Combinations

There are 100 letter tiles in the popular word game Scrabble®. Of these, 42 are vowels. A player chooses seven tiles to begin the game. What is the probability that the player will select only vowels?

Key Terms

combination   Pascal's triangle
null set   binomial theorem

Literacy Strategy

You can solve counting problems using powers, permutations, and/or combinations. As you work through this chapter, you will learn the difference between permutations and combinations. Make a summary and a flowchart to help you decide whether you should solve a problem using powers, permutations, the rule of sum, or the fundamental counting principle. Include simple examples to support your summary.

Career Link

Management Science

Eleanor has a career in management science; she uses advanced analytics to improve decision making. Her position requires university courses in mathematics, including combinatorics. One project she worked on involved designing a panel for a bank of elevators in a 75-storey building. A passenger enters the destination floor into the software and it analyses the variables and combinations of routes. The software then indicates which elevator will be quickest to reach the destination. For example, if there are people on eight different floors requesting elevators, and three elevators can reach each floor, how many different ways can the elevators pick up the passengers?

Chapter Problem

Counting Story

Children's stories are often about a problem that needs to be solved. Some problems may involve math. For example,

> When Goldilocks entered the Three Bears' home, she went into the kitchen. There, she saw three pots of porridge (a blue pot, a red pot, and a green pot). She wanted to try the porridge and looked in the cupboard for a bowl. Goldilocks found a small bowl, a medium bowl, and a large bowl. In how many ways could she serve herself some porridge?

At the end of this chapter, your task will be to rewrite a children's story by including counting techniques.

Factorials

1. Evaluate.

   a) $8!$

   b) $\dfrac{9!}{3!}$

   c) $3! \times 4!$

   d) $_{10}P_{6}$

   e) $_{12}P_{3}$

   f) $\dfrac{_{7}P_{3}}{3!}$

   g) $\dfrac{_{11}P_{4}}{4!}$

   h) $\dfrac{14!}{2!\,5!\,6!}$

2. a) Define $n!$ in words and with a formula.

   b) Define $_{n}P_{r}$ in words and with a formula.

3. Express each permutation in factorial form.

   a) $_{7}P_{3}$

   b) $_{100}P_{92}$

   c) $_{n}P_{6}$

   d) $_{15}P_{r}$

Permutations

4. How many ways are there to arrange

   a) 8 objects?

   b) 5 of 8 objects?

   c) 3 of 13 objects?

5. a) In how many ways could five girls and six boys line up in one row?

   b) In how many ways could they line up with the girls in the front row and the boys in the back row?

6. a) How many permutations are there of all the letters in TRIANGLE?

   b) How many arrangements are there of any three of the letters in TRIANGLE?

Pascal's Triangle

7. This array of numbers is called Pascal's triangle.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

   a) Investigate the terms and describe how to determine the next row. Continue Pascal's triangle for two more rows.

   b) Describe any patterns you see in the triangle.

Probability

8. A coin is flipped three times. Calculate the probability of each event.

   a) heads, heads, heads

   b) heads, heads, tails

   c) heads, tails, heads

   d) two heads and one tail in any order

9. Two cards are dealt from a standard deck. Determine each probability.

   a) The first card is a king and the second card is an ace.

   b) The first card is red and the second card is black.

   c) The first card is a heart or the second card is a king.

10. There are 18 students in a class. Their names are drawn, at random, to determine the order in which they will present their projects. Determine each probability.

   a) Jacob is first.

   b) Caryn is last.

   c) Jacob is first or Caryn is last.

   d) The names are in alphabetical order.

Tree Diagrams

11. The tree diagram illustrates all the possible outcomes when a standard die and a coloured die are rolled.
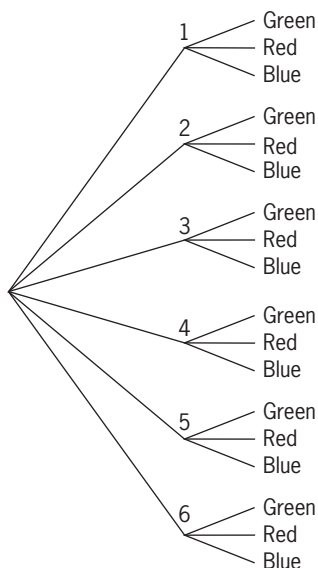
a) Describe the faces of the coloured die.

b) How many different outcomes are there?

c) Determine the probability $P(5, \text{Red})$.

d) What is the probability $P(\text{Green or Blue})$?

Principle of Inclusion and Exclusion

12. Use the Venn diagram to help you explain the principle of inclusion and exclusion.
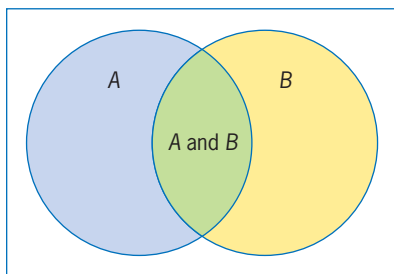
13. In a game of cards, a hand contains five red cards and four face cards. Two cards are red face cards. How many cards are in the hand?

14. Draw a Venn diagram to illustrate each of the following.

a) Face cards and numbered cards from a standard deck.

b) Prime and even whole numbers. Remember, 0 is neither prime nor even.

c) Vowels and consonants. *What can be said about "Y"?*

d) The set of integers and the set of natural numbers.

Simplifying Expressions

15. Simplify.

a) $(x^2)^3$

b) $(2a)^2$

c) $(5m^3)^2$

d) $(3k^3)^4$

16. Expand and simplify.

a) $(x + y)^2$

b) $(a + b)^3$

c) $(2p + q)^2$

17. Simplify by reducing.

a) $\dfrac{(n-1)(n-2)(n-3)(n-4)}{(n-3)(n-4)}$

b) $\dfrac{n(n-1)(n-2)\cdots 3 \times 2 \times 1}{(n-2)(n-3)(n-4)\cdots 3 \times 2 \times 1}$

c) $\dfrac{n!}{(n-1)!}$

Permutations With Non-Ordered Elements

Learning Goals

I am learning to

- recognize the advantages of different counting techniques
- make connections between situations that involve permutations and combinations

Minds On...

In the previous chapter, the objects in each set were always different. This is not always the case. Sometimes, there are objects that are identical.

- Do you think this will increase (or decrease) the number of different arrangements? Why?

- There may be some objects that cannot be rearranged. Do you think this will increase (or decrease) the number of arrangements? Why?

Action!

Investigate Permutations With Identical Elements

Materials

- 6 coloured blocks or linking cubes (3 of one colour, 2 of a second colour, 1 of a third colour)

1. Select the coloured blocks as indicated. Set the others aside. For each set of blocks, answer the following questions:
   - What are the possible permutations?
   - How many arrangements of blocks in a row are there?

   a) two of one colour, one of another colour

   b) two of one colour, two of different colours (e.g., blue blue yellow red)

   c) two of one colour, two of another colour

   d) two of one colour, three of another colour

2. What would you multiply by to get the total number of arrangements if the blocks were all different?

3. Reflect Raj counted all of the permutations of 3 red, 2 blue, and 2 yellow blocks and found there were 210 possible arrangements. Explain how the number of arrangements relates to the types of blocks in the set.

4. Extend Your Understanding Assume you know the number of arrangements of n objects when p, q, r, … of them are alike. Make a hypothesis of what you would multiply by to get the total number of arrangements if the objects were all different. Test your hypothesis.

Example 1

Permutations With Like Elements

Compare the number of arrangements of the sets of letters A_1A_2BC and AABC.

> **Solution**

List all of the possible arrangements of A_1A_2BC and AABC.

Arrangements of A_1A_2BC		Arrangements of AABC
A_1A_2BC	A_2A_1BC	AABC
A_1A_2CB	A_2A_1CB	AACB
A_1BA_2C	A_2BA_1C	ABAC
A_1CA_2B	A_2CA_1B	ACAB
A_2BCA_1	A_1BCA_2	ABCA
A_1CBA_2	A_2CBA_1	ACBA
BA_1A_2C	BA_2A_1C	BAAC
CA_1A_2B	CA_2A_1B	CAAB
BA_1CA_2	BA_2CA_1	BACA
CA_1BA_2	CA_2BA_1	CABA
BCA_1A_2	BCA_2A_1	BCAA
CBA_1A_2	CBA_2A_1	CBAA

In each case, there are 2! permutations of A_1A_2BC for each AABC arrangement. This is because there are 2! permutations of A_1A_2.

Your Turn

Compare the number of arrangements of the sets of letters.

a) $AB_1B_2B_3$ and ABBB

b) $A_1A_2B_1B_2$ and AABB

Permutations With Like Objects

In Example 1, you can multiply the number of arrangements of AABC by 2! to determine the number of arrangements of A_1A_2BC. You can also divide the number of arrangements of A_1A_2BC by 2! to determine the number of arrangements of AABC.

The number of permutations of n elements, when p of one type are identical, q of another type are identical, r of another type are identical, and so on, is $n(A) = \dfrac{n!}{p!\, q!\, r! \ldots}$.

Example 2

Permutations With Several Identical Elements

A hockey team ended its season with 12 wins, 8 losses, and 4 ties. In how many orders could these outcomes have happened?

Solution

Apply the formula.

$n = 12 + 8 + 4 \qquad p = 12 \qquad q = 8 \qquad r = 4$
  $= 24$

Number of permutations $= \dfrac{24!}{12!\, 8!\, 4!}$

When using a calculator, enter the denominator in brackets.

  $= 1\ 338\ 557\ 220$

These outcomes could have happened in 1 338 557 220 ways.

Your Turn

Compare the number of orders of this hockey team's wins, losses, and ties with those of a team that had eight wins, eight losses, and eight ties. Would you expect the number of orders to be higher or lower in the second scenario? Why?

Example 3

Distinct Objects in a Fixed Order

How many ways are there to arrange the letters in the word NUMBER if the consonants must remain in the original order?

Solution

You can place the consonants in any position but you must keep them in the order NMBR.

Some of the possible arrangements include:

NMBEUR
UNEMBR
NUMEBR
NEMUBR

Since you cannot arrange the consonants in a different order, treat them as like elements, using C for each consonant. Rewrite the letters as CUCCEC.

$n = 6$
$p = 4$

The number of arrangements of CUCCEC is

$\frac{6!}{4!} = 30$

There are 30 ways to arrange the letters, keeping the consonants in the original order.

Your Turn

How many permutations are there of the letters in the word EXPLAIN if the vowels must be in alphabetical order?

Consolidate and Debrief

Key Concepts

- The number of permutations of n objects, when p of one type are identical, q of another type are identical, r of another type are identical, and so on, is $n(A) = \frac{n!}{p!\,q!\,r!\ldots}$.
- If a number of distinct objects need to remain in a specific order in a permutation, divide by the factorial of that number.

Reflect

R1. Explain why you need to divide by 4! when calculating the number of arrangements of the digits 1, 2, 2, 2, 2, 3, 4.

R2. Is the number of permutations of three girls and four boys the same as the number of permutations of three red balls and four green balls? Explain.

R3. Why is it easier to use the formula $n(A) = \dfrac{n!}{p!\,q!\,r!\,...}$ than to use a tree diagram or chart?

Practise

Choose the best answer for #2 and #3.

1. Simplify.

a) $\dfrac{10!}{2!\,3!\,5!}$

b) $\dfrac{9!}{3!\,3!\,3!}$

c) $\dfrac{7!}{2!\,3!}$

d) $\dfrac{120!}{115!\,3!\,2!}$

2. What is the number of arrangements of five small tiles and three large tiles?

A 20

B 56

C 720

D 40 320

3. Dana has 12 pens. There are four blue, three red, and the others are different colours. Which set of values for the variables in a permutation calculation is correct?

A $n = 12$, $p = 4$, $q = 3$, $r = 5$

B $n = 7$, $p = 4$, $q = 3$

C $n = 12$, $p = 4$, $q = 3$, $r = 0$

D $n = 12$, $p = 4$, $q = 3$

4. How many permutations are there of all the letters in each name?

a) WATERLOO

b) TORONTO

c) MISSISSAUGA

d) OTTAWA

5. How many five-digit numbers can be formed using each set of numbers?

a) 1, 2, 2, 3, 4

b) 1, 2, 2, 2, 3

c) 1, 1, 2, 3, 3

d) 1, 2, 2, 2, 2

Apply

6. Sam has four different types of fruit. He has three pieces of each type. In how many ways could he arrange them on a platter

a) in a line?

b) in three rows of four?

c) in two rows of six?

7. In a panel of eight light switches, half are on and half are off. In how many ways could this be done?

8. In one of her tricks, a clown rearranges two identical quarters, three identical loonies, and five identical toonies in a row. In how many ways can the clown arrange the coins?

9. How many arrangements are there of 15 flags in a row if five are red, four are green, two are blue, and four are yellow?

10. Communication When applying the formula $n(A) = \dfrac{n!}{p!\,q!\,r!\ldots}$, will the result ever not be a natural number? Justify your explanation. Hint: A natural number is a whole number greater than zero.

11. Application When travelling from home to school, Minh travels five blocks north and six blocks west. How many different routes can he take? What assumptions did you make?

12. Thinking In how many ways could 12 volleyball players be assigned to
   a) four triple rooms?
   b) six double rooms?

13. In how many ways could the letters in the word PROBLEM be arranged if the consonants must remain in the original order?

14. In how many ways could the digits in the number 458 978 be arranged if the prime digits must remain in the original position?

✔ **Achievement Check**

15. In the opening credits of each episode of the TV show *Fawlty Towers*, the sign on the front of the hotel rearranged the letters in the hotel's name.
   a) How many arrangements are there of all the letters in FAWLTY TOWERS?
   b) How many arrangements of these letters are possible if the A, Y, O, and E must remain in their original order?
   c) How many arrangements of these letters are possible if the vowels must remain in the second, sixth, eighth, and tenth positions?
   d) How many arrangements of these letters are possible if the consonants (excluding Y) must remain in alphabetical order?

16. Thinking An Ontario licence plate contains four letters followed by three digits. Derek noticed that the four letters on his licence plate were all different and in alphabetical order. Similarly, the three digits were all different and in numerical order.
   a) How many licence plates have all different letters and digits and just the letters in alphabetical order?
   b) How many licence plates have all different letters and digits and just the digits in numerical order?
   c) How many have all different letters and digits and both the letters in alphabetical order and the digits in numerical order?
   d) Mathematically, how does the number of licence plates in part c) compare to the total number of licence plates with no restrictions?

17. Open Question Design an example that has $\dfrac{12!}{2!\,3!\,4!\,3!}$ as its solution. Justify your reasoning.

Extend

18. How many 7-digit even numbers can be formed using all of the digits 0, 1, 1, 2, 3, 4, 5?

19. How many four-letter "words" can be made from the letters of the word APPLE?

20. A dart board has the numbers from 1 to 20 around the circumference. In how many ways could the numbers be arranged if pairs of numbers on opposite sides of the board must add to the same value?
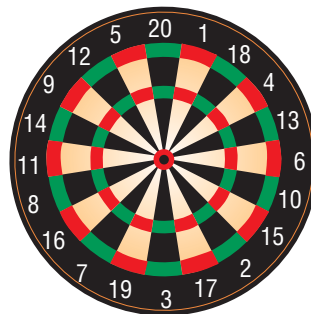
3.2

Combinations

Learning Goals

I am learning to

- recognize the advantages of using permutations and combinations over other counting techniques
- apply combinations to solve counting problems
- express combinations in standard notation, $C(n, r)$, $_nC_r$, $\binom{n}{r}$

Minds On…

Often a set of objects is selected from a larger set, and the order is not important. Selecting pizza toppings and forming teams for a sports game are two examples. Can you describe a situation in which the order of selecting objects does matter and one in which the order does not matter for these activities?

- cooking
- forming teams
- cleaning
- packing for a trip

Action!

Investigate Non-Ordered Selections

1. **a)** List all of the arrangements of two letters in the word MATH.

   b) List all the combinations of two letters in the word MATH, with no regard to order.

   c) Compare the results.

2. Sanjit, Dennis, Mia, and Kelsey volunteer to be on a dance committee.

   a) Make a list of all possible ways a chair, a secretary, and a treasurer could be chosen.

   b) Make a list of all possible ways the committee could be formed without assigning positions.

   c) Compare the results.

3. **Reflect**

   a) Write the formula for the number of arrangements of r objects, taken from a set of n elements.

   b) How would you change the formula to calculate the number of ways of selecting r objects without regard to order?

4. Extend Your Understanding You have a toonie, a loonie, a quarter, a dime, and a nickel. How many different sums of money could you form by selecting three of the coins?

Combinations

In the previous section, you divided the permutation formula by the number of ways of arranging the identical items to account for identical items. Similarly, when choosing r items from a set of n items, without regard to order, divide the permutation formula by $r!$.

The number of **combinations** of r objects chosen from a set of n items is

$$
\begin{aligned}
{}_nC_r &= \frac{{}_nP_r}{r!} \\
&= \frac{\dfrac{n!}{(n-r)!}}{r!} \\
&= \frac{n!}{(n-r)!\,r!}
\end{aligned}
$$

Other standard combination notation is $\binom{n}{r}$ and $C(n, r)$. This is read as "n choose r."

combination

- a selection from a group of objects without regard to order

Example 1

Choose Items From a Set

How many ways can a five-card hand be dealt from a standard deck?

Solution

Order is not important when combining cards to form a hand. For example, king, queen, 9, 8, 3 is the same hand as 3, queen, 9, king, 8.

$$
\begin{aligned}
{}_nC_r &= \frac{n!}{(n-r)!\,r!} \\
{}_{52}C_5 &= \frac{52!}{(52-5)!\,5!} \\
&= \frac{52!}{47!\,5!} \\
&= 2\,598\,960
\end{aligned}
$$

Explore how to use the combinations key on your calculator.

A five-card hand can be dealt in 2 598 960 ways.

Your Turn

In a competition, junior chefs make a gourmet soup by selecting from 10 different ingredients. How many different soups can the chefs make if the soup must include

a) four of the ingredients? **b)** five of the ingredients?
c) six of the ingredients?

Example 2

Choose More Than One Group

A committee of 3 men and 3 women is formed from a group of 8 men and 10 women. How many ways are there to form the committee?

Solution

Choose each group of three men and three women separately. Since the two groups are being selected together, the fundamental counting principle applies.

Choices for men × choices for women = $_8C_3 \times {}_{10}C_3$ Because you are choosing
$= 56 \times 120$ 3 men *and* 3 women,
$= 6720$ multiply.

The committee can be chosen in 6720 ways.

Your Turn

Erica is making a platter of four types of cheese and four types of crackers. She has seven different cheeses and six different crackers. In how many ways can Erica make the platter?

Example 3

Interpret a Diagram

The seven points represent cabins at a lodge. How many paths can be drawn by joining pairs of cabins?
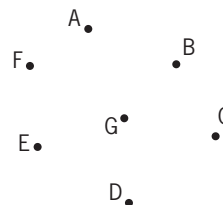
Solution

Choosing any two points from the seven will produce a unique path. Note: Path AE is the same as path EA.

$_7C_2 = \dfrac{7!}{(7-2)!\,2!}$ Why is it important that no
        three points are collinear?
$= \dfrac{7!}{5!\,2!}$
$= 21$
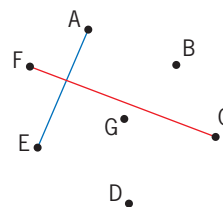
A total of 21 different paths can be made between cabins in the lodge.

Your Turn

How many triangles can be drawn using the seven points as vertices?

Key Concepts

- A combination is a set of items taken from another set in which order does not matter. In a permutation, the order of the items matters.
- The number of combinations of r items taken from a set of n items is

$$_nC_r = \frac{n!}{(n-r)!\,r!}.$$

Reflect

R1. Five people are chosen from a group of eight people. Describe a situation for this set that involves

a) permutations

b) combinations

R2. Describe two everyday situations in which items are chosen and the order of the selections does not matter.

R3. Which situation has a greater number of possibilities, one in which order matters or one in which order does not matter? Explain why.

Practise

Choose the best answer for #2 and #3.

1. Convert to factorial form, then evaluate.

a) $_9C_5$

b) $_8C_4$

c) $C(12,3)$

d) $\binom{11}{5}$

e) $_7C_2 \times _6C_3$

f) $\binom{101}{98} \times \binom{101}{3}$

2. Which is an incorrect way of writing $_{10}C_3$?

A $\dfrac{_{10}P_3}{3!}$

B $\dfrac{_{10}P_3}{7!}$

C $\dfrac{10!}{7!\,3!}$

D $\dfrac{10!}{3!\,7!}$

3. How many three-member committees can be formed from a group of nine people?

A 27

B 84

C 504

D 729

4. In how many ways could 6 online magazine subscriptions be chosen from a set of 10 magazines?

5. In how many ways could you choose 4 packages of pasta from a bin containing 11 different packages of pasta?

6. In how many ways could you reach into a bag containing 10 marbles and pull out none of them?

7. Refer to #6. Evaluate each combination by first writing in factorial form. Remember, $0! = 1$.

a) $_1C_0$

b) $_2C_0$

c) $_3C_0$

d) $_{15}C_0$

e) $_nC_0$

Apply

8. Application On an English exam, students need to answer six out of eight questions in Part A and two out of four questions in Part B. The order in which they answer the questions does not matter. In how many ways could a student answer the questions on this exam?

9. From a standard deck, how many five-card hands contain the following?

   a) only black cards

   b) all face cards

   c) no hearts

   d) two red and three black cards

   e) one face card

10. **Communication** Juries are chosen from large pools of people selected at random from the local population. A jury pool has 40 people.

    a) How many ways are there to form a 12-person jury in a criminal case?

    b) How many ways are there to form a 6-person jury in a civil case?

    c) Which situation gives a larger number of ways? Explain why this is to be expected.

11. A dealership has six models of trucks and five models of cars for sale. Wayne sells four vehicles this week. How many of the following combinations of four can be formed?

    a) no restrictions

    b) two trucks and two cars

    c) three cars and one truck

    d) only cars

    e) only trucks

    f) How are the answers to parts b) to e) related to part a)? Explain why this is true.

12. **Communication** Fourteen family members have a one-on-one video chat with each other to wish each person happy new year.

    a) Is this situation a permutation or a combination? Explain.

    b) How many video chats will occur? Express your answer using standard combinatorial notation and then evaluate.

13. Ten points are drawn on the circumference of a circle. Using these points as vertices,

    a) how many quadrilaterals can be drawn?

    b) how many pentagons can be drawn?

    c) how many polygons of n sides can be drawn?

14. **Thinking**

    a) Compare each pair of values.

       i) $_7C_2$ and $_7C_5$

       ii) $_4C_3$ and $_4C_1$

       iii) $_{12}C_4$ and $_{12}C_8$

    b) State a hypothesis from your observations in part a). Explain why this makes sense when you are thinking about groups.

    c) Prove your hypothesis algebraically, using $_nC_r$ in your proof.

15. **Communication** In a drama class of 18 students, nine are selected to be actors in a play, five will build sets, and four will be stage hands. In how many ways could the class be divided up?

    a) Make your calculations by selecting the actors first.

    b) Make your calculations by selecting the set builders first.

    c) Compare your answers. Explain the results.

16. a) How many diagonals are there in any convex octagon?

Literacy Link

Convex means curved out. *Concave* means curved in. A convex octagon has all vertices pointing outward.
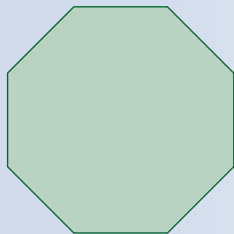
b) How many diagonals are there in a convex polygon with n sides? Explain your reasoning.

✔ **Achievement Check**

17. Ten identical playing pieces are placed on a 5 by 5 game board.
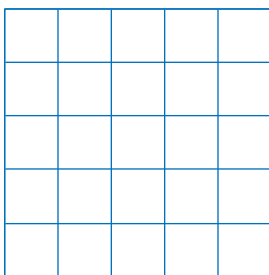
a) In how many ways could 10 playing pieces be placed on the board if there are no restrictions?

b) In how many ways could 10 playing pieces be placed on the board if there must be two pieces in each row?

c) Describe how the results would change if the playing pieces were all different.

18. In how many ways can 15 people be divided into three identical groups of five?

19. In how many ways can a team of 20 hockey players be accommodated in 10 two-person hotel rooms? Assume that the order of assigning the rooms does not matter.

20. Compare your technique in #18 and #19 to the one you used in #12 in section 3.1 on page 109.

21. Show that the number of ways of dividing a class of 30 students into six teams of five members is the same as the number of ways of arranging five red, five green, five purple, five blue, five white, and five black balls.

22. Communication For $r > 0$, will there always be more r-permutations of n items or r-combinations of n items? Why?

23. Thinking Five points are drawn horizontally, and four points are drawn vertically, with the top one overlapping the point on the right side. How many triangles can be formed using the points as vertices?
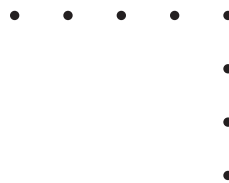
Processes

Problem Solving
How would you solve a simpler version of this problem?

Extend

24. a) Show that the product of three consecutive numbers is divisible by 3!.

b) Show that the product of r consecutive numbers is divisible by $r!$.

25. Solve for n in $n! = 12 \times {}_nC_2$.

26. How many ways are there to choose three numbers from 1 to 20 so that no two are consecutive?

Problem Solving With Combinations

Learning Goals

I am learning to

- distinguish situations that use permutations from those that use combinations
- solve counting problems using the rule of sum and the fundamental counting principle

Minds On...

You have learned techniques for counting items in which order does and does not matter. How many possible sums are there if you have

- one nickel and one toonie
- three dimes and two quarters

How can you be sure you did not miss any amounts? Explain.

Action!

Investigate The Number of Sums of Money

1. For each set of coins,
   - list the different sums of money you could make,
   - record the number of sums, and
   - verify your answer using combinations and the rule of sum.
   a) dime, quarter

   b) nickel, dime, quarter

   c) nickel, dime, quarter, loonie

2. Describe the pattern in the results.

3. **Reflect** Predict the total number of sums of money (including no money) that could be made from a nickel, a dime, a quarter, a loonie, and a toonie.

4. **Extend Your Understanding** How would you change your methods if you are not allowed to choose zero coins?

Total Number of Subsets of a Set

When building subsets of a given set, you can choose from 0 to n elements to be in the subset. The total number of subsets of a set of n elements is $_nC_0 + {_nC_1} + \ldots + {_nC_n}$.

Another way of looking at this is that each given element of a set could be either included or excluded, which can be counted as two different ways. Since there are n elements, this could be done in $2 \times 2 \times 2 \ldots \times 2 = 2^n$ ways. The total number of subsets of a set of n elements is 2^n.

Null sets have zero elements. If the null set is excluded from the tally, the total number of subsets would be $2^n - 1$.

null set

• a set with no elements

Example 1

Determine the Total Number of Subsets

Apples, grapes, peaches, plums, and strawberries are available for dessert. How many different combinations of fruit can be made for dessert?

Solution

Method 1: Use Combinations

Since dessert must include at least one fruit, determine the number of combinations of 1, 2, 3, 4, and 5 fruits.

$_5C_1 + {_5C_2} + {_5C_3} + {_5C_4} + {_5C_5}$

$= \dfrac{5!}{(5-1)!\,1!} + \dfrac{5!}{(5-2)!\,2!} + \dfrac{5!}{(5-3)!\,3!} + \dfrac{5!}{(5-4)!\,4!} + \dfrac{5!}{(5-5)!\,5!}$

$= \dfrac{5!}{4!\,1!} + \dfrac{5!}{3!\,2!} + \dfrac{5!}{2!\,3!} + \dfrac{5!}{1!\,4!} + \dfrac{5!}{0!\,5!}$ *What does 0! equal?*

$= 5 + 10 + 10 + 5 + 1$

$= 31$

Dessert can be made in 31 ways.

Processes

Connecting
Why is the rule of sum applied here?

Method 2: Use the Indirect Method

Dessert must include at least one fruit. So, subtract 1 from the total number of possible desserts. The 1 represents the number of desserts with no fruit (null set).

$2^5 - 1 = 32 - 1$ *The value 2^5 indicates that each of the 5 fruits has the possibility of*
$\qquad = 31$ *being selected in 2 ways, either in the dessert or not in the dessert.*

Dessert can be made in 31 ways.

Processes

Selecting Tools and Computational Strategies
Which method is more efficient for this problem? When might you choose to use method 1? method 2?

Your Turn

A bag contains eight different-coloured marbles. Use two methods to determine the number of ways to reach into the bag and pull out one or more marbles.

Example 2

Count Cases

The card game euchre uses only the 9s, 10s, jacks, queens, kings, and aces. Five-card hands are dealt to the players. How many euchre hands contain

a) at least three queens?

b) at least two black cards?

Solution

There are 6 cards in each suit and 24 cards in total.

a) The hand can have either three queens or four queens.

Case 1: Three Queens and Two Other Cards

$$
\begin{aligned}
\text{choices for queens} \times \text{choices for other cards} &= {}_4C_3 \times {}_{20}C_2 \\
&= \frac{4!}{(4-3)!\,3!} \times \frac{20!}{(20-2)!\,2!} \\
&= \frac{4!}{1!\,3!} \times \frac{20!}{18!\,2!} \\
&= 4 \times 190 \\
&= 760
\end{aligned}
$$

Case 2: Four Queens and One Other Card

$$
\begin{aligned}
{}_4C_4 \times {}_{20}C_1 &= \frac{4!}{0!\,4!} \times \frac{20!}{19!\,1!} \\
&= 1 \times 20 \\
&= 20
\end{aligned}
$$

The first factor represents the number of ways of selecting 4 queens. The second factor represents the number of ways of selecting 1 card from the 20 remaining cards used in euchre.

Because either Case 1 **or** Case 2 applies, use the rule of sum.

$$
\begin{aligned}
\text{Total number of possibilities} &= 760 + 20 \\
&= 780
\end{aligned}
$$

In euchre, 780 hands contain at least three queens.

b) Use the indirect method by subtracting the number of hands with 0 or 1 black card from the total number of hands. Since euchre uses 24 cards, the total number of hands possible is ${}_{24}C_5 = 42\,504$.

Case 1: Zero Black Cards and Five Red Cards

$$
\begin{aligned}
\text{Choices for black cards} \times \text{choices for red cards} &= {}_{12}C_0 \times {}_{12}C_5 \\
&= \frac{12!}{12!\,0!} \times \frac{12!}{7!\,5!} \\
&= 1 \times 792 \\
&= 792
\end{aligned}
$$

Why does ${}_{12}C_0$ represent only one choice for the black cards?

Case 2: One Black Card and Four Red Cards

$$
\begin{aligned}
{}_{12}C_1 \times {}_{12}C_4 &= \frac{12!}{11!\,1!} \times \frac{12!}{8!\,4!} \\
&= 12 \times 495 \\
&= 5940
\end{aligned}
$$

The number of possible hands is $42\,504 - 792 - 5940 = 35\,772$.

In euchre, 35 772 hands contain at least two black cards.

Processes

Reflecting
Could you solve part a) using the indirect method? Which method do you predict will be easier? Why?

Your Turn

In the game of hearts, the entire deck of cards is dealt. If you have a hand with 13 cards, in how many ways could the hand contain

a) at least two hearts?

b) at least ten hearts?

c) five clubs and five spades?

d) three diamonds?

e) five clubs or five spades?

Example 3

Choose, Then Arrange

Christine has 10 pictures of family and 8 pictures of friends to put on her wall. She installs shelves to display the pictures. The shelves can fit only four of the family pictures and three of the friends pictures. In how many ways can Christine arrange the pictures on the shelves?

Solution

First, determine the number of ways of choosing the two types of pictures. The number of ways to choose the four family pictures and the three friends pictures is $_{10}C_4 \times {}_8C_3$.

Once Christine chooses the seven pictures, she must decide how to arrange them. This can be done 7! ways.

The total number of arrangements is

$$_{10}C_4 \times {}_8C_3 \times 7! = \frac{10!}{(10-4)!\,4!} \times \frac{8!}{(8-3)!\,3!} \times 7!$$

Multiply each factor because Christine chooses 4 family pictures and 3 friends pictures. Then, arrange all of them.

$$= \frac{10!}{6!\,4!} \times \frac{8!}{5!\,3!} \times 7!$$

$$= 59\ 270\ 400$$

There are 59 270 400 ways to arrange four of Christine's family pictures and three of her friends pictures on the shelves.

Your Turn

a) How many five-letter codes can be formed from two different vowels and three different consonants? Consider Y a vowel.

b) How many of these codes contain the letter C?

Key Concepts

- The total number of subsets of a set of n elements is $_nC_0 + {_nC_1} + \dots + {_nC_n} = 2^n$.
- In some cases the null set is not considered. In such cases, $_nC_1 + {_nC_2} + \dots + {_nC_n} = 2^n - 1$.
- Consider using the indirect method, especially if it involves fewer cases, such as when you need to choose at least one or two items.
- If the order is important, consider selecting the items first and then arranging them in order.

Reflect

R1. When determining the total number of subsets of a set, you add the number of possibilities in each case. Explain why you add instead of multiply.

R2. When using cases to determine the number of ways of selecting objects from different sets, do you multiply or add? Explain your reasoning.

R3. You can solve counting problems using powers, permutations, combinations, or both. Make a summary and a flowchart to help decide which method(s) to use. Include simple examples to support your summary.

Practise

Choose the best answer for #2 and #3.

1. How many different sums of money can be made from a $5 bill, a $10 bill, a $20 bill, and a $50 bill?

2. In how many ways could a group of 10 people form a committee with at least 8 people on it?

   A 45   **B** 56   **C** 450   **D** 1016

3. If a set has 12 elements, how many subsets can be formed?

   A 12   **B** 24   **C** 4095   **D** 4096

4. A judging panel will have 6 members chosen from 8 teachers and 10 students. There must be at least 3 students on the panel. In how many ways could there be

   a) 3 students on the panel?

   b) 4 students on the panel?

   c) 5 students on the panel?

   d) least 3 students on the panel?

Apply

5. **Communication** Identify whether the following situations involve permutations, combinations, or both. Justify your choice.

   a) forming a committee of 5 people from a group of 12 people

   b) choosing a president, a vice president, and a treasurer from a committee of 12 members

   c) choosing 4 men and 4 women to be on a basketball team from among 6 men and 6 women, and assembling the athletes for a team photo

   d) naming 3 people from among 15 contestants to win 3 different prizes

6. You receive requests to connect with people every day on your social media account. If you have 15 requests to be "friends" with people, in how many ways could you respond by either accepting or rejecting each request?

7. Tonya has the following toppings available for her sandwich: lettuce, tomatoes, onions, olives, sprouts, peppers, mustard, and shredded cheese. She can use up to three toppings. How many different sandwiches can Tonya make?

8. Rohan is shopping for new pants. Six different styles are available. How many different purchases could Rohan make?

9. You can factor the number 210 into prime factors as $2 \times 3 \times 5 \times 7$. The products of prime factors form divisors (e.g., $2 \times 3 = 6$). Determine the total number of divisors of 210.

10. A board of directors needs to assign a chair, vice chair, treasurer, secretary and communications officer. There are four women and six men on the board. There will be two women and three men on the executive. In how many ways could this be done?

11. **Thinking** In cribbage, each player is dealt six cards from a standard deck. In how many ways could a hand contain
  a) at least two queens?
  b) more than three red cards?
  c) at least two hearts and at least two spades?

12. **Application** A telemarketer will call 12 people from a list of 20 men and 25 women. In how many ways could he select
  a) 12 men or 12 women?
  b) 6 men and 6 women?

13. A cabin has two rooms with three single beds each, one room with four single beds, and one room with two single beds. Six girls and six boys are assigned to rooms with people of the same gender. In how many ways can the rooms be assigned?

14. Six students from each of grades 9 to 12 have been pre-selected to win eight different prizes as students of the month. In how many ways could two students from each grade be selected to win these prizes?

15. **Thinking** Given the numbers –6, –5, –4, –3, –2, –1, 1, 2, 3, 4, 5, in how many ways could four different numbers be chosen so that their product is negative?

✔ Achievement Check

16. On a crown and anchor wheel, a crown, an anchor, and the four suits from a deck of cards are displayed in slots around the wheel.
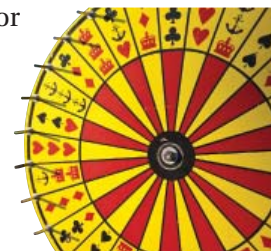
  a) Each three-of-a-kind (e.g., 👑👑👑) occurs twice. Calculate the number of slots with three-of-a-kind.
  b) Determine the number of slots with two-of-a-kind.
  c) The following restrictions are in place when all three symbols are different:
  • A crown and an anchor do not occur together (e.g., 👑⚓♦ cannot occur).
  • Three different suits do not occur together (e.g., ♦♥♠ cannot occur).
  • If a crown occurs with two different suits, an anchor may not also occur with the same two suits, and vice versa (only one of ♠⚓♦ or ♠👑♦ can occur).
  Calculate the number of slots with three different symbols. Use your calculations to verify the total number of slots on the wheel.

Extend

17. There are 10 points in a plane. No three points are collinear. How many convex polygons can be drawn using these points as vertices?

18. Five men and five women are selected from eight men and nine women and then seated around a circular table. In how many ways can this be done if their particular seat at the table does not matter?

Combinations and Pascal's Triangle

Learning Goal

I am learning to

- make connections between combinations and Pascal's triangle

> **Minds On...**

Scholars in many different cultures have known about Pascal's triangle for thousands of years. The modern version is attributed to Blaise Pascal, a 17th-century mathematician and philosopher. He discovered numerous patterns in the triangle, including those relating it to combinatorics and probability.

Pascal's triangle

- a triangular array of numbers in which each term is the sum of the two terms above it

The top rows of Pascal's triangle are shown, along with the term references. The terms are designated by $t_{n,r}$, where n is the row number, starting at zero, and r is the diagonal number, also starting at zero.

1					**Row 0**			$t_{0,0}$		
1		1			**Row 1**		$t_{1,0}$		$t_{1,1}$	
1	2		1		**Row 2**	$t_{2,0}$		$t_{2,1}$		$t_{2,2}$

Row 0   1 $t_{0,0}$

Row 1   1   1 $t_{1,0}$ $t_{1,1}$

Row 2   1   2   1 $t_{2,0}$ $t_{2,1}$ $t_{2,2}$

Row 3   1   3   3   1 $t_{3,0}$ $t_{3,1}$ $t_{3,2}$ $t_{3,3}$

Row 4   1   4   6   4   1 $t_{4,0}$ $t_{4,1}$ $t_{4,2}$ $t_{4,3}$ $t_{4,4}$

- Describe how to generate any given row.
- What are the terms of row 5?

> **Action!**

Investigate Connecting Pascal's Triangle With Combinations

1. **a)** Build a triangular array of terms by calculating each combination.

$${}_0C_0$$
$${}_1C_0 \quad {}_1C_1$$
$${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$$
$${}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3$$
$${}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4$$

b) Describe any observations.

c) Use two methods to determine the next row of Pascal's triangle.

2. How do combinations of the form $_nC_r$ relate to the terms in row 4 of Pascal's triangle?

3. a) Determine the sum of each row.

   b) **Reflect** Relate the results to a previous observation in this chapter.

4. **Extend Your Understanding**

   a) What is the sum of the terms in row 10? How do you know?

   b) What is the sum of the terms in row n? How do you know?

Pascal's Method

The terms of Pascal's triangle are generated by adding two adjacent terms and placing the result immediately below them in the next row.

$$t_{n,r} + t_{n,r+1} = t_{n+1,r+1}$$

Using combinations, $_nC_r + _nC_{r+1} = _{n+1}C_{r+1}$

Example 1

Diagonal Patterns in Pascal's Triangle

Calculate the sum of the first four terms of diagonal 2. Find this sum in Pascal's triangle and relate it to $_nC_r$.

Solution

For diagonal 2, which is outlined in the diagram, $r = 2$.
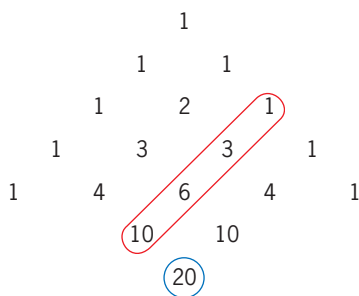
The sum of the first four terms is $1 + 3 + 6 + 10 = 20$. This value is circled in the diagram.

Comparing the terms in Pascal's triangle to combinations gives $_2C_2 + _3C_2 + _4C_2 + _5C_2 = _6C_3$.

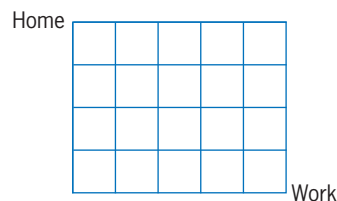### Your Turn

Calculate the sum of the first five terms of diagonal 6. Find this value in Pascal's triangle and relate it to $_nC_r$.

Example 2

Routes on a Grid

To get from home to work, Hannah travels four blocks south and five blocks east. How many different routes can she take, travelling only south and east?

Solution

Method 1: Use Pascal's Method

Each point of intersection can be reached only when Hannah travels south or east. Add the number of paths to the adjacent grid points to determine the number of paths to the given point. Continue until the diagram is complete.
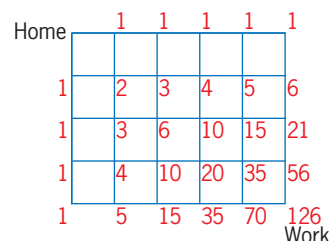
Hannah can take 126 different routes to work.

Method 2: Use Combinations

Hannah needs to travel nine blocks to work. Select any four of these nine blocks to travel southbound. The remaining five blocks will be eastbound.

$$_9C_4 \times {}_5C_5 = \frac{9!}{5!\,4!} \times \frac{5!}{0!\,5!}$$

Why do you not simply use $_{14}C_9$?

$$= 126$$

Hannah can take 126 different routes to work.

Your Turn

Bill rides his bike to school and travels four blocks west and six blocks north. Use two methods to determine the number of different routes Bill could take to school if he travels only north and west.

Example 3

Pascal's Triangle and the Binomial Theorem

a) Expand and simplify $(x + y)^2$ and $(x + y)^3$.

b) Relate the coefficients in the binomial expansion to Pascal's triangle and to combinations.

c) State the degree of each term and describe the pattern in the exponents.

Solution

a)

$(x + y)^2 = (x + y)(x + y)$
$= x^2 + 2xy + y^2$

$(x + y)^3 = (x + y)(x + y)(x + y)$
$= (x^2 + 2xy + y^2)(x + y)$
$= x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3$
$= x^3 + 3x^2y + 3xy^2 + y^3$

b) For $(x + y)^2$, the coefficients 1, 2, and 1 are the terms in row 2.

These terms correspond to combinations $_2C_r$, where $r = 0$ to 2.

$(x + y)^2 = {_2C_0}x^2y^0 + {_2C_1}xy + {_2C_2}x^0y^2$

For $(x + y)^3$, the coefficients 1, 3, 3, and 1 are the terms in row 3.

These terms correspond to combinations $_3C_r$, where $r = 0$ to 3.

$(x + y)^3 = {_3C_0}x^3y^0 + {_3C_1}x^2y + {_3C_2}x^1y^2 + {_3C_3}x^0y^3$

c) For $(x + y)^2$, the degree of each term is 2, which is the exponent of the binomial. For $(x + y)^3$, the degree of each term is 3, which is the exponent of the binomial.

For the first term, the exponent of x is n and the exponent of y is 0. For each successive term, the exponent of x decreases by 1, while the exponent of y increases by 1.

Binomial Theorem

You can use the binomial theorem to expand any power of a binomial expression.

$(x + y)^n = {_nC_0}x^ny^0 + {_nC_1}x^{n-1}y^1 + {_nC_2}x^{n-2}y^2 + \ldots + {_nC_{n-1}}x^1y^{n-1} + {_nC_n}x^0y^n$

The general term is $_nC_r x^{n-r}y^r$.

Your Turn

Use the binomial theorem to expand each binomial, relating it to both Pascal's triangle and combinations. State the degree of each term.

a) $(a + b)^4$ **b)** $(p + q)^5$

Consolidate and Debrief

Key Concepts

- The terms in row n of Pascal's triangle correspond to the combinations $t_{n,r} = {_nC_r}$.

- A given term in Pascal's triangle equals the sum of the two terms directly above it in the previous row. They can be generated using the relationship $t_{n,r} + t_{n,r+1} = t_{n+1,r+1}$. This is known as Pascal's method.

- Using combinations, $_nC_r + {_nC_{r+1}} = {_{n+1}C_{r+1}}$.

- The coefficients in the binomial expansion of $(x + y)^n$ are found in row n of Pascal's triangle.

- According to the binomial theorem,
  $(x + y)^n = {_nC_0}x^ny^0 + {_nC_1}x^{n-1}y^1 + {_nC_2}x^{n-2}y^2 + \ldots + {_nC_{n-1}}x^1y^{n-1} + {_nC_n}x^0y^n$.

- Pascal's method can be applied to counting paths in arrays.

Reflect

R1. Explain why the term labels of Pascal's triangle begin at $t_{0,0}$.

R2. Describe how Pascal's triangle and combinations are related. Use one row as a reference.

R3. Sam is working through Example 2. He calculates the number of arrangements of the letters SSSSEEEEE (S for south, E for east) to solve the problem. Is this a valid solution? Justify your reasoning.

Practise

Choose the best answer for #3 and #4.

1. **a)** Write each term in row 9 of Pascal's triangle using $_nC_r$.

   b) Write the first five terms in diagonal 4 of Pascal's triangle using $_nC_r$.

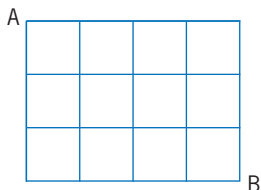2. Use Pascal's method to complete the array.

78	286	b
a	1001	
c		

3. The first three terms in the expansion of $(x + y)^7$ are

   A x^7, x^6y, x^5y^2

   B x^7, x^6, x^5

   C $xy^7, 7xy^6, 21xy^5$

   D $x^7, 7x^6y, 21x^5y^2$

4. Point B is four blocks east and three blocks south of point A. How many routes are possible from A to B, travelling only south and east?

   A 220

   B 35

   C 12

   D 7

5. Write the first nine rows of Pascal's triangle. Circle the given terms in Pascal's triangle, then circle the correct answer to help use Pascal's method to rewrite each of the following:

   a) $_5C_2 + _5C_3$ **b)** $_7C_3 + _7C_4$

   c) $_5C_4 - _4C_4$ **d)** $_8C_6 - _7C_5$

Apply

6. **Communication**

   a) Evaluate each of the following:

      i) $_2C_2 + _3C_2$

      ii) $_3C_2 + _4C_2$

      iii) $_4C_2 + _5C_2$

   b) Describe the results.

   c) Identify the terms from part a) in Pascal's triangle.

   d) Summarize the results, in general, as they apply to combinations and Pascal's triangle.

7. **a)** Calculate the sum of the first four terms of diagonal 7 (diagonals begin at zero). Locate the sum in Pascal's triangle and relate it to $_nC_r$.

   b) Generalize by stating the sum of the first k terms of diagonal r in Pascal's triangle and relating it to $_nC_r$.

8. **Application** A checker is placed on a checkerboard in the top right corner. The checker can move diagonally downward. Determine the number of routes to the bottom of the board.

9. A black checker is placed in the bottom-right corner of a checkerboard. The checker can move diagonally upward. The black checker cannot enter the square occupied by the red checker, but can jump over it. How many routes are there for the black checker to the top of the board?
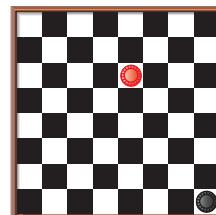
10. Determine the number of paths from A to B, travelling downward and to the right.
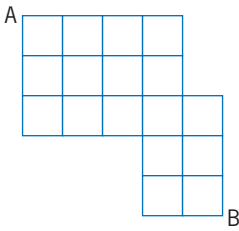
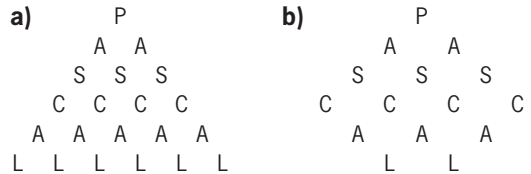**11.** Determine the number of paths that spell PASCAL. Can combinations be used to solve each question?

a)
```
        P
      A   A
    S   S   S
  C   C   C   C
A   A   A   A   A
L   L   L   L   L   L
```

b)
```
          P
       A     A
     S    S     S
  C    C    C     C
   A     A     A
      L     L
```

12. **Application** Investigate the sums of the first n natural numbers. For example, $1 + 2 = 3$, $1 + 2 + 3 = 6$, …

a) Locate the results in Pascal's triangle.
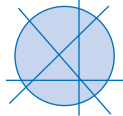
b) Summarize the results using combinations.

c) Refer to Example 1 on page 123. Explain why the results are similar.

13. Use Pascal's triangle to expand and simplify.

a) $(x + y)^8$ **b)** $(x - y)^5$

c) $(2a + b)^4$ **d)** $(x^2 - 2)^3$

14. Drawing a line through a circle divides it into two regions.

a) If n lines are drawn through a circle, what is the maximum number of regions formed? Develop a formula using Pascal's triangle.

b) Twenty lines are drawn through a circle. What is the maximum number of regions inside the circle?

15. a) Investigate the sum of the squares of the natural numbers from 1 to n by copying and completing the chart up to $n = 6$.
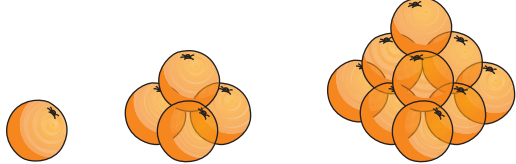
n	Sum of Squares $1^2 + 2^2 + … + n^2$	$t_{n+1,3} + t_{n+2,3}$
1		
2		

b) Describe the results.

c) Generalize using combinations.

d) State the sum of the squares of the first 50 natural numbers.

16. **Thinking** Investigate the number of oranges needed to stack the fruit in a tetrahedron.

a) Complete a chart showing the total number of oranges needed relative to the number of layers.

b) Identify the results in Pascal's triangle and describe your findings.

c) Write a relationship involving $_nC_r$.

d) How many oranges are needed for a 10-layer stack in a tetrahedral shape?

✔ **Achievement Check**

17. a) Calculate the sum of the squares of the terms in rows 2 to 5 of Pascal's Triangle.

Row n	Sum of Squares	Term in Pascal's Triangle
2		
3		

b) Identify the sums from part a) in Pascal's triangle. Describe your findings.

c) Rewrite each of the sums using combinations.

d) Generalize the relationship using a formula involving $_nC_r$.

18. **Thinking** Explain how $(h + t)^5$ could be used to show the different combinations of heads and tails when a coin is tossed repeatedly.

Extend

19. Expand and simplify.

a) $\left(p - \dfrac{1}{p}\right)^5$

b) $\left(3m^2 + \dfrac{2}{m^2}\right)^4$

Probabilities Using Combinations

Learning Goal

I am learning to

- solve probability problems using counting principles

Minds On...

Experienced card players usually consider probability. For example, a player has the king of spades, king of hearts, queen of diamonds, jack of clubs, and 9 of hearts. She needs to determine whether it is more likely that the next card will be another king, for three-of-a-kind, or a 10 for a five-card run (i.e., 9, 10, J, Q, K). How can she determine the likelihood of drawing each card?

Action!

Investigate Winning a Lottery

To win a particular lottery, your ticket must match five different numbers from 1 to 30, without regard to order.

1. How many winning combinations are there?

2. How many ways are there of choosing five different numbers from 1 to 30?

3. What is the probability of winning this lottery?

4. **Reflect** Research a lottery based on matching numbers.
   a) How does the probability of winning this lottery compare to the one analysed above?
   b) Would the probability of winning affect which lottery ticket you would buy? Explain.

5. **Extend Your Understanding** Some lotteries sell a specified number of tickets, and the winning ticket is drawn from those sold. Does the probability of winning increase as you buy more tickets? Make up an example to explain your answer.

Example 1

Chances of Winning

A scratch and win contest at a store allows you to scratch five numbers. If all of your numbers match the winning set of five numbers, chosen from 1 to 25 without regard to order, you win the grand prize.

a) What is the probability of winning the grand prize?

b) To win second prize, four of the five winning numbers must match. What is the probability of winning second prize?

Solution

a) The total number of possible outcomes is $n(S) = {}_{25}C_5$. There is only one successful outcome: matching all five of the winning numbers.

$$n(A) = {}_5C_5$$
$$= 1$$

$$P(\text{all five selected}) = \frac{n(A)}{n(S)}$$
$$= \frac{1}{{}_{25}C_5}$$
$$= \frac{1}{53\ 130}$$
$$\approx 0.000\ 018\ 821$$
$$\approx 0.001\ 882\%$$

There is approximately a 0.001 882% chance of having all five winning numbers.

b) The total number of possible outcomes is $n(S) = {}_{25}C_5$. Four of your numbers will match the winners, but the fifth does not, so it needs to be chosen from the remaining 20 numbers.

$$n(A) = {}_5C_4 \times {}_{20}C_1$$

$$P(\text{four successes}) = \frac{{}_5C_4 \times 20}{{}_{25}C_5}$$
$$= \frac{100}{53\ 130}$$
$$\approx 0.001\ 882$$
$$\approx 0.1882\%$$

There is approximately a 0.1882% chance of winning second prize.

Your Turn

To win the grand prize in a fundraising draw, you need to match seven numbers from 1 to 27, without regard to order.

a) What is the percent chance of winning the grand prize?

b) What is the percent chance of winning second prize, which involves matching six of the seven winning numbers?

c) What is the probability of not winning first or second prize?

d) Comment on how difficult it is to win lottery prizes.

Example 2

Choose From Groups

A university task force of 8 people is to be formed from 16 members of the student government and 10 professors. Each person is equally likely to be chosen.

a) What is the probability that there is an equal number of students and professors?

b) What is the probability that at least six members are students?

c) Which outcome is more likely to occur?

Solution

Eight members are chosen, without regard to order, from a total of 26 people: $n(S) = {}_{26}C_8$.

a) Four students and four professors are chosen.

> The use of **and** tells you to multiply.

$$n(A) = {}_{16}C_4 \times {}_{10}C_4$$

$$P(\text{equal number}) = \frac{{}_{16}C_4 \times {}_{10}C_4}{{}_{26}C_8}$$

> The first factor represents the choices for students. The second factor represents the choices for professors.

$$\approx 0.244\ 64$$

The probability that an equal number of students and professors is chosen is approximately 0.24.

b) There could be six, seven, or eight students. The remaining members are professors.

> The use of **or** tells you to use the rule of sum.

$$n(A) = ({}_{16}C_6 \times {}_{10}C_2) + ({}_{16}C_7 \times {}_{10}C_1) + ({}_{16}C_8 \times {}_{10}C_0)$$

$$P(\text{at least 6 students}) = \frac{({}_{16}C_6 \times {}_{10}C_2) + ({}_{16}C_7 \times {}_{10}C_1) + ({}_{16}C_8 \times {}_{10}C_0)}{{}_{26}C_8}$$

$$= \frac{360\ 360 + 114\ 400 + 12\ 870}{1\ 562\ 275}$$

$$\approx 0.312\ 12$$

The probability that there are at least six students on the task force is approximately 0.31.

c) $P(\text{at least 6 students}) > P(\text{equal number})$. Therefore, it is more likely that there will be at least six students than an equal number of students and professors.

Your Turn

A teacher uses a random name generator to select six students to present their projects. In a class of 23 students, 12 are male and 11 are female.

a) What is the probability that an equal number of male and female students will present?

b) What is the probability that more female than male students will present?

c) Which outcome is more likely?

Example 3

Apply Pascal's Triangle to Probability

In the game of Plinko, a disc is dropped into a slot at the top of a board. When it hits a peg, it falls to the left or right as it travels down the board. State the probability of the disc ending up in each slot at the bottom of the board.
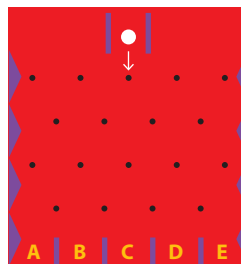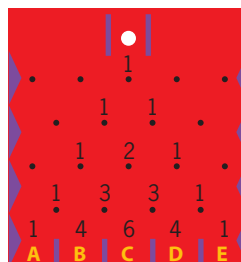
Solution

Using Pascal's triangle, the total number of paths to the bottom of the board is $1 + 4 + 6 + 4 + 1 = 16$.

$$P(C) = \frac{6}{16}$$

$$P(B) = P(D)$$
$$= \frac{4}{16}$$

$$P(A) = P(E)$$
$$= \frac{1}{16}$$

The probability of the disc ending up in each slot from the left is $\frac{1}{16}, \frac{4}{16}, \frac{6}{16}, \frac{4}{16}, \frac{1}{16}$.

Processes

Reflecting
Could you have used combinations to solve this problem?

Your Turn

State the probability of the disc ending up in each slot at the bottom of this Plinko board.
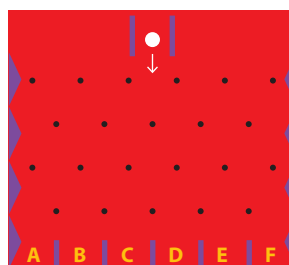
Consolidate and Debrief

Key Concepts

- You can sometimes use combinations or Pascal's triangle to determine probabilities.

- The numerator, $n(A)$, represents the number of successful outcomes, usually involving restrictions.

- The denominator, $n(S)$, represents the total number of outcomes, with no restrictions.

Reflect

R1. Using cards from a standard deck, provide an example of a situation in which permutations are used to calculate probabilities. Change the example so it involves combinations.

R2. Tim and Ginny are solving the following problem:

In a race of eight runners, what is the probability that Jake and Hamid are the top two finishers?

Tim solves the problem using permutations: $P(A) = \dfrac{_2P_2}{_8P_2}$

Ginny solves it using combinations: $P(A) = \dfrac{_2C_2}{_8C_2}$

Are both solutions valid? Why?

## Practise

Choose the best answer for #3 and #4.

1. What is the probability that a hand of five cards contains only

  a) hearts?

  b) red cards?

  c) face cards?

2. In the story of the Three Little Pigs, the big bad wolf was able to blow down the two houses made of straw and of sticks, but not the house made of bricks. If the wolf chose two houses at random, what is the probability that it would be able to blow them both down?

3. A department has 10 employees. Two will be chosen at random to attend a conference. What is the probability that both Sarah and Dan will be selected?

  A $\dfrac{1}{5}$ **B** $\dfrac{1}{45}$

  C $\dfrac{2}{45}$ **D** $\dfrac{1}{90}$

4. To win the grand prize in a hospital lottery, you must match six different numbers chosen from the numbers 1 to 45. What is the probability of winning the grand prize?

  A $\dfrac{1}{5\,864\,443\,200}$

  B $\dfrac{6}{5\,864\,443\,200}$

  C $\dfrac{1}{8\,145\,060}$

  D $\dfrac{6}{8\,145\,060}$

Apply

5. Application In the game of rummy, a player wins with a hand of three-of-a-kind or a run of three or more cards in the same suit. What is the probability that a hand of seven cards will be dealt

  a) three kings?

  b) three-of-a-kind?

  c) the 4, 5, and 6 of spades?

  d) a run of exactly three cards?

6. In the game of hearts, each of four players receives 13 cards. What is the probability that each player receives 13 cards of the same suit?

7. Six girls and five boys wish to join a committee. Four of them will be selected. What is the probability that three girls and one boy will be selected?

8. Four different numbers are selected at random from 1 to 15. What are the odds in favour of exactly two of the numbers being divisible by 5?

9. A spider walks from point A in its web always moving outward from the centre, until it reaches the perimeter. What is the probability that the spider will reach point B?
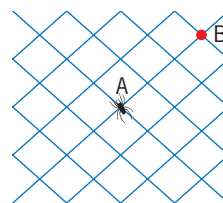
10. Communication Kayla is playing checkers on her smartphone. A checker is in the bottom square of the third column. It will randomly move diagonally left or right one space forward, for a total of seven moves.
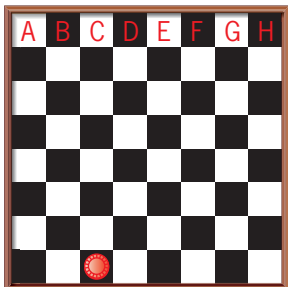
a) Which space has the greatest probability as the destination for the checker?

b) If the checker begins in a different position, will it affect the results? Explain.

11. A four-member curling team is randomly chosen from six grade 11 students and nine grade 12 students. What is the probability that the team has at least one grade 11 student?

12. Thinking A Plinko board has six rows of pegs. The top slots are numbered 1–6. The bottom slots are labelled A–G. Contestants choose the slot into which they drop the disc. What is the best strategy for releasing the disc and for predicting its landing location?
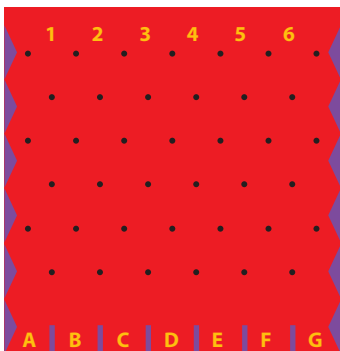
13. Five children are selected at random from a group of eight children. What is the probability that Fariba or Sana, but not both, is selected?

14. Adrian has 15 classmates. He selects four of them to form a study group. What is the probability that Reg or Carlos is selected?

✔ Achievement Check

15. A fast-food restaurant is running a scratch and win contest. Customers scratch six squares on a card and try to match the six numbers, randomly selected from 1 to 20, that are printed across the top of the card.

- First prize is awarded if all six numbers match.
- Second prize is awarded if five of the six numbers match.
- Third prize is awarded if four of the six numbers match.

a) What is the probability of winning first prize?

b) What is the probability of winning second prize?

c) What is the probability of winning third prize?

d) What are the odds against winning a prize?

e) Predict the number of fast-food meals a person would need to buy in order to have a relatively good chance of winning a prize. Do you think this strategy is sensible? Why?

Extend

16. In the binomial expansion of $(x - y)^8$, what is the probability that a randomly selected term is divisible by x^2?

17. Three different numbers are randomly chosen from the numbers 2 to 10. What is the probability that the product is

a) even?   b) divisible by 2?

c) divisible by 6?

18. An unknown card has been lost from a standard deck. Two cards are dealt and both turn out to be spades. What is the probability that the lost card is also a spade?

Chapter 3 Review

Learning Goals

Section	After this section, I can
3.1	• recognize the advantages of different counting techniques • make connections between situations that involve permutations and combinations
3.2	• recognize the advantages of using permutations and combinations over other counting techniques • apply combinations to solve counting problems • express combinations in standard notation, $C(n, r)$, $_nC_r$, $\binom{n}{r}$
3.3	• distinguish situations that use permutations from those that use combinations • solve counting problems using the rule of sum and the fundamental counting principle
3.4	• make connections between combinations and Pascal's triangle
3.5	• solve probability problems using counting principles

3.1 Permutations With Non-Ordered Elements, pages 104–109

1. How many ways are there to arrange four boys and three girls in a line for a photo if the girls must be in order of height from shortest to tallest?

2. How many arrangements are there of the letters in each word?
   a) ANAGRAM
   b) EXPRESSIONS
   c) ENGINEERING

3.2 Combinations, pages 110–115

3. The rooms in Kendra's apartment have 14 walls in total. She has enough paint to cover 10 of these walls in one colour and the rest in another colour. In how many ways could Kendra paint her apartment?

4. a) Determine the value for r that has the greatest number of combinations.
   i) $C(8, r)$ ii) $C(10, r)$
   iii) $C(7, r)$ iv) $C(15, r)$
   b) Generalize your findings.

5. A school choir has 10 sopranos, eight altos, seven tenors, and five basses. How many ways are there to select
   a) an octet of three sopranos, two altos, two tenors, and a bass?
   b) a barbershop quartet of two tenors and two basses?

6. A committee of four people is to be chosen from a list of 10 people.
   a) In how many ways could this be done?
   b) Rewrite the problem so that it requires permutations in its solution.
   c) Solve the new problem.

3.3 Problem Solving With Combinations, pages 116–121

7. In how many ways could five different envelopes be distributed into three mailboxes?

8. You have one each of $5, $10, $20, $50, and $100 bills in your wallet. How many different sums of money could you form by reaching into your wallet and pulling out some bills?

9. You are selecting an 8-character password using 26 letters and numbers 0 through 9. In how many ways could your password contain

   a) at least two letters?

   b) at least two numbers?

   c) at least two letters and two numbers?

3.4 Combinations and Pascal's Triangle, pages 122–127

Refer to Pascal's triangle on page 122.

10. Some entries of two rows of Pascal's triangle are given. Determine the unknown entries.

   330   462   b
       a   924

11. A circle is drawn with n points situated on its circumference.

   a) Using these points as vertices, how many quadrilaterals can be formed if

      i) $n = 4$?

      ii) $n = 5$?

      iii) $n = 6$?

   b) Identify these numbers in Pascal's triangle. Describe their location.

   c) Relate these numbers to combinations in terms of $_nC_r$.

   d) How many quadrilaterals can be formed if $n = 12$?

12. Determine the row in Pascal's triangle that has a sum of

   a) 512

   b) 4096

13. Stephen's school is four blocks west and seven blocks south of his home. Use two methods to determine the number of routes he could take to school, travelling west or south at all times.

14. Use Pascal's triangle and combinations to expand and simplify.

   a) $(a + b)^5$

   b) $(2x + y)^4$

3.5 Probabilities Using Combinations, pages 128–133

15. A poker hand of five cards is dealt from a standard deck.

   a) What is the probability that the jack, queen, and king of hearts, but no other hearts, are in the same hand?

   b) What is the probability that the hand contains five hearts?

16. Five students are randomly selected from seven boys and six girls to join a ski trip. What is the probability that

   a) all are girls?

   b) there are more boys than girls?

17. A total of 25 photos have been submitted for a photo competition, three of which were submitted by you. From the submissions, five photos will be chosen as finalists.

   a) What is the probability that all of your photos will be chosen as finalists?

   b) What is the probability that none of your photos will be chosen as a finalist?

   c) What is the probability that at least one of your photos will be chosen as a finalist?

| Category	Knowledge/ Understanding	Thinking	Communication	Application
Questions	1, 2, 3, 4,	14, 15, 16, 17	9, 13, 15	5, 6, 7, 8, 10, 11, 12

Multiple Choice

Choose the best answer for #1 to #4. Refer to Pascal's triangle on page 122 as needed.

1. How many ways are there to select four people from a group of nine people, without regard to order?

A 36   **B** 262 144

C 126   **D** 3024

2. What is the total number of subsets of a set of 10 elements?

A 1024   **B** 1023   **C** 100   **D** 20

3. Using Pascal's method, what is $_7C_3 + _7C_4$?

A $_8C_3$ **B** $_8C_4$ **C** $_8C_5$ **D** $_7C_5$

4. What is the number of arrangements of three red and two green blocks?

A $\dfrac{5!}{3!\,2!}$ **B** $3! \times 2!$

C $5!$ **D** $\dfrac{6!}{3!\,2!}$

Short Answer

5. In how many ways could a 6-member committee be formed from a 16-member club, if the president and secretary must be on the committee?

6. You found seven library books that you would like to take out, but the maximum is four. In how many ways could you select the four books?

7. How many quadrilaterals can be formed from the vertices of an octagon?
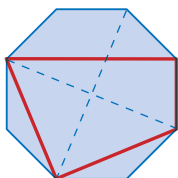
8. How many permutations are there of the letters in the word RELATIONS, if the vowels must be in alphabetical order?

9. a) How are $_8C_3$ and $_8P_3$ related?

b) Explain this relationship. Include an example to support your explanation.

10. Two balls are selected from a bag with five white and nine black balls. What is the probability that both balls are black?

11. What is the coefficient of p^4q^6 in the expansion of $(p + q)^{10}$?

Extended Response

12. Use two methods to show the number of ways 18 members of a rugby team could be assigned to six triple hotel rooms.

13. a) Describe the relationship between Pascal's triangle and combinations.

b) Write Pascal's method as it relates to both the entries in Pascal's triangle and to combinations.

14. Alternately subtract and add successive terms in a row of Pascal's triangle. For example, in row 4, $1 - 4 + 6 - 4 + 1$.

a) Investigate a few rows and describe the results.

b) Relate the results to combinations and provide a formula in terms of $_nC_r$.

15. Mario orders a pizza with 3 toppings, chosen from 15 available toppings.

a) In how many ways could mushrooms or olives be included in his toppings?

**b)** Would the result in part a) be greater or less if he orders 4 toppings? Explain.

16. A package of 50 computer chips contains 45 that are perfect and 5 that are defective. If 2 chips are selected at random, what is the probability that

a) neither is defective?

b) both are defective?

c) only one is defective?

17. The tens, jacks, queens, kings, and aces are removed from a standard deck of cards. From these cards, four are chosen. What is the probability that

a) all are queens?

b) all are red?

c) two are face cards?

d) there is at least one ace?

e) there are at least one ace and one king?

Chapter Problem

Counting Story

One day, when Goldilocks was out for a walk in the woods, she came upon a house and wondered if anybody was home. The door was not locked so she went inside. As soon as she closed the door, Goldilocks smelled something wonderful and it made her feel very hungry.

She went into the kitchen and found porridge in three pots (one blue, one red, and one green) on the stove. There were three empty bowls (one small, one medium, and one large) on the table. Goldilocks wanted some porridge but realized there were so many choices available to her. She could have porridge from the blue pot, the red pot, or the green pot; in a small bowl, a medium bowl, or a large bowl. Goldilocks couldn't think straight, so she drew a diagram.
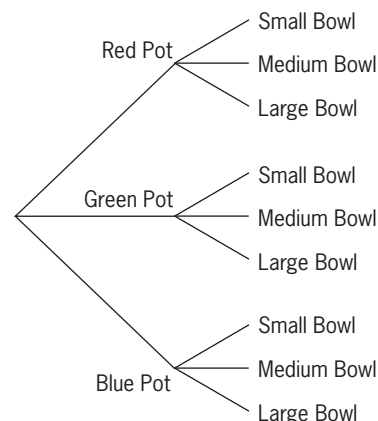
There were so many decisions to make! Goldilocks could serve herself porridge in $3 \times 3 = 9$ different ways.

a) Rewrite a children's story such as Rumpelstiltskin or the Three Little Pigs to involve permutations, combinations, or Pascal's triangle. Make the math an integral part of the story line, where the character is facing difficult decisions involving large numbers. Carry the problem through the story plot. Use diagrams to illustrate the choices to be made, but do not use combinatorial symbols, such as factorials, $_nP_r$ or $_nC_r$, in the story.

b) Then, write a mathematical summary of your story, connecting the story to permutations, combinations, or Pascal's triangle. In this summary, use combinatorial symbols.

Chapters 1 to 3 Cumulative Review

1. In a taste-test survey, 140 people out of 210 picked Koala Cola over Brand X. What is the experimental probability that a randomly selected taster will pick

a) Koala Cola?

b) Brand X?

2. Selia likes to listen to blues, country, and hard rock music. The table shows the songs loaded on her MP3 player.

| Type of Song	Number of Songs
Blues	20
Country	30
Hard Rock	50

Selia's player is set to random shuffle, which means that the player randomly plays a song. What are the odds in favour of the player randomly playing

a) a blues song?

b) a hard rock song?

3. a) A standard die is rolled 10 times. Explain why it is impossible that the experimental and theoretical probabilities will be perfectly equal.

b) How can this experiment be changed so that they become very close to being equal?

4. Determine the probability of rolling a sum of 7 or 11 using a standard pair of dice.
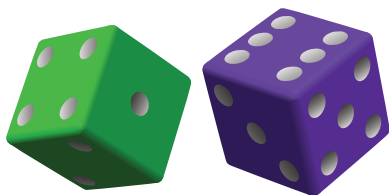
5. What is the probability of randomly drawing an ace or a red card from a standard deck of playing cards?

6. Consider the bag of marbles.
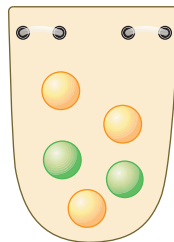
a) What is the probability of selecting a yellow marble followed by another yellow marble if the first marble is replaced?

b) How does this answer change if the first marble chosen is not replaced?

c) Explain why these answers are different.

7. Kaan lives in Orillia and will be flying to Jamaica on vacation. To get to Toronto Pearson International Airport, he can drive, take a bus, or take a taxi. He can fly non-stop to Jamaica or he can go via New York, Miami, Atlanta, or Philadelphia. Draw a map, a tree diagram, and make a list of all possible routes Kaan can take to Jamaica.

8. How many different outcomes are there when rolling

a) three standard dice?

b) four standard dice?

c) two 8-sided dice?

d) three 12-sided dice?

9. a) In how many ways could four adjacent countries on a map be coloured if eight colours are available? Adjacent countries must be different colours.

b) Why is it important that the countries are adjacent?

10. A total of 500 people enter a draw in which there is a first prize, a second prize, and a third prize. In how many ways could the prizes be awarded?

11. In how many ways could a president, vice president, secretary, and treasurer be elected from a condominium board that has 8 members?

12. In how many ways could 12 people be seated at a rectangular table if the two hosts must not sit together?

13. To win a school fundraising lottery, you need to correctly select five different digits in the correct order.

  a) What is the probability of winning?

  b) Would winning be more or less probable if the digits could be repeated? Why?

14. Find the prime factors of 255 255. What is the total number of divisors of 255 255, excluding 1 and itself?

15. Before a school dance, students tweeted requests to the DJ for five hip hop, seven R&B, eight rock, and nine pop songs. The DJ will play three requested songs from each genre. How many different playlists could the DJ generate?


16. To get to school, you travel six blocks west and four blocks south.

  a) Use permutations, combinations, or Pascal's method to determine the number of routes you could take to school.

  b) Relate your method to one of the other methods.

17. **Communication** Copy the table and extend it by six rows.

  a) Use a calculator or refer to Pascal's triangle to complete the chart up to $n = 9$.

n	$_nC_2 \div {_nC_1}$	Result
2		
3		

  b) For which values of n is $_nC_2$ divisible by $_nC_1$?

  c) Generalize your findings as they relate to combinations and Pascal's triangle.

  d) Is $_{15}C_2$ divisible by $_{15}C_1$? How do you know, without actually calculating it?

18. In how many ways can eight tickets be put into two envelopes if one envelope is to contain five tickets and the other envelope is to contain three tickets?

19. How many different sums of money can be made from a $5, a $10, a $20, a $50, and a $100 bill?

  a) Use the direct method.

  b) Use the indirect method.

20. A spider walks from one corner of a cube to the diagonally opposite corner.
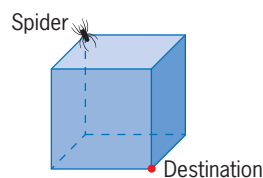
  a) If the spider walks along the edges only, and never backtracks, how many different paths can it take?

  b) Can combinations be used to solve this problem? Why or why not?

21. A jury of 12 people is chosen from 20 men and 30 women. What is the probability that

  a) there is an equal number of men and women on the jury?

  b) there are at least two men on the jury?