to support the type of building planned and blueprints have to be checked for the structure's stability. As the building goes up, Christopher checks the stability, wiring, and safety features. A good knowledge of trigonometry is important when buildings are meant to last and when people's safety is at stake. To prepare for his job academically, Christopher completed a three-year diploma in architecture and construction engineering technology at Conestoga College.

4.2

Co-terminal and Related Angles

Trigonometric ratios can be used to model quantities such as the alternating-current electricity that powers electric motors, among other electrical devices. When solving problems involving trigonometric quantities, there is almost always more than one solution—sometimes an infinite number of solutions. It is important to find all possible solutions and then select which solutions are appropriate for the problem.

In this section, you will learn how to identify different angles that have the same trigonometric ratio, as well as learn how they are related.

Tools

- grid paper

Investigate

How can you find different angles with the same trigonometric ratios?

1. a) In the first quadrant, 45° has a sine of $\frac{1}{\sqrt{2}}$. Explain why.

b) Draw a unit circle. Add a terminal arm to represent 45°. Label the angle.

2. Label the x- and y-coordinates of the point where the terminal arm intersects the unit circle.

3. Find another point on the unit circle that has the same y-coordinate. What is the x-coordinate of this point?

4. a) Draw a second angle in standard position with its terminal arm through the point you found in step 3.

b) What is the measure of this second angle?

5. Which of the trigonometric ratios for the two angles are the same? Which are different?

6. Reflect For any chosen point on the unit circle, how many other points will have the same y-coordinate? Explain why.

7. a) Which angle in the first quadrant has a cosine of $\frac{1}{2}$?

b) Draw a unit circle and plot the point where the terminal arm of the angle intersects the circle. Then, draw the terminal arm of this angle.

8. Label the x- and y-coordinates of the point in step 7.

9. Find another point on the unit circle that has the same x-coordinate. What is the y-coordinate of this point?

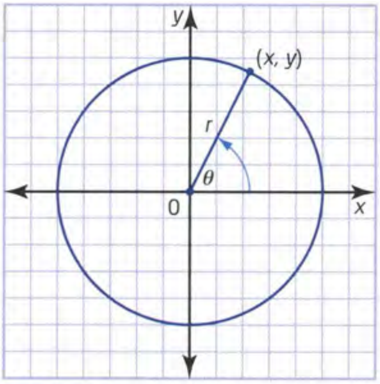
10. a) Draw a second angle in standard position with its terminal arm through the point you found in step 9.

    b) What is the measure of this second angle?

11. Which of the trigonometric ratios for the two angles are the same? Which are different?

12. **Reflect** For any chosen point on the unit circle, how many other points will have the same x-coordinate? Explain how they are related.

13. Suppose that the circle being used is not a unit circle, but rather a circle with radius r. The coordinates (x, y) of a point on the circle are no longer the cosine and sine of the angle, but are related to the cosine and sine. Study the circle shown. Write expressions for $\sin \theta$, $\cos \theta$, and $\tan \theta$ in terms of x, y, and r.



Example 1

Find Primary Trigonometric Ratios and Angles Using Any Circle

a) Given that $\sin A = \frac{3}{5}$ and that $\angle A$ lies in the first quadrant, determine exact values for $\cos A$ and $\tan A$.

b) Determine the primary trigonometric ratios for another angle between 0° and 360° that has the same sine value.

c) Draw a diagram showing the locations of the two angles. How are the two angles related?

d) Use a calculator to help you find the two angles, to the nearest degree.

Solution

a) Since $\sin A = \frac{3}{5}$, possible values of y and r are 3 and 5. Therefore, let $y = 3$ and $r = 5$.

$x^2 + y^2 = r^2$ — Use the Pythagorean theorem to find the value of x.

$x^2 + 3^2 = 5^2$

$x^2 = 16$

$x = \pm 4$

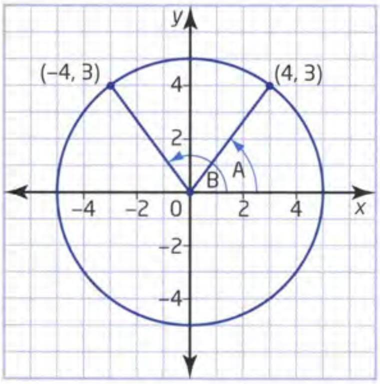
Since $\angle A$ lies in the first quadrant, $x = 4$.

$$\cos A = \frac{x}{r} \qquad \tan A = \frac{y}{x}$$
$$= \frac{4}{5} \qquad = \frac{3}{4}$$

b) The sine ratio is positive in the first and second quadrants. The point that defines an angle with the same sine is $(-4, 3)$. Let $\angle B$ represent the angle.

$$\sin B = \frac{y}{r} \qquad \cos B = \frac{x}{r} \qquad \tan B = \frac{y}{x}$$
$$= \frac{3}{5} \qquad = \frac{-4}{5} \qquad = \frac{3}{-4}$$
$$= -\frac{4}{5} \qquad = -\frac{3}{4}$$

c) From the diagram, $\angle B = 180° - \angle A$.



d) Ensure that the calculator is set to degree measure.

$$\sin A = \frac{3}{5}$$
$$\angle A = \sin^{-1}\left(\frac{3}{5}\right)$$
$$\doteq 37°$$

$$\angle B = 180° - \angle A$$
$$\doteq 180° - 37°$$
$$= 143°$$

Technology Tip

Keystrokes vary. On some calculators, press 2nd SIN to access the $\sin^{-1}$ function. Type 3 ÷ ENTER 5, and then press ENTER.

Example 2

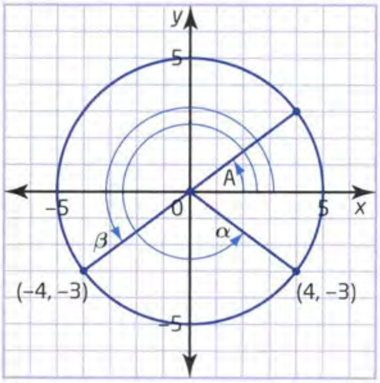
Relations for the Cosine and Tangent Ratios

a) Determine another angle between 0° and 360° that has the same cosine as $\angle A$ in Example 1. What is the relationship between this angle and $\angle A$?

b) Determine another angle between 0° and 360° that has the same tangent as $\angle A$ in Example 1. What is the relationship between this angle and $\angle A$?

Solution

a) Determine another angle, α, such that $\cos \alpha = \frac{4}{5}$. Since the cosine ratio is positive in the fourth quadrant, the coordinates of the required point are $(4, -3)$.



From the diagram,

$$\angle \alpha = 360° - \angle A$$
$$\doteq 360° - 37°$$
$$= 323°$$

b) Determine another angle, β, such that $\tan \beta = \frac{3}{4}$. Since the tangent ratio is positive in the third quadrant, the coordinates of the required point are $(-4, -3)$.

From the diagram,

$$\begin{aligned}\beta &= 180° + \angle A\\ &\doteq 180° + 37°\\ &= 217°\end{aligned}$$

Example 3

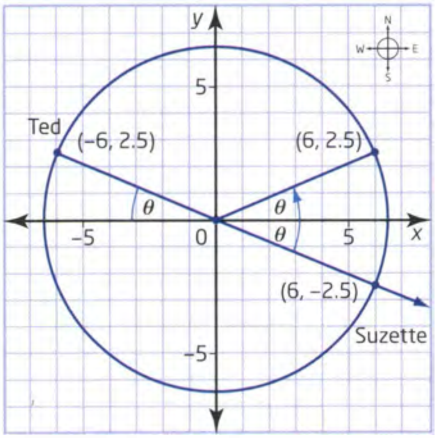
Solve a Map Problem

The city plan of Port Foghorn uses a Cartesian grid, with each grid mark representing a distance of 1 km. The plan places City Hall at the origin of the grid. Ted's house is at grid point $(-6, 2.5)$.

a) The angle of rotation for Suzette's house has the same tangent ratio as the angle of rotation for Ted's house. Where is Suzette's house?

b) Find the angles in standard position if the lines drawn from City Hall to each of the two houses are terminal arms. Round your answers to the nearest degree.

Solution



a) Ted's house is in the second quadrant, so the tangent is negative. If Suzette's house has the same tangent ratio, it must be in the fourth quadrant at $(6, -2.5)$.

b) Ted's house is at a reflection of $(6, 2.5)$ in the y-axis. Suzette's house a reflection of $(6, 2.5)$ in the x-axis. The reference angle associated with these points is θ, where

$$\begin{aligned}\tan \theta &= \frac{2.5}{6}\\ \theta &\doteq 23°\end{aligned}$$

Ted's house is at approximately $180° - 23° = 157°$.

Suzette's house is at approximately $360° - 23° = 337°$.

Example 4

Co-terminal Angles

co-terminal angles

- angles in standard position that have the same terminal arm

a) Find three other positive angles that have the same terminal arm as 30°.

b) Find three negative angles that have the same terminal arm as 30°.

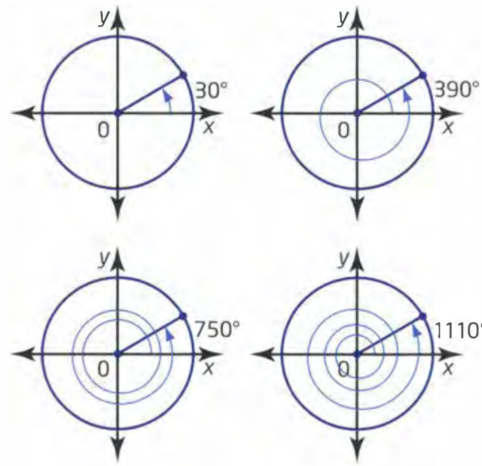
Solution

a) Turn in a counterclockwise direction from 30°. If you continue for 360°, you arrive back at the same terminal arm. $30° + 360° = 390°$, so 30° and 390° are co-terminal angles.

Two other positive angles that are co-terminal with 30° are as follows:

$30° + 2(360°) = 750°$

$30° + 3(360°) = 1110°$



Connections

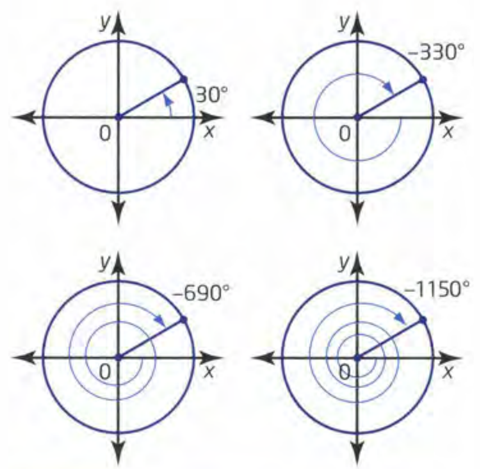
A positive angle is generated when the terminal arm moves in a counterclockwise direction. A negative angle is generated when the terminal arm moves in a clockwise direction.

b) Starting at the positive x-axis and proceeding in a clockwise direction defines a negative angle. You will reach the terminal arm of 30° after a rotation of −330°. Hence, −330° is co-terminal with 30°.

Two other negative angles that are co-terminal with 30° are as follows:

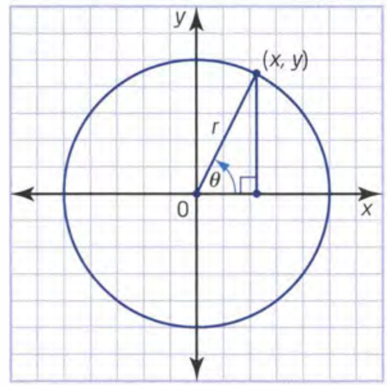
$30° - 2(360°) = -690°$

$30° - 3(360°) = -1050°$



Key Concepts

- The primary trigonometric ratios for the angle θ in standard position that has a point (x, y) on its terminal arm can be calculated as
  $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, and $\tan \theta = \frac{y}{x}$, where $r = \sqrt{x^2 + y^2}$.
- For any given sine ratio, two distinct angles between 0° and 360° have this sine ratio.
- For any given cosine ratio, two distinct angles between 0° and 360° have this cosine ratio.
- For any given tangent ratio, two distinct angles between 0° and 360° have this tangent ratio.
- Pairs of related angles can be found using the coordinates of the endpoints of their terminal arms. Use a reference angle in the first quadrant.
- Co-terminal angles are angles with the same terminal arm. They can be positive or negative.



Communicate Your Understanding

C1 Explain why there are exactly two angles between 0° and 360° that have a given sine ratio.

C2 The terminal arm of an angle is in the first quadrant. What kind of reflection will give the terminal arm of an angle that has the same sine ratio? the same cosine ratio? the same tangent ratio?

C3 How many co-terminal angles can you find for an angle of 30°? Explain.

C4 How do the trigonometric ratios for 30° relate to the trigonometric ratios for 390°? Is this true for all co-terminal angles? Explain.

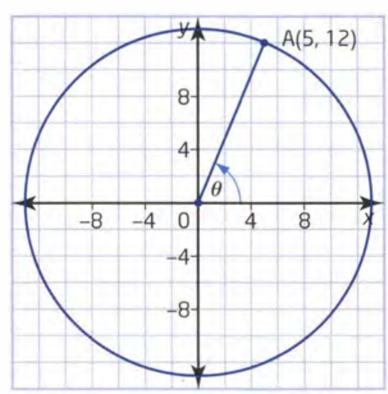
A Practise

Note: Unless otherwise specified, assume that all angles are between 0° and 360°.

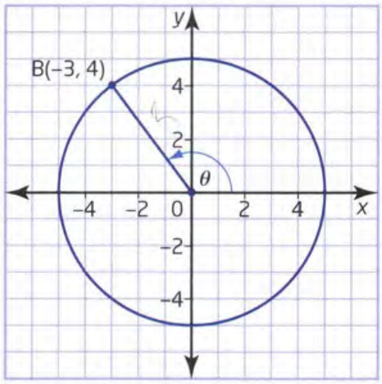
For help with questions 1 and 2, refer to Example 1.

1. The coordinates of a point on the terminal arm of an angle θ are shown. Determine the exact primary trigonometric ratios for θ.

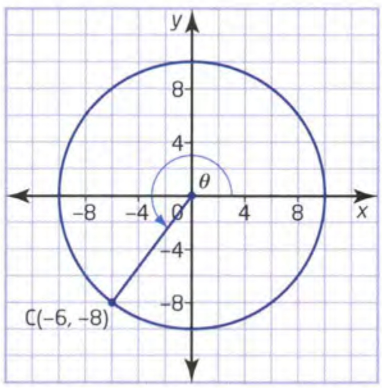
   a) A(5, 12)



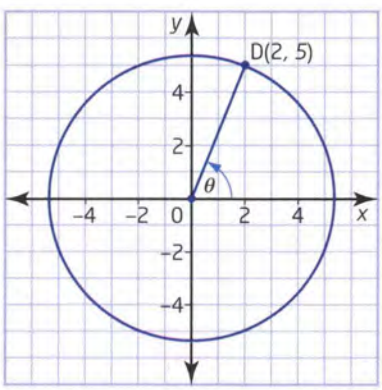
b) B(−3, 4)



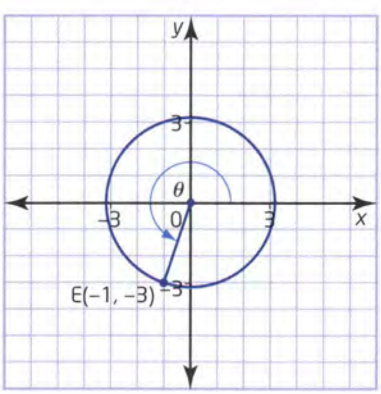
c) C(−6, −8)



d) D(2, 5)



e) E(−1, −3)



2. The coordinates of a point on the terminal arm of an angle θ are given. Determine the exact primary trigonometric ratios for θ.

a) G(−8, 6) **b)** H(3, −4)
c) I(−15, −8) **d)** J(3, −5)
e) K(1, 2) **f)** L(6, −2)

For help with questions 3 and 4, refer to Examples 1 and 2.

3. One of the primary trigonometric ratios for an angle is given, as well as the quadrant in which the terminal arm lies. Find the other two primary trigonometric ratios.

a) $\sin A = \frac{8}{17}$, first quadrant

b) $\cos B = \frac{3}{5}$, fourth quadrant

c) $\tan C = -\frac{5}{12}$, second quadrant

d) $\sin D = -\frac{2}{3}$, third quadrant

e) $\cos E = -\frac{5}{6}$, second quadrant

f) $\tan F = \frac{12}{7}$, first quadrant

4. Determine another angle that has the same trigonometric ratio as each given angle. Draw a sketch with both angles labelled.

a) cos 45° **b)** sin 150°
c) tan 300° **d)** sin 100°
e) cos 230° **f)** tan 350°

For help with questions 5 and 6, refer to Example 4.

5. a) Determine any three positive angles that are co-terminal with 120°.

b) Determine any three negative angles that are co-terminal with 330°.

6. Determine the exact primary trigonometric ratios for each angle. You may wish to use a unit circle to help you.

a) $\angle A = -45°$ **b)** $\angle B = -120°$
c) $\angle C = 540°$ **d)** $\angle D = -315°$
e) $\angle E = 420°$ **f)** $\angle F = -270°$

B Connect and Apply

7. Without using a calculator, determine two angles between 0° and 360° that have a cosine of $-\frac{\sqrt{3}}{2}$.

8. Two angles between 0° and 360° have a tangent of -1. Without using a calculator, determine the angles.

9. The cosine of each of two angles between 0° and 360° is $\frac{1}{\sqrt{2}}$. Without using a calculator, determine the angles.

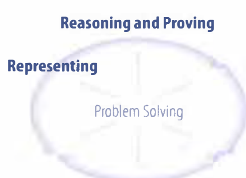
10. Two angles between 0° and 360° have a tangent that is undefined. What are the angles? Why is the tangent undefined for each of these?

11. The point P(-4, 9) is on the terminal arm of ∠A.

a) Determine the primary trigonometric ratios for ∠A and ∠B, such that ∠B has the same sine as ∠A.

b) Use a calculator and a diagram to determine the measures of ∠A and ∠B, to the nearest degree.

12. The point R(-3, -5) is on the terminal arm of ∠E.



a) Determine the primary trigonometric ratios for ∠E and ∠F such that ∠F has the same tangent as ∠E.

b) Use a calculator and a diagram to determine the measures of ∠E and ∠F, to the nearest degree.

13. Use Technology Open *The Geometer's Sketchpad®*.

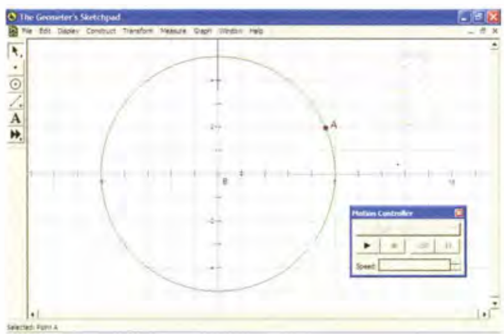
a) Draw a circle with a radius of 5 grid units. Plot a point A on the circle in the first quadrant.

b) Measure the coordinate distance between point A and the origin. Change the label to *r*. Measure the *x*- and *y*-coordinates of point A.

c) Construct formulas to calculate the sine, cosine, and tangent of ∠A defined by terminal arm OA, using the measures of *x*, *y*, and *r*.

d) Right-click on point A and select **Animate Point**. Observe the values of the trigonometric ratios as A moves around the circle. Pause the animation at selected points at which you know the ratios, and compare the values on the screen to your knowledge.

e) Try the controls in the **Motion Controller**. Determine what each of them does. You can also control the animation from the **Display** menu.



f) As the sine increases in the first quadrant, what happens to the cosine? What happens to the tangent? What happens in the second quadrant?

14. An acute angle θ has the point A(p, q) on its terminal arm.

a) Find an expression for the distance OA in terms of p and q.

b) Write exact expressions for the primary trigonometric ratios for θ.

c) Locate the angle $90° - \theta$. Sketch the terminal arm for this angle. Determine the coordinates of a point B on the terminal arm of $90° - \theta$ in terms of p and q.

d) Write exact expressions for the primary trigonometric ratios for $90° - \theta$.

e) Compare the expressions that you found in parts b) and d).

15. Chapter Problem For the second leg of your orienteering course, calculate the direction and distance. Carefully draw the leg on your map, and label all distances and angles. Note: You may not use a calculator.

Direction: Face south. Turn left through an angle with a cosine of 0 and a sine of 1.

Distance: Find two angles between 0° and 360° with a sine of $-\frac{1}{\sqrt{2}}$. Subtract the smaller angle from the larger angle. Find the sine of the resulting angle. Multiply by 40 to obtain the distance.

✔ Achievement Check

16. Consider $\angle C$ such that $\sin C = \frac{7}{25}$.

a) What are the possible quadrants in which $\angle C$ may lie?

b) If you know that $\cos C$ is negative, how does your answer to part a) change?

c) Sketch a diagram to represent $\angle C$ in standard position, given that the condition in part b) is true.

d) Find the coordinates of a point P on the terminal arm of $\angle C$.

e) Write exact expressions for the other two primary trigonometric ratios for $\angle C$.

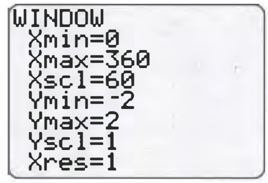
C Extend

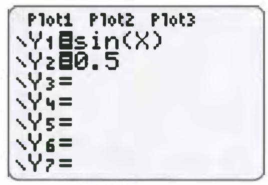
17. The side length of a rhombus is s. One of its diagonals has the same length. Determine an exact expression for the length of the other diagonal.

18. A regular octagon has side length ℓ. A line segment is drawn joining two of its vertices to form a triangle and an irregular heptagon. Determine an expression for the exact length of this line segment.

19. Use Technology How can you use a graphing calculator to check answers when determining two angles with the same trigonometric ratio? For example, suppose you are asked to determine the values of θ such that $\sin \theta = 0.5$.

a) Set the window variables as shown. Enter the left side as **Y1** and the right side as **Y2**.





b) Press GRAPH. Press 2nd [CALC]. Select **5:intersect**. Use the **Intersect** operation twice to determine the values that satisfy the condition.

20. Math Contest A clock face is in the form of a coordinate system with the origin at the centre of the face. The minute hand passes through the point $(1, \sqrt{3})$. The hour hand is somewhere between the 4 and the 5. The time is

A 4:05 **B** 4:10 **C** 4:15
D 4:07 **E** 4:12

21. Math Contest Given A(0, 0), B$(3, 3\sqrt{3})$, and C$(2\sqrt{3}, 2)$, what is the area, in square units, of $\triangle ABC$?

A 12 **B** $\frac{5\sqrt{3}}{2}$ **C** 6
D $3\sqrt{3}$ **E** 3

22. Math Contest If $4x = 7 - 5y$, what is the value of $12x + 15y$?

A 3 **B** 7 **C** 21
D 27 **E** 31

Use a Computer Algebra System to Find Exact Trigonometric Ratios and Angles

A computer algebra system (CAS) found on calculators such as the TI-Nspire™ CAS graphing calculator can display exact values of the trigonometric ratios for the special angles you learned about in Section 4.1. With its equation-solving power, a CAS can also find all angles that satisfy a given function, unlike an ordinary calculator that gives only one answer.

This activity is written for the CAS on a TI-Nspire™ CAS graphing calculator. Other systems can be used.

Tools

- TI-Nspire™ CAS graphing calculator

A: Display Exact Values

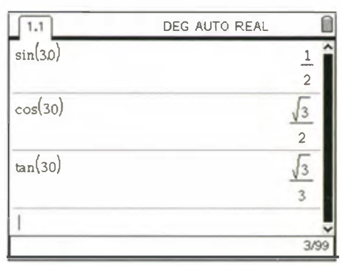
1. Turn on the TI-Nspire™ CAS graphing calculator.

- Press and select **8:System Info**.
- Select **2:System Settings....**
- Use the tab key to scroll down to **Angle**, and ensure that it is set to **Degree**.

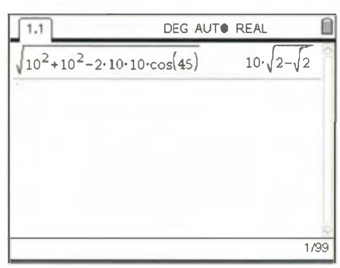
Continue on to **Auto or Approx** and ensure that it is set to **Auto**. Continue down to **OK**, and press twice.

2. Press and select **6:New Document**. Select **1:Add Calculator**.

3. Enter sin(30) and press enter. Notice how the result is displayed. Investigate cos(30) and tan(30). Do you see anything unusual about the display? Explain why this is equivalent to the value that you determined using a unit circle.

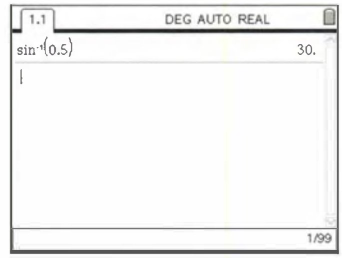


4. You can use a CAS to obtain exact answers to trigonometric problems. For example, in $\triangle ABC$, $b = 10$ cm, $c = 10$ cm, and $\angle A = 45°$. Find the exact measure of c using the cosine law.

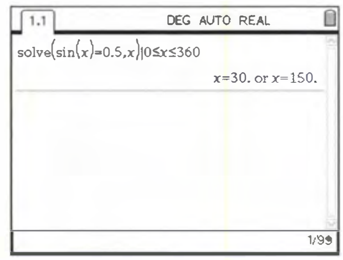


B: Find Angles

1. When you use an ordinary calculator to find an angle θ such that $\sin \theta = 0.5$, you only get one answer. If you use the **sin⁻¹** operation on the TI-Nspire™ CAS graphing calculator, the same thing happens. Try it.



2. Now, use the equation-solving power of the CAS.
   - Press menu—
   - Select **3:Algebra**.
   - Select **1:solve**.
   - Type sin(x) = 0.5, x) | 0≤x≤360.
   - Press enter.



Notice that the CAS has found the angles between 0° and 360° that have a sine of 0.5. The domain of the solution needs to be restricted to answers between 0° and 360°. This is done with the expression following the "with" operator (|).

Technology Tip

To obtain the ≤ sign, press ctrl and then <.

3. Try the same thing with the equation $\cos \theta = 0.5$. Before using the CAS, predict the answers. Then, use the CAS to see if your predictions are correct.

Notice that, unlike an ordinary calculator, the CAS does not use a negative angle for the fourth quadrant.

Technology Tip

You can force the calculator to give you an approximate answer without resetting the mode by pressing ctrl before pressing enter.

4. The CAS can also solve more complex trigonometric equations. Are there any angles between 0° and 360° that satisfy the equation $\sin \theta + \cos \theta = 1$? Write down your prediction, and justify why you think you are correct. Then, use the CAS to check your prediction.

5. Write a trigonometric equation that you think has a solution. Trade equations with a classmate and solve each other's equation. Discuss the results.

4.3

Reciprocal Trigonometric Ratios

The primary trigonometric ratios have many uses, from solving triangles in surveying and navigation to working with music theory and electronics, especially in connection with music synthesizers. The **reciprocal** trigonometric ratios are related to the primary trigonometric ratios and have many applications, including radar antenna design.

In this section, you will learn what the reciprocal trigonometric ratios are, how to calculate them, and how they behave.

reciprocals
- two expressions that have a product of 1 (e.g., 4 and $\frac{1}{4}$ or x and $\frac{1}{x}$)

Investigate

How can you use a calculator to calculate the reciprocal trigonometric ratios?

1. The reciprocal of the sine ratio is called the **cosecant ratio**. It is defined in a right triangle as

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

a) What is the exact value of sin 30°? Predict the exact value of csc 30°.

b) Use a calculator to determine sin 30°. Then, press the reciprocal key. The key is usually labelled $\frac{1}{x}$ or x^{-1}. Does the result confirm your prediction?

cosecant ratio
- reciprocal of the sine ratio: $\csc\theta = \frac{1}{\sin\theta}$

2. The reciprocal of the cosine ratio is called the **secant ratio**. It is defined in a right triangle as

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

a) What is the exact value of cos 60°? Predict the exact value of sec 60°.

b) Use the calculator to determine cos 60°. Then, press the reciprocal key. Does the result confirm your prediction?

secant ratio
- reciprocal of the cosine ratio: $\sec\theta = \frac{1}{\cos\theta}$

3. The reciprocal of the tangent ratio is called the **cotangent** ratio. It is defined in a right triangle as

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

a) What is the exact value of tan 30°? Predict the exact value of cot 30°.

b) Use the calculator to determine tan 30°. Then, press the reciprocal key. Does the result agree with your prediction?

cotangent ratio
- reciprocal of the tangent ratio: $\cot\theta = \frac{1}{\tan\theta}$

4. Use your knowledge of special angles to copy and complete the table. Use exact values.

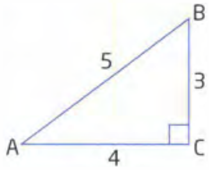
θ	$\sin\theta$	$\csc\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
0°						
30°						
45°						
60°						
90°						

5. **Reflect** As the measure of angle θ increases, how does the value of $\sin\theta$ compare to the value of $\csc\theta$? How does the value of $\cos\theta$ compare to the value of $\sec\theta$? How does the value of $\tan\theta$ compare to the value of $\cot\theta$?

Example 1

Determine Reciprocal Trigonometric Ratios Using a Triangle

Consider a right triangle with sides of length 3 units, 4 units, and 5 units. Determine the six trigonometric ratios for ∠A. Then, determine the six trigonometric ratios for ∠B.



Solution

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

$$\csc B = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4} \qquad \sec B = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3} \qquad \cot B = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}$$

Example 2

Determine the Angle Given the Reciprocal Trigonometric Ratio

Each angle is in the first quadrant. Determine the measure of each angle, to the nearest degree.

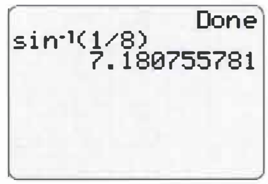
a) $\csc A = 8$ **b)** $\sec B = \frac{5}{2}$ **c)** $\cot C = \frac{5}{16}$

Solution

a) $\csc A = 8$

$\sin A = \frac{1}{8}$

$\angle A \doteq 7°$



Technology Tip

Use the function $\sin^{-1}$ to determine $\angle A$. You may need to press another key, such as 2nd, to access this function.

b) $\sec B = \frac{5}{2}$

$\cos B = \frac{2}{5}$

$\angle B \doteq 66°$

c) $\cot C = \frac{5}{16}$

$\tan C = \frac{16}{5}$

$\angle C \doteq 73°$

Example 3

Determine Angles in the Unit Circle That Have a Given Reciprocal Trigonometric Ratio

Determine two angles between 0° and 360° that have a cosecant of -2.

Solution

$\csc \theta = -2$

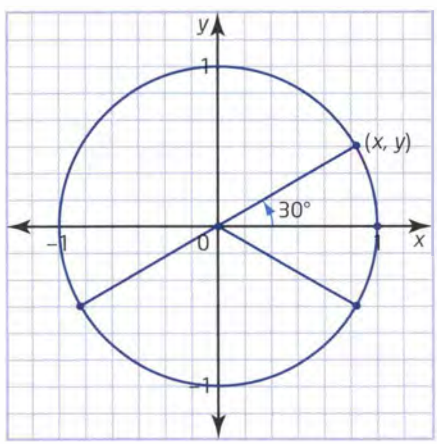
Take the reciprocal of both sides.

$\sin \theta = \frac{1}{-2}$

$= -\frac{1}{2}$

Since $\sin 30° = \frac{1}{2}$, the reference angle is 30°.

The sine ratio is negative in the third and fourth quadrants. Look for reflections of 30° that lie in these quadrants.



One possible value of θ is $180° + 30° = 210°$. The other possible value of θ is $360° - 30° = 330°$.

Two angles between 0° and 360° that have a cosecant of −2 are 210° and 330°.

Key Concepts

- The reciprocal trigonometric ratios are defined as follows:

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan\theta}$$

Communicate Your Understanding

C1 For angles restricted to the first quadrant, what are the maximum and minimum values that $\sin\theta$ may have? What are the maximum and minimum values that $\csc\theta$ may have?

C2 Write expressions for $\csc\theta$, $\sec\theta$, and $\cot\theta$ in terms of x, y, and r if the terminal arm of angle θ intersects a circle of radius r at the point (x, y).

C3 Are there any values of θ, $0° \leq \theta \leq 90°$, for which $\sec\theta$ is undefined? If so, determine them. If not, explain why not.

A Practise

For help with question 1, refer to the Investigate.

1. Use a calculator to determine the six trigonometric ratios for each angle, to three decimal places.

a) 20° **b)** 42° **c)** 75°
d) 88° **e)** 153° **f)** 289°

For help with questions 2 to 4, refer to Example 1.

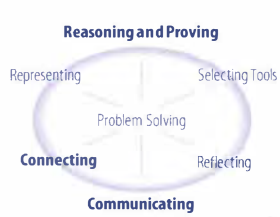
2. Determine exact expressions for the six trigonometric ratios for 315°. Hint: Draw a diagram of the angle in standard position. Then, use special triangles to determine the exact values.

3. Determine exact expressions for the six trigonometric ratios for 120°.

4. Determine exact expressions for the six trigonometric ratios for 270°.

For help with question 5, refer to Example 2.

5. Find the measure, to the nearest degree, of an angle in the first quadrant that satisfies each ratio.



If there is no such angle, explain why.

a) $\sin A = \frac{2}{3}$ **b)** $\cos B = \frac{3}{5}$
c) $\tan C = \frac{12}{5}$ **d)** $\csc D = \frac{9}{8}$
e) $\sec E = \frac{4}{3}$ **f)** $\cot F = \frac{3}{4}$
g) $\csc G = -\frac{4}{3}$ **h)** $\sec H = \frac{2}{5}$

For help with questions 6 and 7, refer to Example 3.

6. Determine two angles between 0° and 360° that have a secant of $-\sqrt{2}$. Use a unit circle to help you. Do not use a calculator.

7. Determine two angles between 0° and 360° that have a cotangent of −1. Use a unit circle to help you. Do not use a calculator.

8. Each point lies on the terminal arm of an angle in standard position. Determine exact expressions for the six trigonometric ratios for the angle.

a) P(−5, 12) **b)** Q(−4, −3)
c) R(−8, 15) **d)** S(24, −7)
e) T(9, 40) **f)** U(−2, −3)
g) V(5, −3) **h)** W(−2, 7)

B Connect and Apply

9. $\triangle PQR$ has a right angle at Q. If $q = 17$ cm and $p = 15$ cm, determine exact expressions for the six trigonometric ratios for $\angle P$.

For questions 10 to 13, round your answers to the nearest degree.

10. Determine two angles between 0° and 360° that have a cosecant of 5.

11. Determine two angles between 0° and 360° that have a secant of -5.

12. Determine two angles between 0° and 360° that have a cotangent of -3.

13. An angle has a cosecant of 1.2. The secant of the same angle is negative. Determine a value for the angle between 0° and 360°.

14. Use Technology Open *The Geometer's Sketchpad®*.

a) Draw a circle with a radius of 5 grid units. Plot a point A on the circle in the first quadrant.

b) Measure the coordinate distance between point A and the origin. Change the label to r. Measure the y-coordinate of point A.

c) Construct formulas to calculate the sine and the cosecant of $\angle A$, defined by terminal arm OA.

d) Animate point A, and observe the values of the sine and cosecant as A moves around the circle.

e) Use the **Motion Controller** to reverse the direction of motion of point A in such a way that it stays in the first quadrant. What happens to the values of sine and cosecant as $\angle A$ increases? as $\angle A$ decreases?

f) Add calculations for cos A and sec A. Repeat the animation while observing these values. Explain why this happens.

g) Add calculations for tan A and cot A. Repeat the animation while observing these values. Explain why this happens.

15. Chapter Problem You are leaving checkpoint #2 on the orienteering course and are on your way to checkpoint #3. Draw this leg of your orienteering challenge on your map. Label all distances and directions.

Direction: Face west. Turn right through an angle with a cosecant of 1.

Distance: The result of evaluating $12(\csc 30° + \sec 300° + \cot 225°)$, rounded to the nearest metre, if necessary.

✓ Achievement Check

16. An angle θ satisfies the relation $\csc \theta \cos \theta = -1$.

a) Use the definition of the reciprocal trigonometric ratios to express the left side in terms of $\sin \theta$ and $\cos \theta$.

b) What is the relation between $\sin \theta$ and $\cos \theta$ for this angle?

c) Determine two possible values for θ. Do not use a calculator.

d) Give an example of other information needed to determine a unique value for θ.

e) If $\sec \theta$ is known to be negative, what is the $\csc \theta$ and value of θ?

C Extend

17. Use expressions for the reciprocal trigonometric ratios in terms of x, y, and r to show that $1 + \tan^2 \theta = \sec^2 \theta$, regardless of the value of θ.

18. Given that $\cot B = -\frac{c}{d}$ and $\angle B$ is in the second quadrant, determine expressions, in terms of c and d, for the other five trigonometric ratios for B. State any restrictions on the values of c and d.

19. Given that $\sec A = \frac{t+1}{t-1}$ and $\angle A$ is in the fourth quadrant, determine an expression for sin A. State any restrictions on the value of t.

20. Parking regulations of a municipality require that each parking space be 3 m wide and 7 m long. Parking along a city block measuring 100 m can be set up for parallel or angle parking.

a) How many parking spaces can be made along the street using parallel parking?

b) How many parking spaces can be made along the street using angle parking at 45°?

c) What area of the roadway is lost to parking if parallel parking is used?

d) Compared to the area of roadway used for parallel parking, predict the area of roadway lost if angle parking is used. Give reasons for your prediction.

e) Calculate the area of roadway lost if angle parking is used. Compare your answer to your prediction in part d) and account for any differences.

21. One of the first recorded attempts to measure the radius of Earth was made by Eratosthenes of Alexandria, who lived from about 276 B.C.E. to 194 B.C.E. He was a Greek mathematician living in what is now Egypt. Eratosthenes' method of measurement was to observe the position of the sun in relation to Earth. He noted that at noon on the summer solstice, the sun was directly overhead the present-day city of Aswan. No shadows were cast. However, at Alexandria, about 800 km north, the sun cast shadows at an angle of 7.2° from the vertical.

a) Represent this problem using a diagram, labelling all information. State any assumptions that you make, and justify why they are reasonable.

b) Use trigonometry to determine a relation among the data given. Then, use the relation to calculate a value for the radius of Earth.

c) Find the accepted value for the radius of Earth using a library or the Internet and compare it to the value that you calculated in part b).

d) Since the sun was directly overhead on the summer solstice, what must the latitude of Aswan be? Use an atlas or the Internet to check.

Connections

In the northern hemisphere, December 21 is usually the shortest day of the year and June 21 is usually the longest. These days are known as the winter and summer solstices. Because of the inclination of Earth's axis, the apparent position of the sun changes as Earth orbits the sun. On the winter solstice, the sun is directly overhead at noon along the Tropic of Capricorn, at 23.5° south latitude. On the summer solstice, the sun is directly overhead at noon along the Tropic of Cancer, at 23.5° north latitude. On the two days of the year when day and night are about equal in duration (usually March 21 and September 21), the sun is directly overhead along the equator at noon. These days are known as the vernal and autumnal equinoxes. Visit the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 4 to find out more about solstices and equinoxes.

22. Math Contest In $\triangle PQR$, $\sec Q = 2.5$, $PQ = 3$, and $QR = 5$. Without using a calculator, find PR.

23. Math Contest Find the smallest natural number that when divided by 3 leaves a remainder of 1, when divided by 4 leaves a remainder of 2, and when divided by 5 leaves a remainder of 3.

24. Math Contest Three ports, Ashtra, Bretha, and Cratha, form a right triangle at Bretha. A ship sailing at 10 km/h makes the trip from Bretha to Ashtra in 2.5 h. The trip from Bretha to Cratha takes 6 h at the same speed. How far is it from Ashtra to Cratha?

4.4

Problems in Two Dimensions

Land surveyors use primary trigonometric ratios, the cosine law, and the sine law to determine distances that are not easily measured directly. For example, a surveyor may need to measure the distance across a river at the location planned for a new bridge. Obstructions on the shore may make it necessary to use a triangle that contains an obtuse angle. Care must be taken when using the sine law when two sides and one opposite angle are given. Sometimes, there are two answers, but only one of them is the desired answer. This is known as the **ambiguous case**.

ambiguous case

- a problem that has two or more solutions

In this section, you will solve problems involving triangles that contain an obtuse angle and deal with the possibility of two answers for the same problem.

Tools

- compasses
- protractor

Optional

- computer with *The Geometer's Sketchpad®*
- grid paper

Investigate

How can you identify the ambiguous case?

A surveyor measures two of the sides and one angle in $\triangle ABC$. She determines that $b = 600$ m, $c = 700$ m, and $\angle B = 45°$. Notice that two sides and one opposite angle are given. This is known as the side-side-angle, or SSA, case. Follow the directions below to construct $\triangle ABC$.

1. Draw a horizontal line segment. Label the left endpoint B.

2. Measure an angle of 45° upward from the line segment, with its vertex at B.

3. Use a suitable scale to draw line segment BA along the terminal arm of the 45° angle. Label point A.

4. Set your compasses to represent the length of the line segment b. With centre A, draw an arc below A such that it cuts the first horizontal line segment in two points. Label these C_1 and C_2, from left to right. Join A to C_1 and C_2.

5. Reflect What is the relationship between $\angle AC_1B$ and $\angle AC_2B$?

6. Reflect Identify two triangles that have side lengths of $b = 600$ m and $c = 700$ m, and $\angle B = 45°$. Which triangle is an acute triangle? Which triangle is an obtuse triangle? Draw these as separate triangles.

7. **Reflect** In $\triangle ABC$, consider that you are given b, c, and $\angle B$.

a) In what circumstances can only one triangle be drawn?

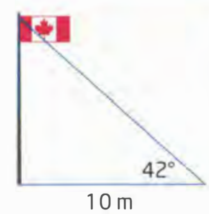
b) When is no triangle possible?

Whenever you are given two sides and one opposite angle and want to use the sine law, you must check for the ambiguous case. Sketch the possible triangles, and calculate an answer for each. Sometimes there are two possible triangles, sometimes there is one possible triangle, and sometimes it is not possible to draw a triangle to match a given set of measurements.

Example 1

Solve a Problem Using Primary Trigonometric Ratios

Basiruddin needs a new rope for his flagpole but is unsure of the length required. He measures a distance of 10 m away from the base of the pole. From this point, the angle of elevation to the top of the pole is 42°.



a) What is the height of the pole, to the nearest tenth of a metre?

b) How much rope should Basiruddin buy? Justify your answer.

Solution

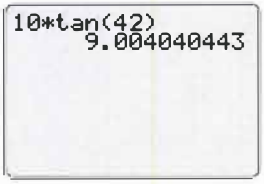
a) It is reasonable to assume that the flagpole is perpendicular to the ground. You can use primary trigonometric ratios. Let h represent the height, in metres, of the flagpole.

$$\tan 42° = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 42° = \frac{h}{10}$$

$$h = 10 \tan 42°$$

$$\doteq 9.0$$



The height of the flagpole is about 9.0 m.

b) Basiruddin should buy about 19 m of rope. The rope must be attached to the flagpole in a loop so that the flag can be lowered and raised. Allow some extra rope for knots and winding around pulleys.

Example 2

Solve an Oblique Triangle Problem

Patina, Quentin, and Romeo are standing on a soccer field. Quentin is 23 m from Romeo. From Quentin's point of view, the others are separated by an angle of 72°. From Patina's point of view, the others are separated by an angle of 55°.

a) Sketch a diagram for this situation. Why is the triangle that is formed an **oblique triangle**?

b) Is it necessary to consider the ambiguous case? Justify your answer.

c) Determine the distance from Patina to Romeo, to the nearest tenth of a metre. If there are two answers, determine both.

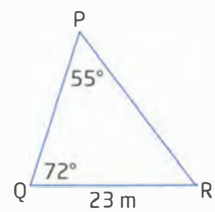
oblique triangle

- a triangle with no right angle

Solution

a) $\angle R = 180° - 72° - 55°$

$= 53°$



Since the triangle has no right angle, it is an oblique triangle.

b) It is not necessary to consider the ambiguous case. Two angles and a side are given, not two sides and one opposite angle.

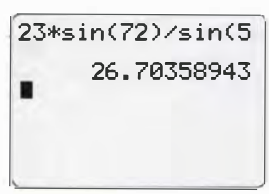
c) Use the sine law.

$$\frac{q}{\sin Q} = \frac{p}{\sin P}$$

$$\frac{q}{\sin 72°} = \frac{23}{\sin 55°}$$

$$q = \frac{23 \sin 72°}{\sin 55°}$$

$$\doteq 26.7$$



Romeo is about 26.7 m from Patina.

Example 3

Use the Sine Law

A lighthouse at point L is 10 km from a yacht at point Y and 8 km from a sailboat at point B. From the yacht, the lighthouse and the sailboat are separated by an angle of 48°.

a) Is it necessary to consider the ambiguous case? Explain.

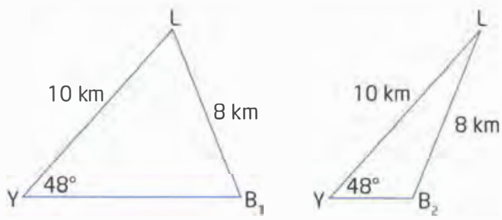
b) Sketch possible diagrams for this situation.

c) Determine the distance from the yacht to the sailboat, to the nearest tenth of a kilometre. If there are two answers, determine both. If there are no answers, explain why.

Solution

a) Two sides and one opposite angle are given, so it is necessary to consider the ambiguous case.

b)



c) Use the sine law in $\triangle LPB_1$ to determine $\angle B_2$.

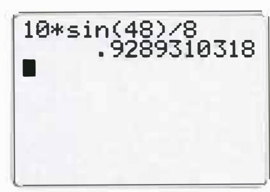
$$\frac{\sin B_1}{b} = \frac{\sin P}{p}$$

$$\frac{\sin B_1}{10} = \frac{\sin 48°}{8}$$

$$\sin B_1 = \frac{10 \sin 48°}{8}$$

$$\doteq 0.9289$$

$$\angle B_1 \doteq 68°$$



$$\angle L \doteq 180° - 48° - 68°$$

$$= 64°$$

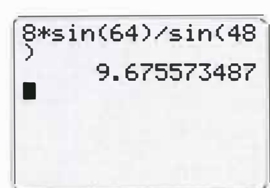
Use the sine law to determine the length of ℓ.

$$\frac{\ell}{\sin L} = \frac{y}{\sin Y}$$

$$\frac{\ell}{\sin 64°} = \frac{8}{\sin 48°}$$

$$\ell = \frac{8 \sin 64°}{\sin 48°}$$

$$\doteq 9.7$$



Using acute $\triangle LYB_1$, the two ships are about 9.7 km apart.

Now use $\triangle LYB_1$.

$$\angle B_2 = 180° - \angle B_1$$

$$\doteq 180° - 68°$$

$$= 112°$$

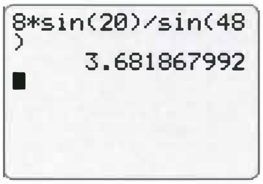
$\angle L \doteq 180° - 48° - 112°$
$= 20°$

$$\frac{\ell}{\sin L} = \frac{y}{\sin Y}$$

$$\frac{\ell}{\sin 20°} = \frac{8}{\sin 48°}$$

$$\ell = \frac{8 \sin 20°}{\sin 48°}$$

$$\doteq 3.7$$



Using the obtuse triangle, the ships are about 3.7 km apart.
There are two possible answers to this problem.

Example 4

Use the Cosine Law

The radar screen at an air traffic control tower shows a Piper Cherokee 15 km from the tower in a direction 30° east of north, and a Cessna Skyhawk 16 km from the tower in a direction 40° east of north, at their closest approach to each other. If the two aircraft are less than 2 km apart, the controller must file a report.

a) Sketch a diagram showing the tower and the two aircraft. Label the given distances and angles.

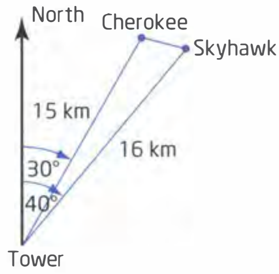
b) From the tower, what is the angle separating the aircraft?

c) Is it necessary to consider the ambiguous case? Justify your answer.

d) Will the controller need to file a report? Explain.

Solution

a)



b) From the tower, the aircraft are $40° - 30° = 10°$ apart.

c) Two sides and a contained angle are given. Since the angle is contained, and not opposite, the ambiguous case need not be considered.

Connections

When driving a car, a "near miss" is considered as coming within less than 1 m of another vehicle. In aviation, because of the speed of jet aircraft, a near miss can often mean coming within 1 km of another aircraft horizontally, or 300 m vertically. Two jetliners at normal cruise approach each other at a combined speed of nearly 2000 km/h. At a distance of 1 km, they could collide in less than 2 s.

d) Let d represent the distance between the two planes. Use the cosine law.

$$d^2 = 15^2 + 16^2 - 2(16)(15)\cos 10°$$
$$\doteq 8.3$$
$$d \doteq 2.9$$

Since d is a distance, only the positive square root applies.

The aircraft are approximately 2.9 km apart at their closest approach. Because the aircraft are more than 2 km apart, the controller does not need to file a report.

Key Concepts

- Primary trigonometric ratios are used to solve triangles that contain a right angle.
- The sine law is used to solve oblique triangles when two angles and a side are given. In the case when two sides and an opposite angle are given, there may be two possible solutions, one solution, or no solution. This is known as the ambiguous case.
- The cosine law is used to solve oblique triangles when two sides and a contained angle or three sides and no angles are given.

Communicate Your Understanding

C1 If two angles and the side between them are given in a triangle, in how many ways can you draw the triangle? Explain why, using a diagram.

C2 Is it possible to have an ambiguous case using the cosine law? If so, show an example. If not, explain why not.

C3 Is it possible to have a triangle such that an unknown side can be determined either with the sine law or the cosine law? If so, show an example. If not, explain why not.

A Practise

For help with questions 1 and 2, see Examples 1, 2, and 4.

1. For each of the following, select the most appropriate trigonometric tool among primary trigonometric ratios, the sine law, and the cosine law. Justify your choice. Do not solve.

a) In $\triangle ABC$, $\angle A = 90°$, $\angle B = 39°$, and $a = 10$ cm. Determine b.

b) In $\triangle PQR$, $\angle P = 35°$, $\angle R = 65°$, and $p = 3$ m. Determine q.

c) In $\triangle DEF$, $\angle D = 60°$, $\angle F = 50°$, and $d = 12$ cm. Determine f.

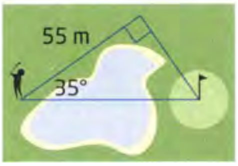
d) In $\triangle XYZ$, $\angle X = 42°$, $y = 25$ km, and $z = 20$ km. Determine x.

2. Determine the indicated unknown quantity for each triangle in question 1. Round each answer to one decimal place.

For help with questions 3 and 4, see Example 1.

3. The shadow of a tree that is 12 m tall measures 9 m in length. Determine the angle of elevation of the sun.

4. There is a water hazard between a golfer's ball and the green. The golfer has two choices. He can hit the ball alongside the water hazard to a point left of the green and play the next shot from there. Or, he can hit directly over the water hazard to the green. The golfer can usually hit an approach shot at least 60 m. Should he attempt the direct shot, or go around the hazard?



For help with question 5, see Example 2.

5. Yolanda flies her ultra-light airplane due east for 100 km. She turns right through an angle of 130°, and flies a second leg. Then, she turns right 110° and returns to her starting point.

a) Represent the flight path using an appropriate diagram, labelling all information.

b) Determine the total length of the flight, to the nearest kilometre.

For help with question 6, see Example 3.

6. For each of the following, draw possible diagrams that match the given measurements. Then, calculate the length of side *c*. If the calculation cannot be made, explain why.

a) In $\triangle ABC$, $a = 13$ cm, $b = 21$ cm, and $\angle A = 29°$.

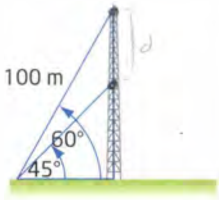
b) In $\triangle ABC$, $a = 24$ m, $b = 21$ m, and $\angle A = 75°$.

For help with question 7, see Example 4.

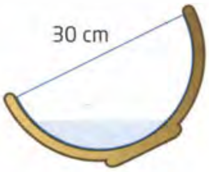
7. Bill and Nadia live across a ravine from each other. Bill walked 180 m to the end of the ravine, turned right through an angle of 45°, and walked another 200 m to Nadia's house. Determine an exact expression for the distance between the two houses.

B Connect and Apply

8. A radio antenna is stabilized by two guy wires. One guy wire is 100 m in length and is attached to the top of the antenna. The wire makes an angle of 60° with the ground. One end of the second guy wire is attached to the ground at the same point as the first guy wire. The other end is attached to the antenna such that the wire makes an angle of 45° with the ground. Determine an exact expression for the distance between the points where the two guy wires are attached to the antenna.

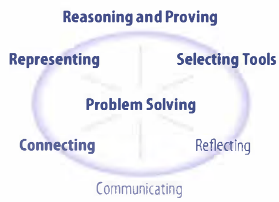


9. A decorative pottery bowl with a diameter of 30 cm is used as a garden ornament. A rain shower fills it with water to a maximum depth of 7 cm. The bowl is slowly tipped to remove the water. What angle will the rim of the bowl make with the horizontal when the water begins to spill out?



10. A Ferris wheel has a radius of 20 m, with 10 cars spaced around the circumference at equal distances. If the cars are numbered in order, how far is it directly from the first car to the fifth car?

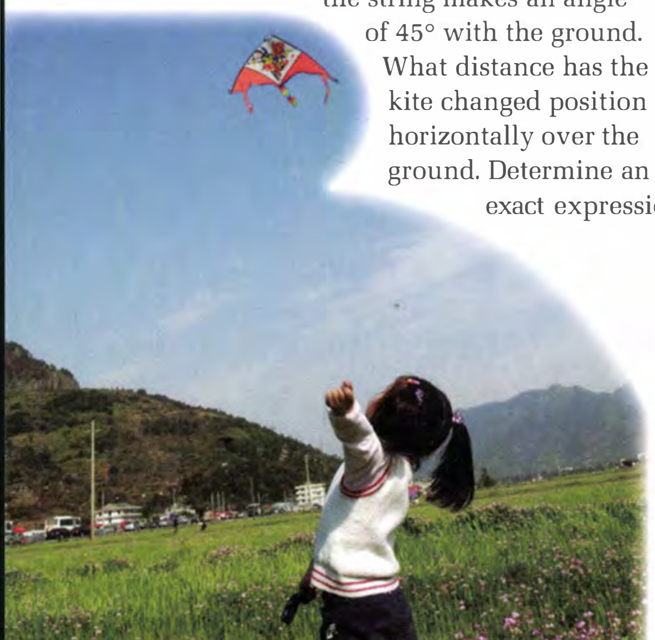
11. Charles leaves the marina and sails his boat 10° west of north for 1.5 h at 18 km/h. He then makes a starboard (right) turn to a heading of 60° east of north, and sails for 1.2 h at 20 km/h.



a) At the end of that time, how far is Charles from his starting point to the nearest kilometre?

b) What is the course required for Charles to return directly to the marina?

12. Giulia unreels her kite string until the kite is flying on a string of length 50 m. A light breeze holds the kite such that the string makes an angle of 60° with the ground. After a few minutes, the wind picks up speed. The wind now pushes the kite until the string makes an angle of 45° with the ground. What distance has the kite changed position horizontally over the ground. Determine an exact expression.



13. **Use Technology** There are several applets available on the Internet to calculate answers using the sine law and the cosine law. Perform a search using the key phrase "sine law calculator" or "cosine law calculator." Use the calculators to check your answers to some of the questions in this exercise.

14. In $\triangle ABC$, $\angle A$ is 40°, $a = 10$ cm, and $b = 12$ cm. Determine c, to the nearest tenth of a centimetre.

a) Explain why the ambiguous case must be considered for this triangle.

b) Sketch diagrams to represent the two possible triangles that match these measurements.

c) Solve for side c in both triangles. How many valid solutions are there?

d) If side a is 12 cm rather than 10 cm, how many solutions are there? Explain why.

e) Determine any solutions in part d).

15. **a)** Refer to your answer to question 14d). If side a is 7 cm rather than 10 cm, how many solutions are there? Explain why.

b) Determine the minimum value of a that results in at least one solution. Calculate your answer to four decimal places.

16. **a)** Create an ambiguous triangle problem that has two solutions. Determine the solutions to ensure that the question is valid.

b) Trade problems with a classmate, and solve.

c) Trade solutions. Write a short critique of your partner's solution, including any comments about proper form.

17. **Chapter Problem** You receive a scientific calculator at checkpoint #3. Determine the direction and distance to checkpoint #4 from the information below. Draw the leg on your map. Include all angles and distances.

Direction: North of west

Use $\angle A$ from $\triangle ABC$. In $\triangle ABC$, $\angle B = 85°$, $a = 41$ m, and $c = 32$ m. Round to the nearest degree, if necessary.

Distance: The measure of b, in $\triangle ABC$, to the nearest metre

18. A shade tree that is 20 m tall is located 30 m from Wok's apartment building, which is 10 m in height. By mid morning, the shadow of the tree falls directly toward the building. The angle of elevation of the sun increases by 15° per hour. Determine the length of time that at least part of the shadow of the tree falls on Wok's building.

19. Many buildings are now designed to use energy as efficiently as possible. One approach involves putting awnings on south-facing windows so that the summer sunlight is shaded from the window but the winter sunlight shines in. The maximum angle of elevation, θ, of the sun at a latitude of L degrees above the equator is given by $\theta = (90° - L) + 23.5°$ for latitudes north of the Tropic of Cancer.

a) Explain why this equation includes the angle 23.5°.

b) Consider a south-facing window that is 1.5 m high in a house built at a latitude of 45° north of the equator. Use a diagram to model an awning that extends a distance d over the window so that the entire window is shaded at noon on the summer solstice.

c) Determine the value of d required in part b).

d) How much of the window receives sunlight at noon on the winter solstice?

20. Albert and Bieta live on the same side of Main Street, 200 m apart. Charmayne lives directly across the street from Albert, and Daniel lives directly across the street from Bieta. From Albert's house, the angle between Bieta's house and Daniel's house is 31°. From Bieta's house, the angle between Albert's house and Charmayne's house is 25°.

a) Represent the positions of the four houses using a diagram. Label all given angles and distances.

b) Outline a method you can use to determine the distance from Charmayne's house to Daniel's house.

c) Use your method to determine the distance.

✓ Achievement Check

21. During a canoe trip, Enrico stops at a point along the south side of a river that flows from east to west. From here he can see a forest lookout tower, located 2.5 km away at a point on the north side of the river. As Enrico continues his trip, he stops at another point on the south side of the river that is 1.8 km from the lookout tower. The angle between the lookout tower, Enrico's first stop, and his second stop is 35°.

a) Explain why there are two possible locations for Enrico's second stop. Draw a diagram to represent each possible location. Label all angles and distances.

b) Determine the possible angles between the lookout tower, Enrico's second stop, and his first stop.

c) Without performing calculations, predict which angle found in part b) will result in the longer distance between Enrico's first and second stops. Explain.

d) Determine the possible distances between Enrico's first and second stops.

e) Suppose the given 35° represents the angle between Enrico's first stop, the lookout tower, and his second stop. How does the solution to the problem change? Solve this new problem.

C Extend

22. A clinometer for measuring angles of elevation for real-world objects can be made using simple materials.

a) Search the Internet for a simple plan for a clinometer.

b) Use the plan to make a clinometer.

c) Select a height in your neighbourhood that is difficult to measure directly. Use the clinometer to help you make this measurement.

d) Write a short report on your methods and calculations.

23. The grand ballroom of a hotel is on the third floor, 15 m above the lobby. It is reached by a spiral staircase that ascends a cylindrical hall of diameter 72 m in one complete turn. Determine the angle of elevation of the staircase.

24. While exploring a flat, desert-like plain on Mars, an astronaut sees a rock column, M, shaped like a mitten, and another rock column, T, shaped like a toy top. He is interested to know how far apart the columns are, but does not have enough oxygen to walk over to find out. The astronaut starts at point A, 600 m from a rock column, B, shaped like a banana. He measures $\angle$MAB as 155° and $\angle$TAB as 32°. He walks 600 m to column B. From here, he measures $\angle$MBA as 20° and $\angle$TBA as 146°.

a) Sketch a diagram to model these data.

b) Explain how you can use trigonometric tools to determine the distance MT.

c) Determine the distance MT.

25. On the vernal equinox (March 21 or 22 in the northern hemisphere), assume that the sun rises at 6:00 a.m., sets at 6:00 p.m., and is directly overhead at noon.

a) A flagpole is 10 m high. At what time will its shadow be 5 m long?

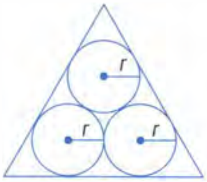
b) Explain why there is more than one answer to part a).

26. Show that the area of $\triangle$PQR can be calculated using the formula $A = \frac{1}{2}pq \sin R$.

27. **Math Contest** Three circles of equal radius are drawn tangent to each other. An equilateral triangle is drawn to circumscribe the three circles. If the radius, r, of each circle is 2 cm, what is the length of one side of the triangle, to the nearest centimetre?

A 10 cm **B** 11 cm **C** 12 cm

D 9 cm **E** 8 cm



28. **Math Contest** A plane is flying at 500 km/h. The angle of depression to a lighthouse on an island in the distance is 6°. After 12 min, as the plane continues to approach the lighthouse, the angle of depression is 15°. How long will it take for the plane to be directly above the lighthouse?

29. **Math Contest** A regular pentagon is inscribed in a circle of diameter 10 cm. What is the area, to the nearest square centimetre, of the part of the circle outside the pentagon?

A 59 cm^2 **B** 19 cm^2

C 79 cm^2 **D** 20 cm^2

E 18 cm^2

Use Technology

Use Geometry Software to Test for the Ambiguous Case

A pipeline is to be drilled from point A to point B through a rock outcrop. To determine the length of pipe needed, a surveyor sends an apprentice to take measurements. The apprentice takes a sighting from A to B and turns left through a measured angle of 40°. He walks along this line for 200 m until he is past the rock outcrop at point C. He then walks directly to B, measuring CB as 150 m. When the apprentice presents the measurements to the surveyor, the surveyor tells him that she does not have enough information.

a) Show why the surveyor's statement is true.

b) Write a note to the apprentice advising him how he might have taken his measurements to avoid this problem.

Tools

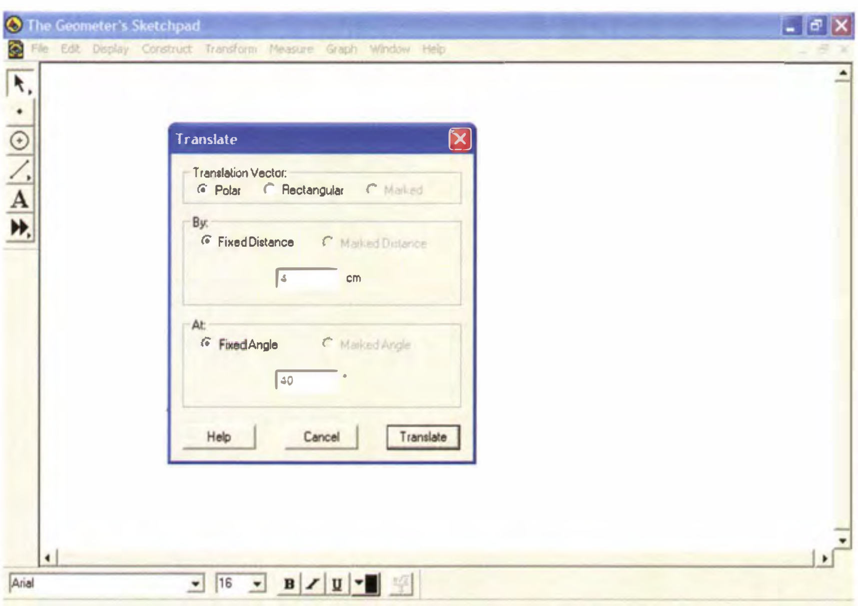
- computer with *The Geometer's Sketchpad®*

Use the following steps to model the problem using *The Geometer's Sketchpad®*.

- Open *The Geometer's Sketchpad®*. Plot a point A. Draw a line segment such that A is the left endpoint.
- Double-click on point A to mark it as a centre. Use a suitable scale. A scale of 1cm:50 m is used here. From the **Transform** menu, select **Translate**. Ensure that the Polar radio button is selected.

Technology Tip

The **Polar** button lets you describe a translation as a distance and an angle. The **Rectangular** button lets you describe a translation as a horizontal distance and a vertical distance.

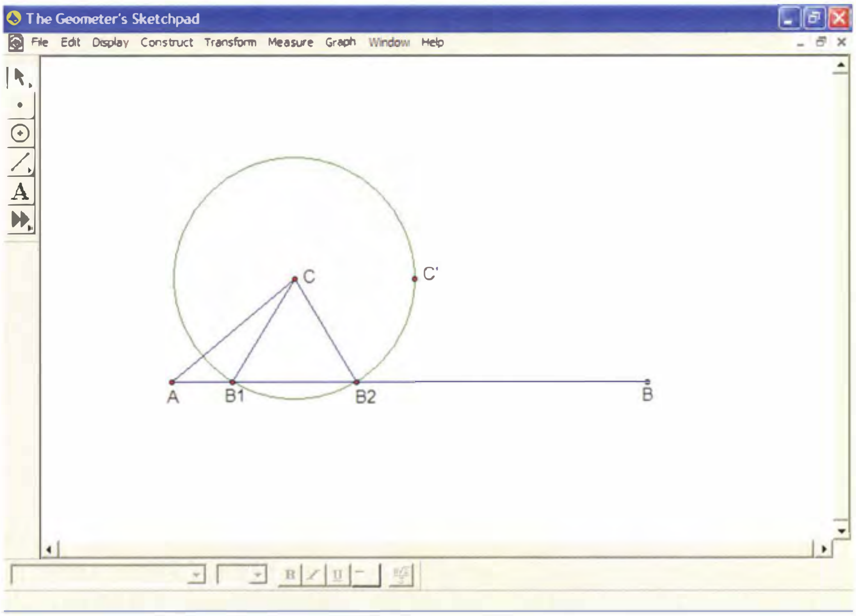


- Enter 4 cm and 40° in the **Distance** and **Angle** boxes.

Technology Tip

You can clean up the screen by hiding objects that were used to draw the sketch, but are now not needed. Select these objects. Then, hold the CTRL computer keyboard key and press h. Alternatively, you can select **Hide Objects** from the **Display** menu.

- Using the scale, point C will be at a fixed distance of 4 cm and a fixed angle of 40°. Press **Translate**. Right-click on the translated point, and use the menu to label it C.
- Use the **Transform** menu again to translate point C a scale distance of 3 cm at an angle of 0°.
- Select point C and then point C′. From the **Construct** menu, select **Circle by Center+Point**.
- Extend the line segment if necessary so that the circle intersects it at two points. Select the circle and the line segment. From the **Construct** menu, select **Intersections**. These points show the possible locations of point B. Relabel them B_1 and B_2. Draw line segments to form $\triangle AB_1C$ and $\triangle AB_2C$. Use your sketch to help you answer part b) above. What angle(s) and length(s) does the apprentice need to measure so the surveyor can calculate the length of the pipeline, B_1B_2?



4.5

Problems in Three Dimensions

Some airports have obstructions such as towers, hills, or tall buildings within a few kilometres of the end of their runways. The elevation of such obstructions often cannot be measured directly. When flying under instrument-only flight rules (for example, when visibility is obscured by fog or snow), the pilot of a departing aircraft cannot see these obstructions. In order to be certified for instrument departures, airports must publish the minimum rate of climb required to clear any possible obstructions.

In this section, you will learn to solve problems in three dimensions using the primary trigonometric ratios, the sine law, and the cosine law.

Investigate

How can you use trigonometry to solve a three-dimensional problem?

1. A hill of unknown height and at an unknown distance is located directly off the departure end of runway 09 (takeoff to the east) at City Airport. From the departure end, the angle of elevation of the hill is 7°. From a point 200 m south of the departure end, the angle formed by the base of the hill and the departure end is 87°. Draw a diagram of this situation.

2. Select the appropriate trigonometric tool to determine the distance from the departure end of the runway to the base of the hill. Determine this distance to the nearest metre.

3. Determine the height of the hill, rounded up to the nearest 10 m.

4. The required rate of climb can be expressed in metres per kilometre, measured from the departure end. What rate of climb is required for an aircraft to just clear the hill?

5. For safety reasons, the rate of climb actually required by regulation must allow the aircraft to clear the obstruction by a specified margin, usually about 330 m. Under this rule, what rate of climb is required?

6. **Reflect** Suggest another context that results in a problem in three dimensions that can be solved using trigonometry.

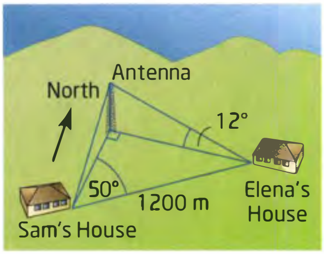
Connections

Airport runways are formally identified by angle directions, using three digits. Starting from north at 000°, turn toward east at 090°, south at 180°, and west at 270°. On airfields, runways are numbered by the first two digits of the runway direction angle, or bearing. Hence, a runway that is landed on facing east is numbered 09. If you landed on this runway from the other end, what number would you see?

Example 1

Use Primary Trigonometric Ratios to Solve a Problem in Three Dimensions

A radio antenna lies due north of Sam's house. Sam walks to Elena's house, a distance of 1200 m, 50° east of north. From Elena's house, the antenna appears due west, with an angle of elevation of 12°. Determine the height of the antenna, to the nearest metre.



Solution

Since Sam's house is south of the antenna and Elena's house is east of the antenna, the triangle formed by the two houses and the base of the antenna is a right triangle. Use primary trigonometric ratios.

Let d represent the distance, in metres, from Elena's house to the base of the antenna.

$$\sin 50° = \frac{d}{1200}$$
$$d = 1200 \sin 50°$$
$$\doteq 919$$

Elena's house is about 919 m from the base of the antenna.

Let h represent the height of the antenna. Use the tangent ratio.

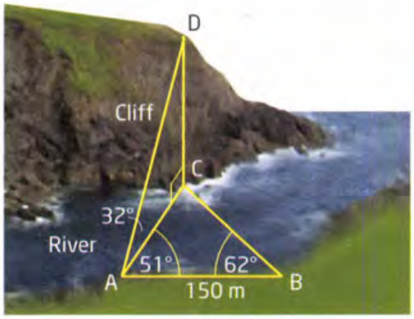
$$\tan 12° = \frac{h}{919}$$
$$h = 919 \tan 12°$$
$$\doteq 195$$

The height of the antenna is about 195 m.

Example 2

Use the Sine Law to Solve a Problem in Three Dimensions

A surveyor is on one side of a river. On the other side is a cliff of unknown height that she wants to measure. The surveyor lays out a baseline AB of length 150 m. From point A, she selects point C at the base of the cliff and measures ∠CAB to be 51°. She selects point D on top of the cliff directly above C and measures an angle of elevation of 32°. She moves to point B and measures ∠CBA as 62°.



a) Are any of the triangles in this problem right triangles? Justify your answer.

b) Which length would you determine first? Select the appropriate trigonometric tool to determine this length. Determine the length, to the nearest metre.

c) Which tool can you use to determine the height of the cliff? Justify your choice. Determine the height of the cliff, to the nearest metre.

Solution

a) Since point C is directly below point D, △ACD is a right triangle.

b) Determine AC in △ABC first. Two angles and a side are given for △ABC. Use the sine law to determine AC. Then, you can use △ACD to determine the height of the cliff CD.
In △ABC,

$$\begin{aligned} \angle C &= 180° - 51° - 62° \\ &= 67° \end{aligned}$$

The sum of the angles in a triangle is 180°.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Use the sine law.

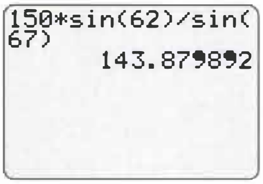
$$\frac{b}{\sin 62°} = \frac{150}{\sin 67°}$$

Substitute the given values.

$$\begin{aligned} b &= \frac{150 \sin 62°}{\sin 67°} \\ &\doteq 144 \end{aligned}$$

Solve for the unknown value.

The length AC is 144 m, to the nearest metre.



Connections

By convention, the side opposite ∠A is called *a*, the side opposite ∠B is called *b*, and the side opposite ∠C is called *c*.

c) Since △ACD is a right triangle, the most appropriate tool is primary trigonometric ratios. Let h represent the height, in metres, of the cliff and use b from △ABC.

$$\begin{aligned} \tan 32° &= \frac{h}{b} \\ &= \frac{h}{144} \\ h &= 144 \tan 32° \\ &\doteq 90 \end{aligned}$$

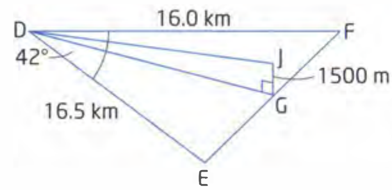
The height of the cliff is approximately 90 m.

When solving three-dimensional problems, it is often helpful to draw two-dimensional triangles for each step in the solution.

Example 3

Use the Cosine Law to Solve a Problem in Three Dimensions

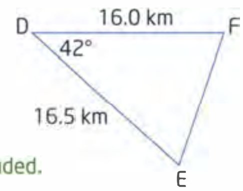
Justine is flying her hot-air balloon. She reports that her position is over a golf course located halfway between Emerytown and Fosterville, at an altitude of 1500 m. Fosterville is 16.0 km east of Danburg, and Emerytown is 16.5 km from Danburg, in a direction 42° south of east. What is the angle of elevation of Justine's balloon as seen from Danburg, to the nearest degree?



Solution

Determine the distance from Emerytown to Fosterville.

Since two sides and a contained angle are given, use the cosine law.



$$d^2 = e^2 + f^2 - 2ef\cos D$$
$$= 16.0^2 + 16.5^2 - 2(16.0)(16.5)\cos 42°$$
$$\doteq 135.87$$
$$d \doteq 11.66$$ Distance is positive, so the negative root is excluded.

The distance from Emerytown to Fosterville is approximately 11.7 km.

The golf course is $\frac{11.7}{2}$ or 5.85 km from Emerytown.

Use the sine law in △DEF to determine the measure of ∠E.

$$\frac{\sin E}{16.0} = \frac{\sin 42°}{11.7}$$
$$\sin E = \frac{16.0 \sin 42°}{11.7}$$
$$\sin E \doteq 0.9151$$
$$\angle E \doteq 66°$$

Use the cosine law in △DEG to determine the measure of DG.

$$e^2 = g^2 + d^2 - 2(g)(d)\cos E$$
$$= 16.5^2 + 5.85^2 - 2(16.5)(5.85)\cos 66°$$
$$\doteq 227.95$$
$$DG \doteq 15.1$$

DG measures approximately 15.1 km.

The altitude of Justine's balloon above the golf course is 1500 m, or 1.5 km. Use primary trigonometric ratios to determine the angle of elevation from Danburg, ∠D in right △DJG.

$$\tan D = \frac{1.5}{15.1}$$
$$\angle D \doteq 6°$$

The angle of elevation of the balloon as seen from Danburg is about 6°.

Key Concepts

- Three-dimensional problems involving triangles can be solved using one or more of the following: the Pythagorean theorem, the six trigonometric ratios, the sine law, and the cosine law.
- The method chosen to solve a triangle depends on the known information.

Communicate Your Understanding

C1 Why are three-dimensional problems more difficult to deal with than two-dimensional problems, given that you are using the same trigonometric tools?

C2 Pieter is visiting the Taj Mahal, in Agra, India. Describe a method he could use to determine the height of the building that could be carried out from the garden area in front, without approaching the building itself. What is the minimum number of measurements that Pieter would need to make? Justify your answer.

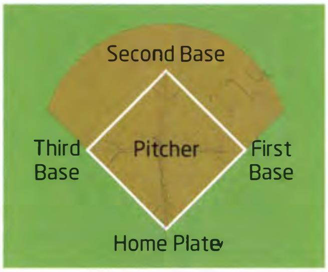
Connections

The Taj Mahal was built as a mausoleum to contain the crypt of one person, the beloved wife of emperor Shah Jahan. It now contains three crypts. To find out more about the Taj Mahal, visit the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 4.

A Practise

For help with questions 1 to 3, refer to Example 1.

1. The bases on a baseball diamond are 27.4 m apart. The pitcher pitches, and the batter hits a fly ball straight up 15 m. What is the maximum angle of elevation of the ball, to the nearest degree, as seen by the pitcher if he is standing at the centre of the diamond?

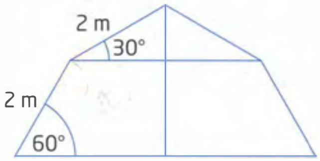


2. A square-based tent has the cross-sectional shape shown. The side wall goes up at an angle of elevation of 60° for 2 m, then continues at an angle of elevation of 30° for another 2 m to the peak.

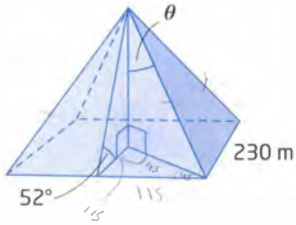
a) Determine an exact value for the height of the tent.

b) Determine an exact value for the side length of the base.

c) Determine an exact value for the length of one of the diagonals of the base.



3. The Great Pyramid of Cheops at Giza in Egypt has a square base of side length 230 m. The angle of elevation of one triangular face is 52°. Determine the measure of the angle θ between the height and one of the edges where two triangular faces meet.



B Connect and Apply

For help with question 4, refer to Example 2.

4. A grass airstrip runs from north to south. Dave notes that there are trees off the southern end at an unknown distance and an unknown height. He would like to determine the height of the trees. From the departure point of the runway, the angle of elevation of the tallest tree is 4°. Dave walks 100 m west and notes that the base of the tree and the departure point of the runway are separated by an angle of 83°.

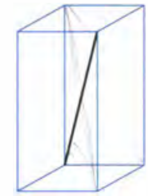
a) Draw a diagram to model this situation, labelling all measurements.

b) What will you calculate first? What tool will you use? Perform the calculation.

c) Determine the height of the tallest tree.

d) Dave consults his aircraft's operating manual and, taking air temperature into account, calculates that his plane can climb at a rate of 75 m per kilometre of horizontal distance flown. Is it safe for Dave to attempt a takeoff and climb out straight ahead? Justify your answer.

5. A box in the shape of a square-based prism has a base of side length 10 cm and a height of 20 cm. A rod dropped into the box lies exactly from one of the bottom corners to the opposite top corner. Determine the angle between the rod and the vertical edge of the box, rounded to one decimal place.



For help with questions 6 and 7, refer to Example 3.

6. The island of Santorini in Greece is a partially sunken volcanic crater, or caldera. Nicos starts at point A and walks 4 km in a straight line to point B. From point A, the volcanic cone at point C makes ∠CAB measuring 53°.

a) Determine the approximate average radius of the central lagoon using the circle shown.

b) How far is it from point C to point D on the nearby smaller island, to the nearest tenth of a kilometre?

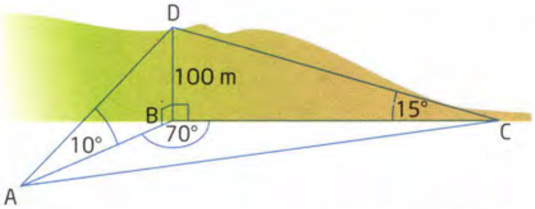
c) Santos is at point C, looking up at an angle of elevation of 3° above point D to the edge of a cliff. What is the height of the cliff, to the nearest metre?

d) What assumptions must you make to solve the problem? Justify whether they are reasonable.

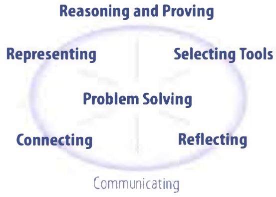
Connections

The island of Santorini suffered a cataclysmic volcanic explosion around 1600 B.C.E. Some writers postulate that this eruption, and the damage it caused to other islands by way of tsunamis, led to the legend of the lost continent of Atlantis.

7. Jodi and Leanna are on top of a cliff at point D, 100 m above the base. They decide to race to a picnic table at A, where lunch is waiting. Jodi runs, at a constant speed of 5 m/s, down the hill from D to C and then directly to A. Leanna climbs down the cliff to B at a constant rate of 1 m/s, and then runs as fast as Jodi to A. Who reaches lunch first?



8. A three-part race consists of swimming, climbing, and zipping. You begin at point A. The finish is at point D, which is 200 m from A in a direction 65° west of north. The first leg is a 600-m swim to an island B due west of A. The second leg is a climb up a vertical cliff to point C. Point C has an angle of elevation of 10°, as seen from point A. The third leg is a zip down a zip line from point C to the finish at point D. You can swim at 1.5 m/s, climb at 0.75 m/s, and zip at 15 m/s.



a) Sketch a diagram to represent this problem, labelling all given measurements on the diagram.

b) How long does it take you to complete the race?

9. A methane molecule, CH_4, consists of four hydrogen atoms at the corners of a tetrahedron and one carbon atom in the centre of the tetrahedron. Use drinking straws and marshmallows, toothpicks and gumdrops, or other suitable manipulatives to make a model of the methane molecule.

a) Explain how you ensured that your "carbon atom" is in the middle of the tetrahedron.

b) Measure the angle formed using two of the hydrogen atoms with the carbon molecule as the vertex.

c) Use the Internet to find a model of the methane atom. What is the exact angle?

d) Suggest ways to refine your model to obtain a more accurate answer.

Connections

Chemists often make three-dimensional models of molecules to help them visualize chemical bonds among atoms. The angles between the atoms can be measured from the models and used to formulate hypotheses. Trigonometry is used to construct mathematical models of molecular structures. The calculations are complicated, but you can use simple manipulatives to obtain reasonable estimates of angles in a molecule.

10. Ranjeet parks his car in a lot on the corner of Park Lane and Main Street. He walks 80 m east to First Avenue, turns 30° to the left, and follows First Avenue for 100 m to the Metro Building, where he takes the elevator to his office on the 15th floor. Each floor in the building is 4 m in height. From his office window, Ranjeet can see his car in the lot.

a) Sketch a diagram to represent this problem, labelling all given measurements on the diagram.

b) How far is Ranjeet from his car, in a direct line?

11. Chapter Problem It is time to move onward to checkpoint #5. Determine the direction and distance from the information given. Draw the leg on your map. Label all angles and distances.

Direction: West of South

Determine two angles between 0° and 360° whose cosecant is $-\frac{2}{\sqrt{3}}$. Add their degree measures, and divide by 12. Use this angle.

Distance: The height of the tree in the following:

William stands some distance from the base of a tree. From his position, the angle of elevation of the top of the tree is 30°. William then moves 40 m in a direction perpendicular to the line from his original position to the base of the tree. At this second point, William observes that his original position and the base of the tree are separated by an angle of 65°. Determine the height of the tree, rounded to the nearest metre, if necessary.

12. A scuba diver drops into the water from a dive boat and descends to a coral reef at a depth of 20 m. She follows the reef in a westerly direction. She estimates that she has travelled 30 m along the reef. The diver continues to follow the reef, which turns 30° to the right, swimming for another 25 m without changing depth. At this point, how far is she from the dive boat?

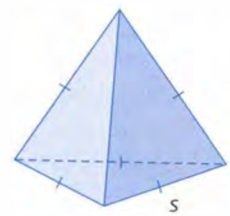
13. a) Pose a problem involving three dimensions. The solution must include at least one use of the sine law or cosine law and require at least three calculations of unknown values. Solve your problem to ensure that it works.

b) Trade your problem with a classmate. Solve each other's problem.

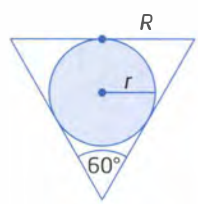
c) Trade solutions, and discuss the validity of the methods used.

C Extend

14. A regular tetrahedron has sides of length s. Show that the surface area, A, of the tetrahedron can be determined using the formula $A = \sqrt{3}s^2$.



15. The paper cup for a snow cone has a radius of R and a vertical angle of 60°. When a ball of radius r is dropped into the cup, the top of the ball is even with the top of the cup. Determine an expression for the distance from the bottom of the ball to the vertex of the cone in terms of R and r.



16. A regular tetrahedron has sides of length s. Determine an expression for the height, h, in terms of s.

17. A pyramid has a base in the shape of a rhombus with side length s. The height of the pyramid is equal to the side length. One diagonal of the rhombus is also equal to the side length. Determine the angle of elevation from the end of the other diagonal to the top of the pyramid.

18. Anwar lives in central Canada. His friend, Bill, lives in the southern United States. Chantal, another friend, lives in France. Each friend determines the exact location of their own house using a Global Positioning System (GPS) unit. Anwar then plots each location on a globe and draws three lines in a triangular shape to connect the locations.

a) When Anwar adds the three angles in the triangle on the globe, the sum is not 180°. Explain why this is so.

b) Is the sum more or less than 180°? Justify your answer.

19. The Global Positioning System (GPS) tracks satellites in orbit around Earth to determine your position on Earth's surface. There are 24 satellites in the GPS constellation, and most GPS receivers will track up to 12 at one time. How many are actually needed to determine a position using trigonometry?

a) When a GPS receiver locks on to a signal from a satellite, the receiver locates itself somewhere on a sphere with that satellite at the centre. When it locks on to the signal of a second satellite, the GPS unit locates itself on the intersection of two spheres. What does the intersection of two spheres look like? Explain.

b) When the GPS receiver locks on to the signal of a third satellite, the receiver locates itself on the intersection of the third sphere with the intersection found in part a). What does this third possible intersection look like? Explain.

c) Why is a fourth satellite necessary to give a precise fix on a location?

20. Math Contest A ship is stranded at sea. Also at sea are a Coast Guard cutter and an ocean liner. A helicopter is 2000 m above the Coast Guard cutter. The angle of depression from the helicopter to the stranded ship is 13°. The angle of elevation of the helicopter from the ocean liner is 23°. The angle formed at the cutter between the other two boats is 55°. If the cutter travels at 25 km/h and the liner at 20 km/h, what is the shortest time in which the stranded vessel will be reached by either the cutter or the liner?

21. Math Contest The product $990y$ is a perfect cube. The least natural number that y can be is

A 980 100 **B** 36 300

C 12 100 **D** 110

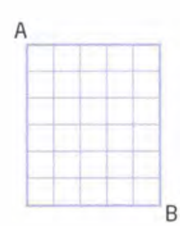
E 31 575

22. Math Contest How many routes are there from A to B if you can only move right or down?

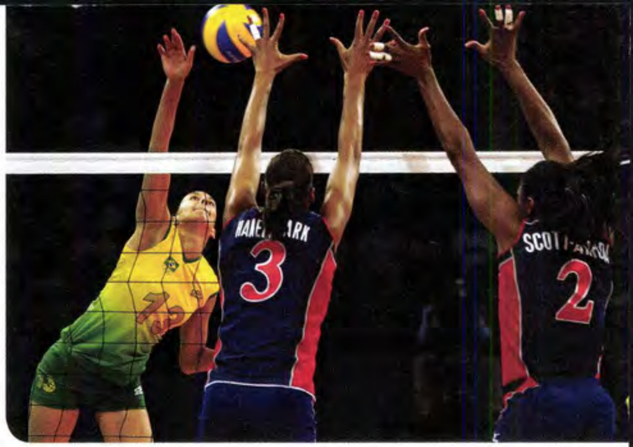
A 460 **B** 30

C 462 **D** 450

E 475



4.6



Trigonometric Identities

Solutions to equations that arise from real-world problems sometimes include trigonometric terms. One example is a trajectory problem. If a volleyball player serves a ball at a speed of 10 m/s, at an angle of θ with respect to the horizontal, the horizontal distance x that the ball will fly before hitting the ground can be modelled by the relation $x = 20 \tan \theta \cos^2 \theta$. The complexity of this equation makes it difficult to determine anything about the flight of the volleyball.

Some relations among trigonometric ratios are always true, regardless of what the angle is. These relations are known as **identities**. In this case, two identities can be used to simplify the above equation to $x = 10 \sin 2\theta$. This form is easier to work with.

identity
- an equation that is always true, regardless of the value of the variable

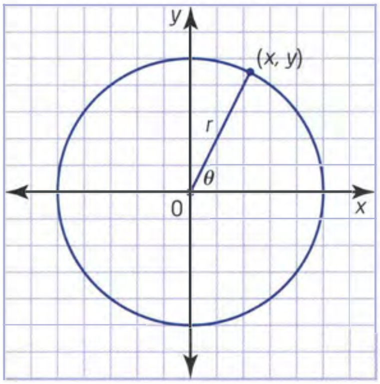
For example, suppose that you want to know the angle of serve that will send the volleyball the greatest distance. The sine ratio has a maximum value of 1 at an angle of 90°. If $2\theta = 90°$, then $\theta = 45°$. Therefore, you should serve the ball at an angle of 45° to send it the greatest distance.

In this section, you will learn the basic trigonometric identities and how to use them to prove other identities.

Investigate

How can you use relationships from a circle to prove an identity?

Earlier in this chapter, you used a circle to relate the trigonometric ratios for an angle θ to a point (x, y) on the terminal arm and the radius, r, of the circle.



1. Write the three primary trigonometric ratios for angle θ in terms of x, y, and r.

2. a) In the expression $\frac{\sin \theta}{\cos \theta}$, substitute the applicable expressions from step 1 and simplify.
   b) **Reflect** What trigonometric ratio is equivalent to this simplified expression?
   c) This ratio and the original expression in part a) are equal. Write this equation.

3. **Reflect** Use your calculator to verify the identity in step 2 for several angles. Select at least one angle from each of the four quadrants.

4. a) In the expression $\sin^2 \theta + \cos^2 \theta$, substitute the applicable expressions from step 1 and simplify.
   b) The original expression and the simplified expression form another identity. Write this equation.

5. **Reflect** Use a calculator to verify the identity in step 4 for several angles. Select at least one angle from each of the four quadrants. If you determine that it works for a finite number of angles, does that constitute a proof? Justify your answer.

6. The identity in step 4 is known as the Pythagorean identity. Explain why it is called this.

The basic identities that you will be using are summarized here:

Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$

Quotient Identity $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$

Reciprocal Identities $\csc \theta = \dfrac{1}{\sin \theta}$ $\quad \sec \theta = \dfrac{1}{\cos \theta}$ $\quad \cot \theta = \dfrac{1}{\tan \theta}$

If it appears that two expressions are always equal, you can form a conjecture that they are an identity. To prove an identity, write down the left side and the right side as shown in Example 1. Work with the left side (L.S.), the right side (R.S.), or both sides until you arrive at the same expression on both sides.

Example 1

Use Basic Identities to Prove an Identity From One Side

Prove that $\tan^2 \theta + 1 = \sec^2 \theta$.

Solution

L.S. $= \tan^2 \theta + 1$ $\qquad$ R.S. $= \sec^2 \theta$

$= \dfrac{\sin^2 \theta}{\cos^2 \theta} + 1$ Use the quotient identity.

$= \dfrac{\sin^2 \theta}{\cos^2 \theta} + \dfrac{\cos^2 \theta}{\cos^2 \theta}$ Use a common denominator.

$= \dfrac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta}$

$= \dfrac{1}{\cos^2 \theta}$ Use the Pythagorean identity.

$= \sec^2 \theta$ Use a reciprocal identity.

L.S. = R.S.

Therefore, $\tan^2 \theta + 1 = \sec^2 \theta$.

One possible strategy is to use the identities to transform one side into terms involving only sines and cosines. Then, simplify.

Connections

$\tan A = \dfrac{\sin A}{\cos A}$, so squaring both sides gives $(\tan A)^2 = \left(\dfrac{\sin A}{\cos A}\right)^2$ or $\tan^2 A = \dfrac{\sin^2 A}{\cos^2 A}$.

Example 2

Use Known Identities to Prove an Identity Using Both Sides

Prove that $1 - \cos^2 \theta = \sin \theta \cos \theta \tan \theta$.

Solution

It is more convenient to work from both sides in this example.

L.S. $= 1 - \cos^2 \theta$

$= \sin^2 \theta$ Use the Pythagorean identity.

R.S. $= \sin \theta \cos \theta \tan \theta$

$= \sin \theta \cos \theta \dfrac{\sin \theta}{\cos \theta}$ Use the quotient identity.

$= \sin^2 \theta$

Therefore, $1 - \cos^2 \theta = \sin \theta \cos \theta \tan \theta$.

Connections

The Pythagorean Identity can be written in different forms.

$\sin^2 \theta + \cos^2 \theta = 1$

$\sin^2 \theta = 1 - \cos^2 \theta$

$\cos^2 \theta = 1 - \sin^2 \theta$

Example 3

Work With Rational Expressions Involving Trigonometric Ratios

Prove that $\dfrac{\sin \theta}{1 + \cos \theta} = \dfrac{1 - \cos \theta}{\sin \theta}$.

Solution

Inspect both sides. Since there are no squared terms, you cannot use the Pythagorean identity directly. However, you can use your knowledge of the difference of squares to create the squared terms. Multiply the numerator and denominator on the left side by $1 - \cos \theta$.

L.S. $= \dfrac{\sin \theta}{1 + \cos \theta}$

R.S. $= \dfrac{1 - \cos \theta}{\sin \theta}$

$= \dfrac{\sin \theta}{1 + \cos \theta} \times \dfrac{1 - \cos \theta}{1 - \cos \theta}$

$= \dfrac{\sin \theta(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$

$= \dfrac{\sin \theta(1 - \cos \theta)}{1 - \cos^2 \theta}$

$= \dfrac{\sin \theta(1 - \cos \theta)}{\sin^2 \theta}$ Use the Pythagorean identity.

$= \dfrac{1 - \cos \theta}{\sin \theta}$ Simplify.

L.S = R.S.

Therefore, $\dfrac{\sin \theta}{1 + \cos \theta} = \dfrac{1 - \cos \theta}{\sin \theta}$.

Connections

The earliest development of trigonometric identities is traced to Claudius Ptolemy, a Greek astronomer and mathematician who lived about 130 B.C.E. Unlike the trigonometry we do today, which is based on relations in right triangles, the trigonometry Ptolemy worked with involved circles, arcs, and chords.

Key Concepts

- A trigonometric identity is a relation among trigonometric ratios that is true for all angles for which both sides are defined.
- The basic identities are the Pythagorean identity, the quotient identity, and the reciprocal identities:

Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$

Quotient Identity $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$

Reciprocal Identities $\csc \theta = \dfrac{1}{\sin \theta}$ $\sec \theta = \dfrac{1}{\cos \theta}$ $\cot \theta = \dfrac{1}{\tan \theta}$

- The basic identities can be used to prove more complex identities.
- Identities can be used to simplify solutions to problems that result in trigonometric expressions.

Communicate Your Understanding

C1 Sabariah says that she has discovered the trigonometric identity $\sin 3\theta = 2 \sin \theta$. As a proof, she points out that the left side equals the right side when $\theta = 30°$. Verify that her solution solves the equation. Does this prove that $\sin 3\theta = 2 \sin \theta$? Justify your answer.

C2 Consider the claimed identity in question C1. What is the difference between an equation and an equation that is an identity?

C3 You can show that an equation is not an identity by finding at least one counterexample. A counterexample is a value of the variable for which the equation is not true. Determine a counterexample for the equation in question C1.

A Practise

For help with question 1, refer to the Investigate.

1. Prove the Pythagorean identity using trigonometric definitions involving the opposite side, adjacent side, and hypotenuse of an angle in a right triangle.

2. Use a graphing calculator or grid paper to plot a graph of the relation $y = \sin^2 x + \cos^2 x$. Explain the shape of the graph.

For help with questions 3 and 4, refer to Example 1.

3. Prove each identity.

a) $\sin \theta = \cos \theta \tan \theta$

b) $\csc \theta = \sec \theta \cot \theta$

c) $\cos \theta = \sin \theta \cot \theta$

d) $\sec \theta = \csc \theta \tan \theta$

4. Prove each identity.

a) $1 + \csc A = \csc A(1 + \sin A)$

b) $\cot B \sin B \sec B = 1$

c) $\cos C(\sec C - 1) = 1 - \cos C$

d) $1 + \sin D = \sin D(1 + \csc D)$

For help with questions 5 and 6, refer to Example 2.

5. Prove that $1 - \sin^2 \theta = \sin \theta \cos \theta \cot \theta$.

6. Prove that $\csc^2 \theta = \cot^2 \theta + 1$.

B Connect and Apply

For help with questions 7 and 8, refer to Example 3.

7. Prove that $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$.

8. Prove that $\dfrac{\cos\theta}{1-\sin\theta} + \dfrac{\cos\theta}{1+\sin\theta} = \dfrac{2}{\cos\theta}$.

9. Prove that $\csc^2\theta\cos^2\theta = \csc^2\theta - 1$.

10. Use Technology The identity in question 8 can be broken into two relations, one for the left side and one for the right side. Use a graphing calculator.

a) Enter the left side in **Y1** and the right side in **Y2**. With your calculator in degree mode, set the window as shown. Set the line style to heavy for **Y2**.

```
WINDOW
 Xmin=-360
 Xmax=360
 Xscl=60
 Ymin=-5
 Ymax=5
 Yscl=1
 Xres=1
```

```
Plot1 Plot2 Plot3
\Y1=cos(X)/(1-si
n(X))+cos(X)/(1+
sin(X))
\Y2=2/cos(X)
\Y3=
\Y4=
\Y5=
```

The graph of **Y1** will appear first. Then, the graph of **Y2** will be drawn. Since it is drawn using a thick line, you can see whether it matches the graph of **Y1**.

b) Compare the graphs. Explain your results.

Technology Tip

Another way to compare two graphs is to toggle them on and off. In the **Y=** editor, move the cursor over the equal sign and press ENTER. When the equal sign is highlighted, the graph is displayed. When the equal sign is not highlighted, the graph is not displayed.

11. Consider the graphing approach used in question 10. Does this constitute a proof of the identity? Justify your answer.

12. Prove that $\tan\theta + \cot\theta = \dfrac{\sec\theta}{\sin\theta}$. Use a graphing calculator to illustrate the identity.

13. Prove that $\cot^2\theta\,(1+\tan^2\theta) = \csc^2\theta$. Use a graphing calculator to illustrate the identity.

14. Chapter Problem You are on the last leg of the orienteering course. Determine the direction and distance from the information given. Draw the leg on your map. Label all angles and distances.

Direction: East of south

Determine the two angles between 0° and 360° such that $\dfrac{\csc\theta}{\sec\theta} = \cot\theta\tan\theta$. Add their degree measures and divide by 9. Use this angle. Hint: Use identities to simplify each side of the equation first.

Distance: The result of evaluating $20(\sec^2\theta\sin^2\theta + \sec^2\theta\cos^2\theta - \tan^2\theta\sin^2\theta - \tan^2\theta\cos^2\theta)$, rounded to the nearest metre if necessary.

15. Draw a right angle. Using the vertex as the centre, draw a quarter-circle that intersects the two arms of the angle. Select any point A on the quarter-circle, other than a point on one of the arms. Draw a tangent line to the quarter-circle at A such that the line intersects one arm at point B and the other arm at point C. Label ∠AOC as θ. Show that the measure of BA divided by the radius of the quarter-circle equals the cotangent of $\angle\theta$.

16. A student writes the following proof for the identity $\cos\theta = \sin\theta\cot\theta$. Critique it for form, and rewrite it in proper form.

$$\cos\theta = \sin\theta\cot\theta$$
$$\cos\theta = \sin\theta\frac{1}{\tan\theta}$$
$$\tan\theta\cos\theta = \sin\theta$$
$$\frac{\sin\theta}{\cos\theta}\cos\theta = \sin\theta$$
$$\sin\theta = \sin\theta$$
$$\text{L.S.} = \text{R.S.}$$

✓ Achievement Check

17. Consider the equation $\tan^2\theta - \sin^2\theta = \sin^2\theta\tan^2\theta$.

a) **Use Technology** Use a graphing calculator to graph each side of the equation. Does it appear to be an identity? Justify your answer.

b) Which of the basic identities will you use first to simplify the left side?

c) Simplify the left side as much as possible. Explain your steps and identify any other identities that you use.

d) Is it necessary to simplify the right side? Explain. If so, simplify it.

e) If the two sides are not the same, go back and try another approach.

C Extend

18. Some trigonometric equations involve multiples of angles. One of these is $\sin 2\theta = 2\sin\theta\cos\theta$.

a) Use a graphing calculator to graph each side of this equation. Does it appear to be an identity?

b) Show that the equation is true for $\theta = 30°$, $45°$, and $90°$.

c) Evaluate each side for an angle from each of the other quadrants.

d) Review the example of the volleyball serve at the beginning of this section (page 270). Assuming the identity to be true, use it to show that $20\tan\theta\cos^2\theta = 10\sin 2\theta$.

19. Some trigonometric identities involve complements of angles. For example, the complement of θ is $90° - \theta$. Consider the equation $\cos\theta = \sin(90° - \theta)$.

a) Use a graphing calculator to graph each side of this equation. Does it appear to be an identity?

b) Show that the equation is true for $\theta = 30°$, $45°$, and $90°$.

c) Use a unit circle to show where this identity comes from.

d) Make a conjecture concerning an identity for $\cos(90° - \theta)$ and an identity for $\tan(90° - \theta)$. Test each conjecture by using a graphing calculator.

20. Some trigonometric identities involve supplements of angles. For example, the supplement of θ is $180° - \theta$. Consider the equation $\sin\theta = \sin(180° - \theta)$.

a) Use a graphing calculator to graph each side of this relation. Does it appear to be an identity?

b) Show that the equation is true for $\theta = 30°$, $45°$, and $90°$.

c) Use a unit circle to show where this identity comes from.

d) Make a conjecture concerning an identity for $\cos(180° - \theta)$. Test your conjecture by graphing. If necessary, adjust your conjecture and retest.

21. **Math Contest** If $\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta = 1$, what does $\cos^2 3\theta + \cos^2 2\theta + \cos^2\theta$ equal?

A 1 **B** 1.5 **C** 2 **D** 2.5 **E** 3

22. **Math Contest** Given $\sin\theta\cot\theta = \frac{\sqrt{3}}{2}$, a possible value for $\sin\theta$ is

A $\frac{\sqrt{3}}{2}$ **B** 30 **C** -0.5

D $-\frac{\sqrt{3}}{2}$ **E** $\frac{1}{\sqrt{3}}$

23. **Math Contest** Given $\cos\theta = 3\sin\theta$, a possible value for $\cos^2\theta$ is

A 0.25 **B** 0.7071 **C** 0.5 **D** 0.9 **E** 1

Chapter 4 Review

4.1 Special Angles, pages 220 to 231

1. Use a unit circle to determine exact values for the primary trigonometric ratios for 210°. Check your results using a calculator.

2. A ship is tied to a dock with a rope of length 10 m. At low tide, the rope is stretched tight, forming an angle of 45° with the horizontal. At high tide, the stretched rope makes an angle of 30° with the horizontal. How much closer to the dock, horizontally, is the ship at low tide than at high tide? Determine an exact expression. Then, use a calculator to determine an approximate answer, correct to the nearest tenth of a metre.

4.2 Co-terminal and Related Angles, pages 232 to 240

3. The coordinates of a point on the terminal arm of an angle θ are shown. Determine the exact primary trigonometric ratios for θ.

   a) A(−5, 12) **b)** B(3, −4) **c)** C(6, −8)
   d) D(−2, −3) **e)** E(1, −5) **f)** F(−7, 4)

4. One of the primary trigonometric ratios for an angle is given, as well as the quadrant that the terminal arm lies in. Determine the other two primary trigonometric ratios.

   a) $\sin A = \frac{4}{5}$, first quadrant

   b) $\cos B = \frac{8}{17}$, fourth quadrant

   c) $\tan C = -\frac{12}{5}$, second quadrant

   d) $\sin D = -\frac{4}{7}$, third quadrant

5. Round your answers in question 5 to the nearest degree.

   a) Solve the equation $\sin \theta = -0.25$ for $0° \leq \theta \leq 360°$.

   b) Solve the equation $\cos \theta = \frac{4}{5}$ for $0° \leq \theta \leq 360°$

   c) Solve the equation $\tan \theta = \frac{5}{8}$ for $0° \leq \theta \leq 360°$.

4.3 Reciprocal Trigonometric Ratios, pages 243 to 248

6. Each point lies on the terminal arm of an angle. Determine the six trigonometric ratios for the angle, rounded to four decimal places.

   a) A(−5, −12) **b)** B(−4, 3)
   c) C(8, 15) **d)** D(7, −24)

7. Determine two angles between 0° and 360° that have a secant of −4. Round your answers to the nearest degree.

8. An angle between 0° and 360° has a cosecant of −1.

   a) Is this enough information to determine a unique solution? If yes, explain why. If no, what other information is required?

   b) Determine the angle or angles.

4.4 Problems in Two Dimensions, pages 249 to 258

9. Marko rides 10 km north of his home on his mountain bike. He reaches an abandoned railroad, turns through an angle of 120° onto the railroad, and then rides another 20 km.

   a) Draw a diagram to model this situation, labelling all distances and angles.

   b) Select the most appropriate trigonometric tool to determine Marko's distance from home. Justify your selection.

   c) Determine an exact value for Marko's distance from home. Then, use a calculator to find the approximate distance.

10. In $\triangle ABC$, $\angle A = 32°$, $a = 15$ m, and $b = 18$ m.

a) Use drawing tools or geometry software to illustrate why the ambiguous case applies to this situation.

b) Sketch diagrams to represent the two possible triangles that match these measurements.

c) Solve for side c in both triangles, to the nearest metre.

4.5 Problems in Three Dimensions, pages 261 to 269

11. An asymmetric pyramid has a base in the shape of a kite, with the longer sides of the base measuring 150 m and the shorter sides measuring 120 m. The angle between the two longer sides measures 70°. The angle of elevation of the top of the pyramid, as seen from the vertex between the longer and shorter sides, is 75°. Determine the height of the pyramid to the nearest tenth of a kilometre.

12. Cate is sailing her boat off the coast, which runs straight north and south. Her GPS confirms that she is 8 km from Haytown and 10 km from Beeville, two towns on the coast. The towns are separated by an angle of 80°, as seen from the boat. A helicopter is hovering at an altitude of 1000 m halfway between Haytown and Beeville.

a) Determine the distance between Haytown and Beeville, to the nearest tenth of a kilometre.

b) Determine the angle of elevation of the helicopter, as seen from the sailboat, to the nearest tenth of a degree.

4.6 Trigonometric Identities, pages 270 to 275

13. Prove that $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

14. Prove that $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.

15. Prove that
$\cot\theta = \cos\theta\sin\theta + \cos^3\theta\csc\theta$.

Chapter Problem WRAP-UP

As you worked through this chapter, you completed six legs of an orienteering course based on trigonometry. It is time to head back to the starting point and finish the course.

a) Refer to your map. Create a problem that involves trigonometry to model the direction back to the starting point from checkpoint #6, similar to the problems that you solved to complete the six legs.

b) Trade your problem with a classmate. Solve each other's problem. Do you both finish back at the starting point?

c) **Use Technology** Select a suitable scale and set up a map of the orienteering course using *The Geometer's Sketchpad®*. Select appropriate tools from the **Transform** menu to draw each of the six legs. Measure the direction and distance back to the starting point from checkpoint #6. How does it compare with the direction and distance that you measured from your map?

Chapter 4 Practice Test

For questions 1 to 5, select the best answer.

1. An angle in the first quadrant has a sine of $\frac{1}{\sqrt{2}}$. The tangent of this angle is

A $\frac{1}{\sqrt{2}}$ **B** 1 **C** $\frac{1}{\sqrt{3}}$ **D** $\sqrt{3}$

2. A 25-m-high pine tree is growing in soft ground. After a storm, the tree leans at an angle of 60° with the ground. A pine cone falls from the top of the tree to the ground. Determine an expression for the exact distance that the pine cone falls.

A $25\sqrt{3}$ m **B** $\frac{25}{2}$ m

C $\frac{\sqrt{3}}{2}$ m **D** $\frac{25\sqrt{3}}{2}$ m

3. An angle θ in the third quadrant has a sine of -0.6133. In which quadrant is another angle with the same sine?

A first quadrant **B** second quadrant

C third quadrant **D** fourth quadrant

4. An angle in the second quadrant has a tangent of $-\frac{3}{4}$. Another angle with the same tangent measures about

A 37° **B** 53° **C** 127° **D** 323°

5. In $\triangle PQR$, $\angle P = 25°$, $\angle R = 65°$, and $q = 12$ cm. To determine the length of p, what is the most appropriate trigonometric tool?

A the sine law

B the cosine law

C primary trigonometric ratios

D reciprocal trigonometric ratios

6. a) $\angle A$ lies in the second quadrant and has a cotangent of $-\frac{5}{7}$. Sketch a diagram showing the position of $\angle A$, including a triangle with the lengths of the sides labelled.

b) Determine expressions for the other five trigonometric ratios for $\angle A$.

7. A hot-air balloon is used to give rides to visitors at a summer fair. The balloon is tethered to the ground by a long cable. The cable is extended to its maximum length of 300 m, and the wind is blowing the balloon such that the cable makes an angle of 60° with the ground. The cable is pulled in to 200 m, but the wind strengthens, decreasing the angle to 45°.

a) Sketch the two positions of the balloon, including distances and angles.

b) Find an exact expression for the horizontal distance that the balloon moves between the two positions.

c) Use a calculator to help you determine whether the balloon moves horizontally toward or away from the tether point.

8. Tanis leaves home and rides her bicycle 12 km north. She turns east and rides another 5 km. Then, she turns onto a forest bicycle path that runs 45° south of east and rides for another 5 km.

a) Sketch a diagram of Tanis's journey.

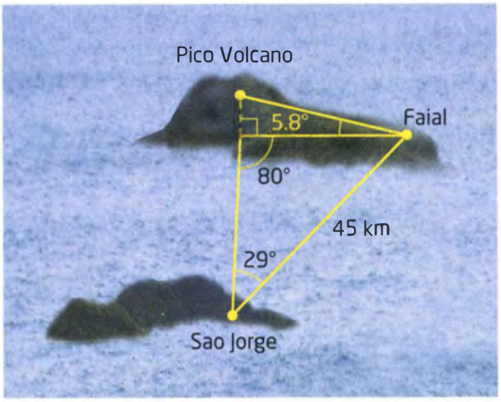
b) What is the most appropriate trigonometric tool to use in determining her distance from home? Justify your answer.

c) How far is Tanis from home at this point?

d) Which direction will take her directly home?

9. Antonio parks his car in the parking lot of a mountain-biking area and then bikes 1.4 km along a level trail in a direction 20° east of north. He turns right through an angle of 140° and rides up a sloping trail with an angle of elevation of 15°. When Antonio reaches the top of a hill, his odometer indicates that he has come another 1.2 km. He notices that there is a path sloping downward directly back to the parking lot. Antonio takes this path, returning to his car. Determine the total distance that he rode his bike, to the nearest tenth of a kilometre.

10. While visiting relatives in the Azores Islands, Juan sails from their home on Sao Jorge to Faial, a 45-km ferry ride. From Faial, he measures the angle of elevation to the Pico volcano as 5.8° and the angle of separation between the base of the volcano and Sao Jorge as 80°. When he returns to Sao Jorge, Juan measures the angle of separation of Faial and the base of the Pico volcano as 29°. Use this information to determine the height of the volcano, to the nearest metre.



11. Use a unit circle to show that $\cos \theta = \cos (360° - \theta)$ is an identity.

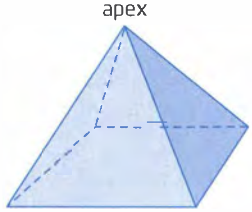
12. Prove that $\dfrac{\csc \theta}{\sec \theta} = \cot \theta$.

13. Prove that $\dfrac{\sin \theta}{1 - \cos \theta} = \csc \theta(1 + \cos \theta)$.

Task

Pyramids and Angles of Elevation

Use eight drinking straws to build a square-based pyramid with equilateral triangular faces.



a) Determine the height of the pyramid.

b) Imagine that you are standing at the midpoint of one of the edges of the base. Calculate the angle of elevation to the apex.

c) Now, imagine walking to one of the vertices of the base. Predict whether the angle of elevation to the apex will change. Use words to justify your prediction.

d) Verify your prediction in part c) mathematically.

e) Develop an algebraic model to calculate the angle of elevation of the apex from any point on the edge of the base.