CHAPTER 6

Introduction to Probability

Specific Expectations	Section
Use Venn diagrams as a tool for organizing information in counting problems.	6.5
Solve problems, using techniques for counting permutations where some objects may be alike.	6.3
Solve problems, using techniques for counting combinations.	6.3
Solve probability problems involving combinations of simple events, using counting techniques.	6.3, 6.4, 6.5, 6.6
Interpret probability statements, including statements about odds, from a variety of sources.	6.1, 6.2, 6.3, 6.4, 6.5, 6.6
Design and carry out simulations to estimate probabilities in situations for which the calculation of the theoretical probabilities is difficult or impossible.	6.3
Assess the validity of some simulation results by comparing them with the theoretical probabilities, using the probability concepts developed in the course.	6.3
Represent complex tasks or issues, using diagrams.	6.1, 6.5
Represent numerical data, using matrices, and demonstrate an understanding of terminology and notation related to matrices.	6.6
Demonstrate proficiency in matrix operations, including addition, scalar multiplication, matrix multiplication, the calculation of row sums, and the calculation of column sums, as necessary to solve problems, with and without the aid of technology.	6.6
Solve problems drawn from a variety of applications, using matrix methods.	6.6

Chapter Problem

Genetic Probabilities

Biologists are studying a deer population in a provincial conservation area. The biologists know that many of the bucks (male deer) in the area have an unusual "cross-hatched" antler structure, which seems to be genetic in origin. Of 48 randomly tagged deer, 26 were does (females), 22 were bucks, and 7 of the bucks had cross-hatched antlers.

Several of the does have small bald patches on their hides. This condition also seems to have some genetic element. Careful long-term study has found that female offspring of does with bald patches have a 65% likelihood of developing the condition themselves, while offspring of healthy does have only a 20% likelihood of developing it. Currently, 30% of the does have bald patches.

1. Out of ten deer randomly captured, how many would you expect to have either cross-hatched antlers or bald patches?

2. Do you think that the proportion of does with the bald patches will increase, decrease, or remain relatively stable?

In this chapter, you will learn methods that the biologists could use to calculate probabilities from their samples and to make predictions about the deer population.

Review of Prerequisite Skills

If you need help with any of the skills listed in purple below, refer to Appendix A.

1. **Fractions, percents, decimals** Express each decimal as a percent.
   - **a)** 0.35
   - **b)** 0.04
   - **c)** 0.95
   - **d)** 0.008
   - **e)** 0.085
   - **f)** 0.375

2. **Fractions, percents, decimals** Express each percent as a decimal.
   - **a)** 15%
   - **b)** 3%
   - **c)** 85%
   - **d)** 6.5%
   - **e)** 26.5%
   - **f)** 75.2%

3. **Fractions, percents, decimals** Express each percent as a fraction in simplest form.
   - **a)** 12%
   - **b)** 35%
   - **c)** 67%
   - **d)** 4%
   - **e)** 0.5%
   - **f)** 98%

4. **Fractions, percents, decimals** Express each fraction as a percent. Round answers to the nearest tenth, if necessary.
   - **a)** $\frac{1}{4}$
   - **b)** $\frac{13}{15}$
   - **c)** $\frac{11}{14}$
   - **d)** $\frac{7}{10}$
   - **e)** $\frac{4}{9}$
   - **f)** $\frac{13}{20}$

5. **Tree diagrams** A coin is flipped three times. Draw a tree diagram to illustrate all possible outcomes.

6. **Tree diagrams** In the game of backgammon, you roll two dice to determine how you can move your counters. Suppose you roll first one die and then the other and you need to roll 9 or more to move a counter to safety. Use a tree diagram to list the different rolls in which
   - **a)** you make at least 9
   - **b)** you fail to move your counter to safety

7. **Fundamental counting principle (section 4.1)** Benoit is going skating on a cold wintry day. He has a toque, a watch cap, a beret, a heavy scarf, a light scarf, leather gloves, and wool gloves. In how many different ways can Benoit dress for the cold weather?

8. **Additive counting principle (section 4.1)** How many 13-card bridge hands include either seven hearts or eight diamonds?

9. **Venn diagrams (section 5.1)**
   - **a)** List the elements for each of the following sets for whole numbers from 1 to 10 inclusive.
     - **i)** E, the set of even numbers
     - **ii)** O, the set of odd numbers
     - **iii)** C, the set of composite numbers
     - **iv)** P, the set of perfect squares
   - **b)** Draw a diagram to illustrate how the following sets are related.
     - **i)** E and O
     - **ii)** E and C
     - **iii)** O and P
     - **iv)** E, C, and P

10. Principle of inclusion and exclusion (section 5.1)

a) Explain the principle of inclusion and exclusion.

b) A gift store stocks baseball hats in red or green colours. Of the 35 hats on display on a given day, 20 are green. As well, 18 of the hats have a grasshopper logo on the brim. Suppose 11 of the red hats have logos. How many hats are red, or have logos, or both?

11. Factorials (section 4.2) Evaluate.

a) $6!$ **b)** $0!$

c) $\dfrac{16!}{14!}$ **d)** $\dfrac{12!}{9!\,3!}$

e) $\dfrac{100!}{98!}$ **f)** $\dfrac{16!}{10! \times 8!}$

12. Permutations (section 4.2) Evaluate.

a) ${}_5P_3$ **b)** ${}_7P_1$

c) $P(6, 2)$ **d)** ${}_9P_9$

e) ${}_{100}P_1$ **f)** $P(100, 2)$

13. Permutations (section 4.2) A baseball team has 13 members. If a batting line-up consists of 9 players, how many different batting line-ups are possible?

14. Permutations (section 4.2) What is the maximum number of three-digit area codes possible if the area codes cannot start with either 1 or 0?

15. Combinations (section 5.2) Evaluate these expressions.

a) ${}_6C_3$ **b)** $C(4, 3)$

c) ${}_8C_8$ **d)** ${}_{11}C_0$

e) $\dbinom{6}{4} \times \dbinom{7}{5}$ **f)** $\dbinom{100}{1}$

g) ${}_{20}C_2$ **h)** ${}_{20}C_{18}$

16. Combinations (section 5.2) A pizza shop has nine toppings available. How many different three-topping pizzas are possible if each topping is selected no more than once?

17. Combinations (section 5.3) A construction crew has 12 carpenters and 5 drywallers. How many different safety committees could they form if the members of this committee are

a) any 5 of the crew?

b) 3 carpenters and 2 drywallers?

18. Matrices (section 1.6) Identify any square matrices among the following. Also identify any column or row matrices.

a) $\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$ **b)** $[0.4 \;\; 0.3 \;\; 0.2]$

c) $\begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \\ 0.8 & 0.6 \end{bmatrix}$ **d)** $\begin{bmatrix} -2 & 3 & 9 \\ 0 & 11 & -4 \\ 3 & 6 & -1 \end{bmatrix}$

e) $\begin{bmatrix} 49 & 63 \\ 25 & 14 \\ 72 & 9 \end{bmatrix}$ **f)** $\begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix}$

19. Matrices (section 1.7) Given $A = [0.3 \;\; 0.7]$ and $B = \begin{bmatrix} 0.4 & 0.6 \\ 0.55 & 0.45 \end{bmatrix}$, perform the following matrix operations, if possible. If the operation is not possible, explain why.

a) $A \times B$ **b)** $B \times A$

c) B^2 **d)** B^3

e) A^2 **f)** $A \times A^t$

6.1 Basic Probability Concepts

How likely is rain tomorrow? What are the chances that you will pass your driving test on the first attempt? What are the odds that the flight will be on time when you go to meet someone at the airport?

Probability is the branch of mathematics that attempts to predict answers to questions like these. As the word *probability* suggests, you can often predict only what *might* happen. However, you may be able to calculate how likely it is. For example, if the weather report forecasts a 90% chance of rain, there is still that slight possibility that sunny skies will prevail. While there are no sure answers, in this case it *probably* will rain.

INVESTIGATE & INQUIRE: A Number Game

Work with a partner. Have each partner take three identical slips of paper, number them 1, 2, and 3, and place them in a hat, bag, or other container. For each trial, both partners will randomly select one of their three slips of paper. Replace the slips after each trial. Score points as follows:

- If the product of the two numbers shown is less than the sum, Player A gets a point.
- If the product is greater than the sum, Player B gets a point.
- If the product and sum are equal, neither player gets a point.

1. Predict who has the advantage in this game. Explain why you think so.

2. Decide who will be Player A by flipping a coin or using the random number generator on a graphing calculator. Organize your results in a table like the one below.

Trial	1	2	3	4	5	6	7	8	9	10
Number drawn by A										
Number drawn by B										
Product										
Sum										
Point awarded to:										

3. a) Record the results for 10 trials. Total the points and determine the winner. Do the results confirm your prediction? Have you changed your opinion on who has the advantage? Explain.
   b) To estimate a probability for each player getting a point, divide the number of points each player earned by the total number of trials.
4. a) Perform 10 additional trials and record point totals for each player over all 20 trials. Estimate the probabilities for each player, as before.
   b) Are the results for 20 trials consistent with the results for 10 trials? Explain.
   c) Are your results consistent with those of your classmates? Comment on your findings.
5. Based on your results for 20 trials, predict how many points each player will have after 50 trials.
6. Describe how you could alter the game so that the other player has the advantage.

The investigation you have just completed is an example of a **probability experiment**. In probability, an experiment is a well-defined process consisting of a number of **trials** in which clearly distinguishable **outcomes**, or possible results, are observed.

The **sample space, *S*,** of an experiment is the set of all possible outcomes. For the sum/product game in the investigation, the outcomes are all the possible pairings of slips drawn by the two players. For example, if Player A draws 1 and Player B draws 2, you can label this outcome (1, 2). In this particular game, the result is the same for the outcomes (1, 2) and (2, 1), but with different rules it might be important who draws which number, so it makes sense to view the two outcomes as different.

Outcomes are often equally likely. In the sum/product game, each possible pairing of numbers is as likely as any other. Outcomes are often grouped into **events**. An example of an event is drawing slips for which the product is greater than the sum, and there are several outcomes in which this event happens. Different events often have different chances of occurring. Events are usually labelled with capital letters.

Example 1 Outcomes and Events

Let event A be a point awarded to Player A in the sum/product game. List the outcomes that make up event A.

Solution

Player A earns a point if the sum of the two numbers is greater than the product. This event is sometimes written as event $A = \{\text{sum} > \text{product}\}$. A useful technique in probability is to tabulate the possible outcomes.

Sums

		Player A		
		1	2	3
Player B	1	2	3	4
	2	3	4	5
	3	4	5	6

Products

		Player A		
		1	2	3
Player B	1	1	2	3
	2	2	4	6
	3	3	6	9

Use the tables shown to list the outcomes where the sum is greater than the product:
(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)
These outcomes make up event A. Using this list, you can also write event A as event $A = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)\}$

The **probability** of event A, $\boldsymbol{P(A)}$, is a quantified measure of the likelihood that the event will occur. *The probability of an event is always a value between 0 and 1.* A probability of 0 indicates that the event is impossible, and 1 signifies that the event is a certainty. Most events in probability studies fall somewhere between these extreme values. Probabilities less than 0 or greater than 1 have no meaning. Probability can be expressed as fractions, decimals, or percents. Probabilities expressed as percents are always between 0% and 100%. For example, a 70% chance of rain tomorrow means the same as a probability of 0.7, or $\frac{7}{10}$, that it will rain.

The three basic types of probability are

- empirical probability, based on direct observation or experiment
- theoretical probability, based on mathematical analysis
- subjective probability, based on informed guesswork

The **empirical probability** of a particular event (also called **experimental** or **relative frequency probability**) is determined by dividing the number of times that the event actually occurs in an experiment by the number of trials. In the sum/product investigation, you were calculating empirical probabilities. For example, if you had found that in the first ten trials, the product was greater than the sum four times, then the empirical probability of this event would be

$$\begin{aligned} P(A) &= \frac{4}{10} \\ &= \frac{2}{5} \text{ or } 0.4 \end{aligned}$$

The **theoretical probability** of a particular event is deduced from analysis of the possible outcomes. Theoretical probability is also called **classical** or ***a priori*** probability. *A priori* is Latin for “from the preceding,” meaning based on analysis rather than experiment.

For example, if all possible outcomes are equally likely, then

$$P(A) = \frac{n(A)}{n(S)}$$

where $n(A)$ is the number of outcomes in which event A can occur, and $n(S)$ is the total number of possible outcomes. You used tables to list the outcomes for A in Example 1, and this technique allows you to find the theoretical probability $P(A)$ by counting $n(A) = 5$ and $n(S) = 9$. Another way to determine the values of $n(A)$ and $n(S)$ is by organizing the information in a tree diagram.

Project Prep

You will need to determine theoretical probabilities to design and analyse your game in the probability project.

Example 2 Using a Tree Diagram to Calculate Probability

Determine the theoretical probabilities for each key event in the sum/product game.

Solution

The tree diagram shows the nine possible outcomes, each equally likely, for the sum/product game.

		product		sum
1	1	1	<	2
	2	2	<	3
	3	3	<	4
2	1	2	<	3
	2	4	=	4
	3	6	>	5
3	1	3	<	4
	2	6	>	5
	3	9	>	6

Let event A be a point for Player A, event B a point for Player B, and event C a tie between sum and product. From the tree diagram, five of the nine possible outcomes have the sum greater than the product. Therefore, the theoretical probability of this event is

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{5}{9} \end{aligned}$$

Similarly,

$$\begin{aligned} P(B) &= \frac{n(B)}{n(S)} \\ &= \frac{3}{9} \end{aligned} \quad \text{and} \quad \begin{aligned} P(C) &= \frac{n(C)}{n(S)} \\ &= \frac{1}{9} \end{aligned}$$

In Example 2, you know that one, and only one, of the three events will occur. The sum of the probabilities of all possible events always equals 1.

$$\begin{aligned} P(A) + P(B) + P(C) &= \frac{5}{9} + \frac{3}{9} + \frac{1}{9} \\ &= 1 \end{aligned}$$

Here, the numerator in each fraction represents the number of ways that each event can occur. The total of these numerators is the total number of possible outcomes, which is equal to the denominator.

Empirical probabilities may differ sharply from theoretical probabilities when only a few trials are made. Such **statistical fluctuation** can result in an event occurring more frequently or less frequently than theoretical probability suggests. Over a large number of trials, however, statistical fluctuations tend to cancel each other out, and empirical probabilities usually approach theoretical values. Statistical fluctuations often appear in sports, for example, where a team can enjoy a temporary winning streak that is not sustainable over an entire season.

In most problems, you will be determining theoretical probability. Therefore, from now on you may take the term *probability* to mean *theoretical probability* unless stated otherwise.

Example 3 Dice Probabilities

Many board games involve a roll of two six-sided dice to see how far you may move your pieces or counters. What is the probability of rolling a total of 7?

Solution

The table shows the totals for all possible rolls of two dice.

		First Die					
		1	2	3	4	5	6
Second Die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

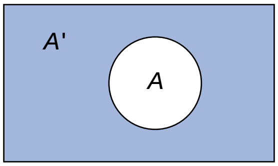
To calculate the probability of a particular total, count the number of times it appears in the table. For event A = {rolling 7},

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{n(\text{rolls totalling 7})}{n(\text{all possible rolls})} \\ &= \frac{6}{36} \\ &= \frac{1}{6} \end{aligned}$$

The probability of rolling a total of 7 is $\frac{1}{6}$.

A useful and important concept in probability is the complement of an event. The **complement** of event A, $\boldsymbol{A'}$ or $\boldsymbol{\sim A}$, is the event that "event A does *not* happen." Thus, whichever outcomes make up A, all the other outcomes make up A'. Because A and A' together include all possible outcomes, the sum of their probabilities must be 1. Thus,

$$P(A) + P(A') = 1 \qquad \text{and} \qquad P(A') = 1 - P(A)$$



The event A' is usually called "A-prime," or sometimes "not-A"; $\sim A$ is called "tilde-A."

Example 4 The Complement of an Event

What is the probability that a randomly drawn integer between 1 and 40 is *not* a perfect square?

Solution

Let event A = {a perfect square}. Then, the complement of A is the event A' = {*not* a perfect square}. In this case, you need to calculate $P(A')$, but it is easier to do this by finding $P(A)$ first. There are six perfect squares between 1 and 40: 1, 4, 9, 16, 25, and 36. The probability of a perfect square is, therefore,

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{6}{40} \\ &= \frac{3}{20} \end{aligned}$$

Thus,

$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - \frac{3}{20} \\ &= \frac{17}{20} \end{aligned}$$

There is a $\frac{17}{20}$ or 85% chance that a random integer between 1 and 40 will not be a perfect square.

Subjective probability, the third basic type of probability, is an estimate of likelihood based on intuition and experience—an educated guess. For example, a well-prepared student may be 90% confident of passing the next data management test. Subjective probabilities often figure in everyday speech in expressions such as "I think the team has only a 10% chance of making the finals this year."

Example 5 Determining Subjective Probability

Estimate the probability that

a) the next pair of shoes you buy will be the same size as the last pair you bought

b) an expansion baseball team will win the World Series in their first season

c) the next person to enter a certain coffee shop will be male

Solution

a) There is a small chance that the size of your feet has changed significantly or that different styles of shoes may fit you differently, so 80–90% would be a reasonable subjective probability that your next pair of shoes will be the same size as your last pair.

b) Expansion teams rarely do well during their first season, and even strong teams have difficulty winning the World Series. The subjective probability of a brand-new team winning the World Series is close to zero.

c) Without more information about the coffee shop in question, your best estimate is to assume that the shop's patrons are representative of the general population. This assumption gives a subjective probability of 50% that the next customer will be male.

WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

For some interesting baseball statistics, visit the above web site and follow the links. Write a problem that could be solved using probabilities.

Note that the answers in Example 5 contain estimates, assumptions, and, in some cases, probability *ranges*. While not as rigorous a measure as theoretical or empirical probability, subjective probabilities based on educated guesswork can still prove useful in some situations.

Key Concepts

- A probability experiment is a well-defined process in which clearly identifiable outcomes are measured for each trial.
- An event is a collection of outcomes satisfying a particular condition. The probability of an event can range between 0 (impossible) and 1 or 100% (certain).
- The empirical probability of an event is the number of times the event occurs divided by the total number of trials.
- The theoretical probability of an event A is given by $P(A) = \frac{n(A)}{n(S)}$, where $n(A)$ is the number of outcomes making up A, $n(S)$ is the total number of outcomes in the sample space S, and all outcomes are equally likely to occur.
- A subjective probability is based on intuition and previous experience.
- If the probability of event A is given by $P(A)$, then the probability of the complement of A is given by $P(A') = 1 - P(A)$.

Communicate Your Understanding

1. Give two synonyms for the word *probability*.

2. a) Explain why $P(A) + P(A') = 1$.

b) Explain why probabilities less than 0 or greater than 1 have no meaning.

3. Explain the difference between theoretical, empirical, and subjective probability. Give an example of how you would determine each type.

4. Describe three situations in which statistical fluctuations occur.

5. a) Describe a situation in which you might determine the probability of event A indirectly by calculating $P(A')$ first.

b) Will this method always yield the same result as calculating $P(A)$ directly?

c) Defend your answer to part b) using an explanation or proof, supported by an example.

Practise

A

1. Determine the probability of
   a) tossing heads with a single coin
   b) tossing two heads with two coins
   c) tossing at least one head with three coins
   d) rolling a composite number with one die
   e) not rolling a perfect square with two dice
   f) drawing a face card from a standard deck of cards

2. Estimate a subjective probability of each of the following events. Provide a rationale for each estimate.
   a) the sun rising tomorrow
   b) it never raining again
   c) your passing this course
   d) your getting the next job you apply for

3. Recall the sum/product game at the beginning of this section. Suppose that the game were altered so that the slips of paper showed the numbers 2, 3, and 4, instead of 1, 2, and 3.
   a) Identify all the outcomes that will produce each of the three possible events
      i) $p > s$ ii) $p < s$ iii) $p = s$
   b) Which player has the advantage in this situation?

Apply, Solve, Communicate

4. The town planning department surveyed residents of a town about home ownership. The table shows the results of the survey.

Residents	At Address Less Than 2 Years	At Address More Than 2 Years	Total for Category
Owners	2000	8000	10 000
Renters	4500	1500	6 000
Total	6500	9500	16 000

Determine the following probabilities.
   a) P(resident owns home)
   b) P(resident rents and has lived at present address less than two years)
   c) P(homeowner has lived at present address more than two years)

B

5. **Application** Suppose your school's basketball team is playing a four-game series against another school. So far this season, each team has won three of the six games in which they faced each other.
   a) Draw a tree diagram to illustrate all possible outcomes of the series.
   b) Use your tree diagram to determine the probability of your school winning exactly two games.
   c) What is the probability of your school sweeping the series (winning all four games)?
   d) Discuss any assumptions you made in the calculations in parts b) and c).

6. **Application** Suppose that a graphing calculator is programmed to generate a random natural number between 1 and 10 inclusive. What is the probability that the number will be prime?

7. **Communication**
   a) A game involves rolling two dice. Player A wins if the throw totals 5, 7, or 9. Player B wins if any other total is thrown. Which player has the advantage? Explain.
   b) Suppose the game is changed so that Player A wins if 5, 7, or doubles (both dice showing the same number) are thrown. Who has the advantage now? Explain.
   c) Design a similar game in which each player has an equal chance of winning.

8. a) Based on the randomly tagged sample, what is the empirical probability that a deer captured at random will be a doe?

b) If ten deer are captured at random, how many would you expect to be bucks?

C

9. Inquiry/Problem Solving Refer to the prime number experiment in question 6. What happens to the probability if you change the upper limit of the sample space? Use a graphing calculator or appropriate computer software to investigate this problem. Let A be the event that the random natural number will be a prime number. Let the random number be between 1 and n inclusive. Predict what you think will happen to $P(A)$ as n increases. Investigate $P(A)$ as a function of n, and reflect on your hypothesis. Did you observe what you expected? Why or why not?

10. Suppose that the Toronto Blue Jays face the New York Yankees in the division final. In this best-of-five series, the winner is the first team to win three games. The games are played in Toronto and in New York, with Toronto hosting the first, second, and if needed, fifth games. The consensus among experts is that Toronto has a 65% chance of winning at home and a 40% chance of winning in New York.

a) Construct a tree diagram to illustrate all the possible outcomes.

b) What is the chance of Toronto winning in three straight games?

c) For each outcome, add to your tree diagram the probability of that outcome.

d) Communication Explain how you found your answers to parts b) and c).

11. Communication Prior to a municipal election, a public-opinion poll determined that the probability of each of the four candidates winning was as follows:

Jonsson 10%

Trimble 32%

Yakamoto 21%

Audette 37%

a) How will these probabilities change if Jonsson withdraws from the race after ballots are cast?

b) How will these probabilities change if Jonsson withdraws from the race before ballots are cast?

c) Explain why your answers to a) and b) are different.

12. Inquiry/Problem Solving It is known from studying past tests that the correct answers to a certain university professor's multiple-choice tests exhibit the following pattern.

Correct Answer	Percent of Questions
A	15%
B	25%
C	30%
D	15%
E	15%

a) Devise a strategy for guessing that would maximize a student's chances for success, assuming that the student has no idea of the correct answers. Explain your method.

b) Suppose that the study of past tests revealed that the correct answer choice for any given question was the same as that of the immediately preceding question only 10% of the time. How would you use this information to adjust your strategy in part a)? Explain your reasoning.

6.2 Odds

Odds are another way to express a level of confidence about an outcome. Odds are commonly used in sports and other areas. Odds are often used when the probability of an event versus its complement is of interest, for example whether a sprinter will win or lose a race or whether a basketball team will make it to the finals.

INVESTIGATE & INQUIRE: Tennis Tournament

For an upcoming tennis tournament, a television commentator estimates that the top-seeded (highest-ranked) player has "a 25% probability of winning, but her odds of winning are only 1 to 3."

1. a) If event A is the top-seeded player winning the tournament, what is A'?
   b) Determine $P(A')$.

2. a) How are the odds of the top-seeded player winning related to $P(A)$ and $P(A')$?
   b) Should the television commentator be surprised that the odds were *only* 1 to 3? Why or why not?

3. a) What factors might the commentator consider when estimating the probability of the top-seeded player winning the tournament?
   b) How accurate do you think the commentator's estimate is likely to be? Would you consider such an estimate primarily a classical, an empirical, or a subjective probability? Explain.

WEB CONNECTION

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For more information about tennis rankings and other tennis statistics, visit the above web site and follow the links. Locate some statistics about a tennis player of your choice. Use odds to describe these statistics.

The **odds in favour** of an event's occurring are given by the ratio of the probability that the event will occur to the probability that it will not occur.

$$\text{odds in favour of } A = \frac{P(A)}{P(A')}$$

Giving odds in favour of an event is a common way to express a probability.

Example 1 Determining Odds

A messy drawer contains three red socks, five white socks, and four black socks. What are the odds in favour of randomly drawing a red sock?

Solution

Let the event A be drawing a red sock. The probability of this event is

$$\begin{aligned} P(A) &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$

The probability of not drawing a red sock is

$$\begin{aligned} P(A') &= 1 - P(A) \\ &= \frac{3}{4} \end{aligned}$$

Using the definition of odds,

$$\begin{aligned} \text{odds in favour of } A &= \frac{P(A)}{P(A')} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{1}{3} \end{aligned}$$

Therefore, the odds in favour of drawing a red sock are $\frac{1}{3}$, or less than 1. You are more likely *not* to draw a red sock. These odds are commonly written as 1:3, which is read as "one to three" or "one in three."

Project Prep

A useful feature you could include in your probability project is a calculation of the odds of winning your game.

Notice in Example 1 that the ratio of red socks to other socks is 3:9, which is the same as the odds in favour of drawing a red sock. In fact, the odds in favour of an event A can also be found using

$$\text{odds in favour of } A = \frac{n(A)}{n(A')}$$

A common variation on the theme of odds is to express the odds *against* an event happening.

$$\text{odds against } A = \frac{P(A')}{P(A)}$$

Example 2 Odds Against an Event

If the chance of a snowstorm in Windsor, Ontario, in January is estimated at 0.4, what are the odds against Windsor's having a snowstorm next January? Is a January snowstorm more likely than not?

Solution

Let event A = {snowstorm in January}.
Since $P(A) + P(A') = 1$,

$$\begin{aligned}\text{odds against } A &= \frac{P(A')}{P(A)} \\ &= \frac{1 - P(A)}{P(A)} \\ &= \frac{1 - 0.4}{0.4} \\ &= \frac{0.6}{0.4} \\ &= \frac{3}{2}\end{aligned}$$

The odds against a snowstorm are 3:2, which is greater than 1:1. So a snowstorm is *less* likely to occur than not.

Sometimes, you might need to convert an expression of odds into a probability. You can do this conversion by expressing $P(A')$ in terms of $P(A)$.

Example 3 Probability From Odds

A university professor, in an effort to promote good attendance habits, states that the odds of passing her course are 8 to 1 when a student misses fewer than five classes. What is the probability that a student with good attendance will pass?

Solution

Let the event A be that a student with good attendance passes. Since

$$\text{odds in favour of } A = \frac{P(A)}{P(A')},$$

$$\frac{8}{1} = \frac{P(A)}{P(A')}$$

$$= \frac{P(A)}{1 - P(A)}$$

$$8 - 8P(A) = P(A)$$

$$8 = 9P(A)$$

$$P(A) = \frac{8}{9}$$

The probability that a student with good attendance will pass is $\frac{8}{9}$, or approximately 89%.

In general, it can be shown that if the odds in favour of $A = \frac{h}{k}$, then $P(A) = \frac{h}{h+k}$.

Example 4 Using the Odds-Probability Formula

The odds of Rico's hitting a home run are 2:7. What is the probability of Rico's hitting a home run?

Solution

Let A be the event that Rico hits a home run. Then, $h = 2$ and $k = 7$, and

$$P(A) = \frac{h}{h+k}$$

$$= \frac{2}{2+7}$$

$$= \frac{2}{9}$$

Rico has approximately a 22% chance of hitting a home run.

Key Concepts

- The odds in favour of A are given by the ratio $\frac{P(A)}{P(A')}$.
- The odds against A are given by the ratio $\frac{P(A')}{P(A)}$.
- If the odds in favour of A are $\frac{h}{k}$, then $P(A) = \frac{h}{h+k}$.

Communicate Your Understanding

1. Explain why the terms *odds* and *probability* have different meanings. Give an example to illustrate your answer.
2. Would you prefer the odds in favour of passing your next data management test to be 1:3 or 3:1? Explain your choice.
3. Explain why odds can be greater than 1, but probabilities must be between 0 and 1.

Practise

A

1. Suppose the odds in favour of good weather tomorrow are 3:2.
   a) What are the odds against good weather tomorrow?
   b) What is the probability of good weather tomorrow?
2. The odds against the Toronto Argonauts winning the Grey Cup are estimated at 19:1. What is the probability that the Argos will win the cup?
3. Determine the odds in favour of rolling each of the following sums with a standard pair of dice.
   a) 12
   b) 5 or less
   c) a prime number
   d) 1
4. Calculate the odds in favour of each event.
   a) New Year's Day falling on a Friday
   b) tossing three tails with three coins
   c) not tossing exactly two heads with three coins
   d) randomly drawing a black 6 from a complete deck of 52 cards
   e) a random number from 1 to 9 inclusive being even

Apply, Solve, Communicate

B

5. Greta's T-shirt drawer contains three tank tops, six V-neck T-shirts, and two sleeveless shirts. If she randomly draws a shirt from the drawer, what are the odds that she will
   a) draw a V-neck T-shirt?
   b) not draw a tank top?
6. **Application** If the odds in favour of Boris beating Elena in a chess game are 5 to 4, what is the probability that Elena will win an upset victory in a best-of-five chess tournament?
7. Chapter Problem
   a) Based on the randomly tagged sample, what are the odds in favour of a captured deer being a cross-hatched buck?
   b) What are the odds against capturing a doe?

WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

Visit the above web site and follow the links for more information about Canadian wildlife.

8. The odds against A, by definition, are equivalent to the odds in favour of A'. Use this definition to show that the odds against A are equal to the reciprocal of the odds in favour of A.

9. **Application** Suppose the odds of the Toronto Maple Leafs winning the Stanley Cup are 1:5, while the odds of the Montréal Canadiens winning the Stanley Cup are 2:13. What are the odds in favour of either Toronto or Montréal winning the Stanley Cup?

10. What are the odds against drawing
   a) a face card from a standard deck?
   b) two face cards?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

11. Mike has a loaded (or unfair) six-sided die. He rolls the die 200 times and determines the following probabilities for each score:

$P(1) = 0.11$
$P(2) = 0.02$
$P(3) = 0.18$
$P(4) = 0.21$
$P(5) = 0.40$

   a) What is $P(6)$?
   b) Mike claims that the odds in favour of tossing a prime number with this die are the same as with a fair die. Do you agree with his claim?
   c) Using Mike's die, devise a game with odds in Mike's favour that an unsuspecting person would be tempted to play. Use probabilities to show that the game is in Mike's favour. Explain why a person who does not realize that the die is loaded might be tempted by this game.

12. George estimates that there is a 30% chance of rain the next day if he waters the lawn, a 40% chance if he washes the car, and a 50% chance if he plans a trip to the beach. Assuming George's estimates are accurate, what are the odds
   a) in favour of rain tomorrow if he waters the lawn?
   b) in favour of rain tomorrow if he washes the car?
   c) against rain tomorrow if he plans a trip to the beach?

C

13. **Communication** A volleyball coach claims that at the next game, the odds of her team winning are 3:1, the odds against losing are 5:1, and the odds against a tie are 7:1. Are these odds possible? Explain your reasoning.

14. **Inquiry / Problem Solving** Aki is a participant on a trivia-based game show. He has an equal likelihood on any given trial of being asked a question from one of six categories: Hollywood, Strange Places, Number Fun, Who?, Having a Ball, and Write On! Aki feels that he has a 50/50 chance of getting Having a Ball or Strange Places questions correct, but thinks he has a 90% probability of getting any of the other questions right. If Aki has to get two of three questions correct, what are his odds of winning?

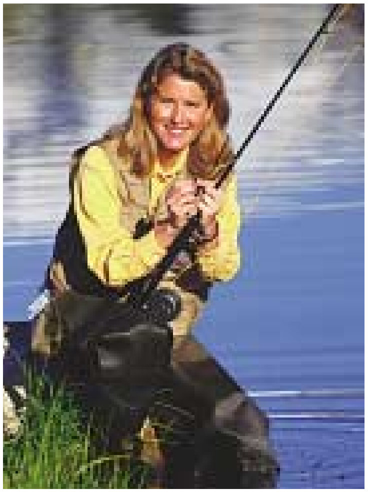
15. **Inquiry / Problem Solving** Use logic and mathematical reasoning to show that if the odds in favour of A are given by $\frac{h}{k}$, then $P(A) = \frac{h}{h+k}$. Support your reasoning with an example.

6.3 Probabilities Using Counting Techniques

How likely is it that, in a game of cards, you will be dealt just the hand that you need? Most card players accept this question as an unknown, enjoying the unpredictability of the game, but it can also be interesting to apply counting analysis to such problems.

In some situations, the possible outcomes are not easy or convenient to count individually. In many such cases, the counting techniques of permutations and combinations (see Chapters 4 and 5, respectively) can be helpful for calculating theoretical probabilities, or you can use a simulation to determine an empirical probability.

INVESTIGATE & INQUIRE: Fishing Simulation



Suppose a pond has only three types of fish: catfish, trout, and bass, in the ratio 5:2:3. There are 50 fish in total. Assuming you are allowed to catch only three fish before throwing them back, consider the following two events:

- event A = {catching three trout}
- event B = {catching the three types of fish, in alphabetical order}

1. Carry out the following probability experiment, independently or with a partner. You can use a hat or paper bag to represent the pond, and some differently coloured chips or markers to represent the fish. How many of each type of fish should you release into the pond? Count out the appropriate numbers and shake the container to simulate the fish swimming around.

2. Draw a tree diagram to illustrate the different possible outcomes of this experiment.

3. Catch three fish, one at a time, and record the results in a table. Replace all three fish and shake the container enough to ensure that they are randomly distributed. Repeat this process for a total of ten trials.

4. Based on these ten trials, determine the empirical probability of event A, catching three trout. How accurate do you think this value is? Compare your results with those of the rest of the class. How can you obtain a more accurate empirical probability?

5. Repeat step 4 for event B, which is to catch a bass, catfish, and trout in order.

6. Perform step 3 again for 10 new trials. Calculate the empirical probabilities of events A and B, based on your 20 trials. Do you think these probabilities are more accurate than those from 10 trials? Explain why or why not.

7. If you were to repeat the experiment for 50 or 100 trials, would your results be more accurate? Why or why not?

8. In this investigation, you knew exactly how many of each type of fish were in the pond because they were counted out at the beginning. Describe how you could use the techniques of this investigation to estimate the ratios of different species in a real pond.

This section examines methods for determining the theoretical probabilities of successive or multiple events.

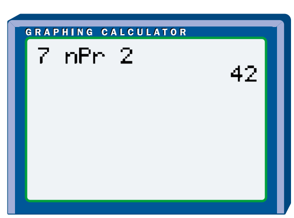
Example 1 Using Permutations

Two brothers enter a race with five friends. The racers draw lots to determine their starting positions. What is the probability that the older brother will start in lane 1 with his brother beside him in lane 2?

Solution

A permutation ${}_nP_r$, or $P(n, r)$, is the number of ways to select r objects from a set of n objects, *in a certain order*. (See Chapter 4 for more about permutations.) The sample space is the total number of ways the first two lanes can be occupied. Thus,

$$\begin{aligned} n(S) &= {}_7P_2 \\ &= \frac{7!}{(7-2)!} \\ &= \frac{7!}{5!} \\ &= \frac{7 \times 6 \times (5!)}{5!} \\ &= 42 \end{aligned}$$



The specific outcome of the older brother starting in lane 1 and the younger brother starting in lane 2 can only happen one way, so $n(A) = 1$. Therefore,

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{1}{42} \end{aligned}$$

The probability that the older brother will start in lane 1 next to his brother in lane 2 is $\frac{1}{42}$, or approximately 2.3%.

Example 2 Probability Using Combinations

A focus group of three members is to be randomly selected from a medical team consisting of five doctors and seven technicians.

a) What is the probability that the focus group will be comprised of doctors only?

b) What is the probability that the focus group will not be comprised of doctors only?

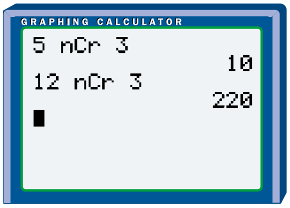
Solution

a) A combination ${}_nC_r$, also written $C(n, r)$ or $\binom{n}{r}$, is the number of ways to select r objects from a set of n objects, *in any order*. (See Chapter 5 for more about combinations.) Let event A be selecting three doctors to form the focus group. The number of possible ways to make this selection is

$$\begin{aligned} n(A) &= {}_5C_3 \\ &= \frac{5!}{3!(5-3)!} \\ &= \frac{5 \times 4 \times 3!}{3! \times 2!} \\ &= \frac{20}{2} \\ &= 10 \end{aligned}$$

However, the focus group can consist of any three people from the team of 12.

$$\begin{aligned} n(S) &= {}_{12}C_3 \\ &= \frac{12!}{3!(12-3)!} \\ &= \frac{12 \times 11 \times 10 \times 9!}{3! \times 9!} \\ &= \frac{1320}{6} \\ &= 220 \end{aligned}$$



The probability of selecting a focus group of doctors only is

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{10}{220} \\ &= \frac{1}{22} \end{aligned}$$

The probability of selecting a focus group consisting of three doctors is $\frac{1}{22}$, or approximately 0.045.

b) Either the focus group is comprised of doctors only, or it is not. Therefore, the probability of the complement of A, $P(A')$, gives the desired result.

$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - \frac{1}{22} \\ &= \frac{21}{22} \end{aligned}$$

So, the probability of selecting a focus group not comprised of doctors only is $\frac{21}{22}$, or approximately 0.955.

Project Prep

When you determine the classical probabilities for your probability project, you may need to apply the counting techniques of permutations and combinations.

Example 3 Probability Using the Fundamental Counting Principle

What is the probability that two or more students out of a class of 24 will have the same birthday? Assume that no students were born on February 29.

Solution 1 Using Pencil and Paper

The simplest method is to find the probability of the complementary event that no two people in the class have the same birthday.

Pick two students at random. The second student has a different birthday than the first for 364 of the 365 possible birthdays. Thus, the probability that the two students have different birthdays is $\frac{364}{365}$. Now add a third student. Since there are 363 ways this person can have a different birthday from the other two students, the probability that all three students have different birthdays is $\frac{364}{365} \times \frac{363}{365}$. Continuing this process, the probability that none of the 24 people have the same birthday is

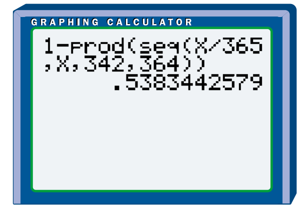
$$\begin{aligned} P(A') &= \frac{n(A')}{n(S)} \\ &= \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots \times \frac{342}{365} \\ &\doteq 0.462 \end{aligned}$$

$$\begin{aligned} P(A) &= 1 - P(A') \\ &= 1 - 0.462 \\ &= 0.538 \end{aligned}$$

The probability that at least two people in the group have the same birthday is approximately 0.538.

Solution 2 Using a Graphing Calculator

Use the iterative functions of a graphing calculator to evaluate the formula above much more easily. The **prod(function** on the LIST MATH menu will find the product of a series of numbers. The **seq(function** on the LIST OPS menu generates a sequence for the range you specify. Combining these two functions allows you to calculate the probability in a single step.



Key Concepts

- In probability experiments with many possible outcomes, you can apply the fundamental counting principle and techniques using permutations and combinations.
- Permutations are useful when order is important in the outcomes; combinations are useful when order is not important.

Communicate Your Understanding

1. In the game of bridge, each player is dealt 13 cards out of the deck of 52. Explain how you would determine the probability of a player receiving

a) all hearts **b)** all hearts in ascending order

2. a) When should you apply permutations in solving probability problems, and when should you apply combinations?

b) Provide an example of a situation where you would apply permutations to solve a probability problem, other than those in this section.

c) Provide an example of a situation where you would apply combinations to solve a probability problem, other than those in this section.

Practise

A

1. Four friends, two females and two males, are playing contract bridge. Partners are randomly assigned for each game. What is the probability that the two females will be partners for the first game?

2. What is the probability that two out of a group of eight friends will have the same birthday?

3. A fruit basket contains five red apples and three green apples. Without looking, you randomly select two apples. What is the probability that

a) you will select two red apples?

b) you will not select two green apples?

4. Refer to Example 1. What is the probability that the two brothers will start beside each other in any pair of lanes?

Apply, Solve, Communicate

B

5. An athletic committee with three members is to be randomly selected from a group of six gymnasts, four weightlifters, and eight long-distance runners. Determine the probability that
   a) the committee is comprised entirely of runners
   b) the committee is represented by each of the three types of athletes

6. A messy drawer contains three black socks, five blue socks, and eight white socks, none of which are paired up. If the owner grabs two socks without looking, what is the probability that both will be white?

7. a) A family of nine has a tradition of drawing two names from a hat to see whom they will each buy presents for. If there are three sisters in the family, and the youngest sister is always allowed the first draw, determine the probability that the youngest sister will draw both of the other two sisters' names. If she draws her own name, she replaces it and draws another.
   b) Suppose that the tradition is modified one year, so that the first person whose name is drawn is to receive a "main" present, and the second a less expensive, "fun" present. Determine the probability that the youngest sister will give a main present to the middle sister and a fun present to the eldest sister.

8. **Application**
   a) Laura, Dave, Monique, Marcus, and Sarah are going to a party. What is the probability that two of the girls will arrive first?
   b) What is the probability that the friends will arrive in order of ascending age?
   c) What assumptions must be made in parts a) and b)?

9. A hockey team has two goalies, six defenders, eight wingers, and four centres. If the team randomly selects four players to attend a charity function, what is the likelihood that
   a) they are all wingers?
   b) no goalies or centres are selected?

10. **Application** A lottery promises to award ten grand-prize trips to Hawaii and sells 5 400 000 tickets.
    a) Determine the probability of winning a grand prize if you buy
       i) 1 ticket
       ii) 10 tickets
       iii) 100 tickets
    b) **Communication** How many tickets do you need to buy in order to have a 5% chance of winning a grand prize? Do you think this strategy is sensible? Why or why not?
    c) How many tickets do you need to ensure a 50% chance of winning?

11. Suki is enrolled in one data-management class at her school and Leo is in another. A school quiz team will have four volunteers, two randomly selected from each of the two classes. Suki is one of five volunteers from her class, and Leo is one of four volunteers from his. Calculate the probability of the two being on the team and explain the steps in your calculation.

12. a) Suppose 4 of the 22 tagged bucks are randomly chosen for a behaviour study. What is the probability that

Chapter Problem

i) all four bucks have the cross-hatched antlers?

ii) at least one buck has cross-hatched antlers?

b) If two of the seven cross-hatched males are randomly selected for a health study, what is the probability that the eldest of the seven will be selected first, followed by the second eldest?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

13. Suppose a bag contains the letters to spell *probability*.

a) How many four-letter arrangements are possible using these letters?

b) What is the probability that Barb chooses four letters from the bag in the order that spell her name?

c) Pick another four-letter arrangement and calculate the probability that it is chosen.

d) What four-letter arrangement would be most likely to be picked? Explain your reasoning.

C

14. Communication Refer to the fishing investigation at the beginning of this section.

a) Determine the theoretical probability of

i) catching three trout

ii) catching a bass, catfish, and trout in alphabetical order

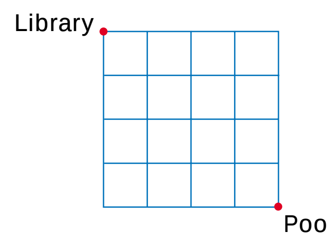
b) How do these results compare with the empirical probabilities from the investigation? How do you account for any differences?

c) Could the random-number generator of a graphing calculator be used to simulate this investigation? If so, explain how. If not, explain why.

d) Outline the steps you would use to model this problem with software such as Fathom™ or a spreadsheet.

e) Is the assumption that the fish are randomly distributed likely to be completely correct? Explain. What other assumptions might affect the accuracy of the calculated probabilities?

15. A network of city streets forms square blocks as shown in the diagram.



Jeanine leaves the library and walks toward the pool at the same time as Miguel leaves the pool and walks toward the library. Neither person follows a particular route, except that both are always moving toward their destination. What is the probability that they will meet if they both walk at the same rate?

16. Inquiry / Problem Solving A committee is formed by randomly selecting from eight nurses and two doctors. What is the minimum committee size that ensures at least a 90% probability that it will not be comprised of nurses only?

6.4 Dependent and Independent Events

If you have two examinations next Tuesday, what is the probability that you will pass both of them? How can you predict the risk that a critical network server and its backup will both fail? If you flip an ordinary coin repeatedly and get heads 99 times in a row, is the next toss almost certain to come up tails?

In such situations, you are dealing with **compound events** involving two or more separate events.

INVESTIGATE & INQUIRE: Getting Out of Jail in MONOPOLY®

While playing MONOPOLY® for the first time, Kenny finds himself in jail. To get out of jail, he needs to roll doubles on a pair of standard dice.

1. Determine the probability that Kenny will roll doubles on his first try.
2. Suppose that Kenny fails to roll doubles on his first two turns in jail. He reasons that on his next turn, his odds are now 50/50 that he will get out of jail. Explain how Kenny has reasoned this.
3. Do you agree or disagree with Kenny's reasoning? Explain.
4. What is the probability that Kenny will get out of jail on his third attempt?
5. After how many turns is Kenny certain to roll doubles? Explain.
6. Kenny's opponent, Roberta, explains to Kenny that each roll of the dice is an independent event and that, since the relatively low probability of rolling doubles never changes from trial to trial, Kenny may never get out of jail and may as well just forfeit the game. Explain the flaws in Roberta's analysis.

In some situations involving compound events, the occurrence of one event has no effect on the occurrence of another. In such cases, the events are **independent**.

Example 1 Simple Independent Events

a) A coin is flipped and turns up heads. What is the probability that the second flip will turn up heads?

b) A coin is flipped four times and turns up heads each time. What is the probability that the fifth trial will be heads?

c) Find the probability of tossing five heads in a row.

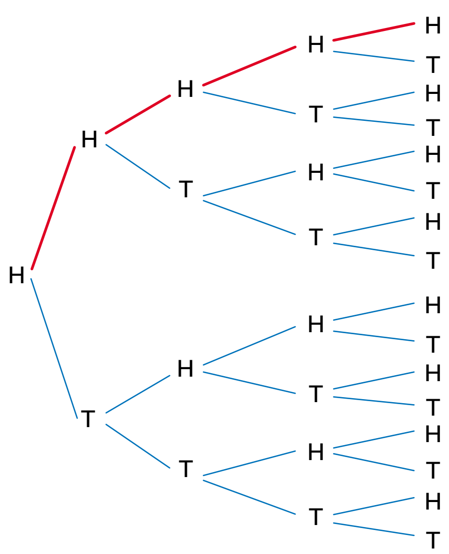
d) Comment on any difference between your answers to parts b) and c).

Solution

a) Because these events are independent, the outcome of the first toss has no effect on the outcome of the second toss. Therefore, the probability of tossing heads the second time is 0.5.

b) Although you might think "tails has to come up sometime," there is still a 50/50 chance on each independent toss. The coin has no memory of the past four trials! Therefore, the fifth toss still has just a 0.5 probability of coming up heads.

c) Construct a tree diagram to represent five tosses of the coin.



There is an equal number of outcomes in which the first flip turns up tails.

The number of outcomes doubles with each trial. After the fifth toss, there are 2^5 or 32 possible outcomes, only one of which is five heads in a row. So, the probability of five heads in a row, prior to any coin tosses, is $\frac{1}{32}$ or 0.031 25.

d) The probability in part c) is much less than in part b). In part b), you calculate only the probability for the fifth trial on its own. In part c), you are finding the probability that every one of five separate events actually happens.

Example 2 Probability of Two Different Independent Events

A coin is flipped while a die is rolled. What is the probability of flipping heads and rolling 5 in a single trial?

Solution

Here, two independent events occur in a single trial. Let A be the event of flipping heads, and B be the event of rolling 5. The notation $P(A \text{ and } B)$ represents the compound, or joint, probability that both events, A and B, will occur simultaneously. For independent events, the probabilities can simply be multiplied together.

$$\begin{aligned} P(A \text{ and } B) &= P(A) \times P(B) \\ &= \frac{1}{2} \times \frac{1}{6} \\ &= \frac{1}{12} \end{aligned}$$

The probability of simultaneously flipping heads while rolling 5 is $\frac{1}{12}$ or approximately 8.3%

In general, the compound probability of two independent events can be calculated using the **product rule for independent events**:

$$P(A \text{ and } B) = P(A) \times P(B)$$

From the example above, you can see that the product rule for independent events agrees with common sense. The product rule can also be derived mathematically from the fundamental counting principle (see Chapter 4).

Proof:

A and B are separate events and so they correspond to separate sample spaces, S_A and S_B.

Their probabilities are thus

$P(A) = \dfrac{n(A)}{n(S_A)}$ and $P(B) = \dfrac{n(B)}{n(S_B)}$.

Call the sample space for the compound event S, as usual.

You know that

$$P(A \text{ and } B) = \frac{n(A \text{ and } B)}{n(S)} \qquad \textbf{(1)}$$

Because A and B are independent, you can apply the fundamental counting principle to get an expression for $n(A \text{ and } B)$.

$$n(A \text{ and } B) = n(A) \times n(B) \qquad \textbf{(2)}$$

Similarly, you can also apply the fundamental counting principle to get an expression for $n(S)$.

$$n(S) = n(S_A) \times n(S_B) \qquad \textbf{(3)}$$

Substitute equations (2) and (3) into equation (1).

$$\begin{aligned} P(A \text{ and } B) &= \frac{n(A)n(B)}{n(S_A)n(S_B)} \\ &= \frac{n(A)}{n(S_A)} \times \frac{n(B)}{n(S_B)} \\ &= P(A) \times P(B) \end{aligned}$$

Example 3 Applying the Product Rule for Independent Events

Soo-Ling travels the same route to work every day. She has determined that there is a 0.7 probability that she will wait for at least one red light and that there is a 0.4 probability that she will hear her favourite new song on her way to work.

a) What is the probability that Soo-Ling will not have to wait at a red light and will hear her favourite song?

b) What are the odds in favour of Soo-Ling having to wait at a red light and not hearing her favourite song?

Solution

a) Let A be the event of Soo-Ling having to wait at a red light, and B be the event of hearing her favourite song. Assume A and B to be independent events. In this case, you are interested in the combination A' and B.

$$\begin{aligned} P(A' \text{ and } B) &= P(A') \times P(B) \\ &= (1 - P(A)) \times P(B) \\ &= (1 - 0.7) \times 0.4 \\ &= 0.12 \end{aligned}$$

There is a 12% chance that Soo-Ling will hear her favourite song and not have to wait at a red light on her way to work.

b)
$$\begin{aligned} P(A \text{ and } B') &= P(A) \times P(B') \\ &= P(A) \times (1 - P(B)) \\ &= 0.7 \times (1 - 0.4) \\ &= 0.42 \end{aligned}$$

The probability of Soo-Ling having to wait at a red light and not hearing her favourite song is 42%.

The odds in favour of this happening are

$$\begin{aligned} \text{odds in favour} &= \frac{P(A \text{ and } B')}{1 - P(A \text{ and } B')} \\ &= \frac{42\%}{100\% - 42\%} \\ &= \frac{42}{58} \\ &= \frac{21}{29} \end{aligned}$$

The odds in favour of Soo-Ling having to wait at a red light and not hearing her favourite song are 21:29.

In some cases, the probable outcome of an event, B, depends directly on the outcome of another event, A. When this happens, the events are said to be **dependent**. The **conditional probability** of B, $\boldsymbol{P(B \mid A)}$, is the probability that B occurs, *given* that A has already occurred.

Example 4 Probability of Two Dependent Events

A professional hockey team has eight wingers. Three of these wingers are 30-goal scorers, or "snipers." Every fall the team plays an exhibition match with the club's farm team. In order to make the match more interesting for the fans, the coaches agree to select two wingers at random from the pro team to play for the farm team. What is the probability that two snipers will play for the farm team?

Solution

Let A = {first winger is a sniper} and B = {second winger is a sniper}. Three of the eight wingers are snipers, so the probability of the first winger selected being a sniper is

$$P(A) = \frac{3}{8}$$

If the first winger selected is a sniper, then there are seven remaining wingers to choose from, two of whom are snipers. Therefore,

$$P(B \mid A) = \frac{2}{7}$$

Applying the fundamental counting principle, the probability of randomly selecting two snipers for the farm team is the number of ways of selecting two snipers divided by the number of ways of selecting any two wingers.

$$\begin{aligned} P(A \text{ and } B) &= \frac{3 \times 2}{8 \times 7} \\ &= \frac{3}{28} \end{aligned}$$

There is a $\frac{3}{28}$ or 10.7% probability that two professional snipers will play for the farm team in the exhibition game.

Notice in Example 4 that, when two events A and B are dependent, you can still multiply probabilities to find the probability that they both happen. However, you must use the conditional probability for the second event. Thus, the probability that both events will occur is given by the **product rule for dependent events**:

$$P(A \text{ and } B) = P(A) \times P(B \mid A)$$

This reads as: "The probability that both A and B will occur equals the probability of A times the probability of B given that A has occurred."

Project Prep

When designing your game for the probability project, you may decide to include situations involving independent or dependent events. If so, you will need to apply the appropriate product rule in order to determine classical probabilities.

Example 5 Conditional Probability From Compound Probability

Serena's computer sometimes crashes while she is trying to use her e-mail program, OutTake. When OutTake "hangs" (stops responding to commands), Serena is usually able to close OutTake without a system crash. In a computer magazine, she reads that the probability of OutTake hanging in any 15-min period is 2.5%, while the chance of OutTake and the operating system failing together in any 15-min period is 1%. If OutTake is hanging, what is the probability that the operating system will crash?

Solution

Let event A be OutTake hanging, and event B be an operating system failure. Since event A can trigger event B, the two events are dependent. In fact, you need to find the conditional probability $P(B \mid A)$. The data from the magazine tells you that $P(A) = 2.5\%$, and $P(A \text{ and } B) = 1\%$. Therefore,

$$
\begin{aligned}
P(A \text{ and } B) &= P(A) \times P(B \mid A) \\
1\% &= 2.5\% \times P(B \mid A) \\
P(B \mid A) &= \frac{1\%}{2.5\%} \\
&= 0.4
\end{aligned}
$$

There is a 40% chance that the operating system will crash when OutTake is hanging.

Example 5 suggests a useful rearrangement of the product rule for dependent events.

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

This equation is sometimes used to define the conditional probability $P(B \mid A)$.

Key Concepts

- If A and B are independent events, then the probability of both occurring is given by $P(A \text{ and } B) = P(A) \times P(B)$.
- If event B is dependent on event A, then the conditional probability of B given A is $P(B \mid A)$. In this case, the probability of both events occurring is given by $P(A \text{ and } B) = P(A) \times P(B \mid A)$.

Communicate Your Understanding

1. Consider the probability of randomly drawing an ace from a standard deck of cards. Discuss whether or not successive trials of this experiment are independent or dependent events. Consider cases in which drawn cards are

a) replaced after each trial

b) not replaced after each trial

2. Suppose that for two particular events A and B, it is true that $P(B \mid A) = P(B)$. What does this imply about the two events? (*Hint*: Try substituting this equation into the product rule for dependent events.)

Practise

A

1. Classify each of the following as independent or dependent events.

	First Event	Second Event
a)	Attending a rock concert on Tuesday night	Passing a final examination the following Wednesday morning
b)	Eating chocolate	Winning at checkers
c)	Having blue eyes	Having poor hearing
d)	Attending an employee training session	Improving personal productivity
e)	Graduating from university	Running a marathon
f)	Going to a mall	Purchasing a new shirt

2. Amitesh estimates that he has a 70% chance of making the basketball team and a 20% chance of having failed his last geometry quiz. He defines a "really bad day" as one in which he gets cut from the team and fails his quiz. Assuming that Amitesh will receive both pieces of news tomorrow, how likely is it that he will have a really bad day?

3. In the popular dice game Yahtzee®, a Yahtzee occurs when five identical numbers turn up on a set of five standard dice. What is the probability of rolling a Yahtzee on one roll of the five dice?

Apply, Solve, Communicate

B

4. There are two tests for a particular antibody. Test A gives a correct result 95% of the time. Test B is accurate 89% of the time. If a patient is given both tests, find the probability that

a) both tests give the correct result

b) neither test gives the correct result

c) at least one of the tests gives the correct result

5. a) Rocco and Biff are two koala bears participating in a series of animal behaviour tests. They each have 10 min to solve a maze. Rocco has an 85% probability of succeeding if he can smell the eucalyptus treat at the other end. He can smell the treat 60% of the time. Biff has a 70% chance of smelling the treat, but when he does, he can solve the maze only 75% of the time. Neither bear will try to solve the maze unless he smells the eucalyptus. Determine which koala bear is more likely to enjoy a tasty treat on any given trial.

b) **Communication** Explain how you arrived at your conclusion.

6. Shy Tenzin's friends assure him that if he asks Mikala out on a date, there is an 85% chance that she will say yes. If there is a 60% chance that Tenzin will summon the courage to ask Mikala out to the dance next week, what are the odds that they will be seen at the dance together?

7. When Ume's hockey team uses a "rocket launch" breakout, she has a 55% likelihood of receiving a cross-ice pass while at full speed. When she receives such a pass, the probability of getting her slapshot away is $\frac{1}{3}$. Ume's slapshot scores 22% of the time. What is the probability of Ume scoring with her slapshot when her team tries a rocket launch?

8. **Inquiry/Problem Solving** Show that if A and B are dependent events, then the conditional probability $P(A \mid B)$ is given by

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}.$$

9. A consultant's study found Megatran's call centre had a 5% chance of transferring a call about schedules to the lost articles department by mistake. The same study shows that, 1% of the time, customers calling for schedules have to wait on hold, only to discover that they have been mistakenly transferred to the lost articles department. What are the chances that a customer transferred to lost articles will be put on hold?

10. Pinder has examinations coming up in data management and biology. He estimates that his odds in favour of passing the data-management examination are 17:3 and his odds against passing the biology examination are 3:7. Assume these to be independent events.
    a) What is the probability that Pinder will pass both exams?
    b) What are the odds in favour of Pinder failing both exams?
    c) What factors could make these two events dependent?

11. **Inquiry/Problem Solving** How likely is it for a group of five friends to have the same birth month? State any assumptions you make for your calculation.

12. Determine the probability that a captured deer has the bald patch condition.

Chapter Problem

13. **Communication** Five different CD-ROM games, Garble, Trapster, Zoom!, Bungie, and Blast 'Em, are offered as a promotion by SugarRush cereals. One game is randomly included with each box of cereal.
    a) Determine the probability of getting all 5 games if 12 boxes are purchased.
    b) Explain the steps in your solution.
    c) Discuss any assumptions that you make in your analysis.

14. **Application** A critical circuit in a communication network relies on a set of eight identical relays. If any one of the relays fails, it will disrupt the entire network. The design engineer must ensure a 90% probability that the network will not fail over a five-year period. What is the maximum tolerable probability of failure for each relay?

C

15. a) Show that if a coin is tossed n times, the probability of tossing n heads is given by $P(A) = \left(\frac{1}{2}\right)^n$.
    b) What is the probability of getting at least one tail in seven tosses?

16. What is the probability of not throwing 7 or doubles for six consecutive throws with a pair of dice?

17. Laurie, an avid golfer, gives herself a 70% chance of breaking par (scoring less than 72 on a round of 18 holes) if the weather is calm, but only a 15% chance of breaking par on windy days. The weather forecast gives a 40% probability of high winds tomorrow. What is the likelihood that Laurie will break par tomorrow, assuming that she plays one round of golf?

18. **Application** The Tigers are leading the Storm one game to none in a best-of-five playoff series. After a playoff win, the probability of the Tigers winning the next game is 60%, while after a loss, their probability of winning the next game drops by 5%. The first team to win three games takes the series. Assume there are no ties. What is the probability of the Storm coming back to win the series?

6.5 Mutually Exclusive Events

The phone rings. Jacques is really hoping that it is one of his friends calling about either softball or band practice. Could the call be about both?

In such situations, more than one event could occur during a single trial. You need to compare the events in terms of the outcomes that make them up. What is the chance that at least one of the events happens? Is the situation "either/or," or can both events occur?

INVESTIGATE & INQUIRE: Baseball Pitches

Marie, at bat for the Coyotes, is facing Anton, who is pitching for the Power Trippers. Anton uses three pitches: a fastball, a curveball, and a slider. Marie feels she has a good chance of making a base hit, or better, if Anton throws either a fastball or a slider. The count is two strikes and three balls. In such full-count situations, Anton goes to his curveball one third of the time, his slider half as often, and his fastball the rest of the time.

1. Determine the probability of Anton throwing his
   a) curveball b) slider c) fastball
2. a) What is the probability that Marie will get the pitch she does not want?
   b) Explain how you can use this information to determine the probability that Marie will get a pitch she likes.
3. a) Show another method of determining this probability.
   b) Explain your method.
4. What do your answers to questions 2 and 3 suggest about the probabilities of events that cannot happen simultaneously?

The possible events in this investigation are said to be **mutually exclusive** (or **disjoint**) since they cannot occur at the same time. The pitch could not be both a fastball and a slider, for example. In this particular problem, you were interested in the probability of *either* of two favourable events. You can use the notation $P(A \text{ or } B)$ to stand for the probability of either A or B occurring.

Example 1 Probability of Mutually Exclusive Events

Teri attends a fundraiser at which 15 T-shirts are being given away as door prizes. Door prize winners are randomly given a shirt from a stock of 2 black shirts, 4 blue shirts, and 9 white shirts. Teri really likes the black and blue shirts, but is not too keen on the white ones. Assuming that Teri wins the first door prize, what is the probability that she will get a shirt that she likes?

Solution

Let A be the event that Teri wins a black shirt, and B be the event that she wins a blue shirt.

$P(A) = \frac{2}{15}$ and $P(B) = \frac{4}{15}$

Teri would be happy if either A or B occurred.
There are $2 + 4 = 6$ non-white shirts, so

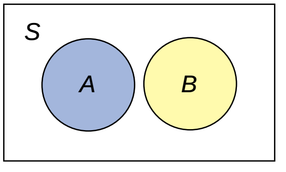
$$\begin{aligned} P(A \text{ or } B) &= \frac{6}{15} \\ &= \frac{2}{5} \end{aligned}$$

The probability of Teri winning a shirt that she likes is $\frac{2}{5}$ or 40%. Notice that this probability is simply the sum of the probabilities of the two mutually exclusive events.

When events A and B are mutually exclusive, the probability that A or B will occur is given by the **addition rule for mutually exclusive events**:

$P(A \text{ or } B) = P(A) + P(B)$

A Venn diagram shows mutually exclusive events as non-overlapping, or disjoint. Thus, you can apply the additive counting principle (see Chapter 4) to prove this rule.



Proof:
If A and B are mutually exclusive events, then

$$\begin{aligned} P(A \text{ or } B) &= \frac{n(A \text{ or } B)}{n(S)} \\ &= \frac{n(A) + n(B)}{n(S)} \\ &= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} \\ &= P(A) + P(B) \end{aligned}$$

A and B are disjoint sets, and thus share no elements.

In some situations, events are **non-mutually exclusive**, which means that they can occur simultaneously. For example, consider a board game in which you need to roll either an 8 or doubles, using two dice.

Notice that in one outcome, rolling two fours, both events have occurred simultaneously. Hence, these events are not mutually exclusive. Counting the outcomes in the diagram shows that the probability of rolling either an 8 or doubles is $\frac{10}{36}$ or $\frac{5}{18}$. You need to take care not to count the (4, 4) outcome twice. You are applying the principle of inclusion and exclusion, which was explained in greater detail in Chapter 5.

		Second die					
		1	2	3	4	5	6
First die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Example 2 Probability of Non-Mutually Exclusive Events

A card is randomly selected from a standard deck of cards. What is the probability that either a heart or a face card (jack, queen, or king) is selected?

Solution

Let event A be that a heart is selected, and event B be that a face card is selected.

$P(A) = \frac{13}{52}$ and $P(B) = \frac{12}{52}$

If you add these probabilities, you get

$$P(A) + P(B) = \frac{13}{52} + \frac{12}{52}$$
$$= \frac{25}{52}$$

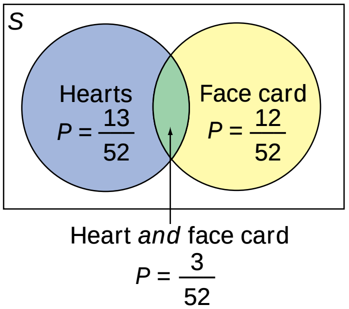
However, since the jack, queen, and king of hearts are in both A and B, the sum $P(A) + P(B)$ actually includes these outcomes *twice*.

A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠

Based on the diagram, the actual theoretical probability of drawing either a heart or a face card is $\frac{22}{52}$, or $\frac{11}{26}$. You can find the correct value by subtracting the probability of selecting the three elements that were counted twice.

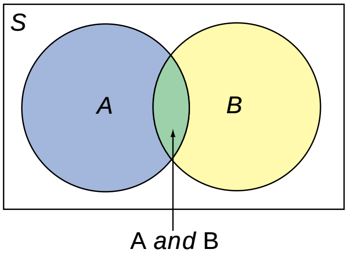
$$P(A \text{ or } B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$$
$$= \frac{22}{52}$$
$$= \frac{11}{26}$$

The probability that either a heart or a face card is selected is $\frac{11}{26}$.



When events A and B are non-mutually exclusive, the probability that A or B will occur is given by the **addition rule for non-mutually exclusive events**:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

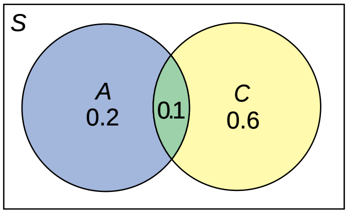


Example 3 Applying the Addition Rule for Non-Mutually Exclusive Events

An electronics manufacturer is testing a new product to see whether it requires a surge protector. The tests show that a voltage spike has a 0.2% probability of damaging the product's power supply, a 0.6% probability of damaging downstream components, and a 0.1% probability of damaging both the power supply and other components. Determine the probability that a voltage spike will damage the product.

Solution

Let A be damage to the power supply and C be damage to other components.



The overlapping region represents the probability that a voltage surge damages both the power supply and another component. The probability that either A or C occurs is given by

$$P(A \text{ or } C) = P(A) + P(C) - P(A \text{ and } C)$$
$$= 0.2\% + 0.6\% - 0.1\%$$
$$= 0.7\%$$

There is a 0.7% probability that a voltage spike will damage the product.

Project Prep

When analysing the possible outcomes for your game in the probability project, you may need to consider mutually exclusive or non-mutually exclusive events. If so, you will need to apply the appropriate addition rule to determine theoretical probabilities.

Key Concepts

- If A and B are mutually exclusive events, then the probability of either A or B occurring is given by $P(A \text{ or } B) = P(A) + P(B)$.
- If A and B are non-mutually exclusive events, then the probability of either A or B occurring is given by $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Communicate Your Understanding

1. Are an event and its complement mutually exclusive? Explain.
2. Explain how to determine the probability of randomly throwing either a composite number or an odd number using a pair of dice.
3. a) Explain the difference between independent events and mutually exclusive events.
   b) Support your explanation with an example of each.
   c) Why do you add probabilities in one case and multiply them in the other?

Practise

A

1. Classify each pair of events as mutually exclusive or non-mutually exclusive.

	Event A	Event B
a)	Randomly drawing a grey sock from a drawer	Randomly drawing a wool sock from a drawer
b)	Randomly selecting a student with brown eyes	Randomly selecting a student on the honour roll
c)	Having an even number of students in your class	Having an odd number of students in your class
d)	Rolling a six with a die	Rolling a prime number with a die
e)	Your birthday falling on a Saturday next year	Your birthday falling on a weekend next year
f)	Getting an A on the next test	Passing the next test
g)	Calm weather at noon tomorrow	Stormy weather at noon tomorrow
h)	Sunny weather next week	Rainy weather next week

2. Nine members of a baseball team are randomly assigned field positions. There are three outfielders, four infielders, a pitcher, and a catcher. Troy is happy to play any position except catcher or outfielder. Determine the probability that Troy will be assigned to play
   a) catcher
   b) outfielder
   c) a position he does not like

3. A car dealership analysed its customer database and discovered that in the last model year, 28% of its customers chose a 2-door model, 46% chose a 4-door model, 19% chose a minivan, and 7% chose a 4-by-4 vehicle. If a customer was selected randomly from this database, what is the probability that the customer
   a) bought a 4-by-4 vehicle?
   b) did not buy a minivan?
   c) bought a 2-door or a 4-door model?
   d) bought a minivan or a 4-by-4 vehicle?

Apply, Solve, Communicate

B

4. As a promotion, a resort has a draw for free family day-passes. The resort considers July, August, March, and December to be "vacation months."

a) If the free passes are randomly dated, what is the probability that a day-pass will be dated within

i) a vacation month?

ii) June, July, or August

b) Draw a Venn diagram of the events in part a).

5. A certain provincial park has 220 campsites. A total of 80 sites have electricity. Of the 52 sites on the lakeshore, 22 of them have electricity. If a site is selected at random, what is the probability that

a) it will be on the lakeshore?

b) it will have electricity?

c) it will either have electricity or be on the lakeshore?

d) it will be on the lakeshore and not have electricity?

6. A market-research firm monitored 1000 television viewers, consisting of 800 adults and 200 children, to evaluate a new comedy series that aired for the first time last week. Research indicated that 250 adults and 148 children viewed some or all of the program. If one of the 1000 viewers was selected, what is the probability that

a) the viewer was an adult who did not watch the new program?

b) the viewer was a child who watched the new program?

c) the viewer was an adult or someone who watched the new program?

7. **Application** In an animal-behaviour study, hamsters were tested with a number of intelligence tasks, as shown in the table below.

Number of Tests	Number of Hamsters
0	10
1	6
2	4
3	3
4 or more	5

If a hamster is randomly chosen from this study group, what is the likelihood that the hamster has participated in

a) exactly three tests?

b) fewer than two tests?

c) either one or two tests?

d) no tests or more than three tests?

8. **Communication**

a) Prove that, if A and B are non-mutually exclusive events, the probability of either A or B occurring is given by $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

b) What can you conclude if $P(A \text{ and } B) = 0$? Give reasons for your conclusion.

9. **Inquiry / Problem Solving** Design a game in which the probability of drawing a winning card from a standard deck is between 55% and 60%.

10. Determine the probability that a captured deer has either cross-hatched antlers or bald patches. Are these events mutually exclusive? Why or why not?

11. The eight members of the debating club pose for a yearbook photograph. If they line up randomly, what is the probability that

a) either Hania will be first in the row or Aaron will be last?

b) Hania will be first and Aaron will not be last?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

12. Consider a Stanley Cup playoff series in which the Toronto Maple Leafs hockey team faces the Ottawa Senators. Toronto hosts the first, second, and if needed, fifth and seventh games in this best-of-seven contest. The Leafs have a 65% chance of beating the Senators at home in the first game. After that, they have a 60% chance of a win at home if they won the previous game, but a 70% chance if they are bouncing back from a loss. Similarly, the Leafs' chances of victory in Ottawa are 40% after a win and 45% after a loss.

a) Construct a tree diagram to illustrate all the possible outcomes of the first three games.

b) Consider the following events:

A = {Leafs lose the first game but go on to win the series in the fifth game}

B = {Leafs win the series in the fifth game}

C = {Leafs lose the series in the fifth game}

Identify all the outcomes that make up each event, using strings of letters, such as *LLSLL*. Are any pairs from these three events mutually exclusive?

c) What is the probability of event A in part b)?

d) What is the chance of the Leafs winning in exactly five games?

e) Explain how you found your answers to parts c) and d).

C

13. A grade 12 student is selected at random to sit on a university liaison committee. Of the 120 students enrolled in the grade 12 university-preparation mathematics courses,

- 28 are enrolled in data management only
- 40 are enrolled in calculus only
- 15 are enrolled in geometry only
- 16 are enrolled in both data management and calculus
- 12 are enrolled in both calculus and geometry
- 6 are enrolled in both geometry and data management
- 3 are enrolled in all three of data management, calculus, and geometry

a) Draw a Venn diagram to illustrate this situation.

b) Determine the probability that the student selected will be enrolled in either data management or calculus.

c) Determine the probability that the student selected will be enrolled in only one of the three courses.

14. **Application** For a particular species of cat, the odds against a kitten being born with either blue eyes or white spots are 3:1. If the probability of a kitten exhibiting only one of these traits is equal and the probability of exhibiting both traits is 10%, what are the odds in favour of a kitten having blue eyes?

15. **Communication**

a) A standard deck of cards is shuffled and three cards are selected. What is the probability that the third card is either a red face card or a king if the king of diamonds and the king of spades are selected as the first two cards?

b) Does this probability change if the first two cards selected are the queen of diamonds and the king of spades? Explain.

16. **Inquiry/Problem Solving** The table below lists the degrees granted by Canadian universities from 1994 to 1998 in various fields of study.

a) If a Canadian university graduate from 1998 is chosen at random, what is the probability that the student is

i) a male?

ii) a graduate in mathematics and physical sciences?

iii) a male graduating in mathematics and physical sciences?

iv) not a male graduating in mathematics and physical sciences?

v) a male *or* a graduate in mathematics and physical sciences?

b) If a male graduate from 1996 is selected at random, what is the probability that he is graduating in mathematics and physical sciences?

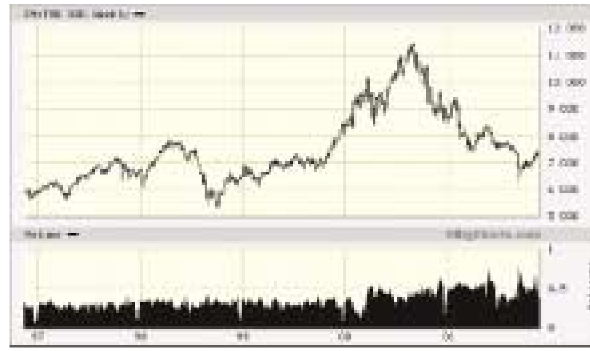
c) If a mathematics and physical sciences graduate is selected at random from the period 1994 to 1996, what is the probability that the graduate is a male?

d) Do you think that being a male and graduating in mathematics and physical sciences are independent events? Give reasons for your hypothesis.

	1994	1995	1996	1997	1998
Canada	178 074	178 066	178 116	173 937	172 076
Male	76 470	76 022	75 106	73 041	71 949
Female	101 604	102 044	103 010	100 896	100 127
Social sciences	69 583	68 685	67 862	66 665	67 019
Male	30 700	29 741	29 029	28 421	27 993
Female	38 883	38 944	38 833	38 244	39 026
Education	30 369	30 643	29 792	27 807	25 956
Male	9093	9400	8693	8036	7565
Female	21 276	21 243	21 099	19 771	18 391
Humanities	23 071	22 511	22 357	21 373	20 816
Male	8427	8428	8277	8034	7589
Female	14 644	14 083	14 080	13 339	13 227
Health professions and occupations	12 183	12 473	12 895	13 073	12 658
Male	3475	3461	3517	3460	3514
Female	8708	9012	9378	9613	9144
Engineering and applied sciences	12 597	12 863	13 068	12 768	12 830
Male	10 285	10 284	10 446	10 125	10 121
Female	2312	2579	2622	2643	2709
Agriculture and biological sciences	10 087	10 501	11 400	11 775	12 209
Male	4309	4399	4756	4780	4779
Female	5778	6102	6644	6995	7430
Mathematics and physical sciences	9551	9879	9786	9738	9992
Male	6697	6941	6726	6749	6876
Female	2854	2938	3060	2989	3116
Fine and applied arts	5308	5240	5201	5206	5256
Male	1773	1740	1780	1706	1735
Female	3535	3500	3421	3500	3521
Arts and sciences	5325	5271	5755	5532	5340
Male	1711	1628	1882	1730	1777
Female	3614	3643	3873	3802	3563

6.6 Applying Matrices to Probability Problems

In some situations, the probability of an outcome depends on the outcome of the previous trial. Often this pattern appears in stock market trends, weather patterns, athletic performance, and consumer habits. Dependent probabilities can be calculated using Markov chains, a powerful probability model pioneered about a century ago by the Russian mathematician Andrei Markov.



INVESTIGATE & INQUIRE: Running Late

Although Marla tries hard to be punctual, the demands of her home life and the challenges of commuting sometimes cause her to be late for work. When she is late, she tries especially hard to be punctual the next day. Suppose that the following pattern emerges: If Marla is punctual on any given day, then there is a 70% chance that she will be punctual the next day and a 30% chance that she will be late. On days she is late, however, there is a 90% chance that she will be punctual the next day and just a 10% chance that she will be late. Suppose Marla is punctual on the first day of the work week.

1. Create a tree diagram of the possible outcomes for the second and third days. Show the probability for each branch.

2. a) Describe two branches in which Marla is punctual on day 3.
   b) Use the product rule for dependent events on page 332 to calculate the compound probability of Marla being punctual on day 2 and on day 3.
   c) Find the probability of Marla being late on day 2 and punctual on day 3.
   d) Use the results from parts b) and c) to determine the probability that Marla will be punctual on day 3.

3. Repeat question 2 for the outcome of Marla being late on day 3.

4. a) Create a 1×2 matrix A in which the first element is the probability that Marla is punctual and the second element is the probability that she is late on day 1. Recall that Marla is punctual on day 1.
   b) Create a 2×2 matrix B in which the elements in each row represent *conditional* probabilities that Marla will be punctual and late. Let the first row be the probabilities after a day in which Marla was punctual, and the second row be the probabilities after a day in which she was late.

c) Evaluate $A \times B$ and $A \times B^2$.

d) Compare the results of part c) with your answers to questions 2 and 3. Explain what you notice.

e) What does the first row of the matrix B^2 represent?

The matrix model you have just developed is an example of a **Markov chain**, a probability model in which the outcome of any trial depends directly on the outcome of the previous trial. Using matrix operations can simplify probability calculations, especially in determining long-term trends.

The 1×2 matrix A in the investigation is an **initial probability vector, $S^{(0)}$**, and represents the probabilities of the initial state of a Markov chain. The 2×2 matrix B is a **transition matrix, P**, and represents the probabilities of moving from any initial state to a new state in any trial.

These matrices have been arranged such that the product $S^{(0)} \times P$ generates the row matrix that gives the probabilities of each state after one trial. This matrix is called the **first-step probability vector, $S^{(1)}$**. In general, the ***n*th-step probability vector, $S^{(n)}$**, can be obtained by repeatedly multiplying the probability vector by P. Sometimes these vectors are also called **first-state** and ***n*th-state vectors**, respectively.

Notice that each entry in a probability vector or a transition matrix is a probability and must therefore be between 0 and 1. The possible states in a Markov chain are always mutually exclusive events, one of which must occur at each stage. Therefore, the entries in a probability vector must sum to 1, as must the entries in each row of the transition matrix.

Example 1 Probability Vectors

Two video stores, Video Vic's and MovieMaster, have just opened in a new residential area. Initially, they each have half of the market for rented movies. A customer who rents from Video Vic's has a 60% probability of renting from Video Vic's the next time and a 40% chance of renting from MovieMaster. On the other hand, a customer initially renting from MovieMaster has only a 30% likelihood of renting from MovieMaster the next time and a 70% probability of renting from Video Vic's.

a) What is the initial probability vector?

b) What is the transition matrix?

c) What is the probability of a customer renting a movie from each store the second time?

d) What is the probability of a customer renting a movie from each store the third time?

e) What assumption are you making in part d)? How realistic is it?

Solution

a) Initially, each store has 50% of the market, so, the initial probability vector is

$$\begin{array}{rc} & \begin{matrix}\text{VV} & \text{MM}\end{matrix} \\ S^{(0)} = & [0.5 \quad 0.5] \end{array}$$

b) The first row of the transition matrix represents the probabilities for the second rental by customers whose initial choice was Video Vic's. There is a 60% chance that the customer returns, so the first entry is 0.6. It is 40% likely that the customer will rent from MovieMaster, so the second entry is 0.4.

Similarly, the second row of the transition matrix represents the probabilities for the second rental by customers whose first choice was MovieMaster. There is a 30% chance that a customer will return on the next visit, and a 70% chance that the customer will try Video Vic's.

$$\begin{array}{rcl} & \begin{matrix}\text{VV} & \text{MM}\end{matrix} & \\ P = & \begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix} & \begin{matrix}\text{VV} \\ \text{MM}\end{matrix} \end{array}$$

Regardless of which store the customer chooses the first time, you are assuming that there are only two choices for the next visit. Hence, the sum of the probabilities in each row equals one.

c) To find the probabilities of a customer renting from either store on the second visit, calculate the first-step probability vector, $S^{(1)}$:

$$\begin{aligned} S^{(1)} &= S^{(0)}P \\ &= [0.5 \quad 0.5]\begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix} \\ &= [0.65 \quad 0.35] \end{aligned}$$

This new vector shows that there is a 65% probability that a customer will rent a movie from Video Vic's on the second visit to a video store and a 35% chance that the customer will rent from MovieMaster.

d) To determine the probabilities of which store a customer will pick on the third visit, calculate the second-step probability vector, $S^{(2)}$:

$$\begin{aligned} S^{(2)} &= S^{(1)}P \\ &= [0.65 \quad 0.35]\begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix} \\ &= [0.635 \quad 0.365] \end{aligned}$$

So, on a third visit, a customer is 63.5% likely to rent from Video Vic's and 36.5% likely to rent from MovieMaster.

e) To calculate the second-step probabilities, you assume that the conditional transition probabilities do not change. This assumption might not be realistic since customers who are 70% likely to switch away from MovieMaster may not be as much as 40% likely to switch back, unless they forget why they switched in the first place. In other words, Markov chains have no long-term memory. They recall only the latest state in predicting the next one.

Note that the result in Example 1d) could be calculated in another way.

$$\begin{aligned} S^{(2)} &= S^{(1)}P \\ &= (S^{(0)}P)P \\ &= S^{(0)}(PP) \quad \text{since matrix multiplication is associative} \\ &= S^{(0)}P^2 \end{aligned}$$

Similarly, $S^{(3)} = S^{(0)}P^3$, and so on. In general, the nth-step probability vector, $S^{(n)}$, is given by

$$S^{(n)} = S^{(0)}P^n$$

This result enables you to determine higher-state probability vectors easily using a graphing calculator or software.

Example 2 Long-Term Market Share

A marketing-research firm has tracked the sales of three brands of hockey sticks. Each year, on average,

- Player-One keeps 70% of its customers, but loses 20% to Slapshot and 10% to Extreme Styx
- Slapshot keeps 65% of its customers, but loses 10% to Extreme Styx and 25% to Player-One
- Extreme Styx keeps 55% of its customers, but loses 30% to Player-One and 15% to Slapshot

a) What is the transition matrix?

b) Assuming each brand begins with an equal market share, determine the market share of each brand after one, two, and three years.

c) Determine the long-range market share of each brand.

d) What assumption must you make to answer part c)?

Solution 1 Using Pencil and Paper

a) The transition matrix is

$$P = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \text{P} & \text{S} & \text{E} \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.25 & 0.65 & 0.1 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{P} \\ \text{S} \\ \text{E} \end{array}$$

b) Assuming each brand begins with an equal market share, the initial probability vector is

$$S^{(0)} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

To determine the market shares of each brand after one year, compute the first-step probability vector.

$$\begin{aligned} S^{(1)} &= S^{(0)}P \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.25 & 0.65 & 0.1 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} \\ &= [0.41\overline{6} \quad 0.\overline{3} \quad 0.25] \end{aligned}$$

So, after one year Player-One will have a market share of approximately 42%, Slapshot will have 33%, and Extreme Styx will have 25%.

Similarly, you can predict the market shares after two years using

$$\begin{aligned} S^{(2)} &= S^{(1)}P \\ &= [0.41\overline{6} \quad 0.\overline{3} \quad 0.25] \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.25 & 0.65 & 0.1 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} \\ &= [0.45 \quad 0.3375 \quad 0.2125] \end{aligned}$$

After two years, Player-One will have approximately 45% of the market, Slapshot will have 34%, and Extreme Styx will have 21%.

The probabilities after three years are given by

$$\begin{aligned} S^{(3)} &= S^{(2)}P \\ &= [0.45 \quad 0.3375 \quad 0.2125] \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.25 & 0.65 & 0.1 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} \\ &= [0.463 \quad 0.341 \quad 0.196] \end{aligned}$$

After three years, Player-One will have approximately 46% of the market, Slapshot will have 34%, and Extreme Styx will have 20%.

c) The results from part b) suggest that the relative market shares may be converging to a steady state over a long period of time. You can test this hypothesis by calculating higher-state vectors and checking for stability.

For example,

$$\begin{aligned} S^{(10)} &= S^{(9)}P \\ &= [0.471 \quad 0.347 \quad 0.182] \end{aligned} \qquad \begin{aligned} S^{(11)} &= S^{(10)}P \\ &= [0.471 \quad 0.347 \quad 0.182] \end{aligned}$$

The values of $S^{(10)}$ and $S^{(11)}$ are equal. It is easy to verify that they are equal to all higher orders of $S^{(n)}$ as well. The Markov chain has reached a **steady state**. A steady-state vector is a probability vector that remains unchanged when multiplied by the transition matrix. A steady state has been reached if

$$\begin{aligned} S^{(n)} &= S^{(n)}P \\ &= S^{(n+1)} \end{aligned}$$

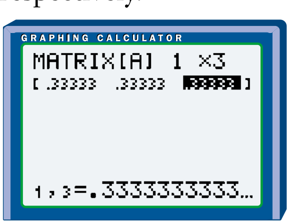
In this case, the steady state vector [0.471 0.347 0.182] indicates that, over a long period of time, Player-One will have approximately 47% of the market for hockey sticks, while Slapshot and Extreme Styx will have 35% and 18%, respectively, based on current trends.

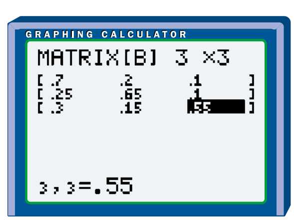
d) The assumption you make in part c) is that the transition matrix does not change, that is, the market trends stay the same over the long term.

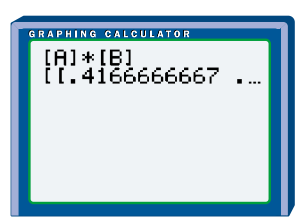
Solution 2 Using a Graphing Calculator

a) Use the MATRX EDIT menu to enter and store a matrix for the transition matrix B.

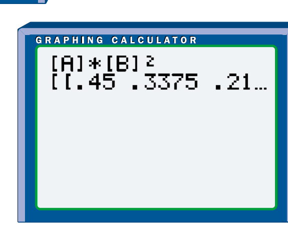
b) Similarly, enter the initial probability vector as matrix A. Then, use the MATRX EDIT menu to enter the calculation $A \times B$ on the home screen. The resulting matrix shows the market shares after one year are 42%, 33%, and 25%, respectively.



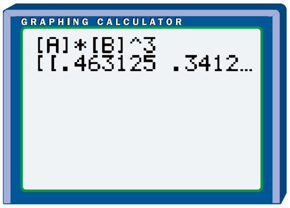




To find the second-step probability vector use the formula $S^{(2)} = S^{(0)}P^2$. Enter $A \times B^2$ using the MATRX NAMES menu and the x^2 key. After two years, therefore, the market shares are 45%, 34%, and 21%, respectively.



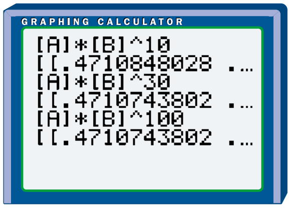
Similarly, enter $A \times B^3$ to find the third-step probability vector. After three years, the market shares are 46%, 34%, and 20%, respectively.



c) Higher-state probability vectors are easy to determine with a graphing calculator.

$$S^{(10)} = S^{(0)}P^{10} = [0.471 \;\; 0.347 \;\; 0.182]$$

$$S^{(100)} = S^{(0)}P^{100} = [0.471 \;\; 0.347 \;\; 0.182]$$



$S^{(10)}$ and $S^{(100)}$ are equal. The tiny difference between $S^{(10)}$ and $S^{(100)}$ is unimportant since the original data has only two significant digits. Thus, $[0.471 \;\; 0.347 \;\; 0.182]$ is a steady-state vector, and the long-term market shares are predicted to be about 47%, 35%, and 18% for Player-One, Slapshot, and Extreme Styx, respectively.

Regular Markov chains always achieve a steady state. A Markov chain is regular if the transition matrix P or some power of P has no zero entries. Thus, regular Markov chains are fairly easy to identify. A regular Markov chain will reach the same steady state *regardless* of the initial probability vector.

Project Prep

In the probability project, you may need to use Markov chains to determine long-term probabilities.

Example 3 Steady State of a Regular Markov Chain

Suppose that Player-One and Slapshot initially split most of the market evenly between them, and that Extreme Styx, a relatively new company, starts with a 10% market share.

a) Determine each company's market share after one year.

b) Predict the long-term market shares.

Solution

a) The initial probability vector is
$S^{(0)} = [0.45 \;\; 0.45 \;\; 0.1]$

Using the same transition matrix as in Example 2,

$$S^{(1)} = S^{(0)}P = [0.45 \;\; 0.45 \;\; 0.1]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.25 & 0.65 & 0.1 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} = [0.4575 \;\; 0.3975 \;\; 0.145]$$

These market shares differ from those in Example 2, where $S^{(1)} = [0.41\overline{6} \;\; 0.\overline{3} \;\; 0.25]$.

b) $S^{(100)} = S^{(0)}P^{100}$
$= [0.471 \quad 0.347 \quad 0.182]$

In the long term, the steady state is the same as in Example 2. Notice that although the short-term results differ as seen in part a), the same steady state is achieved in the long term.

The steady state of a regular Markov chain can also be determined analytically.

Example 4 Analytic Determination of Steady State

The weather near a certain seaport follows this pattern: If it is a calm day, there is a 70% chance that the next day will be calm and a 30% chance that it will be stormy. If it is a stormy day, the chances are 50/50 that the next day will also be stormy. Determine the long-term probability for the weather at the port.

Solution

The transition matrix for this Markov chain is

$$\begin{array}{cc} & \begin{array}{cc} \text{C} & \text{S} \end{array} \\ P = & \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{C} \\ \text{S} \end{array}$$

The steady-state vector will be a 1×2 matrix, $S^{(n)} = [p \quad q]$.

The Markov chain will reach a steady state when $S^{(n)} = S^{(n)}P$, so

$$\begin{aligned}[p \quad q] &= [p \quad q]\begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \\ &= [0.7p + 0.5q \quad 0.3p + 0.5q]\end{aligned}$$

Setting first elements equal and second elements equal gives two equations in two unknowns. These equations are dependent, so they define only one relationship between p and q.

$p = 0.7p + 0.5q$
$q = 0.3p + 0.5q$

Subtracting the second equation from the first gives

$$\begin{aligned} p - q &= 0.4p \\ q &= 0.6p \end{aligned}$$

Now, use the fact that the sum of probabilities at any state must equal 1,

$$\begin{aligned} p + q &= 1 \\ p + 0.6p &= 1 \\ p &= \frac{1}{1.6} \\ &= 0.625 \\ q &= 1 - p \\ &= 0.375 \end{aligned}$$

So, the steady-state vector for the weather is [0.625 0.375]. Over the long term, there will be a 62.5% probability of a calm day and 37.5% chance of a stormy day at the seaport.

Key Concepts

- The theory of Markov chains can be applied to probability models in which the outcome of one trial directly affects the outcome of the next trial.
- Regular Markov chains eventually reach a steady state, which can be used to make long-term predictions.

Communicate Your Understanding

1. Why must a transition matrix always be square?
2. Given an initial probability vector $S^{(0)} = [0.4 \ \ 0.6]$ and a transition matrix $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$, state which of the following equations is easier to use for determining the third-step probability vector:
   $S^{(3)} = S^{(2)}P$ or $S^{(3)} = S^{(0)}P^3$
   Explain your choice.
3. Explain how you can determine whether a Markov chain has reached a steady state after k trials.
4. What property or properties must events A, B, and C have if they are the only possible different states of a Markov chain?

Practise

A

1. Which of the following cannot be an initial probability vector? Explain why.

a) $[0.2 \;\; 0.45 \;\; 0.25]$

b) $[0.29 \;\; 0.71]$

c) $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$

d) $[0.4 \;\; -0.1 \;\; 0.7]$

e) $[0.4 \;\; 0.2 \;\; 0.15 \;\; 0.25]$

2. Which of the following cannot be a transition matrix? Explain why.

a) $\begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.1 & 0 & 0.9 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$

b) $\begin{bmatrix} 0.2 & 0.8 \\ 0.65 & 0.35 \end{bmatrix}$

c) $\begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.3 & 0.22 & 0.48 \end{bmatrix}$

3. Two competing companies, ZapShot and E-pics, manufacture and sell digital cameras. Customer surveys suggest that the companies' market shares can be modelled using a Markov chain with the following initial probability vector $S^{(0)}$ and transition matrix P.

$$S^{(0)} = [0.67 \;\; 0.33] \qquad P = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$$

Assume that the first element in the initial probability vector pertains to ZapShot. Explain the significance of

a) the elements in the initial probability vector

b) each element of the transition matrix

c) each element of the product $S^{(0)}P$

Apply, Solve, Communicate

B

4. Refer to question 3.

a) Which company do you think will increase its long-term market share, based on the information provided? Explain why you think so.

b) Calculate the steady-state vector for the Markov chain.

c) Which company increased its market share over the long term?

d) Compare this result with your answer to part a). Explain any differences.

5. For which of these transition matrices will the Markov chain be regular? In each case, explain why.

a) $\begin{bmatrix} 0.2 & 0.8 \\ 0.95 & 0.05 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.33 & 0.3 & 0.37 \\ 0.5 & 0 & 0.5 \end{bmatrix}$

6. Gina noticed that the performance of her baseball team seemed to depend on the outcome of their previous game. When her team won, there was a 70% chance that they would win the next game. If they lost, however, there was only a 40% chance that they would their next game.

a) What is the transition matrix of the Markov chain for this situation?

b) Following a loss, what is the probability that Gina's team will win two games later?

c) What is the steady-state vector for the Markov chain, and what does it mean?

7. Application Two popcorn manufacturers, Ready-Pop and ButterPlus, are competing for the same market. Trends indicate that 65% of consumers who purchase Ready-Pop will stay with Ready-Pop the next time, while 35% will try ButterPlus. Among those who purchase ButterPlus, 75% will buy ButterPlus again and 25% will switch to Ready-Pop. Each popcorn producer initially has 50% of the market.

a) What is the initial probability vector?

b) What is the transition matrix?

c) Determine the first- and second-step probability vectors.

d) What is the long-term probability that a customer will buy Ready-Pop?

8. Inquiry/Problem Solving The weather pattern for a certain region is as follows. On a sunny day, there is a 50% probability that the next day will be sunny, a 30% chance that the next day will be cloudy, and a 20% chance that the next day will be rainy. On a cloudy day, the probability that the next day will be cloudy is 35%, while it is 40% likely to be rainy and 25% likely to be sunny the next day. On a rainy day, there is a 45% chance that it will be rainy the next day, a 20% chance that the next day will be sunny, and a 35% chance that the next day will be cloudy.

a) What is the transition matrix?

b) If it is cloudy on Wednesday, what is the probability that it will be sunny on Saturday?

c) What is the probability that it will be sunny four months from today, according to this model?

d) What assumptions must you make in part c)? Are they realistic? Why or why not?

9. Application On any given day, the stock price for Bluebird Mutual may rise, fall, or remain unchanged. These states, R, F, and U, can be modelled by a Markov chain with the transition matrix:

$$\begin{array}{cc} \begin{array}{ccc} \mathrm{R} & \mathrm{F} & \mathrm{U} \end{array} & \\ \begin{bmatrix} 0.75 & 0.15 & 0.1 \\ 0.25 & 0.6 & 0.15 \\ 0.4 & 0.4 & 0.2 \end{bmatrix} & \begin{array}{c} \mathrm{R} \\ \mathrm{F} \\ \mathrm{U} \end{array} \end{array}$$

a) If, after a day of trading, the value of Bluebird's stock has fallen, what is the probability that it will rise the next day?

b) If Bluebird's value has just risen, what is the likelihood that it will rise one week from now?

c) Assuming that the behaviour of the Bluebird stock continues to follow this established pattern, would you consider Bluebird to be a safe investment? Explain your answer, and justify your reasoning with appropriate calculations.

10. Assume that each doe produces one female offspring. Let the two states be D, a normal doe, and B, a doe with bald patches. Determine

a) the initial probability vector

b) the transition matrix for each generation of offspring

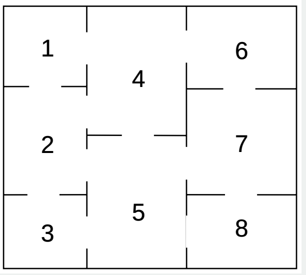
c) the long-term probability of a new-born doe developing bald patches

d) Describe the assumptions which are inherent in this analysis. What other factors could affect the stability of this Markov chain?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

11. When Mazemaster, the mouse, is placed in a maze like the one shown below, he will explore the maze by picking the doors at random to move from compartment to compartment. A transition takes place when Mazemaster moves through one of the doors into another compartment. Since all the doors lead to *other* compartments, the probability of moving from a compartment back to the same compartment in a single transition is zero.



a) Construct the transition matrix, P, for the Markov chain.

b) Use technology to calculate P^2, P^3, and P^4.

c) If Mazemaster starts in compartment 1, what is the probability that he will be in compartment 4 after

i) two transitions?

ii) three transitions?

iii) four transitions?

d) Predict where Mazemaster is most likely to be in the long run. Explain the reasoning for your prediction.

e) Calculate the steady-state vector. Does it support your prediction? If not, identify the error in your reasoning in part d).

C

12. **Communication** Refer to Example 4 on page 351.

a) Suppose that the probability of stormy weather on any day following a calm day increases by 0.1. Estimate the effect this change will have on the steady state of the Markov chain. Explain your prediction.

b) Calculate the new steady-state vector and compare the result with your prediction. Discuss any difference between your estimate and the calculated steady state.

c) Repeat parts a) and b) for the situation in which the probability of stormy weather following either a calm or a stormy day increases by 0.1, compared to the data in Example 4.

d) Discuss possible factors that might cause the mathematical model to be altered.

13. For each of the transition matrices below, decide whether the Markov chain is regular and whether it approaches a steady state. (*Hint*: An irregular Markov chain could still have a steady-state vector.)

a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$

14. Refer to Example 2 on page 347.

a) Using a graphing calculator, find P^{100}. Describe this matrix.

b) Let $S^{(0)} = [a \;\; b \;\; c]$. Find an expression for the value of $S^{(0)}P^{100}$. Does this expression depend on $S^{(0)}$, P, or both?

c) What property of a regular Markov chain can you deduce from your answer to part b)?

15. **Inquiry/Problem Solving** The transition matrix for a Markov chain with steady-state vector of $\begin{bmatrix} \frac{7}{13} & \frac{6}{13} \end{bmatrix}$ is $\begin{bmatrix} 0.4 & 0.6 \\ m & n \end{bmatrix}$.
Determine the unknown transition matrix elements, m and n.

Career Connection

Investment Broker

Many people use the services of an investment broker to help them invest their earnings. An investment broker provides advice to clients on how to invest their money, based on their individual goals, income, and risk tolerance, among other factors. An investment broker can work for a financial institution, such as a bank or trust company, or a brokerage, which is a company that specializes in investments. An investment broker typically buys, sells, and trades a variety of investment items, including stocks, bonds, mutual funds, and treasury bills.

An investment broker must be able to read and interpret a variety of financial data including periodicals and corporate reports. Based on experience and sound mathematical principles, the successful investment broker must be able to make reasonable predictions of uncertain outcomes.

Because of the nature of this industry, earnings often depend directly on performance. An investment broker typically earns a commission, similar to that for a sales representative. In the short term, the investment broker can expect some fluctuations in earnings. In the long term, strong performers can expect a very comfortable living, while weak performers are not likely to last long in the field.

Usually, an investment broker requires a minimum of a bachelor's degree in economics or business, although related work experience in investments or sales is sometimes an acceptable substitute. A broker must have a licence from the provincial securities commission and must pass specialized courses in order to trade in specific investment products such as securities, options, and futures contracts. The chartered financial analyst (CFA) designation is recommended for brokers wishing to enter the mutual-fund field or other financial-planning services.

WEB CONNECTION

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Visit the above web site and follow the links to find out more about an investment broker and other careers related to mathematics.

Review of Key Concepts

6.1 Basic Probability Concepts

Refer to the Key Concepts on page 311.

1. A bag of marbles contains seven whites, five blacks, and eight cat's-eyes. Determine the probability that a randomly drawn marble is
   a) a white marble
   b) a marble that is not black

2. When a die was rolled 20 times, 4 came up five times.
   a) Determine the empirical probability of rolling a 4 with a die based on the 20 trials.
   b) Determine the theoretical probability of rolling a 4 with a die.
   c) How can you account for the difference between the results of parts a) and b)?

3. Estimate the subjective probability of each event and provide a rationale for your decision.
   a) All classes next week will be cancelled.
   b) At least one severe snow storm will occur in your area next winter.

6.2 Odds

Refer to the Key Concepts on page 317.

4. Determine the odds in favour of flipping three coins and having them all turn up heads.

5. A restaurant owner conducts a study that measures the frequency of customer visits in a given month. The results are recorded in the following table.

Number of Visits	Number of Customers
1	4
2	6
3	7
4 or more	3

Based on this survey, calculate
   a) the odds that a customer visited the restaurant exactly three times
   b) the odds in favour of a customer having visited the restaurant fewer than three times
   c) the odds against a customer having visited the restaurant more than three times

6.3 Probabilities Using Counting Techniques

Refer to the Key Concepts on page 324.

6. Suppose three marbles are selected at random from the bag of marbles in question 1.
   a) Draw a tree diagram to illustrate all possible outcomes.
   b) Are all possible outcomes equally likely? Explain.
   c) Determine the probability that all three selected marbles are cat's-eyes.
   d) Determine the probability that none of the marbles drawn are cat's-eyes.

7. The Sluggers baseball team has a starting line-up consisting of nine players, including Tyrone and his sister Amanda. If the batting order is randomly assigned, what is the probability that Tyrone will bat first, followed by Amanda?

8. A three-member athletics council is to be randomly chosen from ten students, five of whom are runners. The council positions are president, secretary, and treasurer. Determine the probability that
   a) the committee is comprised of all runners
   b) the committee is comprised of the three eldest runners
   c) the eldest runner is president, second eldest runner is secretary, and third eldest runner is treasurer

6.4 Dependent and Independent Events

Refer to the Key Concepts on page 333.

9. Classify each of the following pairs of events as independent or dependent.

	First Event	Second Event
a)	Hitting a home run while at bat	Catching a pop fly while in the field
b)	Staying up late	Sleeping in the next day
c)	Completing your calculus review	Passing your calculus exam
d)	Randomly selecting a shirt	Randomly selecting a tie

10. Bruno has just had job interviews with two separate firms: Golden Enterprises and Outer Orbit Manufacturing. He estimates that he has a 40% chance of receiving a job offer from Golden and a 75% chance of receiving an offer from Outer Orbit.

a) What is the probability that Bruno will receive both job offers?

b) Is Bruno applying the concept of theoretical, empirical, or subjective probability? Explain.

11. Karen and Klaus are the parents of James and twin girls Britta and Kate. Each family member has two shirts in the wash. If a shirt is pulled from the dryer at random, what is the probability that the shirt belongs to

a) Klaus, if it is known that the shirt belongs to one of the parents?

b) Britta, if it is known that the shirt is for a female?

c) Kate, if it is known that the shirt belongs to one of the twins?

d) Karen or James

12. During a marketing blitz, a telemarketer conducts phone solicitations continuously from 16 00 until 20 00. Suppose that you have a 20% probability of being called during this blitz. If you generally eat dinner between 18 00 and 18 30, how likely is it that the telemarketer will interrupt your dinner?

6.5 Mutually Exclusive Events

Refer to the Key Concepts on page 340.

13. Classify each pair of events as mutually exclusive or non-mutually exclusive.

	First Event	Second Event
a)	Randomly selecting a classical CD	Randomly selecting a rock CD
b)	Your next birthday occurring on a Wednesday	Your next birthday occurring on a weekend
c)	Ordering a hamburger with cheese	Ordering a hamburger with no onions
d)	Rolling a perfect square with a die	Rolling an even number with a die

14. a) Determine the probability of drawing either a 5 or a black face card from a standard deck of cards.

b) Illustrate this situation with a Venn diagram.

15. In a data management class of 26 students, there are 9 with blonde hair, 7 with glasses, and 4 with blonde hair and glasses.

a) Draw a Venn diagram to illustrate this situation.

b) If a student is selected at random, what is the probability that the student will have either blonde hair or glasses?

16. Of 150 students at a school dance, 110 like pop songs and 70 like heavy-metal songs. A third of the students like both pop and heavy-metal songs.

a) If a pop song is played, what are the odds in favour of a randomly selected student liking the song?

b) What are the odds in favour of a student disliking both pop and heavy-metal songs?

c) Discuss any assumptions which must be made in parts a) and b).

17. The four main blood types are A, B, AB, and O. The letters A and B indicate whether two factors (particular molecules on the surface of the blood cells) are present. Thus, type AB blood has both factors while type O has neither. Roughly 42% of the population have type A blood, 10% have type B, 3% have type AB, and 45% have type O. What is the probability that a person

a) has blood factor A?

b) does not have blood factor B?

6.6 Applying Matrices to Probability Problems

Refer to the Key Concepts on page 352.

18. Alysia, the star on her bowling team, tends to bowl better when her confidence is high. When Alysia bowls a strike, there is a 50% probability that she will bowl a strike in the next frame. If she does not bowl a strike, then she has a 35% probability of bowling a strike in the next frame. Assume that Alysia starts the first game with a strike.

a) What is the initial probability vector for this situation?

b) What is the transition matrix?

c) Determine the probability that Alysia will bowl a strike in the second, third, and tenth frames.

d) There are ten frames in a game of bowling. What is the probability that Alysia will bowl a strike in the first frame of the second game?

e) What is the long-term probability that Alysia will bowl a strike?

f) What assumptions must be made in parts d) and e)?

19. A year-long marketing study observed the following trends among consumers of three competing pen manufacturers.

- 20% of Blip Pens customers switched to Stylo and 10% switched to Glyde-Wryte.
- 15% of Stylo customers switched to Glyde-Wryte and 25% switched to Blip Pens.
- 30% of Glyde-Wryte customers switched to Blip Pens and 5% switched to Stylo.

a) What is the transition matrix for this situation?

b) Determine the steady-state vector.

c) What is the expected long-term market share of Glyde-Wryte if these trends continue?

Chapter Test

ACHIEVEMENT CHART				
Category	Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application
Questions	All	11	5, 10, 11	3, 4, 6, 7, 8, 9, 10, 11

1. A coin is tossed three times.

a) Draw a tree diagram to illustrate the possible outcomes.

b) Determine the probability that heads will appear each time.

2. A jumbled desk drawer contains three pencils, four pens, and two markers. If you randomly pull out a writing instrument,

a) what is the probability that it is not a pencil?

b) what are the odds in favour of pulling out a pen?

3. A die is rolled ten times. What is the probability that a prime number will be rolled every time?

4. If Juanita bumps into Troy in the hallway between periods 2 and 3, there is a 25% chance that she will be late for class. If she does not bump into Troy, she will make it to class on time. If there is a 20% chance that Juanita will bump into Troy, how likely is it that she will be late on any given day?

5. Of 150 workers surveyed in an industrial community, 65 worked in the paper mill and 30 worked in the water-treatment plant.

a) What is the probability that a worker surveyed at random works

i) in either the paper mill or the water-treatment plant?

ii) somewhere other than the paper mill or the water-treatment plant?

b) What assumptions must you make in part a)?

6. The gene for blood type A is dominant over the one for type O blood. To have type O blood, a child must inherit type O genes from both parents. If the parents of a child both have one blood type A gene and one blood type O gene, what are the odds in favour of the child having type O blood?

7. Of the members of a track-and-field club, 42% entered track events at the most recent provincial meet, 32% entered field events, and 20% entered both track and field events.

a) Illustrate the club's entries with a Venn diagram.

b) What is the probability that a randomly selected member of the club

i) entered either a track event or a field event at the provincial meet?

ii) did not compete at the meet?

8. Five siblings, Paula, Mike, Stephanie, Kurt, and Emily, are randomly seated along one side of a long table. What is the probability that the children are seated

a) with the three girls in the middle?

b) in order of age?

9. Naomi, a fan of alternative music, has 12 CDs.

Band	Number of CDs
Nine Inch Nails	3
Soundgarden	4
Monster Magnet	2
Pretty & Twisted	1
Queensrÿche	2

If Naomi randomly loads her player with five CDs, what is the probability that it will hold

a) no Soundgarden CDs?

b) exactly one Monster Magnet CD?

c) three Nine Inch Nails CDs or three Soundgarden CDs?

d) one CD from each band?

10. Ursula, an electrical engineer in a quality-control department, checks silicon-controlled rectifiers (SCRs) for manufacturing defects. She has noticed that when a defective SCR turned up on the assembly line, there was a 0.07 probability that the next unit would also be defective. If, however, an SCR passed inspection, then there was just a 0.004 likelihood that the next unit would fail inspection.

a) Assuming that Ursula has just found a defective SCR, find

i) the initial probability vector

ii) the transition matrix

iii) the probability that the next two SCRs will both fail inspection

b) Is this Markov chain regular? Explain why or why not.

c) What does your answer to part b) imply about the long-term probability of an SCR failing inspection? Quantify your answer.

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

11. Candice owns a chocolate shop. One of her most popular products is a box of 40 assorted chocolates, 5 of which contain nuts.

a) If a person selects two chocolates at random from the box, what is the probability that

i) both of the chocolates contain nuts?

ii) at least one of the chocolates contains nuts?

iii) only one of the chocolates contains nuts?

iv) neither of the chocolates contain nuts?

b) Describe how you could simulate choosing the two chocolates. Outline a method using

i) a manual technique

ii) appropriate technology

c) Suppose you ran 100 trials with either of your simulations. Would you expect empirical probabilities based on the results of these trials to match the probabilities you computed in part a)? Why or why not?

d) Due to the popularity of the chocolates with nuts, Candice is planning to double the number of them in each box. She claims that having 10 of the 40 chocolates contain nuts will double the probability that one or both of two randomly selected chocolates will contain nuts. Do you agree with her claim? Support your answer with probability calculations.

Wrap-Up

Implementing Your Action Plan

1. Determine the probability of winning your game. If the game is simple enough, you can present both a theoretical and an empirical probability. For complex games, you may have to rely on empirical probability alone. If practical, use technology to simulate your game, and run enough trials to have confidence in your results.

2. Develop a winning strategy for your game. If no winning strategy is possible, explain why.

3. Prepare all the components for the game, such as a board, tokens, instruction sheets, or score cards.

4. Have other students, and perhaps your teacher, try your game. Note their comments and any difficulties they had with the game.

5. Keep track of the outcomes and analyse them using technology, where appropriate. Determine whether you need to make any adjustments to the rules or physical design of your game.

6. Record any problems that arise as you implement the plan, and outline how you deal with them.

Suggested Resources

- Toy stores
- Web sites for manufacturers and players' groups
- CD-ROM games
- Books on games of chance
- Statistical software
- Spreadsheets
- Graphing calculators

Evaluating Your Project

1. Assess your game in terms of
   - clarity of the rules
   - enjoyment by the players
   - originality
   - physical design, including attractiveness and ease of use

2. Consider the quality and extent of your mathematical analysis. Have you included all appropriate theoretical and empirical probabilities? Have you properly analysed the effectiveness of possible strategies for winning? Are there other mathematical investigations that you could apply to your game? Have you made appropriate use of technology? Have you documented all of your analysis?

3. If you were to do this project again, what would you do differently? Why?

4. Are there questions that arose from your game that warrant further investigation? How could you address these issues in a future project?

Presentation

1. Explain and demonstrate the game you have created.
2. Discuss comments made by the players who tested your game.
3. Outline the probabilities that apply to your game.
4. Present the outcomes as analysed and organized data.
5. Outline possible winning strategies and comment on their effectiveness.
6. Discuss the positive aspects of your project, its limitations, and how it could be extended or improved.
7. Listen to presentations by your classmates and ask for clarification or suggest improvements, where appropriate. Consider how you might be able to apply both the strengths and weaknesses of other presentations to improve your project.

Preparing for the Culminating Project

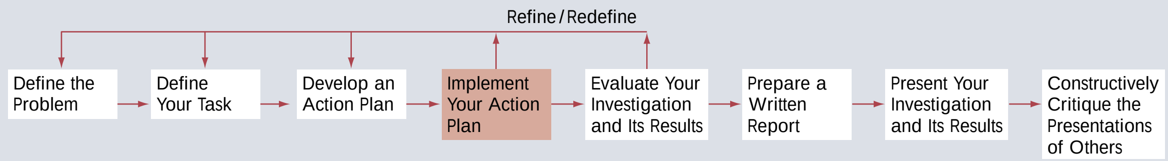
Applying Project Skills

In the course of this probability project, you will have developed skills that could be essential for your culminating project:

- developing and carrying out an action plan
- applying techniques for determining probabilities
- working with others to test your ideas
- evaluating your own work
- presenting results
- critiquing the work of other students

Keeping on Track

At this point, you should have implemented enough of the action plan for your culminating project to determine what data you need and how to analyse these data. You should have considered whether you need to refine or redefine your project. For example, you may have found that the data you require are not readily available or your method of analysis is not practical. In such cases, you should discuss your revised action plan with your teacher. Section 9.3 provides suggestions for developing and implementing your action plan.



Cumulative Review: Chapters 4 to 6

1. Evaluate.

a) $7!$ **b)** ${}_7P_1$ **c)** ${}_7C_1$

d) $P(7, 7)$ **e)** $\binom{7}{2}$ **f)** $C(7, 2)$

2. Use the binomial theorem to

a) expand $(3x - 2y)^5$

b) factor $2a^4 - 8a^3b + 12a^2b^2 - 8ab^3 + 2b^4$

3. If upper-case and lower-case letters are considered as different letters, how many six-letter computer passwords are possible

a) with no repeated letters?

b) with at least one capital letter?

4. In how many ways can 12 different cars be parked in the front row of a used-car lot if the owner does not want the red one beside the orange one because the colours clash?

5. What is the probability that a random integer between 1 and 50, inclusive, is not a prime number?

6. A computer expert estimates that the odds of a chess grand master defeating the latest chess-playing computer are 4:5. What is the probability that the chess master will win a match against the computer?

7. a) How many divisors of 4725 are there?

b) How many of these divisors are divisible by 5?

8. Eight friends, three of whom are left-handed, get together for a friendly game of volleyball. If they split into two teams randomly, what is the probability that one team is comprised of

a) all right-handed players?

b) two right-handed and two left-handed players?

9. A manager interviews in random order five candidates for a promotion. What is the likelihood that the most experienced candidate will be interviewed first, followed by the second most experienced candidate?

10. If four decks of cards are shuffled together, what is the probability of dealing a 13-card hand that includes exactly two black 3s?

11. At Inglis Park in Owen Sound, you can see adult salmon jumping over a series of logs as they swim upstream to spawn. The salmon have a 0.6 probability of a successful jump if they rest prior to the jump, but only a 0.3 probability immediately after jumping the previous log. If the fish are rested when they come to the first log, what is the probability that a salmon will clear

a) both of the first two logs on the first try without resting?

b) all of the first four logs on the first try if it rests after the second jump?

12. The weather forecast calls for a 12% chance of rain tomorrow, but it is twice as likely that it will snow. What is the probability that it will neither rain nor snow tomorrow?

13. Sasha and Pedro meet every Tuesday for a game of backgammon. They find that after winning a game, Sasha has a 65% probability of winning the next game. Similarly, Pedro has a 60% probability of winning after he has won a game. Pedro won the game last week.

a) What are the probabilities of each player winning this week?

b) What is the probability of Pedro winning the game two weeks later?

c) If Pedro and Sasha play 100 games, how many games is each player likely to win?

Endangered Species

Background

The Canadian Nature Federation has identified 380 endangered species in Canada. Of these, 115 are near extinction. Worldwide, there are over 9485 endangered species. Some species, such as whooping cranes or giant pandas, have populations of only a few hundred.

There is often debate about the state of a particular species. In 2000, the Red List compiled by the World Conservation Union (IUCN) listed 81 species of whale that are classified as extinct, endangered, or vulnerable. Most nations have stopped all commercial whale hunting. However, Japan and Norway continue whaling, and Japan even increased the number of whale species that it hunts.

In southern Africa, elephant herds are often reduced by killing whole families of elephants. The governments involved claim that such culling keeps the herds within the limits of the available food supply and prevents the elephants from encroaching on farmers' fields. However, critics claim that the culling is done primarily for the profit derived from selling the elephants' ivory tusks and reduces the number of elephants dangerously. Thirty years ago, there were three million elephants in Africa. Estimates today place the number at less than 250 000.

In 2001, the government of British Columbia reintroduced hunting of grizzly bears, claiming that their numbers have rebounded enough that hunting will not endanger the species. The government places the number of adult grizzlies at over 13 000. The Canadian Nature Federation claims the number of grizzlies is at most 6000, and could be as low as 4000.

As you can see from these examples, obtaining accurate statistics is vital to the management of endangered species.

Your Task

Collect and analyse data about a species of your choice to determine whether it is endangered.

Developing an Action Plan

You will need to select a species that you believe may be endangered, find reliable sources of data about this species, and outline a method for analysing the data and drawing conclusions from them.