CHAPTER 2

Permutations

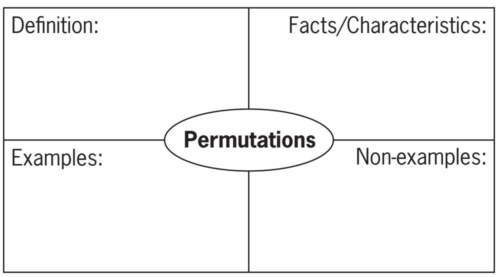
You can think of a permutation as a way to arrange objects. We use the power of permutations on a regular basis in everyday life. Encryption is a way of coding messages so that only people who know the code can read the message. Companies use encryption to ensure online payments and passwords are secure. Randomization of electronic game content forces players to make wise choices when playing. Route planners need to determine the most efficient route for garbage and recycling pickup, as well as street cleaning. Toys such as the Rubik's Cube® rely on permutations to make them challenging. Composers explore the permutations of rhythms and notes to create music. Brainstorm other ways in which people use permutations on a regular basis.

Key Terms

- fundamental counting principle
- arrangement
- factorial
- permutation
- indirect method

Literacy Strategy

A Frayer model is a visual organizer that helps you understand key words and concepts. Copy and complete this organizer as you learn about permutations in this chapter.



Career Link

Computer Programmer

Public key encryption allows people to encrypt and decrypt messages without sharing passwords. Encryption is used in automated banking machine (ABM) security, cell phone security, Internet purchases, smart card applications, and hard disk protection. Its use has allowed Internet commerce to flourish. Dan is a computer programmer who specializes in encryption code. He has a degree in computer science and has taken mathematics courses in combinatorics and set theory. For a debit card, how many four-digit passwords are possible? What about for an eight-character password that includes digits, capital letters, and lower-case letters?

Chapter Problem

Password Encryption

Consider the passwords you use for your bank card, the Internet, and so on. What are the rules for creating them? What must be included? What may be included? What must not be included? What is the probability of someone guessing a password on the first try? How long, on average, do you think it would take for a good password cracking program to break each of your passwords? In this chapter problem, you will analyse passwords and learn how to create a good password.

Prerequisite Skills

Decimals and Fractions

1. Order each list from least to greatest.

a) 0.5, 0.24, 0.718, 0.039

b) 3.78, 3.078, 3.0078

c) $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}$

d) $\frac{5}{8}, \frac{3}{4}, \frac{5}{6}, \frac{7}{12}$

2. Convert to a percent.

> **Example:** To change 1.5 to a percent, multiply by 100.
>
> $1.5 \times 100 = 150\%$
>
> To change $\frac{3}{5}$ to a percent, determine the decimal and multiply by 100.
>
> $3 \div 5 = 0.6$
>
> $0.6 \times 100 = 60\%$

a) 0.275

b) 4.9

c) 125.62

d) $\frac{2}{5}$

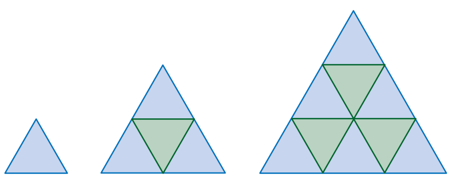
e) $\frac{57}{12}$

3. Find the next number in the sequence $\frac{2}{8}, \frac{7}{12}, \frac{33}{36}, \frac{20}{16}$.

Number Patterns

34 Describe each pattern and extend it for three more terms.

a)

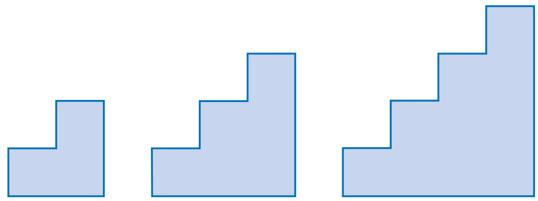


b) 12, 9, 6, 3, …

c) $n - 2, n - 3, n - 4, \ldots$

d) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

5. You can use the diagram to illustrate patterns.



a) Identify at least two number patterns from the diagram.

b) Extend each sequence of numbers by two more terms.

Order of Operations

6. Use the order of operations to evaluate.

a) $(12)(11)(10) - (9)(8)(7)$

b) $\frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} + \frac{6 \times 5 \times 4}{4 \times 3}$

c) $9(9 - 1)(9 - 2)(9 - 3)(9 - 4)$

d) $5^5 - (3^5 + 2^5)$

e) $1 - \left(\frac{2}{3}\right)^4$

f) $\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^3$

Evaluating Expressions

7. Evaluate each expression.

a) $n(n - 1)(n - 2)(n - 3)$ for $n = 6$

b) $(a + 3)(a + 2)(a + 1)$ for $a = 10$

c) $\frac{x(x - 1)(x - 2)(x - 3)(x - 4)}{(x + 1)(x + 2)}$ for $x = 7$

d) $n^r \times m^q$ for $m = 0.4$, $n = 0.6$, $r = 3$, and $q = 4$

8. Evaluate for $n = 5$ and $m = 3$.

a) $n(n - 1)(n - 2)$

b) $(n + 2)(m + 6)$

c) $\frac{n(n - 1)(n - 2)}{m + 1}$

d) $(n + 5)(n + 4)(n + 3) + (m - 1)(m - 2)$

Simplifying Expressions

9. Simplify.

a) $x(x - 1)(x - 2)$

b) $(x + 1)(x + 2) + (x - 1)(x - 2)$

c) $\dfrac{(x + 5)(x + 4)}{x + 4}$

d) $\dfrac{x(x - 1)(x - 2)(x - 3)}{x(x - 1)}$

10. Evaluate each expression as written. Then, simplify the expression before evaluating.

a) $\dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$

b) $\dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 2}$

c) $\dfrac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4}$

d) $\dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$

Probability

11. State the probability of each event.

a) Rolling a 4 on a single die.

b) Rolling a sum of 6 on a pair of dice.

c) Flipping tails with a coin.

d) Flipping two heads with two coins.

e) Selecting a blue ball from a bag containing a red, a blue, a green, a yellow, a brown, and a purple ball.

12. State whether the events are independent. Justify your reasoning.

a) Flipping a coin and rolling a die.

b) Dealing a card to one person and a second card to another person.

c) Rolling two dice.

d) Randomly selecting a date from a calendar. Randomly selecting someone's name from a list.

13. An icosahedron die has 20 faces labelled from 1 to 20. When rolled, what is the probability that the upper face is

a) 3?

b) 4?

c) 3 or 4?

d) even?

e) a prime number?

f) greater than 6?

14. Classify each pair of events as mutually exclusive or non-mutually exclusive.

	Event A	Event B
a)	rolling a 3 with a single die	rolling an even number with a single die
b)	randomly selecting a student with blue eyes	randomly selecting a student with brown hair
c)	selecting a face card from a deck	selecting a numbered card from a deck
d)	selecting a red sweater	selecting a wool sweater
e)	randomly selecting a vowel from the alphabet	randomly selecting "A" or "E" from the alphabet

Drawing Diagrams

15. There are four daily flights available from Waterloo region to Ottawa, and two from Ottawa to Sudbury. There is also one direct flight from Waterloo to Sudbury. Draw a route map and a tree diagram illustrating a passenger's choices to fly from Waterloo to Sudbury.

16. Make a chart and draw a tree diagram illustrating the sums when rolling two standard dice.

2.1

Organized Counting

Learning Goal

I am learning to

- make lists, charts, and tree diagrams to organize counting

Minds On...

A store is offering a promotion on cell phones. Three different models are on sale. Each model is available in two different colours. How many different choices are available?

Action!

Investigate Illustrating and Counting Outcomes

On a TV game show, a contestant must pick one of three doors, labelled 1, 2, and 3. Behind each door are two boxes to choose from. Boxes A and B are behind Door 1, boxes C and D are behind Door 2, and boxes E and F are behind Door 3.

1. Illustrate all the possible outcomes in each of the following ways:
   a) a chart
   b) a set of ordered pairs
2. How many possible outcomes are there?
3. Draw and describe what a tree diagram would look like for this situation.
4. **Reflect** How does the tree diagram show the number of possible outcomes?
5. **Extend Your Understanding** Describe how the tree diagram would change if there were a third set of choices after the contestant selects the box.

Example 1

Illustrating Possible Outcomes

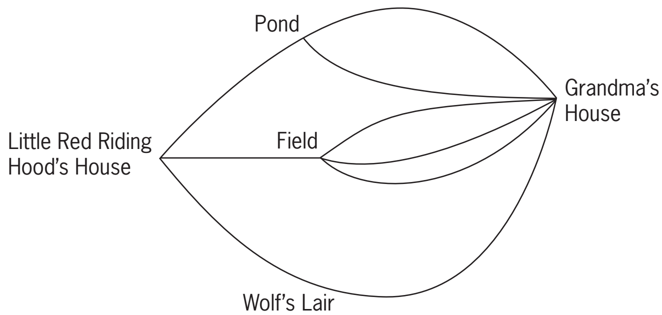
Little Red Riding Hood knows there are three paths from her house: one going to a pond, one going to a field, and one going to the wolf's lair. From the pond, there are two paths to Grandma's house. From the field, there are three paths, and from the wolf's lair there is one path.

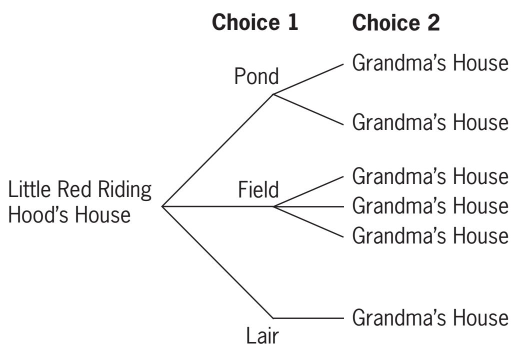
a) Illustrate Little Red Riding Hood's choices with a map and a tree diagram.

b) How many different routes are there for Little Red Riding Hood to take to Grandma's house?

Solution

a)





b) There are six branches in the final stage. So, there are six different routes that Little Red Riding Hood could take.

Your Turn

Abby makes two stops on her way to school. She stops to pick up her friend Regan, and then stops to get breakfast at a coffee shop. There are three routes from her house to Regan's house, and two routes from Regan's house to the coffee shop. There is one route between the coffee shop and school.

a) Make a map and a tree diagram illustrating all the possible routes Abby can take to school.

b) How many possible routes are there?

Example 2

Illustrating Replacement and Non-Replacement of Items

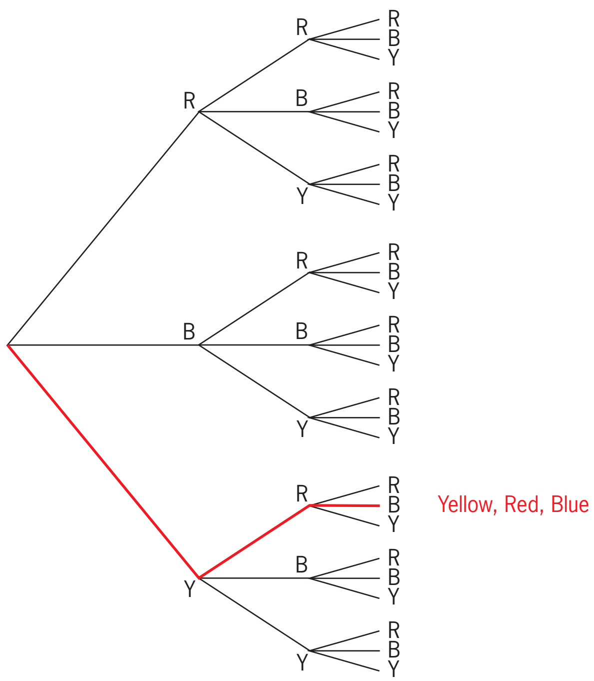
A jar contains a red, a blue, and a yellow ball. A student removes three balls one after the other. Draw a tree diagram to illustrate and count the number of outcomes. Highlight the path illustrating (yellow, red, blue) if

a) the balls are replaced after each selection.

b) the balls are not replaced after each selection.

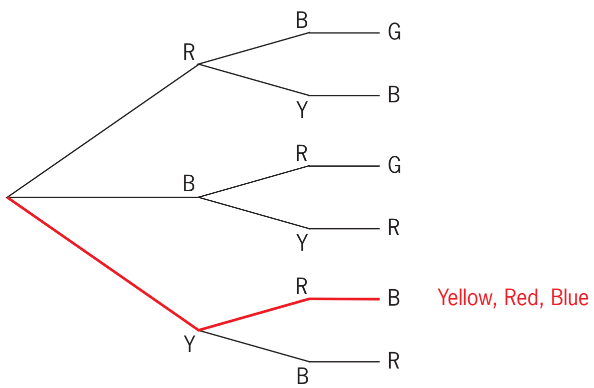
Solution

a) **Selection 1** **Selection 2** **Selection 3**



There are 27 possible outcomes.

b) **Selection 1** **Selection 2** **Selection 3**



Without replacement, there is one fewer choice at each stage.

There are 6 possible outcomes.

Your Turn

A family has four boys, Zach, Dylan, Ben, and Rhys. Each morning, there are four chores to do: set the table, help make breakfast, wash the dishes, and sweep the floor. Each boy does one chore.

a) Make a tree diagram illustrating the different options for doing the chores.

b) Highlight the path that illustrates Zach setting the table, Ben helping make breakfast, Rhys washing the dishes, and Dylan sweeping the floor.

c) What is the total number of different arrangements for doing the chores?

Consolidate and Debrief

Key Concepts

- You can illustrate a sequence of events using multiple methods, including a list, a chart, and a tree diagram.
- In a tree diagram, each stage in the event is illustrated with a new set of branches extending from the end of each branch in the previous stage.
- To identify a given outcome in a tree diagram, read across a distinct path.
- To determine the number of outcomes in a tree diagram, count the number of distinct end paths across the tree diagram.

Reflect

R1. Consider a spinner with four sections. Is a tree diagram an efficient way of illustrating the outcomes of three spins? Explain.

R2. a) Ken shows all the outcomes of rolling two dice using a table of values. Barb uses a tree diagram. Which method do you prefer? Why?

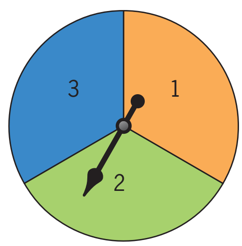
b) When is a chart less efficient than a tree diagram?

Practise

Choose the best answer for #3 and #4.

1. Make a list and a tree diagram illustrating the outcomes on a true or false quiz with five questions. How many different ways are there to answer the five questions?

2. A spinner is divided into three equal sections. Make a tree diagram and state the number of outcomes if the spinner is spun

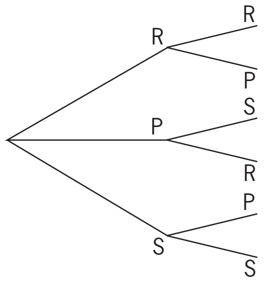


a) twice

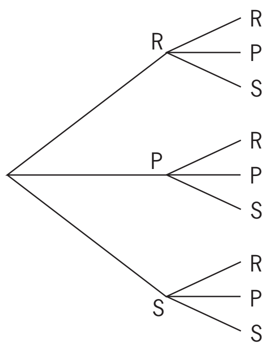
b) three times

3. In the game Rock-Paper-Scissors, two competitors use hand signals to indicate either rock, paper, or scissors. Consider each competitor as a stage. Which tree diagram correctly illustrates the outcomes of the game?

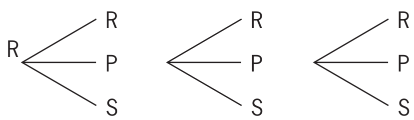
A



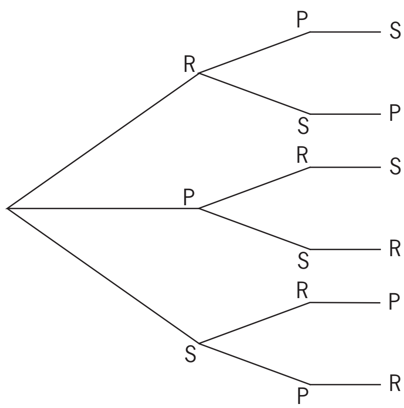
B



C



D



4. A single die is rolled twice. What is the best description of the tree diagram illustrating this event?

   A six stages, two branches per stage, 36 outcomes

   B two stages, six branches per stage, 12 outcomes

   C two stages, six branches per stage, 36 outcomes

   D six stages, six branches per stage, 36 outcomes

Apply

5. Draw a family tree, starting with your parents. Work back four generations to illustrate the number of direct ancestors you have.

6. **Application** In a best-of-five hockey playoff, the winner of the series is declared once a team wins three games. Ties are not allowed.
   a) Draw a tree diagram illustrating all the possible outcomes in a best-of-five series.
   b) List all the outcomes.
   c) How many outcomes are possible?

7. Two people will be chosen to be on a dance committee from Alicia, Benoit, Chantel, Daniqua, Eddie, and Farid.
   a) Make a chart to illustrate this situation.
   b) How many outcomes are possible?
   c) **Communication** How many outcomes are truly different if order is not important? Explain.

8. **Communication** For her probability project, Ming designs a game that requires a player to roll a die and flip a coin. Does it matter whether the die is rolled first or the coin is flipped first? Support your decision with a diagram.

9. A teacher allows his students to write a test up to three times. Draw a tree diagram to illustrate the possible sets of results for a given student. How many different sets of results are there?

10. To get from her house to the shopping mall, Cathie can walk along Main Street, through the park, or along a path out of her subdivision. From Main Street, there are two streets that go to the mall. From the park, there are four streets that go to the mall. From the path, there is only one street that goes to the mall.

   a) Illustrate Cathie's choices with a map and a tree diagram.

   b) How many different routes are there for Cathie to get to the shopping mall?

✓ Achievement Check

11. The jack, queen, king, and ace of diamonds are removed from a standard deck of cards. One card is selected at random from these four cards, returned to the deck, and then another is selected.

   a) Illustrate the outcomes using a list, a tree diagram, or a chart.

   b) Repeat with repetition of cards not permitted.

   c) Explain the difference in the resulting number of outcomes.

12. **Thinking** Pinetree Homes builds new homes with white, grey, or tan siding. The trim is white, black, almond, brown, or grey. Garage doors are painted white, grey, or tan.

   a) How many different colour configurations does Pinetree Homes offer?

   b) Which would increase the number of choices by a greater amount—an additional siding colour or an additional trim colour? Justify your answer.

Processes

Selecting Tools and Computational Strategies
Which method is the best for solving this problem? How did you decide?

13. A coin is flipped five times. How many results are possible in which there are

   a) no consecutive flips of heads or tails?

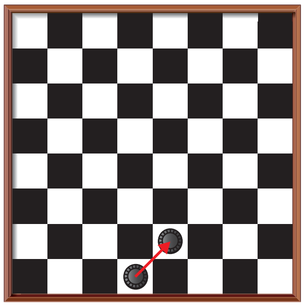
   b) at least two consecutive flips of tails?

   c) two flips of two consecutive tails?

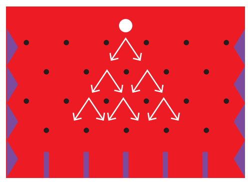
Extend

14. A telephone number has 10 digits and consists of a 3-digit area code, a 3-digit local prefix, and 4 more digits. From her house, Sarah can make local calls to area codes 519 and 226. Within area code 519, there are 80 local prefixes. Within area code 226, there are 39 local prefixes. How many different local phone numbers can Sarah call?

15. A checkerboard is an 8 by 8 grid. You can move a checker diagonally left or right forward one square until it reaches the opposite side of the board. For a checker in the fourth square on the near side of the board, draw a tree diagram to determine the total number of possible moves to the opposite side of the checkerboard.



16. In the game of Plinko, a disc is fed into a slot at the top of a board and can go either left or right as it proceeds down the board, as shown.



   a) How many paths are there to the bottom of the board?

   b) How many paths would there be if the board were extended to five rows?

   c) How many paths would there be if the board were extended to n rows?

2.2

The Fundamental Counting Principle

Learning Goal

I am learning to

- use the fundamental counting principle for counting and to solve problems

Minds On...

There are many different types of random number generators. At their simplest, dice and spinners are used in many board games. On a graphing calculator, randInt($lower$,$upper$) returns a random integer between the lower and upper values.

- How many different outcomes (ordered pairs) are possible when rolling two dice?
- How many different outcomes are possible when spinning the pictured spinner twice?
- How many different outcomes are possible when generating a random number twice on a calculator using randInt(1,100)?
- What operation did you use to do your calculations?

Action!

Investigate Determining the Number of Outcomes

Materials

- 1 die
- 1 coin
- spinner with four equal sections

1. **a)** Roll one die and flip a coin. Make a list or a diagram to illustrate all the possible outcomes.

   b) How many outcomes are possible?

2. Roll one die, flip a coin, and spin a spinner with four equal sections. Without making a list or diagram, determine how many outcomes are possible. Describe your method.

3. Randomly select a number between 1 and 100, and a letter from the alphabet. How many outcomes are possible? How did you know this without counting all the outcomes?

4. **Reflect** Events A and B are independent. Event A has m outcomes. Event B has n outcomes. How many outcomes are possible, in total, if both events occur together?

5. **Extend Your Understanding** Explain why your method in step 4 works.

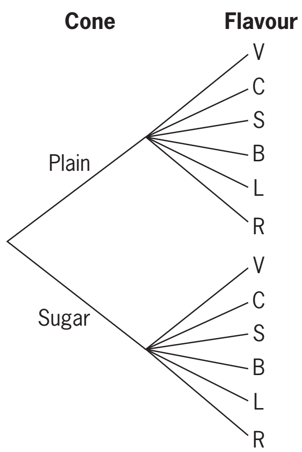
Example 1

Use the Fundamental Counting Principle

At an ice-cream stand, customers have a choice of a plain cone or a sugar cone. There are six choices for ice-cream flavours: vanilla, chocolate, strawberry, butterscotch, lemon, and raspberry. How many different single-scoop ice-cream cones can be made?

Solution

You can use a tree diagram to illustrate the outcomes. The flavours are labelled with their initial letters.



For each of the two types of cone, there are six choices for flavours of ice cream. You can use the **fundamental counting principle**, sometimes called the multiplicative counting rule, to determine the number of outcomes. Multiply to get the total number of choices.

fundamental counting principle

- if one event can occur in m ways and a second event can occur in n ways, then together they can occur in $m \times n$ ways

$2 \times 6 = 12$

There are 12 different ways to make a single-scoop ice-cream cone.

Your Turn

When buying a new smartphone, Li Ming has the following choices:

- 2 GB, 4 GB, or 8 GB of memory
- 64 GB or 128 GB of storage
- 10 colours

How many different configurations of the smartphone are available?

Example 2

Counting Repeated Independent Trials

You roll a standard die. How many outcomes are possible with

a) two rolls? **b)** three rolls?

Solution

A standard die has six faces. For each die, there are six choices for the upper face. Use squares to help visualize the number of trials.

a) | 6 | 6 |

$6 \times 6 = 36$

There are 36 different outcomes with two rolls.

b) | 6 | 6 | 6 |

$6 \times 6 \times 6 = 216$

There are 216 different outcomes with three rolls.

Processes

Representing
Why would a tree diagram be hard to use here?

Your Turn

A password consists of six letters of the alphabet, with repetition permitted.

a) How many different passwords are possible?

b) How many passwords are possible if the letters can be capitals or lower case?

Example 3

Counting Repeated Trials Without Replacement

Two cards are chosen from a standard deck without replacement. How many possible outcomes are there?

Solution

A deck of cards contains 52 cards. So, there are 52 choices for the first card. Since the card is not replaced for the second draw, there are now only 51 cards remaining.

| 52 | 51 |

$52 \times 51 = 2652$

There are 2652 possible outcomes when choosing two cards without replacement.

Your Turn

From a class of 25 students, in how many ways could three of them be selected to attend a workshop—one as a speaker, one as a videographer, and one to take notes?

Consolidate and Debrief

Key Concepts

- If one event can occur in m ways and a second event can occur in n ways, then together they can occur in $m \times n$ ways. This is called the fundamental counting principle.
- The fundamental counting principle can be used for multiple trials. For example, events can occur in $m \times n \times p \ldots$ ways.

Reflect

R1. Johnny wants to go see a movie in Toronto. He is considering four different movies and notices they are each showing at eight different theatres, and at three different times. Johnny thinks he has 15 choices in total, because $4 + 8 + 3 = 15$. Is he right or wrong? Explain.

R2. Explain the fundamental counting principle, using an example to support your explanation.

Practise

Choose the best answer for #5 and #6.

1. Determine the number of possible outcomes when a coin is tossed

a) twice

b) three times

c) four times

d) n times

2. A committee has 15 people.

a) In how many ways could a president and vice president be chosen?

b) In how many ways could a president, vice president, and secretary be chosen?

3. When selecting patio stones, the customer has 10 choices for the type of bricks, 8 choices for colours, and 3 choices for layout. How many choices does the customer have in total?

4. A set, three-course menu in a restaurant allows the customer to select from four appetizers, five main courses, and three desserts.

a) How many options are there?

b) How many choices are there for each option?

c) What is the total number of meal choices for the customer?

5. A computer randomly selects three different numbers from between 1 and 100. In how many ways can this be done?

A 3^{100}

B 100^3

C $100 \times 99 \times 98$

D 3×100

6. On a TV game show, a contestant spins a spinner to randomly select a letter of the alphabet. At the same time, the contestant rolls a standard die. What is the total number of possible outcomes?

   A 32

   B 156

   C 308 915 776

   D 52

7. How many two-digit numbers can be formed from the digits 1, 2, 3, 4, 5 if repetition is

   a) permitted? **b)** not permitted?

Apply

8. How many different outcomes are possible when rolling

   a) two 4-sided dice?

   b) three 4-sided dice?

   c) two 8-sided dice?

   d) four 8-sided dice?

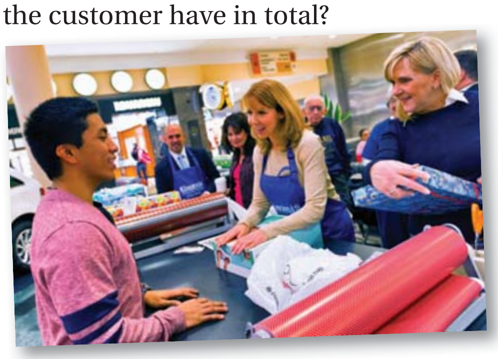
   e) two 12-sided dice?

   f) five 12-sided dice?

   g) k n-sided dice?

9. A business card design software package provides 25 templates, 38 fonts, and 20 colour combinations. How many different business card designs are available to the user?

10. Tonya has a job wrapping gifts during the holiday season. There are five colours of paper, six choices for ribbon, and three choices for bows. How many choices does the customer have in total?



11. In the game of Yahtzee, five dice are rolled. How many outcomes are there for rolling the five dice once?

12. **Application** A radio station plays a winning song once per hour at a randomly selected time (in minutes). During an announcer's four-hour show, how many different arrangements of winning times could occur if

    a) repetition of times during each hour is permitted?

    b) repetition of times during each hour is not permitted?

13. A combination lock uses the numbers from 0 to 59. Three numbers are dialled in the correct sequence. How many unique lock combinations are possible

    a) if repetition is permitted?

    b) if repetition is not permitted?

✓ Achievement Check

14. An eight-character password has been randomly assigned, containing digits and capital and lower-case letters, with repetition permitted.

    a) How many passwords are available in total?

    b) In how many ways could the password begin with four different capital letters, followed by four different digits?

    c) In how many ways could the password contain one digit and seven letters?

15. Licence plates consist of letters and/or digits. Calculate the number of licence plates that could be formed in each province or territory. Assume all numbers and letters are possible.

    a) Ontario, with four letters followed by three digits

    b) Québec, with three letters followed by three digits

    c) Northwest Territories, with six digits

16. Alberta licence plates have three letters followed by four digits. Is this approximately the same number of licence plates as Ontario? Explain without calculating the total number of Alberta plates.

17. When flying from Halifax to Vancouver on his preferred airline, Angus needs to stop over in Toronto. He has a choice of four morning flights to Toronto and six connecting afternoon flights to Vancouver. In how many ways could Angus travel from Halifax to Vancouver via Toronto?

18. **Communication** Will the number of outcomes for the following events be the same or different? Explain.
- A red die, a green die, and a white die are rolled at the same time.
- Three white dice are rolled at the same time.
- A die is rolled three times.

19. a) **Application** Simulate trying to break a security code using a graphing calculator. Think of a three-digit security code. On a graphing calculator, press MATH. From the PRB menu, choose **5: randInt**. Enter randInt(0,9,3) and press ENTER repeatedly until your security code comes up. How does that compare with the total number of possible three-digit security codes?

b) How long do you think it would take to break a five-digit security code?

20. **Thinking** At Triple Pizza, every pizza has three different toppings. Triple Pizza advertises that you can choose from 4080 different pizzas. How could this be so?

21. **Application** Each question on a 10-question multiple choice test has four possible answers. In how many ways could the questions be answered if

a) all questions must be answered?

b) the student is permitted to leave answers blank?

Extend

22. **Thinking** An Ontario licence plate consists of four letters followed by three digits. Plates are assigned in numeric order, then alphabetic order. Assume all letters of the alphabet and all digits can be used. How many licence plates were assigned between the ones shown?

23. If repetition is not permitted, how many even three-digit numbers can be formed from the digits

a) 1, 2, 3, 4, 5, 6?

b) 0, 1, 2, 3, 4, 5?

24. There are three grade 9, five grade 10, six grade 11, and nine grade 12 students on a student council. A committee is being formed with one student from each grade, plus an additional student from either grade 11 or 12. In how many ways could this committee be formed?

25. How many strings of five different letters can be formed from the alphabet if they must begin with a vowel and end with a consonant?

2.3

Permutations and Factorials

Learning Goals

I am learning to

- see how using permutations has advantages over other counting techniques
- solve simple problems using techniques for counting permutations
- write permutation solutions using proper mathematical notation

Minds On...

A game involves placing three different coloured balls under three different cups, then quickly mixing them up. The player must correctly identify the final order of the three colours. In how many different orders could the colours finish? Describe at least two ways to figure this out.

Action!

Investigate The Number of Arrangements of Coloured Blocks

Materials

- 5 different coloured blocks or linking cubes

arrangement

- an ordered list of items

1. Select any three blocks and place the others aside.

a) Use the blocks to make a list or a tree diagram of all the possible arrangements of three blocks in a row.

b) How many arrangements are there?

c) Use the fundamental counting principle to verify your answer in part b).

2. Select any four blocks and place the other aside.

a) Make a list or a tree diagram to show all of the possible arrangements of four blocks in a row.

b) How many arrangements are there?

c) Use the fundamental counting principle to verify your answer in part b).

3. How many arrangements are there with five blocks?

4. **Reflect** Explain a method of calculating the number of arrangements of n different items.

5. **Extend Your Understanding** Use your method to determine the number of arrangements of 15 different items.

Example 1

Evaluating Factorial Expressions

Evaluate each **factorial**.

a) $3!$ **b)** $5!$ **c)** $10!$ **d)** $\frac{6!}{4!}$

factorial
- a product of sequential natural numbers with the form $n! = n(n-1)(n-2) \cdots \times 2 \times 1$
- $n!$ is read "n factorial"

Solution

a) $3! = 3 \times 2 \times 1$
$= 6$

Explore how to use the factorial key on your calculator.

b) $5! = 5 \times 4 \times 3 \times 2 \times 1$
$= 120$

c) $10! = 10 \times 9 \times 8 \times \ldots \times 1$
$= 3\ 628\ 800$

Why do the values increase quickly as the value of n increases?

d) $\frac{6!}{4!} = \frac{6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1}}$
$= 6 \times 5$
$= 30$

Your Turn

Evaluate each factorial.

a) $4!$ **b)** $6!$ **c)** $\frac{11!}{7!}$ **d)** $\frac{6! \times 4!}{5!}$

Permutations of *n* Items

When an arrangement of items needs to appear in order, it is called a **permutation**.

There are n ways of selecting the first item, $n - 1$ ways of selecting the second item, $n - 2$ ways of selecting the third item, and so on, until there is only one way of selecting the last remaining item. Using the fundamental counting principle, multiply these numbers together: $n(n-1)(n-2)(n-3) \ldots \times 3 \times 2 \times 1$, which is n factorial.

The number of permutations of n items is ${}_nP_n = n!$.

permutation
- an arrangement of n distinct items in a definite order
- the total number of these permutations is written ${}_nP_n$ or $P(n, n)$

Example 2

Counting Permutations

A photographer lines up six people. How many different arrangements could she make?

Solution

Since all six people are to be arranged, use a factorial.

$$
\begin{aligned}
6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\
&= 720
\end{aligned}
$$

The photographer can arrange the people in 720 ways.

Your Turn

A half-hour TV show has eight 30-second advertisement time slots. In how many ways could the eight advertisements be assigned a time?

Example 3

Permutations of Some Items in a Set

There are 12 people on a swim team. Four will be chosen to take part in a relay, racing in a given order. In how many ways could the four swimmers be selected?

Solution

Method 1: Use the Fundamental Counting Principle

There are four positions to fill.

12	11	10	9

$12 \times 11 \times 10 \times 9 = 11\ 880$

The positions can be filled in 11 880 ways.

Method 2: Use Factorials

Start with 12! because there are 12 people on the swim team. Divide by the number of arrangements of the non-chosen swimmers, 8!.

$$
\begin{aligned}
\frac{12!}{8!} &= \frac{12 \times 11 \times 10 \times 9 \times \cancel{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}} \\
&= 12 \times 11 \times 10 \times 9 \\
&= 11\ 880
\end{aligned}
$$

The positions can be filled in 11 880 ways.

Your Turn

Forty athletes are entered in a triathlon. Medals are presented to the top three finishers. In how many ways could the gold, silver, and bronze medals be awarded?

Permutations of r Items Out of n Items

The number of permutations of r items from a collection of n items is

$$\begin{aligned} {}_nP_r &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Note that $n \geq r$.

Applying the formula to Example 3,

$$\begin{aligned} {}_{12}P_4 &= \frac{12!}{(12-4)!} \\ &= \frac{12!}{8!} \\ &= 11\ 880 \end{aligned}$$

If your calculator has an ${}_nP_r$ function, explore how to use it.

Processes

Reasoning and Proving
How could you develop the permutation formula algebraically?

Example 4

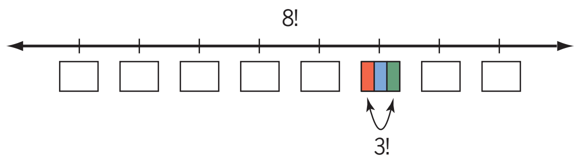
Permutations With Restrictions

A librarian wants to display 10 books by Canadian authors on a bookshelf. There are three books by Joseph Boyden, and the rest are by different authors. In how many ways could he arrange the books if the Joseph Boyden books must remain side-by-side?

Solution

Use the fundamental counting principle. First, arrange the books with the three Joseph Boyden books together. Consider the Joseph Boyden books as a single book with the others. This can be done in 8! ways.

The three Joseph Boyden books can be arranged in 3! ways.



$8! \times 3! = 241\ 920$

The books can be arranged in 241 920 ways.

Your Turn

Six team photos are hanging on the wall outside a high school gym. Two of the photos are of the junior and senior football teams. In how many ways could they be arranged in a straight line if the two football photos must be beside each other?

Project Prep

This example may help you with your probability project. How can you count the results when there is a restriction on the selection of items?

Consolidate and Debrief

Key Concepts

- The number of permutations of n items is n factorial,

  $n! = n(n-1)(n-2)(n-3) \ldots \times 3 \times 2 \times 1$

- You can use factorials as a counting technique when repetition is not permitted.
- The number of r-permutations of n items can be calculated by

$$\begin{aligned} {}_nP_r &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Reflect

R1. Which would have more possibilities, arranging r people from a group of n people **with** regard to order or **without** regard to order? Explain your reasoning.

R2. Use your calculator to determine the value of 0!. Explain why it would have this value. Include an example to support your explanation.

Practise

Choose the best answer for #4 and #5.

1. Evaluate.

a) 9!

b) $\frac{12!}{5!}$

c) ${}_7P_7$

d) ${}_8P_5$

2. Write in factorial form.

a) ${}_6P_4$

b) ${}_{15}P_6$

c) $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

d) $8 \times 7 \times 6 \times 5$

e) $n(n-1)(n-2)(n-3)$

f) $(n+1) \times (n) \times (n-1) \times \ldots \times 3 \times 2 \times 1$

3. Express in the form ${}_nP_r$.

a) 6!

b) $91 \times 90 \times 89 \times 88 \times 87 \times 86$

c) $\frac{18!}{12!}$

4. Which is the correct simplification of $\frac{96!}{24!}$?

A 4!

B 4

C ${}_{96}P_{72}$

D ${}_{96}P_{24}$

5. Which is the correct number of permutations of five items from a list of nine items?

A 126

B 15 120

C 45

D 59 049

6. There are 15 teams competing in a synchronized swimming competition. In how many ways could first, second, and third place be awarded?

7. A club has 18 members. In how many ways could a president, vice president, treasurer, and secretary be elected?

Apply

8. There are 22 players on a baseball team. In how many ways could the batting order of nine players be assigned?

9. Write in simplest factorial form.
   a) $10 \times 9 \times 8 \times 7!$
   b) $99 \times 98 \times 97!$
   c) $90 \times 8!$
   d) $n(n - 1)!$
   e) $(n + 2)(n + 1)n!$

10. **Application** A salesperson needs to visit 15 different offices during the week.
    a) In how many ways could this be done?
    b) In how many ways could she visit four different offices on Monday?
    c) In how many ways could she visit three different offices each day from Monday to Friday?

11. a) How many 10-digit numbers are there with no digits repeated?
    b) How many 7-digit numbers are there with no digits repeated?

12. Caleb needs to create an 8-digit password using only numbers. How many different passwords are there if he wants to use 00 exactly once?

✓ Achievement Check

13. The six members of the student council executive are lined up for a yearbook photo.
    a) In how many ways could the executive line up?
    b) In how many ways could this be done if the president and vice president must sit together?
    c) In how many ways could this be done if the president and vice president must sit together in the middle of the group?

14. How many ways are there to seat six boys and seven girls in a row of chairs so that none of the girls sit together?

15. **Thinking** Twenty figure skaters are in a competition. In the final round, the bottom five competitors skate first in a random order. The next five do likewise, and so on until the top five skate last in a random order. In how many ways could the skating order be assigned?

Extend

16. Solve for n.
    a) ${}_nP_2 = 110$
    b) $P(n, 3) = 5!$

17. Ten couples are being seated in a circle. How many different seating arrangements are there if each couple must sit together?

18. The names of the Knights of the Round Table at Winchester, UK, were engraved on the table, but they are no longer visible. There are 23 knights, plus King Arthur himself. In how many ways could King Arthur and the knights be seated at the Round Table?

19. A double factorial represents the product of all odd, or even, integers up to a given odd number, n. For example, $9!! = 1 \times 3 \times 5 \times 7 \times 9$.
    a) Express $9!!$ as a quotient of factorials.
    b) Express $(2k + 1)!!$ as a quotient of factorials.
    c) Simplify $(2n)!!$, writing it in simple factorial form.

20. Without using a calculator, determine how many zeros occur at the end of $30!$.

2.4

The Rule of Sum

Learning Goal

I am learning to

- use the rule of sum to solve counting problems

Minds On...

Many grade 12 students consider attending university. Sidney is applying to Waterloo for math or engineering, Queen's for physics, chemistry, or engineering, and Laurentian for science or engineering. How many program choices does Sidney have in total?

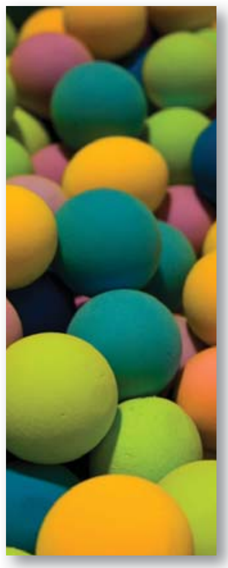
Jenna is applying to Windsor for business or economics, York for commerce or business, and Ottawa for economics, political science, or business. How many program choices does Jenna have in total?

Describe how you calculated the number of choices Sidney and Jenna each have.

Action!

Investigate The Rule of Sum

The fundamental counting principle is used when one event **and** another event occur together. This activity investigates when one event **or** another event occurs.



1. Make a tree diagram of the outcomes for the following activity.

   Imagine you are picking coloured balls out of bins.
   - For your first selection, you can choose a red or a green ball.
   - If your first ball is red, you can choose blue, yellow, or orange for your second ball.
   - If your first ball is green, you can choose white or black.
   - Regardless of the colour of the first or second ball, you can choose orange, pink, or brown for your third ball.

2. How many outcomes are possible if you choose red as your first selection?

3. How many outcomes are possible if you choose green as your first selection?

4. How many outcomes are there in total?

5. What is the relationship that connects your answers to steps 2, 3, and 4?

6. **Reflect** If you know the number of outcomes for each of events A **and** B, how many outcomes are there for event A **or** B?

7. **Extend Your Understanding**
   a) If you are allowed to either roll a die or toss a coin, how many possible outcomes are there?
   b) Compare the results in part a) to the number of outcomes when you roll a die and toss a coin.

Example 1

Use the Rule of Sum

At an international conference, either eight or nine countries may attend. In how many different arrangements could the countries' flags be flown?

Solution

Use the rule of sum because either eight **or** nine countries' flags will be flown.

$8! + 9! = 403\ 200$

The flags could be arranged in 403 200 ways.

Your Turn

At the conference in Example 1,

a) in how many different arrangements could the flags be flown if seven, eight, or nine countries attend?

b) in how many different arrangements could the flags be flown if the host country's flag is always on the far left?

Literacy Link

In chapter 1, you learned about the *rule of sum*, which is also called the additive principle for mutually exclusive events. It states that if one mutually exclusive event can occur in *m* ways, and a second can occur in *n* ways, then one or the other can occur in *m* + *n* ways.

Example 2

Use the Principle of Inclusion and Exclusion

Three players are playing the card game Pass the Ace. Each player receives one card. In how many ways could the cards all be face cards or red cards?

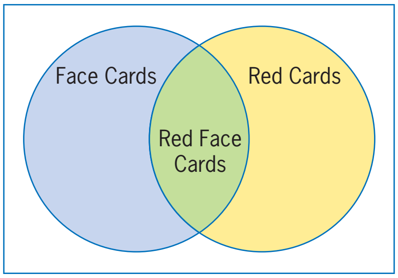
Solution

Because each player receives a different card, order is important.

There are ${}_{12}P_3$ ways to select three face cards.

There are ${}_{26}P_3$ ways to select three red cards.

The events "face cards" and "red cards" are not mutually exclusive, since there are six red face cards, which have been counted twice.



Apply the principle of inclusion and exclusion, $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$, to determine the number of red cards or face cards.

There are ${}_{6}P_3$ ways to deal three red face cards.

$$\begin{aligned}\text{Number of ways to deal three face cards or red cards} &= {}_{12}P_3 + {}_{26}P_3 - {}_{6}P_3 \\ &= 1320 + 15\,600 - 120 \\ &= 16\,800\end{aligned}$$

There are 16 800 ways to deal three face cards or three red cards.

Your Turn

Three players each cut one card from a standard deck. If order is important, in how many ways could they be

a) all hearts?

b) all aces?

c) all aces or hearts?

Example 3

Use the Indirect Method

The 12 members of a basketball team are lining up for their medals after a tournament. In how many ways can this be done

a) if there is no restriction?

b) if the captain and assistant captain must be together?

c) if the captain and assistant captain must not be together?

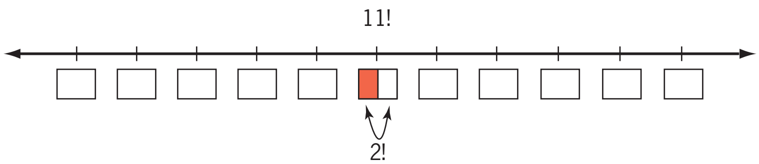
Solution

a) With no restriction, it is a permutation of all 12 players.

$12! = 479\ 001\ 600$

They could line up in 479 001 600 ways.

b) Treat the captain and assistant captain as a single person, making 11 players. Then arrange the two of them among themselves.



$11! \times 2! = 79\ 833\ 600$

They could line up in 79 833 600 ways.

c) There are numerous positions in which the captain and assistant captain could be apart, so the solution is the sum of many different scenarios. You can use the **indirect method** to subtract the number of ways they are together from the total without restrictions. In other words, part a) minus part b).

$$\begin{aligned}\text{Total} - \text{together} &= 12! - 11! \times 2!\\ &= 479\ 001\ 600 - 39\ 916\ 800 \times 2\\ &= 479\ 001\ 600 - 79\ 833\ 600\\ &= 399\ 168\ 000\end{aligned}$$

They can be apart in 399 168 000 ways.

Your Turn

In how many ways could the letters in the word FACTOR be arranged so that the vowels are not together?

How is the indirect method is similar to the probability of a complement that was introduced in chapter 1?

indirect method

- subtract the number of unwanted outcomes from the total number of outcomes without restrictions

Processes

Connecting
How could you determine the number of girls in a class?

Directly: count the girls

Indirectly: subtract the number of boys from the total number in the class

Consolidate and Debrief

Key Concepts

- The rule of sum states that if one mutually exclusive event can occur in m ways, and a second can occur in n ways, then one **or** the other can occur in $m + n$ ways.
- If two events are not mutually exclusive, the principle of inclusion and exclusion needs to be considered:

  $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$.
- To reduce calculations, consider using the indirect method, which involves subtracting the unwanted event from the total number of outcomes in the sample space: $n(A) = n(S) - n(A')$.

Reflect

R1. A student council has five executive positions: president, vice president, secretary, treasurer, and assistant treasurer. There are six female and seven male candidates. It is important that at least one male and one female be on the executive. Explain why the indirect method is useful in determining the number of possible outcomes.

R2. Write a general guideline explaining when to use the fundamental counting principle and when to use the rule of sum for permutations. Include a similar example for each.

Practise

Choose the best answer for #3 and #4.

1. Determine the total number of arrangements of three or four toys from a basket of eight different toys.

2. a) How many ways are there to roll a sum of 7 or 11 on two dice?

b) How many ways are there to roll doubles or a sum divisible by three on two dice?

3. A game has players roll either one or two standard dice. Which is the total number of possible different outcomes?

A 42

B 36

C 18

D 12

4. Which is the total number of arrangements of the digits 1, 2, 3, 4, 5, if the even digits must not be together?

A 120

B 24

C 48

D 72

Apply

5. a) How many even numbers can be formed from the digits 1, 2, 3, 4, 5?

b) How many of these numbers are greater than 3000?

6. **Application** A motorcycle licence plate consists of two or three letters followed by four digits. How many licence plates can be made?

7. A security code consists of either five or six different letters. How many distinct security codes are possible?

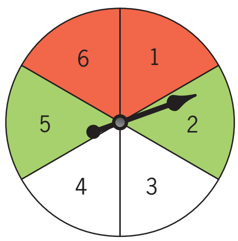
8. **Communication** Suppose a country has a rule that a newborn child may have either one, two, or three names.
   a) If parents were to choose from a list of 50 names, how many choices would they have when naming their child?
   b) What if they could choose from 100 names?
   c) Explain why the total in part b) is more than 2 times the total in part a).

9. **Open Question** Five speakers, P, Q, R, S, and T, have been booked to address a meeting.
   a) In how many ways could the speakers be ordered if speaker P must go before speaker Q?
   b) Make up your own problem about these five speakers. Solve it, share it with a classmate, and check his or her solution.

10. How many five-digit numbers include the digits 4 or 6 or both?

11. Ten names are placed into a hat. In how many ways could they be pulled from the hat so they are not in alphabetical order?

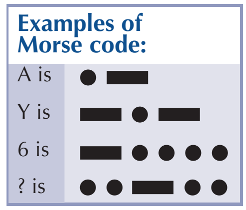
12. A spinner has six equally spaced sections numbered 1 to 6 as shown. You spin the spinner four times.



   a) In how many ways could the spinner result in the same colour on all four spins?
   b) In how many ways could the spinner result in an even number or the same colour on all four spins?

13. **Thinking** In the game of Monopoly, you can get out of jail by rolling doubles. If you are unsuccessful on the first roll you may try again, up to a total of three attempts. In how many ways could this occur? Explain your solution.

14. **Communication** Morse code uses dots and dashes to represent letters, digits, and eight punctuation symbols. Use the fundamental counting principle and the rule of sum to help explain why a maximum of six characters is needed.



✓ Achievement Check

15. A password must be 6, 7, or 8 characters long, and may include capital letters, lower-case letters, or digits. In how many ways could this be done
    a) with no restriction?
    b) with no repetition permitted?
    c) if at least one of the characters in part a) must be a digit?

Extend

16. How many different numbers can be formed by multiplying some or all of the numbers 2, 3, 4, 5, 6, 7, 8?

17. A derangement is a permutation of a set of numbers in which no item remains in its original position. For the set $\{1, 2, 3\}$, the derangements are $\{2, 3, 1\}$ and $\{3, 1, 2\}$. The permutation $\{1, 3, 2\}$ is not a derangement because 1 is in its original position. Determine the number of derangements of each set.
    a) $\{1, 2, 3, 4\}$
    b) $\{1, 2, 3, 4, 5\}$

18. The labels from six different cans of soup have come off. If you were to replace them at random, in how many ways could this be done so that
    a) none of the cans will be labelled correctly?
    b) at least one of the cans will be labelled correctly?
    c) all of the cans will be labelled correctly?

2.5

Probability Problems Using Permutations

Learning Goal

I am learning to

- solve probability problems using counting principles for situations with equally likely outcomes

Minds On...

In chapter 1, you investigated simple probabilities. Match each event with its appropriate description.

Event	Description
hoping for a king to be dealt, given that a king has already been dealt	Independent
Jonathan Toews has a 17.6% probability of scoring	Dependent
two rolls of a single die	Mutually exclusive
a number is even, a number is odd	Complements
dealing three cards from a standard deck	Statistical
the results on a green die or a red die	Rule of sum
the results on a green die and a red die	Fundamental counting principle
a card is a heart, a card is a spade	Mutually exclusive

Describe a different event that would be categorized by
- independent events
- dependent events
- mutually exclusive events
- complements

Action!

In this section, you will be extending probability to more complex problems. It will be important to list or describe the event and the sample space and then calculate the number of elements in each. Multiple choice tests are used not only by teachers, but also by testing sites for things like driver's licences. How would probability play a role in multiple choice tests?

Example 1

Independent Trials

Software for generating multiple choice tests randomly assigns A, B, C, or D as the correct answer. On a 10-question test, what is the probability that all 10 questions have C as the correct answer?

Solution

Method 1: Use the Theoretical Probability

There are four choices for each correct answer. There are 10 questions, so there are 10 trials.

$$\begin{aligned} n(S) &= 4^{10} \\ &= 1\,048\,576 \end{aligned}$$

There is only one successful outcome: all Cs. So, $n(A) = 1$.

$$\begin{aligned} P(\text{all Cs}) &= \frac{n(A)}{n(S)} \\ &= \frac{1}{1\,048\,576} \end{aligned}$$

The probability that all 10 questions have C as the correct answer is $\frac{1}{1\,048\,576}$.

Method 2: Use the Independent Event Probability

These trials are independent, so the probability of a C on each question is $\frac{1}{4}$.

$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

There are 10 trials, so

$$\begin{aligned} P(\text{all Cs}) &= \left(\frac{1}{4}\right)^{10} \\ &= \frac{1}{1\,048\,576} \end{aligned}$$

The probability that all 10 questions have C as the correct answer is $\frac{1}{1\,048\,576}$.

> **Processes**
>
> **Selecting Tools and Computational Strategies**
> How are these strategies the same? How are they different?

Your Turn

A street illusionist asks five people to each secretly write a number between 1 and 100 on a card. Incredibly, they all write the same number.

a) What is the probability of this occurring?

b) Relate your answer to part a) to the probability of rolling a six on a standard die five times in a row.

Example 2

Dependent Trials

Eight people on a waiting list for advance tickets to a concert have been selected to choose their seats. What is the probability they will have been notified in order from youngest to oldest?

Solution

Method 1: Use Factorials

The trials are dependent, since a person cannot be selected more than once.

You can use factorials.

$n(S) = 8!$

There is only one successful outcome, the single order from youngest to oldest, so $n(A) = 1$.

$$\begin{aligned} P &= \frac{1}{8!} \\ &= \frac{1}{40\,320} \end{aligned}$$

The probability of notifying the people in order from youngest to oldest is $\frac{1}{40\,320}$.

Method 2: Use Individual Probabilities

Since the people are notified without replacement, the events are dependent.

The probability of selecting the youngest first is $\frac{1}{8}$.

The probability of notifying the next youngest on the second choice is $\frac{1}{7}$.

Following a similar process for all eight people gives the probability of notifying people in order from youngest to oldest of

$$\begin{aligned} P &= \frac{1}{8} \times \frac{1}{7} \times \frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1 \\ &= \frac{1}{40\,320} \end{aligned}$$

The probability of notifying the people in order from youngest to oldest is $\frac{1}{40\,320}$.

Your Turn

Four students, one from each of grades 9, 10, 11, and 12, line up to pose for a photograph. What is the probability that they will be in order of their grades?

Example 3

Ordered Selections

Logan selects three cards in order, without replacement, from a standard deck. What is the probability that he selects a king, then two queens?

Solution

Since the Logan selects the cards without replacement, the trials are dependent.

$n(S) = {}_{52}P_3$

Select one king from 4 and two queens from 4.

$n(A) = {}_4P_1 \times {}_4P_2$

$$P(\text{king, queen, queen}) = \frac{{}_4P_1 \times {}_4P_2}{{}_{52}P_3}$$
$$= \frac{4 \times 12}{132\,600}$$
$$= \frac{48}{132\,600}$$
$$= \frac{2}{5525}$$

The probability of selecting a king and then two queens is $\frac{2}{5525}$.

Processes

Representing

Why does $\frac{4}{52} \times \frac{4}{51} \times \frac{3}{50}$ give the same answer?

Your Turn

Kylie selects five cards.

a) What is the probability that she selects three aces followed by two jacks?

b) What is the probability that Kylie selects two hearts followed by three clubs?

Example 4

The Birthday Problem

There are 30 students in Wayne's class.

a) What is the probability that no two people have the same birthday?

b) What is the probability that at least two students share the same birthday?

Solution

a) There are 365 days in one year. So, for the sample space, each of the 30 students has 365 choices for their birthday. There are 30 trials, so $n(S) = 365^{30}$.

If everyone must have a different birthday, the successful outcomes are dependent and 30 days are selected from 365 days.

So, $n(A) = {}_{365}P_{30}$.

$$P(\text{all different}) = \frac{{}_{365}P_{30}}{365^{30}}$$
$$\approx 0.2937$$

The probability that no two people have the same birthday is approximately 0.2937.

b) Use the indirect method, $P(A) = 1 - P(A')$.

$P(\text{at least two the same}) = 1 - P(\text{no two the same})$

$$n(A') = {}_{365}P_{30}$$
$$P(A) = 1 - \frac{{}_{365}P_{30}}{365^{30}}$$
$$\approx 0.7063$$

Why would the probability be this great with only 30 people?

It is likely someone in the classroom has the same birthday as you?

The probability that at least two students have the same birthday is approximately 0.7063.

Your Turn

From a group of 16 people, what is the probability that

a) none share a birthday?

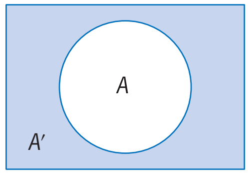
b) at least two of them share the same birthday?

Consolidate and Debrief

Key Concepts

- You can calculate the probability of an event using $P = \frac{n(A)}{n(S)}$, where $n(A)$ is the number of successful outcomes and $n(S)$ is the number of outcomes in the sample space.
- If the trials are dependent, you can use permutations in the calculations.
- To use the indirect method, subtract the probability of the complement from 1.

  $P(A) = 1 - P(A')$



$P(A) = 1 - P(A')$

Reflect

R1. A tour guide bet her group of 27 tourists that at least two people have the same birthday. Should they accept the wager? Explain.

R2. Describe the clues you should look for to identify that a probability problem involves permutations. Include an example.

R3. How is ${}_{12}P_3$ different from ${}_{12}P_1 \times 3$? Use examples to help distinguish between the two.

Practise

Choose the best answer for #3 and #4.

1. Three cards are drawn from a deck without replacement. What is the probability that they will be a king, a queen, and a jack, in that order?

2. Abby, Chantral, Dougie, Kajan, Minh, and Zara are all in a race and are considered to be equally fast. What is the probability that Abby and Chantral will be the first two finishers?

3. Five names are selected at random from a list of 25 names. What is the probability that they will be in alphabetical order?

A $\frac{1}{{}_{25}P_5}$ **B** $\frac{5!}{{}_{25}P_5}$

C $\frac{1}{25^5}$ **D** $\frac{5}{25!}$

4. A standard die is rolled four times. What is the probability that it shows a number divisible by three all four times?

A $\frac{1}{3}$ **B** $\frac{1}{6}$

C $\frac{1}{81}$ **D** $\frac{1}{12}$

Apply

5. There are 15 numbered balls on a pool table. What are the odds against them falling in order from 1 to 15? Remember, odds against is the ratio $P(A'):P(A)$.

6. **Communication** A charity lottery uses a random number generator to choose three different days from the calendar. These are days on which grand prizes will be awarded.

a) What is the probability that all three days fall in the month of April?

b) Explain the method you used.

7. In the game of backgammon, when you roll doubles with two dice you can double the total on the dice.
   a) What is the probability of rolling doubles?
   b) What is the probability of rolling doubles on two consecutive rolls?
   c) Which has a greater probability: rolling consecutive doubles, or rolling consecutive sums of 7 on two rolls of the dice?

8. What is the probability that a family has all boys, in a family of
   a) 3 children?
   b) 4 children?
   c) 5 children?
   d) n children?

9. A four-letter word jumble is being formed from the letters in the word LOGARITHM.
   a) What is the probability it spells MATH?
   b) What is the probability it includes the letters M, A, T, and H?
   c) What is the probability it includes the letter M?

10. **Application** One card is dealt from a standard deck to each of seven players.
    a) What is the probability that the cards are dealt in ascending order?
    b) What is the probability that none of the cards are of the same denomination?

11. What is the probability that two or more people in a party with 20 people will have the same birthday?

12. **Thinking** How many students are needed in a class for the probability of the "birthday problem" in Example 4 on page 92 to reach 0.5?

13. Your MP3 player is set to random and will play 10 of your favourite songs. What is the probability that
    a) the songs are played in your order of preference?
    b) your two favourite songs are first and second?

14. Five people each choose a card from a standard deck. They replace the card after making their choice.
    a) What are the odds against at least two people choosing the same card?
    b) What are the odds against at least two people choosing the same denomination?

Processes

Selecting Tools and Computational Strategies
How does the tool you selected help you understand the birthday problem?

15. To simulate the results of the birthday problem in Example 4 on page 92, use either a spreadsheet or a graphing calculator to generate 30 random integers between 1 and 365. Repeat this 10 times and determine the number of classes in which at least two people share the same birthday.
    - In a spreadsheet, enter **=randbetween(1,365)** in cell A1. Fill down to cell A30. To identify the most frequent number, or mode, in cell A31, enter **=mode(A1:A30)**.
    - Using a graphing calculator, press STAT then select **1:Edit**. Place the cursor in the heading for list **L1**. Press MATH then select **PRB** and **5:randInt(1,365,30)**. Press ENTER. Sort the data by pressing STAT, then **2:SortA(L1)** and ENTER. Press STAT, then **1:Edit**. Scroll down to see which numbers repeat.

16. a) Find the probability of cracking a combination lock on a safe if five different numbers are used from
       i) 1 to 35
       ii) 1 to 40
       iii) 1 to 45
    b) **Communication** Compare the results and explain the differences.

✓ Achievement Check

17. On a TV game show, the contestant is asked to pick one of three doors. Behind each door are two large boxes to choose from. The grand prize is in one of the boxes behind Door 1. There are good prizes in one box behind Door 2 and one box behind Door 3. The other boxes all contain gag prizes.

a) Make a tree diagram showing the possible outcomes.

b) Assign a probability to each branch in the tree diagram.

c) What is the probability of winning the grand prize?

d) What is the probability of winning a good prize?

e) What is the probability of winning a gag prize?

f) What is the sum of all the probabilities? Explain the result.

18. Which of the following scenarios could be modelled using the "birthday problem"? Solve it using the appropriate techniques.

- The probability that at least two people receive hearts when each of six people are dealt five cards.
- The probability that at least two people roll double sixes, from a group of 10 people.
- The probability that at least two people, from a group of 25 people, have the same birthday as you.

19. **Thinking** Which is more likely?

a) Throwing a sum of 7, or not throwing a sum of 7, on six consecutive rolls of a pair of dice.

b) Five different digits being arranged in descending order, or three different letters being arranged in alphabetical order.

c) At least two out of 20 friends having the same birthday, or at least two out of five friends having the same birth month.

Extend

20. A lottery ticket contains five numbers chosen from the numbers 1 to 40. The winning ticket is the one that matches all five numbers in the correct order. The second prize winner matches four of the five numbers in the correct order. What is the probability of winning the first or second prize?

21. **Open Question** Create your own probability example that has $\frac{1}{{}_{15}P_7}$ as its solution. Provide a rationale.

22. A computer screen is divided into a 16 by 9 grid with grid points defined by ordered pairs, using whole numbers, from (0, 0) to (16, 9). A segment is drawn joining two randomly chosen points.

a) What is the probability the segment is horizontal?

b) What is the probability the segment is on one of the screen's diagonals?

23. A game involves making a 3 by 3 grid with nine cards from a standard deck. You win if three cards in a row (horizontally, vertically, or diagonally) are the same denomination or are consecutive (in any order).

a) What is the probability that there is exactly one winning set of the same denomination?

b) What is the probability that there is exactly one winning set of consecutive cards?

24. In the card game, Six in a Row, six cards are dealt in a row. Points are given for the number of consecutive cards. What is the probability that the six cards are

a) consecutive and in order (e.g. 4, 5, 6, 7, 8, 9)?

b) consecutive, but in any order?

Chapter 2 Review

Learning Goals

Section	After this section, I can
2.1	• make lists, charts, and tree diagrams to organize counting
2.2	• use the fundamental counting principle for counting and to solve problems
2.3	• see how using permutations has advantages over other counting techniques • solve simple problems using techniques for counting permutations • write permutation solutions using proper mathematical notation
2.4	• use the rule of sum to solve counting problems
2.5	• solve probability problems using counting principles for situations with equally likely outcomes

2.1 Organized Counting, pages 64–69

1. Draw a tree diagram showing all the possible outcomes (win, loss, or tie) in three games between two hockey teams. How many possible outcomes are there?

2. Octahedral dice have eight faces. Make a chart showing the sums of the faces on two dice. Which sum occurs most frequently? Which sums occur least frequently?

3. The heart honour cards (10, J, Q, K, A) are removed from a standard deck. Three cards are randomly selected from the heart honour cards without replacement.

a) Illustrate all the possible outcomes using a tree diagram.

b) Highlight the path that indicates the run queen of hearts, king of hearts, ace of hearts).

c) How many possible outcomes are there?

2.2 The Fundamental Counting Principle, pages 70–75

4. A home security code requires five digits to be entered on a keypad.

a) How many distinct security codes are possible?

b) Sarah reset her security code but has forgotten it. If it takes her eight seconds per attempt, what is the maximum time it would take for her to find the correct code?

5. When ordering a gaming computer online, Ryan has three choices for processors, four choices for size of RAM, five choices for the video card, three choices for the hard drive, and two choices for the sound card.

a) How many choices does Ryan have when configuring his computer?

b) If there were an additional choice for the video card, how would it affect the total number of choices? Explain the difference.

6. Barb knits socks for a charity supporting homeless and low income people. She likes to make striped socks and selects from six different colours. The top stripe can be any colour. The second stripe may not match the first colour. The bottom colour may not match the second, but may match the first. How many distinct pairs of socks could Barb make?

2.3 Permutations and Factorials, pages 76–81

7. How many ways are there for a company to assign three different jobs to three of its five employees?

8. a) Evaluate each permutation and place them in the array as shown.

$$\begin{array}{ccccccccc} & & & & {}_1P_1 & & & & \\ & & & {}_2P_1 & & {}_2P_2 & & & \\ & & {}_3P_1 & & {}_3P_2 & & {}_3P_3 & & \\ & {}_4P_1 & & {}_4P_2 & & {}_4P_3 & & {}_4P_4 & \\ {}_5P_1 & & {}_5P_2 & & {}_5P_3 & & {}_5P_4 & & {}_5P_5 \end{array}$$

b) Extend the array by one row without using factorials. Explain how you did this.

c) Identify and describe two other patterns in the array.

9. A bookstore clerk is arranging seven novels, four plays, and five poetry books in a display case. Each type of book remains in its own group, but the groups can be in any order. In how many ways could she arrange the books?

2.4 The Rule of Sum, pages 82–87

10. In how many ways could the letters in the word STORAGE be arranged if the vowels must remain in

a) even positions?

b) odd positions?

c) even or odd positions?

11. In how many ways can you rearrange the letters in the word NATIVE if the vowels must not be together?

12. How many five-digit even numbers can be formed using all the digits 0, 1, 2, 3, and 4?

2.5 Probability Problems Using Permutations, pages 88–95

13. For a gift exchange, 10 people's names are written on slips of paper and placed in a bowl. The slips of paper are mixed up, and each person selects one name.

a) What is the probability that everyone selects their own name?

b) What is the probability that nobody selects his or her own name?

14. Six different coloured balls are placed in a box. Kendra and Abdul each select a ball without replacement.

a) What is the probability that Kendra selects the green ball and Abdul selects the red ball?

b) What is the probability that Kendra selects the green ball and Abdul does not select the red ball?

c) What is the probability that Kendra does not select the green ball and Abdul does not select the red ball?

15. If five people each select a letter from the alphabet with repetition permitted, what is the probability that they are

a) all the same?

b) all different?

Chapter 2 Test Yourself

✓ Achievement Chart

Category	Knowledge/ Understanding	Thinking	Communication	Application
Questions	1, 2, 3, 4, 7, 8	11, 13, 14	4, 14	5, 6, 9, 10, 12

Multiple Choice

Choose the best answer for #1 to #3.

1. How many orders of faces are possible when a standard die is rolled four times?

A 16

B 24

C 1296

D 4096

2. Which of the following is equivalent to ${}_{101}P_{98}$?

A 3!

B $101 \times 100 \times 99 \times 98$

C $\dfrac{101!}{98!}$

D $\dfrac{101!}{3!}$

3. When flipping a coin five times, what is the probability that heads turns up every time?

A $\dfrac{1}{32}$

B $\dfrac{5}{32}$

C $\dfrac{1}{10}$

D $\dfrac{1}{25}$

4. Which of the following is not defined? Explain your reasoning.

- ${}_{12}P_8$
- ${}_{9}P_{10}$
- ${}_{7}P_0$
- ${}_{100}P_{100}$

Short Answer

5. Rosa is getting dressed and has decided that her shirt, pants, and socks are not to be the same colour. She has red, green, black, and blue of each.

a) Draw a tree diagram illustrating her choices.

b) How many choices does she have if she starts with a red pair of pants?

6. A hockey team has four left wingers, three right wingers, four centres, three left defence, four right defence, and two goalies. To create a starting lineup, a coach needs one player in each position. In how many ways could the starting lineup be chosen?

7. How many ways are there to assign five different roles in a play to the 12 members of a drama club?

8. There are three Canadians in the finals at a ski competition. Assuming all eight competitors are equally likely to win, what is the probability that the three Canadians will win gold, silver, and bronze?

Justine Dufour-Lapointe

Extended Response

9. a) How many arrangements are there of the letters in the word COMPUTER?
   b) How many of them begin with a consonant?

10. In how many ways could the 11 members of a soccer team line up if the captain and assistant captain must remain apart?

11. There are 25 men and 20 women who belong to a club. An executive panel consisting of a president, vice president, secretary, and treasurer is being chosen.
    a) In how many ways could the executive panel be chosen with no restrictions?
    b) In how many ways could the executive panel be chosen if it must include at least one woman and one man?
    c) In how many ways could the executive panel be chosen if the president and vice president must have different genders?

12. Four letters are randomly selected from the alphabet. What is the probability that they are A, B, C, and D, in that order,
    a) if repetition is permitted?
    b) if repetition is not permitted?

13. Ten people each randomly select a number between 1 and 20. What is the probability that at least two of them select the same number?

14. To determine who should be the first dealer in a card game, one card is dealt to each of five players. The player with the card of the highest denomination gets to deal first.
    a) How many different results are possible when dealing to the five players?
    b) In how many ways could all players receive cards of different denominations?
    c) What is the probability that four players receive cards of the same denomination?
    d) How would the solution to part c) change if players each chose a card from a full deck instead of being dealt one?

Chapter Problem

Password Encryption

Consider four passwords you use, for bank cards, websites, and so on.

a) What are the rules for each? What must be included? What may be included? What must not be included?

b) What is the probability of someone guessing each password on the first try?

c) At 90 000 codes per second, how long, on average, would it take for a good password cracking program to break each of your passwords?

d) Some passwords require at least one digit and at least one capital letter. Why?

e) Develop a set of guidelines to identify a good, poor, or average password. Back it up with examples and calculations.