Chapter 5

Exponential and Logarithmic Functions

In this chapter, you will investigate the rate of change of exponential functions and discover some interesting properties of the numerical value e. You will find that this value frequently appears in the natural sciences and business, and you will see how it is related to a special type of logarithmic function. You will extend your understanding of differential calculus by exploring and applying the derivatives of exponential functions.

By the end of this chapter, you will

- determine, through investigation using technology, the graph of the derivative $f'(x)$ or $\frac{dy}{dx}$ of a given exponential function, and make connections between the graphs of $f(x)$ and $f'(x)$ or y and $\frac{dy}{dx}$
- determine, through investigation using technology, the exponential function $f(x) = a^x$ ($a > 0$, $a \neq 1$) for which $f'(x) = f(x)$, identify the number e to be the value of a for which $f'(x) = f(x)$, and recognize that for the exponential function $f(x) = e^x$ the slope of the tangent at any point on the function is equal to the value of the function at that point
- recognize that the natural logarithmic function $f(x) = \log_e x$, also written as $f(x) = \ln x$, is the inverse of the exponential function $f(x) = e^x$, and make connections between $f(x) = \ln x$ and $f(x) = e^x$
- verify, using technology, that the derivative of the exponential function $f(x) = a^x$ is $f'(x) = a^x \ln a$ for various values of a
- solve problems, using the product and chain rules, involving the derivatives of polynomial functions, sinusoidal functions, exponential functions, rational functions, radical functions, and other simple combinations of functions
- make connections between the graphical or algebraic representations of derivatives and real-world applications
- solve problems, using the derivative, that involve instantaneous rates of change, including problems arising from real-world applications, given the equation of a function
- solve optimization problems involving polynomial, simple rational, and exponential functions drawn from a variety of applications, including those arising from real-world situations
- solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate the results

Prerequisite Skills

Graphing Exponential and Logarithmic Functions

1. **a)** Graph the function $y = 2^x$.

b) Graph its inverse on the same grid, by reflecting the curve in the line $y = x$.

c) What is the equation of the inverse? Explain how you know.

2. Identify the following key features of the graphs of $y = 2^x$ and its inverse from question 1.

a) domain and range

b) any x-intercepts and y-intercepts

c) intervals for which the function is increasing or decreasing

d) the equations of any asymptotes

3. Use the graphs from question 1 to estimate the following values.

a) 2^3 **b)** $2^{3.5}$ **c)** $2^{1.5}$

d) $\log_2 10$ **e)** $\log_2 7$ **f)** $\log_2 4.5$

4. Check your answers to question 3 using a calculator.

Changing Bases of Exponential and Logarithmic Expressions

5. Rewrite each exponential function using a base of 2.

a) $y = 8^x$

b) $y = 4^{2x}$

c) $y = 16^{\frac{x}{2}}$

d) $y = \left(\frac{1}{4}\right)^{-2x}$

CONNECTIONS

Logarithms in base a can be converted to logarithms in base c using the formula $\log_a b = \dfrac{\log_c b}{\log_c a}$.

6. Express each logarithm in terms of common logarithms (base-10 logarithms), and then use a calculator to evaluate. Round answers to three decimal places.

a) $\log_2 5$ **b)** $\log_4 66$

c) $\log_3 10$ **d)** $\log_2 7$

e) $\log_3 75$ **f)** $\log_5\left(\frac{1}{10}\right)$

g) $\log_{\frac{1}{2}} \frac{1}{4}$ **h)** $\log_{0.5} 5$

Applying Exponent Laws and Laws of Logarithms

7. Simplify. Express answers with positive exponents.

a) $(h^2k^3)(hk^{-2})$ **b)** $(a^3)(ab^3)^2$

c) $\dfrac{(x)(y^3)^{-2}}{(x^3y^3)^4}$ **d)** $\dfrac{8u^3v^{-2}}{4uv^{-1}}$

e) $(g^2)(gh^3)^{-2}$ **f)** $x^2x^4 + (x^2)^3$

g) $\dfrac{2^x4^y}{4^{-x}}$ **h)** $\dfrac{a^xb^{2x}}{ab^x}$

8. Evaluate, by applying the laws of logarithms.

a) $\log 5 + \log 2$

b) $\log_2 24 - \log_2 3$

c) $\log_5 50 - \log_5 0.08$

d) $\log(0.01)^3$

e) $\log\sqrt{1000} + \log\sqrt[3]{100}$

f) $2\log 2 + 2\log 5$

9. Simplify.

a) $\log a - \log 2a$

b) $\log ab + \log a - \log ab^2$

c) $4\log a^2 - 4\log a$

d) $3\log a^2b + 3\log ab^2$

e) $\log 2a^2b + \log 2b^2$

Solving Exponential and Logarithmic Equations

10. Solve for x.

a) $2^x = 4^{x+1}$

b) $4^{2x+1} = 64^x$

c) $3^{2x-5} = \sqrt{27}$

d) $\log x - \log 2 = \log 5$

e) $\log 5 + \log x = 3$

f) $x - 3\log 5 = 3\log 2$

11. Solve for x. Round your solution to two decimal places where required.

a) $2 = 1.06^x$ **b)** $50 = 5^{2x}$

c) $10 = \left(\frac{1}{2}\right)^x$ **d)** $75 = 225(2)^{-\frac{x}{4}}$

Constructing and Applying an Exponential Model

12. A bacterial culture whose initial population is 50 doubles in population every 3 days.

a) Determine the population after

i) 3 days

ii) 6 days

iii) 9 days

b) Which of the following equations correctly gives the population, P, as a function of time, t, in days?

A $P = 50(2)^t$

B $P = 50(2)^{3t}$

C $P = 50(2)^{\frac{t}{3}}$

D $P = 50\left(\frac{1}{2}\right)^{\frac{t}{3}}$

E $P = 50\left(\frac{1}{2}\right)^t$

F $P = 50\left(\frac{1}{2}\right)^{3t}$

c) Use mathematical reasoning to justify your answer to part b).

13. A radioactive substance with an initial mass of 100 g has a half-life of 5 min.

a) Copy and complete the following table.

Time (min)	Amount Remaining (g)
0	100
5	50
10	
15	
20	

b) Write an equation that expresses the amount remaining, A, in grams, as a function of time, t, in minutes.

c) Use the equation to determine the amount of radioactive material remaining after

i) half an hour

ii) half a day

CHAPTER PROBLEM

Sheona is a second-year electrical engineering student at university. As part of a cooperative education program she has been placed to work in the research and development department of a high-tech firm. Her supervisor has assigned her tasks that involve the testing and troubleshooting of various electrical circuits and components. Sheona will be required to apply concepts related to exponential and logarithmic functions during her work term. What other applications of exponential and logarithmic functions might she encounter?

5.1 Rates of Change and the Number *e*

Exponential functions occur frequently in various areas of study, such as science and business. Consider the following situations.

Suppose the number of rabbits in the picture were doubling every n days. The population of rabbits could be expressed as $P(t) = P_0(2)^{\frac{t}{n}}$, where P_0 is the initial population, and $P(t)$ is the size of the population after t days.

The mass of radioactive material remaining after x half-life periods can be written as $M = M_0\left(\frac{1}{2}\right)^x$, where M_0 is the initial mass, and M is the mass remaining, in grams.

The value of a \$500 investment earning 8% compound interest per year is given by the equation $A = 500(1.08)^t$, where A is the value of the investment after t years.

What do all of these functions have in common? Think about what happens to the dependent variable each time you increase the independent variable by 1. How quickly do the dependent variables grow as time passes?

As with other types of functions, the rate of change of an exponential function is an important concept. Analysing this rate of change will lead to the development of the derivative of an exponential function and its use in applications.

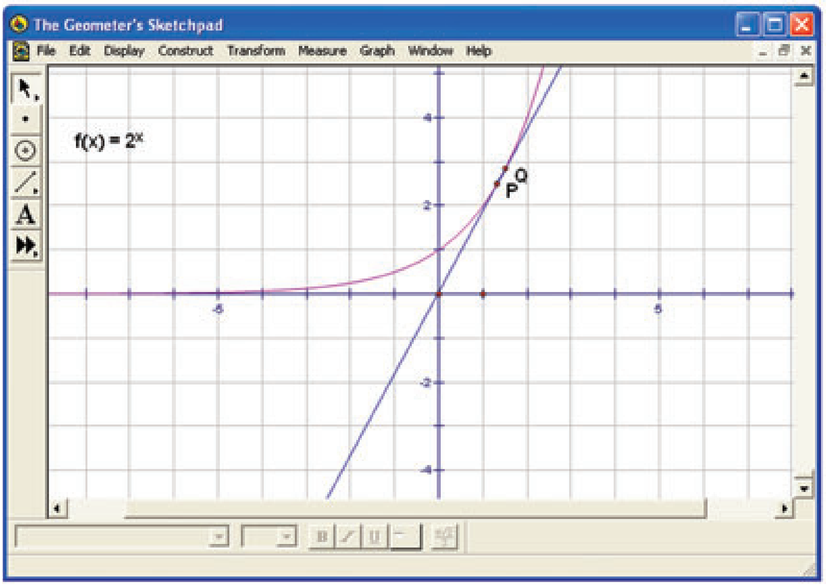
Investigate A What is the behaviour of the rate of change of an exponential function?

Tools

- computer with *The Geometer's Sketchpad*®

A: Find the Rate of Change of an Exponential Function

1. Graph the function $f(x) = 2^x$.
2. Construct a secant that is almost tangent to the graph of f.
   - Construct two points, P and Q, on f.
   - Construct a line through P and Q.
   - Drag the points so that they are very close together, but still far enough apart to distinguish.



Technology Tip

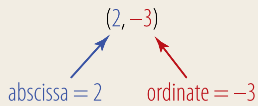
Refer to the Technology Appendix for instructions on how to perform some basic functions using *The Geometer's Sketchpad*®, such as constructing points and lines.

3. **a)** Measure and plot the rate of change of this function as a function of x.
   - Measure the **Abscissa (x)** of point P.
   - Measure the slope of the line passing through PQ.
   - Select these two measures, in order, and from the **Graph** menu, choose **Plot as (x, y)**.
   - From the **Display** menu, choose **Trace Plotted Point**.
   - Select points P and Q. Drag them along f to trace out the rate of change.

   b) Describe the shape of the function that gives the rate of change of f as a function of x.

CONNECTIONS

The x-coordinate of a point is called the abscissa, and the y-coordinate is called the ordinate.



4. **a)** Repeat steps 1 to 3 for a variety of functions of the form $f(x) = b^x$.
   - $f(x) = 1.5^x$
   - $f(x) = 2.5^x$
   - $f(x) = 3^x$
   - $f(x) = 3.5^x$

   b) Focus on the first quadrant of the graphs you have just created. For which of these functions is the rate of change a function.
   - below f?
   - above f?

5. **Reflect** Analyse the instantaneous rate of change of the function $f(x) = b^x$, and classify the instantaneous rate of change according to its shape. What conclusion can you draw about the nature of the derivative of an exponential function?

   The Geometer's Sketchpad® has a built-in feature that can calculate derivatives, which can be used to verify these findings.

Technology Tip

Follow these steps to graph the derivative of a function:

- Select the equation for $f(x)$.
- From the **Graph** menu, choose **Derivative**.
- From the **Graph** menu, choose **Plot Function**.

Technology Tip

Follow these steps to create a parameter:

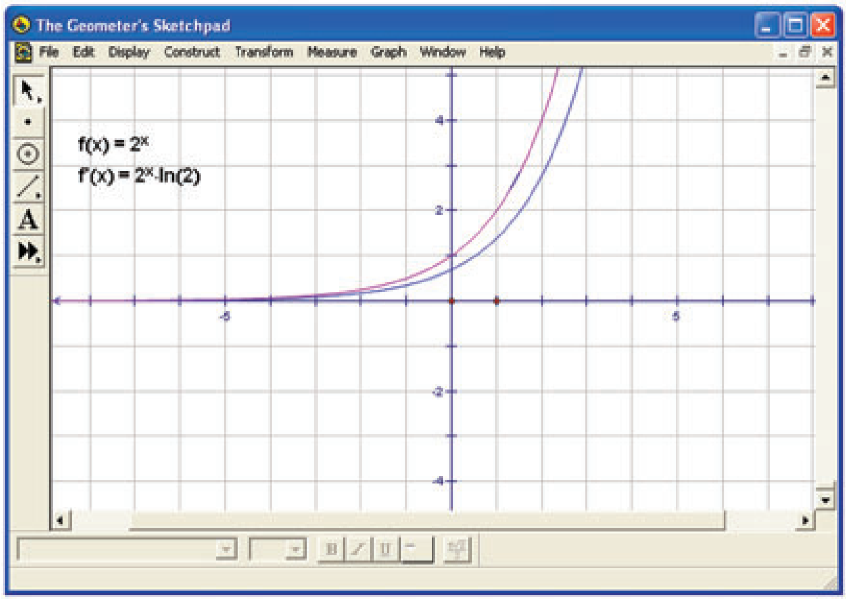
- From the **Graph** menu, choose **New Parameter**. Call the parameter k.
- From the **Graph** menu, choose **Plot New Function**.
- Plot the function $g(x) = k*f(x)$.

The parameter k can be adjusted manually either by right-clicking on it and choosing **Edit Parameter** or by selecting it and pressing + or −.
The parameter k can be adjusted dynamically by right-clicking on it and choosing **Animate Parameter**. This will enable the **Motion Controller**.

B: Confirm the Nature of the Rate of Change

1. a) Do you think that the rate of change of an exponential function is exponential? Justify your prediction.

b) Graph the function $f(x) = 2^x$. Graph its derivative.



The unfamiliar constant number, $\ln(2)$, will be investigated in the next section.

2. Can $f(x)$ be stretched or compressed vertically onto $f'(x)$?

a) Multiply $f(x)$ by an adjustable parameter, k.

b) Try to adjust the value of k so that $kf(x)$ matches $f'(x)$ exactly.

3. Repeat steps 1 and 2 for exponential functions having different bases. Can functions of the form $y = b^x$ be mapped onto their derivatives by applying a vertical stretch or compression? For what bases is a vertical stretch needed? For what bases is a vertical compression needed?

4. Reflect Explain why this confirms that the derivative of an exponential function is also exponential.

The two parts of Investigate A seem to suggest that there exists a value for the base b that is between 2.5 and 3 for which the rate of change of $f(x) = b^x$ as a function of x is identical to $f'(x)$. In the next Investigate you will determine this value, known as e.

Investigate B What is the value of the number *e*?

A: Rough Approximations of *e*.

1. Evaluate the expression $\left(1+\frac{1}{n}\right)^n$ for
   - **a)** $n = 1$
   - **b)** $n = 2$
   - **c)** $n = 3$
   - **d)** $n = 4$
   - **e)** $n = 10$
   - **f)** $n = 100$
2. **Reflect** What value do these expressions seem to be approaching?

B: Better Approximations of *e*.

The actual value of e can be determined by evaluating the following limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

1. How can the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ be evaluated? Think about the following tools:
   - *The Geometer's Sketchpad*®
   - spreadsheet
   - scientific calculator
   - graphing calculator
   - computer algebra system (CAS)

Choose two of these tools and write down a strategy for how you could use each tool to determine a reasonably accurate approximate value of e.

2. **a)** Use one of the tools and strategies from step 1 to determine e, correct to two decimal places.

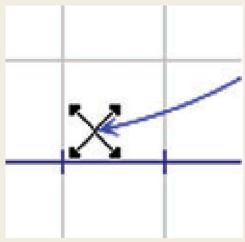
   b) Use a different tool and strategy to confirm your result. Be sure to include at least one graphical approach and one numerical approach.

Tools

- computer with *The Geometer's Sketchpad*®
- graphing calculator
- computer algebra system (CAS)

Technology Tip

When working with *The Geometer's Sketchpad*®, sometimes you can extend the graph of a function that is truncated. Make the cursor hover over the arrow that appears at the end of the function until the cursor becomes a multidirectional arrow, and then click and drag.



3. **Reflect**

a) Identify the advantages and disadvantages of each tool and strategy you used to find an approximate value of e.

b) Using the strategy of your choice, determine the value of e, correct to four decimal places.

c) Explain how you know that your value is correct.

4. **Reflect** Verify this result by graphing the function $y = e^x$ using the value of e you discovered, and then graphing its derivative. What do you notice?

KEY CONCEPTS

- The symbol e is an irrational number whose value is defined as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. The value of this limit lies between 2.71 and 2.72.
- The rate of change of an exponential function is also exponential in nature.
- The derivative of an exponential is a function that is a vertical stretch or compression of the original function.
- The derivative of the function $y = e^x$ is the same function, i.e., $y' = e^x$.

Communicate Your Understanding

C1 a) List the features that are common to all exponential functions.

b) What features can be different among exponential functions?

C2 Describe the nature of the rate of change of an exponential function. Use an example to illustrate your answer.

C3 a) What is the approximate value of the number e?

b) What is significant about the function $y = e^x$?

A Practise

1. Without using technology, sketch the following graphs on the same axes.

a) $f(x) = 2^x$

b) $f(x) = 3^x$

c) $f(x) = 5^x$

d) $f(x) = 10^x$

2. Sketch the graphs of the derivatives of the functions in question 1.

3. a) What is the domain of each function?

$f(x) = 2^x$

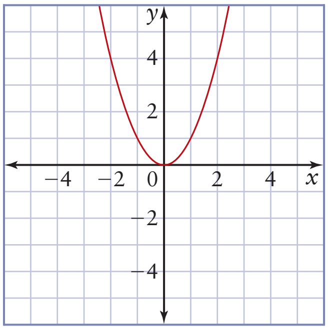
$f(x) = e^x$

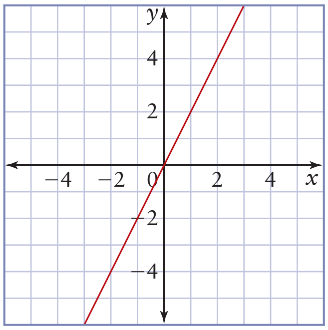
b) Is it possible for 2^x to equal a negative number for some value of x?

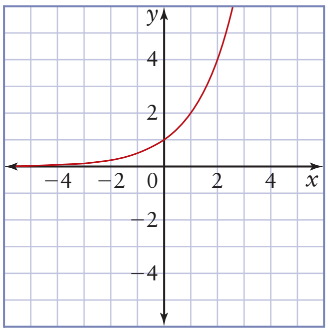
c) Is it possible for e^x to equal a negative number for some value of x?

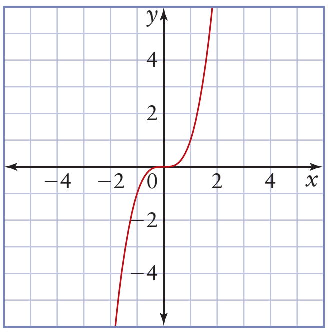
4. Match each graph with the graph of its derivative function. Justify your choices.

Function

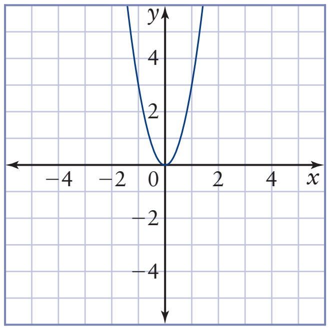
a)


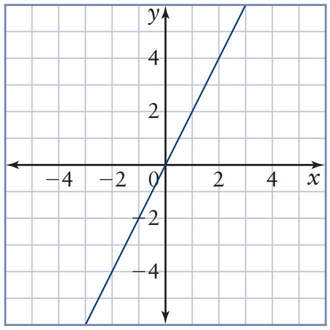
b)


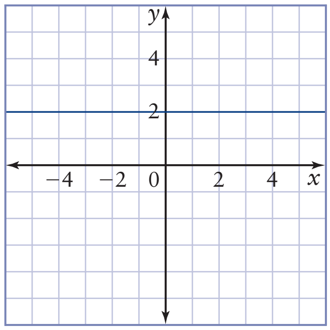
c)


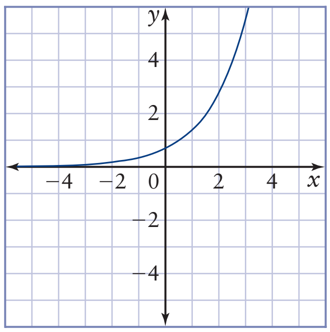
d)


Derivative Function

A


B


C


D


5. For what range of values of b is the graph of the derivative of $f(x) = b^x$

a) a vertical stretch of $f(x)$?

b) a vertical compression of $f(x)$?

B Connect and Apply

6. a) Sketch the graph of $y = \left(\frac{1}{2}\right)^x$.

b) Predict the shape of the graph that is the rate of change of this function with respect to x.

c) Use graphing technology to check your prediction.

7. What is different about the rate of change of an exponential function of the form $f(x) = b^x$ in cases where $0 < b < 1$, compared to cases where $b > 1$? Support your answer with examples and sketches.

8. **Use Technology**

a) Graph the function $f(x) = 4^x$.

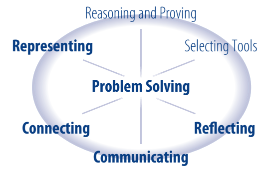
b) Graph the derivative of $f(x)$.

c) Predict the shape of the graph of the combined function $g(x) = \frac{f'(x)}{f(x)}$.

d) Graph $g(x)$ and compare its shape to the one you predicted. Explain why this shape makes sense.

9. Refer to question 8.

a) Will the shape of the graph of g change if f has a base other than 4? Explain, using words and diagrams.



b) What is special about $g(x)$ when $f(x) = e^x$?

Achievement Check

10. Let $f(x) = \left(\frac{1}{b}\right)^x$ for $b > 1$, and let $g(x) = \frac{f'(x)}{f(x)}$.

a) Predict the shape of the graph of $g(x)$.

b) Test your prediction by exploring two specific cases.

c) Summarize your findings, using words and diagrams.

d) For what value of b will $g(x) = -1$?

C Extend and Challenge

11. Let $f(x) = b^x$ for $b > 0$, and let $g(x) = \frac{f'(x)}{f(x)}$.

a) Predict the equation and shape of the function $g'(x)$.

b) Check your prediction using graphing technology.

c) Explain why this result does not depend on the value of b.

12. Consider $f(x) = \frac{e^x}{e^x + c}$ where c is a constant greater than 0.

a) What is the domain of the function?

b) What is the range of the function?

c) How does the value of c affect the graph?

13. Carry out an Internet search on the number e. Write a brief summary that includes

- which mathematician(s) the number has been named for
- when it was first identified
- other interesting facts

14. **Math Contest** A function f satisfies the property $f(x)f(y) = f(x + y)$. If $f(1) = k \neq 0$, then for any positive integer n, $f(-n)$ equals

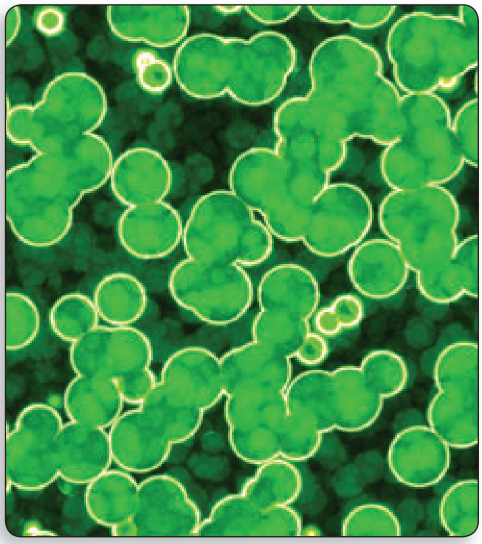
A $-kn$ **B** k^{-n} **C** n^{-k} **D** $-k^n$ **E** $-n^k$

15. **Math Contest** $\lim_{n\to\infty}\left(1 + \frac{3}{n}\right)^n =$

A e **B** $3e$ **C** $\frac{3}{e}$ **D** e^3 **E** $3e^3$

5.2 The Natural Logarithm

The number e is an irrational number, similar in nature to π. Its non-terminating, non-repeating value is $e = 2.718\ 281\ 828\ 459\ \ldots$



Like π, e also occurs frequently in natural phenomena. In fact, of the three most commonly used bases in exponential functions—2, e, and 10—e is used most frequently. Why is this? What could this unusual number have to do with such things as a bacterial culture?

The symbols e and π are both examples of **transcendental numbers**: real numbers that cannot be roots of a polynomial equation with integer coefficients.

As it turns out, e has some interesting properties. For example, the instantaneous rate of change of $y = e^x$ as a function of x produces the exact same graph; therefore, every higher-level derivative of $y = e^x$ also produces the same graph.

Example 1 The Natural Logarithm

a) Graph the function $y = e^x$ and its inverse using technology.

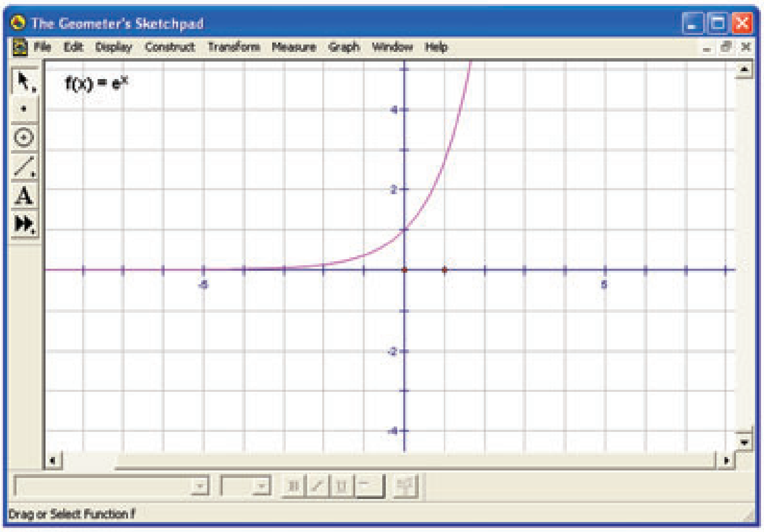
b) Identify the key features of the graphs.

Solution

a) Use graphing technology.

Method 1: ***Use The Geometer's Sketchpad®***

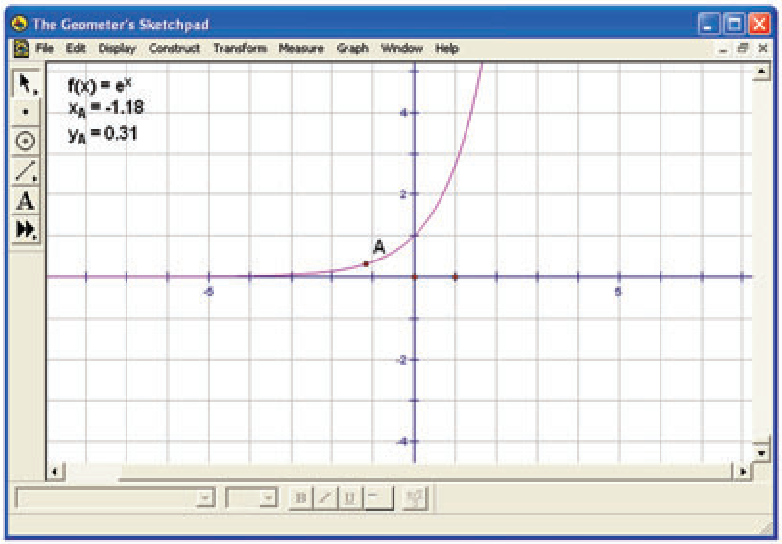
Plot the function $y = e^x$.



Technology Tip

You can access the value e from the **Values** menu in the **New Function** dialog box.

Construct a point on the graph and measure its abscissa (x-coordinate) and ordinate (y-coordinate).

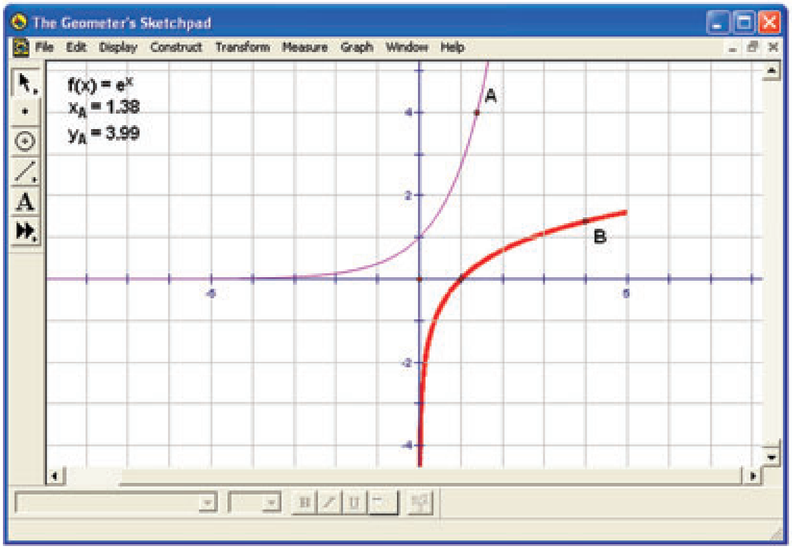


Technology Tip

You can graph a point on the inverse function by reversing the roles of x and y.

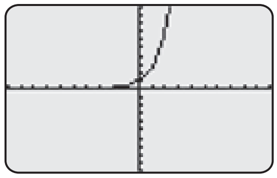
Select the graphs, and plot and trace the inverse.

- Click on the ordinate (y) and abscissa (x), in order.
- From the **Graph** menu, choose **Plot as (x, y)**.
- From the **Display** menu, choose **Trace Plotted Point**.
- Click and drag the point on the function plot to trace out the inverse of $y = e^x$.



Method 2: ***Use a Graphing Calculator***

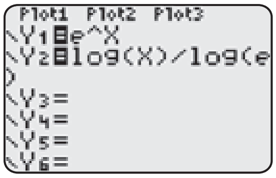
Graph the function $y = e^x$.

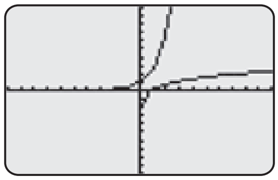


The inverse of the exponential function is the logarithmic function. The **log** function on the graphing calculator graphs logarithmic functions with base 10. To graph $y = \log_e x$, rewrite this in terms of base 10:

$$\begin{aligned} y &= \log_e x \\ &= \frac{\log x}{\log e} \end{aligned}$$

Graph this function.



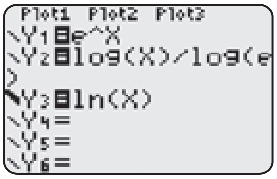


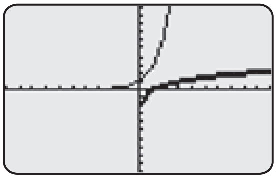
Notice that these graphs are reflections of each other in the line $y = x$, confirming that they are indeed inverse functions of each other.
The logarithmic function having base e occurs very frequently, and has a special name.

The **natural logarithm** of x is defined as $\ln x = \log_e x$.

The left side of this equation is the natural logarithm of x, and is read as "lon x."

You can confirm that $y = \ln x$ is the same function as $y = \log_e x$ by graphing both functions together, using different line styles:





b) The following table lists the key features of each graph.

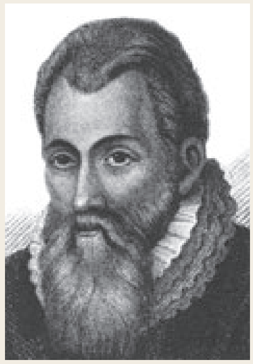
$y = e^x$	$y = \ln x$
Domain: $x \in \mathbb{R}$	Domain: $\{x \mid x > 0, x \in \mathbb{R}\}$
Range: $\{y \mid y > 0, y \in \mathbb{R}\}$	Range: $y \in \mathbb{R}$
Increasing on its domain	Increasing on its domain
y-intercept $= 1$	No y-intercept
No x-intercept	x-intercept $= 1$
Horizontal asymptote at $y = 0$ (x-axis)	Vertical asymptote at $x = 0$ (y-axis)
No minimum or maximum point	No minimum or maximum point
No point of inflection	No point of inflection

CONNECTIONS

Recall that $\log_a b = \frac{\log_c b}{\log_c a}$.

CONNECTIONS

Natural logarithms are also sometimes called Naperian logarithms, named after the Scottish mathematician and philosopher John Napier (1550–1617).



Napier is also famous for inventing the decimal point, as well as a very primitive form of a mechanical calculator.

Example 2 Evaluate e^x

Evaluate, correct to three decimal places.

a) e^3

b) $e^{-\frac{1}{2}}$

Solution

Use a scientific or graphing calculator to find an accurate value. These calculators have a dedicated button for e.

a) $e^3 \doteq 20.086$

b) $e^{-\frac{1}{2}} = e^{-0.5}$

$\doteq 0.607$

Example 3 Evaluate ln *x*

Evaluate, correct to two decimal places.

a) $\ln 10$ **b)** $\ln(-5)$ **c)** $\ln e$

Solution

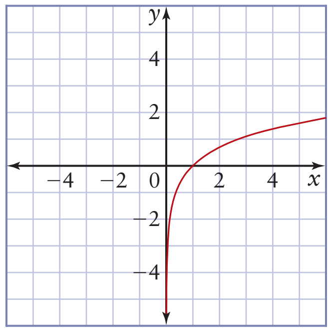
Scientific and graphing calculators have a button that can be used to evaluate natural logarithms, labelled LN.

a) $\ln 10 \doteq 2.30$

b) $\ln(-5)$ is undefined. Recall that the domain of all logarithmic functions, including $y = \ln x$, is $\{x \mid x > 0, x \in \mathbb{R}\}$, so natural logarithms can only be found for positive numbers.

c) Recall that $\ln e = \log_e e$ and $\log_b b^x = x$.

Therefore, $\ln e = 1$.



$$\ln e^x = x \text{ and } e^{\ln x} = x$$

These properties are useful when you are solving equations involving exponential and logarithmic functions.

Example 4 Bacterial Growth

The population of a bacterial culture as a function of time is given by the equation $P(t) = 200e^{0.094t}$, where P is the population after t days.

a) What is the initial population of the bacterial culture?

b) Estimate the population after 3 days.

c) How long will the bacterial culture take to double its population?

d) Rewrite this function as an exponential function having base 2.

Solution

a) To determine the initial population, set $t = 0$.

$$\begin{aligned} P(0) &= 200e^{0.094(0)} \\ &= 200e^{0} \\ &= 200(1) \\ &= 200 \end{aligned}$$

The initial population is 200.

b) Set $t = 3$ to determine the population after 3 days.

$$\begin{aligned} P(3) &= 200e^{0.094(3)} \\ &= 200e^{0.282} \\ &= 200(1.325\,7\ldots) \\ &\doteq 265 \end{aligned}$$

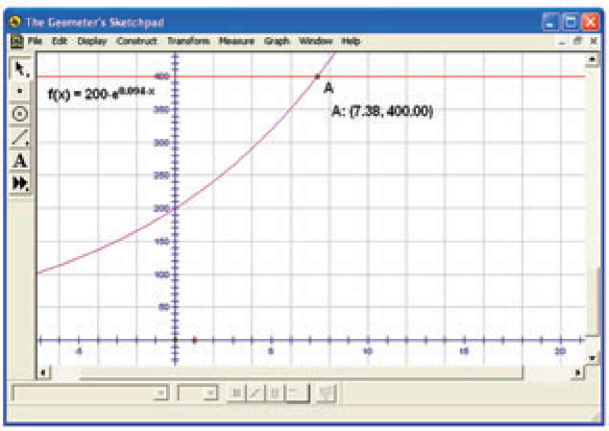
After 3 days, the bacterial culture will have a population of approximately 265.

c) To find the time required for the population to double, determine t when $P(t) = 400$.

$$400 = 200e^{0.094t}$$

Method 1: ***Use Graphical Analysis***

Graph the function using graphing technology to identify the value of t when $P(t) = 400$.



The graph shows that the population will double after about 7.4 days.

Method 2: ***Use Algebraic Reasoning***

Use natural logarithms to solve this equation algebraically.

$$\begin{aligned}
400 &= 200e^{0.094t} \\
2 &= e^{0.094t} \\
\ln 2 &= \ln e^{0.094t} && \text{Take the natural logarithm of both sides.} \\
\ln 2 &= 0.094t \\
\frac{\ln 2}{0.094} &= t \\
t &\doteq 7.4
\end{aligned}$$

Therefore, the bacterial culture will double after approximately 7.4 days.

d) Since the bacterial culture has an initial population of 200 and doubles after 7.4 days, the relationship between population and time can be approximated by the function

$$P(t) = (200)2^{\frac{t}{7.4}},$$

where P is the population after t days.

Note that $\frac{t}{7.4}$ expresses time in terms of the number of doubling periods.

KEY CONCEPTS

- The value of e, correct to several decimal places, is $e = 2.718\ 281\ 828\ 459 \ldots$.
- $\ln x = \log_e x$
- The functions $\ln x$ and e^x are inverses.
- Many naturally occurring phenomena can be modelled using base e exponential functions.

Communicate Your Understanding

C1 How can you determine the inverse of the function $y = e^x$

a) graphically? **b)** algebraically?

C2 What is unique about the function $f(x) = e^x$ compared to exponential functions having bases other than e?

C3 The following two equations were used in Example 4:

$P(t) = 200e^{0.094t}$ $\qquad$ $P(t) = (200)2^{\frac{t}{7.4}}$

where P represents a population of bacteria after t days. Why do these two functions yield slightly different results?

A Practise

Use this information to answer questions 1 to 3.

Let $f(x) = -e^x$ and $g(x) = -\ln x$.

1. a) Use technology to graph $f(x)$.
   b) Identify the following key features of the graph.
      i) domain
      ii) range
      iii) any x-intercepts or y-intercepts
      iv) the equations of any asymptotes
      v) intervals for which the function is increasing or decreasing
      vi) any minimum or maximum points
      vii) any inflection points

2. Repeat question 1 for $g(x)$.

3. Are $f(x)$ and $g(x)$ inverse functions? Justify your answer with mathematical reasoning.

4. Estimate the value of each exponential function, without using a calculator.
   a) e^4 b) e^5 c) e^2 d) e^{-2}

5. Evaluate each expression in question 4, correct to three decimal places, using a calculator.

6. Evaluate, if possible, correct to three decimal places, using a calculator.
   a) $\ln 7$ b) $\ln 200$ c) $\ln \frac{1}{4}$ d) $\ln(-4)$

7. What is the value of $\ln 0$? Why is this reasonable?

B Connect and Apply

8. Simplify.
   a) $\ln(e^{2x})$
   b) $\ln(e^x) + \ln(e^x)$
   c) $e^{\ln(x+1)}$
   d) $(e^{\ln(3x)})(\ln(e^{2x}))$

9. Solve for x, correct to three decimal places.
   a) $e^x = 5$
   b) $1000 = 20e^{\frac{x}{4}}$
   c) $\ln(e^x) = 0.442$
   d) $7.316 = e^{\ln(2x)}$

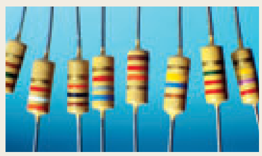
10. a) Solve $3^x = 15$ by taking natural logarithms of both sides.
    b) Solve $3^x = 15$ by taking common logarithms (base 10) of both sides.
    c) What do you conclude?

11. **Chapter Problem** Sheona's supervisor has given her some capacitors to analyse. When one of the charged capacitors is connected to a resistor to form an RC (resistor-capacitor) circuit, the capacitor discharges according to the equation $V(t) = V_{\max}e^{-\frac{t}{4}}$, where V is the voltage, t is time, in seconds, and $V_{\max}$ is the initial voltage. Determine how long it will take for a capacitor in this type of circuit to discharge to
    a) half of its initial charge
    b) 10% of its initial charge

CONNECTIONS

Capacitors are useful for storing and releasing electric charges. They can come in a variety of shapes, styles, and sizes, and are used in a number of devices, such as surge protectors, audio amplifiers, and computer electronics.

Resistors dissipate energy, often in a useful form such as heat or light. They also come in a variety of forms.



12. Use Technology A pizza is removed from the oven at $t = 0$ min at a temperature of 200°C. The temperature, T, measured at the end of each minute for the next 10 min is given in the table.

Time (min)	Temperature (°C)
0	200
1	173
2	150
3	130
4	113
5	98
6	85
7	74
8	64
9	55
10	48

a) Using exponential regression, determine a value of k so that $T(t) = 200e^{-\frac{t}{k}}$ models the temperature as a function of time.

b) Show that your function correctly predicts the temperature at $t = 10$ min.

c) Predict the temperature at $t = 15$ min and also after a long period of time.

13. a) Evaluate, using a calculator, $\ln 2 + \ln 3$.

b) Evaluate $\ln 6$. Compare these results.

c) What law of logarithms does this seem to verify? Recall that $\ln 2 = \log_e 2$. Rewrite this law of logarithms using natural logarithms.

14. Carbon-14 (C-14) is a radioactive substance with a half-life of approximately 5700 years. Carbon dating is a method used to determine the age of ancient fossilized organisms, by comparing the ratio of the amount of radioactive C-14 to stable carbon-12 (C-12) in the sample to the current ratio in the atmosphere, according to the equation $N(t) = N_0 e^{-\frac{(\ln 2)t}{5700}}$, where $N(t)$ is the ratio of C-14 to C-12 at the time when the organism died, N_0 is the ratio of C-14 to C-12 presently in the atmosphere, and t is the age of the fossil, in years.

a) Calculate the approximate age of a fossilized sample that was found to have a C-14 : C-12 ratio of

i) 10% of today's level

ii) 1% of today's level

iii) half of today's level

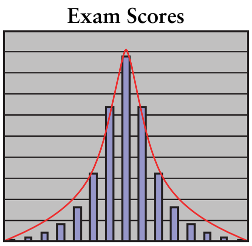
b) Do you need the equation to find all of the results in part a)? Explain your reasoning.

c) Rearrange this formula to express it explicitly in terms of t (isolate t on one side of the equation).

C Extend and Challenge

15. Use Technology If you study data management or statistics, you will learn about the **normal distribution** curve.

University exam scores often follow a normal distribution. The normal distribution curve is also sometimes called the **bell curve**.



a) Graph the function $y = e^{-x^2}$.

b) Describe the shape of the graph.

c) What is the maximum value of this function, and where does it occur?

d) Estimate the total area between the curve and the x-axis. Use a trapezoid to approximate the curve.

e) Estimate the fraction of this area that occurs between $x = -1$ and $x = 1$.

16. Math Contest $e^{\log_{e^2} x} =$

A $2x$ **B** $\frac{x}{2}$ **C** x^2 **D** $\sqrt{x}$ **E** $\ln x$

17. Math Contest If $\log_x(e^a) = \log_a e$, where $a \neq 1$ is a positive constant, then $x =$

A a **B** $\frac{1}{a}$ **C** a^a **D** a^{-a} **E** $a^{\frac{1}{a}}$

5.3 Derivatives of Exponential Functions

As you discovered previously in this chapter, the derivative of an exponential function is also an exponential function. If $y = b^x$, then $\frac{dy}{dx} = kb^x$, where k is some constant. Furthermore, you found that when $y = e^x$, $\frac{dy}{dx} = e^x$. In other words, when $b = e$, $k = 1$. How can the value of k be determined for other bases?

To explore this, apply the first principles definition of the derivative to the exponential function.

Investigate What is the derivative of $f(x) = b^x$?

Tools

- computer with *The Geometer's Sketchpad*®

CONNECTIONS

Function notation is convenient to use when you are determining a derivative from first principles. It is wise to be comfortable in working with various notations for the derivative:

$f'(x)$, $\frac{dy}{dx}$, y' and $D_x y$ are all notations that represent the first derivative of a function.

Let $f(x) = b^x$ for some constant, b.

1. Copy each step of the following algebraic argument. Beside each step write a brief explanation of what is happening. The first and last steps have been done for you.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \text{First principles definition.} \\ &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} \\ &= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} \\ &= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h} \\ &= b^x \lim_{h \to 0} \frac{(b^h - 1)}{h} && b^x \text{ does not depend on } h. \text{ It can be factored out of the limit.} \end{aligned}$$

2. Explain how the limit in the last line is related to k in the equation $\frac{dy}{dx} = kb^x$, introduced in the section introduction.

3. a) Explore this limit when $b = 2$.

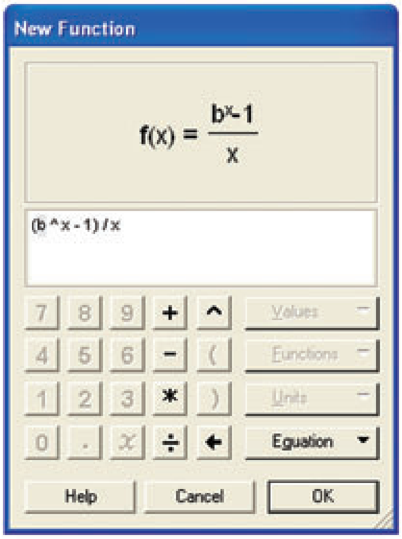
- Open *The Geometer's Sketchpad*® and begin a new sketch.
- Create a parameter, b, and use it to plot the graph of $y = \frac{b^x - 1}{x}$.

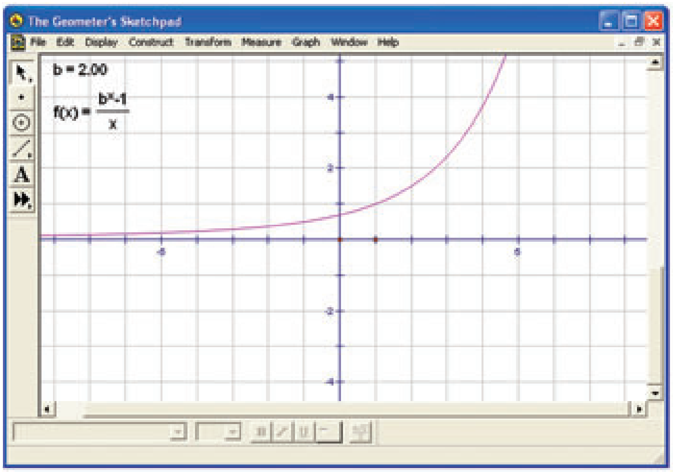
Note that *The Geometer's Sketchpad*® requires that x be the independent variable; hence, x replaces h for now.

Technology Tip

To plot the function $y = \dfrac{b^x - 1}{x}$, follow these steps.

- From the **Graph** menu, choose **New Parameter**. Call it *b* and set its initial value to 2.
- From the **Graph** menu, choose **Plot New Function** and enter the equation $y = \dfrac{b^x - 1}{x}$, using the parameter *b*.





CONNECTIONS

The approximate value of $\lim\limits_{x \to 0} \dfrac{2^x - 1}{x}$ does not seem to be a remarkable number until we use a calculator and realize that it is close to ln 2. Further investigation would find other natural logarithms appearing as limits.

$$\lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2$$

b) Approximate the limit $\lim\limits_{x \to 0} \dfrac{2^x - 1}{x}$.

- Select the graph. From the **Construct** menu, choose **Point on Function Plot.**
- From the **Measure** menu, choose **Abscissa** (**x**).
- From the **Measure** menu, choose **Ordinate** (**y**).
- Click and drag the point as close to the *y*-axis as possible.

What is the approximate value of $\lim\limits_{x \to 0} \dfrac{2^x - 1}{x}$? Explain how you can tell.

c) What is the value of *k* in the equation $\dfrac{dy}{dx} = k2^x$?

4. Explore the value of $\lim\limits_{x \to 0} \dfrac{b^x - 1}{x}$ for various values of the parameter b.

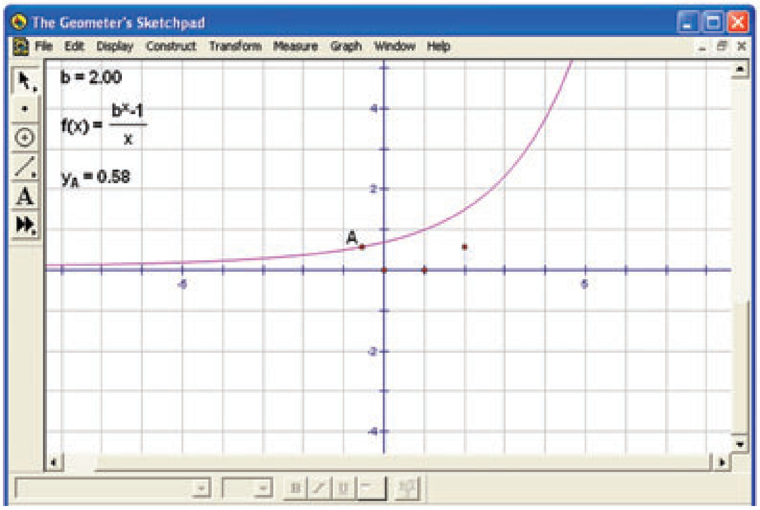
a) Copy and complete the following table.

b	$\lim\limits_{x \to 0} \dfrac{b^x - 1}{x}$	$y = b^x$	$\dfrac{dy}{dx} = kb^x$
2			
3			
4			
5			
e			

b) Describe any patterns that you see in the table.

5. Plot and trace the value of k for many different values of b.

- Select the **Parameter b** measure and the **Ordinate** (**y**) measure, in order. From the **Graph** menu, choose **Plot as** (**x, y**). A point should appear. Describe the significance of this point.



- With the new point selected, from the **Display** menu, choose **Trace Plotted Point.**

- Select and right-click on the **Parameter b** measure, and choose **Animate Parameter**. When the **Motion Controller** comes up, use the control buttons to trace the y-intercept as a function of b.

6. Reflect

a) Describe the shape of the traced curve that appears. This shape corresponds to a function. What function do you think this is?

b) Plot the function from part a) to test your prediction.

c) What does this suggest about the value of k in the equation $\dfrac{dy}{dx} = kb^x$?

The derivative of the exponential function $y = b^x$ is $\frac{dy}{dx} = (\ln b)b^x$.

Example 1 Differentiate an Exponential Function

Determine the derivative of each function. Graph each function and its derivative.

a) $y = 2^x$

b) $y = e^x$

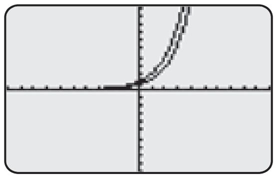
Solution

a) $y = 2^x$

Apply the rule for the derivative of an exponential function.

$$\frac{dy}{dx} = (\ln 2)2^x$$

To see the graphs of $y = 2^x$ and its derivative, use graphing technology.



Which graph corresponds to which function? Evaluate ln 2.

$$\begin{aligned}\frac{dy}{dx} &= (\ln 2)2^x \\ &\doteq (0.693)2^x\end{aligned}$$

This is a vertical compression of the function $y = 2^x$, so the derivative must be the lower function.

b) $y = e^x$

$$\begin{aligned}\frac{dy}{dx} &= (\ln e)e^x \\ &= (1)e^x \\ &= e^x\end{aligned}$$

This algebraic result confirms the earlier graphical discovery that the instantaneous rate of change of $y = e^x$ produces the exact same function.

Example 2 Equation of a Tangent Line

Find the equation of the line tangent to the curve $y = 2e^x$ at $x = \ln 3$.

Solution

Method 1: ***Use Paper and Pencil***

To write the equation of the tangent line, we need to know the slope and the coordinates of one point on the line. Substitute $x = \ln 3$ into the equation and solve for y.

$$\begin{aligned} y &= 2e^{\ln 3} \\ &= 2(3) \qquad e^{\ln x} = x \\ &= 6 \end{aligned}$$

The point $(\ln 3, 6)$ is on the tangent line.

To find the slope, differentiate the function and evaluate the derivative at $x = \ln 3$.

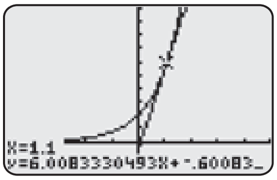
$$\begin{aligned} y &= 2e^x \\ y' &= 2e^x \qquad \text{Apply the constant multiple rule.} \end{aligned}$$

When $x = \ln 3$,

$$\begin{aligned} y' &= 2e^{\ln 3} \\ &= 2(3) \\ &= 6 \end{aligned}$$

The slope of the tangent line is 6. Substitute into the equation of a line and simplify.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 6 &= 6(x - \ln 3) \\ y &= 6x + 6 - 6\ln 3 \end{aligned}$$



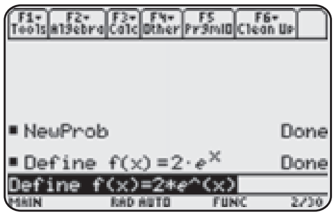
Method 2: ***Use a Computer Algebra System (CAS)***

Turn on the CAS. If necessary, press the HOME key to go to the **HOME** screen. Clear the variables and set the **MODE**.

- Clear the CAS variables by pressing 2ND F1 to access the **F6** menu.
- Select **2:NewProb**. Press ENTER.
- Press MODE key. Scroll down to **Exact/Approx**, and ensure that **AUTO** is selected.

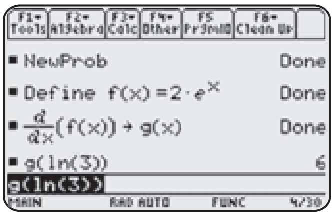
Enter the function and store it as $f(x)$.

- From the **F4** menu, select **1:Define**. Enter the function $f(x) = 2*e\verb|^|(x)$.
- Press ENTER.



Determine the value of the derivative at $x = \ln 3$.

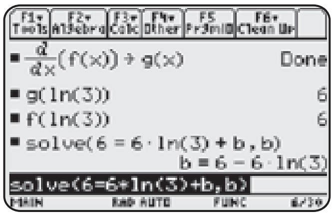
- From the **F3** menu, select **1:*d*(differentiate**.
- Enter $f(x)$, x). Your screen will show **d(f(x), x)**.
- Press the STO key. Store the derivative as $g(x)$, and press ENTER.
- Enter $g(\ln(3))$, and press ENTER.



The slope of the tangent is equal to 6 at $x = \ln 3$.

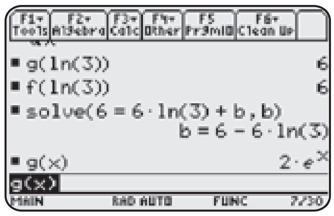
Find the point of tangency.

- Enter $f(\ln(3))$ and press ENTER.



The equation of a straight line is $y = mx + b$. Substitute $m = 6$, $x = \ln 3$, and $y = 6$, and solve for b.

- Press F2 and select **1:solve(**.
- Enter 6 = 6*ln(3) + b, b)
- Press ENTER.



The equation of the tangent to the curve at $(\ln 3, 6)$ is $y = 6x + 6 - 6\ln 3$.

To view the form of the derivative, enter $g(x)$, and press ENTER.

Example 3 Insect Infestation

A biologist is studying the increase in the population of a particular insect in a provincial park. The population triples every week. Assume the population continues to increase at this rate. Initially, there are 100 insects.

a) Determine the number of insects present after 4 weeks.

b) How fast is the number of insects increasing

i) when they are initially discovered?

ii) at the end of 4 weeks?

Solution

a) Write an equation that gives the number of insects, N, as a function of time, t, in weeks.

$N(t) = (100)3^t$

To find the number of insects present after 4 weeks, substitute $t = 4$.

$$\begin{aligned} N(4) &= (100)3^4 \\ &= (100)81 \\ &= 8100 \end{aligned}$$

There are 8100 insects after 4 weeks.

b) To find how fast the insects are increasing at any time, differentiate the function with respect to time.

$$\begin{aligned} N(t) &= (100)3^t \\ N'(t) &= (100)(\ln 3)3^t \end{aligned}$$

i) To find how fast the insects are increasing initially, evaluate $N'(0)$.

$$\begin{aligned} N'(0) &= (100)(\ln 3)3^0 \\ &\doteq (100)(1.1)3^0 \\ &\doteq (110)1 \\ &\doteq 110 \end{aligned}$$

The number of insects is increasing at a rate of approximately 110 per week at the beginning of the first week.

ii) To find how fast the insect population is growing at the end of 4 weeks, evaluate $N'(4)$.

$$\begin{aligned} N'(4) &= (100)(\ln 3)3^4 \\ &= (100)(\ln 3)81 \\ &= (8100)(\ln 3) \\ &\doteq 8899 \end{aligned}$$

At the end of 4 weeks, the insects are increasing in number at a rate of approximately 8899 per week.

KEY CONCEPTS

- The derivative of an exponential function is also exponential.

  $$\frac{d}{dx}b^x = (\ln b)b^x$$

- Derivatives of exponential functions can be used to solve problems involving growth of populations or investments.

Communicate Your Understanding

C1 Let $f(x) = e^x$ and $g(x) = \dfrac{f'(x)}{f(x)}$. Which of the following statements do you agree or disagree with? Explain your answer in each case.

a) $g(x)$ is a function.

b) $g(x)$ is a linear function.

c) $g(x)$ is a constant function.

d) $g(x)$ is the same function as its inverse.

C2 Consider the function $y = 2e^x$ and the point where $x = \ln 3$. What do you notice about the y-coordinate and the slope of the tangent at this point? Do you think this is always true for exponential functions? Explain your thinking.

C3 Consider the function $N(t) = (100)2^t$, where $N(t)$ represents the number of bacteria in a population after t weeks. Why is the initial growth rate of the bacteria not 200 per week?

A Practise

1. Determine the derivative with respect to x for each function.

a) $g(x) = 4^x$ **b)** $f(x) = 11^x$ **c)** $y = \left(\dfrac{1}{2}\right)^x$

d) $N(x) = -3e^x$ **e)** $h(x) = e^x$ **f)** $y = \pi^x$

2. a) Find the first, second, and third derivatives of the function $f(x) = e^x$.

b) What is the nth derivative of $f(x)$, for any $n \in \mathbb{N}$?

3. Calculate the instantaneous rate of change of the function $y = 5^x$ when $x = 2$.

4. Determine the slope of the graph of $y = \dfrac{1}{2}e^x$ at $x = 4$.

5. Determine the equation of the line tangent to $y = 8^x$ at the point on the curve where $x = \dfrac{1}{2}$.

6. A fruit fly infestation is doubling every day. There are ten flies when the infestation is first discovered.

a) Write an equation that relates the number of flies to time.

b) Determine the number of flies present after 1 week.

c) How fast is the fly population increasing after 1 week?

d) How long will it take for the fly population to reach 500?

e) How fast is the fly population increasing at this point?

B Connect and Apply

7. Refer to question 6.

a) At which point is the fly population increasing at a rate of

i) 20 flies per day? **ii)** 2000 flies per day?

b) How can this information help to plan an effective extermination strategy?

8. Determine the equation of the line perpendicular to the tangent line to the function $f(x) = \frac{1}{2}e^x$ at the point on the curve where $x = \ln 3$.

CONNECTIONS

The line perpendicular to the tangent line to a function is also called the **normal line**.

9. **Use Technology** Refer to question 8. Solve this problem using a computer algebra system.

10. Let $f(x) = kb^x$ for some positive base b and some constant k.

a) Predict the shape of the graph of $g(x)$, where $g(x)$ is defined as

$$g(x) = \frac{f'(x)}{f(x)}.$$

b) Provide mathematical reasoning for your answer to part a).

c) What is the simplified equation for $g(x)$? Explain.

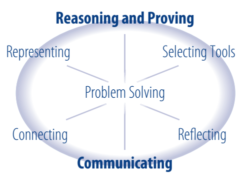
11. **Use Technology** Refer to question 10. Use graphing technology to verify your answers.

12. Refer to question 10.

a) What does your result in part a) simplify to when $b = e$? Explain.

b) Use graphing technology to verify your answer.

13. a) Determine a formula for finding the nth derivative of the function $f(x) = b^x$, where b is a constant greater than zero.



b) Use mathematical reasoning to explain your result.

14. **Use Technology** Consider the following two functions over the restricted domain $\{x \mid 4 \le x \le 16, x \in \mathbb{R}\}$:

$f(x) = x^2 \qquad g(x) = 2^x$

a) How similar are these functions over this domain?

b) Do you think their derivatives will be similar? Explain.

c) Use graphing technology to check your prediction. Comment on what you notice.

d) Assuming $x \in \mathbb{R}$, will there exist any x-values at which the slope of $f(x)$ is the same as the slope of $g(x)$? If so, find the x-value(s).

15. Evangelista Torricelli (1608–1647) developed a formula for calculating the barometric pressure at various altitudes. The altimeter of an aircraft uses a similar formula to determine the altitude of the aircraft above sea level. The formula has the form $P = 101.3e^{-kh}$ for the atmosphere at standard temperature and pressure, where P is the barometric pressure, in kilopascals, h is the altitude, in metres, and k is a constant.

a) If the barometric pressure at an altitude of 1000 m is 95.6 kPa, determine the value of k.

b) Show that the value of k from part a) correctly predicts a pressure of 90.2 kPa at an altitude of 2000 m.

c) The vertical speed indicator on an aircraft uses the change in barometric pressure with respect to time to determine the rate of climb of the aircraft. Determine the first derivative of the barometric pressure formula.

d) Determine the rate of change of the pressure with respect to height at an altitude of 1500 m.

CONNECTIONS

Most of the time, the atmosphere is not at a standard temperature of 20°C and a sea-level pressure of 101.3 kPa. Pilots must obtain correct values for temperature and pressure, and correct the readings on their instruments accordingly.

16. The total number of visitors to a particular Web site is doubling every week. When the Web master starts monitoring, there have been 50 visitors to the site.

a) Write an equation that relates the number of visitors to time. Identify the variables and clearly explain what they represent.

b) How many visitors will there be after

i) 4 weeks? ii) 12 weeks?

c) Find the rate of growth of visitors at each of these times.

d) Will this trend continue indefinitely? Justify your response using a graph.

C Extend and Challenge

17. How fast is Earth's human population growing?

a) Collect data to answer this question.

b) Develop a model that expresses population as a function of time. Express your model as

i) an equation ii) a graph

c) Using your model, predict the size of the human population in the year

i) 2025 ii) 2500 iii) 3000

d) Is your model sustainable over the long term? Could other factors affect this trend?

e) How might the equation look if it accounts for the factors you suggest in part d)?

CONNECTIONS

To see population estimates from ancient times up to 1950 go to *www.mcgrawhill.ca/links/calculus12* and follow the links to Section 5.3.

18. **Use Technology** Refer to question 17.

a) Use regression analysis to estimate when Earth's population was

i) 1000 ii) 100 iii) 2

b) Do your answers seem reasonable? Explain.

19. **Use Technology** Use graphing software to explore the instantaneous rate of change and the derivative of the function $y = \ln x$. Write a brief report on what you discover.

20. **Math Contest** If $\lim_{h\to 0} \dfrac{a^{x+h} - a^x}{h} = a^x \ (x \neq 0)$, then $a =$

A a **B** 1 **C** e **D** 0 or e **E** 1 or e

21. **Math Contest** If $f(x) = \dfrac{e^x - e^{-x}}{2}$, then $f'(\ln 2) =$

A 0 **B** 2 **C** $\ln 2$ **D** $\dfrac{3}{4}$ **E** $\dfrac{5}{4}$

CAREER CONNECTION

Katrina completed a 4-year Bachelor of Science honours degree in geophysics at the University of Western Ontario. As a petroleum geophysicist, she studies the structure and composition of Earth. Her goal is to find the location of oil and natural gas deposits below Earth's surface. Katrina uses the seismic method to build a picture of where deposits may be located. In this technique, shock waves that are set off at Earth's surface penetrate Earth like sonar. They reflect off various rock layers under Earth's surface, and the returning echo or signal is recorded using sophisticated equipment. Katrina spends her time processing and interpreting these seismic data.

5.4 Differentiation Rules for Exponential Functions

Exponential models and their derivatives occur frequently in engineering, science, mathematics, and business studies. As the models become more complex, it is important to remember and apply the rules of differentiation as they are needed.

What could the chain rule and exponential modelling have to do with the value of the motorcycle pictured here?

Example 1 Differentiation Rules

Find the derivative of each function.

a) $y = xe^x$ **b)** $y = e^{2x+1}$ **c)** $y = e^x - e^{-x}$

d) $y = 2e^x \cos x$ **e)** $y = x^2 10^x$

Solution

a)
$$y = xe^x$$
$$\frac{dy}{dx} = xe^x + (1)e^x \quad \text{Apply the product rule.}$$
$$= e^x(x+1)$$

b)
$$y = e^{2x+1}$$
$$\frac{dy}{dx} = e^{2x+1}(2) \quad \text{Apply the chain rule.}$$
$$= 2e^{2x+1}$$

c)
$$y = e^x - e^{-x}$$
$$\frac{dy}{dx} = e^x - (-1)e^{-x} \quad \text{Apply the difference rule and the chain rule.}$$
$$= e^x + e^{-x}$$

d)
$$y = 2e^x \cos x$$
$$\frac{dy}{dx} = 2e^x(-\sin x) + \cos x(2e^x) \quad \text{Apply the product rule.}$$
$$= 2e^x(\cos x - \sin x)$$

e)
$$y = x^2 10^x$$
$$\frac{dy}{dx} = x^2 \ln 10(10^x) + (2x)10^x \quad \text{Apply the product rule.}$$
$$= x10^x(x \ln 10 + 2)$$

CONNECTIONS

You explored the rules of differentiation in Chapter 2 Derivatives.

Product Rule

$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$

Chain Rule

If $f(x) = g(h(x))$, then $f'(x) = g'[h(x)]\,h'(x)$

Difference Rule

$(f-g)'(x) = f'(x) - g'(x)$

The **extrema** of a function are the points where $f(x)$ is at either a local maximum or a local minimum.

Example 2 Extreme Values

Identify the local extrema of the function $f(x) = x^2e^x$.

Solution

To determine where the local extreme (maximum and minimum) values occur, differentiate the function and set it equal to 0, because the slope of the tangent at a maximum or minimum is equal to 0.

$$\begin{aligned} f'(x) &= x^2e^x + 2xe^x \\ &= x^2e^x + 2xe^x \end{aligned}$$

Set the derivative equal to 0.

$$\begin{aligned} f'(x) &= 0 \\ x^2e^x + 2xe^x &= 0 \\ xe^x(x + 2) &= 0 \end{aligned}$$

This equation has three factors whose product is 0. Consider cases to determine possible solutions.

Case 1

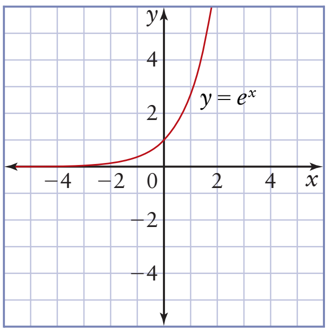
$x = 0$

The first factor gives $x = 0$ as a solution.

Case 2

$e^x = 0$

The range of $y = e^x$ is $y > 0$.



Therefore $e^x = 0$ has no solution.

Case 3

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

Therefore, there are two solutions: $x = 0$ and $x = -2$.

This divides the domain into three intervals: $x < -2$, $-2 < x < 0$, and $x > 0$. Test $f'(x)$ with a point in each interval.

Evaluate the derivative of the function for a point to the left of $x = -2$, say $x = -3$.

$$\begin{aligned} f'(x) &= e^x(x^2 + 2x) \\ f'(-3) &= e^{-3}[(-3)^2 + 2(-3)] \\ &= 0.1 \end{aligned}$$

The slope of the tangent is positive.

Evaluate the derivative of the function for a point between $x = -2$ and $x = 0$, say $x = -1$.

$$\begin{aligned} f'(x) &= x^2e^x + 2xe^x \\ &= e^x(x^2 + 2x) \\ f'(-1) &= e^{-1}[(-1)^2 + 2(-1)] \\ &= -0.4 \end{aligned}$$

The slope of the tangent is negative.

Evaluate the derivative of the function for a point to the right of $x = 0$, say $x = 1$.

$$\begin{aligned} f'(x) &= e^x(x^2 + 2x) \\ f'(1) &= e^1[(1)^2 + 2(1)] \\ &= 8.2 \end{aligned}$$

The slope of the tangent is positive.

	$x < -2$	$-2 < x < 0$	$x > 0$
Test Values	-3	-1	1
Sign of $f'(x)$	$+$	$-$	$+$
Nature of $f(x)$	increasing $\nearrow$	decreasing $\searrow$	increasing $\nearrow$

Because the graph changes directions at $x = -2$ and $x = 0$, these points represent local maxima or minima.

Substitute $x = 0$ into the function to solve for its y-coordinate.

$$\begin{aligned} f(x) &= x^2 e^x \\ f(0) &= 0^2 e^0 \\ &= 0(1) \\ &= 0 \end{aligned}$$

Therefore (0, 0) is a local minimum.

Substitute $x = -2$ into the function to solve for its y-coordinate.

$$\begin{aligned} f(x) &= x^2 e^x \\ f(-2) &= (-2)^2 e^{-2} \\ &= 4e^{-2} \\ &= \frac{4}{e^2} \end{aligned}$$

Therefore $\left(-2, \frac{4}{e^2}\right)$ is a local maximum.

The behaviour of the function can be summarized in a table.

Example 3 Motorcycle Depreciation

Laura has just bought a new motorcycle for \$10 000. The value of the motorcycle depreciates over time. The value can be modelled by function $V(t) = 10\,000e^{-\frac{t}{4}}$, where V is the value of the motorcycle after t years.

a) At what rate is the value of the motorcycle depreciating the instant Laura drives it off of the dealer's lot?

b) Laura decides that she will stop insurance coverage for collision once the motorcycle has depreciated to one quarter of its initial value. When should Laura stop her collision coverage?

c) At what rate is the motorcycle depreciating at the time determined in part b)?

CONNECTIONS

Collision insurance provides reimbursement for the repair or replacement of your vehicle in the event of an accident.

Solution

a) To find the rate of change of the value of Laura's motorcycle, differentiate the value function.

$$\begin{aligned} V'(t) &= -\frac{10\,000}{4}e^{-\frac{t}{4}} \\ &= -2500e^{-\frac{t}{4}} \end{aligned}$$

Apply the constant multiple rule and the chain rule.

To determine the depreciation rate when Laura first drives her bike off the lot, substitute $t = 0$ in the derivative.

$$\begin{aligned} V'(0) &= -2500e^{-\frac{0}{4}} \\ &= -2500e^0 \\ &= -2500(1) \\ &= -2500 \end{aligned}$$

When Laura drives off the dealer's lot, her motorcycle is depreciating at a rate of \$2500 per year.

b) To determine when Laura should stop collision coverage on her insurance plan, find the time when the value of her motorcycle reaches one quarter of its original value, or \$2500.

$$\begin{aligned} 2500 &= 10\,000e^{-\frac{t}{4}} \\ \frac{2500}{10\,000} &= e^{-\frac{t}{4}} \\ 0.25 &= e^{-\frac{t}{4}} \\ \ln 0.25 &= \ln\left(e^{-\frac{t}{4}}\right) && \text{Take the natural logarithm of both sides.} \\ -1.386 &\doteq -\frac{t}{4} \\ t &\doteq -1.386(-4) \\ &\doteq 5.5 \end{aligned}$$

Therefore, Laura should stop her collision coverage after approximately 5.5 years.

c) To find the rate of depreciation at this time, evaluate $V'(5.5)$.

$$\begin{aligned} V'(5.5) &= -2500e^{-\frac{5.5}{4}} \\ &= -2500e^{-1.375} \\ &\doteq -2500(0.25) \\ &\doteq -632 \end{aligned}$$

After 5.5 years, Laura's motorcycle is depreciating at a rate of approximately \$632 per year.

KEY CONCEPTS

- The differentiation rules apply to functions involving exponentials.
- The first and second derivative tests can be used to analyse functions involving exponentials.
- If $y = e^{f(x)}$, then $y' = e^{f(x)}f'(x)$.

Communicate Your Understanding

C1 Which differentiation rule(s) would you apply in order to determine the derivative of each function? Justify your answers.

a) $y = 2e^x + x^2$ **b)** $f(x) = e^{-2x+3}$

c) $y = e^{2x}x^3$ **d)** $g(x) = e^x \sin x - e^{-3x}\cos x$

C2 Consider an exponential function of the form $y = b^x$.

a) Is it possible to identify local extrema without differentiating the function and setting the derivative equal to zero? Justify your answer using both algebraic and geometric reasoning.

b) Does your answer to part a) depend on the value of b? Explain.

C3 In Example 3, the derivative of the function used to model the value of Laura's motorcycle is $-2500e^{-\frac{t}{4}}$. What is the significance of each of the negative signs in this expression?

A Practise

1. a) Rewrite the function $y = b^x$ with base e.

b) Find the derivative of your function in part a) and simplify.

2. Differentiate with respect to x.

a) $y = e^{-3x}$ **b)** $f(x) = e^{4x-5}$

c) $y = e^{2x} - e^{-2x}$ **d)** $y = 2^x + 3^x$

e) $f(x) = 3e^{2x} - 2^{3x}$ **f)** $y = 4xe^x$

g) $y = 5^x e^{-x}$ **h)** $f(x) = xe^{2x} + 2e^{-3x}$

3. Determine the derivative with respect to x for each function.

a) $y = e^{-x}\sin x$ **b)** $y = e^{\cos x}$

c) $f(x) = e^{2x}(x^2 - 3x + 2)$ **d)** $g(x) = 2x^2 e^{\cos 2x}$

4. Identify the coordinates of any local maximum or minimum values of the function $y = e^x - e^{2x}$.

5. Use algebraic reasoning to explain why the function $y = e^x + e^{2x}$ has no local extrema.

6. Use Technology Use graphing technology to confirm your results in questions 4 and 5. Sketch the graphs of these functions and explain how they confirm your results.

B Connect and Apply

7. A bacterial colony's population is modelled by the function $P(t) = 50e^{0.5t}$, where P is the number of bacteria after t days.

a) What is the bacterial population after 3 days?

b) How long will it take for the population to reach 10 times its initial level?

c) Rewrite this function as an exponential function having a base of 10.

d) Determine the bacterial population after 5 days using your equation from part c).

e) Use the function from the initial question to find the bacterial population in part d). Why do your answers differ?

8. The value of an investment is modelled by $A(t) = A_0e^{0.065t}$, where A is the amount the investment is worth after t years, and A_0 is the initial amount invested.

a) If the initial investment was \$3000, what is that value of the investment after

i) 2 years? **ii)** 5 years? **iii)** 25 years?

b) How long will it take for this investment to double in value?

c) At what rate is the investment growing at the time when its value has doubled?

9. a) Determine the first, second, third, fourth, fifth, and sixth derivatives of the function $f(x) = e^x \sin x$.

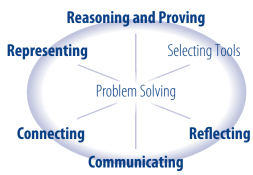
b) Describe any patterns you notice.

c) Predict the

i) seventh derivative **ii)** eighth derivative

Check your predictions by finding these derivatives.

d) Write one or two rules for finding the nth derivative of $f(x)$. (Hint: Consider cases).



10. Consider the rate of change function for the value of Laura's motorcycle in Example 3: $V'(t) = -2500e^{-\frac{t}{4}}$, where $V'(t)$ is the rate at which the value of Laura's motorcycle depreciates as a function of time, t, in years. When does Laura's motorcycle depreciate in value the fastest? Justify your answer with mathematical reasoning.

11. A pond has a population of algae of 2000. After 15 min, the population is 4000. This population can be modelled by an equation of the form $P = P_0(a^t)$, where P is the population after t hours, and P_0 is the initial population.

a) Determine the values of P_0 and a.

b) Find the algae population after 10 min.

c) Find the rate of change of the algae population after

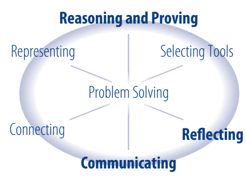
i) 1 h **ii)** 3 h

12. Cheryl has missed a few classes and tried to catch up on her own. She asks you for help with differentiating the following function.

Cheryl's solution	**Reasoning**
$y = 10^x$	I remember the power rule. I will apply that.
$\dfrac{dy}{dx} = x10^{x-1}$	Multiply by the value of the exponent, and reduce the exponent by 1.

a) What is the flaw in Cheryl's reasoning?

b) Why do you think she has made this error?



c) Explain how to correct her solution.

13. Find the points of inflection of $y = e^{-x^2}$.

14. a) How many local extrema occur for the function $y = e^x \cos x$ over the interval $0 \leq x \leq 2\pi$?

b) How do the coordinates of the local extrema in part a) compare with those of $y = \cos x$ over the same interval?

15. Chapter Problem Sheona is charging some battery cells and monitoring the charging process. A battery charger restores the voltage of a battery cell according to the equation

$V(t) = V_{max}\left(1 - e^{-\frac{t}{8}}\right)$ where V is the voltage of the cell at time t, in hours, and V_{max} is the cell's peak charge voltage.

a) Determine the time required to restore a dead cell's voltage to 75% of its peak charge.

b) Determine an equation that expresses the rate of charging as a function of time.

C Extend and Challenge

16. The hyperbolic sine function, $y = \sinh x$, is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

a) Graph this function using graphing technology. Describe the shape of the graph.

b) Predict the shape of its derivative. Sketch your prediction.

17. The hyperbolic cosine function is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

a) Graph this function using graphing technology. Describe the shape of the graph.

b) Predict the shape of its derivative. Sketch your prediction.

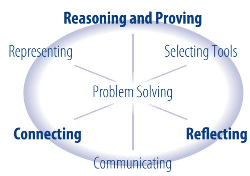
18. Refer to questions 15 and 16.

a) Use algebraic reasoning to show that

i) the derivative of $y = \sinh x$ is $\frac{dy}{dx} = \cosh x$

ii) the derivative of $y = \cosh x$ is $\frac{dy}{dx} = \sinh x$

b) Compare these results to the predictions that you made. How close were your predictions?



CONNECTIONS

Hyperbolic trigonometric functions have various applications in areas of science and engineering, such as optics, radioactivity, electricity, heat transfer, and fluid dynamics.

The St. Louis Gateway Arch is an architectural structure whose shape can be modelled by a hyperbolic cosine function.

You will study hyperbolic trigonometric functions in more depth in university mathematics, science, and/or engineering.

19. a) Given the function $y = \ln x$, how would you find its derivative? Hint: remember that $y = \ln x$ can also be written $x = e^y$

b) Find the derivative of $y = \ln x$.

20. Math Contest If μ is a root of the quadratic equation $ax^2 + bx + c = 0$ and $f(x) = e^{\mu x}$, then $af''(x) + bf'(x) + cf(x) =$

A 0 **B** 1 **C** μx

D $e^{\mu x}$ **E** $\mu e^{\mu x}$

21. Math Contest If $y = \log_c(ax + b)$, then $\frac{dy}{dx} =$

A $\dfrac{a}{c^x \ln c}$

B $\dfrac{1}{c^x \ln c}$

C $\dfrac{1}{ax + b}$

D $\dfrac{a}{ax + b}$

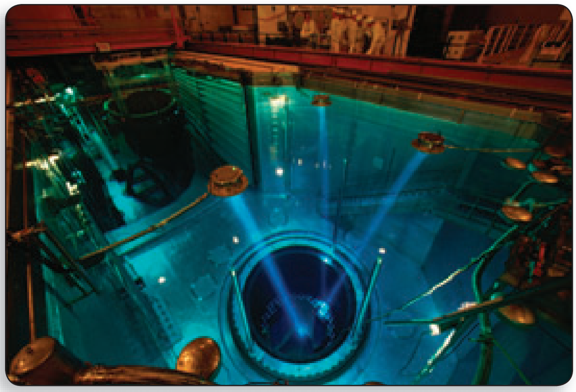
E $\dfrac{a}{(ax + b)\ln c}$

5.5 Making Connections: Exponential Models

Uranium-233 (U-233) is a commonly used radioactive material in nuclear power generators, because when it decays, it produces a wealth of harvestable energy. This isotope, however, does not occur naturally. To produce it, a series of nuclear reactions, called a breeding chain, must be induced. Thorium-233 becomes protactinium-233, which then becomes uranium-233. The following illustrates part of this such breeding chain:

(Th-233) $\rightarrow$ (Pa-233) $\rightarrow$ (U-233)

Analysis of exponential decay functions is important in helping us understand how nuclear reactors work.



The amount of a radioactive material as a function of time is given by the standard function $N(t) = N_0 e^{-\lambda t}$, where N is the number of radioactive nuclei at time t, N_0 is the initial number of radioactive nuclei, and λ is the disintegration constant.

Example 1 Medical Treatment

A radioactive isotope of gold, Au-198, is used in the diagnosis and treatment of liver disease. Suppose that a 6.0-mg sample of Au-198 is injected into a liver, and that this sample decays to 4.6 mg after 1 day. Assume the amount of Au-198 remaining after t minutes is given by $N(t) = N_0 e^{-\lambda t}$.

a) Determine the disintegration constant for Au-198.

b) Determine the half-life of Au-198.

c) Write the equation that gives the amount of Au-198 remaining as a function of time, in terms of its half-life.

d) How fast is the sample decaying after 3 days?

Solution

a) The amount of Au-198 remaining decays exponentially over time according to the formula $N(t) = N_0 e^{-\lambda t}$, where t is time, in days. At $t = 1$, the sample has decayed from its initial amount, $N_0 = 6.0$, to $N(1) = 4.6$. Substitute these values into the equation and solve for λ.

$$4.6 = 6.0e^{-\lambda(1)}$$

$$\frac{4.6}{6} = e^{-\lambda} \qquad \text{Divide both sides by 6.}$$

$$\ln\left(\frac{4.6}{6}\right) = \ln e^{-\lambda} \qquad \text{Take the natural logarithm of both sides.}$$

$$-0.266 \doteq -\lambda$$

$$\lambda \doteq 0.266$$

CONNECTIONS

The Greek letter λ (lambda) is commonly used in physics. For example, in wave mechanics, λ is often used to denote the wavelength of a signal. In the example shown, λ indicates the disintegration constant, which is related to how fast a radioactive substance decays.

The disintegration constant is approximately 0.27. The equation for the amount of Au-198 remaining as a function of time can be written as $N(t) = 6e^{-0.27t}$, where t is time, in days.

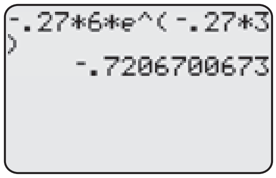
b) To determine the half-life of Au-198, determine the time required for the 6.0-mg sample to decay to half of this amount. Set $N(t) = 3$ and solve for t.

$$\begin{aligned} 3 &= 6e^{-0.27t} \\ 0.5 &= e^{-0.27t} \\ \ln 0.5 &= \ln e^{-0.27t} \\ -0.693 &\doteq -0.27t \\ t &\doteq 2.6 \end{aligned}$$

The half-life of Au-198 is approximately 2.6 days.

c) The equation that gives the amount of Au-198 as a function of time, in terms of its half-life, is $N(t) = 6\left(\frac{1}{2}\right)^{\frac{t}{2.6}}$, where t is measured in days.

d) To find the rate of decay after 3 days, differentiate the function modelling the amount of Au-198 remaining and evaluate at $t = 3$.

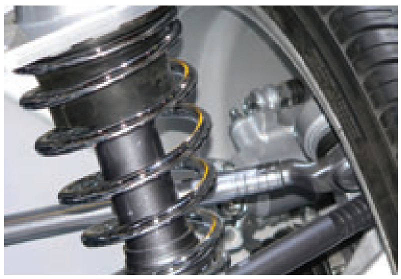


$$\begin{aligned} N(t) &= 6e^{-0.27t} \\ N'(t) &= -0.27(6)e^{-0.27t} \\ N'(3) &= -0.27(6)e^{-0.27(3)} \\ &\doteq -0.72 \end{aligned}$$

After 3 days the sample of Au-198 is decaying at a rate of approximately 0.72 mg/day.

Composite functions involving exponentials occur in a variety of engineering and scientific fields of study. The next example is related to mechanical engineering.

Example 2 Automotive Shock Absorbers



A pendulum is an example of a **harmonic oscillator;** a moving object whose motion repeats over regular time intervals. When the amplitude of a harmonic oscillator diminishes over time due to friction, the motion is called **damped harmonic motion**.

The vertical displacement of a sport vehicle's body after running over a bump is modelled by the function $h(t) = e^{-0.5t}\sin t$, where h is the vertical displacement, in metres, at time t, in seconds.

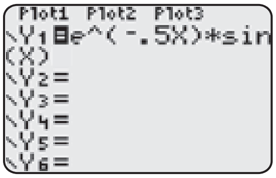
a) Graph the function and describe the shape of the graph.

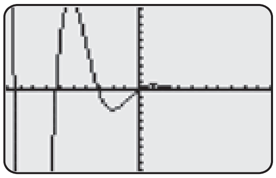
b) Determine when the maximum displacement of the sport vehicle's body occurs.

c) Determine the maximum displacement.

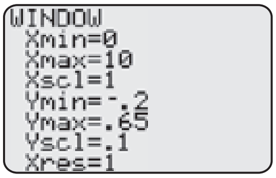
Solution

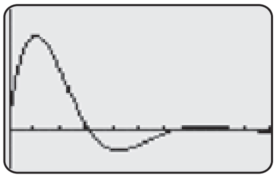
a) Use graphing technology.





The graph has no physical meaning for $t < 0$, since that represents the time prior to when the car hit the bump. Zoom in on the graph to the right of the vertical axis. Adjust the **window** settings to view as much of the graph as possible.





The graph is sinusoidal with diminishing amplitude.

b) To determine when the maximum vertical displacement occurs, differentiate the function and set the derivative equal to zero.

$$h(t) = e^{-0.5t}\sin t$$

$$h'(t) = e^{-0.5t}\cos t + (-0.5e^{-0.5t})\sin t$$

$$= e^{-0.5t}(\cos t - 0.5\sin t)$$

Set $h'(t) = 0$ and solve for t.

$$0 = e^{-0.5t}(\cos t - 0.5\sin t)$$

Set one of the factors equal to zero.

$e^{-0.5t} = 0$ This has no solution.

Set the other factor equal to zero.

$$\cos t - 0.5\sin t = 0$$

$$0.5\sin t = \cos t$$

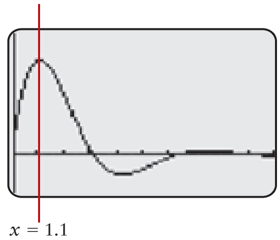
$0.5\tan t = 1$ Divide both sides by $\cos t$ and apply the identity $\frac{\sin t}{\cos t} = \tan t$.

$$\tan t = 2$$

$t = \tan^{-1}(2)$ Make sure that your calculator is set to Radian mode.

$$\doteq 1.1$$

There are actually an infinite number of solutions to this trigonometric equation, due to the periodic nature of the tangent function.



From the graph of $h(t)$, it is clear that $t = 1.1$ is when the vertical displacement is a maximum.

The maximum displacement occurs approximately 1.1 s after hitting the bump.

c) To determine the maximum displacement, find the value of $h(1.1)$.

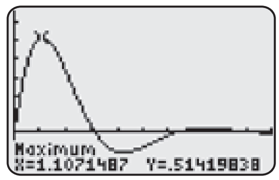
Method 1: ***Inspect the graph***

Use a graphing calculator to determine the value of the local maximum occurring at $t = 1.1$.

- Press [2ND] [TRACE] to access the **CALCULATE** menu.
- Choose **1:value**.
- Press **1.1** [ENTER].

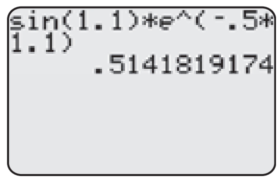
Following is another method for finding the local maximum.

- Press [2ND] [TRACE] to access the **CALCULATE** menu.
- Choose **4:maximum**.
- Move the cursor to the left side of the local maximum. Press [ENTER].
- Move the cursor to the right side of the local maximum. Press [ENTER] twice.



The graph shows that the maximum vertical displacement is approximately 0.51 m.

Method 2: ***Evaluate the function***



Substitute $t = 1.1$ into the displacement function and evaluate.

$$h(1.1) = e^{-0.5(1.1)}\sin(1.1)$$
$$\doteq 0.5142$$

The function equation verifies that the maximum vertical displacement is approximately 0.51 m.

KEY CONCEPTS

- Exponential functions and their derivatives are important modelling tools for a variety of fields of study, such as nuclear engineering, mechanical engineering, electronics, biology, and environmental science.
- Exponential models can be represented in different ways. The choice of representation can depend on the nature of the problem being solved.

Communicate Your Understanding

C1 Example 1 showed that both of the following equations represent the amount of Au-198 as a function of time:

$$N(t) = 6e^{-0.27t} \qquad N(t) = 6\left(\frac{1}{2}\right)^{\frac{t}{2.6}}$$

a) Are the two functions equivalent?

b) How could you verify your answer?

C2 Can the function $N(t) = 6\left(\frac{1}{2}\right)^{\frac{t}{2.6}}$ be used to determine how fast a sample is decaying after 3 days? How does this differ from the solution you found in Example 1?

C3 In Example 2, the motion of the shock absorber was modelled by the function $h(t) = e^{-0.5t}\sin t$.

a) Identify the factor that generates the periodic nature of the graph.

b) Identify the dampening factor that causes the amplitude to diminish over time.

c) What is the impact if the dampening factor is set equal to 1?

A Practise

1. A 100-mg sample of thorium-233 (Th-233) is placed into a nuclear reactor. After 10 min, the sample has decayed to 73 mg. Use the equation $N(t) = N_0e^{-\lambda t}$ to answer the following questions.

a) Determine the disintegration constant λ for Th-233.

b) Determine the half-life of Th-233.

c) Write the equation that gives the amount of Th-233 remaining as a function of time, in terms of its half-life.

d) How fast is the sample decaying after 5 min?

Use the following information to answer questions 2 to 4.

Radon-222 (Rn-222) is a radioactive element that spontaneously decays into polonium-218 (Po-218) with a half-life of 3.8 days. The atoms of these two substances have approximately the same mass. Suppose that the initial sample of radon has a mass of 100 mg.

2. The mass of radon, in milligrams, as a function of time is given by the function $M_{Rn}(t) = M_0\left(\frac{1}{2}\right)^{\frac{t}{3.8}}$, where M_0 is the initial mass of radon, and M_{Rn} is the mass of radon at time t, in days.

   a) How much radon will remain after

      i) 1 day? **ii)** 1 week?

   b) How long will it take for a sample of radon to decay to 25% of its initial mass?

   c) At what rate is the radon decaying at each of these times?

3. As radon decays, polonium is produced. The mass of polonium, M_{Po}, in milligrams, as a function of time is given by the function

$$M_{Po}(t) = M_0\left[1 - \left(\frac{1}{2}\right)^{-\frac{t}{3.8}}\right],$$ where M_0 is the initial mass of radon and t is time, in days.

   a) How much polonium is there

      i) initially?

      ii) after 1 day?

   b) Find the first derivative of this function. Explain what it means.

B Connect and Apply

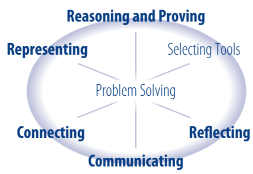
4. **Use Technology** Refer to questions 2 and 3.

   a) Graph the two functions, $M_{Rn}(t)$ and $M_{Po}(t)$, on the same grid. Are these functions inverses of each other? Explain your answer.

   b) Find the point of intersection of these functions. Identify the coordinates and explain what they mean.

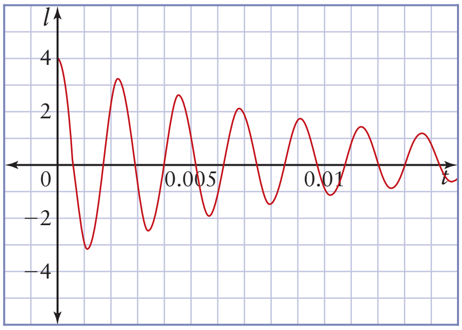
   c) How are the derivatives of each function related to each other at the point of intersection? Explain why this makes sense from a physical perspective.

   d) Graph the sum of the two functions, $M_{Rn}(t) + M_{Po}(t)$. What is the shape of this graph? Explain why this makes sense from a physical perspective.



Use the following information to answer questions 5 to 7.

The following graph shows the intensity of sound produced when a certain guitar string is plucked and a palm-muting technique is used.



CONNECTIONS

To perform a palm mute, the guitarist lightly places the palm of his or her picking hand on the strings near the bridge of the guitar while picking one or more strings.

This effect produces a percussive, chugging sound that is used in many types of music.

5. **a)** Is the function in the graph an example of damped harmonic motion? Explain how you can tell.

b) Determine the period, in

i) milliseconds

ii) seconds

c) Determine the frequency, f, in hertz.

d) Substitute the frequency and the initial intensity into the equation $I(t) = I_0 \cos(2\pi ft)\, e^{-kt}$.

CONNECTIONS

The **hertz** is a unit of measurement of frequency, equal to one cycle per second.

6. a) Determine the value of the decay constant, k, and substitute it into the equation.

b) Explain how you found the value of k.

c) Graph the equation you produced in part a) to see if it matches the given graph.

7. Pitch decay occurs when the frequency of a sound diminishes over time.

a) Explain how you can tell that the graph shown above does not exhibit pitch decay.

b) Sketch what the graph would look like for a sound experiencing pitch decay.

8. a) Consider a car shock absorber modelled by the equation $h(t) = e^{-0.5t}\sin t$, where $h(t)$ represents the vertical displacement, in metres, as a function of time, t, in seconds. Determine when the maximum vertical velocity, in metres per second, occurs and its value, given that $\text{velocity} = \dfrac{\text{displacement}}{\text{time}}$.

b) The greatest force felt by passengers riding in the rear seat of the car occurs when the acceleration is at a maximum. Use the relationship $F = ma$ to determine the greatest force felt by a passenger whose mass is 60 kg, where F is the force, in newtons, m is the mass of the passenger, in kilograms, and a is the acceleration, in metres per second per second.

9. **Use Technology** Refer to question 8. Due to wear and tear, after a couple of years, the equation for the vertical displacement of the shock absorber becomes $h(t) = e^{-0.2t}\sin t$.

a) Graph this function and compare it to the one given in Example 2.

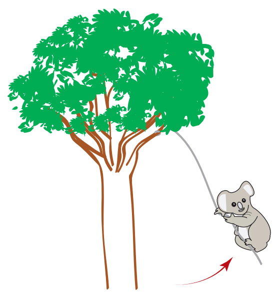
b) What does this suggest about the automobile's rear shock absorbers? Use mathematical reasoning to justify your answer.

c) Explain why the equation modelling a new shock absorber would change over time.

10. Rocco and Biff are two koala bears that are foraging for food together in a eucalyptus tree. Suddenly a gust of wind causes Rocco to lose his grip and begin to fall. He quickly grabs a nearby vine and begins to swing away from the tree. Rocco's horizontal displacement as a function of time is given by the equation

$$x(t) = 5\cos\left(\frac{\pi t}{2}\right)e^{-0.1t}$$

where x is Rocco's horizontal displacement from the bottom of his swing arc, in metres, at time t, in seconds.



a) Is Rocco's motion an example of damped harmonic motion? Explain how you can tell.

b) Biff can grab Rocco if Rocco swings back to within 1 m from where he started falling. Will Biff be able to rescue Rocco? Explain, using mathematical reasoning.

c) The other option Rocco has is to let go of the vine at the bottom of one of the swing arcs and drop to the ground. But Rocco will only feel safe doing this if his horizontal velocity at the bottom of the swing is less than 2 m/s. Assuming that Biff is unable to save his friend, how many times must Rocco swing back and forth on the vine before he can safely drop to the ground?

d) Sketch a graph of this function.

11. Refer to question 10.

a) Determine the length of vine that Rocco used to save himself.

b) How would the shape of Rocco's position graph change if the vine were

i) shorter?

ii) longer?

c) Describe the potential impact each of these conditions could have on Rocco's conditions for being saved.

CONNECTIONS

Recall from Chapter 4 that the period of a pendulum is given by the function $T = 2\pi\sqrt{\frac{l}{g}}$, where T is time, in seconds, l is the length of the pendulum in metres, and $g = 9.8$ m/s^2 is the acceleration due to gravity.

12. Chapter Problem Sheona is measuring the current through a resistor-inductor (RL) circuit that is given by the function

$$I = I_{pk}\left[1 - e^{-\left(\frac{R}{L}\right)t}\right]$$

where I is the current, in amperes, as a function of t, time after the switch is closed, in seconds, I_{pk} is the peak current, R is the value of the resistor, in ohms (Ω), and L is the value of the inductor, in henries (H). The circuit Sheona is analysing has a resistor with a value of 1000 Ω and an inductor with a value of 200 H.

a) Once the switch is closed, determine how long it will take for the circuit to reach

i) 50% of its peak current

ii) 90% of its peak current

b) Determine the rate at which the circuit is charging at these times.

Achievement Check

13. One Monday, long ago, in a secondary school far, far away, all the students in the Calculus and Vectors class decided to start a rumour about a new law called "The Homework Abolishment Act"! The spreading of this rumour throughout the school could be modelled by the function

$$n(t) = \frac{800}{1 + 39e^{-t}}$$

where n is the number of students who have heard the rumour as a function of time, t, in days.

a) Assuming that every student at the school eventually hears the rumour, determine

i) the number of students in the Calculus and Vectors class who started the rumour

ii) the student population of the school

iii) how long it will take for the rumour to reach half of the school's population

iv) the day on which the rumour was spreading the fastest

b) Sketch a graph of this function.

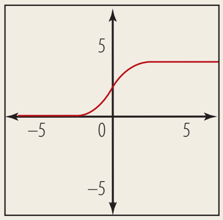
c) Why do you think this curve has the shape it does? Explain why the shape of the curve is reasonable in the context of this question. Why does it not keep rising exponentially?

CONNECTIONS

Certain types of growth phenomena follow a pattern that can be modelled by a **logistic function**, which can take the form

$f(x) = \frac{c}{1 + ae^{-bx}}$

where a, b, and c are constants related to the conditions of the phenomenon. The logistic curve is sometimes called the **S-curve**, because of its shape. Logistic functions occur in diverse areas such as biology, environmental studies, and business, in situations where resources for growth are limited and/or where conditions for growth vary over time.



To learn more about logistic functions and their applications, go to *www.mcgrawhill.ca/links/calculus12* and follow the links to Section 5.5.

C Extend and Challenge

Use the following information to answer questions 14 to 17.

The following table gives the rabbit population over time in a wilderness reserve.

Year	Population
0	52
1	78
2	125
3	206
4	291
5	378
6	465
7	551
8	620
9	663
10	701
11	726
12	735

14. **Use Technology**

a) Create a scatter plot of population versus time. Describe the shape of the graph.

b) Perform a logistic regression analysis. Record the equation of the curve of best fit.

Technology Tip

To perform a logistic regression and store the equation with a graphing calculator

- Enter the data into **L1** and **L2**.
- From the **STAT** menu, choose **CALC**, and then choose **B:Logistic**.
- Press L1, L2, Y1 ENTER. This will store the equation in **Y1**.

The equation will appear. To view the scatter plot and curve of best fit, press ZOOM and choose **9:ZoomStat**.

c) How well does the curve fit the data?

d) Extend the viewing window to locate an asymptote for this curve. What is the equation of this asymptote?

e) What is the significance of the asymptote, as it relates to the rabbit population?

15. **Use Technology**

a) Plot the function from question 14 corresponding to the curve of best fit, using *The Geometer's Sketchpad®*.

b) Graph the derivative of this function.

- Select the function equation.
- From the **Graph** menu, choose **Derivative**.

Describe the shape of this graph and explain why it has this shape.

c) Determine the time at which the rabbit population was growing the fastest.

d) How fast was the rabbit population growing at this time?

16. Refer to your result from question 15 part b). Verify the equation of this derivative algebraically.

17. Suppose that after 15 years, a small population of wolves is introduced into the reserve, where there previously were no wolves. Create some data for the wolf and rabbit populations to model this situation.

a) Sketch graphs of the wolf and rabbit populations over the next several years.

b) Explain the shapes of your graphs.

c) Use regression analysis or a curve-fitting process (e.g., use of sliders) to develop curves of best fit for your sketches. Record the equations of the curves of best fit, and explain what they mean.

18. **Math Contest** $f^{(n)}(x) = \left(\frac{d}{dx}\left(\frac{d}{dx}\left(\ldots\frac{d}{dx}(f(x))\right)\right)\right)$ denotes the nth derivative of the function $f(x)$. If $f(x) = e^x \sin x$, then $f^{(1000)}(x) =$

A $-2^{500} e^x \sin x$

B $2^{500} e^x \sin x$

C $e^x \sin x$

D $2^{1000} e^x \sin x$

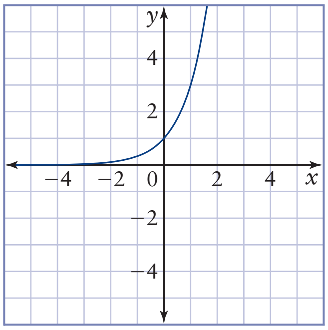
E $-2^{1000} e^x \sin x$

Chapter 5 REVIEW

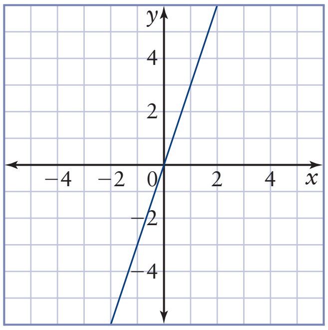
5.1 Rate of Change of an Exponential Function and the Number *e*

1. Each graph represents the rate of change of a function. Determine a possible equation for the function.

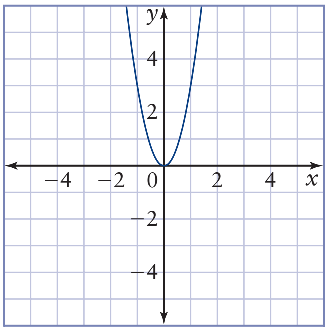
a)



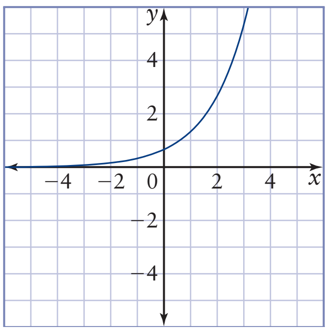
b)



c)



d)



2. a) Describe one method that you could use to estimate the value of e, assuming you do not know its value, and without using the $\boxed{e}$ button on a calculator. Use only the following definition:

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n, n \in \mathbb{N}$$

b) Use your method to estimate the value of e, correct to two decimal places.

5.2 The Natural Logarithm

3. a) Graph the function $y = e^{-x}$.

b) Graph the inverse of this function by reflecting the curve in the line $y = x$.

c) What is the equation of this inverse function? Explain how you know.

4. Evaluate, correct to three decimal places.

a) e^{-3}

b) $\ln(6.2)$

c) $\ln\left(e^{\frac{3}{4}}\right)$

d) $e^{\ln(0.61)}$

5. Solve for x, correct to two decimal places.

a) $9 = 3e^x$

b) $\ln x = -5$

c) $x = 10e^{-\frac{3}{2}}$

d) $10 = 100e^{-\frac{x}{4}}$

6. The population of a bacterial culture as a function of time is given by the equation $P(t) = 50e^{0.12t}$, where P is the population after t days.

a) What is the initial population of the bacterial culture?

b) Estimate the population after 4 days.

c) How long will it take for the population to double?

d) Rewrite $P(t)$ as an exponential having base 2.

5.3 Derivatives of Exponential Functions

7. a) Differentiate each function with respect to x. $f(x) = \left(\frac{1}{2}\right)^x$ $g(x) = -2e^x$

b) Graph each function and its derivative from part a) on the same grid.

8. Find the equation of the line tangent to the curve $y = 2(3^x)$ at $x = 1$.

9. Find the equation of the line tangent to the curve $y = -3e^x$ at $x = \ln 2$.

10. An investor places \$1000 into an account whose value increases according to the function $A = 1000(2)^{\frac{t}{9}}$, where A is the investment's value after t years.

a) Determine the value in the account after 5 years.

b) How long will it take for this investment to

i) double in value?

ii) triple in value?

c) Find the rate at which the investment is growing at each of these times.

5.4 Differentiation Rules for Exponential Functions

11. Determine each derivative.

a) $y = e^{3x^2-2x+1}$ **b)** $f(x) = (x-1)e^{2x}$

c) $y = 3x + e^{-x}$ **d)** $y = e^x \cos(2x)$

e) $g(x) = \left(\frac{1}{3}\right)^{4x} - 2e^{\sin x}$

12. Identify the coordinates of any local extreme values of the function $y = e^{x^2}$ and classify each as either a local maximum or minimum.

13. Identify the coordinates of any local extreme values of the function $y = 2e^x$ and classify each as either a local maximum or minimum.

14. A certain computer that is purchased today depreciates in value according to the function $V(t) = \$900e^{-\frac{t}{3}}$, where t represents time, in years.

a) What was the purchase price of the computer?

b) What is its value after 1 year?

c) How long will it take for the computer's value to drop to half of its original value?

d) At what rate will the computer's value be depreciating at this time?

5.5 Making Connections: Exponential Models

15. An 80-mg sample of protactinium-233 (Pa-233) is placed into a nuclear reactor. After 5 days, the sample has decayed to 70 mg. The amount of protactinium remaining in the reactor can be modelled by the function

$$N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

where $N(t)$ represents the amount of Pa-233, in milligrams, as a function of time, t, in days, N_0 represents the initial amount of Pa-233, in milligrams, and h represents the half-life of Pa-233, in days.

a) Determine the half-life of Pa-233.

b) Write the equation that gives the amount of Pa-233 remaining as a function of time, in terms of its half-life.

c) How fast is the sample decaying after 5 min?

16. After Lee gives his little sister Kara a big push on a swing, her horizontal position as a function of time is given by the equation $x(t) = 3\cos(t)\,e^{-0.05t}$, where $x(t)$ is her horizontal displacement, in metres, from the lowest point of her swing, as a function of time, t, in seconds.

a) From what horizontal distance from the bottom of Kara's swing did Lee push his sister?

b) Determine the highest speed Kara will reach and when this occurs.

c) How long will it take for Kara's maximum horizontal displacement at the top of her swing arc to diminish to 1 m? After how many swings will this occur?

d) Sketch the graph of this function.

Chapter 5 PRACTICE TEST

For questions 1 to 4, choose the best answer.

1. Which of the following is the derivative of $y = 5^x$?

A $\frac{dy}{dx} = (\ln 5)5^x$

B $\frac{dy}{dx} = e^x$

C $\frac{dy}{dx} = 5e^x$

D $\frac{dy}{dx} = 5(5)^x$

2. What is the value of $\ln(e^{-2x})$ when $x = 2$?

A e^{-4}

B $\ln(-4)$

C -4

D $-2e^{-2}$

3. What is the derivative of $f(x) = e^{-x}\cos x$?

A $f'(x) = -e^{-x}(\cos x + \sin x)$

B $f'(x) = -e^{-x}(\cos x - \sin x)$

C $f'(x) = -e^{-x}(\sin x - \cos x)$

D $f'(x) = -e^{-x}\sin^2 x$

4. What is the solution to $50 = 25e^{-2x}$?

A $x = -4$

B $x = -1$

C $x = 2\ln 2$

D $x = -\frac{\ln 2}{2}$

5. Differentiate each function.

a) $y = -2e^{-\frac{1}{2}x}$

b) $f(x) = x^3e^{2x} - x^2e^{-2x}$

6. Determine the coordinates of any local extreme values of the function $y = x^2(e^{-2x})$ and classify each as either a local maximum or a local minimum.

7. An influenza virus is spreading according to the function $P(t) = 50(2)^{\frac{t}{2}}$, where P is the number of people infected after t days.

a) How many people had the virus initially?

b) How many will be infected in 1 week?

c) How fast will the virus be spreading at the end of 1 week?

d) How long will it take until 1000 people are infected?

8. a) Graph $f(x) = -2e^x$ and its inverse on the same grid.

b) Identify the key features of both graphs in part a).

9. Determine the equation of the tangent line that meets $f(x) = -2e^x$ when $x = \ln 2$.

10. A sample of uranium-239 (U-239) decays into neptunium-239 (Np-239), according to the standard decay function $N(t) = N_0e^{-\lambda t}$. After 10 min, the sample has decayed to 64% of its initial amount.

a) Determine the value of the disintegration constant, λ.

b) Determine the half-life of U-239.

c) Write the equation that gives the amount of U-239 remaining as a function of time, in terms of its half-life.

d) Suppose the initial sample had a mass of 25 mg. How fast is the sample decaying after 15 min?

11. The value of a treasury bond is given by the function $V = 1000(1.05)^t$, where V is the value, in dollars, after t years.

a) What is the value of the bond after 10 years?

b) How long will it take the bond to double in value? Round your answer to the nearest tenth of a year.

c) Determine the derivative of the value function with respect to t.

d) How fast is the value of the bond changing at the end of 10 years?

12. Determine the derivative of each function with respect to x.

a) $y = e^x \sin^2 x$

b) $y = (x^2 + 1)e^{-x}$

c) $y = x^2 e^{\sin x}$

13. Joel Schindall at the Massachusetts Institute of Technology (MIT) has invented a supercapacitor that can replace conventional chemical batteries. It recharges in a few seconds, and has a lifetime of at least a decade. The voltage in volts drops as a function of t in hours as the supercapacitor is used to power a load, such as a laptop computer, according to the relation $V = 20e^{-\frac{t}{16}}$.

a) A laptop will fail if the voltage drops below 15 volts. How long will this take?

b) Determine the derivative of the voltage function with respect to t.

c) How fast is the voltage dropping after 1 h?

d) Predict whether the voltage will be dropping faster, slower, or at the same rate as in part c), after 2 h. Justify your answer.

e) Provide calculations to test your prediction in part d).

14. A student grew bacteria in a petri dish as a science project. He estimated the initial population as 1000 bacteria. After 1 day, the population N was estimated to be 1500 bacteria.

a) Assume that the bacterial growth follows an exponential relation of the form $N = N_0 e^{kt}$. Determine the values of N_0 and k, assuming that t is measured in days.

b) How long will it take for the population to double to 2000 bacteria?

c) Differentiate the population function with respect to t.

d) How fast is the population growing at $t = 5$ days?

15. When an aircraft is properly trimmed, it will return to its original flight path if displaced according to the relation $d = d_0 e^{-t}$, where t is the time in seconds, and d_0 is its initial displacement, in metres.

a) Suppose that the aircraft is displaced by 5 m. Write the function that predicts d as a function of t.

b) Graph the function in part a) over the interval $0 \leq t \leq 10$.

c) What is the displacement after 1 s? 2 s?

d) Determine the derivative of the function in part a) with respect to time.

e) How fast is the displacement changing at $t = 1$ s?

16. A pulse in a spring can be modelled with the relation $A = 15e^{-x^2}$, where A is the amplitude of the pulse in cm. How fast is the amplitude A changing when $x = 1$ cm?

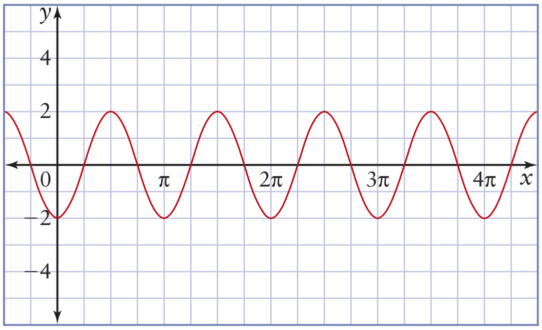
CHAPTER

PROBLEM WRAP-UP

Sheona has finished her work term and returned to university to learn more about the theory and applications of electrical engineering. Were you surprised by the amount of mathematics required for Sheona's cooperative education placement?
Perform some research in the field of electronics or in another area of science or engineering of interest to you. Identify a topic not treated in this chapter in which exponential and/or logarithmic functions are used. Write a brief report of your findings that includes

- the field of study and topic that you researched
- a couple of equations that relate to the topic, with explanations for all of the variables and constants of the equations
- two problems, posed and solved, based on the material you found

Chapter 4



1. a) Consider the graph of a function $f(x)$ shown. Write the equation of $f(x)$ assuming that $f(x)$ is a cosine function.

   b) Determine a point on the graph where the instantaneous rate of change of $f(x)$ is zero. Justify your answer.

   c) Determine a point on the graph where the instantaneous rate of change of $f(x)$ is a maximum. Justify your answer.

   d) Determine a point on the graph where the instantaneous rate of change of $f(x)$ is a minimum. Justify your answer.

   e) Determine a point on the graph that is a point of inflection. Justify your answer. If there are no points of inflection, explain how you know.

2. a) Determine the derivative of the function $f(x) = x + \sin x$ with respect to x.

   b) Determine the equation of the tangent to $f(x)$ at $x = \pi$.

   c) Will the tangent to $f(x)$ at $x = 3\pi$ have the same equation? Explain why or why not.

3. a) Determine the derivative of $f(x) = \sin 2x$ with respect to x.

   b) Determine the derivative of $g(x) = 2\sin x \cos x$ with respect to x.

   c) Show that the derivatives in parts a) and b) are equal.

   d) Explain why the derivatives in parts a) and b) should be equal.

4. A musician has adjusted his synthesizer to produce a sinusoidal output such that the amplitude of the sound follows the function $A = \sin t + 2\cos t$. Show that the rate of change of the amplitude never exceeds the maximum value of the amplitude itself.

5. A carousel has a diameter of 16 m, and completes a revolution every 30 s.

   a) Model the north–south position of a rider on the outside rim of the carousel using a sine function.

   b) Differentiate the function in part a).

   c) Determine the maximum north–south speed of the rider.

   d) What is the position of the rider on the carousel when this maximum north–south speed is reached? If it occurs at more than one position, determine all such positions in one revolution.

6. Sam has a cottage on the ocean. He measured the depth of the water at the end of his dock, and found that it reached a maximum of 11 ft at 9:00 a.m., and a minimum of 7 ft at 3:00 p.m.

   a) Assuming that the water depth as a function of time can be modelled using a sinusoidal function, determine a possible model.

   b) Determine the derivative of the function in part a).

   c) At what time is the depth increasing the fastest? Justify your answer.

   d) How fast is the water depth increasing at the time in part c)?

7. Television consists of 30 complete pictures displayed on the screen every second, giving the illusion of motion. The video signal that creates the picture must include a pulse between pictures to synchronize one picture to the next. This "sync" signal is modelled using a function of the form $A = 20\sin^2 \dfrac{t}{1000\pi}$, where A is the amplitude in volts, and t is the time in seconds.

a) Determine the derivative of the sync function with respect to t.

b) What is the first value of t greater than zero for which the derivative is equal to zero?

c) Graph the function and a tangent to verify your answer to part b).

Chapter 5

8. Consider the function $f(x) = 2^x$.

a) Use a limiting process to estimate the rate of change of the function at (0, 1).

b) Suppose that the base of the function is increased to 3. Predict the effect on the rate of change at the point (0, 1). Justify your prediction.

c) Use a limiting process to test your prediction in part b).

9. Suppose that you deposit \$1.00 into an account that pays 100% interest annually.

a) How much will you have in the bank at the end of 1 year?

b) Suppose that the interest is compounded twice each year. How much will you have in the bank at the end of 1 year?

c) Use technology to calculate the amount in the bank at the end of 1 year with more compounding periods: 3, 4, 5, … up to 100.

d) What number does the amount in part c) seem to be approaching as the number of compounding periods increases?

10. Simplify each expression.

a) $\ln(e^{\sin^2 x}) + \ln(e^{\cos^2 x})$

b) $(\ln e^{5x})\left(e^{\ln\left(\frac{1}{x}\right)}\right)$

11. A smoke detector contains about 0.2 mg of americium-241, a radioactive element that decays over t years according to the relation $m = 0.2(0.5)^{\frac{t}{432.2}}$, where m is the mass remaining in milligrams after t years.

a) The smoke detector will no longer work when the amount of americium drops below half its initial value. Is it likely to fail while you own it? Justify your answer.

b) If you buy a smoke detector today, how much of the americium will remain after 50 years?

c) How long will it take for the amount of americium to drop to 0.05 mg?

12. Differentiate each of the following functions with respect to x.

a) $f(x) = 12^x$

b) $g(x) = \left(\dfrac{3}{4}\right)^x$

c) $h(x) = -5e^x$

d) $i(x) = \theta^x$, where θ is a constant

TASK

Headache Relief? Be Careful!

Codeine phosphate is a drug used as a painkiller. Generally, it is mixed with acetaminophen in tablet form. It is rapidly absorbed into the bloodstream from the gastrointestinal tract and is gradually eliminated from the body via the kidneys. A common brand contains 30 mg of codeine. Since it is physically addictive and has other unwanted side effects, it is important to avoid an overdose while helping to relieve pain symptoms such as those caused by a headache.

Samples of blood were taken at regular time intervals from a patient who had taken a pill containing 30 mg of codeine. The amount of codeine in the bloodstream was determined every 30 min for 3 h. The data are shown in the table below.

Time After Consumption (min)	Amount of Codeine in Blood (mg)
30	27.0
60	23.5
90	21.2
120	18.7
150	16.6
180	14.5

a) Create a scatter plot of the data and determine a suitable equation to model the amount of codeine in the bloodstream t min after taking the pill. Justify your choice of models.

b) Use the model to determine the instantaneous rate of change in the amount of codeine at each time given in the chart. How does it relate to the amount of codeine in the blood?

c) It is recommended that a second pill be taken when 90% of the codeine is eliminated from the body. When would this occur?

d) Assume that the same model applies to the second pill as to the first. Suppose the patient took a second pill one hour after consuming the first pill.

- Create a model for the amount of codeine in the patient's bloodstream t min after taking the first pill.
- Determine the maximum amount of codeine in the patient's bloodstream.
- Determine when 90% of the maximum amount would be eliminated from the body.

e) If the patient were to delay taking the second pill, how would it affect the results from part d)?