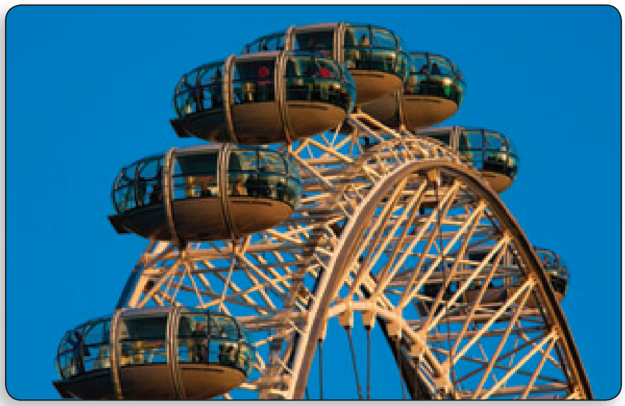
Chapter 4

Derivatives of Sinusoidal Functions



The passengers in the photo are experiencing a phenomenon that can be modelled as periodic motion, or motion that repeats itself on a regular interval. The world around us is filled with phenomena that are periodic in nature. They include rhythms of the daily rotation of the Earth, the seasons, the tides, and the weather or the body rhythms of brain waves, breathing, and heartbeats. Periodic behaviour and sinusoidal functions are also involved in the study of many aspects of physics including electricity, electronics, optics, and music.

All of these situations can be modelled by sinusoidal functions; that is combinations of the basic periodic functions, the sine and cosine functions. An advanced theorem in calculus states that everything periodic can be expressed as an algebraic combination of sine and cosine curves.

In this chapter you will explore the instantaneous rates of change of sinusoidal functions and apply the rules of differentiation from earlier chapters to model and solve a variety of problems involving periodicity.

By the end of this chapter, you will

- determine, through investigation using technology, the graph of the derivative $f'(x)$ or $\frac{dy}{dx}$ of a given sinusoidal function
- solve problems, using the product and chain rules, involving the derivatives of polynomial functions, sinusoidal functions, exponential functions, rational functions, radical functions, and other simple combinations of functions
- solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate the results

Prerequisite Skills

Angle Measure and Arc Length

1. Use the relationship π rad $= 180°$ to express the following angle measures in radian measure.

a) $360°$

b) $90°$

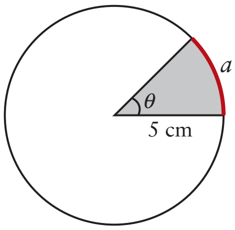
c) $-45°$

d) $29.5°$

e) $115°$

f) $240°$

2. Use the relationship $a = r\theta$ to find the arc length, a, in centimetres, subtended by an angle, θ, in radians, of a circle having a radius, r, of 5 cm.



a) π rad

b) 2.0 rad

c) $60°$

d) $11.4°$

e) $\dfrac{\pi}{2}$ rad

f) $173°$

Trigonometric Functions and Radian Measure

3. Graph each function over the interval $-2\pi \le x \le 2\pi$.

a) $y = \sin x$

b) $y = 4\cos x$

4. For each graph in question 3, determine the amplitude and period of the function.

5. **Use Technology** **a)** Without graphing, describe how the graph of $f(x) = \cos x$ can be transformed to produce the graph of $y = 3f(2x)$.

b) For the function $y = 3f(2x)$, determine

i) the minimum value of the function

ii) the maximum value of the function

c) Refer to part b). Use set notation to describe the points at which $y = 3f(2x)$ is

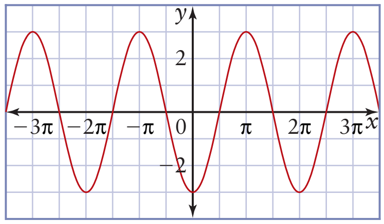
i) a maximum

ii) a minimum

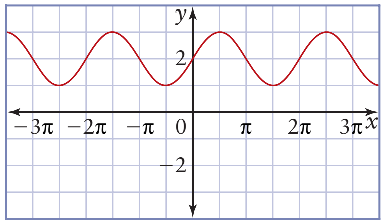
d) Use a graphing calculator to graph both functions and verify your answers.

6. Consider the graphs shown here.

A



B



For each function:

a) Determine the maximum and minimum values.

b) Determine one possible equation of a function corresponding to the graph.

c) Explain whether the function you produced in part b) is the only possible solution. If not, produce another function with the same graph.

CONNECTIONS

The reciprocal trigonometric functions are

$\csc x = \dfrac{1}{\sin x}$ $\quad \sec x = \dfrac{1}{\cos x}$ $\quad \cot x = \dfrac{1}{\tan x}$

7. Use special triangles to express the following as exact values.

a) $\sin\left(\dfrac{\pi}{3}\right)$

b) $\cos\left(\dfrac{\pi}{4}\right)$

c) $\sin\left(\dfrac{\pi}{2}\right) + \cos\left(\dfrac{\pi}{3}\right)$

d) $\sin^2\left(\dfrac{\pi}{4}\right) - \sin\left(\dfrac{\pi}{6}\right)$

e) $\sec\left(\frac{\pi}{4}\right)$

f) $\cot\left(\frac{\pi}{2}\right)$

g) $\csc\left(\frac{\pi}{3}\right)$

h) $\sec^2\left(\frac{\pi}{4}\right)$

Differentiation Rules

8. Differentiate each function with respect to the variable indicated.

a) $y = 5x + 7$

b) $y = -2x^3 + 4x^2$

c) $f(t) = \sqrt{t^2 - 1}$

d) $f(x) = 2x^{-2}\sqrt{x - 3}$

9. Let $f(x) = x^2$ and $g(x) = 3x + 4$. Find the derivative of each function.

a) $f(g(x))$

b) $g(f(x))$

c) $f(f(x))$

d) $f(x)g(x)$

Applications of Derivatives

10. Find the slope of the line tangent to the curve $y = -3x^2 + 5x - 11$ at $x = -4$.

11. Find the equation of the line tangent to the curve $y = \frac{x^2}{2} + 6x$ at $x = -2$.

12. Determine the coordinates of all local maxima and minima for the function $y = x^3 + 5x^2 + 3x - 3$ and state whether each is a local maximum or minimum.

Trigonometric Identities

13. Replace x and y with either $\sin a$ or $\cos a$ to complete the following identities.

a) $\sin(a + b) = x(\cos b) + y(\sin b)$

b) $\sin(a - b) = x(\cos b) - (\cos a)(\sin b)$

c) $\cos(a + b) = y(\cos b) - x(\sin b)$

14. Prove each identity.

a) $\sin^2\theta = 1 - \cos^2\theta$

b) $\tan(-\theta)\cos(-\theta) = -\sin\theta$

c) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

d) $\cot\theta = \frac{\csc\theta}{\sec\theta}$

15. Simplify each expression so that no denominators remain.

a) $\frac{\sin x}{\tan x}$

b) $\frac{1 - \sin^2\theta}{\cos^2\theta}$

c) $\frac{\sin x}{\sin^2 x}$

16. a) Use a sum or difference identity to prove that $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$.

b) Use transformations of the graph of $y = \cos x$ to illustrate this identity.

17. a) Use a sum or difference identity to prove that $\sin(\theta + \pi) = -\sin\theta$.

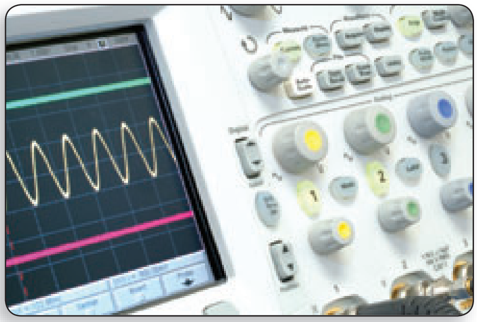
b) Use transformations of the graph of $y = \sin x$ to sketch a graph illustrating this identity.

CHAPTER PROBLEM

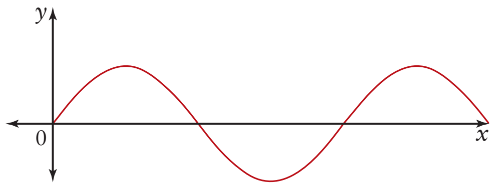
Roller coasters come in all sorts of shapes and sizes. Think of the types of mathematical functions that can be used to model some of their curves. How could sinusoidal functions be useful in roller coaster design? How could you design a new roller coaster that would be functional, fun, and safe?

4.1 Instantaneous Rates of Change of Sinusoidal Functions

The electronic device pictured is called an oscilloscope. This powerful tool allows a technician or engineer to observe voltage or current signals in electrical circuits. Notice that the shape of the signal is that of a sine or cosine wave.



Sinusoidal functions are functions whose graphs have the shape of a sine wave. Electromagnetic waves including light, x-ray, infrared, radio, cell phone, and television are all examples of phenomena that can be modelled by sinusoidal functions. A microwave oven uses electromagnetic waves to cook your food. Exploration of the instantaneous rate of change of a sinusoidal function can reveal the nature of it derivative.



Investigate — What is the nature of the instantaneous rate of change of a sinusoidal function?

Tools

- computer with *The Geometer's Sketchpad®*

A: The Sine Function

1. Open *The Geometer's Sketchpad®* and begin a new sketch.

2. Sketch a graph of $y = \sin x$. Accept the option to change the units to radians.

3. a) Explain why $y = \sin x$ is a periodic function.

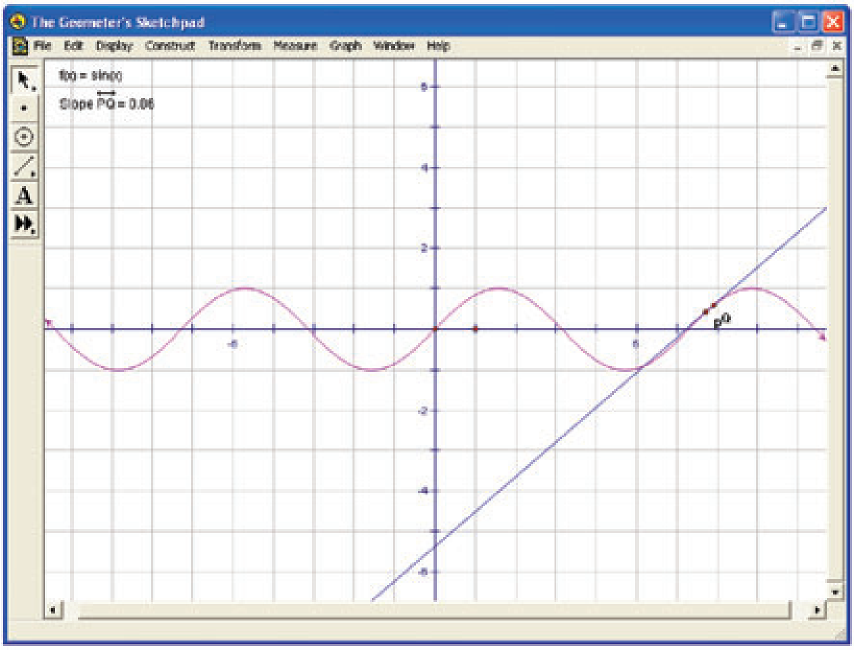
b) Use the **Segment Tool** to draw approximate tangent lines at several points along one period of the curve. Estimate the slope at each of these points.

c) What type of pattern do you think the slopes will follow? Explain your reasoning.

4. a) Construct a secant that is almost tangent to the curve:

- Select the curve only. From the **Construct** menu, choose **Point on Function Plot.**
- Construct another point on the curve. Label these points P and Q.
- Construct a line that passes through P and Q.
- Select the line only. From the **Measure** menu, choose **Slope**.

- Measure the slope of this line.
- Move Q so that P and Q are very close together.



b) Explain why the slope of the line passing through P and Q is approximately equal to the rate of change of the sine function at Q.

c) How could you improve the accuracy of this measure?

5. Explore what happens at various points along the curve.

a) Select both points P and Q. Click and drag, and explain what happens to

i) the line

ii) the slope of the line

b) Identify points where the slope is

i) zero

ii) a maximum

iii) a minimum

c) What are the maximum and minimum values of the slope?

6. Trace the rate of change of the slope of PQ as you follow the sine curve.

- Select the point P only. From the **Measure** menu, choose **Abscissa (x)**.
- Select the line PQ only. From the **Measure** menu, choose **Slope**.
- Select the measures of x_P and m_{PQ}, in that order.
- From the **Graph** menu, choose **Plot As (x, y)**.

CONNECTIONS

Recall the geometric interpretations of secant lines and tangent lines.

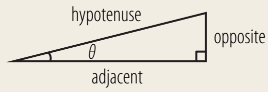
secant line

- a line passing through two different points on a curve

tangent line

- a line that touches, but does not cross, a curve at only one point

These terms have a different interpretation in trigonometry. For a right triangle,



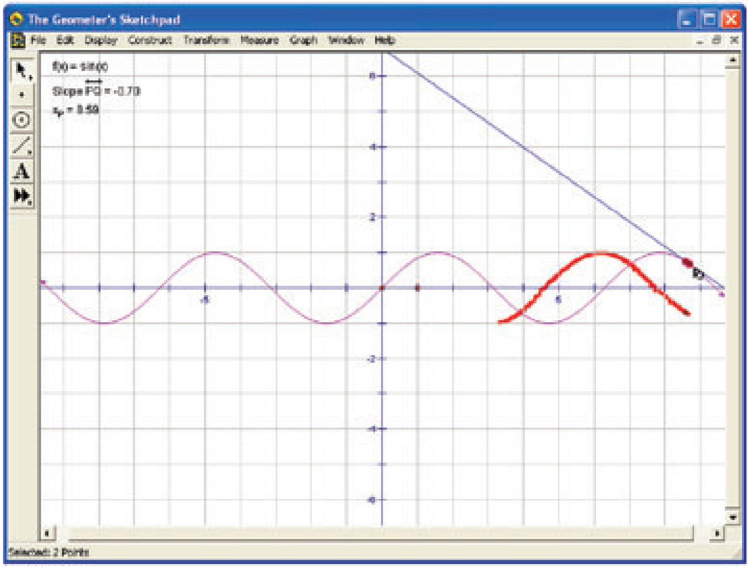
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

To learn more about relationships between trigonometric functions and where their names come from, go to *www.mcgrawhill.ca/links/calculus12* and follow the links.

A point having coordinates (x_P, m_{PQ}) will appear.

- Plot m_{PQ} as a function of x.
- Select the point (x_P, m_{PQ}).
- From the **Display** menu, choose **Trace Plotted Point**.
- Select points P and Q. Click and drag these points to trace out the instantaneous rates of change.



7. Reflect a) What function does the trace plot look like?

b) Identify intervals for which the instantaneous rate of change is

i) increasing

ii) decreasing

iii) zero

c) What do these results suggest about the derivative of $y = \sin x$? Explain your reasoning.

d) Check your prediction from part c) by graphing the derivative of $\sin x$.

- Select the equation $f(x) = \sin x$.
- From the **Graph** menu, choose **Derivative**.

Was your prediction correct?

B: The Cosine Function

8. a) Predict the derivative of $y = \cos x$. Give reasons for your prediction.

b) Without using the **Derivative** command, design and carry out an investigation using *The Geometer's Sketchpad*® to test your prediction. Use the **Derivative** command only to check your final result.

9. Reflect Summarize your results. Were your predictions in steps 7 c) and 8 a) correct? Explain.

KEY CONCEPTS

- The rate of change of a sinusoidal function is periodic in nature.
- The derivative of a sinusoidal function is also a sinusoidal function.

Communicate Your Understanding

C1 Describe how the instantaneous rate of change varies along a sinusoidal curve.

C2 Consider the curve $y = \sin x$ on the domain $\{x \mid 0 < x < 2\pi, x \in \mathbb{R}\}$. How many

a) local maxima are there?

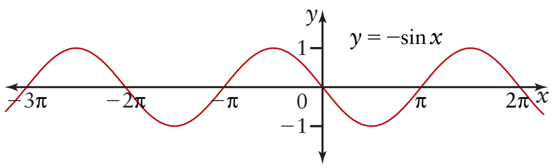
b) local minima are there?

Explain your answers using a diagram.

C3 Refer to the previous question. How do these answers change when the domain is all of $\mathbb{R}$? Explain why.

B Connect and Apply

1.



a) Identify all points where the slope of $y = -\sin x$ is

i) zero

ii) a local maximum

iii) a local minimum

b) Over which intervals is the curve

i) concave up? **ii)** concave down?

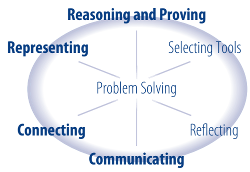
c) What are the maximum and minimum values of the slope?

d) Sketch a graph of the instantaneous rate of change of $y = -\sin x$ as a function of x.

2. a) Sketch a graph of the function $y = -4\cos x$.

b) Sketch a graph of the instantaneous rate of change of $y = -4\cos x$ as a function of x.

3. Does a sinusoidal curve have any points of inflection? Use geometric reasoning to support your answer. Sketch a diagram to show where these points occur.



C Extend and Challenge

4. Use Technology

a) Sketch a graph of $y = \csc x$.

b) Determine the instantaneous rate of change at several points.

c) Sketch a graph of the instantaneous rate of change of $y = \csc x$ as a function of x. Identify the key features of the graph.

5. Repeat question 4 for $y = \sec x$.

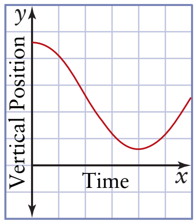
6. Repeat question 4 for $y = \cot x$.

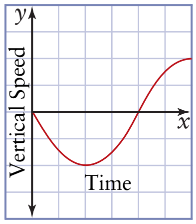
7. Math Contest The period of the function $3\sin 4x + 2\sin 6x$ is

A $\frac{\pi}{12}$ **B** $\frac{\pi}{2}$ **C** π

D 2π **E** 12π

4.2 Derivatives of the Sine and Cosine Functions





In the last section, you discovered that the graph of the instantaneous rate of change of a sinusoidal function is periodic in nature and also appears to be a sinusoidal function. Algebraic and graphical reasoning can be applied to determine the precise nature of the derivative of a sinusoidal function.

Apply the first principles definition of a derivative to find the derivative of $y = \sin x$. You will need to use the identity $\sin(a + b) = (\sin a)(\cos b) + (\sin b)(\cos a)$ in your work.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}\\
&= \lim_{h\to 0}\left[\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\sin h}{h}\right]\\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h} \qquad \text{The limit of a sum is the sum of the limits.}
\end{aligned}$$

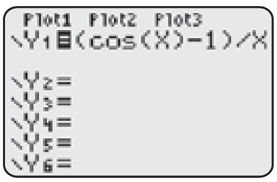
Notice that both $\sin x$ and $\cos x$ do not depend on h. Since h varies but x is fixed in the limit process, $\sin x$ and $\cos x$ can be moved as factors to the left of the limit signs.

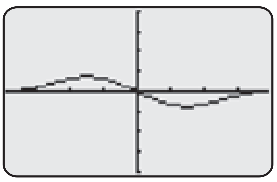
$$\frac{dy}{dx} = \sin x\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$

Use a graphing calculator to determine these two limits. You have not evaluated limits like these before. Direct substitution in either limit leads to an indeterminant form $\frac{0}{0}$.

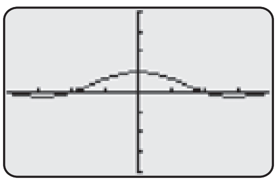
Technology Tip

When viewing some trigonometric functions, it is useful to use the **Zoom 7:ZTrig** option, which adjusts the scales on the axes for better viewing.





The graph shows that $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$.



The graph shows that $\lim_{h\to 0}\frac{\sin h}{h} = 1$.

Substitute these limits. Then, simplify the expression for the derivative with respect to x.

$$\begin{aligned}\frac{dy}{dx} &= \sin x(0) + \cos x(1)\\ &= \cos x\end{aligned}$$

The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$.

Use this result to determine the derivative of $y = \cos x$ with respect to x.

$$\begin{aligned}y &= \cos x\\ &= \sin\left(\frac{\pi}{2} - x\right) \quad \text{Use the difference identity } \sin\left(\frac{\pi}{2} - x\right) = \cos x\end{aligned}$$

Differentiate this function by using the result above and applying the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \underbrace{\cos\left(\frac{\pi}{2} - x\right)}_{\sin x}\underbrace{\frac{d}{dx}\left(\frac{\pi}{2} - x\right)}_{-1}\\ &= (\sin x)(-1)\\ &= -\sin x\end{aligned}$$

The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$.

CONNECTIONS

The trigonometric difference identity $\sin(a - b) = (\sin a)(\cos b) - (\cos a)(\sin b)$ can be derived from the sum identity $\sin(a + b) = (\sin a)(\cos b) + (\cos a)(\sin b)$.

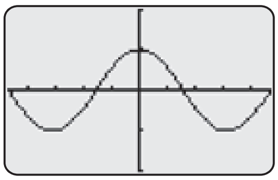
Investigate Why use radian measure?

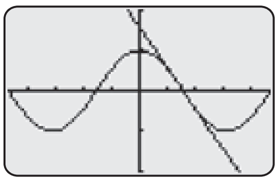
Tools

- graphing calculator

Consider the function $y = \cos x$.

1. Calculate the value of this function at $x = \frac{\pi}{2}$.
2. Calculate $\frac{dy}{dx}$ and evaluate the derivative at $x = \frac{\pi}{2}$.
3. Use a graphing calculator to graph $y = \cos x$. Make sure that the calculator is set to use radians.
4. From the **Draw** menu, choose **5:Tangent(**, and construct the tangent to $y = \cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

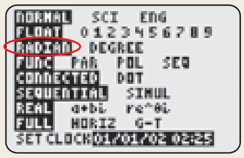




Technology Tip

To set your graphing calculator to radian mode:

- Press MODE.
- Scroll down to the line containing RADIAN.



- Select RADIAN.
- Press Y= to return to the function screen.

5. Find an equation of the line passing through the point $\left(\frac{\pi}{2}, 0\right)$ with a slope of -1. Is this the equation of the tangent line to $y = \cos x$ at $\left(\frac{\pi}{2}, 0\right)$? Explain.

6. Convert $\frac{\pi}{2}$ rad to degrees. If the scale of the horizontal axis of your graph were expressed in degrees, what would be the coordinates of the point corresponding to $\left(\frac{\pi}{2}, 0\right)$ on your original graph?

7. Change the scale of the horizontal axis from radians to degrees:

- Change the **MODE** to **DEGREE**.
- Change the **WINDOW** variables to $x \in [-270, 270]$.
- Redraw the tangent line at (90°, 0).

Examine the tangent line you have constructed. Has the y-intercept of that line changed? Has the x-intercept changed? Explain.

8. Find an equation of the tangent line to the graph of this function that passes through the point (90°, 0). What is the slope of this line?

9. **Reflect** Explain why relabelling the horizontal axis of a graph will change the slope of any lines you have plotted. If you change the labelling of the horizontal axis of the graph of $y = \cos x$ from radians to degrees, will the function $y = -\sin x$ give the slope of the tangent at a point? Explain.

CONNECTIONS

You explored the various differentiation rules in Chapter 2 Derivatives.

The rules for differentiation that you learned earlier, such as the constant multiple rule, $\frac{d}{dx}[cf(x)] = c\left[\frac{d}{dx}f(x)\right]$, can be applied to sinusoidal functions.

Example 1 Constant Multiple Rule

Find each derivative with respect to x.

a) $y = 2\sin x$ **b)** $f(x) = -3\cos x$

Solution

a)

$$
\begin{aligned}
y &= 2\sin x \\
\frac{dy}{dx} &= \frac{d}{dx}(2\sin x) \\
&= 2\frac{d}{dx}(\sin x) && \text{Apply the constant multiple rule.} \\
&= 2\cos x
\end{aligned}
$$

b)

$$
\begin{aligned}
f(x) &= -3\cos x \\
f'(x) &= -3(-\sin x) && \text{Apply the constant multiple rule.} \\
&= 3\sin x
\end{aligned}
$$

The sum and difference rules state that $(f \pm g)'(x) = f'(x) \pm g'(x)$.

Example 2 Sum and Difference Rules

Differentiate with respect to x.

a) $y = \sin x + \cos x$

b) $y = 2\cos x - 4\sin x$

Solution

a)
$$y = \sin x + \cos x$$
$$\frac{dy}{dx} = \cos x + (-\sin x)$$
Apply the sum rule.
$$= \cos x - \sin x$$

b)
$$y = 2\cos x - 4\sin x$$
$$\frac{dy}{dx} = 2(-\sin x) - 4\cos x$$
Apply the difference rule and constant multiple rule.
$$= -2\sin x - 4\cos x$$

Example 3 Slope at a Point

Find the slope of the tangent line to the graph of $f(x) = 3\sin x$ at the point where $x = \frac{\pi}{4}$.

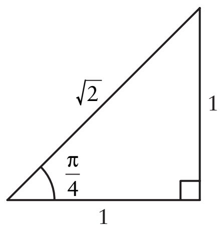
Solution

To find the slope, differentiate the given function with respect to x. Then, evaluate the derivative at $x = \frac{\pi}{4}$.

$f(x) = 3\sin x$

$f'(x) = 3\cos x$

The value of the derivative gives the slope of the tangent passing through the point on $f(x)$ where $x = \frac{\pi}{4}$.

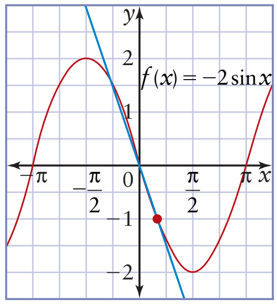


$$f'\left(\frac{\pi}{4}\right) = 3\cos\left(\frac{\pi}{4}\right)$$
$$= 3\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{3}{\sqrt{2}}$$

The slope, or instantaneous rate of change, when $x = \frac{\pi}{4}$ is $\frac{3}{\sqrt{2}}$.

Example 4 Equation of a Tangent Line

Find the equation of the tangent line to the curve $f(x) = -2\sin x$ at the point where $x = \frac{\pi}{6}$.



Solution

Method 1: ***Use Paper and Pencil***

To find the equation of a line, you need its slope and a point on the line.

Calculate the point of tangency with x-coordinate $\frac{\pi}{6}$.

$$f(x) = -2\sin x$$

$$f\left(\frac{\pi}{6}\right) = -2\sin\left(\frac{\pi}{6}\right)$$

$$= -2\left(\frac{1}{2}\right)$$

$$= -1$$

The required point is $\left(\frac{\pi}{6}, -1\right)$.

To find the slope, differentiate the function with respect to x. Then, evaluate the derivative at $x = \frac{\pi}{6}$.

$$f(x) = -2\sin x$$

$$f'(x) = -2\cos x$$

$$f'\left(\frac{\pi}{6}\right) = -2\cos\left(\frac{\pi}{6}\right)$$

$$= -2\left(\frac{\sqrt{3}}{2}\right)$$

$$= -\sqrt{3}$$

The slope of the tangent line is $-\sqrt{3}$.

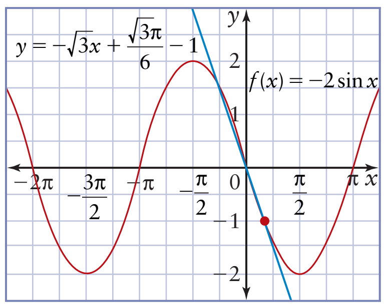
To find the equation of the tangent line, substitute the point and the slope into the point-slope form equation of a line.

The equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 1 = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

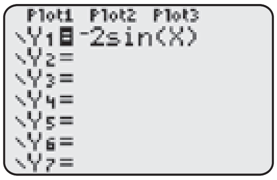
$$y = -\sqrt{3}x + \frac{\sqrt{3}\pi}{6} - 1$$

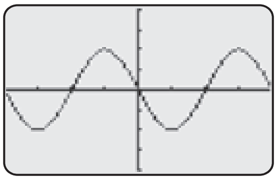


Method 2: ***Use a Graphing Calculator***

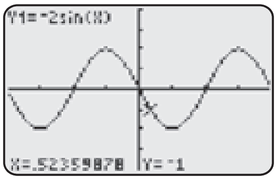
To find the equation of the tangent line, find the point of tangency and the slope through that point.

Graph the curve $y = -2\sin x$.



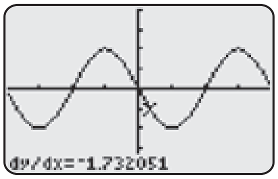


Determine the value of y when $x = \frac{\pi}{6}$ by using the **Calculate** menu.



The tangent point is $\left(\frac{\pi}{6}, -1\right)$.

Use the **Calculate** menu to find the slope of the derivative at $x = \frac{\pi}{6}$.



When $\left(\frac{\pi}{6}, -1\right)$, the approximate value of the slope of the tangent line is -1.732.

To find the y-intercept of the tangent line, substitute the point $\left(\frac{\pi}{6}, -1\right)$ and the slope into the point-slope form equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y + 1 = -1.732\left(x - \frac{\pi}{6}\right)$$

$$y = -1.732x - 0.093$$

So, the approximate equation of the tangent line at $x = \frac{\pi}{6}$ is $y = -1.732x - 0.093$.

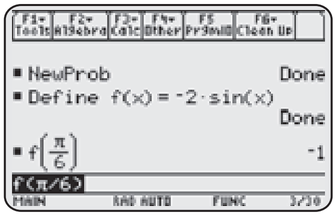
Method 3: ***Use a Computer Algebra System (CAS)***

- Turn on the CAS and select **2:NewProb**, then press ENTER.
- Press the MODE key. Scroll down to **Angle**, and ensure that **RADIAN** is selected.

Technology Tip

When you choose to begin a new problem with the CAS, any variables and values in the memory that may have been left by a previous user are cleared. If you do not begin a new problem, you may experience unexpected or incorrect results.

- Scroll down to **Exact/Approx**, and ensure that **AUTO** is selected.
- Press [F4] and select **1:Define**. Enter the function $f(x) = -2\sin x$.
- Evaluate the function at $x = \frac{\pi}{6}$.

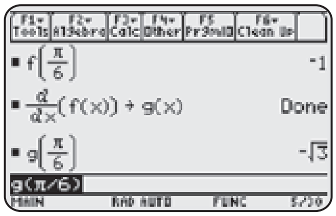


The CAS shows that $f\left(\frac{\pi}{6}\right) = -1$.

Determine the derivative of the function, and evaluate the derivative at $x = \frac{\pi}{6}$.

- Press [F3] and select **1:*d*(differentiate**.
- Enter [ALPHA] [|] [(] [X] [,] [X] [)].
- Press the [STO] key and store the derivative as $g(x)$.

Determine the slope by evaluating the derivative at $x = \frac{\pi}{6}$.

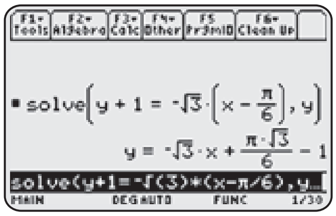


When $x = \frac{\pi}{6}$, the exact value of the slope of the tangent line is $-\sqrt{3}$. Note that the CAS displays the exact value.

To find the equation of the tangent line, substitute the point $\left(\frac{\pi}{6}, -1\right)$ and the slope into the point-slope form equation of a line.

- Press [F2] and select **1:solve**.
- Enter [Y] [+] 1 [=] [(-)] [2ND] [×] 3 [)] [×] [(] [X] [−] [2ND] [^] [÷] 6 [)] [,] [Y] [)] and press [ENTER].

The exact equation of the tangent line at $x = \frac{\pi}{6}$ is $y = -\sqrt{3}x + \frac{\pi\sqrt{3}}{6} - 1$.



Compare this equation to the equation produced using Method 2.

You can set the CAS parameters to display approximate rather than exact values. Press the [MODE] key, scroll down to **Exact/Approx**, and select **APPROXIMATE**.

KEY CONCEPTS

- The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$.
- The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$.
- The constant multiple, sum, and difference differentiation rules apply to sinusoidal functions.
- Given the graph of a sinusoidal function expressed in radians, you can find the slope of the tangent at a point on the graph by calculating the derivative of the function at that point.
- Given the graph of a sinusoidal function expressed in radians, you can find the equation of the tangent line at a point on the graph by determining the slope of the tangent at that point and then substituting the slope and point into the equation of a line.

Communicate Your Understanding

C1 **a)** Is the derivative of a sinusoidal function always periodic? Explain why or why not.

b) Is this also true for the

i) second derivative? **ii)** fifth derivative?

Explain how you can tell.

C2 **a)** Over the interval $\{x \mid 0 \le x \le 2\pi, x \in \mathbb{R}\}$, how many points on the graph of the function $y = \sin x$ have a tangent line with slope

i) 0? **ii)** 1?

Use diagrams to help explain your answers.

b) How do these answers change for $x \in \mathbb{R}$? Explain why.

C3 In Example 4, three methods were used to find the equation of a tangent line: paper and pencil, graphing calculator, and CAS.

a) Describe the advantages and disadvantages of each solution method.

b) Describe another method that could be used to solve that problem.

A Practise

1. Match each function with its derivative.

a) $y = \sin x$ **A** $\frac{dy}{dx} = \sin x$

b) $y = \cos x$ **B** $\frac{dy}{dx} = \cos x$

c) $y = -\sin x$ **C** $\frac{dy}{dx} = -\sin x$

d) $y = -\cos x$ **D** $\frac{dy}{dx} = -\cos x$

2. Find the derivative with respect to x for each function.

a) $y = 4\sin x$

b) $y = \pi\cos x$

c) $f(x) = -3\cos x$

d) $g(x) = \frac{1}{2}\sin x$

e) $f(x) = 0.007\sin x$

3. Differentiate with respect to x.

a) $y = \cos x - \sin x$

b) $y = \sin x + 2\cos x$

c) $y = x^2 - 3\sin x$

d) $y = \pi\cos x + 2x + 2\pi\sin x$

e) $y = 5\sin x - 5x^3 + 2$

f) $y = \cos x + 7\pi\sin x - 3x$

4. Differentiate with respect to θ.

a) $f(\theta) = -3\cos\theta - 2\sin\theta$

b) $f(\theta) = \frac{\pi}{2}\sin\theta - \pi\cos\theta + 2\pi$

c) $f(\theta) = 15\cos\theta + \theta - 6$

d) $f(\theta) = \frac{\pi}{4}\cos\theta - \frac{\pi}{3}\sin\theta$

B Connect and Apply

5. a) Find the slope of the graph of $y = 5\sin x$ at $x = \frac{\pi}{2}$.

b) How does the slope of this graph at $x = \frac{\pi}{2}$ compare to the slope of the graph of $y = \sin x$ at $x = \frac{\pi}{2}$? Explain.

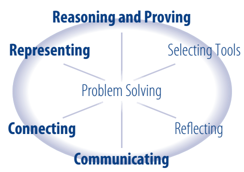
6. Find the slope of the graph of $y = 2\cos\theta$ at $\theta = \frac{\pi}{6}$.

7. a) Show that the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is on the curve of $y = \cos x$.

b) Find the equation of the line tangent to the function $y = \cos x$ and passing through the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.

8. Find the equation of the line tangent to the function $y = -4\sin x$ at $x = \frac{\pi}{4}$.

9. a) Sketch a graph of $y = \sin x$ and its derivative on the same grid, over the interval from 0 to 4π.



b) Use transformations to explain how these graphs are related.

c) Find the second derivative of $y = \sin x$ and sketch it on the same grid. How is this graph related to the other two?

d) Predict the graph of the third derivative and sketch it on the same grid. Find this derivative and check your prediction.

e) Use these results to predict the

i) fourth derivative of $y = \sin x$

ii) tenth derivative of $y = \sin x$

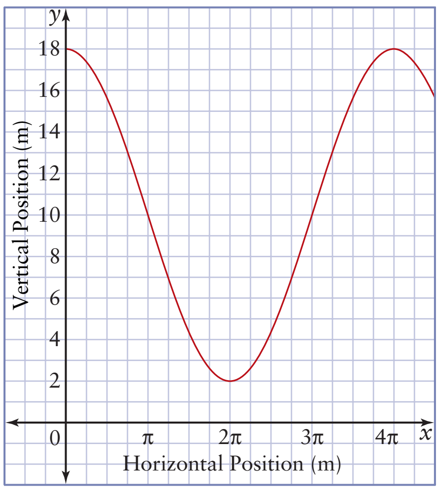
10. Find the fifteenth derivative of $y = \cos x$. Explain your method.

11. **Use Technology** Create two sinusoidal functions of your choice. Use *The Geometer's Sketchpad*® or other graphing software to demonstrate that the sum and difference differentiation rules hold true for the functions you have chosen.

12. a) Find an equation of a line tangent to $y = -\cos x$ whose slope is -1.

b) Is there more than one solution? Explain.

13. **Chapter Problem** The following graph models a section of a roller coaster ride.



a) What are the maximum and minimum heights above the ground for the riders on this section of the ride?

b) Write an equation that relates vertical and horizontal position.

c) What is the maximum slope of this section of the ride?

Achievement Check

14. a) Find an equation of a line tangent to $y = 2\sin x$ whose slope is a maximum value.

b) Is there more than one solution? Explain.

C Extend and Challenge

15. Use Technology a) Sketch a graph of $y = \tan x$. Is this function periodic? Explain.

b) Predict the shape of the graph of the derivative of $y = \tan x$. Justify your reasoning.

c) Use *The Geometer's Sketchpad*® or other graphing software to check your prediction. Was the result what you expected? Explain.

16. Refer to question 15.

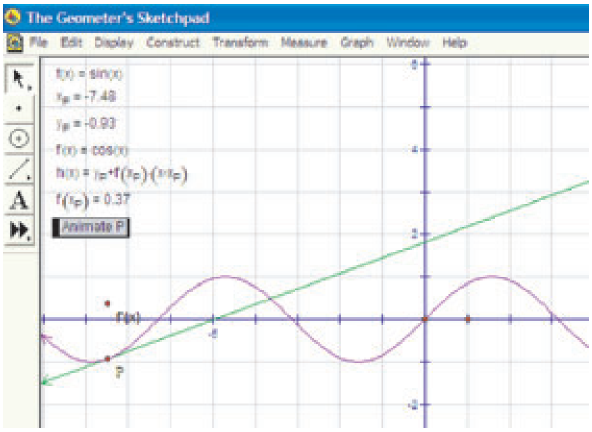
a) What happens to the graph of the derivative of $y = \tan x$ as $x \to \frac{\pi}{2}$

i) from the left?

ii) from the right?

b) What does this imply about the value of the derivative of $y = \tan x$ at $x = \frac{\pi}{2}$? Explain.

17. Use Technology Consider this sketch created using *The Geometer's Sketchpad*®.



a) What do you think this sketch illustrates about the function $f(x) = \cos x$? Explain.

b) Describe what you think will happen if you press the **Animate P** button.

c) Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to **Dynamic Derivative of Sine.gsp**. Open the sketch and test your predictions. Describe your observations.

d) Use *The Geometer's Sketchpad*® to create a dynamic derivative of cosine sketch of your own.

e) Demonstrate your sketch to a classmate, family member, or friend. Describe what your sketch illustrates and how you created it.

18. Use Technology Explore the graph of one of the reciprocal trigonometric functions and its derivative. What can you determine about the following features of that function and its derivative?

a) domain and range

b) maximum/minimum values

c) periodicity

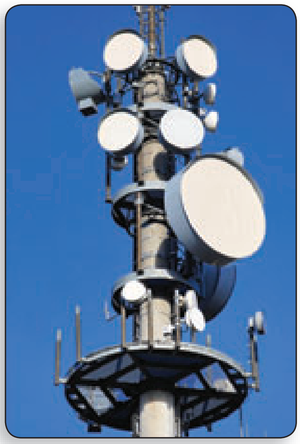
d) asymptotes

e) graph

19. Math Contest Estimate $\sin 37° - \sin 36°$ by making use of the fact that $\cos 36° = \frac{1+\sqrt{5}}{4}$.

4.3 Differentiation Rules for Sinusoidal Functions

Sinusoidal patterns occur frequently in nature. Sinusoidal functions and compound sinusoidal functions are used to describe the patterns found in the study of radio-wave transmission, planetary motion, and particle behaviour.



Radio waves, for example, are electromagnetic fields of energy that consist of two parts: a signal wave and a carrier wave. Both parts periodically alternate in a manner that can be modelled using sinusoidal functions.

Radio waves transmitted from a base station are transformed by a receiver into pulses that cause a thin membrane in a speaker to vibrate. This in turn causes molecules in the air to vibrate, allowing your ear to hear music.

Investigate How do the differentiation rules apply to sinusoidal functions?

Tools

- computer with *The Geometer's Sketchpad*®

Optional

- graphing software
- graph paper

CONNECTIONS

When expressing the power of a trigonometric function, observe the placement of the exponent:

- $\sin^2 x = (\sin x)^2$
  $= (\sin x)(\sin x)$
- $\sin^2 x \neq \sin(\sin x)$

1. Consider the following functions:

$y = \sin 2x \quad y = \sin^2 x \quad y = \sin(x^2) \quad y = x^2 \sin x$

a) Do you think these functions will have the same derivative? Explain your reasoning.

b) Predict the derivative of each function.

2. Reflect Use *The Geometer's Sketchpad*® or other graphing software to graph these functions and their derivatives. Compare the results with your predictions. Were your predictions correct?

3. Which differentiation rules would you apply to produce the derivative of each function in step 1?

4. a) Create a sinusoidal function of your own that requires the power of a function rule in order to find its derivative. Find the derivative of your function.

b) Repeat part a) for the chain rule.

c) Repeat part a) for the product rule.

5. Reflect

a) Check your results in step 4 using *The Geometer's Sketchpad*®.

b) Do the rules of differentiation apply to sinusoidal functions? Explain.

Example 1 Chain Rule

Find the derivative with respect to x for each function.

a) $y = \cos 3x$ **b)** $f(x) = 2\sin \pi x$

Solution

a) $y = \cos 3x$

Let $y = \cos u$ and $u = 3x$.

$$\frac{dy}{du} = -\sin u \text{ and } \frac{du}{dx} = 3$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (-\sin u) \times (3)\\ &= -3\sin u\\ &= -3\sin 3x\end{aligned}$$

b) $f(x) = 2\sin \pi x$

$f'(x) = 2\pi\cos \pi x$

CONNECTIONS

Recall the differentiation rules you learned in Chapter 2 Derivatives:

The Chain Rule

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

The Power of a Function Rule

If $g(x)$ is a differentiable function, then $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\frac{d}{dx}g(x)$.

The Product Rule

If $f(x)$ and $g(x)$ are differentiable, then $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$.

Example 2 Power of a Function Rule

Differentiate with respect to x.

a) $y = \cos^3 x$ **b)** $y = 2\sin^3 x - 4\cos^2 x$

Solution

a) $y = \cos^3 x$

$$\begin{aligned}\frac{dy}{dx} &= (3\cos^2 x)(-\sin x)\\ &= -3\sin x\cos^2 x\end{aligned}$$

b) $y = 2\sin^3 x - 4\cos^2 x$

$$\begin{aligned}\frac{dy}{dx} &= 2(3\sin^2 x)\cos x - 4(2\cos x)(-\sin x)\\ &= 6\cos x\sin^2 x + 8\sin x\cos x\\ &= 2\cos x\sin x(3\sin x + 4)\end{aligned}$$

Apply the power of a function rule and the difference rule.

Notice that the answer in part b) can be further simplified if you apply a double angle identity, which gives $\frac{dy}{dx} = (\sin 2x)(3\sin x + 4)$.

CONNECTIONS

One double angle identity is $\sin 2x = 2\sin x\cos x$.

Example 3 Product Rule

Find each derivative with respect to t.

a) $y = t^3 \cos t$ **b)** $h(t) = \sin(4t)\cos^2 t$

Solution

a) $y = t^3 \cos t$

Let $u = t^3$ and $v = \cos t$.

Then $u' = 3t^2$ and $v' = -\sin t$.

$$\begin{aligned} y' &= uv' + u'v \\ &= (t^3)(-\sin t) + (3t^2)(\cos t) \\ &= 3t^2 \cos t - t^3 \sin t \\ &= t^2(3\cos t - t\sin t) \end{aligned}$$

b) $h(t) = \sin 4t \cos^2 t$

$$\begin{aligned} \frac{dh}{dt} &= (\sin 4t)(-2\sin t\cos t) + (4\cos 4t)(\cos^2 t) \\ &= 4\cos^2 t\cos 4t - 2\sin t\cos t\sin 4t \\ &= 2\cos t(2\cos t\cos 4t - \sin t\sin 4t) \end{aligned}$$

Apply the product rule in conjunction with other rules.

KEY CONCEPTS

- The power, chain, and product differentiation rules apply to sinusoidal functions.
- The derivative of a composite or compound sinusoidal function is often easier to calculate if you first express that function in terms of simpler functions.

Communicate Your Understanding

C1 Identify the differentiation rule needed in order to find the derivative of each of the following functions with respect to x.

a) $y = \sin x \cos x$ **b)** $y = \sin(\cos x)$

c) $y = \sin^2 x$ **d)** $y = \sin x + \cos x$

C2 Let $f(x) = x\sin x$.

a) Describe the steps you would use to find $f'(\pi)$.

b) Explain how you might use the graph of $f(x)$ to check your answer.

C3 Let $y = \sin^2 x$. Describe the steps you would use to find the equation of the line tangent to the graph of this function at the point where $x = \frac{\pi}{4}$.

A Practise

1. Find each derivative with respect to x.

   a) $y = \sin 4x$
   b) $y = \cos(-\pi x)$
   c) $f(x) = \sin(2x + \pi)$
   d) $f(x) = \cos(-x - \pi)$

2. Differentiate with respect to θ.

   a) $y = -2\sin 3\theta$
   b) $y = -\cos\left(5\theta - \dfrac{\pi}{2}\right)$
   c) $f(\theta) = \dfrac{1}{2}\cos 2\pi\theta$
   d) $f(\theta) = -3\sin(2\theta - \pi)$

3. Find each derivative with respect to x.

   a) $y = \sin^2 x$
   b) $y = \dfrac{1}{3}\cos^3 x$
   c) $f(x) = \cos^2 x - \sin^2 x$
   d) $f(x) = 2\cos^3 x + \cos^4 x$

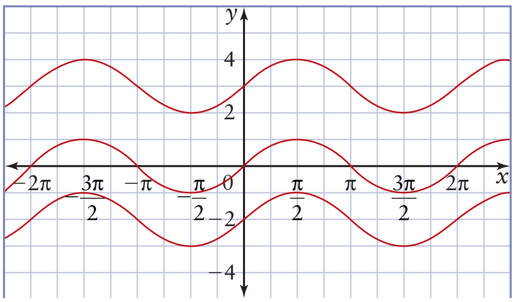
4. Differentiate with respect to t.

   a) $y = 3\sin^2(2t - 4) - 2\cos^2(3t + 1)$
   b) $f(t) = \sin(t^2 + \pi)$
   c) $y = \cos(\sin t)$
   d) $f(t) = \sin^2(\cos t)$

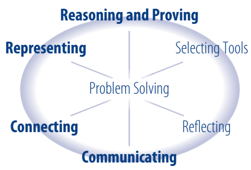
5. Determine the derivative of each function with respect to the variable indicated.

   a) $y = x\cos 2x$
   b) $f(x) = -x^2 \sin(3x - \pi)$
   c) $y = 2\sin\theta\cos\theta$
   d) $f(\theta) = \sin^2\theta\cos^2\theta$
   e) $f(t) = 3t\sin^3(2t - \pi)$
   f) $y = x^{-1}\cos^2 x$

B Connect and Apply

6. 

   a) How are the derivatives of the three functions shown related to each other?



   b) Write equations for these functions. Use algebraic or geometric reasoning to explain how you produced your equations.

7. Find the slope of the function $y = 2\cos x \sin 2x$ at $x = \dfrac{\pi}{2}$.

8. Find the equation of the line tangent to $y = x^2 \sin 2x$ at $x = -\pi$.

9. a) Is $y = \sin x$ an even function, an odd function, or neither? Explain your reasoning.

   b) Use this property to show that $\dfrac{dy}{dx}[\sin(-x)] = -\cos x$.

CONNECTIONS

Recall that a function is even if $f(-x) = f(x)$, and it is odd if $f(-x) = -f(x)$, for all x in its domain.

10. a) Is $y = \cos x$ an even function, an odd function, or neither? Explain your reasoning.

    b) Use your answer to part a) to show that $\dfrac{dy}{dx}\cos(-x) = -\sin x$.

11. a) Use the differentiation rules to show that $y = \sin^2 x + \cos^2 x$ is a constant function, a function of the form $y = c$ for some $c \in \mathbb{R}$.

    b) Use a trigonometric identity to verify your answer in part a).

12. Determine $\dfrac{d^2y}{dx^2}$ for $y = x^2 \cos x$.

13. Let $f(x) = \cos^2 x$.

a) Use algebraic reasoning to explain why, over the interval $0 \leq x < 2\pi$, this function has half as many zeros as its derivative.

b) Graph $f(x)$ and its derivative to support your explanation.

14. Create a composite function that consists of a sinusoidal and a cubic. Find the first and second derivatives of the function.

15. a) Write $y = \csc x$ in terms of $\sin x$ as a reciprocal function.

b) Write the function in terms of a negative power of $\sin x$.

c) Use the power rule and chain rule to find the derivative of $y = \csc x$.

d) Identify the domain of both $y = \csc x$ and its derivative.

CONNECTIONS

Trigonometric functions involving negative exponents are written differently than those with positive exponents. For example, $\sin x$ raised to the power of -1 is written as $(\sin x)^{-1}$, which is equal to $\dfrac{1}{\sin x}$. This is to avoid confusion with $\sin^{-1} x$, which means the inverse sine of x. The function $\csc x$ is often used to avoid confusion.

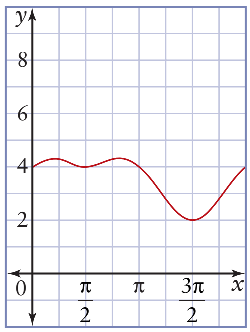
16. Use Technology

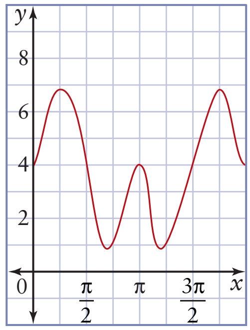
What impact will a horizontal shift have on the derivative of a sinusoidal function? Make a conjecture. Then, use graphing software or a computer algebra system (CAS) to explore and test your conjecture. Summarize your findings.

17. Chapter Problem

Experiment with different sinusoidal functions and combinations of sinusoidal functions. Use a variety of tools and strategies, for example, sliders, regression, and systematic trial.

a) Find a function that models each of the following roller coaster segments.





b) Identify the maximum slope in the interval $0 \leq x \leq 2\pi$ for each of the functions in part a).

C Extend and Challenge

18. a) Find the derivative of $y = \sec x$ with respect to x.

b) Find the derivative of $y = \dfrac{\sec x}{\cos^2 x}$ with respect to x.

19. Consider the function $y = \tan x$. Show that $\dfrac{dy}{dx} = 1 + \tan^2 x$.

20. Use algebraic reasoning to find the derivative of $y = \cot x$ with respect to x.

21. Create a compound sinusoidal function that will require the chain rule to be used twice in order to calculate the function's derivative. Find the derivative of the function you have created.

22. Use Technology

a) Find the derivative of $f(x) = \sin^2 x \cos x$.

b) Graph $f(x)$ using *The Geometer's Sketchpad®*.

c) Use *The Geometer's Sketchpad®* to graph $f'(x)$. Did the software produce the equation you found in part a)?

23. Math Contest Consider the infinite series $S(x) = 1 - \tan^2 x + \tan^4 x - \tan^6 x + \ldots$, where $0 < x < \dfrac{\pi}{4}$. The derivative $S'(x) =$

A $\sin 2x$ **B** $\cos 2x$ **C** $-\tan 2x$

D $-\sin 2x$ **E** $-\cos 2x$

4.4 Applications of Sinusoidal Functions and Their Derivatives

The motion of a simple pendulum can be modelled using a sinusoidal function. The first derivative of the function will give the bob's velocity and the second derivative will give the bob's acceleration.

Pendulums have been used in clocks for hundreds of years because their periodic motion is so regular. The Foucault pendulum shown here will oscillate for long periods of time. It appears to change direction over time but, in fact, it is the Earth beneath it that is turning.

Sinusoidal functions and their derivatives have many applications including the study of alternating electric currents, engine pistons, and oscillating springs.

Investigate How can sinusoidal functions be used to model periodic behaviour?

Tools

- simple pendulum (string plus a weight)
- metre stick

Optional

- computer with Internet access

You can construct a pendulum by simply attaching a weight to a piece of string. There are also a number of pendulum simulations available on the Internet.

1. Construct a pendulum or load a computer simulation.
2. Release the pendulum from some initial angle and observe its motion as it swings back and forth. Is this motion approximately periodic? Explain.
3. At which points does the pendulum have
   a) maximum speed?
   b) minimum speed?

   Make a sketch to illustrate your answers.
4. Stop the pendulum and restart it from a different initial angle. What impact does increasing or decreasing the initial angle have on
   a) the period of motion?
   b) the maximum speed?
5. Explore and describe what happens to the period and maximum speed when you vary the length of the pendulum.
6. **Reflect** Do you think that the motion of a pendulum can be described using a sinusoidal function? Explain why or why not.

CONNECTIONS

To explore a pendulum simulation, go to *www.mcgrawhill.ca/links/calculus12* and follow the links.

Example 1 An AC-DC Coupled Circuit

A power supply delivers a voltage signal that consists of an alternating current (AC) component and a direct current (DC) component. Where t is the time, in seconds, the voltage, in volts, at time t is given by the function $V(t) = 5\sin t + 12$.

a) Find the maximum and minimum voltages. At which times do these values occur?

b) Determine the period, T, in seconds, frequency, f, in hertz, and amplitude, A, in volts, for this signal.

Solution

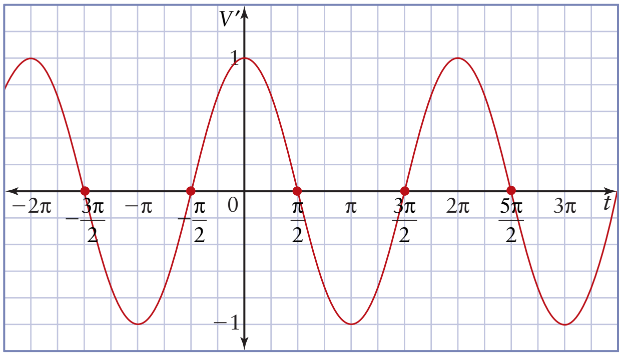
a) Method 1: ***Use Differential Calculus***

The extreme values of this function can be found by setting the first derivative equal to zero and solving for t.

Calculate the derivative with respect to time.

$$\begin{aligned} V(t) &= 5\sin t + 12 \\ \frac{dV}{dt} &= 5\cos t + 0 \\ &= 5\cos t \\ 5\cos t &= 0 \\ \cos t &= 0 \end{aligned}$$

Identify the values of t for which $\cos t = 0$. Consider the graph of $y = \cos t$.



The zeros occur at $\left\{\ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\right\}$, which can be expressed using set notation as $\left\{t \mid t = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\right\}$.

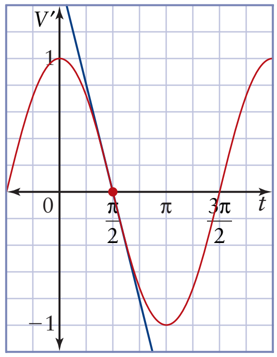
CONNECTIONS

AC-DC coupled circuits are used in the operational amplifiers found in (analog) computers, and in some high-power distribution networks. You will learn more about AC and DC circuits if you study electricity at college or university.

CONNECTIONS

The German word for number is *Zahlen*, and is the reason why the symbol $\mathbb{Z}$ is used to represent the integers.

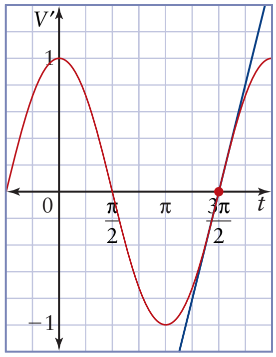
To determine whether these are maximum or minimum points, inspect the graph at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.



The tangent to the first derivative at $t = \frac{\pi}{2}$ has a negative slope, which implies that the second derivative is negative. This means that $t = \frac{\pi}{2}$ s will produce a maximum value. To find this maximum, substitute $t = \frac{\pi}{2}$ into the equation and solve for voltage.

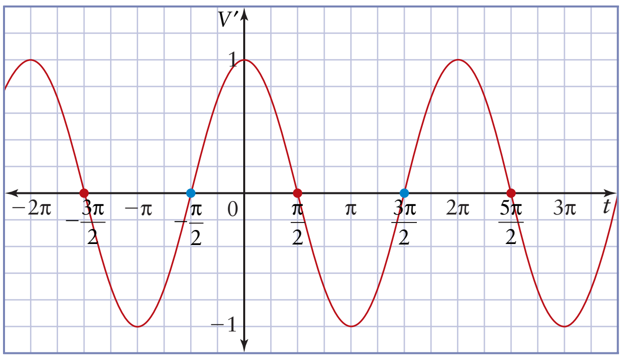
$$\begin{aligned} V(t) &= 5\sin t + 12 \\ V\left(\frac{\pi}{2}\right) &= 5\sin\left(\frac{\pi}{2}\right) + 12 \\ &= 5(1) + 12 \\ &= 17 \end{aligned}$$

The tangent to the first derivative at $t = \frac{3\pi}{2}$ has a positive slope, which implies that the second derivative is positive. This means that $t = \frac{3\pi}{2}$ s will produce a minimum value. To find this minimum, substitute $t = \frac{3\pi}{2}$ into the equation and solve for voltage.



$$\begin{aligned} V(t) &= 5\sin t + 12 \\ V\left(\frac{3\pi}{2}\right) &= 5\sin\left(\frac{3\pi}{2}\right) + 12 \\ &= 5(-1) + 12 \\ &= 7 \end{aligned}$$

Inspection shows that the maxima and minima alternate.



The maximum voltage 17 V occurs at $\left\{t \mid t = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\right\}$ seconds, and the minimum voltage 7 V occurs at $\left\{t \mid t = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z}\right\}$ seconds.

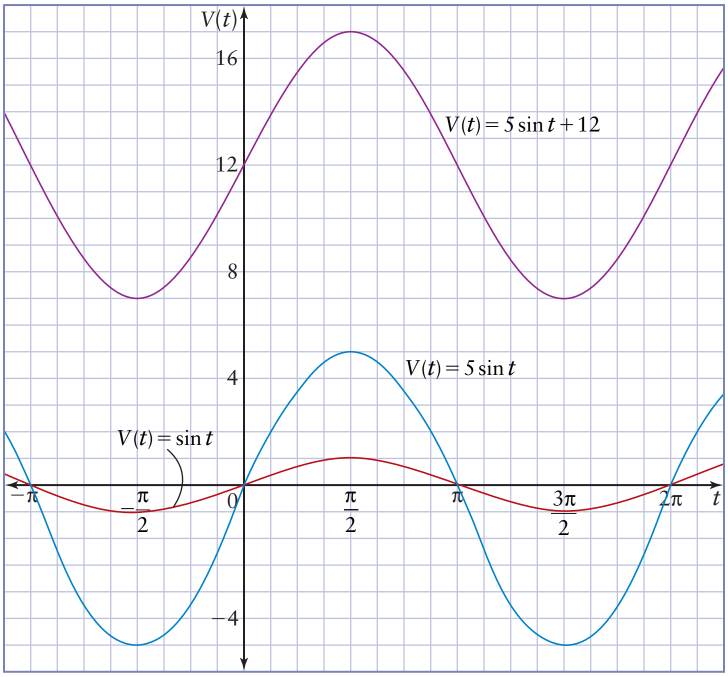
Method 2: ***Apply Transformations***

Apply transformations to the graph of $V(t) = \sin t$ to graph the voltage function $V(t) = 5\sin t + 12$. Then, use the graph to identify the key values of the voltage function.

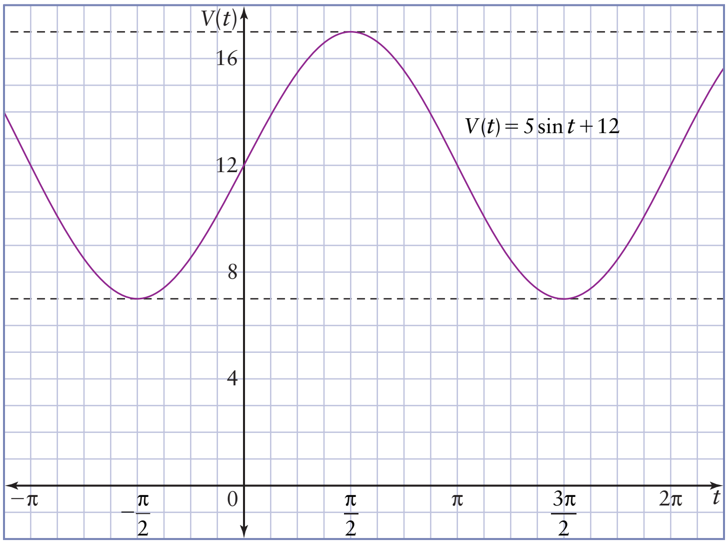
$$V(t) = 5\sin t + 12$$

Vertical stretch by a factor of 5

Vertical shift of 12 units up



Inspect the graph to locate the maximum and minimum values and when they occur.



The graph shows that the maximum voltage is 17 V, occurring at $\left\{t \,\middle|\, t = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\right\}$, and the minimum voltage is 7 V, occurring at $\left\{t \,\middle|\, t = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z}\right\}$.

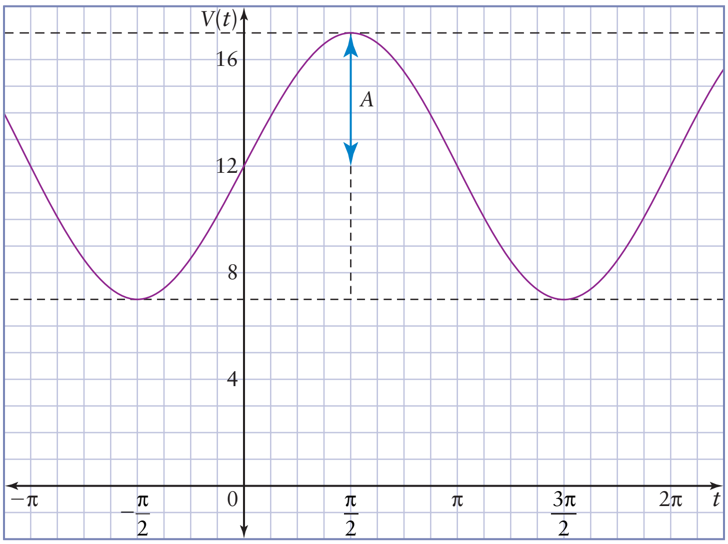
b) The period is the time required for one complete cycle. Since successive maxima and successive minima are each 2π apart, the period is $T = 2\pi$ s.

The frequency is the reciprocal of the period.

$$\begin{aligned} f &= \frac{1}{T} \\ &= \frac{1}{2\pi} \end{aligned}$$

The frequency is $\frac{1}{2\pi}$ Hz.

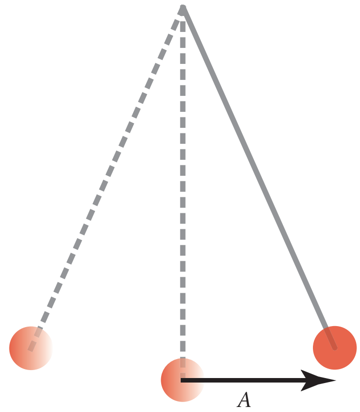
The amplitude of a sinusoidal function is half the difference between the maximum and minimum values.



$$\begin{aligned} A &= \frac{1}{2}(V_{max} - V_{min}) \\ &= \frac{1}{2}(17 - 7) \\ &= 5 \end{aligned}$$

The amplitude is 5 V. Notice that this value appears as the coefficient of $\sin t$ in the equation $V(t) = 5\sin t + 12$.

Example 2 A Simple Pendulum



For small amplitudes, and ignoring the effects of friction, a pendulum is an example of **simple harmonic motion**. Simple harmonic motion is motion that can be modelled by a sinusoidal function, and the graph of a function modelling simple harmonic motion has a constant amplitude. The period of a simple pendulum depends only on its length and can be found using the relation $T = 2\pi\sqrt{\frac{l}{g}}$, where T is the period, in seconds, l is the length of the pendulum, in metres, and g is the acceleration due to gravity. On or near the surface of Earth, g has a constant value of 9.8 m/s^2.

Under these conditions, the horizontal position of the bob as a function of time can be described by the function $h(t) = A\cos\left(\frac{2\pi t}{T}\right)$, where A is the amplitude of the pendulum, t is time, in seconds, and T is the period of the pendulum, in seconds.

Find the maximum speed of the bob and the time at which that speed first occurs for a pendulum having a length of 1.0 m and an amplitude of 5 cm.

Solution

To find the position function for this pendulum, first find the period.

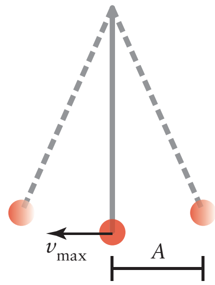
$$\begin{aligned} T &= 2\pi\sqrt{\frac{l}{g}} \\ &= 2\pi\sqrt{\frac{1.0}{9.8}} \\ &\doteq 2.0 \end{aligned}$$

The period of the pendulum is approximately 2.0 s.

Substitute the period and amplitude into the position function.

$$\begin{aligned} h(t) &= A\cos\left(\frac{2\pi t}{T}\right) \\ &= 5\cos\left(\frac{2\pi t}{2.0}\right) \\ &= 5\cos \pi t \end{aligned}$$

Velocity is the rate of change of position with respect to time. The velocity of an object also gives information about the direction in which the object is moving. Since the bob is initially moving in a direction opposite the initial displacement, the first point at which the bob reaches its maximum speed will have a negative velocity.



Let $v(t)$ be the horizontal velocity of the bob at time t.

$$\begin{aligned} v(t) &= \frac{dh}{dt} \\ &= -5\pi \sin \pi t \end{aligned}$$

Acceleration is the rate of change of velocity with respect to time. Let $a(t)$ be the horizontal acceleration of the bob at time t.

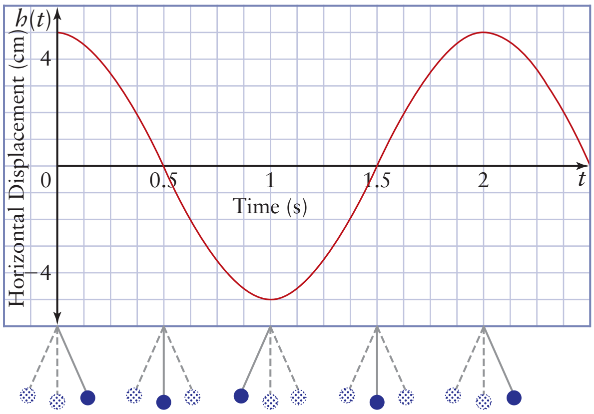
$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= -5\pi^2 \cos \pi t \end{aligned}$$

To find the maximum speed, set the acceleration equal to zero.

$$\begin{aligned} -5\pi^2 \cos \pi t &= 0 \\ \cos \pi t &= 0 \end{aligned}$$

The least positive value for which cosine is 0 is $\frac{\pi}{2}$, and so the equation is satisfied when $\pi t = \frac{\pi}{2}$, which gives $t = 0.5$ s.

So, the maximum speed first occurs when $t = 0.5$ s, or after one quarter of the pendulum's period has elapsed. Think about the pendulum scenario and explain why this result makes sense.



Substitute $t = 0.5$ s into the velocity function to find the maximum speed of the bob.

$$\begin{aligned} v(t) &= -5\pi \sin \pi t \\ v(0.5) &= -5\pi \sin 0.5\pi \\ &= -5\pi \sin\left(\frac{\pi}{2}\right) \\ &= -5\pi \\ &\doteq -15.7 \end{aligned}$$

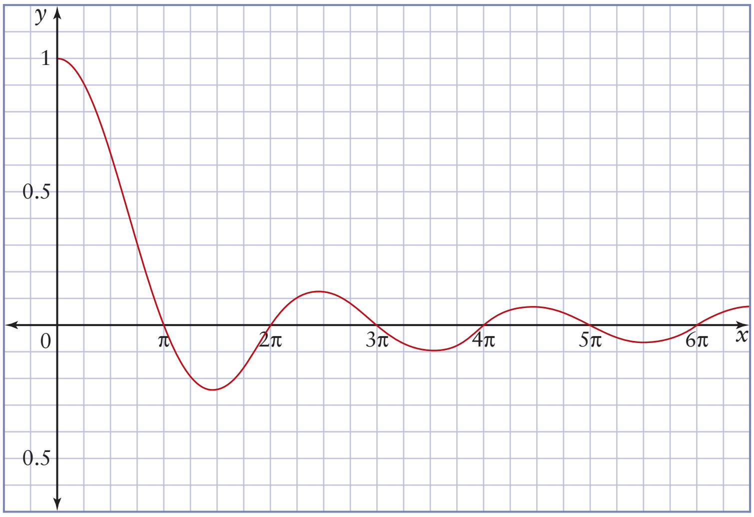
The maximum speed of the pendulum is approximately 15.7 cm/s.

When friction is present, the amplitude of an oscillating pendulum diminishes over time. Harmonic motion for which the amplitude diminishes over time is called **damped harmonic motion**.

CONNECTIONS

You will explore the behaviour of damped harmonic motion in Chapter 5 Exponential and Logarithmic Functions, and if you choose to study physics.

Damped Harmonic Motion



KEY CONCEPTS

- Periodic motion occurs in a variety of physical situations. These situations can often be modelled by sinusoidal functions.
- Investigation of the equations and graphs of sinusoidal functions and their derivatives can yield a lot of information about the real-world situations they model.

Communicate Your Understanding

C1 Refer to Example 1. The DC component of the signal is represented by the constant term of the function $V(t) = 5\sin t + 12$. Suppose this DC component were increased. What impact would this have on

a) the maximum and minimum voltages?

b) the amplitude of the signal?

c) the first derivative of the voltage function?

Explain. Use algebraic reasoning and diagrams in your explanations.

C2 Refer to Example 2. Do velocity and acceleration always have the same sign? Give a real-world example to illustrate your answer.

A Practise

1. An AC-DC coupled circuit produces a current described by the function $I(t) = 60\cos t + 25$, where t is time, in seconds, and $I(t)$ is the current, in amperes, at time t.
   a) Find the maximum and minimum currents, and the times at which they occur.
   b) For the given current, determine
      i) the period, T, in seconds
      ii) the frequency, f, in hertz
      iii) the amplitude, A, in amperes

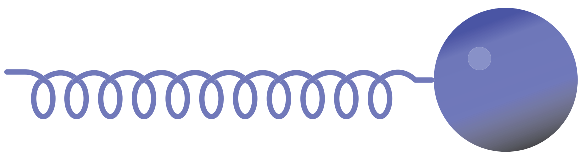
2. The voltage signal from a standard North American wall socket can be described by the equation $V(t) = 170\sin 120\pi t$, where t is time, in seconds, and $V(t)$ is the voltage, in volts, at time t.
   a) Find the maximum and minimum voltage levels, and the times at which they occur.
   b) For the given signal, determine
      i) the period, T, in seconds
      ii) the frequency, f, in hertz
      iii) the amplitude, A, in volts

3. Consider a simple pendulum that has a length of 50 cm and a maximum horizontal displacement of 8 cm.
   a) Find the period of the pendulum.
   b) Determine a function that gives the horizontal position of the bob as a function of time.
   c) Determine a function that gives the velocity of the bob as a function of time.
   d) Determine a function that gives the acceleration of the bob as a function of time.

B Connect and Apply

4. Refer to the situation described in question 3.
   a) Find the maximum velocity of the bob and the time at which it first occurs.
   b) Find the maximum acceleration of the bob and the time at which it first occurs.
   c) Determine the times at which
      i) the displacement equals zero
      ii) the velocity equals zero
      iii) the acceleration equals zero
   d) Describe how the answers in part c) are related in terms of when they occur. Explain why these results make sense.

5. A marble is placed on the end of a horizontal oscillating spring.



If you ignore the effect of friction and treat this situation as an instance of simple harmonic motion, the horizontal position of the marble as a function of time is given by the function $h(t) = A\cos 2\pi ft$, where A is the maximum displacement from rest position, in centimetres, f is the frequency, in hertz, and t is time, in seconds. In the given situation, the spring oscillates every 1 s and has a maximum displacement of 10 cm.

   a) What is the frequency of the oscillating spring?
   b) Write the simplified equation that expresses the position of the marble as a function of time.
   c) Determine a function that expresses the velocity of the marble as a function of time.
   d) Determine a function that expresses the acceleration of the marble as a function of time.

6. Refer to question 5.

a) Sketch a graph of each of the following relations over the interval from 0 to 4 s. Align your graphs vertically.

i) displacement versus time

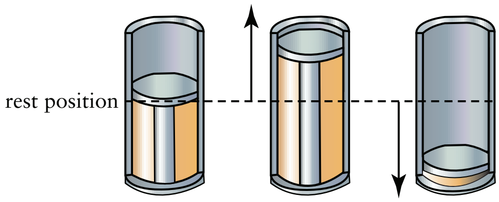
ii) velocity versus time

iii) acceleration versus time

b) Describe any similarities and differences between the graphs.

c) Find the maximum and minimum values for displacement. When do these values occur? Refer to the other graphs and explain why these results make sense.

7. A piston in an engine oscillates up and down from a rest position as shown.



The motion of this piston can be approximated by the function $h(t) = 0.05\cos 13t$, where t is time, in seconds, and $h(t)$ is the displacement of the piston head from rest position, in metres, at time t.

a) Determine an equation for the velocity of the piston head as a function of time.

b) Find the maximum and minimum velocities, and the times at which they occur.

8. A high-power distribution line delivers an AC-DC coupled voltage signal whose

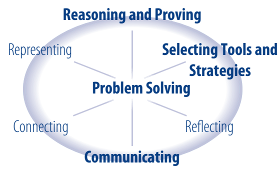
- AC component has an amplitude, A, of 380 kV
- DC component has a constant voltage of 120 kV
- frequency, f, is 60 Hz

a) Add the AC component, V_{AC}, and DC component, V_{DC}, to determine an equation that relates voltage, in kilovolts, to time, t, in seconds. Use the equation $V_{AC}(t) = A\sin 2\pi ft$ to determine the AC component.

b) Determine the maximum and minimum voltages, and the times at which they occur.

9. A **differential equation** is an equation involving a function and one or more of its derivatives. Determine whether the function $y = \pi\sin\theta + 2\pi\cos\theta$ is a solution to the differential equation $\frac{d^2y}{d\theta^2} + y = 0$.

10. a) Determine a function that satisfies the differential equation $\frac{d^2y}{dx^2} = 4y$.



b) Explain how you found your solution.

11. a) Create a differential equation that is satisfied by a sinusoidal function, and show that the function you have created satisfies that equation.

b) Explain how you found your answer to part a).

CONNECTIONS

You will study differential equations in depth at university if you choose to major in engineering or the physical sciences.

Achievement Check

12. An oceanographer measured a set of sea waves during a storm and modelled the vertical displacement of a wave, in metres, using the equation $h(t) = 0.6\cos 2t + 0.8\sin t$, where t is the time in seconds.

a) Determine the vertical displacement of the wave when the velocity is 0.8 m/s.

b) Determine the maximum velocity of the wave and when it first occurs.

c) When does the wave first change from a "hill" to a "trough"? Explain.

C Extend and Challenge

13. **Potential energy** is energy that is stored, for example, the energy stored in a compressed or extended spring. The amount of potential energy stored in a spring is given by the equation $U = \frac{1}{2}kx^2$, where

- U is the potential energy, in joules
- k is the spring constant, in newtons per metre
- x is the displacement of the spring from rest position, in metres

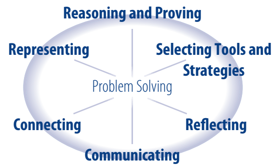
Use the displacement equation from question 5 to find the potential energy of an oscillating spring as a function of time.

> **CONNECTIONS**
>
> The computer simulation referred to in this section's Investigate provides a useful illustration of how potential and kinetic energy are related in the motion of a pendulum.
>
> To further explore the connections between energy and harmonic motion, including the damping effect of friction, go to *www.mcgrawhill.ca/links/calculus12* and follow the links.

14. **Kinetic energy** is the energy of motion. The kinetic energy of a spring is given by the equation $K = \frac{kv^2T^2}{8\pi^2}$, where K is the kinetic energy, in joules; k is the spring constant, in newtons per metre; v is the velocity as a function of time, in metres per second; and T is the period, in seconds.

Use the velocity equation from question 5 to express the kinetic energy of an oscillating spring as a function of time.

15. **Use Technology**



Refer to questions 13 and 14. An oscillating spring has a spring constant of 100 N/m, an amplitude of 0.02 m, and a period of 0.5 s.

a) Graph the function relating potential energy to time in this situation. Find the maxima, minima, and zeros of the potential energy function, and the times at which they occur.

b) Repeat part a) for the function relating kinetic energy to time.

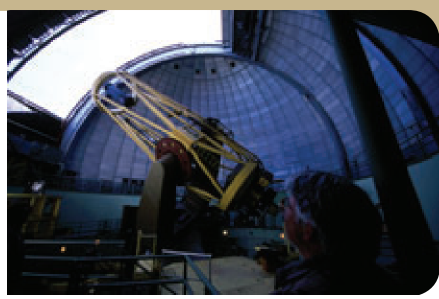
c) Explain how the answers to parts a) and b) are related.

16. **Math Contest** For any constants A and B, the local maximum value of $A\sin x + B\cos x$ is

A $\frac{1}{2}|A + B|$ **B** $|A + B|$ **C** $\frac{1}{2}(|A| + |B|)$

D $|A| + |B|$ **E** $\sqrt{A^2 + B^2}$

CAREER CONNECTION

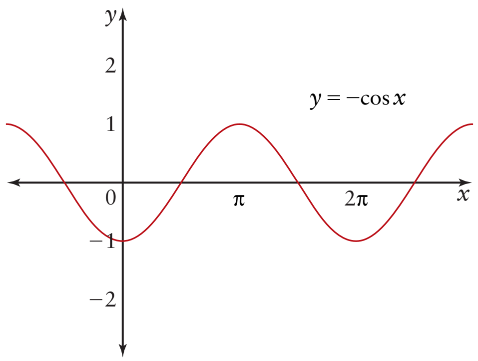


Michael took a 4-year bachelor's degree in astronomy at a Canadian university. As an observational astronomer, he collects and analyses data about stars, planets, and other celestial bodies. Cameras attached to telescopes record a series of values, which correspond to the amount of light objects in the sky are emitting, the type of light emitted, etc. Michael uses trigonometry, calculus, and other branches of mathematics to interpret the values. Michael pays particular attention to objects, like meteors, that might collide with Earth. He also works with meteorologists to predict weather patterns based on solar activity.

Chapter 4 REVIEW

4.1 Investigate Instantaneous Rates of Change of Sinusoidal Functions

1. The graph of a sinusoidal function is shown below.



a) Identify the values in the interval $0 \le x \le 2\pi$ where the slope is

i) zero

ii) a local maximum

iii) a local minimum

b) Sketch a graph of the instantaneous rate of change of this function with respect to x.

2. a) Sketch a graph of the function $y = -3\sin x$.

b) Sketch a graph of the instantaneous rate of change of this function with respect to x.

4.2 Derivatives of the Sine and Cosine Functions

3. Find the derivative of each function with respect to x.

a) $y = \cos x$

b) $f(x) = -2\sin x$

c) $y = \cos x - \sin x$

d) $f(x) = 3\sin x - \pi\cos x$

4. Determine the slope of the function $y = 4\sin x$ at $x = \frac{\pi}{3}$.

5. a) Find the equation of the line tangent to the curve $y = 2\sin\theta + 4\cos\theta$ at $\theta = \frac{\pi}{4}$.

b) Find the equation of the line tangent to the curve $y = 2\cos\theta - \frac{1}{2}\sin\theta$ at $\theta = \frac{3\pi}{2}$.

c) Graph each function and its tangent line from parts a) and b).

4.3 Differentiation Rules for Sinusoidal Functions

6. Differentiate with respect to the variable indicated.

a) $y = -\cos^2 x$

b) $y = \sin 2\theta - 2\cos 2\theta$

c) $f(\theta) = -\frac{\pi}{2}\sin(2\theta - \pi)$

d) $f(x) = \sin(\sin x)$

e) $f(x) = \cos(\cos x)$

f) $f(\theta) = \cos 7\theta - \cos 5\theta$

7. Determine the derivative of each function with respect to the variable indicated.

a) $y = 3x\sin x$

b) $f(t) = 2t^2\cos 2t$

c) $y = \pi t\sin(\pi t - 6)$

d) $y = \cos(\sin\theta) + \sin(\cos\theta)$

e) $f(x) = \cos^2(\sin x)$

f) $f(\theta) = \cos 7\theta - \cos^2 5\theta$

8. a) Find an equation of a line tangent to the curve $f(x) = 2\cos 3x$ whose slope is a maximum.

b) Is this the only possible solution? Explain. If there are other possible solutions, how many solutions are there?

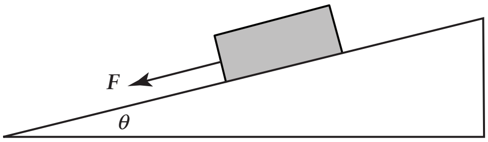
4.4 Applications of Sinusoidal Functions and Their Derivatives

9. The voltage of the power supply in an AC-DC coupled circuit is given by the function $V(t) = 130\sin 5t + 18$, where t is time, in seconds, and $V(t)$ is the voltage, in volts, at time t.

a) Find the maximum and minimum voltages, and the times at which they occur.

b) Determine the period, T, in seconds, the frequency, f, in hertz, and the amplitude, A, in volts, for this signal.

10. A block is positioned on a frictionless ramp as shown below.



The gravitational force component acting on the block directed along the ramp as a function of the angle of inclination, θ, is $F = mg\sin\theta$, where m is the mass of the block and g is the acceleration due to gravity.

a) For what angle of inclination, $0 \le \theta \le \frac{\pi}{2}$, will this force be

i) a maximum?

ii) a minimum?

b) Explain how you found your answers in part a). Explain why these answers make sense.

CONNECTIONS

Force and momentum can both be modelled using vectors, and it is often useful to express these vectors as a sum of vector components. You will learn more about vectors and their components in Chapter 6 Geometric Vectors and Chapter 7 Cartesian Vectors, and in physics.

11. Newton's second law of motion was originally written as $F = \frac{dp}{dt}$, where

- F is the force acting on a body, in newtons
- p is the momentum of the body, in kilogram metres per second, given by the equation $p = mv$ (where m is the mass of the body in kilograms and v the velocity of the body in metres per second)
- t is time, in seconds

a) Assuming a body's mass is constant, use this definition to show that Newton's second law can also be written as $F = ma$, where a is the acceleration of the body.

b) Suppose a body is oscillating such that its velocity, in metres per second, at time t is given by the function $v(t) = 2\cos 3t$, where t is time, in seconds. Find the times when the force acting on the body is zero.

c) What is the speed of the body at these times?

CHAPTER PROBLEM WRAP-UP

Use Technology Design a roller coaster that consists of combinations of sine and cosine functions. Include at least three different segments. Your design must satisfy the following criteria:

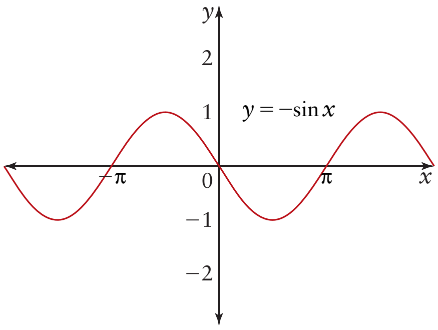
- Riders must be at least 2 m above the ground throughout the ride, and never higher than 20 m above the ground.
- There must be one segment in which the maximum slope has a value between 3 and 5.
- The slope must never exceed 6.
- The segments must join together in a reasonably continuous way (i.e., no sudden bumps or changes in direction).

Write equations to model each segment of your roller coaster. Use algebraic and graphical reasoning to show that your roller coaster meets all of the design requirements.

Chapter 4 PRACTICE TEST

For questions 1 to 8, select the best answer.

1. Where do the minimum instantaneous rates of change occur for the function shown here?



A $(2k + 1)\pi,\ k \in \mathbb{Z}$ **B** $2k\pi,\ k \in \mathbb{Z}$

C $\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ **D** $\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$

2. If $f(x) = \sin 3x + \cos 2x$, then $f'(x) =$

A $\cos 3x - \sin 2x$ **B** $3\sin x + 2\cos x$

C $3\cos 3x - 2\sin 2x$ **D** $5\cos 5x$

3. If $g(x) = 3\cos 2x$, then $g'(x) =$

A $6\cos x$ **B** $-6\sin x$

C $-6\sin 2x$ **D** $-6\cos 2x$

4. If $h(x) = x\sin x$, then $h'(x) =$

A $\cos x$ **B** $\sin x - \cos x$

C $\sin x + x - \cos x$ **D** $\sin x + x\cos x$

5. Which is the second derivative of the function $y = -\cos x$ with respect to x?

A $\frac{d^2y}{dx^2} = \sin x$ **B** $\frac{d^2y}{dx^2} = -\sin x$

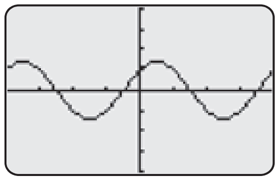
C $\frac{d^2y}{dx^2} = \cos x$ **D** $\frac{d^2y}{dx^2} = -\cos x$

6. What is the slope of the curve $y = 2\sin x$ at $x = \frac{\pi}{3}$?

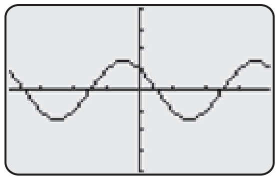
A $-\sqrt{3}$ **B** $\sqrt{3}$ **C** -1 **D** 1

7. Which graph best shows the derivative of the function $y = \sin x + \cos x$ when the window variables are $x \in [-2\pi, 2\pi]$, $\text{Xscl} = \frac{\pi}{2}$, $y \in [-4, 4]$?

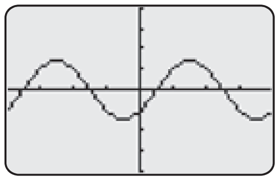
A



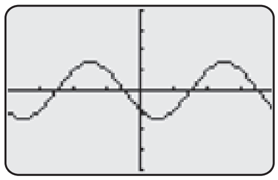
B



C



D



8. Which is the derivative of $y = \cos\theta \sin\theta$ with respect to θ?

A $\frac{dy}{d\theta} = \cos^2\theta + \sin^2\theta$

B $\frac{dy}{d\theta} = \cos^2\theta - \sin^2\theta$

C $\frac{dy}{d\theta} = \sin^2\theta - \cos^2\theta$

D $\frac{dy}{d\theta} = -\sin^2\theta - \cos^2\theta$

9. Differentiate with respect to the variable indicated.

   a) $y = \cos x - \sin x$

   b) $y = 3\sin 2\theta$

   c) $f(x) = -\frac{\pi}{2}\cos^2 x$

   d) $f(t) = 3t^2 \sin t$

10. Differentiate with respect to θ.

   a) $y = \sin\left(\theta + \frac{\pi}{4}\right)$

   b) $y = \cos\left(\theta - \frac{\pi}{4}\right)$

   c) $y = \sin^4\theta$

   d) $y = \sin\theta^4$

11. Find the slope of the line tangent to the curve $y = 2\sin x\cos x$ at $x = \frac{\pi}{4}$.

12. Find the equation of the line tangent to the curve $y = 2\cos^3 x$ at $x = \frac{\pi}{3}$.

13. The voltage signal from a standard European wall socket can be described by the equation $V(t) = 325\sin 100\pi t$, where t is time, in seconds, and $V(t)$ is the voltage at time t.

   a) Find the maximum and minimum voltage levels, and the times at which they occur.

   b) Determine

      i) the period, T, in seconds

      ii) the frequency, f, in hertz

      iii) the amplitude, A, in volts

14. Refer to question 13. Compare the standard wall socket voltage signals of Europe and North America. Recall that in North America, the voltage is described by the function $V(t) = 170\sin 120\pi t$.

   a) Repeat question 13 using the function for the North American wall socket voltage.

   b) Discuss the similarities and differences between these functions.

15. Differentiate both sides of the double angle identity $\sin 2x = 2\sin x\cos x$ to determine an identity for $\cos 2x$.

16. Let $f(x) = \sin x\cos x$.

   a) Determine the first derivative, $f'(x)$, and the second derivative, $f''(x)$.

   b) Determine the third, fourth, fifth, and sixth derivatives. Describe any patterns that you notice.

   c) Make a prediction for the seventh and eighth derivatives. Calculate these and check your predictions.

   d) Write an expression for each of the following derivatives of $f(x)$:

      i) $f^{(2n)}(x)$, $n \in \mathbb{N}$

      ii) $f^{(2n+1)}(x)$, $n \in \mathbb{N}$

   e) Use these results to determine

      i) $f^{(12)}(x)$

      ii) $f^{(15)}(x)$

17. **Use Technology** Use *The Geometer's Sketchpad*®, other graphing software, or a computer algebra system (CAS).

   a) Find a function $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = \frac{d^5y}{dx^5}$, where $\frac{d^5y}{dx^5}$ is the fifth derivative of $f(x)$.

   b) Find two other functions that satisfy this differential equation.

   c) How many solutions to this differential equation exist? Explain your reasoning.

CONNECTIONS

Can you think of a function that satisfies the following differential equation: $f(x) = f'(x)$? You will study such a function in Chapter 5 Exponential and Logarithmic Functions.

TASK

Double Ferris Wheel

Some amusement parks have a double Ferris wheel, which consists of two vertically rotating wheels that are attached to each other by a bar that also rotates. There are eight gondolas equally spaced on each wheel. Riders experience a combination of two circular motions that provide a sensation more thrilling than the classic single Ferris wheel. In particular, riders experience the greatest sensation when their rate of change in height is the greatest.

- Each of the two wheels is 6 m in diameter and revolves every 12 s.
- The rotating bar is 9 m long. The ends of the bar are attached to the centres of the wheels.
- The height from the ground to the centre of the bar is 8 m. The bar makes a complete revolution every 20 s.
- A rider starts seated at the lowest position and moves counterclockwise.
- The bar starts in the vertical position.

Consider the height of a rider who begins the ride in the lowest car.

a) Write a function $f(t)$ that expresses the height of the rider relative to the centre of the wheel at time t seconds after the ride starts. Write a second function $g(t)$ that expresses the position of the end of the bar (the centre of the rider's wheel) relative to the ground at time t seconds.

b) Explain how the sum of these two functions gives the rider's height above the ground after t seconds.

c) Use technology to graph the two functions and their sum for a 2-min ride.

d) What is the maximum height reached by the rider? When does this occur?

e) What is the maximum vertical speed of the rider? When does this occur?

f) Design your own double Ferris wheel. Determine the position function for a rider on your wheel. What is the maximum speed experienced by your riders? Is there a simple relationship between the dimensions of the Ferris wheel and the maximum heights or speeds experienced?