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Date: May 28 2015

UNIT 6 TEST: Algebraic Vectors
The Final Frontier

Instructions:
1. Read the instructions carefully.
2. Attempt all problems and don't spend all your time on one problem. Other problems will feel left out ☹
3. Show all work in the space provided. Three (3) marks will be given to the overall mathematical form on this test.
4. **Leave all answers as exact answers unless indicated otherwise.**
5. ENJOY ☺

PART A: KNOWLEDGE AND UNDERSTANDING

1. Given $\vec{a} = (-3,1,0)$ and $\vec{b} = (5,2,-6)$,

a) determine $|\vec{a}|$

$|\vec{a}| = \sqrt{(-3)^2 + (1)^2 + (0)^2}$

$= \sqrt{10}$ units ✓

b) determine $|\vec{b}|$ [2]

$|\vec{b}| = \sqrt{(5)^2 + (2)^2 + (-6)^2}$

$= \sqrt{65}$ units ✓

$\vec{a} \cdot \vec{b}$
$= u_1v_1 + u_2v_2 + u_3v_3$
$= (-3)(5) + (1)(2) + (0)(-6)$
$= -15 + 2$
$= -13$

c) determine $\vec{a} \cdot \vec{b}$

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$u_1v_1 + u_2v_2 + u_3v_3 = (\sqrt{10})(\sqrt{65}) \cos\theta$

$(-3)(5) + (1)(2) + (0)(-6) = (\sqrt{10})(\sqrt{65}) \cos\theta$

$\theta = 120°$

$\vec{a} \cdot \vec{b} = (\sqrt{10})(65) \cos 120$

$= -(\sqrt{10})(65)\frac{1}{2}$

$= -\frac{(\sqrt{10})(65)}{2}$ ✗

d) determine $\vec{a} \times \vec{b}$ [2]

minus

$\vec{a} \times \vec{b} = (-6-0, 0+18, -6-5)$

$= (-6, 18, -11)$

$\vec{a} \times \vec{b} = (-6-0)\hat{i} - (18)\hat{j} + (-6-5)\hat{k}$

$= (-6)\hat{i} - (18)\hat{j} + (-11)\hat{k}$ $\sin 120°$

$= -35$ units ✗

e) determine the angle between $\vec{a}$ and $\vec{b}$ (round to 1 decimal place if necessary) [1]

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$u_1v_1 + u_2v_2 + u_3v_3 = (\sqrt{10})(\sqrt{65}) \cos\theta$

$(-3)(5) + (1)(2) + (0)(-6) = (\sqrt{10})(\sqrt{65}) \cos\theta$

$\frac{-13}{(\sqrt{10})(\sqrt{65})} = \cos\theta$

$\theta \doteq 120.7°$ ✓

f) determine $\hat{a} = -3\hat{i} + \hat{j}$ ✗

g) determine $|\hat{a}|$ [2]

$= \sqrt{(-3)^2 + (1)^2}$

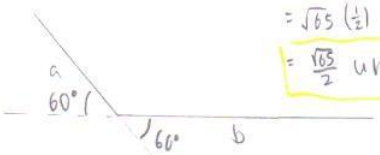
$= \sqrt{10}$ units ✗

h) determine $|\text{proj}_{\vec{b}}\vec{a}|$

$= |\vec{b}| \cos 60$

$= \sqrt{65}\left(\frac{1}{2}\right)$

$= \frac{\sqrt{65}}{2}$ units ✗

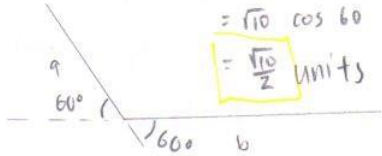


i) determine $\text{proj}_{\vec{a}}\vec{b}$ [2]

$= |\vec{a}| \cos 60°$

$= \sqrt{10} \cos 60$

$= \frac{\sqrt{10}}{2}$ units ✗



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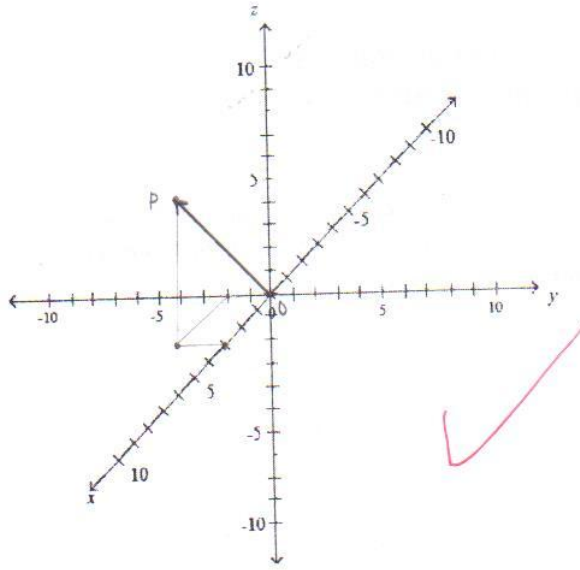
2. a) Draw the position vector $\overrightarrow{OP}$ of the point $P(3,-2,5)$. [4]



3/4 ✓

b) Determine the algebraic vector representing $\overrightarrow{PQ}$ given the point $Q(-2,-3,-6)$.

$\overrightarrow{PQ} = -2\hat{i} - 3\hat{j} - 6\hat{k}$ ✗

c) Determine the vector parallel to $\overrightarrow{PQ}$.

$-4\hat{i} - 6\hat{j} - 12\hat{k}$ ✓ ok...

d) Determine a vector that is collinear with $\overrightarrow{PQ}$.

$4\hat{i} + 6\hat{j} + 12\hat{k}$ ✓ ok...

3. Given that $\vec{u} = 2\hat{i} - 5\hat{j} + 3\hat{k}$ and $\vec{v} = 3\hat{i} + 4\hat{j}$

a) determine $\vec{u} + \vec{v}$ [2]

$= (2\hat{i} - 5\hat{j} + 3\hat{k} + 3\hat{i} + 4\hat{j})$

$= 5\hat{i} - \hat{j} + 3\hat{k}$ ✓

4/4

b) determine $\vec{u} - \vec{v}$

$= (2\hat{i} - 5\hat{j} + 3\hat{k}) - (3\hat{i} + 4\hat{j})$

$= -\hat{i} - 9\hat{j} + 3\hat{k}$ ✓✓

c) determine $3\vec{u} + 2\vec{v}$ [2]

$= 3(2\hat{i} - 5\hat{j} + 3\hat{k}) + 2(3\hat{i} + 4\hat{j})$

$= 6\hat{i} - 15\hat{j} + 9\hat{k} + 6\hat{i} + 8\hat{j}$

$= 12\hat{i} - 7\hat{j} + 9\hat{k}$ ✓✓

4/4

PART B: APPLICATIONS

4. Given $\vec{a} = (2,3,7)$ and $\vec{b} = (-4, y, -14)$,

a) for what values of y are the vectors collinear? [2]

$y = -6$ ✓✓

b) for what values of y are the vectors perpendicular? [2]

$0 = (2)(-4) + (3)(y) + (7)(-14)$

$= -8 + 3y + -98$

$106 = 3y$

$y = \frac{106}{3}$ ✓✓

7+4 ✓

X

5. Given $\hat{a}$ and $\hat{b}$ are unit vectors, if the angle between them is $60°$, calculate $(6\hat{a}+\hat{b})\cdot(\hat{a}-2\hat{b})$ [4]

$\vec{u}$ $\vec{v}$

$(6,1)$ $(1,-2)$

$|\vec{u}| = \sqrt{36+1}$ $|\vec{v}| = \sqrt{1+4}$

$= \sqrt{37}$ units $= \sqrt{5}$ units

$(6\hat{a}+\hat{b})\cdot(\hat{a}-2\hat{b}) = (\sqrt{37})(\sqrt{5})\cos 60°$ X

$= \frac{(\sqrt{37})(\sqrt{5})}{2}$ units

1/4

6. Determine the volume of the parallelepiped determined by the vectors $\vec{a}=(2,-5,-1)$, $\vec{b}=(4,0,1)$ and $\vec{c}=(3,-1,-1)$. [4]

$A_p = |(\vec{a}\times\vec{b})\cdot\vec{c}|$

$= |[(-5)(1)+(-4-2)+(0+20)]\cdot(3,-1,-1)|$

$= |[(-4)+(-6)+(20)]\cdot(3,-1,-1)|$

$= |(-12)+(6)+(-20)|$ ✓ ok

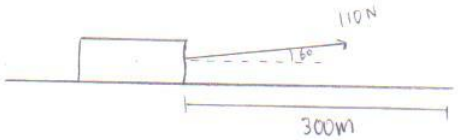
$= |-26 \text{ units}^3|$ ✓ ok

magnitude! therefore, the volume is X -26 units^3

$\vec{a}\times\vec{b}$ must be vector!

2/4

7. A pedicab is pulled a distance of 300 m by a force of 110 N applied at an angle of 6° to the roadway. Calculate the work done. (round to 1 decimal place if necessary) [2]



$|\vec{F}||\vec{d}|\cos\theta$

$W = |\vec{F}|\cos\theta$

$= (110\text{N})\cos 6°$ X

$\doteq 109.4$ Nm

$W = |\vec{F}||\vec{d}|\cos\theta$

$= (110\text{ N})(300\text{m})\cos 6°$

$= 32819.2$ Nm

therefore, work done is 109.4 N·m

✓ ok...

1/2

1/4

8. A 50-N force is applied to the end of a 20-cm wrench and makes an angle of 30° with the handle of the wrench.

a) What is the torque on a bolt at the other end of the wrench? [2]

0.2m wrench
50N Force
30° angle

$\vec{\tau} = (50N)(0.2m)(\sin 30°)$
$= 5\ N\,m$

therefore, the torque is 5 N m

4/4

b) What is the maximum torque that can be exerted by a 50-N force on this wrench and how can it be achieved? [2]

$\sin 90° = 1$

$\max \vec{\tau} = (50N)(0.2m)(\sin 90°)$
$= 10\ N\,m$

∴ max torque that can be exerted is 10 N m

what angle? (state here!)

PART C: COMMUNICATION

9. Explain whether the following expressions are vectors, scalars, or meaningless.

a) $(\vec{a} + \vec{b}) \cdot \vec{c}$ [2]

= (vector) · (vector) ✓
= scalar ✓

b) $(\vec{a} + \vec{b}) \cdot (\vec{c} \cdot \vec{c})$ [2]

= (vector) · (scalar) ✓ ok. meaningless
= vector X vector can't dot scalars.

3/6

c) $|\vec{a}|(\vec{b} \times \vec{c})$ [2]

= |a| (scalar) X vector!
= meaningless X

4+3\.

10. What is the relationship between $\vec{u}$, $\vec{v}$ and $\vec{w}$ if $\vec{u} \times \vec{v} \cdot \vec{w} = 0$? Explain. [2]

$\hat{i} \times \hat{j} \cdot \hat{k}$
$= \hat{k} \cdot \hat{k}$
$= 0$

$\vec{u}$ and $\vec{v}$ or $\vec{w}$ must be perpendicular to each other, so multiplying would make the answer zero

PART D: THINKING

11. Given that θ is the angle between two vectors $\vec{a}$ and $\vec{b}$ in three-space, prove that [3]

$$(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

12. Prove that, for any two vectors $\vec{a}$ and $\vec{b}$ in three-space, $|\vec{a} \times \vec{b}| = \sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$ [5]

$a_1\ a_2\ a_3\ a_1\ a_2\ a_3$
$b_1\ b_2\ b_3\ b_1\ b_2\ b_3$

$|\vec{a} \times \vec{b}| = [(b_3)(a_2) - (b_2)(a_3)] + [(b_1)(a_3) - (b_3)(a_1)] + [(a_1)(b_2) - (b_1)(a_2)]$

$\sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$

$= \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (|\vec{a}||\vec{b}| \cos\theta)^2}$

$= \sqrt{|\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)}$

$= \sqrt{|\vec{a}|^2 |\vec{b}|^2 \sin^2\theta}$

$= |\vec{a}||\vec{b}| \sin\theta$

$\sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$

$= \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{b})}$

$= \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (|\vec{a}||\vec{b}| \cos\theta)^2}$

$= \sqrt{|\vec{a}|^2 |\vec{b}|^2 -$

4/5

0+1

13. State whether each of the following statements as **TRUE** or **FALSE** for any non-collinear $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$. [6]
If the statement is *true*, provide a proof. If the statement is *false*, give a counterexample.

a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ TRUE

$|\vec{a}| = 5$
$|\vec{b}| = 4$
$\theta = 60°$

Left side
$\vec{a} \cdot \vec{b}$
$= |\vec{a}||\vec{b}| \cos 60°$
$= 20\ (\frac{1}{2})$
$= 10$ units

Right side
$\vec{b} \cdot \vec{a}$
$= |\vec{b}||\vec{a}| \cos 60°$
$= 20\ (\frac{1}{2})$
$= 10$ units

need proof -1

$\therefore LS = RS$
$\therefore$ it is true

b) $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$ FALSE

$\vec{a} = (1,1,1)$ $\vec{b} = (2,3,4)$

$\vec{a} \times \vec{b}$
$= (4-6) + (4-4) + (3-2)$
$= -2 + 0 + 1$
$= -1$

$\vec{b} \times \vec{a}$
$= (6-4) + (4-4) + (2-3)$
$= 2 + 0 - 1$
$= 1$

diff answers X

1 2 1 1
3 4 2 3

3 4 2 3
1 2 1 1

cross product should give vector -1

2/6

c) $\vec{a} \cdot \vec{a} \times \vec{b} = 0$ and $\vec{b} \cdot \vec{a} \times \vec{b} = 0$

$= |\vec{a}|^2 \times \vec{b}$
$= 25 \times \vec{b}$

FALSE X

$|\vec{a}| = 5$
$|\vec{b}| = 4$
$\theta = 60°$

$\vec{b} \cdot \vec{a} \times \vec{b}$
$= 20 \cos 60 \times \vec{b}$
$= 10 \times \vec{b}$ different

-2

☺ The end ☺ ☺ The end ☺

21.