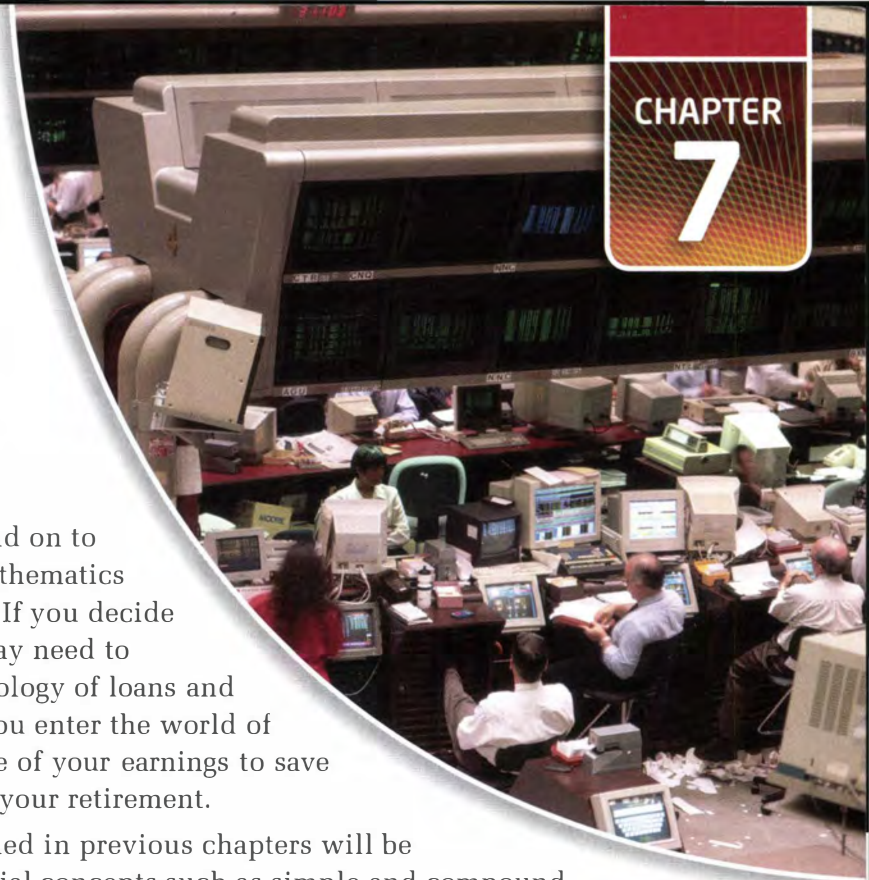
CHAPTER 7



Financial Applications

As you move through high school and on to post-secondary studies, financial mathematics will become increasingly important. If you decide to go to university or college, you may need to understand the concepts and terminology of loans and interest payments. Similarly, once you enter the world of work, you may decide to invest some of your earnings to save for a car, a mortgage, or, eventually, your retirement.

Many of the concepts you have learned in previous chapters will be useful when you explore new financial concepts such as simple and compound interest, present value, and annuities.

By the end of this chapter, you will

- make and describe connections between simple interest, arithmetic sequences, and linear growth, through investigation with technology
- make and describe connections between compound interest, geometric sequences, and exponential growth, through investigation with technology
- solve problems, using a scientific calculator, that involve the calculation of the amount, A; the principal, P; or the interest rate per compounding period, i, using the compound interest formula in the form $A = P(1 + i)^n$
- determine, through investigation using technology, the number of compounding periods, n, using the compound interest formula in the form $A = P(1 + i)^n$; describe strategies for calculating this number; and solve related problems
- explain the meaning of the term *annuity*, and determine the relationships between ordinary simple annuities, geometric series, and exponential growth, through investigation with technology
- determine, through investigation using technology, the effects of changing the conditions of ordinary simple annuities
- solve problems, using technology, that involve the amount, the present value, and the regular payment of an ordinary simple annuity

Prerequisite Skills

Refer to the Prerequisite Skills Appendix on pages 478 to 495 for examples of the topics and further practice.

Linear and Exponential Growth

1. a) Graph the equation $y = 40x + 400$. What type of relationship is this?

b) Identify the slope and the y-intercept.

c) Make a table of values using x-values of 0, 1, 2, 3, and 4.

d) Calculate the first differences and describe their pattern.

2. a) Graph the equation $y = 100(1.05)^x$. What type of relationship is this?

b) Identify the y-intercept.

c) Make a table of values using x-values of 0, 1, 2, 3, and 4.

d) Calculate the first and second differences and describe any patterns you see.

e) Calculate the common ratios by dividing consecutive y-values, and describe their pattern.

3. Determine whether each table of values represents an exponential function. Justify your reasoning using finite differences, a graph, or common ratios.

a)

x	y
−2	0
1	4.5
0	9.0
1	13.5
2	18.0

b)

x	y
0	100
1	90
2	81
3	72.9
4	65.61

Direct Variation and Partial Variation

4. Identify each relation as a direct variation, a partial variation, or neither. Justify your answer.

a) $y = 2x$ **b)** $y = 3x + 1$

c) $C = 5n + 25$ **d)** $d = 4t$

5. The cost of a taxi ride is \$3 plus \$1 per kilometre.

a) Write an equation to relate the total cost, C, in dollars, of the ride to the trip distance, d, in kilometres.

b) Identify the fixed part and the variable part of this relation.

c) Graph the relation.

d) Determine the slope and the vertical intercept of the graph.

e) Explain how your answers to parts b) and d) are related.

Arithmetic Sequences and Series

6. a) Explain why the sequence 7, 10, 13, 16, 19, ... is arithmetic.

b) Determine the first term, a, and the common difference, d.

c) Write the formula for the general term, or nth term, t_n.

7. The formula for the general term of an arithmetic sequence is $t_n = 5n - 7$.

a) Write the first four terms of the sequence.

b) Determine the first term, a, and the common difference, d.

8. Determine the sum of the first 100 terms of the series $3 + 8 + 13 + 18 + \cdots$.

Geometric Sequences and Series

9. a) Explain why the sequence 3, 6, 12, 24, 48, ... is geometric.

b) Determine the first term, a, and the common ratio, r.

c) Write a formula for the general term, or nth term, t_n.

10. The formula for the general term of a geometric sequence is $t_n = -2(3)^{n-1}$.

a) Write the first four terms of the sequence.

b) Determine the first term, a, and the common ratio, r.

11. Determine the sum of the first 15 terms of the series $1 + 1.05 + 1.1025 + 1.157\,625 + \cdots$.

12. The sum of the first 10 terms of a geometric series is 699 050. The common ratio is 4. Determine the first term.

Solve Equations

13. Solve for each variable.
  a) $600 = 40 + 7n$
  b) $840 = I + 5(12)$
  c) $250 = P[1 + 0.08(4)]$
  d) $496.50 = 300(1 + i)^8$

14. Solve for x. Round answers to four decimal places, if necessary.
  a) $200 = x(1 + 0.045)^6$
  b) $500 = \dfrac{40}{x + 1}$
  c) $300 = \dfrac{320}{(1 + x)^2}$
  d) $240 = 200x^6$

Time

15. Convert each time to the unit specified.
  a) 3 years to months
  b) 15 weeks to years
  c) 130 days to years
  d) 9 months to weeks
  e) 100 days to months
  f) 25 weeks to months

16. How many intervals are in each time?
  a) weekly intervals in 2 years
  b) annual intervals in 4 years
  c) monthly intervals in 3.5 years
  d) quarterly intervals in 3 years
  e) semi-annual intervals in 2 years
  f) semi-annual intervals in 3.5 years

Chloe, a student just entering high school, has inherited $10 000. She decides to invest the money until she is ready for university. To help her make wise decisions, she and her parents hire a financial advisor. The advisor recommends that she invest her money in a diversified portfolio, in other words, put her money into a balanced mix of investment options.

You will encounter problems related to Chloe's investments throughout this chapter.

7.1

Simple Interest

Whether you are investing or borrowing, there is almost always a fee charged to the borrower, typically referred to as *interest*. Interest can be either paid or earned. When you take out a loan, *you* are the borrower and you *pay* interest. When you deposit money at a financial institution, the *bank* is the borrower and you *earn* interest.

simple interest

- interest calculated only on the original principal using the formula $I = Prt$, where I is the interest, in dollars; P is the principal, in dollars; r is the annual rate of interest, as a decimal; and t is the time, in years

principal

- amount of money initially invested or borrowed

annual rate of interest

- rate at which interest is charged, as a percent, per year
- expressed as a decimal for calculations

amount

- the value of an investment or loan at the end of a time period
- calculated by adding the principal and interest

Tools

- graphing calculator

or

- computer with graphing or spreadsheet software

or

- grid paper

The simplest form of interest is called **simple interest**. Simple interest, I, can be calculated by multiplying the **principal**, P, by the **annual rate of interest**, r, expressed as a decimal, and by the time, t, for which the money is lent, in years.

$I = Prt$

At the end of the lending period, the **amount**, A, repaid by the borrower to the lender is the sum of the principal and the interest.

$A = P + I$

While simple interest is rarely used in bank accounts and loans today, it is important to understand its fundamental nature before learning the more advanced concepts to be introduced in the next section.

Investigate

How can you represent simple interest mathematically?

Alexis receives a $1000 gift for her grade 8 graduation. She decides to invest the money at 5% per year, simple interest.

1. a) Copy and complete the table, which relates the amount of the investment to time.

Time, *n* (years)	Amount, *A* ($)	First Differences
1	1050	
2	1100	1100 − 1050 = 50
3	1150	
4		
5		

b) Is this relationship linear? Explain how you know.

2. a) Graph the amount versus time. Describe the trend.

b) Does this support your answer to step 1b)? Explain.

3. a) Determine the slope and the vertical intercept of the graph.

b) Write an equation to relate the amount, A, to the time, n.

c) Is this an example of direct variation or is it an example of partial variation? Explain.

4. a) Examine the values in the Amount column of the table. Explain why this is an arithmetic sequence.

b) Identify the first term, a, and the common difference, d.

c) Write a simplified equation for the nth term of this sequence, using the formula $t_n = a + (n - 1)d$.

d) Compare your equation in part c) to the one from step 3b). How are they alike? different?

5. Reflect Summarize the different mathematical ways that simple interest can be represented.

Example 1

Calculate Simple Interest

a) How much interest is earned if $1200 is invested at 5% per year simple interest for 3 years?

b) How much interest is paid if $400 is borrowed at 8% per annum simple interest for 7 months?

c) How much interest is earned if $900 is invested at 4.25% annual simple interest for 90 days?

Connections

Per year, per annum, yearly, and *annually* are all commonly used terms with the same meaning—per year.

Solution

Use the formula $I = Prt$. Express the interest rate as a decimal and the time in years.

a) $I = Prt$

$= 1200(0.05)(3)$

$= 180$

The interest earned is $180.

b) $I = Prt$

$= 400(0.08)\left(\frac{7}{12}\right)$ Seven months represents $\frac{7}{12}$ of a year.

$\doteq 18.67$ Round money values to the nearest cent.

The interest paid is $18.67.

c) $I = Prt$

$= 900(0.0425)\left(\frac{90}{365}\right)$ There are 365 days in a year.

$\doteq 9.43$

The interest earned is $9.43.

Example 2

Develop a Linear Model for Simple Interest

Robert deposits \$500 into a guaranteed investment certificate (GIC) that earns 6% per year, simple interest.

a) Develop a linear model to relate the amount to time. Identify the fixed part and the variable part. Graph the function.

b) How long will it take, to the nearest month, for the investment to double?

c) What annual rate of interest must be earned so that the investment doubles in 8 years?

Connections

The linear model representing the amount in an account earning simple interest can also be written using slope *y*-intercept form, $y = mx + b$, as $A = 30t + 500$.

Solution

a) The amount varies partially with time.

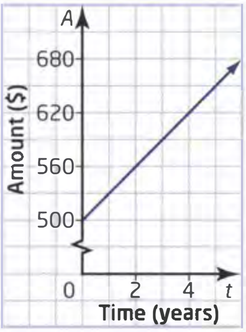
$A = P + I$

The fixed part is the principal, \$500. The variable part is the interest, which can be found by substituting the known values into the formula $I = Prt$.

$$\begin{aligned} I &= Prt \\ &= 500(0.06)t \\ &= 30t \end{aligned}$$

Therefore, the amount is given by $A = 500 + 30t$.

The graph is a straight line with slope 30 and vertical intercept 500.



b) Method 1: Apply Graphical Analysis

To determine how long it will take for the amount to double from \$500, use graphing technology to locate the point where the graph of $A = 500 + 30t$ intersects the graph of $A = 1000$.

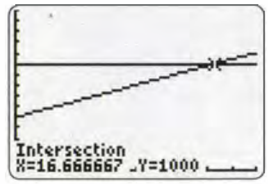
Use a graphing calculator to graph the functions as **Y1** and **Y2**. Apply number sense and systematic trial to set reasonable window settings.

```
Plot1 Plot2 Plot3
\Y1=500+30X
\Y2=1000
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
 Xmin=0
 Xmax=20
 Xscl=2
 Ymin=0
 Ymax=1500
 Yscl=100
 Xres=1
```

Use the **Intersect** operation to identify the coordinates of the point of intersection.

- Press 2nd [CALC] to display the **CALCULATE** menu, and select **5:intersect**.
- Press ENTER when prompted for the first curve, second curve, and guess.



The solution to this linear system indicates that it will take $16\frac{2}{3}$ years for the investment to double.

To express the time in years and months, convert the fraction part of the answer to months:

$\frac{2}{3} \times 12 = 8$

The time required for this investment to double is 16 years 8 months.

Method 2: Apply Algebraic Reasoning

To determine how long it will take for the amount to double from \$500, substitute $A = 1000$ and solve for t.

$$
\begin{aligned}
1000 &= 500 + 30t \\
500 &= 30t \\
t &= 16\frac{2}{3}
\end{aligned}
$$

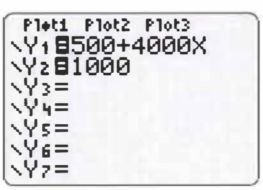
It will take $16\frac{2}{3}$ years, or 16 years 8 months, for the investment to double.

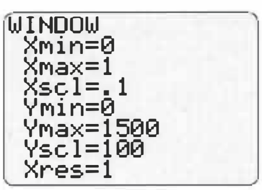
c) Method 1: Apply Graphical Analysis

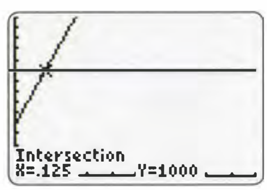
To determine the rate of interest for \$500 to double in 8 years, substitute $P = 500$ and $t = 8$ into the amount equation to express A in terms of r.

$$
\begin{aligned}
A &= P + I \\
&= P + Prt \\
&= 500 + 500r(8) \\
&= 500 + 4000r
\end{aligned}
$$

Use the **Intersect** operation on a graphing calculator to locate the point where this function intersects the graph of $A = 1000$.







The solution to this linear system indicates that for the amount to double after 8 years, the annual rate of interest must be 0.125, or 12.5%.

Connections

Note that in part b) the independent variable in the graphing calculator, X, represents time, t. In part c), X represents the interest rate, r. It is important to understand the variables being compared in each situation.

Method 2: Apply Algebraic Reasoning

To determine the rate of interest to double \$500 in 8 years, substitute $A = 1000$, $P = 500$, and $t = 8$ into the equation $A = P + Prt$ and solve for r.

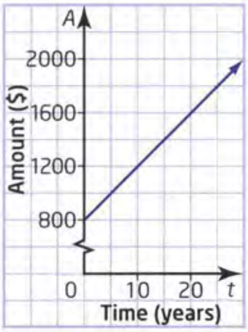
$$\begin{aligned} A &= P + Prt \\ 1000 &= 500 + 500r(8) \\ 500 &= 4000r \\ r &= 0.125 \end{aligned}$$

For the amount to double after 8 years, the annual rate of interest must be 12.5%.

Example 3

Analyse a Simple Interest Scenario

The graph shows the amount of an investment earning simple interest.



a) What is the principal?

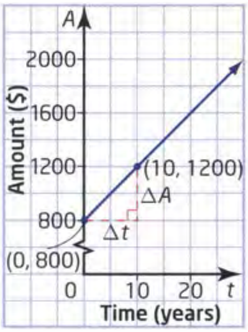
b) What is the annual interest rate?

c) Write an equation to relate the amount of the investment to time.

Solution

a) The principal is the amount of the initial investment. This occurs when $t = 0$. From the graph, the vertical intercept is 800. So, the principal is \$800.

b) Determine the interest rate from the slope of the line.



$$\begin{aligned} m &= \frac{\Delta A}{\Delta t} \\ &= \frac{1200 - 800}{10 - 0} \\ &= \frac{400}{10} \\ &= 40 \end{aligned}$$

Use the slope formula.

The slope is 40, which means that \$40 interest is earned each year. Express this as a percent of the principal.

$$i = \frac{40}{800}$$
$$= 0.05$$

Therefore, the annual interest rate is 5%.

c) This is the graph of a partial variation, so its equation is of the form $A = mt + b$.

The equation for the amount of the investment as a function of time is $A = 40t + 800$.

Key Concepts

- Simple interest, I, in dollars, can be calculated by multiplying the principal, P, in dollars, by the annual interest rate, r, expressed as a decimal, and by the time, t, in years.
  $I = Prt$
- The amount, A, of an account earning simple interest is the sum of the principal, P, and the interest, I.
  $A = P + I$
- The amount in an account earning simple interest can be represented using
  - a table of values
  - a partial variation equation
  - a linear graph
  - an arithmetic sequence

Communicate Your Understanding

C1 Explain how you can represent each time period in terms of years.

a) 4 months

b) 75 days

c) 15 weeks

C2 The table shows the amount in a simple interest account.

Time (years)	Amount (\$)
0	100
1	108
2	116
3	124
4	132

a) Look at the values in the Amount column. Is this an arithmetic sequence? Explain.

b) What is the annual rate of simple interest? How do you know?

C3 An account with an initial value of \$600 earns 5% simple interest annually.

a) Describe the shape of the graph of amount versus time.

b) Determine the slope and vertical intercept of the graph. What do they represent?

A Practise

For help with question 1, refer to Example 1.

1. Determine the simple interest earned on each investment.

a) \$450 is deposited for 4 years and earns 6.5% per year simple interest.

b) \$750 is deposited for 5 months at 7% per year simple interest.

c) \$500 is invested at 4.75% annual simple interest for 35 weeks.

d) \$1100 is invested at 7.8% per year, simple interest, for 60 days.

For help with questions 2 to 4, refer to Example 2.

2. Connor deposits \$200 into an account that earns 6% simple interest annually.

a) Determine the amount of the investment after 1, 2, 3, 4, and 5 years.

b) Identify the first term, a, and the common difference, d, of this arithmetic sequence.

c) Write an equation to represent the nth term of this sequence. What is the significance of the nth term?

3. The table shows the amount of a simple interest GIC over a period of several years.

Time (years)	Amount (\$)
1	689
2	728
3	767
4	806
5	845

a) Calculate the first differences. What do these values represent?

b) What is the principal of this investment? How do you know?

c) What is the annual rate of simple interest?

4. Refer to the table in question 3.

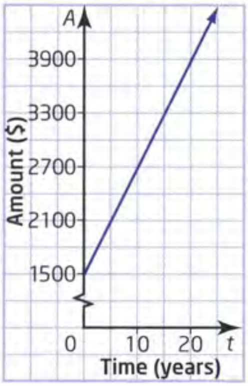
a) Develop a linear model to represent the amount in the GIC versus time.

b) Explain why the model from part a) is a partial variation. Identify the fixed part and the variable part.

c) How long will it take, to the nearest month, for this investment to double from its initial value?

For help with questions 5 and 6, refer to Example 3.

5. The graph shows the amount of an investment earning simple interest.



a) What is the principal?

b) What is the annual interest rate?

c) Write an equation to relate the amount to time.

d) Use your equation from part c) to determine how long it will take, to the nearest month, for the original investment to double.

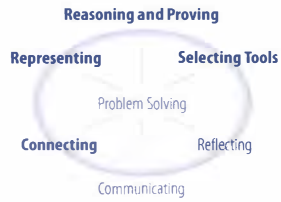
B Connect and Apply

6. Refer to question 5.

a) Write an equation to relate the interest to time.

b) Use your equation from part a) to determine how long it will take for the original investment to double. Compare this result with your answer to question 5d).

7. To save for a new pair of skis, Sven deposits $250 into a savings bond that earns 4.5% per year, simple interest.



a) Write an equation to relate the amount of the investment to time.

b) Graph the function.

c) How long will it take, to the nearest month, for the amount to reach $300?

d) What interest rate is required for the amount to reach $300 in 2 years less than your answer in part c)?

8. Rita borrows $500 at an annual rate of 8.25% simple interest to enrol in a driver's education course. She plans to repay the loan in 18 months.

a) What amount must she pay back?

b) How much interest will she pay?

c) How much sooner should Rita repay the loan if she wants to pay no more than $50 in interest charges?

9. José borrows $1400 to buy a road racing bicycle. He repays the loan 2 years later in the amount of $1700. What annual rate of simple interest was José charged?

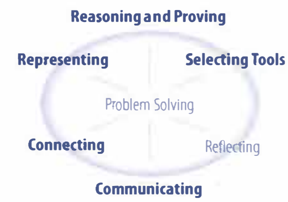
10. Chapter Problem Chloe's financial advisor has recommended that she invest at least 20% of her money in treasury bonds, at a fixed rate of return, or interest rate. Following this recommendation, she invests $2000 in a treasury bond for 4 years at a rate of 3.5% per year simple interest. Chloe cannot access this money before the end of the 4 years without paying a financial penalty.

a) Determine the interest earned over the term of the bond.

b) Determine the amount of the investment at the end of 4 years.

11. Tamara took out a loan for $940 at an annual rate of 11.5% simple interest. When she repaid the loan, the amount was $1100. How long did Tamara hold this loan?

12. Dmitri wants to borrow $5500 to buy a used car. He is considering two options:



- Borrow from the bank at 12.4% per year simple interest.
- Borrow from the car dealership at 11% per year simple interest, plus a $200 administration fee due upon the repayment date.

a) For each option, write an equation to relate the amount, A, to time, t, in years.

b) Graph the amount payable versus time for each option on the same set of axes.

c) Which option is the better deal? Explain.

C Extend

13. a) Use algebraic reasoning to derive an equation to express the annual simple interest rate, r, in terms of the principal, P; the amount, A; and the time, t, of a simple interest investment.

b) Use your formula from part a) to determine the annual simple interest rate earned by an account that grows from $860 to $1000 in 3 years.

c) Verify your result in part b) using another method of your choice.

14. a) Use algebraic reasoning to derive an equation to express the time, t, in years, of an investment, in terms of the principal, P; the amount, A; and the annual rate of simple interest, r.

b) Pose and solve a problem related to the formula you developed.

7.2

Compound Interest

Gaston is a talented young musician who earns a living by playing at coffee houses, university campuses, and special events such as weddings. Before he buys a new guitar, Gaston wants to compare various borrowing options that are available to him.

In Section 7.1, you learned that when simple interest is earned, the amount grows linearly as a function of time. However, most investments and loans are based on **compound interest**. This means that at the end of each **compounding period**, the interest is added to the principal for the next compounding period so that interest is then paid on this new total amount.

compound interest
- interest that is calculated at regular compounding periods
- added to the principal for the following compounding period

compounding period
- time interval after which compound interest is calculated

Tools
- graphing calculator

or
- computer with graphing software

Investigate

How can you calculate compound interest?

Recall from the Investigate in Section 7.1 that Alexis receives \$1000 for her grade 8 graduation. Under the simple interest scenario, the interest earned at 5% per year for 5 years is \$250.

Suppose that Alexis deposits the money into an account that pays the same annual interest rate of 5%, but compounded annually. How does this affect the amount of interest she will earn?

1. Copy the table. Calculate the interest using the formula $I = Prt$ and complete the table. Leave room for two more columns to the right.

Year	Balance at Start of Year (\$)	Interest Earned During the Year (\$)	Balance at End of Year (\$)
1	1000	1000(0.05)(1) = 50	1000 + 50 = 1050
2	1050	1050(0.05)(1) = 52.50	1050 + 52.50 = 1102.50
3			
4			
5			

2. **a)** Look at the values in the Interest Earned column. Describe what you notice.

   b) **Reflect** What is an advantage of compound interest over simple interest?

3. **a)** Add a column to the right of your table from step 1. Label the column "First Differences." Calculate the first differences by subtracting successive balances at the end of the year. Record the values in the table.

   b) What do the first differences represent?

   c) Is the balance at the end of the year versus time a linear relationship? Explain.

4. **a)** Add a column to the right of your table from step 3. Label the column "Common Ratios." Calculate the common ratios by dividing successive balances at the end of the year. For example, calculate 1102.50 ÷ 1050. Record these values in the table.

   b) What do you notice about the pattern of common ratios?

   c) What type of function does this represent? Explain how you can tell.

5. **a)** Graph the Balance at the End of Year versus Year, from step 1, using graphing technology.

   b) Does the shape of the graph support your answer to step 4c)? Explain.

   c) Graph the amount in the simple interest investment given by the equation $A = 1000 + 50t$ on the same set of axes as your graph in part a). How are these graphs alike? different?

   d) What happens to these graphs after a long period of time? Why does this happen?

6. **Reflect**

   a) Is the interest earned per year the same for every year under simple interest conditions? under compound interest conditions? Explain.

   b) Suppose that Alexis decides to use her investment to help her buy a car when she graduates from university. How much more interest will she have earned in the compound interest account than in the simple interest account, assuming that the money is invested for 8 years?

The amount of a compound interest investment or loan can be found recursively as follows.

Let i represent the annual interest rate per compounding period. The amount after one compounding period can be determined by adding the principal to the interest for that compounding period.

$$\begin{aligned} A &= P + I \\ &= P + Pit \\ &= P + Pi(1) && \text{After one compounding period, } t = 1. \\ &= P + Pi \\ &= P(1 + i) && \text{Factor the expression.} \end{aligned}$$

Connections

You studied recursive relationships in Chapter 6 Discrete Functions.

Connections

It is conventional to use r to represent the annual interest rate in situations involving simple interest and i to represent the interest rate per compounding period in situations involving compound interest. Both r and i refer to the interest rate expressed as a decimal.

Therefore, the amount after the first compounding period is the product of the principal at the beginning of the period and the factor $(1 + i)$. This amount becomes the new principal for the second compounding period. This process can be continued, as shown in the table.

Compounding Period	Principal for the Period	Amount Calculation	Amount at End of Period
1	P	$A = P(1 + i)$	$P(1 + i)$
2	$P(1 + i)$	$A = P(1 + i)(1 + i) = P(1 + i)^2$	$P(1 + i)^2$
3	$P(1 + i)^2$	$A = P(1 + i)^2(1 + i) = P(1 + i)^3$	$P(1 + i)^3$
4	$P(1 + i)^3$	$A = P(1 + i)^3(1 + i) = P(1 + i)^4$	$P(1 + i)^4$
⋮	⋮	⋮	⋮

Examine the pattern of terms in the last column.

$P(1 + i), P(1 + i)^2, P(1 + i)^3, P(1 + i)^4, \ldots$

This is a geometric sequence with first term $a = P(1 + i)$ and common ratio $r = 1 + i$. Substitute these into the formula for the nth term of a geometric sequence.

$$\begin{aligned} t_n &= ar^{n-1} \\ &= [P(1 + i)](1 + i)^{n-1} \\ &= P(1 + i)^{1 + (n-1)} \\ &= P(1 + i)^n \end{aligned}$$

Apply the product rule of exponents.

Note that the nth term corresponds to the amount, A, at the end of the nth compounding period. So, replace t_n with A.

$A = P(1 + i)^n$

This result is the compound interest formula.

The amount, A, of a compound interest investment or loan can be determined using the formula $A = P(1 + i)^n$.

P represents the principal.

i represents the interest rate per compounding period, expressed as a decimal.

n represents the number of compounding periods.

Example 1

Calculate Compound Interest

To buy a new guitar, Gaston borrows \$650, which he plans to repay in 5 years. The bank charges 12% per annum, compounded annually.

a) Determine the amount that Gaston must repay.

b) How much interest will Gaston have to pay?

c) Compare this to the amount of interest he would have to pay if the bank charged simple interest.

Solution

a) List the given information. Then, apply the compound interest formula.

$P = 650$
$i = 0.12$
$n = 5$

$$\begin{aligned} A &= P(1 + i)^n \\ &= 650(1 + 0.12)^5 \\ &= 650(1.12)^5 \\ &\doteq 1145.52 \end{aligned}$$

650 (×) **1.12** (y^x) **5** (=)

Gaston must repay \$1145.52 after 5 years.

b) To calculate the interest paid, subtract the principal from the amount.

$$\begin{aligned} I &= A - P \\ &= 1145.52 - 650 \\ &= 495.52 \end{aligned}$$

Gaston will pay \$495.52 in interest charges.

c) If the bank charges 12% simple interest, then the interest can be found by applying the simple interest formula.

$$\begin{aligned} I &= Prt \\ &= 650(0.12)(5) \\ &= 390 \end{aligned}$$

Gaston would pay only \$390 in simple interest, compared to \$495.52 in compound interest.

Connections

Securing a simple interest loan for a significant period of time is rare in today's financial institutions.

Example 2

Vary the Compounding Period

Financial institutions often use other compounding periods rather than annual. For example, semi-annual compounding means every 6 months, while quarterly compounding means every 3 months, and so on. Refer to Example 1. What will be the impact on the interest Gaston pays if the interest is compounded

a) semi-annually?

b) monthly?

Solution

a) When interest is charged semi-annually (twice a year), the number of compounding periods is doubled and the interest rate per compounding period is halved.

$P = 650$

$i = \frac{0.12}{2}$ Divide by the number of compounding periods in a year, 2.

$= 0.06$

$n = 5 \times 2$ Multiply by the number of compounding periods in a year, 2.

$= 10$

Substitute the known values into the compound interest formula.

$A = P(1 + i)^n$

$= 650(1 + 0.06)^{10}$

$= 650(1.06)^{10}$

$\doteq 1164.05$

Calculate the interest paid.

$I = A - P$

$= 1164.05 - 650$

$= 514.05$

Gaston will pay \$514.05 in interest charges if interest is compounded semi-annually.

b) When interest is charged monthly, the number of compounding periods and the interest rate per compounding period can be calculated as follows.

$P = 650$

$i = \frac{0.12}{12}$ Divide by the number of compounding periods in a year, 12.

$= 0.01$

$n = 5 \times 12$ Multiply by the number of compounding periods in a year, 12.

$= 60$

$A = P(1 + i)^n$

$= 650(1 + 0.01)^{60}$

$= 650(1.01)^{60}$

$\doteq 1180.85$

Calculate the interest paid.

$I = A - P$

$= 1180.85 - 650$

$= 530.85$

Gaston will pay \$530.85 in interest charges if interest is compounded monthly.

The table summarizes the effects of the various compounding periods.

Compounding Period	Amount, A (\$)	Interest, I (\$)
annual	1145.52	495.52
semi-annual	1164.05	514.05
monthly	1180.85	530.85

Note that as the compounding period becomes shorter, the interest charges increase.

Example 3

Determine the Interest Rate

Nina is starting a small business. She applies for an \$8000 loan, which she plans to repay in 4 years. She is told by the loan officer that the amount payable when the loan is due is \$11 501.24. What rate of interest, compounded annually, is Nina being charged?

Solution

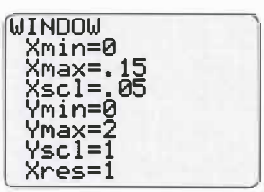
List the given information. Then, apply the compound interest formula.

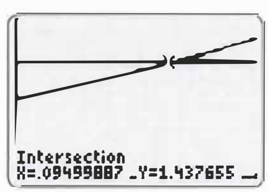
$P = 8000$
$A = 11\ 501.24$
$n = 4$

$$A = P(1 + i)^n$$
$$11\ 501.24 = 8000(1 + i)^4$$
$$\frac{11\ 501.24}{8000} = (1 + i)^4$$
$$1.437\ 655 = (1 + i)^4$$

Method 1: Apply Graphical Analysis

To determine when $(1 + i)^4 = 1.437\ 655$, graph each side of this equation as a separate function using a graphing calculator. Then, use the **Intersect** operation.





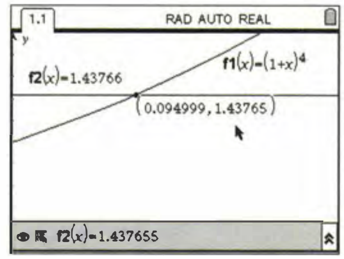
Technology Tip

To use the **Intersection Point(s)** operation on a TI-Nspire™ CAS graphing calculator:

- Press (menu). Select **6:Points & Lines**.
- Select **3:Intersection Point(s)**. Move the cursor to the first graph and press (enter). Move the cursor to the second graph and press (enter).

The coordinates of the intersection point will be displayed.

Similarly, you can use the **Intersection Point(s)** operation on a TI-Nspire™ CAS graphing calculator.



The solution is approximately (0.095, 1.437 655), which indicates that the interest rate is 9.5%, compounded annually.

Method 2: Apply Algebraic Reasoning

To solve the equation $1.437\,655 = (1 + i)^4$ for i, apply the opposite operations to isolate the variable.

$$\sqrt[4]{1.437\,655} = \sqrt[4]{(1 + i)^4}$$ Take the fourth root of both sides.

$$\sqrt[4]{1.437\,655} = 1 + i$$ Apply the properties of powers and roots.

$$\sqrt[4]{1.437\,655} - 1 = i$$ **1.437655** (2nd) $[\sqrt[x]{y}]$ **4** (−) **1** (=)

$$0.095 \doteq i$$

The interest rate is approximately 9.5%, compounded annually.

Key Concepts

- Compound interest investments or loans add the interest from one compounding period to the previous principal and use the sum as the principal for the next compounding period.
- The compounding effect causes an amount to grow exponentially over time. The amounts after each compounding period produce a geometric sequence.
- The compound interest formula $A = P(1 + i)^n$ can be used to calculate the amount, A, if the principal, P; the interest rate per compounding period, i; and the number of compounding periods, n, are known.
- The table shows common methods of compounding.

Frequency of Compounding	Number of Times Interest Is Added During a Year
annual	1 (every year)
semi-annual	2 (every 6 months)
quarterly	4 (every 3 months)
monthly	12 (every month)
bi-weekly	26 (every 2 weeks)
daily	365 (every day)

Communicate Your Understanding

C1 Explain the advantage of compound interest over simple interest, from an investor's point of view. Use an example to illustrate your answer.

C2 The table shows the amount in an annually compounded interest account over time.

Time (years)	Amount ($)
0	400.00
1	416.00
2	432.64
3	449.95
4	467.94

a) Look at the values in the Amount column. Is this an arithmetic or a geometric sequence? Explain.

b) What is the annual rate of compound interest? How do you know?

C3 An account with an initial value of $800 earns 6% interest per year, compounded annually.

a) Describe the shape of the graph of amount versus time.

b) What is the vertical intercept of the graph? What does it represent?

c) Describe what happens to the slope of the graph.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Darlene invests $500 for 6 years at 4% interest per year, compounded annually.

a) Determine the amount in the account after 6 years.

b) How much interest will Darlene earn?

2. Wes borrows $850 at a rate of 9.5% interest per year, compounded annually, for 4 years.

a) Determine the amount to be repaid after 4 years.

b) How much interest will Wes have to pay?

For help with questions 3 to 7, refer to Example 2.

3. For each compounding condition, determine the interest rate per compounding period, expressed as a decimal.

a) 9% per year, compounded monthly

b) 8% per annum, compounded quarterly

c) 6% annual interest, compounded semi-annually

d) 13% per year, compounded bi-weekly

4. For each compounding condition, determine the number of compounding periods.

a) quarterly compounding for 3 years

b) semi-annual compounding for 4 years

c) monthly compounding for $\frac{3}{4}$ of a year

d) daily compounding for 2 weeks

e) annual compounding for 6 years

5. Determine the total number, n, of compounding periods and the interest rate, i, as a decimal, per compounding period for each scenario.

a) 8.75% per year, compounded annually, for 6 years

b) 6% per annum, compounded quarterly, for 3 years

c) 2.4% per year, compounded monthly, for 2 years

d) 4.5% per annum, compounded, semi-annually for 7.5 years

6. Manjinder invests \$1400 in a GIC that earns 5.75% interest per year, compounded quarterly, for 3 years.

   a) Determine the amount in the account after 3 years.

   b) How much interest was earned?

   c) Compare this to the amount of interest that would have been earned if simple interest had been earned at the same rate.

7. Susan's chequing account earns 1.25% interest per year, compounded daily. How much interest will she earn if she has \$2000 in the account for 30 days?

B Connect and Apply

8. Karin borrows \$600 for 5 years at 12% interest per year.

   a) Compare the interest charges under each condition. Then, rank these scenarios from best to worst from Karin's perspective.

      i) simple interest

      ii) annual compounding

      iii) quarterly compounding

      iv) monthly compounding

   b) Explain the effect of the compounding period on this loan.

For help with questions 9 and 10, refer to Example 3.

9. Ramone borrows \$10 000 as start-up capital for his new business. He plans to repay the loan in 4 years, at which point he will owe \$15 180.70. What rate of interest is Ramone being charged, assuming that it is compounded annually?

10. Pat borrows \$350 for 4 years at an interest rate that is compounded quarterly. At the end of the 4 years, she repays \$480.47. What annual interest rate, compounded quarterly, was Pat charged?

11. Suppose you have \$1500 to invest for 5 years. Two options are available:

   • First Provincial Bank: earns 5% per year, compounded semi-annually

   • Northern Credit Union: earns 4.8% per year, compounded monthly

   Which investment would you choose and why?

12. **Chapter Problem** Chloe places \$2000 in a chequing account that earns 1.8% interest per year, compounded monthly, for 4 years. Chloe can access this money at any time with no penalty except lost interest.

   a) Determine the amount in this account after 4 years, assuming no other transactions take place.

   b) How much interest was earned?

   c) Compare this with the interest earned by Chloe's investment in question 10 in Section 7.1. Why might Chloe open this chequing account?

13. The Rule of 72 states that the number of years required for an investment to double when interest is compounded annually can be estimated by dividing 72 by the annual interest rate.

   a) Use the Rule of 72 to determine how many years it will take for an amount to double for each interest rate, compounded annually.

      i) 8% **ii)** 9% **iii)** 12%

   b) Verify your results in part a) using the compound interest formula. Is the Rule of 72 exact? Explain.

14. Pavel invested \$720 in an account that earned 7.25% interest per year, compounded quarterly. When he closed the account, it contained \$985. For how long did Pavel invest his money?

Achievement Check

15. Consider these investment options:

Option A: \$800 invested at 10% per year, simple interest

Option B: \$800 invested at 7% interest per year, compounded annually

a) Write an equation to represent the amount of each investment as a function of time.

b) Graph both equations on the same set of axes.

c) Which is the better investment option? Explain.

d) How does your answer change if the compounding period in Option B changes? Provide a detailed explanation.

C Extend

16. Refer to question 13. Is the Rule of 72 valid for compounding periods other than annual compounding? Explore this question and use mathematical reasoning to justify your response.

Connections

Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 7 to learn more about the Rule of 72, doubling your money, and more.

17. There are several compound interest calculators available on the Internet. Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 7 to see one such calculator. Use this compound interest calculator or an alternative one to answer the following.

a) Why is the online calculator useful?

b) Explain what each field means.

c) Perform a sample calculation using the online tool.

d) Verify the accuracy of the result using the compound interest formula.

e) Discuss the advantages and the disadvantages of the online tool.

18. Math Contest The nominal interest rate is the rate quoted. The effective rate is the actual rate charged if it was simple interest. Some credit cards charge 19% per annum, compounded daily. What is the effective rate, to two decimal places?

A 20.92 **B** 28.00

C 69.35 **D** 19.00

19. Math Contest Sumeet invested \$1200 for 5 years. He put part of the money into a high-yielding risky investment at 18% per annum, compounded semi-annually. He put the rest into another investment at 7% per annum, compounded annually. At the end of 5 years, he had earned \$869 in interest. In which interval does the interest, I, earned at 18% per annum, compounded semi-annually, fall?

A $I < 530$ **B** $530 < I < 540$

C $540 < I < 550$ **D** $I > 550$

20. Math Contest The constant π is used in many applications in mathematics. Another mathematical constant is e. One of the applications for e is continuous compounding. This means the compounding period is infinitely small. The formula for continuous compounding is $A = Pe^{rt}$, where A represents the amount; P represents the principal; $e \doteq 2.718\,281\,828\,459$; r represents the annual rate of interest, as a decimal; and t represents the number of years. Determine the amounts for \$1000 invested for 5 years at 10% for each compounding period.

a) annual **b)** monthly

c) daily **d)** continuous

7.3

Present Value

Do you plan to buy a car some day? Having your own vehicle can provide tremendous freedom and independence. On the downside, owning a vehicle has all sorts of costs associated with it. Aside from the purchase price, you must consider costs such as fuel, insurance, and repairs. It is important to choose the right time to buy, and even to decide whether you should buy a car at all.

Suppose that you have some money to invest today to buy a car at some future date. If you know how much money you need in the future, you can decide how much to invest today at a given interest rate, compounding frequency, and duration.

Investigate

How can you determine the principal that must be invested today to have a known amount in the future?

Eric's parents plan to invest some money so that they can buy a car for their son in 5 years. They estimate that, 5 years from now, they will need \$10 000 to buy a good used car.

1. One bank offers 6% interest, compounded annually. Which of the variables in the compound interest formula $A = P(1 + i)^n$ are known?

2. a) Substitute the known information into the formula and simplify the equation, if possible.

b) Identify the unknown variable. Describe a strategy to solve for the variable.

c) Solve for the unknown variable.

3. Reflect How much should Eric's parents invest under the given conditions, and why?

When the principal, P, needed to generate a known future amount is unknown, the compound interest formula can be rearranged to isolate P.

$$A = P(1 + i)^n$$

$$\frac{A}{(1 + i)^n} = P \qquad \text{Divide both sides by } (1 + i)^n.$$

$$P = \frac{A}{(1 + i)^n}$$

In this form of the formula, the principal is referred to as the **present value** and the amount is referred to as the **future value**. The formula can be rewritten to include these terms.

The present value, PV, of a compound interest account or loan is related to its future value, FV, by the formula $PV = \frac{FV}{(1 + i)^n}$, where i represents the interest rate, as a decimal, per compounding period and n represents the number of compounding periods.

present value

- the principal invested or borrowed today to result in a given future amount, with given interest and time conditions

future value

- the amount that a principal invested or borrowed will grow to, with given interest and time conditions

Example 1

Calculate Present Value

Sahar wants to invest money today to have $1000 in 6 years. If she invests her money at 5.75% per year, compounded quarterly, how much does she need to invest?

Solution

Method 1: Use a Scientific Calculator

Identify the given information.

$FV = 1000$

$$i = \frac{0.0575}{4}$$

Divide by the number of compounding periods in a year, 4.

$$= 0.014\,375$$

$$n = 6 \times 4$$

Multiply by the number of compounding periods in a year, 4.

$$= 24$$

Substitute the known information into the formula for the present value.

$$PV = \frac{FV}{(1 + i)^n}$$

$$= \frac{1000}{(1 + 0.014\,375)^{24}}$$

$$= \frac{1000}{1.014\,375^{24}}$$

1000 $\boxed{\div}$ **1.014375** $\boxed{y^x}$ **24** $\boxed{=}$

$$\doteq 709.96$$

Sahar must invest $709.96 today to have $1000 in 6 years.

Connections

The present value formula can also be written with a negative exponent, as either $P = A(1 + i)^{-n}$ or $PV = FV(1 + i)^{-n}$.

Method 2: Use a TVM Solver

To access the Time Value of Money (TVM) Solver on a graphing calculator, press APPS, select **1:Finance**, and then select **1:TVM Solver...**. Enter the values in the fields as shown.

```
N=6
I%=5.75
PV=0
PMT=0
FV=1000
P/Y=1
C/Y=4
PMT:END BEGIN
```

- N represents the time period of the investment (or loan), in years.
- I% represents the annual interest rate, as a percent.
- PV represents the present value. Do not enter a value, since it is unknown.
- PMT represents the payment. PMT = 0 because there are no additional payments after the initial investment.
- FV represents the future value.
- P/Y represents the number of payments per year. P/Y = 1 because the investment is one time only.
- C/Y represents the number of compounding periods per year.

Technology Tip

Notice that in the TVM Solver you enter the number of years, N, and the annual interest rate, I, as a percent. The TVM Solver automatically converts these using the number of compounding periods per year, C/Y.

To solve for the unknown present value, move the cursor to the **PV** field and press ALPHA ENTER for [SOLVE]. The present value will be calculated.

```
N=6
I%=5.75
▪PV=-709.9620061
PMT=0
FV=1000
P/Y=1
C/Y=4
PMT:END BEGIN
```

Note that the present value is negative. This means that the initial amount is paid out, as opposed to received.

Connections

The fields for PMT and P/Y are not required for this type of calculation. You will use them when you learn about annuities in Sections 7.4 and 7.5.

Sahar must invest $709.96 today to have $1000 in 6 years.

Example 2

Determine the Number of Compounding Periods

Tamara received $250 for her 14th birthday, which she invested at 6% per year. On Tamara's 18th birthday, her investment is worth $317.62. How frequently was the interest compounded?

Solution

Method 1: Apply Graphical Analysis

Let x represent the number of compounding periods per year. Identify the given information.

$PV = 250$

$FV = 317.62$

$i = \frac{0.06}{x}$

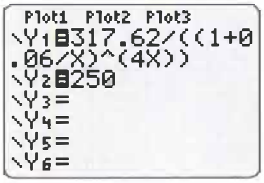
$n = 4x$

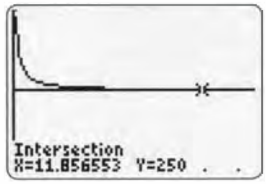
Substitute these values and expressions into the present value formula.

$$PV = \frac{FV}{(1 + i)^n}$$

$$250 = \frac{317.62}{\left(1 + \frac{0.06}{x}\right)^{4x}}$$

This equation is difficult to solve for x using standard algebraic techniques. Solve for x graphically. Graph each side of this equation as a separate function using a graphing calculator, and use the **Intersect** operation.

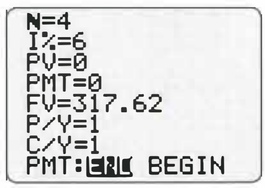




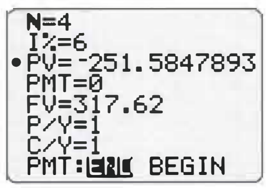
The point of intersection is approximately (12, 250), which means that when interest is compounded 12 times per year, the present value of this account is \$250. Therefore, interest in this account was compounded monthly.

Method 2: Use a TVM Solver and Systematic Trial

Access the TVM Solver and enter the values in the fields as shown. The number of compounding periods per year is unknown, so try different values for C/Y and see which gives a present value, PV, closest to \$250.

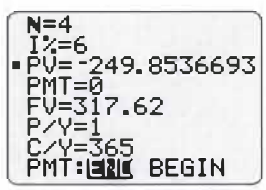


Start with annual compounding and use a systematic approach. With C/Y set to 1, move the cursor to the **PV** field and press (ALPHA) (ENTER) for [SOLVE].



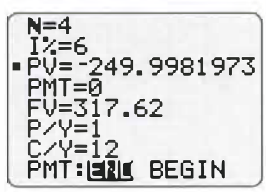
This present value, \$251.58, is more than \$250, which suggests that interest was earned faster. Try daily interest.

Daily interest means 365 compounding periods per year.



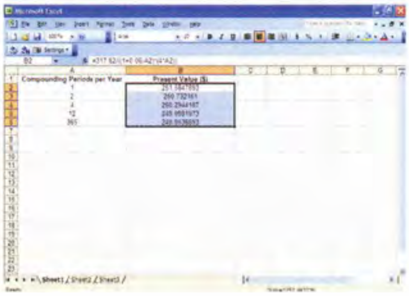
This is very close. Try monthly compounding.

Monthly interest means 12 compounding periods per year.



Monthly compounding gives a present value of \$250.00, correct to the nearest cent. Therefore, the interest in this account was compounded monthly.

Method 3: Use a Spreadsheet



Set up a table using a spreadsheet, as shown. Include the most common compounding scenarios (annual, semi-annual, quarterly, monthly, and daily) in column A. To calculate the present value for each scenario, enter the present value formula =317.62/((1+0.06/A2)^(4*A2)) in cell B2. Use **Fill Down** to calculate the remaining present values.

Monthly compounding gives a present value of \$250.00, correct to the nearest cent. Therefore, the interest in this account was compounded monthly.

Example 3

Choose a Loan

Lamar wants to borrow some money now to buy a motorcycle and some riding gear. He estimates that in 5 years he can pay back an amount of \$12 000. Which of the two loan options should he choose, and why?

Regal Bank: charges 8.9% interest, compounded semi-annually

Suburbia Credit Union: charges 8.4% interest, compounded monthly

Solution

At a glance, the Suburbia Credit Union loan seems to charge a lower interest rate, but the interest is compounded more frequently than with the Regal Bank loan. Calculate and compare the present values of the loans to decide which is the better option.

Regal Bank:

$FV = 12\ 000$

$$i = \frac{0.089}{2} = 0.0445$$

$$n = 5 \times 2 = 10$$

Substitute these values into the present value formula.

$$PV = \frac{12\ 000}{(1 + 0.0445)^{10}} \doteq 7764.20$$

Suburbia Credit Union:

$FV = 12\ 000$

$$i = \frac{0.084}{12} = 0.007$$

$$n = 5 \times 12 = 60$$

Substitute these values into the present value formula.

$$PV = \frac{12\ 000}{(1 + 0.007)^{60}} \doteq 7896.11$$

The Suburbia Credit Union loan has a greater present value than the Regal Bank loan. For Lamar's situation, the Suburbia Credit Union loan is the better option.

Key Concepts

- Present value refers to the principal that must be invested today to grow to a known future amount under specified interest and time conditions.
- The formula $PV = \frac{FV}{(1 + i)^n}$ can be used to calculate the present value, PV, if the future value, FV; number of compounding periods, n; and interest rate, i, as a decimal, per compounding period are known.

Communicate Your Understanding

C1 Explain how the following terms are related. Include examples to illustrate.

a) principal and amount

b) principal and present value

c) present value, future value, and interest

C2 An investment earns 9% annual interest, compounded semi-annually, for 4 years, at which point it is worth \$284.42. Which equation will give the correct present value, and why? Explain what is wrong with each of the other equations.

A $PV = 284.42(1 + 0.045)^8$

B $PV = \frac{284.42}{(1 + 0.09)^4}$

C $PV = \frac{284.42}{(1 + 0.18)^2}$

D $PV = \frac{284.42}{(1 + 0.045)^8}$

A Practise

For help with questions 1 to 4, refer to Example 1.

1. Determine the present value of each future amount for the given conditions.

a) In 5 years, an investment earning 5% per year, compounded annually, will have a value of \$700.

b) In 3 years, an investment earning 4.8% annual interest, compounded quarterly, will have a value of \$1021.86.

2. In 4 years, an investment will be worth \$504.99. If interest is earned at a rate of 6% per year, compounded annually, what is the present value of this investment?

3. In 6 years, money invested at 7.5% per annum, compounded quarterly, will grow to \$807.21.

a) How much money was invested?

b) How much interest will be earned in 6 years?

4. Pina receives a financial gift from her grandparents, which she invests at 8% annual interest, compounded semi-annually. She is advised that the investment will be worth \$3421.40 in 4 years.

a) What is the amount of the gift?

b) How much interest will Pina's investment earn?

B Connect and Apply

For help with questions 5 and 6, refer to Example 2.

5. A bond will be worth \$500 when it becomes due in 5 years. If the bond was purchased today for \$450 at 2.13% per year, determine how frequently the interest was compounded.

6. Serge invests \$700 at 5.75% per year, compounded quarterly. When the account is closed, its value will be \$950. How long will Serge's money be invested?

For help with questions 7 and 8, refer to Example 3.

7. Tracey would like to have \$10 000 in 4 years to use as a down payment for a house. She is considering two investment options:

Investment A: 6.6% annual interest, compounded semi-annually

Investment B: 6.2% annual interest, compounded monthly

a) Compare the present values of the two options.

b) Which investment is the better choice for Tracey? Explain your reasoning.

8. Jacques needs to borrow money to buy dress clothes for his new sales job. He estimates that he can repay \$1600 in 6 months.

He is considering two bank offers:

Bank A: 8.5% annual interest, compounded monthly

Bank B: 9% simple interest

Which bank should Jacques borrow from, and why?

9. Five years ago, money was invested at 7% per year, compounded annually. Today the investment is worth \$441.28.

a) How much money was originally invested?

b) How much interest was earned?

10. Four and a half years ago, some money was deposited into an account that paid an annual interest rate of 3.2%, compounded semi-annually. Today, the account has a value of \$821.36. What was the amount of the original deposit?

11. Lydia borrows \$500 to buy a television. She agrees to repay \$610 in a year and a half. What annual rate of interest, compounded monthly, is Lydia being charged?

12. a) An investment reaches a future value of \$2500 in 5 years. Explore and compare the effects of varying the compounding period on the present value under each condition.

i) annual compounding

ii) semi-annual compounding

iii) quarterly compounding

iv) monthly compounding

b) What implication do your results have from an investor's perspective?

13. A loaf of bread costs \$3.50 in 2009.

a) What was the price of a similar loaf of bread in 1990, assuming an average inflation rate of 3% per year, compounded annually?

b) How much did a similar loaf cost in 1900?

Connections

The consumer price index (CPI) provides a broad measure of the cost of living in Canada. The CPI measures the average price of consumer goods and services bought by households, including food, housing, transportation, furniture, clothing, and recreation.

14. The future value of a loan due to a financial institution in 10 years is \$50 000. The financial institution is willing to sell the debt today discounted at 6% per year, compounded semi-annually. What is the value of the debt today?

Connections

A financial institution considers money owed to it an investment. When a financial institution discounts an investment, it sells the investment to another creditor at a discounted value that is equal to the present value.

15. Chapter Problem Chloe has \$6000 in her portfolio left to invest. On the advice of her advisor, she puts the money into a moderate-risk mutual fund. She is hoping that the total future value of her investments will be \$12 000 at the end of 4 years.

a) Refer to Section 7.1, question 10, and Section 7.2, question 12. What are the future values of Chloe's other investments?

b) Subtract these from \$12 000. What does this answer represent?

c) Determine the minimum interest rate that the mutual fund must earn for Chloe to achieve her goal, assuming that interest is compounded annually.

16. Explore the concept of present value as a function of time. Suppose an account earning 6% per year, compounded annually, has a future value of \$800.

a) Write an equation to represent the present value of the account as a function of time.

b) Use Technology Graph the function. Describe the shape of the graph.

c) Interpret the horizontal scale of the graph.

d) Describe what information the graph provides for $t > 0$.

e) Does the graph have meaning for $t < 0$? If it does, explain what it means. If it does not, explain why not.

✓ Achievement Check

17. Tanya wants to have \$1200 in 2.5 years to go to Mexico.

a) How much money must she invest today at 6.4% annual interest, compounded semi-annually, to have enough money?

b) Tanya only has \$970 today. What interest rate must she obtain to have enough money for her vacation?

c) Suppose that Tanya cannot find a better rate than 6.4%. What other options does she have?

C Extend

18. Refer to question 16. Discuss the effects of reflecting this function in the vertical axis, from a geometric perspective and a time-line perspective.

19. Use algebraic reasoning to develop a formula for the present value, PV, of a simple interest account, in terms of its future value, FV; the simple annual rate of interest, r; and time, t, in years.

20. Math Contest A debt can be paid off in three equal instalments: \$1000 now, \$1000 in 3 years, and \$1000 in 6 years. What single payment can pay off the loan 4 years from now, if interest is 10% per annum, compounded semi-annually?

21. Math Contest A circle passes through the points A(5, 7) and B(−3, 11). Which of the following is not a possible centre for the circle?

A (1, 9) **B** (0, 7)

C (11, 29) **D** (−2, 11)

7.4

Annuities

How would you like to be a millionaire without working all your life to earn it? Perhaps if you were lucky enough to win a lottery or have an amazing run on a television game show, it would happen. For most people, however, fantasies such as these are not likely to come true. However, what if you saved money from a very early age? Is it possible to accumulate a million dollars in your lifetime?

When people invest, they usually do not simply deposit one lump sum and wait several years for it to earn interest. Most wise investors make **regular payments**, often deducted directly from their paycheques. Investments of this type are called **annuities**. In this chapter, you will encounter only **ordinary simple annuities**.

regular payments
- payments of equal value made at equal time periods

annuity
- a sum of money paid as a series of regular payments

ordinary annuity
- an annuity for which the payments are made at the end of each payment period

simple annuity
- an annuity for which the compounding and payment periods are the same

Investigate

How can you determine the amount of an annuity?

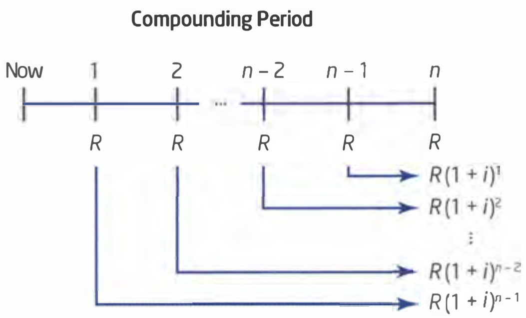
Colin is awarded \$500 per year as long as he maintains a certain average mark for each of his 4 years of high school. Colin plans to deposit his award at the end of each school year into an account that pays 4% per year, compounded annually.

1. a) Assuming that Colin maintains the necessary average, how many times will he receive \$500?
   b) Will each of these \$500 deposits earn the same interest, in dollars? Explain.
2. a) Determine a method of calculating the total amount of Colin's investment at graduation.
   b) Carry out the method. How much will be in Colin's account when he graduates?
3. **Reflect** Consider the method that you used to solve this problem. Would your method be efficient if the number of regular payments was very large? Why or why not?

Situations like the one described in the Investigate can be represented using a **time line**.

time line
- a diagram used to illustrate the cash flow of an annuity

The time line shows that a regular payment, R, in dollars, is deposited into an account at the end of each compounding period, for n periods. Because these deposits are made at different times, they will each earn different amounts of interest. For example, the last payment will earn no interest, because it will be received at the end of the annuity. To determine the amount that each of the other deposits will earn, apply the compound interest formula $A = P(1 + i)^n$ to each payment individually.



The amount, A, of the annuity can be determined by adding the amounts of all the payments.

$$A = R + R(1 + i) + R(1 + i)^2 + \cdots + R(1 + i)^{n-2} + R(1 + i)^{n-1}$$

Since this is a geometric series with first term $a = R$ and common ratio $r = 1 + i$, use the formula for the sum of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{R[(1 + i)^n - 1]}{(1 + i) - 1}$$

$$= \frac{R[(1 + i)^n - 1]}{i}$$

Connections

You studied geometric series in Chapter 6 Discrete Functions.

The total amount, A, at the time of the last payment of an annuity can be determined using the formula $A = \frac{R[(1 + i)^n - 1]}{i}$, where R represents the regular payment, in dollars; i represents the interest rate per compounding period, as a decimal; and n represents the number of compounding periods.

This equation can also be written in terms of future value as $FV = \frac{R[(1 + i)^n - 1]}{i}$.

Connections

In this course, you will study only ordinary simple annuities. You will study more complex annuities with, for example, different compounding and payment periods, if you choose to study business at university or college.

Example 1

Amount of an Annuity

At the end of every month, Hoshi deposits \$100 in an account that pays 6% per year, compounded monthly. He does this for 3 years.

a) Draw a time line to represent this annuity.

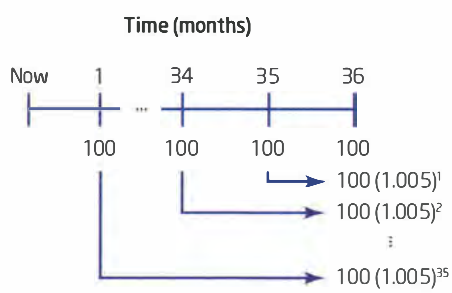
b) Determine the amount in the account after 3 years.

c) How much interest will the annuity have earned?

Solution

a) Determine the interest per compounding period and the number of compounding periods before drawing the time line.

$$i = \frac{0.06}{12}$$
$$= 0.005$$
$$n = 3 \times 12$$
$$= 36$$
$$R = 100$$



b) Method 1: Use a Scientific Calculator

Substitute the known values into the formula for the amount of an annuity and evaluate.

$$A = \frac{R[(1 + i)^n - 1]}{i}$$
$$= \frac{100[(1 + 0.005)^{36} - 1]}{0.005}$$
$$= \frac{100(1.005^{36} - 1)}{0.005}$$

100 × (**1.005** y^x **36** − **1**) ÷ **0.005** =

$$\doteq 3933.61$$

The amount in Hoshi's account after 3 years will be \$3933.61.

Method 2: Use a TVM Solver

Access the TVM Solver on a graphing calculator and enter the values, as shown.

```
N=36
I%=6
PV=0
PMT=-100
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN
```

When solving for an annuity, enter the number of payments for N.

The negative sign indicates that payments are being paid, not received.

Both the number of payments and the compounding periods per year are 12.

Move the cursor to the **FV** field and press (ALPHA) [SOLVE].

```
N=36
I%=6
PV=0
PMT=-100
FV=3933.610496
P/Y=12
C/Y=12
PMT:END BEGIN
```

The future value of the sum of Hoshi's payments is \$3933.61. This is the amount in his account after 3 years.

c) To determine the interest earned, calculate the difference between the actual dollar sum of Hoshi's payments and the future value of the annuity. Hoshi made 36 payments of \$100, for a total of \$3600.

$$\begin{aligned}\text{Interest} &= 3933.61 - 3600 \\ &= 333.61\end{aligned}$$

Hoshi's annuity earned \$333.61 in interest.

Example 2

Determine the Regular Payment

Sadia needs \$4000 for university tuition when she graduates in 2 years. She plans to make deposits into an account that earns 6.5% per year, compounded bi-weekly.

a) Draw a time line to represent this annuity.

b) How much should she deposit bi-weekly?

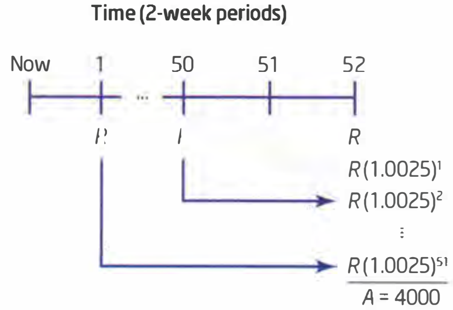
Solution

a) Bi-weekly means every 2 weeks. Since there are 52 weeks in a year, the number of compounding periods per year is 52 ÷ 2, or 26.

$$\begin{aligned}i &= \frac{0.065}{26} \\ &= 0.0025\end{aligned}$$

$$\begin{aligned}n &= 2 \times 26 \\ &= 52\end{aligned}$$

The regular payment, R, is unknown, but the total future value of the annuity, A, is \$4000. A time line representing this annuity is shown.



b) Use the formula for the amount of an annuity to solve for the regular payment, R.

Method 1: Substitute and Then Rearrange

$$A = \frac{R[(1 + i)^n - 1]}{i}$$

$$4000 = \frac{R[(1 + 0.0025)^{52} - 1]}{0.0025} \qquad \text{Substitute the known values.}$$

$$10 = R(1.0025^{52} - 1) \qquad \text{Multiply both sides by 0.0025.}$$

$$R = \frac{10}{1.0025^{52} - 1} \qquad \text{Divide both sides by } 1.0025^{52} - 1.$$

$$R \doteq 72.13$$

Sadia should deposit \$72.13 bi-weekly to have \$4000 in 2 years.

Method 2: Rearrange and Then Substitute

$$A = \frac{R[(1 + i)^n - 1]}{i}$$

$$Ai = R[(1 + i)^n - 1] \qquad \text{Multiply both sides by } i.$$

$$R = \frac{Ai}{(1 + i)^n - 1} \qquad \text{Divide both sides by } (1 + i)^n - 1.$$

$$R = \frac{4000(0.0025)}{(1 + 0.0025)^{52} - 1} \qquad \text{Substitute the known values.}$$

$$R \doteq 72.13$$

Sadia should deposit \$72.13 every 2 weeks to have \$4000 in 2 years.

Example 3

Determine the Interest Rate

Oliver plans to invest \$2000 quarterly for 5 years. His financial advisor informs him that his investment will grow to \$45 682.40 after the 5 years. What annual rate of interest, compounded quarterly, is Oliver's annuity earning?

Solution

List the known information for this ordinary simple annuity.

$A = 45\ 682.40$

$R = 2000$

$n = 5 \times 4$ Payments are made 4 times a year for 5 years.

$\quad = 20$

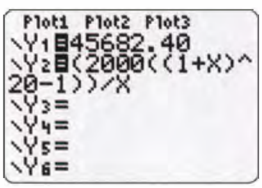
Substitute the known values into the equation for the amount of an annuity and solve for i.

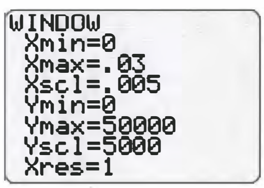
$$45\ 682.40 = \frac{2000[(1 + i)^{20} - 1]}{i}$$

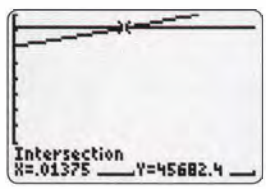
This equation is difficult to solve using standard algebraic techniques. A method of systematic trial could be used, but it would be time-consuming.

Method 1: Apply Graphical Analysis

Graph each side of the equation $45\ 682.40 = \frac{2000[(1 + i)^{20} - 1]}{i}$ as a separate function using a graphing calculator, and use the **Intersect** operation.







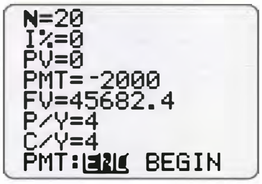
The point of intersection is (0.013 75, 45 682.4), which means that the account will grow to \$45 682.40 at the end of the annuity when the interest rate per compounding period is 0.013 75, or 1.375%. To determine the annual interest rate, multiply by the number of compounding periods per year.

$1.375\% \times 4 = 5.5\%$

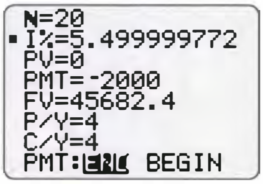
Oliver's annuity is earning interest at a rate of 5.5%, compounded quarterly.

Method 2: Use a TVM Solver

Access the TVM Solver on a graphing calculator and enter the values, as shown.



Move the cursor to the **I%** field and press ALPHA [SOLVE].

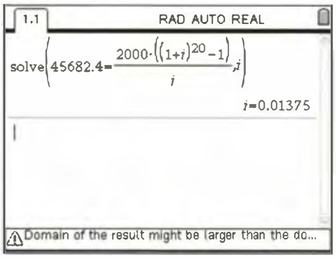


Recall that I% represents the annual interest rate, as a percent. Oliver's annuity is earning interest at a rate of 5.5%, compounded quarterly.

Method 3: Use a TI-Nspire™ CAS Graphing Calculator

Use the **solve** function of the TI-Nspire™ CAS graphing calculator to solve for the interest rate.

- From the home screen, select **6:New Document**.Then, select **1:Add Calculator**.
- Press menu. Select **3:Algebra**, and then select **1:Solve**.
- Type the equation, and then press , i enter.



The interest rate per compounding period is 0.013 75, or 1.375%. To determine the annual interest rate, multiply by the number of compounding periods per year.

$1.375\% \times 4 = 5.5\%$

Therefore, Oliver's annuity is earning interest at a rate of 5.5%, compounded quarterly.

Example 4

Vary the Conditions of an Annuity

Felicia plans to invest $2600 at 6% per year, compounded annually, for the next 15 years. Compare the effects on the final amount if the deposits are made and compounding periods are

- annual
- quarterly
- monthly
- weekly

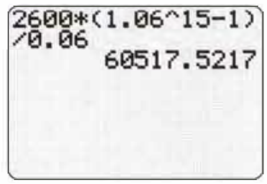
Solution

Method 1: Use a Graphing Calculator

Use a table to organize the values of the variables needed to use the formula for the amount of an annuity $A = \dfrac{R[(1 + i)^n - 1]}{i}$. Note that the regular payment, R, must be divided by the number of payments per year in each scenario.

Compounding Period	R	i	n
annual	2600	0.06	15
quarterly	650	0.015	60
monthly	216.67	0.005	180
weekly	50	0.001 154	780

Calculate the amount for the first scenario, annual compounding, using a graphing calculator.



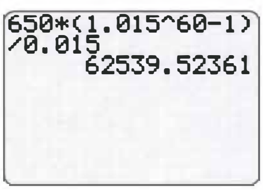
Felicia will have $60 517.52 at the end of 15 years if interest is compounded annually.

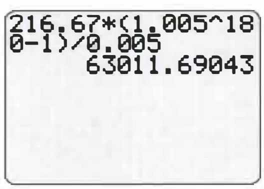
Repeat the calculation for the other scenarios.

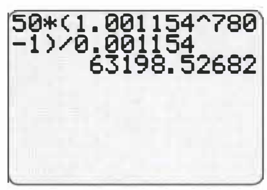
- Press 2nd [ENTRY] to recall the previous calculation.
- Use the cursor keys, the **DEL** key, and the **INS** key to modify the equation to fit each scenario.
- Press ENTER to perform the new calculation.

Technology Tip

Press CLEAR to start each calculation with a new window screen.







The results of these calculations are summarized in the table.

Compounding Period	Amount ($)
annual	60 517.52
quarterly	62 539.52
monthly	63 011.69
weekly	63 198.53

The amount of the annuity increases as the compounding interval becomes more frequent. The difference between weekly and annual compounding is $63 198.53 − $60 517.52, or $2681.01.

Method 2: Use a Spreadsheet

Set up a table in a spreadsheet.

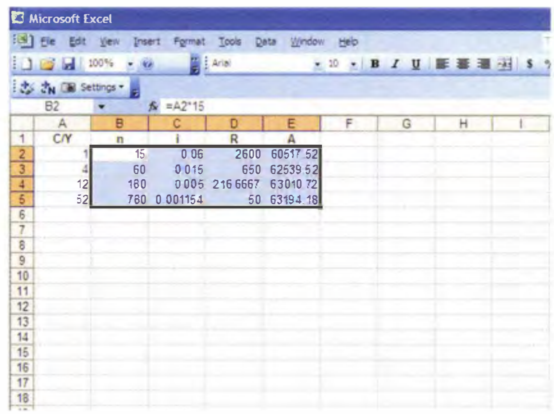
- Type the headings *C/Y* (compounding periods per year), *n* (number of compounding periods or payments), *i* (interest rate per compounding period), *R* (regular payment), and *A* (amount).
- Enter the values 1, 4, 12, and 52 in the C/Y column, to represent the number of compounding periods in each scenario.
- Enter the formulas, starting in cell B2 and working to the right:

B2 =A2*15	$n = 15 \times$ number of payments per year
C2 =0.06/A2	$i = 0.06 \div$ number of payments per year
D2 =2600/A2	$R = \$2600 \div$ number of payments per year
E2 =(D2*((1+C2)^B2-1))/C2	Calculate the amount using the formula $A = \dfrac{R[(1 + i)^n - 1]}{i}$.

- Use **Fill Down** to evaluate the remaining calculations.

The results of these calculations are summarized in the table.

Compounding Period	Amount ($)
annual	60 517.52
quarterly	62 539.52
monthly	63 010.72
weekly	63 194.18



The amount of the annuity increases as the compounding interval becomes more frequent. The difference between weekly and annual compounding is $63 194.18 − $60 517.52, or $2676.66.

Technology Tip

To enter a mathematical expression in a cell in Microsoft® Excel, click on the equal sign and type the expression.

Connections

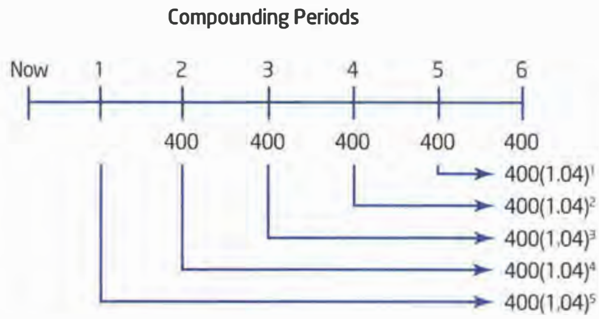
Some of the amounts are slightly different in Method 1 and Method 2. This is due to rounding in the calculation entry steps. Discrepancies such as this become increasingly important when large sums of money are involved. In general, rounding errors can be avoided or minimized by leaving rational values in fraction form or carrying additional decimal places in the calculation process.

Key Concepts

- An annuity is an investment in which regular payments are deposited into an account.
- An ordinary simple annuity is one in which payments are made at the end of every payment period and interest is compounded at the end of the same payment period.
- The amount, A, of an annuity can be calculated using the formula $A = \dfrac{R[(1 + i)^n - 1]}{i}$, where R represents the regular payment; i represents the interest rate per compounding period, as a decimal; and n represents the number of compounding periods.

Communicate Your Understanding

C1 The time line shows an annuity with an annual interest rate of 8%.



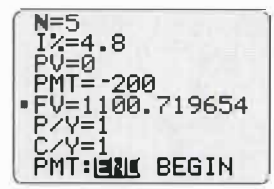
a) How often is interest compounded? How can you tell?

b) What is the duration of the annuity? How can you tell?

c) Explain why this annuity can be represented as a geometric series.

d) Identify the first term, a, and the common ratio, r, of the geometric series.

C2 The graphing calculator screen of a solution using the TVM Solver is shown.



a) What is the duration of the annuity? How can you tell?

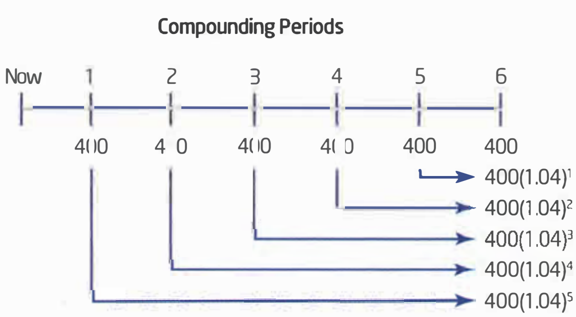
b) Why is the payment value negative?

c) Why is the future value positive?

A Practise

For help with questions 1 to 3, refer to Example 1.

1. Calculate the amount of the annuity shown in the time line.



2. To help her granddaughter with university costs, Sasha's grandmother puts $250 into an account that earns 4.5% per year, compounded annually, at the end of every year for 6 years.

a) Draw a time line to represent this annuity.

b) Determine the amount of the annuity.

c) How much interest was earned?

3. At the end of every week, for 2 years, Carlo puts $35 into an account that earns 5.2% per year, compounded weekly.

a) Draw a time line to represent this annuity.

b) Determine the amount of the annuity.

c) How much interest was earned?

For help with questions 4 and 5, refer to Example 2.

4. How much must be invested at the end of each year, for 4 years, to achieve an amount of $10 000, if interest is earned at a rate of 6.25% per year, compounded annually?

5. Lucy wants to have $18 000 in her account 3 years from now to buy a car. How much must she invest per month, if her account earns 7.2% annual interest, compounded monthly?

B Connect and Apply

For help with questions 6 and 7, refer to Example 3.

6. Donna invests \$75 every 2 weeks in an account that earns compound interest bi-weekly. If she does this for 7 years, she will end up with \$16 939.83 in the account.

a) How much total interest will have been earned?

b) Determine the annual rate of interest, compounded bi-weekly.

7. Refer to question 6. By how much must the interest rate be increased for the amount to grow to \$18 000 after 7 years?

For help with questions 8 to 10, refer to Example 4. Use the following information. Maurice is planning to deposit \$160 per month into an account that earns 4.8% annual interest, compounded monthly, for 15 years.

8. a) Determine the amount in the account at the end of this annuity.

b) How much interest will have been earned?

9. Maurice's financial advisor suggests that he would improve the value of his annuity if he changed his payments to \$40 per week, at the same interest rate, compounded weekly. Do you agree or disagree with the financial advisor? Justify your response with mathematical reasoning.

10. A competing bank offers Maurice 5% per annum, compounded monthly, for his monthly deposits of \$160. Which option should Maurice choose? Justify your answer with mathematical reasoning.

- Stick with his current arrangement.
- Follow his financial advisor's suggestion about increasing the frequency of his deposits.
- Switch to the competing bank.

11. Lee would like to retire at age 60 and is considering two investment options:
Option A: Invest \$500 per month beginning at age 20.
Option B: Invest \$1000 per month beginning at age 40.
In both cases, the interest is 6% per annum, compounded monthly. Which option pays more interest, and by how much?

12. Pinder wants to be a millionaire before he retires. He plans to save a certain amount every week for 40 years.

a) If he puts money in an investment that earns 7% annual interest, compounded weekly, what amount must Pinder deposit weekly?

b) What other strategies could Pinder use to achieve his goal? Discuss any assumptions you must make.

C Extend

13. Use Technology Use a graphing calculator or graphing software.

a) Graph the function $A = \dfrac{100(1.05^n - 1)}{0.05}$.
Describe the shape of the graph.

b) Interpret this function, assuming that it is related to the amount of an annuity.

c) Assuming that interest is compounded annually, identify the regular payment and the annual interest rate.

d) Pose and solve two problems related to this function.

14. In question 13, the function $A = \frac{100(1.05^n - 1)}{0.05}$ represents the amount of an annuity.

a) Write a function to describe the total principal invested after n compounding periods.

b) Graph the amount and the principal functions on the same set of axes. Describe the shape of the principal function.

c) Describe how the graphs of these two functions represent the interest earned over time. Write a function to represent the interest earned after n compounding periods.

15. Math Contest Bethany starts investing \$300 per month at 6% per annum, compounded monthly, for 5 years. After 3 years, the interest increases to 9% per annum, compounded monthly. Determine the amount of her investment after 5 years.

A \$19 657.37 **B** \$21 975.21

C \$21 453.45 **D** \$3975.21

16. Math Contest In △ABC, $a = 10$ mm, $b = 26$ mm, and $c = 24$ mm. If D is the midpoint of AC and BC is extended to E such that DE = 24 mm, determine the measure of ∠CED without using a calculator.

A 30° **B** 60° **C** 45° **D** 67°

17. Math Contest Given that $a^2 + (a + b)^2 = 100$ and a and b are whole numbers, determine all possible ordered pairs (a, b) that solve this equation.

18. Math Contest Given $x^2 - y^2 = 2311$, $2311 = (2)(3)(5)(7)(11) + 1$, and $x > y > 0$, where x and y are integers, which of the following is true?

A only x is divisible by 11

B only y is divisible by 11

C both are divisible by 11

D neither is divisible by 11

Career Connection

Felicity is an investment trader for a large firm in Toronto. She uses her company's money to buy and sell stocks, bonds, and shares to make a profit. Because the stock market is so volatile, Felicity must always have access to the newest information from around the world. Success in her job depends on the use of computers that can perform fast mathematical calculations. Waiting too long to make a trade could cost her company a lot of money. Felicity trained for her career by taking a 4-year bachelor of administrative studies degree at York University.

7.5

Present Value of an Annuity

Owen and Anna are approaching retirement and are putting their finances in order. They have worked hard and invested their earnings so that they now have a large amount of money on which to live. They hire a financial advisor, and together they consider whether Owen and Anna have enough money to allow them to live comfortably for the rest of their lives by making **regular withdrawals** from an account. To do this, they calculate the **present value of an annuity** based on Owen and Anna's projected living expenses.

regular withdrawals

- withdrawals of equal value drawn at equal periods

present value of an annuity

- the amount of money needed to finance a series of regular withdrawals

Investigate

How can you determine the present value of an annuity?

Owen and Anna have estimated that they will need to withdraw $1000 per month for living expenses for the next 20 years, and wonder if they will have enough to finance this. The amount in their account will earn 9% annual interest, compounded monthly.

1. a) How many withdrawals have Owen and Anna planned for? How do you know?

b) Multiply the total number of withdrawals by $1000. Do Owen and Anna need to have this much in their life savings account on the day they retire? Explain why or why not.

2. Suppose Owen and Anna retire at the end of December. They will make their first withdrawal from their life savings account at the end of January.

a) Determine the present value of this first withdrawal, using the formula $PV = \frac{FV}{(1 + i)^n}$.

b) Determine the present value of the second withdrawal, which Owen and Anna will make at the end of February.

c) Are the present values of each withdrawal equal? Explain why or why not.

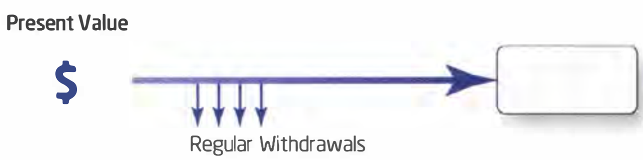
d) Predict whether the present value of the third withdrawal will be greater than or less than these values. Explain your prediction. Check your prediction by calculating.

3. Will the pattern of present values observed in step 2 continue? Explain your thinking.

4. **Reflect** Suggest a method to determine the sum of the present values of all of the withdrawals that Owen and Anna plan to make after they retire.

In Section 7.4, problems were posed in which regular payments are made into an account that grows to a large future amount.

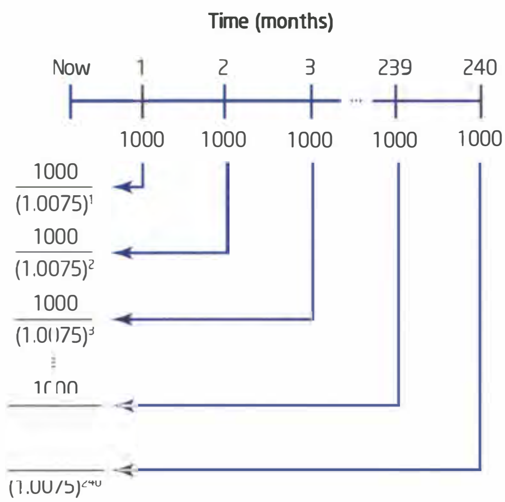
In this section, problems will be posed in which regular withdrawals will be made from an account that begins with a large balance.



To determine the present value required to finance a retirement plan such as the one in the Investigate, it is necessary to calculate the present value of each withdrawal using the present value formula

$PV = \frac{FV}{(1 + i)^n}$.

The present value of the annuity can be determined by adding the present values of all the withdrawals.



$$PV = \frac{1000}{1.0075^1} + \frac{1000}{1.0075^2} + \frac{1000}{1.0075^3} + \cdots + \frac{1000}{1.0075^{239}} + \frac{1000}{1.0075^{240}}$$

Since this is a geometric series with first term $a = \frac{1000}{1.0075^1}$ and common ratio $r = \frac{1}{1.0075}$, the formula for the sum of the first n terms of a geometric series $S_n = \frac{a(r^n - 1)}{r - 1}$ can be used.

This process can be generalized to produce a simplified result for the present value of an annuity. You can derive this result in question 15.

The present value, PV, of an annuity can be determined using the formula $PV = \frac{R[1 - (1 + i)^{-n}]}{i}$, where R represents the regular withdrawal; i represents the interest rate per compounding period, as a decimal; and n represents the number of compounding periods.

Example 1

Present Value of an Annuity

Josh is putting his summer earnings into an annuity from which he can draw living expenses while he is at university. He will need to withdraw \$900 per month for 8 months. Interest is earned at a rate of 6%, compounded monthly.

a) Draw a time line to represent this annuity.

b) How much does Josh need to invest at the beginning of the school year to finance the annuity?

Solution

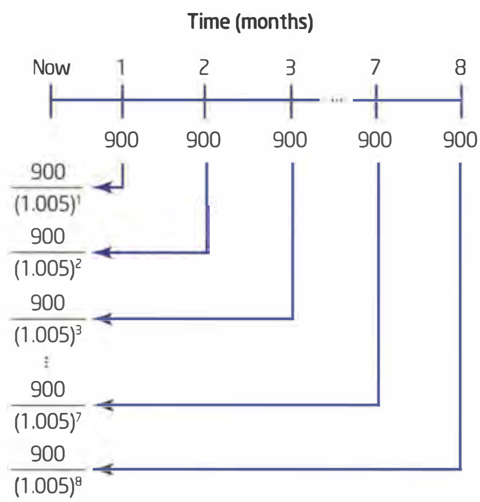
a) Represent the known information on a time line.

$R = 900$

$n = 8$

$i = \frac{0.06}{12}$

$= 0.005$



b) Method 1: Use a Scientific Calculator

Substitute the known values into the formula for the present value of an annuity and evaluate.

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i}$$

$$= \frac{900[1 - (1 + 0.005)^{-8}]}{0.005}$$

$$= \frac{900(1 - 1.005^{-8})}{0.005}$$

900 [×] [(] 1 [−] 1.005 [y^x] 8 [+/−] [)] [÷] 0.005 [=]

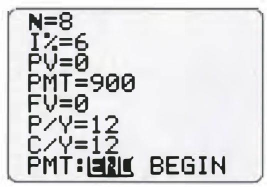
$$\doteq 7040.66$$

Calculator key strokes may vary.

Josh needs to invest \$7040.66 at the beginning of the school year to finance the annuity.

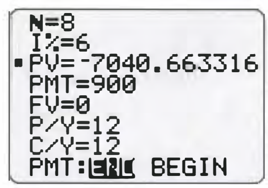
Method 2: Use a TVM Solver

Access the TVM Solver on a graphing calculator and enter the values, as shown.



Move the cursor to the **PV** field and press (ALPHA) [SOLVE].

The present value of this annuity is \$7040.66, which is the amount that Josh must invest at the beginning of the school year.



Example 2

Determine the Regular Withdrawal

Fiona's life savings total \$300 000 when she decides to retire. She plans an annuity that will pay her quarterly for the next 30 years. If her account earns 5.2% annual interest, compounded quarterly, how much can Fiona withdraw each quarter?

Solution

Determine the number of compounding periods and the interest rate per compounding period.

$$n = 30 \times 4 \qquad i = \frac{0.052}{4} \qquad PV = 300\ 000$$

$$= 120 \qquad = 0.013$$

Use the formula for the present value of an annuity to solve for the regular withdrawal, R.

Method 1: Substitute and Then Rearrange

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i}$$

$$300\ 000 = \frac{R[1 - (1 + 0.013)^{-120}]}{0.013}$$ Substitute the known values.

$$3900 = R(1 - 1.013^{-120})$$ Multiply both sides by 0.013.

$$R = \frac{3900}{(1 - 1.013^{-120})}$$ Divide both sides by $1 - 1.013^{-120}$.

$$R \doteq 4950.87$$

Fiona can withdraw $4950.87 every quarter for 30 years.

Method 2: Rearrange and Then Substitute

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i}$$

$$i(PV) = R[1 - (1 + i)^{-n}]$$ Multiply both sides by i.

$$R = \frac{i(PV)}{1 - (1 + i)^{-n}}$$ Divide both sides by $1 - (1 + i)^{-n}$.

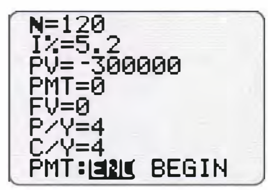
$$R = \frac{0.013(300\ 000)}{1 - (1 + 0.013)^{-120}}$$ Substitute the known values.

$$R \doteq 4950.87$$

Fiona can withdraw $4950.87 every quarter for 30 years.

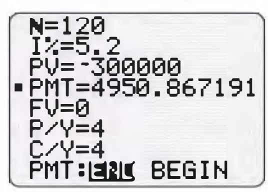
Method 3: Use a TVM Solver

Access the TVM Solver on a graphing calculator and enter the values, as shown. Note that the present value is negative, indicating that this amount is paid into the account.



Move the cursor to the **PMT** field and press (ALPHA) [SOLVE].

The amount of the regular withdrawals is $4950.87. Note that this value is positive, indicating that Fiona will receive these payments.



Technology Tip

When working with annuities using a TVM Solver, you must specify the actual number of payments or withdrawals, N. The graphing calculator software does not automatically assume that this coincides with the number of compounding intervals.

Key Concepts

- The present value of an annuity is the total amount that can finance a series of regular withdrawals over a specific period of time.
- The present value, PV, of an annuity can be calculated using the formula $PV = \frac{R[1 - (1 + i)^{-n}]}{i}$, where R represents the regular withdrawal; i represents the interest rate per compounding period, as a decimal; and n represents the number of compounding periods.

Communicate Your Understanding

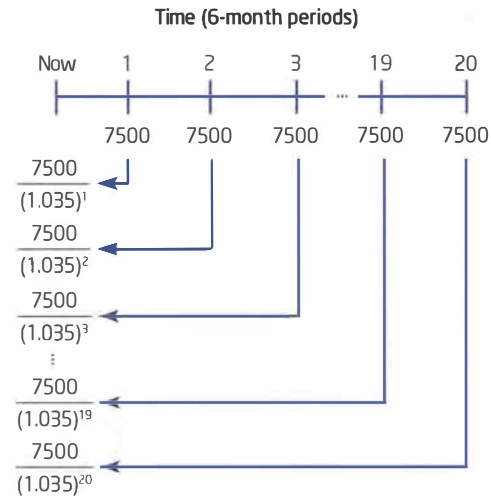
C1 The time line shows an annuity from which semi-annual withdrawals are made for 10 years. Interest is compounded semi-annually.

a) What is the annual rate of interest? How can you tell?

b) How many withdrawals will be made, in total? How can you tell?

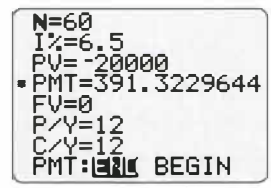
c) Explain why this annuity can be represented as a geometric series.

d) Identify the first term, a, and the common ratio, r, of the geometric series.



C2 The graphing calculator screen of a solution using the TVM Solver is shown.

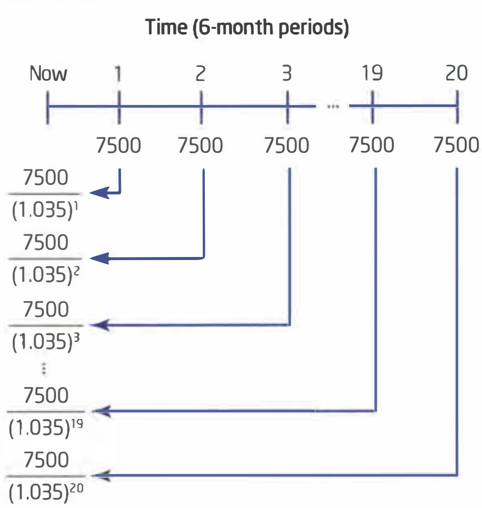
Describe this annuity fully.



A Practise

For help with questions 1 to 3, refer to Example 1.

1. Calculate the present value of the annuity shown.



2. Brandon plans to withdraw \$1000 at the end of every year, for 4 years, from an account that earns 8% interest, compounded annually.

a) Draw a time line to represent this annuity.

b) Determine the present value of the annuity.

3. Lauren plans to withdraw \$650 at the end of every 3 months, for 5 years, from an account that earns 6.4% interest, compounded quarterly.

a) Draw a time line to represent this annuity.

b) Determine the present value of the annuity.

c) How much interest is earned?

For help with questions 4 and 5, refer to Example 2.

4. An annuity has an initial balance of $8000 in an account that earns 5.75% interest, compounded annually. What amount can be withdrawn at the end of each of the 6 years of this annuity?

5. After graduating from high school, Karen works for a few years to save $40 000 for university. She deposits her savings into an account that will earn 6% interest, compounded quarterly. What quarterly withdrawals can Karen make for the 4 years that she will be at university?

B Connect and Apply

6. How much should be in an account today so that withdrawals in the amount of $15 000 can be made at the end of each year for 20 years, if interest in the account is earned at a rate of 7.5% per year, compounded annually?

7. Julie just won $200 000 in a lottery! She estimates that to live comfortably she will need to withdraw $5000 per month for the next 50 years. Her savings account earns 4.25% annual interest, compounded monthly.

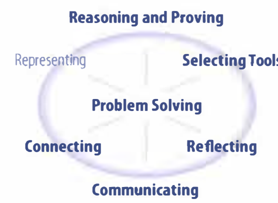
a) Can Julie afford to retire and live off her lottery winnings?

b) What is the minimum amount that Julie must win to retire in comfort immediately? Discuss any assumptions you must make.

8. An annuity has an initial balance of $5000. Annual withdrawals are made in the amount of $800 for 9 years, at which point the account balance is zero. What annual rate of interest, compounded annually, was earned over the duration of this annuity?

9. Shen has invested $15 000 into an annuity from which she plans to withdraw $500 per month for the next 3.5 years. If at the end of this time period the balance of the annuity is zero, what annual rate of interest, compounded monthly, did this account earn?

10. Jordan has $6000 to invest in an annuity from which he plans to make regular withdrawals over the next 3 years. He is considering two options:



Option A: Withdrawals are made every quarter and interest is earned at a rate of 8%, compounded quarterly.

Option B: Withdrawals are made every month and interest is earned at a rate of 7.75%, compounded monthly.

a) Determine the regular withdrawal for each option.

b) Determine the total interest earned for each option.

c) Discuss the advantages and disadvantages of each option from Jordan's perspective.

11. Ronald and Benjamin need to take out a small loan to help expand their pet-grooming business, Fluff 'n' Shine. They estimate that they can afford to pay back $250 monthly for 3 years. If interest is 6%, compounded monthly, how much of a loan can Ronald and Benjamin afford?

12. Chapter Problem Chloe has achieved her initial financial goal of growing her investments to $12 000 after 4 years. She deposits this amount into a new account that earns 5% per year, compounded quarterly. Now her intention is to make regular withdrawals from this account every 3 months for the next 2 years. How much will Chloe's regular withdrawals be?

13. Josie plans to invest $10 000 at the end of each year for the 25 years leading up to her retirement. After she retires, she plans to make regular withdrawals for 25 years. Assume that the interest rate over the next 50 years remains constant at 7% per year, compounded annually.

a) Once she retires, which amount do you predict that Josie will be able to withdraw per year?

- less than $10 000
- $10 000
- more than $10 000

Explain your answer.

b) Estimate how much she will be able to withdraw. Provide reasoning for your estimate.

c) Determine the amount of Josie's investment annuity on the day she retires.

d) Use this amount to determine the regular withdrawal she can make at the end of each year for 25 years after retirement.

e) Compare your answer in part d) to your estimate in part b). How close was your estimate?

Achievement Check

14. Abraham's grandparents plan to set up an annuity to help him when he moves into an apartment to attend college. Abraham will be able to withdraw $3000 at the end of each year for 4 years. The first withdrawal will be made 1 year from now, when Abraham begins college. If the annuity earns 7% interest, compounded annually, how much should Abraham's grandparents invest now to finance the annuity? Use two different methods to solve this problem.

C Extend

15. For a simple annuity with regular payment R and interest rate, i, as a decimal, per compounding period, use the formula for the sum of a geometric series,

$$S_n = \frac{a(r^n - 1)}{r - 1},$$ to derive the formula for the present value of the annuity,

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i}.$$

16. By law, mortgage interest rates must be stated as an annual rate compounded semi-annually. However, payments are usually made monthly, so the interest rate must be converted to a monthly rate. A $200 000 mortgage has an amortization period of 25 years, at an interest rate of 5%, compounded semi-annually.

a) Use the compound interest formula to determine the equivalent interest rate compounded monthly.

b) Determine the monthly payment required to pay off this mortgage over 25 years.

Connections

A mortgage is a type of loan by which a house or other real estate property is used as a guarantee of repayment of the debt. The total time over which the loan is repaid is called the amortization period.

17. A mortgage of $150 000 is amortized over 25 years with an interest rate of 6.7%, compounded semi-annually.

a) What is the monthly payment?

b) Suppose you choose to make weekly payments instead of monthly payments. What is the weekly payment?

c) Calculate the total interest paid with the weekly payments.

Chapter 7 Review

7.1 Simple Interest, pages 418 to 425

1. Louise borrows $720, which she plans to repay in a year and a half. She is charged 9.5% simple interest.
   a) How much interest must Louise pay?
   b) What is the total amount that she must pay back?

2. Yuri deposits $850 into an account that earns 6.25% per year simple interest. How long will it take for the amount in this account to reach $1000?

3. Nicola borrowed $750 for 4 years. The amount she repaid was $945.
   a) Use the given information to draw a graph of the amount in the account as a function of time.
   b) Explain why the relationship is linear.
   c) Determine the vertical intercept of the graph and explain what it means.
   d) Determine the slope of the graph and explain what it means.

7.2 Compound Interest, pages 426 to 435

4. Steve deposits $835 into an account that earns 8.25% per year, compounded annually.
   a) Determine the amount in the account after 5 years.
   b) How much interest will have been earned?

5. An account with an initial value of $1000 earns 3% interest per year, compounded semi-annually.
   a) Describe the shape of the graph of amount versus time.
   b) What is the vertical intercept of the graph? What does it represent?
   c) Describe what happens to the slope of the graph.

6. Elise deposits $500 into an account that earns 6.5% annual interest, compounded quarterly.
   a) How long will it take, to the nearest month, for this amount to double?
   b) Does the doubling time change if the principal changes? Explain.

7.3 Present Value, pages 436 to 443

7. What amount should be invested today so that there will be $1000 in an account in 3 years, if interest is earned at a rate of 7% per annum, compounded annually?

8. Dwayne needs $45 000 in 6 years to buy a new car. His investment earns interest at a rate of 4.8% per year, compounded monthly.
   a) Determine the present value needed in the account so that Dwayne can afford the car.
   b) How much interest will be earned?

9. The present value of an account worth $3823 in 4 years is $3000. If interest is compounded semi-annually, determine the annual rate of interest.

10. The future value of a $200 deposit in an account that earns 6.25% annual interest is $272.71 after 5 years. Determine the compounding period for this investment.

7.4 Annuities, pages 444 to 455

11. At the end of every year for 4 years, Jacqueline deposits $2400 into an account that earns 4.3% per annum, compounded annually.
    a) Draw a time line to illustrate this annuity.
    b) Explain why this annuity can be represented as a geometric series.
    c) Determine the amount of the annuity.
    d) How much interest will be earned?

12. Marko deposits \$400 into an account at the end of every month for 8 years. Interest is earned at a rate of 5.5%, compounded monthly.

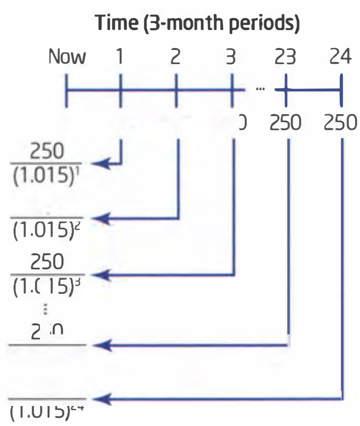
a) Determine the amount of the annuity.

b) How much interest will be earned?

13. Latisha decides to make regular deposits every 2 weeks into an account that earns 7.8% annual interest, compounded bi-weekly. She hopes to have \$30 000 at the end of 3 years. Determine the amount of the regular payment that she will need to make.

7.5 Present Value of an Annuity, pages 456 to 463

14. Examine the time line for the annuity shown.



a) What is the duration of this annuity? How can you tell?

b) Determine the annual rate of interest and the number of compounding periods per year.

c) Determine the present value of this annuity.

d) Determine the total interest earned.

15. Suki would like to withdraw \$800 per month for the next 20 years. The interest in her account is 6.25% per year, compounded monthly. How much must Suki deposit today to finance this annuity?

16. Mario has saved \$250 000 for his retirement, which he has deposited into an account that earns 7.2% per year, compounded monthly. He plans to make regular monthly withdrawals for the next 25 years. What is the maximum monthly amount that Mario can withdraw?

In Section 7.1, question 10; Section 7.2, question 12; and Section 7.3, question 15, you explored Chloe's investment portfolio.

a) What was the interest rate earned by each investment?

b) What is the total interest earned by all of Chloe's investments, assuming that she just met her financial goal?

c) The effective rate of return is the annual simple interest rate earned on all investments. Divide the total interest earned by the total principal and by the number of years to determine the effective rate of return of Chloe's investment portfolio.

d) How much more money could Chloe have earned had she invested her entire principal in the highest-yield investment?

e) Why do you suppose Chloe's advisor did not recommend the strategy in part d)?

Chapter 7 Practice Test

For questions 1 to 5, select the best answer.

1. Karl borrows \$500 for 4 years at an annual simple interest rate of 12% per year. What is the amount that must be repaid?

A \$60

B \$240

C \$560

D \$740

2. Jasmine invests \$400 at 8% annual interest, compounded annually. How much interest will be earned after 5 years?

A \$160

B \$187.73

C \$560

D \$587.73

3. If an amount is invested at 6.5% per year, compounded semi-annually, for 3 years, determine the number of compounding periods and the interest rate per compounding period.

A $n = 3, i = 0.065$

B $n = 6, i = 0.0325$

C $n = 1.5, i = 0.13$

D $n = 12, i = 0.01625$

4. If the annual interest rate is 3.9% and the interest per compounding period is 0.975%, what is the compounding period?

A weekly

B monthly

C quarterly

D semi-annual

5. An amount is deposited into an account that earns 9% per year, compounded quarterly. After 6 years, the amount in the account is \$597.02. What is the present value?

A \$350.00

B \$355.98

C \$404.24

D \$421.36

6. After 1 year, Nadia's investment is worth \$256.80. After 2 years, the amount has reached \$290.40.

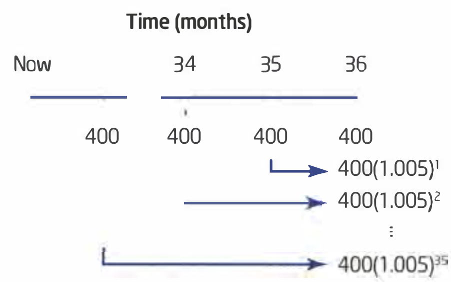
a) How much simple interest is Nadia's investment earning per year?

b) What is the principal?

c) What is the annual simple interest rate per year?

7. Deanna invests \$500 at 8% per year simple interest. She puts money in the bank on July 1 and takes it out December 3. How much money does she take out?

8. The time line for an annuity is shown.



a) What is the duration of this annuity? How can you tell?

b) Determine the annual rate of interest and the number of compounding periods per year.

c) Determine the amount of this annuity.

d) Determine the total interest earned.

9. Leon has \$3000 that he wants to invest for 6 years. Which option should he choose, and why?

Option A: 5.2% annual interest, compounded quarterly

Option B: 5% annual interest, compounded monthly

10. Colette takes out a loan for $2800 to buy a scooter. She plans to repay the loan in 3 years. The amount payable when the loan is due is $3420.51. What rate of interest, compounded annually, is Colette being charged?

11. You invest $1000 at 6% per year, compounded quarterly, for 3 years. What interest rate, compounded monthly, will give the same results?

12. An account paying 7.25% annual interest, compounded semi-annually, has a future value of $1429 in 8 years.

a) What is the present value of the account?

b) How much more interest will have been earned than if simple interest was paid?

13. To have $5000 at the end of 8 years, how much do you need to invest today, at 6% per annum, compounded semi-annually?

14. Jerry deposited $300 into an account that earns 6.7% annual interest, compounded daily. When he closed the account, the amount had grown to $348.56. How long was the money invested?

15. Heather deposits $200 per week for 20 years into an account that earns 2.6% annual interest, compounded weekly.

a) Draw a time line to represent this annuity.

b) Determine the amount of the annuity.

c) How much interest will be earned?

16. After 20 years of investing, Heather decides to retire and use the amount of her annuity to finance her retirement for the next 20 years. Use the amount from question 15b) and the same interest conditions.

a) Draw a time line to represent Heather's retirement annuity.

b) Determine the maximum monthly withdrawal that she can make.

c) Determine the total amount of interest earned over the 40 years spanning the two annuities.

17. Niki needs $5200 for university tuition when she graduates from high school in 2 years. She plans to make deposits into an account that earns 6.5% per year, compounded bi-weekly.

a) Draw a time line to represent this annuity.

b) How much should she deposit every 2 weeks?

18. Shira has invested $18 000 in an annuity from which she plans to withdraw $650 per month for the next 4.5 years. If at the end of this time period the balance of the annuity is zero, what annual rate of interest, compounded monthly, did the account earn?

19. Instead of investing $3000 at the end of 5 years and $4000 at the end of 10 years, Steve wishes to make regular monthly payments that will amount to the same total after 10 years. Determine the monthly payment if interest is compounded monthly at an annual rate of 4%.

Chapters 6 and 7 Review

Chapter 6 Discrete Functions

1. Write the first three terms of each sequence. Describe the pattern in words.

a) $t_n = 3n - 2$

b) $f(n) = 4^n + 1$

c) $t_n = 5n^2 - 16$

d) $f(n) = \frac{n^3}{2} + 2$

e) $t_1 = 1, t_n = 2t_{n-1} + 5$

f) $t_1 = -2, t_n = (t_{n-1})^2 - 8$

2. Graph the first eight terms of each sequence in question 1.

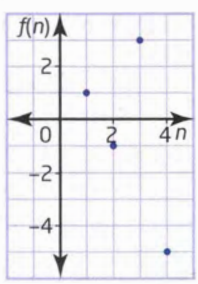
3. The value of a new car bought for \$45 000 depreciates at a rate of 15% in the first year and 5% every year after that.

a) Determine the value of the car at the end of the first year, the second year, and the third year. Write these values as a sequence.

b) Determine an explicit formula for the value of the car at the end of year n.

c) What is the value of the car at the end of year 20? Is this realistic? Explain your thinking.

4. Given the graph, write the sequence of terms and determine a recursion formula using function notation.



5. Write a recursion formula and an explicit formula for each sequence.

a) 5, 7, 9, 11, ...

b) 2, 4, 16, 256, ...

6. Investigate the prime-numbered rows of Pascal's triangle. Describe the property that is common to these rows but not common to the rows that are not prime-numbered.

7. Expand and simplify each binomial.

a) $(2x + 5)^7$

b) $(a^2 - 3b)^5$

c) $\left(\frac{2}{x} + x^2\right)^6$

d) $\left(5 - \frac{3}{\sqrt{n}}\right)^4$

8. State whether each sequence is arithmetic, geometric, or neither. Determine a defining equation for those that are arithmetic or geometric, and find the 12th term.

a) 6, 11, 17, 29, ...

b) −3, 1, 5, 9, ...

c) 3, 12, 48, 192, ...

d) 2 657 205, −885 735, 295 245, −98 415, ...

9. The fourth term of an arithmetic sequence is 6 and the seventh term is 27. Determine the first and second terms.

10. How many terms are in the geometric sequence 7, 21, 63, ..., 3 720 087?

11. A new golf course has 480 lifetime members 3 weeks after it opens and 1005 lifetime members 6 weeks after it opens. Assume the membership increase is arithmetic.

a) Determine the general term that represents the sequence for this situation.

b) How many members were there at the end of the fifth week?

c) When will there be over 2000 members?

12. Determine the sum of the first 10 terms of each series.

a) $2 + 9 + 16 + 23 + \cdots$

b) $5 - 25 + 125 - 625 + \cdots$

c) $256 + 128 + 64 + 32 + \cdots$

d) $-\frac{1}{3} - \frac{5}{6} - \frac{4}{3} - \frac{11}{6} - \cdots$

13. A ball is dropped from a height of 160 cm. Each time it drops, it rebounds to 80% of its previous height.

a) What is the height of the rebound after the 8th bounce?

b) What is the total distance travelled at the time of the 15th bounce?

Chapter 7 Financial Applications

14. Define each term.

a) principal

b) amount

c) simple interest

d) compound interest

e) annuity

f) present value

g) compounding period

15. Determine the interest earned for each investment.

a) \$1000 is invested for 8 months at 5% per year simple interest.

b) \$800 is placed into an account that pays 2.5% annual simple interest for 40 weeks.

c) A 90-day \$10 000 treasury bill earns simple interest at a rate of 4.8% per year.

16. Alex deposited \$500 into an account at 3% simple interest.

a) Write an equation to relate the amount in the account to time.

b) Sketch a graph of this relation for 1 year.

c) How long does it take to earn \$10 in interest?

17. Sarah invests \$750 in a term deposit, at 4.5% per annum, compounded semi-annually, for 5 years. How much interest will Sarah earn?

18. Abdul decides to invest some money so that he can have \$10 000 for a down payment on a new car in 5 years. He is considering two investment options:

Account A pays 3.5% per annum, compounded semi-annually.

Account B pays 3.2% per annum, compounded monthly.

a) Compare the present values of the two options.

b) Which account is the better choice for Abdul? Explain your reasoning.

19. To save for her university education, Shaquilla will deposit \$50 into an account at the end of each month for the next 3 years. She expects the interest rate to be 1.5% per year, compounded monthly, over that time. How much will she have saved after 3 years?

20. Wayne is 16 years old. To become a millionaire by the time he is 50 years old, how much does Wayne need to invest, at the end of every 6 months, at 4% per year, compounded semi-annually?

21. What annual interest rate, compounded monthly, is needed for Eva to accumulate \$20 000 by depositing \$300 at the end of each month for 5 years?

22. A rental contract calls for a down payment of \$1000, and \$500 to be paid at the end of each month for 3 years. If interest is at 4.5% per year, compounded monthly, what is the present value of this rental contract?

23. A retirement account contains \$100 000. Barb would like to withdraw equal amounts of money at the end of every 3 months for 15 years. If interest is 5% per year, compounded quarterly, what will the size of Barb's withdrawals be?

Task

Loans and Annuities Due

a) i) Ali is paying off a loan by making \$50 payments at the end of every month for 2 years, at 6%, compounded monthly. What is the value of the loan today?

ii) Ken is paying off a loan by making a \$50 down payment and then \$50 payments at the end of every month for 2 years, at 6%, compounded monthly. What is the value of the loan today?

iii) Maria is paying off a loan by making \$50 payments at the beginning of every month for 2 years, at 6%, compounded monthly. What is the value of the loan today?

iv) Explain the factors in each situation that account for the differences in the results.

b) An ordinary annuity consists of payments at the end of each payment period, as you have seen in this chapter. An *ordinary annuity due* consists of the same number of payments at the beginning, rather than the end, of each payment period.

i) Which part of part a) is an annuity due? Explain.

ii) Use the results of your calculations to develop a formula for the present value of an ordinary annuity due.

iii) Use your formula to determine the present value of a loan, with \$200 payments at the beginning of every 3 months for 2 years, with interest at 7%, compounded quarterly.

iv) Use your formula to determine the monthly payments at the beginning of each month on a \$15 000 car loan, at 3%, compounded monthly, for 3 years.