Name: Uni Lee

KU: 21 / 26 TH: 7 / 9 COMM: 9+3 / 10+3 APPS: 15 /15 Date: Feb 24 2015

87%

UNIT 1 TEST: Limits and Rates of Change
A limited Edition

Instructions:
1. Read the instructions carefully.
2. Attempt all problems and don't spend all your time on one problem. Other problems will feel left out ☹
3. Show all work in the space provided. Three (3) marks will be given to the overall mathematical form on this test.
4. ENJOY ☺

PART A: KNOWLEDGE AND UNDERSTANDING

Multiple Choice – Circle the BEST answer. [6]

1. Evaluate $\lim_{x\to 0} \pi$

a. 0 b. 1 c. π d. undefined e. DNE (LL≠RL)

2. Evaluate $\lim_{x\to 2} \frac{x^2-4}{x^2+4}$

a. 0 b. 1 c. -1 d. undefined e. DNE (LL≠RL)

3. Evaluate $\lim_{x\to +\infty} 5^{\frac{1}{x}}$

a. 0 b. 1 c. 5 d. undefined e. DNE ($+\infty$)

4. Evaluate $\lim_{x\to 4} \sqrt{x-4}$

a. 0 b. 2 c. $\sqrt{-4}$ d. undefined e. DNE (LL≠RL)

5. If $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -2$, then $\lim_{x\to 4} [f(x) - g(x)]$ is equal to:

a. 0 b. -2 c. 2 d. 4 e. undefined

6. Given $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$. In order for $y = f(x)$ to be continuous at $x = a$, the $\lim_{x\to a} f(x)$ must equal:

a. 0 b. a c. x d. $f(a)$ e. $f(x)$

/4

Complete solutions must be shown for full marks.

7. Use the graph of $y = f(x)$ below to answer the following questions. [6]

3/6

a) $f(-3) =$ ___1___ ✓

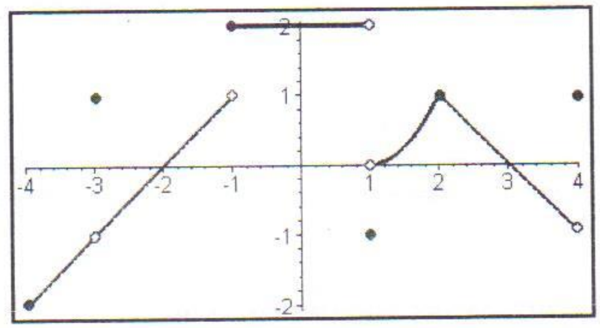
b) $\lim_{x \to -3} f(x) =$ ___1___ ✗ -1

c) $\lim_{x \to 1^+} f(x) =$ ___0___ ✓

d) $\lim_{x \to 1} f(x) =$ ___-1___ ✗ DNE LL≠RL

e) $\lim_{h \to 0} \frac{f(3+h)-f(3)}{h} =$ ___-1___ ✓

f) $\lim_{h \to 0} \frac{f(-3+h)-f(-3)}{h} =$ ___+1___ ✗ DNE



8. Evaluate the following limits algebraically. Leave answers as **exact** answers. NO decimals!

a) $\lim_{x \to \frac{-4}{3}} \left[\frac{27x^3+64}{3x+4}\right]$ [2]

$= \lim_{x \to -4/3} \frac{\cancel{(3x+4)}(9x^2-12x+16)}{\cancel{3x+4}}$

$= \lim_{x \to -4/3} 9x^2 - 12x + 16$

$= 9\left(-\frac{4}{3}\right)^2 - 12\left(-\frac{4}{3}\right) + 16$

$= 9\left(\frac{16}{9}\right) + \frac{48}{3} + 16$

$\boxed{= 48}$ ✓

b) $\lim_{x \to +\infty} \left[\frac{x^3-2x+9}{7x^3-5}\right]$ [2]

$\boxed{= \frac{1}{7}}$ ✓

10/10

c) $\lim_{x \to 4} \left[\frac{x-4}{\sqrt{x+5}-3}\right]$ [3]

$= \lim_{x \to 4} \left[\frac{x-4}{\sqrt{x+5}-3}\right]\left(\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}\right)$

$= \lim_{x \to 4} \frac{(x-4)(\sqrt{x+5}+3)}{x+5-9}$

$= \lim_{x \to 4} \frac{\cancel{(x-4)}(\sqrt{x+5}+3)}{\cancel{(x-4)}}$

$= \lim_{x \to 4} (\sqrt{x+5}+3)$

$= \sqrt{4+5} + 3$

$= \sqrt{9} + 3$

$\boxed{= 6}$ ✓

d) $\lim_{x \to 1} \left[\frac{\frac{1}{x+2} - \frac{1}{3}}{x-1}\right]$ [3]

$= \lim_{x \to 1} \left(\frac{3}{(3)x+2} - \frac{x+2}{3(x+2)}\right) \times \frac{1}{x-1}$

$= \lim_{x \to 1} \left(\frac{3-x-2}{3x+6}\right) \times \frac{1}{x-1}$

$= \lim_{x \to 1} \frac{-\cancel{(x-1)}}{\cancel{(x-1)}(3x+6)}$

$= \lim_{x \to 1} -\frac{1}{3x+6}$

$= -\frac{1}{3(1)+6}$

$\boxed{= -\frac{1}{9}}$ ✓

13

9. Using the method of first principles, find the derivative of $f(x) = \sqrt{3x-2}$. [4]

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-2} - (\sqrt{3x-2})}{h}$

$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-2} - (\sqrt{3x-2})}{h}\left(\frac{\sqrt{3(x+h)-2}+\sqrt{3x-2}}{\sqrt{3(x+h)-2}+\sqrt{3x-2}}\right)$

$= \lim_{h\to 0} \frac{(3(x+h)-2)-(3x-2)}{h(\sqrt{3(x+h)-2}+\sqrt{3x-2})}$

$= \lim_{h\to 0} \frac{\cancel{3x}+3h-\cancel{2}-\cancel{3x}+\cancel{2}}{h(\sqrt{3x+3h-2}+\sqrt{3x-2})}$

$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}(\sqrt{3x+3h-2}+\sqrt{3x-2})}$

$= \lim_{h\to 0} \frac{3}{\sqrt{3x+3h-2}+\sqrt{3x-2}}$

$= \frac{3}{\sqrt{3x+3(0)-2}+\sqrt{3x-2}}$

$= \frac{3}{\sqrt{3x-2}+\sqrt{3x-2}}$

$= \frac{3}{2\sqrt{3x-2}}$

$\therefore f'(x) = \frac{3}{2\sqrt{3x-2}}$

4/4.

PART B: APPLICATIONS

All derivatives must be determined from first principles, otherwise a mark of zero will be awarded

10. At time t, in seconds, a particle is h meters above the ground. If $h(t) = 28t - 2t^2$, where $t \geq 0$, determine:

a) the **average velocity** for the first 5 seconds. [2]

careful!

$h(t) = 28 - 2t^2$

$\vec{V}_{avg} = \frac{\Delta h}{\Delta t}$

at zero seconds:
$28-2(0)^2$
$= 28 - 0$
$= 28\text{ m}$

at five seconds:
$28 - 2(5)^2$
$= 28 - 50$
$= -22\text{ m}$

$= \frac{(-22\text{m})-(+28\text{m})}{5\text{s}-0\text{s}}$

$= \frac{-50\text{m}}{5\text{s}}$

$= -10$ m/s

$= 10$ m/s [downwards]

Therefore, the average velocity for the first 5s is 10 m/s [downwards]

1/2

$\vec{V}_{avg} = 18$ m/s

OK.

b) the **instantaneous velocity** at $t = 5$ seconds. let $f(t)$ represent $h(t)$ [3]

$\vec{V}_{int} = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$

$= \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$

$= \lim_{h\to 0} \frac{28(5+h)-2(5+h)^2-(28(5)-2(5)^2)}{h}$

$= \lim_{h\to 0} \frac{140+28h-2(25+10h+h^2)-28(5)+2(25)}{h}$

$= \lim_{h\to 0} \frac{\cancel{140}+28h-\cancel{50}-20h-2h^2-\cancel{140}+\cancel{50}}{h}$

$= \lim_{h\to 0} \frac{8h-2h^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(8-2h)}{\cancel{h}}$

$= \lim_{h\to 0} (8-2h)$

$= 8 - 2(0)$

$= 8$ m/s

$\therefore$ the instantaneous velocity at 5s is 8 m/s

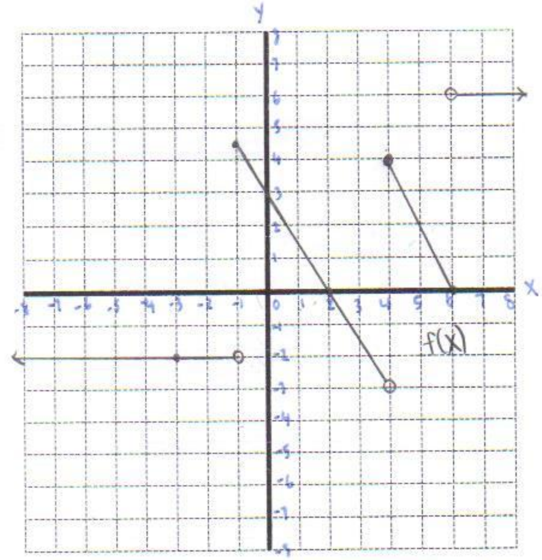
3/3

/4+4

11. Mr. Wong has asked you to draw a graph of **ONE** function, $y = f(x)$ such that the graph $y = f(x)$ must have the following characteristics: [5]

- x–intercepts of 2 and 6
- y–intercept of 3
- $\lim_{x \to +\infty} f(x) = 6$
- $\lim_{x \to -3} f(x) = -2$
- $\lim_{x \to 4^+} f(x) = 4$
- $\lim_{x \to 4} f(x) = \text{DNE } (LL \neq RL)$

$\frac{5}{5} + 1$



12. Find all the **points** on the curve $y = x^3 - 3x$ at which the tangent line is parallel to the x-axis. [5]

$$y' = \lim_{h \to 0} \frac{(x+h)^3 - 3(x+h) - [x^3 - 3x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h - x^3 + 3x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 3$$

$$= 3x^2 + 3x(0) + (0)^2 - 3$$

$$= 3x^2 - 3 \quad ①$$

5/5

parallel to x-axis = slope is 0

sub ① into ②

$0 = 3x^2 - 3$ ②

$3 = 3x^2$

$1 = x^2$

$\pm\sqrt{1} = \sqrt{x^2}$

③ $x = -1$ or $x = +1$ ④

sub ③ into original equation

$y = x^3 - 3x$

$= (-1)^3 - 3(-1)$

$= -1 + 3$

$= 2$

Point: $(-1, 2)$

sub ④ into original equation

$y = x^3 - 3x$

$= (+1)^3 - 3(+1)$

$= 1 - 3$

$= -2$

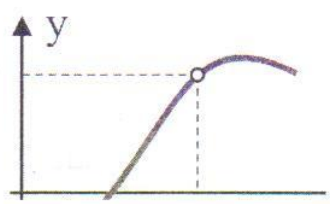
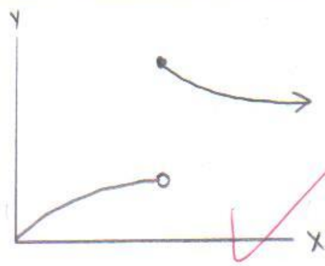
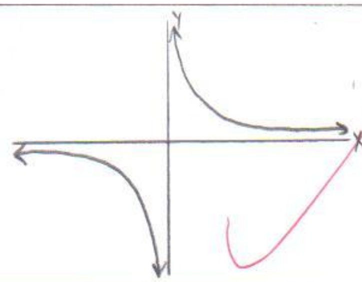
Point $(1, -2)$

Therefore, at points $(-1, 2)$ and $(1, -2)$, their tangent line is parallel to x-axis

11

PART C: COMMUNICATION

13. Complete the following chart regarding discontinuities in a function. [6]

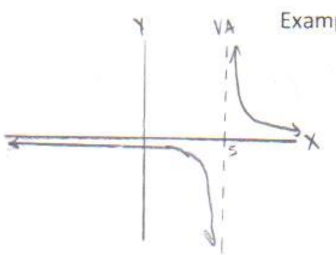
Type of Discontinuity	Graphical Representation (Sketch is sufficient)	Algebraic Representation
removable removal discontinuity		$f(x) = \frac{(x-6)(x+5)}{(x-6)}$
Jump discontinuity		$f(x) = \begin{cases} x^2, & x > 2 \\ x-2, & x \leq 2 \end{cases}$
infinite discontinuity		$f(x) = \frac{1}{x+4}$

6/6

14. Give two examples of functions where the derivative fails to exist at a particular point for a <u>different reason</u>. For each example, give a graphical representation, as well as an explanation why the function fails to have a derivative at that particular point. [4]

3/4

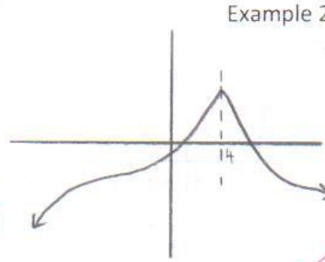
Example 1



fails to exist at $x=5$ because vertical asymptote. it'll be ~~indetermined~~ discontinuous

-1 limit doesn't exist!

Example 2



at $x=4$, it is a corner. the slope there will be $\frac{\text{number}}{0}$ which will be undefined.

/9

PART D: THINKING

15. Find the area of the triangle created by the x-axis, y-axis and the tangent to $y = -4x^2$ at $x = 2$. [5]

$y' = \lim_{h\to 0} \frac{-4(x+h)^2 - (-4x^2)}{h}$

$= \lim_{h\to 0} \frac{-4(x^2+2xh+h^2)+4x^2}{h}$

$= \lim_{h\to 0} \frac{-4x^2-8xh-4h^2+4x^2}{h}$

$= \lim_{h\to 0} \frac{-8xh-4h^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(-8x-4h)}{\cancel{h}}$

$= \lim_{h\to 0} (-8x-4h)$

$= -8x-4(0)$

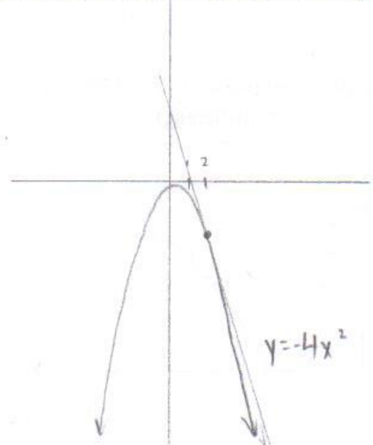
$= -8x$ sub in $x=2$

$= -8(2)$

$= -16$ slope of tangent

sub in poin $(2,-16)$

5/5



$y=mx+b$

$-16 = -16(2)+b$

$-16 = -32+b$

$b = 16$

y-int $= (0,16)$

equation $y = -16x+16$

at $x=2$

$y = -4(2)^2$

$= -4(4)$

$= -16$

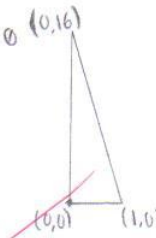
point $(2,-16)$

find x-int, sub $y=0$

$0 = -16x+16$

$-16 = -16x$

$x = 1$



$A = (b \times h) \div 2$

$= (1 \times 16) \div 2$

$= 8$ units^2

$\boxed{\text{Therefore, the area is } 8 \text{ units}^2}$

16. Find the values of constants $\boldsymbol{a}$ and $\boldsymbol{b}$ so that the function [4]

$$f(x) = \begin{cases} \dfrac{x^2 - ax - 6}{x - 2} & \text{for } x > 2 \\ x^2 + b & \text{for } x \le 2 \end{cases}$$

will be continuous for all values of x. Justify your answer.

why?

2 because continuous $\longrightarrow$ $2 = \frac{x^2-ax-6}{x-2}$

$2(x-2) = x^2-ax-6$ ①

$2x-4 = x^2-ax-6$

$0 = x^2-ax-2x-2$

$3 = \frac{x^2-ax-6}{x-2}$

$3x-6 = x^2-ax-6$ ②

sub ① into ②

$3x-6 = 2x-4$

$1x = 2$

$x = 2$ ③

sub ③ into ②

$3(2)-6 = (2)^2-a(2)-6$

$0 = 4-2a-6$

$2a = -2$

$a = -1$

ok

2/4

$2 = x^2+b$

$\sqrt{2-b} = \sqrt{x^2}$

④ $x = +\sqrt{2-b}$ ⑤ $x = -\sqrt{2-b}$

sub ③ into ④

$2 = +\sqrt{2-b}$

$2^2 = 2-b$

$b = -2$

sub ③ into ④

$2^2 = (-\sqrt{2-b})^2$

$4 = 2-b$

$b = -2$

Therefore, the constants a can be -1, and b can be -2

$\therefore a = -1$

$\therefore b = 1$

☺ The end ☺

YAY!!! ^_^