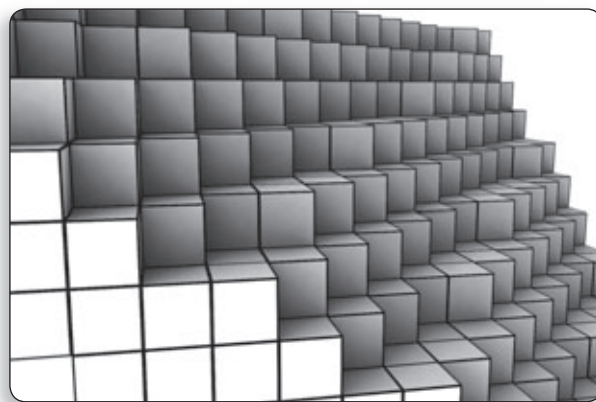


Cartesian Vectors

Simple vector quantities can be expressed geometrically. However, as the applications become more complex, or involve a third dimension, you will need to be able to express vectors in Cartesian coordinates, that is, x -, y -, and z -coordinates. In this chapter, you will investigate these Cartesian vectors and develop a three-dimensional coordinate system. You will also develop vector products and explore their applications.



By the end of this chapter, you will

- represent a vector in two-space geometrically and algebraically, with directions expressed in different ways, and recognize vectors with the same magnitude and direction but different positions as equal vectors
- determine, using trigonometric relationships, the Cartesian representation of a vector in two-space given as a directed line segment, or the representation as a directed line segment of a vector in two-space given in Cartesian form
- recognize that points and vectors in three-space can both be represented using Cartesian coordinates, and determine the distance between two points and the magnitude of a vector using their Cartesian representations
- perform the operations of addition, subtraction, and scalar multiplication on vectors represented in Cartesian form in two-space and three-space
- determine, through investigation with and without technology, some properties of the operations of addition, subtraction, and scalar multiplication of vectors
- solve problems involving the addition, subtraction, and scalar multiplication of vectors, including problems arising from real-world applications
- perform the operation of dot product on two vectors represented as directed line segments and in Cartesian form in two-space and three-space, and describe applications of the dot product
- determine, through investigation, properties of the dot product
- perform the operation of cross product on two vectors represented in Cartesian form in three-space, determine the magnitude of the cross product, and describe applications of the cross product
- determine, through investigation, properties of the cross product
- solve problems involving dot product and cross product, including problems arising from real-world applications

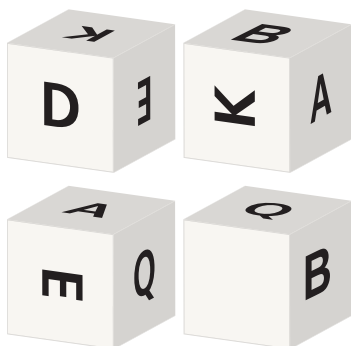
Prerequisite Skills

Visualize Three Dimensions

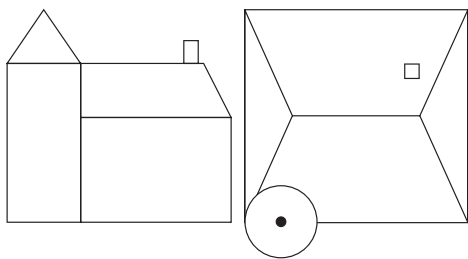
- Four views of the same alphabet block are shown. Study these views and decide which letter belongs on the blank face and which way the letter should face. Then, draw a net of the block.

CONNECTIONS

A net is a flat diagram that contains the faces of a polyhedron. The net can be folded up to form the solid.



- Given the front and top views of the building, draw the back, left, and right views.



Convert a Bearing to an Angle in Standard Position

- Convert each bearing to an angle in standard position on a Cartesian graph. Sketch each angle.

- | | | |
|-------------------|-------------------|-------------------|
| a) 072° | b) 168° | c) 90° |
| d) 180° | e) 210° | f) 300° |
| g) 360° | h) N 20° E | i) S 15° E |
| j) S 50° W | k) N 80° W | l) S 10° W |

Distance Between Two Points

- Find the distance between the points in each pair.
 - A(5, 6), B(3, 1)
 - C(-4, 3), D(6, 7)
 - E(-1, 0), F(-5, 8)
 - G(5, -2), H(-3, -9)

Properties of Addition and Multiplication

- Use an example to verify each property of real-number operations, given that a , x , y , and $z \in \mathbb{R}$. Describe each property in words.
 - Commutative property for addition:
 $x + y = y + x$
 - Commutative property for multiplication:
 $x \times y = y \times x$
 - Associative property for addition:
 $(x + y) + z = x + (y + z)$
 - Associative property for multiplication:
 $(x \times y) \times z = x \times (y \times z)$
 - Distributive property of multiplication over addition:
 $a(x + y) = ax + ay$
 - Non-commutative property of subtraction:
 $x - y \neq y - x$

Solve Systems of Equations

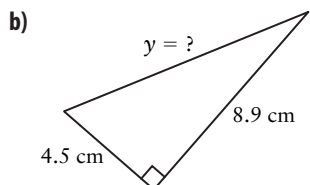
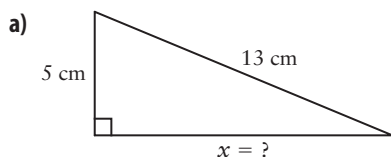
- Solve each linear system by elimination.
 - $5x + 3y = 11$
 $2x + y = 4$
 - $2x + 6y = 14$
 $x - 4y = -14$
 - $3x - 5y = -5$
 $-6x + 2y = 2$
 - $-1.5x + 3.2y = 10$
 $0.5x + 0.4y = 4$
- Which of the following is *not* equivalent to $9x - 6y = 18$?
 - $3x - 2y = 6$
 - $y = \frac{2}{3}x - 3$
 - $2y - 3x - 6 = 0$
 - $x - \frac{2}{3}y = 2$

Evaluating the Sine and Cosine of an Angle

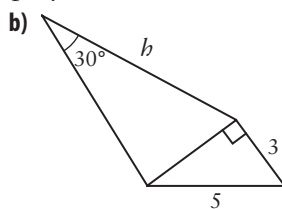
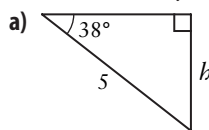
8. Evaluate. Leave in exact form.
- a) $\sin 30^\circ$ b) $\cos 60^\circ$ c) $\cos 30^\circ$
d) $\sin 45^\circ$ e) $\cos 45^\circ$ f) $\cos 90^\circ$
g) $\cos 0^\circ$ h) $\sin 180^\circ$ i) $\cos 120^\circ$
j) $\sin 300^\circ$ k) $\sin 90^\circ$ l) $\cos 180^\circ$
9. Evaluate using a calculator. Round your answers to one decimal place.
- a) $\sin 20^\circ$ b) $\cos 48^\circ$ c) $\cos 127^\circ$
d) $\sin 245^\circ$ e) $\sin 50^\circ$ f) $\cos 35^\circ$

Sides and Angles of Right Triangles

10. Find the indicated side length in each triangle. Round your answers to one decimal place, if necessary.



11. Express h in terms of the given information. It is not necessary to simplify.

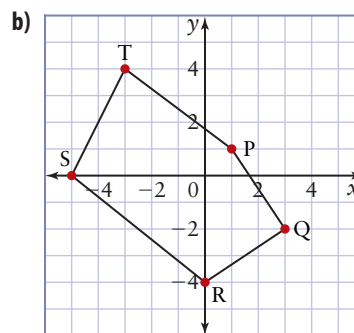


Simplifying Expressions

12. Expand and simplify.
- a) $(a_1 + b_1)^2$
b) $(a_1 + b_1)(a_1 - b_1)$
c) $(a_1^2 + a_2^2)(b_1^2 + b_2^2)$
d) $(a_1b_1 + a_2b_2 + a_3b_3)^3$
13. Suppose a pentagon has vertices (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , (x_4, y_4) , and (x_5, y_5) . It can be shown that the area of the pentagon is
- $$A = \frac{1}{2} |(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_5 - x_5y_4 + x_5y_1 - x_1y_5)|.$$

Find the area of each pentagon.

- a) ABCDE with A(0, 0), B(0, 2), C(2, 2), D(3, 1), and E(3, -1)



- c) VWXYZ with V(4, -10), W(5, -7), X(3, 2), Y(0, 5), and Z(-2, 3)

14. Refer to question 13. Are there similar formulas for triangles and hexagons? If so, state them and give an example for each. If not, explain why not.

PROBLEM

CHAPTER

Automobiles have many moving parts, and are moving objects themselves. Vectors can be used to analyse their design and motion. How can vectors help us to understand what happens to momentum in a car accident or what mechanical work is done by the tow truck as it removes a disabled vehicle?"

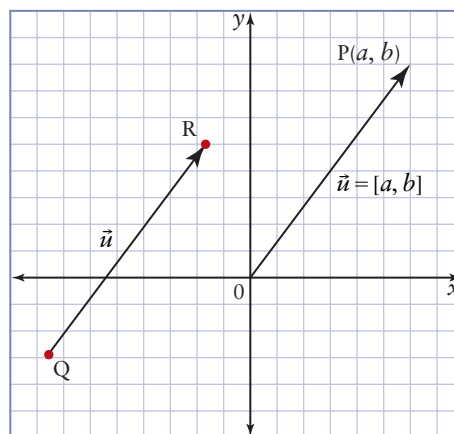


Cartesian Vectors

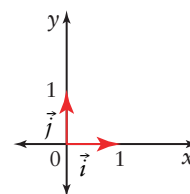
The mathematicians René Descartes and Pierre de Fermat linked algebra and geometry in the early 17th century to create a new branch of mathematics called analytic geometry. Their ideas transform geometric concepts such as vectors into an algebraic form that makes operations with those vectors much easier to perform. Without this new geometry, also called Cartesian or coordinate geometry, it would not have been possible for Newton and Leibniz to make such rapid progress with the invention of calculus.



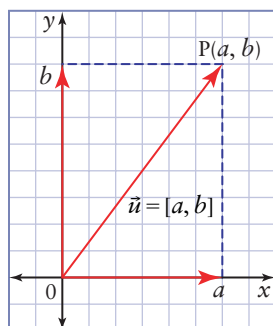
Suppose \vec{u} is any vector in the plane with endpoints Q and R. We identify \overrightarrow{QR} as a **Cartesian vector** because its endpoints can be defined using Cartesian coordinates. If we translate \vec{u} so that its tail is at the origin, O, then its head will be at some point $P(a, b)$. Then, we define this special Cartesian vector as the **position vector** $[a, b]$. We use square brackets to distinguish between the point (a, b) and the related position vector $[a, b]$. $[a, b]$ is also used to represent any vector with the same magnitude and direction as \overrightarrow{OP} .



The **unit vectors** \vec{i} and \vec{j} are the building blocks for Cartesian vectors. Unit vectors have magnitude 1 unit. \vec{i} and \vec{j} are special unit vectors that have their tails at the origin. The head of vector \vec{i} is on the x -axis at $(1, 0)$ and the head of vector \vec{j} is on the y -axis at $(0, 1)$. In the notation for Cartesian vectors, $\vec{i} = [1, 0]$ and $\vec{j} = [0, 1]$.



If we resolve \vec{u} into its horizontal and vertical vector components, we get two vectors that add to \vec{u} .



From the graph on the previous page, the magnitude of the horizontal vector is a , and it is collinear to \vec{i} . Similarly, the magnitude of the vertical vector is b , and it is collinear to \vec{j} .

Thus,

$$\vec{u} = a\vec{i} + b\vec{j}$$

Also, from the graph, $a\vec{i} = [a, 0]$ and $b\vec{j} = [0, b]$. Since $\vec{i} = [1, 0]$ and $\vec{j} = [0, 1]$, we also have $[a, 0] = a[1, 0]$ and $[0, b] = b[0, 1]$. This will be important in later proofs.

Thus,

$$\vec{u} = [a, 0] + [0, b]$$

$$\text{Since } \vec{u} = [a, b],$$

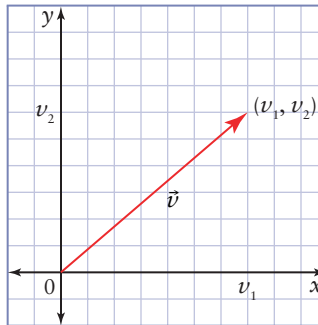
$$[a, b] = [a, 0] + [0, b]$$

Thus, any vector $[a, b]$ can be written as the sum of its vertical and horizontal vector components, $[a, 0]$ and $[0, b]$.

Magnitude of a Vector

Any Cartesian vector $\vec{v} = [v_1, v_2]$ can be translated so its head is at the origin, $(0, 0)$ and its tail is at the point (v_1, v_2) . To find the magnitude of the vector, use the formula for the distance between two points.

$$\begin{aligned} |\vec{v}| &= \sqrt{(v_1 - 0)^2 - (v_2 - 0)^2} \\ &= \sqrt{v_1^2 + v_2^2} \end{aligned}$$



Adding Vectors

To add two Cartesian vectors $\vec{u} = [u_1, u_2]$ and $\vec{v} = [v_1, v_2]$, use the unit vectors \vec{i} and \vec{j} .

$$\begin{aligned} \vec{u} + \vec{v} &= [u_1, u_2] + [v_1, v_2] \\ &= [u_1, 0] + [0, u_2] + [v_1, 0] + [0, v_2] \\ &= u_1\vec{i} + u_2\vec{j} + v_1\vec{i} + v_2\vec{j} \\ &= (u_1 + v_1)\vec{i} + (u_2 + v_2)\vec{j} \\ &= (u_1 + v_1)[1, 0] + (u_2 + v_2)[0, 1] \\ &= [u_1 + v_1, 0] + [0, u_2 + v_2] \\ &= [u_1 + v_1, u_2 + v_2] \end{aligned}$$

Write \vec{u} and \vec{v} as the sum of their horizontal and vertical vector components.

Write the vector components in terms of \vec{i} and \vec{j} .

Collect like terms and factor.

Substitute $[1, 0]$ and $[0, 1]$ for \vec{i} and \vec{j} .

Expand.

Write the sum as a single vector.

The steps in this proof depend on the properties of geometric vectors that we proved in Chapter 6 involving addition, scalar multiplication, and the distributive property for vectors.

Multiplying a Vector by a Scalar

Let $\vec{u} = [u_1, u_2]$ and $k \in \mathbb{R}$.

Then,

$$\begin{aligned}
 k\vec{u} &= k[u_1, u_2] \\
 &= k([u_1, 0] + [0, u_2]) && \text{Write } \vec{u} \text{ as the sum of its horizontal and vertical vector components.} \\
 &= k(u_1\vec{i} + u_2\vec{j}) && \text{Write the vector components as multiples of the unit vectors } \vec{i} \text{ and } \vec{j}. \\
 &= ku_1\vec{i} + ku_2\vec{j} && \text{Expand.} \\
 &= ku_1[1, 0] + ku_2[0, 1] && \text{Substitute } [1, 0] \text{ and } [0, 1] \text{ for } \vec{i} \text{ and } \vec{j}. \\
 &= [ku_1, 0] + [0, ku_2] && \text{Expand.} \\
 &= [ku_1, ku_2] && \text{Write the sum as a single vector.}
 \end{aligned}$$

The product of a vector $\vec{u} = [u_1, u_2]$ and a scalar $k \in \mathbb{R}$ is $k\vec{u} = [ku_1, ku_2]$.

Remember from Chapter 6 that any scalar multiple $k\vec{u}$ of a vector \vec{u} is collinear with \vec{u} .

Subtracting Cartesian Vectors

In Chapter 6, you learned that to subtract vectors, add the opposite. Thus, for $\vec{u} = [u_1, u_2]$ and $\vec{v} = [v_1, v_2]$,

$$\begin{aligned}
 \vec{u} - \vec{v} &= \vec{u} + (-\vec{v}) \\
 &= [u_1, u_2] + (-[v_1, v_2]) \\
 &= [u_1, u_2] + [-v_1, -v_2] \\
 &= [u_1 + (-v_1), u_2 + (-v_2)] \\
 &= [u_1 - v_1, u_2 - v_2]
 \end{aligned}$$

Example 1 Operations With Cartesian Vectors

Given the vectors $\vec{a} = [5, -7]$ and $\vec{b} = [2, 3]$, determine each of the following.

- an expression for \vec{a} in terms of its horizontal and vertical vector components
- an expression for \vec{b} in terms of \vec{i} and \vec{j}
- $3\vec{a}$
- $\vec{a} + \vec{b}$
- $2\vec{a} - 4\vec{b}$
- two unit vectors collinear with \vec{a}
- $|\vec{a} - \vec{b}|$

Solution

a) $\vec{a} = [5, 0] + [0, -7]$

b) $\vec{b} = [2, 3]$
 $= 2\vec{i} + 3\vec{j}$

c) $3\vec{a} = 3[5, -7]$
 $= [3(5), 3(-7)]$
 $= [15, -21]$

d) $\vec{a} + \vec{b} = [5, -7] + [2, 3]$
 $= [5 + 2, -7 + 3]$
 $= [7, -4]$

e) $2\vec{a} - 4\vec{b} = 2[5, -7] - 4[2, 3]$
 $= [10, -14] - [8, 12]$
 $= [10 - 8, -14 - 12]$
 $= [2, -26]$

f) A vector collinear with \vec{a} is $k\vec{a}$, where $k \in \mathbb{R}$. Find a value for k so that $k\vec{a}$ has magnitude one.

$$|k\vec{a}| = |k[5, -7]|$$

$$1 = |[5k, -7k]|$$

$$1 = \sqrt{(5k)^2 + (-7k)^2}$$

$$1 = 74k^2$$

$$\frac{1}{74} = k^2$$

$$k = \pm \frac{1}{\sqrt{74}}$$

Thus, two unit vectors collinear with \vec{a} are $\left[\frac{5}{\sqrt{74}}, -\frac{7}{\sqrt{74}} \right]$ and

$\left[-\frac{5}{\sqrt{74}}, \frac{7}{\sqrt{74}} \right]$. Check that the two vectors have magnitude one.

$$\begin{aligned} \left| \left[\frac{5}{\sqrt{74}}, -\frac{7}{\sqrt{74}} \right] \right| &= \sqrt{\left(\frac{5}{\sqrt{74}} \right)^2 + \left(-\frac{7}{\sqrt{74}} \right)^2} \\ &= \sqrt{\frac{25}{74} + \frac{49}{74}} \\ &= 1 \end{aligned}$$

Similarly, the magnitude of $\left[-\frac{5}{\sqrt{74}}, \frac{7}{\sqrt{74}} \right]$ is one.

g) $|\vec{a} - \vec{b}| = |[5 - 2, -7 - 3]|$
 $= |[3, -10]|$
 $= \sqrt{3^2 + (-10)^2}$
 $= \sqrt{109}$

Cartesian Vector Between Two Points

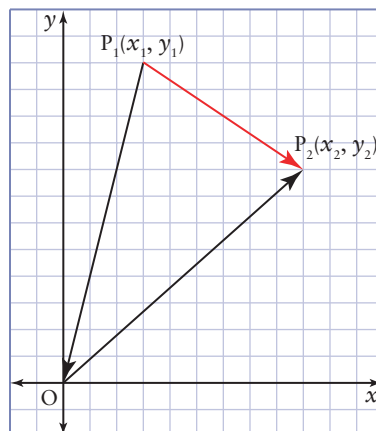
Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the coordinate plane.

Then,

$$\begin{aligned}\overrightarrow{P_1P_2} &= \overrightarrow{P_1O} + \overrightarrow{OP_2} \\ &= \overrightarrow{OP_2} - \overrightarrow{OP_1} \\ &= [x_2, y_2] - [x_1, y_1] \\ &= [x_2 - x_1, y_2 - y_1]\end{aligned}$$

To find the magnitude of $\overrightarrow{P_1P_2}$, use the formula for the distance between two points:

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 2

Find the Coordinates and the Magnitude of Cartesian Vectors Between Two Points

Find the coordinates and the magnitude of each vector.

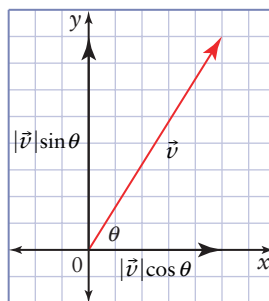
- a) \overrightarrow{AB} , for $A(1, 3)$ and $B(7, 2)$
- b) \overrightarrow{CD} , for $C(-10, 0)$ and $D(0, 10)$
- c) \overrightarrow{EF} , for $E(4, -3)$ and $F(1, -7)$

Solution

- a)
$$\begin{aligned}\overrightarrow{AB} &= [x_2 - x_1, y_2 - y_1] \\ &= [7 - 1, 2 - 3] \\ &= [6, -1] \\ |\overrightarrow{AB}| &= \sqrt{6^2 + (-1)^2} \\ &= \sqrt{37}\end{aligned}$$
- b)
$$\begin{aligned}\overrightarrow{CD} &= [x_2 - x_1, y_2 - y_1] \\ &= [0 - (-10), 10 - 0] \\ &= [10, 10] \\ |\overrightarrow{CD}| &= \sqrt{10^2 + 10^2} \\ &= \sqrt{200}\end{aligned}$$
- c)
$$\begin{aligned}\overrightarrow{EF} &= [x_2 - x_1, y_2 - y_1] \\ &= [1 - 4, -7 - (-3)] \\ &= [-3, -4] \\ |\overrightarrow{EF}| &= \sqrt{(-3)^2 + (-4)^2} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

To write a geometric vector \vec{v} in Cartesian form, you need to use trigonometry.

In Chapter 6, we found that the magnitude of the horizontal component is $|\vec{v}|\cos\theta$, and the magnitude of the vertical component is $|\vec{v}|\sin\theta$, where θ is the angle \vec{v} makes with the horizontal, or the positive x -axis. Thus, $\vec{v} = [|\vec{v}|\cos\theta, |\vec{v}|\sin\theta]$.



Example 3 Force as a Cartesian Vector

Write a force of 200 N at 20° to the horizontal in Cartesian form.

Solution

$$\begin{aligned}\vec{F} &= [|\vec{F}|\cos\theta, |\vec{F}|\sin\theta] \\ &= [200\cos 20^\circ, 200\sin 20^\circ] \\ &\doteq [187.9, 68.4]\end{aligned}$$

The force can be written in Cartesian form as approximately $[187.9, 68.4]$.

Example 4 Velocity and Bearings

A ship's course is set to travel at 23 km/h, relative to the water, on a heading of 040° . A current of 8 km/h is flowing from a bearing of 160° .

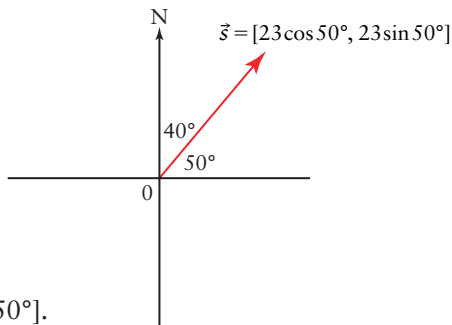
- Write each vector as a Cartesian vector.
- Determine the resultant velocity of the ship.

Solution

Headings and bearings are measured clockwise from north, which is equivalent to clockwise from the positive y -axis. These have to be converted to angles with respect to the positive x -axis.

- The ship is travelling on a heading of 040° , which is 40° clockwise from the positive y -axis. This is equivalent to an angle of 50° from the positive x -axis. Let \vec{s} represent the velocity of the ship.

The velocity of the ship can be expressed as $\vec{s} = [23\cos 50^\circ, 23\sin 50^\circ]$.



The current is flowing from a bearing of 160° , which is equivalent to an angle of 110° with the positive x -axis. Let \vec{c} represent the velocity of the current.

The velocity of the current can be expressed as $\vec{c} = [8 \cos 110^\circ, 8 \sin 110^\circ]$.

b) Let \vec{R} represent the resultant velocity.

$$\begin{aligned}\vec{R} &= \vec{s} + \vec{c} \\ &= [23 \cos 50^\circ, 23 \sin 50^\circ] + [8 \cos 110^\circ, 8 \sin 110^\circ] \\ &= [23 \cos 50^\circ + 8 \cos 110^\circ, 23 \sin 50^\circ + 8 \sin 110^\circ] \\ &\doteq [12.048, 25.137]\end{aligned}$$

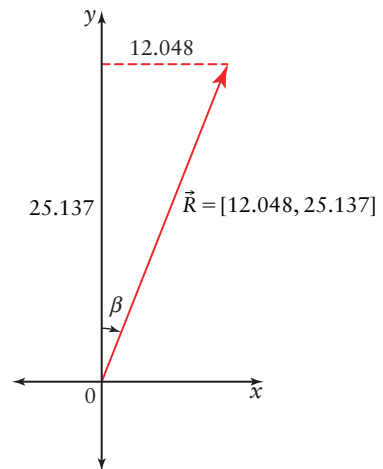
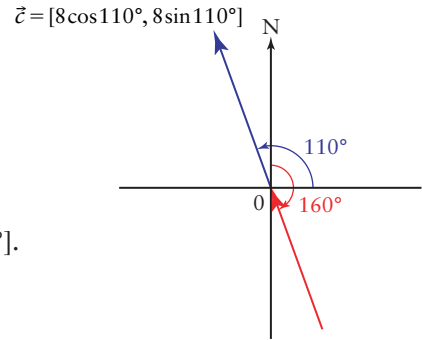
Find the magnitude of the resultant velocity \vec{R} .

$$\begin{aligned}|\vec{R}| &\doteq \sqrt{12.048^2 + 25.137^2} \\ &\doteq 27.875\end{aligned}$$

The bearing is the angle, β , the vector makes with the positive y -axis. From the graph,

$$\begin{aligned}\beta &\doteq \tan^{-1}\left(\frac{12.048}{25.137}\right) \\ &\doteq 25.6^\circ\end{aligned}$$

The ship is travelling at a velocity of approximately 27.9 km/h at a bearing of 025.6° .



KEY CONCEPTS

- The unit vectors $\vec{i} = [1, 0]$ and $\vec{j} = [0, 1]$ have magnitude 1 unit and tails at the origin and point in the directions of the positive x - and y -axes, respectively.
- A Cartesian vector is a representation of a vector on the Cartesian plane. Its endpoints are defined using Cartesian coordinates.
- If a Cartesian vector \vec{u} is translated so that its tail is at the origin, $(0, 0)$, and its tip is at the point (u_1, u_2) , the translated vector is called the position vector of \vec{u} . The position vector, and any other vector with the same magnitude and direction, is represented by the ordered pair $[u_1, u_2]$.
- The magnitude of $\vec{u} = [u_1, u_2]$ is $|\vec{u}| = \sqrt{u_1^2 + u_2^2}$.
- Any Cartesian vector $[u_1, u_2]$ can be written as the sum of its vertical and horizontal vector components, $[u_1, 0]$ and $[0, u_2]$.

- For vectors $\vec{u} = [u_1, u_2]$ and $\vec{v} = [v_1, v_2]$ and scalar $k \in \mathbb{R}$,
 - $\vec{u} + \vec{v} = [u_1 + v_1, u_2 + v_2]$
 - $\vec{u} - \vec{v} = [u_1 - v_1, u_2 - v_2]$
 - $k\vec{v} = [kv_1, kv_2]$
- The Cartesian vector between two points, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, is $\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1]$. Its magnitude is $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A geometric vector \vec{v} can be written in Cartesian form as $\vec{v} = [|\vec{v}|\cos\theta, |\vec{v}|\sin\theta]$, where θ is the angle \vec{v} makes with the positive x -axis.

Communicate Your Understanding

- C1** Given two points, A(5, 3) and B(-7, 2), describe how to find the coordinates and the magnitude of the Cartesian vector \overrightarrow{AB} .
- C2** What are the coordinates of the zero vector, $\vec{0}$?
- C3** How does a position vector differ from a Cartesian vector?
- C4** Which is easier: adding vectors in Cartesian form or adding vectors in geometric form? Explain.
- C5** What is the difference between $[2, 3]$ and $(2, 3)$?

A Practise

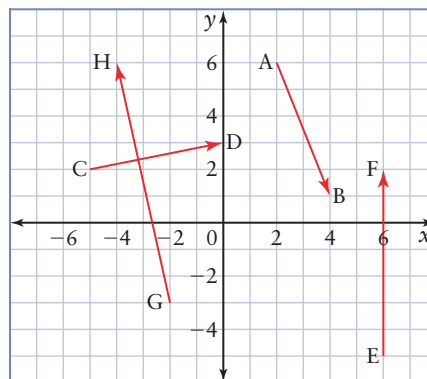
1. Express each vector in terms of \vec{i} and \vec{j} .

- | | |
|---------------|-------------------|
| a) $[2, 1]$ | b) $[3, -5]$ |
| c) $[-3, -6]$ | d) $[5, 0]$ |
| e) $[9, -7]$ | f) $[0, -8]$ |
| g) $[-6, 0]$ | h) $[-5.2, -6.1]$ |

2. Express each vector in the form $[a, b]$.

- | | |
|---------------------------|-------------------------------|
| a) $\vec{i} + \vec{j}$ | b) $-4\vec{i}$ |
| c) $2\vec{j}$ | d) $3\vec{i} + 8\vec{j}$ |
| e) $-5\vec{i} - 2\vec{j}$ | f) $7\vec{i} - 4\vec{j}$ |
| g) $-8.2\vec{j}$ | h) $-2.5\vec{i} + 3.3\vec{j}$ |

3. Write the coordinates of each Cartesian vector.



4. Determine the magnitude of each vector in question 3.

5. You are given the vector $\vec{v} = [5, -1]$.
- State the vertical and horizontal vector components of \vec{v} .
 - Find two unit vectors that are collinear with \vec{v} .
 - An equivalent vector \overrightarrow{PQ} has its initial point at $P(-2, -7)$. Determine the coordinates of Q .
 - An equivalent vector \overrightarrow{LM} has its terminal point at $M(5, 8)$. Determine the coordinates of L .
6. Given the points $P(-6, 1)$, $Q(-2, -1)$, and $R(-3, 4)$, find
- \overrightarrow{QP}
 - $|\overrightarrow{RP}|$
 - the perimeter of $\triangle PQR$
7. If $\vec{u} = [4, -1]$ and $\vec{v} = [2, 7]$, find
- $8\vec{u}$
 - $-8\vec{u}$
 - $\vec{u} + \vec{v}$
 - $\vec{v} - \vec{u}$
 - $5\vec{u} - 3\vec{v}$
 - $-4\vec{u} + 7\vec{v}$
8. Which vector is *not* collinear with $\vec{a} = [6, -4]$?
- $\vec{b} = [3, -2]$
 - $\vec{c} = [-6, -4]$
 - $\vec{d} = [-6, 4]$
 - $\vec{e} = [-9, 6]$

B Connect and Apply

9. Write each force as a Cartesian vector.
- 500 N applied at 30° to the horizontal
 - 1000 N applied at 72° to the vertical
 - 125 N applied upward
 - 230 N applied to the east
 - 25 N applied downward
 - 650 N applied to the west
10. A ship's course is set at a heading of 192° , with a speed of 30 knots. A current is flowing from a bearing of 112° , at 14 knots. Use Cartesian vectors to determine the resultant velocity of the ship.
11. Let $\vec{a} = [4, 7]$ and $\vec{b} = [2, -9]$.
- Plot the two vectors.
 - Which is greater, $|\vec{a} + \vec{b}|$ or $|\vec{a}| + |\vec{b}|$?
 - Will this be true for all pairs of vectors? Justify your answer with examples.
12. Let $\vec{a} = [-1, 6]$ and $\vec{b} = [7, 2]$.
- Plot the two vectors.
 - Which is greater, $|\vec{a} + \vec{b}|$ or $|\vec{a} - \vec{b}|$?
 - Will this be true for all pairs of vectors? Justify your answer with examples.



13. Let $\vec{u} = [u_1, u_2]$, $\vec{v} = [v_1, v_2]$, and $\vec{w} = [w_1, w_2]$, and let $k \in \mathbb{R}$. Prove each property using Cartesian vectors.
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
 - $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$
 - $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
 - $(k + m)\vec{u} = k\vec{u} + m\vec{u}$

CONNECTIONS

In Chapter 6, these properties of vectors were proved geometrically.

14. A person pulls a sleigh, exerting a force of 180 N along a rope that makes an angle of 30° to the horizontal. Write this force in component form as a Cartesian vector.
15. A person pushes a lawnmower with a force of 250 N. The handle makes an angle of 35° with the ground. Write this force in component form as a Cartesian vector.
16. An airplane is flying at 550 km/h on a heading of 080° . The wind is blowing at 60 km/h from a bearing of 120° . Determine the ground velocity of the airplane. Compare your answer to that of Section 6.4, question 2.

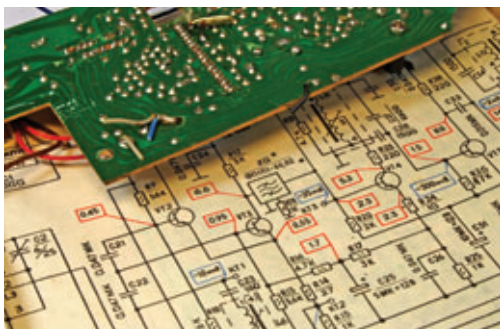


17. Emily and Clare kick a soccer ball at the same time. Emily kicks it with a force of 120 N and Clare kicks it with a force of 200 N. The angle between the forces is 60° . Calculate the magnitude and direction of the resultant force. Compare your answer to that of Section 6.4, question 11.
18. Three basketball players are fighting over the ball. Sam is pulling with a force of 608 N, Jason is pulling with a force of 550 N, and Nick is pulling with a force of 700 N. The angle between Sam and Jason is 120° , and the angle between Jason and Nick is 150° . Determine the resultant force on the basketball.
19. **Chapter Problem** In a collision, a car has momentum $15\,000\text{ kg}\cdot\text{m/s}^2$ and strikes another car that has momentum $12\,000\text{ kg}\cdot\text{m/s}^2$. The angle between their directions of travel is 15° . Determine the total resultant momentum of these cars after the collision. (Hint: Place one of the momentum vectors along the x -axis.)



C Extend and Challenge

20. Cartesian vectors are also used to represent components in electric circuits. The tuner in a radio selects the desired station by adjusting the reactance of a circuit. Resistive reactance is represented by a vector along the positive x -axis, capacitive reactance is represented by a vector along the positive y -axis, and inductive reactance is represented by a vector along the negative y -axis. You can determine the total reactance by adding the three vectors.
- Consider a radio circuit with a resistive reactance of $4\ \Omega$ (ohms), a capacitive reactance of $2\ \Omega$, and an inductive reactance of $5\ \Omega$.
- What is the magnitude of the total reactance of the circuit?
 - What is the angle between the total reactance and the positive x -axis?
21. A boat's destination is 500 km away, at a bearing of 048° . A 15-km/h current is flowing from a bearing of 212° . What velocity (magnitude and direction) should the captain set in order to reach the destination in 12 h?
22. Solve for x .
- $\vec{u} = [x, 3x]$, $|\vec{u}| = 9$
 - $\vec{u} = [2x, x]$, $\vec{v} = [x, 2x]$, $|\vec{u} + \vec{v}| = 6$
 - $\vec{u} = [3x, 7]$, $\vec{v} = [5x, x]$, $|\vec{u} + \vec{v}| = 10x$
23. Is it possible for the sum of two unit vectors to be a unit vector? Justify your response.
24. **Math Contest** The circle $(x - 7)^2 + (y - 4)^2 = 25$ and the parabola $y = 3x^2 - 42x + k$ have the same x -intercepts. Determine the distance from the vertex of the parabola to the centre of the circle.
25. **Math Contest** Prove (without using a calculator) that $\sin 20^\circ + \sin 40^\circ = \sin 80^\circ$. (Hint: One method starts with an equilateral triangle.)



7.2

Dot Product

The operations you have performed with vectors are addition, subtraction, and scalar multiplication. These are familiar from your previous work with numbers and algebra. You will now explore a new operation, the dot product, that is used only with vectors. One of the applications of the dot product is to determine the **mechanical work** (or, simply, the work) performed. Mechanical work is the product of the magnitude of the displacement travelled by an object and the magnitude of the force applied in the direction of motion. The units are newton-metres ($\text{N} \cdot \text{m}$), also known as joules (J).



CONNECTIONS

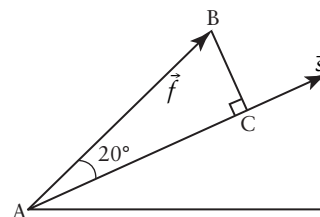
James Joule (1818–1892) studied heat and its relationship to mechanical work. The unit of energy, the joule, is named after him. Go to www.mcgrawhill.ca/links/Calculus12 and follow the links to discover more about James Joule.

CONNECTIONS

Energy and work use the same units because energy is the capacity for doing work. To do 50 J of work, you must have expended 50 J of energy.

Example 1 Find the Work Performed

Max is pulling his sled up a hill with a force of 120 N at an angle of 20° to the surface of the hill. The hill is 100 m long. Find the work that Max performs.



Solution

There are two vectors to consider: the force, \vec{f} , on the sled and the displacement, \vec{s} , along the hill. To find the work, we need the magnitude of the component of the force along the surface of the hill. Label the endpoints of vector \vec{f} as A and B. Draw a perpendicular from B to the surface of the hill at C. Then, the length of AC is equal to the magnitude of the force along the surface of the hill.

From the triangle,

$$\begin{aligned}\frac{AC}{AB} &= \cos 20^\circ \\ AC &= AB \cos 20^\circ \\ &= 120 \cos 20^\circ\end{aligned}$$

From the definition of work,

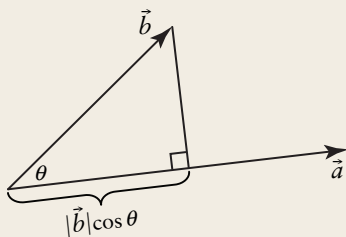
$$\begin{aligned}\text{work} &= |\vec{s}|(AC) \\ &= (100)(120 \cos 20^\circ) \\ &\doteq 11\,276\end{aligned}$$

The work done by Max in pulling his sled up the hill is 11 276 J.

The new product for geometric vectors in Example 1 is called the **dot product**.

The Dot Product

For two vectors \vec{a} and \vec{b} , the dot product is defined as $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, where θ is the angle between \vec{a} and \vec{b} when the vectors are arranged tail to tail, and $0 \leq \theta \leq 180^\circ$. The dot product is a scalar, not a vector, and the units depend on the application.

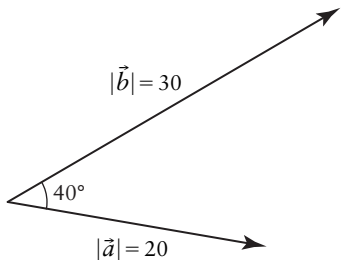


If either \vec{a} or \vec{b} is the zero vector, then $\vec{a} \cdot \vec{b} = 0$.

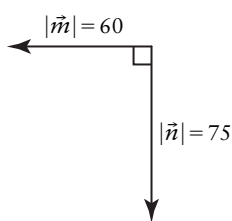
Example 2 Dot Products

Determine the dot product of each pair of vectors.

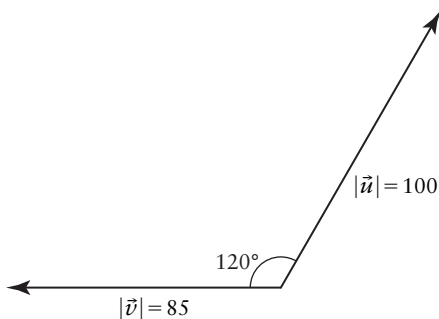
a)



b)



c)



Solution

$$\begin{aligned} \text{a) } \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos \theta \\ &= (20)(30) \cos 40^\circ \\ &\doteq 459.6 \end{aligned}$$

$$\begin{aligned} \text{b) } \vec{m} \cdot \vec{n} &= |\vec{m}| |\vec{n}| \cos \theta \\ &= (60)(75) \cos 90^\circ \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{c) } \vec{u} \cdot \vec{v} &= |\vec{u}| |\vec{v}| \cos \theta \\ &= (100)(85) \cos 120^\circ \\ &= -4250 \end{aligned}$$

Investigate

How can you prove some properties of the dot product?

Tools (optional)

- computer with *The Geometer's Sketchpad*®

Investigate the properties of the dot product using paper and pencil, geometry software, or a tool of your choice.

1. Investigate the dot product for different angles $0^\circ \leq \theta \leq 180^\circ$ between vectors \vec{a} and \vec{b} . What happens when $\theta < 90^\circ$? What happens when $\theta > 90^\circ$?
2. What happens when the vectors \vec{a} and \vec{b} are perpendicular? Outline the steps you would take to find the dot product.
3. Is the dot product commutative? That is, for any vectors \vec{a} and \vec{b} , does $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ hold? Outline the steps you would take to prove or disprove this property.
4. What happens when you take the dot product of a vector with itself? That is, for a vector \vec{a} , what does $\vec{a} \cdot \vec{a}$ equal?
5. Is there an associative property for the dot product?
 - a) Can you find the dot product of three vectors? Explain.
 - b) Explore the associative property with two vectors \vec{a} and \vec{b} and a scalar $k \in \mathbb{R}$. That is, does $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$ hold?
 - c) Explore whether $k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$ holds.
6. Explore the distributive property for dot product over addition. That is, for vectors \vec{a} , \vec{b} , and \vec{c} , what does $\vec{a} \cdot (\vec{b} + \vec{c})$ equal?
7. **Reflect** Summarize your findings in steps 1 to 6.

Properties of the Dot Product

1. For non-zero vectors \vec{u} and \vec{v} , \vec{u} and \vec{v} are perpendicular **if and only if** $\vec{u} \cdot \vec{v} = 0$. That is, if \vec{u} and \vec{v} are perpendicular, then $\vec{u} \cdot \vec{v} = 0$, and if $\vec{u} \cdot \vec{v} = 0$, then \vec{u} and \vec{v} are perpendicular.
2. For any vectors \vec{u} and \vec{v} , $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. This is the **commutative property** for the dot product.
3. For any vector \vec{u} , $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.
4. For any vectors \vec{u} and \vec{v} and scalar $k \in \mathbb{R}$, $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$. This is the **associative property** for the dot product.
5. For any vectors \vec{u} , \vec{v} , and \vec{w} , $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ and, because of the commutative property, $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$. This is the **distributive property** for the dot product.

Example 3 Properties of the Dot Product

Use the properties of the dot product to expand and simplify each expression.

a) $(k\vec{u}) \cdot (\vec{u} + \vec{v})$ b) $(\vec{r} + \vec{s}) \cdot (\vec{r} - \vec{s})$

Solution

a) $(k\vec{u}) \cdot (\vec{u} + \vec{v}) = (k\vec{u}) \cdot \vec{u} + (k\vec{u}) \cdot \vec{v}$ Distributive property
 $= k(\vec{u} \cdot \vec{u}) + k(\vec{u} \cdot \vec{v})$ Associative property
 $= k|\vec{u}|^2 + k(\vec{u} \cdot \vec{v})$

b) $(\vec{r} + \vec{s}) \cdot (\vec{r} - \vec{s}) = (\vec{r} + \vec{s}) \cdot \vec{r} + (\vec{r} + \vec{s}) \cdot (-\vec{s})$ Distributive property
 $= \vec{r} \cdot \vec{r} + \vec{s} \cdot \vec{r} + \vec{r} \cdot (-\vec{s}) + \vec{s} \cdot (-\vec{s})$ Distributive property
 $= |\vec{r}|^2 + \vec{s} \cdot \vec{r} - 1(\vec{r} \cdot \vec{s}) - 1(\vec{s} \cdot \vec{s})$ Associative property
 $= |\vec{r}|^2 + \vec{r} \cdot \vec{s} - \vec{r} \cdot \vec{s} - |\vec{s}|^2$ Commutative property
 $= |\vec{r}|^2 - |\vec{s}|^2$

The Dot Product for Cartesian Vectors

The dot product can also be defined for Cartesian vectors.

Let $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$. Then, from the geometric definition of the dot product,

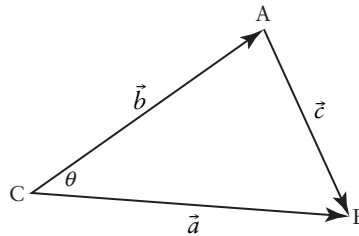
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

From the triangle,

$$\begin{aligned}\vec{c} &= \vec{a} - \vec{b} \\ &= [a_1 - b_1, a_2 - b_2]\end{aligned}$$

By the cosine law,

$$\begin{aligned}|\vec{c}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta \\ 2|\vec{a}||\vec{b}|\cos\theta &= |\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2 \\ |\vec{a}||\vec{b}|\cos\theta &= \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2}{2} \\ \vec{a} \cdot \vec{b} &= \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2}{2} \\ &= \frac{a_1^2 + a_2^2 + b_1^2 + b_2^2 - [(a_1 - b_1)^2 + (a_2 - b_2)^2]}{2} \\ &= \frac{\cancel{a_1^2} + \cancel{a_2^2} + \cancel{b_1^2} + \cancel{b_2^2} - \cancel{a_1^2} + 2a_1b_1 - \cancel{b_1^2} - \cancel{a_2^2} + 2a_2b_2 - \cancel{b_2^2}}{2} \\ &= \frac{2a_1b_1 + 2a_2b_2}{2} \\ &= a_1b_1 + a_2b_2\end{aligned}$$



Thus, the dot product of two Cartesian vectors $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$ is $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$.

Example 4 Calculate Dot Products of Vectors

Calculate $\vec{u} \cdot \vec{v}$.

a) $\vec{u} = [5, -3], \vec{v} = [4, 7]$

b) $\vec{u} = [-2, 9], \vec{v} = [-1, 0]$

Solution

$$\begin{aligned} \text{a) } \vec{u} \cdot \vec{v} &= u_1v_1 + u_2v_2 \\ &= 5(4) + (-3)(7) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{b) } \vec{u} \cdot \vec{v} &= u_1v_1 + u_2v_2 \\ &= -2(-1) + 9(0) \\ &= 2 \end{aligned}$$

Example 5 Prove a Dot Product Property

a) Use an example to verify the property that $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ for any vector \vec{a} .

b) Prove the property algebraically.

Solution

a) Let $\vec{a} = [2, 5]$.

$$\begin{array}{ll} \text{L.S.} = \vec{a} \cdot \vec{a} & \text{R.S.} = |\vec{a}|^2 \\ = [2, 5] \cdot [2, 5] & = |[2, 5]|^2 \\ = 2(2) + 5(5) & = (\sqrt{2^2 + 5^2})^2 \\ = 4 + 25 & = (\sqrt{29})^2 \\ = 29 & = 29 \end{array}$$

$$\text{L.S.} = \text{R.S.}$$

Thus, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ for $\vec{a} = [2, 5]$.

b) Let $\vec{a} = [a_1, a_2]$.

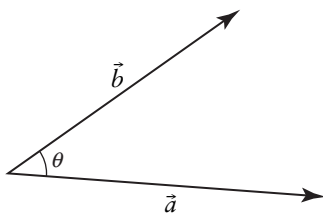
$$\begin{array}{ll} \text{L.S.} = \vec{a} \cdot \vec{a} & \text{R.S.} = |\vec{a}|^2 \\ = [a_1, a_2] \cdot [a_1, a_2] & = |[a_1, a_2]|^2 \\ = a_1^2 + a_2^2 & = (\sqrt{a_1^2 + a_2^2})^2 \\ & = a_1^2 + a_2^2 \end{array}$$

$$\text{L.S.} = \text{R.S.}$$

Thus, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ for any vector \vec{a} .

KEY CONCEPTS

- The dot product is defined as $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, where θ is the angle between \vec{a} and \vec{b} , $0 \leq \theta \leq 180^\circ$.
- For any vectors \vec{u} , \vec{v} , and \vec{w} and scalar $k \in \mathbb{R}$,
 - $\vec{u} \neq 0$ and $\vec{v} \neq 0$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$
 - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative property)
 - $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$ (associative property)
 - $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ and $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$ (distributive property)
 - $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
 - $\vec{u} \cdot \vec{0} = 0$
- If $\vec{u} = [u_1, u_2]$ and $\vec{v} = [v_1, v_2]$, then $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$.

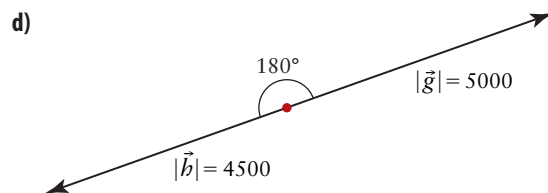
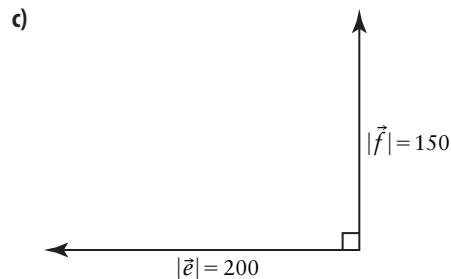
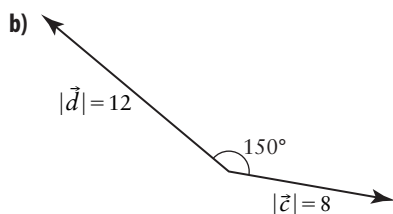
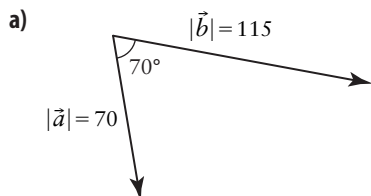


Communicate Your Understanding

- C1** In the dot product definition, $0 \leq \theta \leq 180^\circ$. What happens if $\theta > 180^\circ$? Explain.
- C2** Explain why the dot product of any vector with the zero vector, $\vec{0}$, is equal to zero.
- C3** Does the equation $\vec{u} \cdot \vec{v} = \vec{w}$ make sense? Explain.
- C4** What does the dot product measure?

A Practise

1. Calculate the dot product for each pair of vectors. Round answers to one decimal place.



2. Calculate the dot product for each pair of vectors. θ is the angle between the vectors when they are placed tail to tail. Round answers to one decimal place.

- a) $|\vec{u}| = 6, |\vec{v}| = 10, \theta = 30^\circ$
 b) $|\vec{s}| = 30, |\vec{t}| = 15, \theta = 120^\circ$
 c) $|\vec{f}| = 5.8, |\vec{g}| = 13.4, \theta = 180^\circ$
 d) $|\vec{q}| = 4.0, |\vec{r}| = 6.1, \theta = 90^\circ$
 e) $|\vec{a}| = 850, |\vec{b}| = 400, \theta = 58^\circ$
 f) $|\vec{m}| = 16, |\vec{p}| = 2|\vec{m}|, \theta = 153^\circ$

3. Use the properties of the dot product to expand and simplify each of the following, where $k, l \in \mathbb{R}$.

- a) $\vec{u} \cdot (k\vec{u} + \vec{v})$
 b) $(k\vec{u} - \vec{v}) \cdot (l\vec{v})$

- c) $(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$
 d) $(\vec{u} + \vec{v}) \cdot (\vec{w} + \vec{x})$

4. Calculate the dot product of each pair of vectors.

- a) $\vec{u} = [2, 4], \vec{v} = [3, -1]$
 b) $\vec{m} = [-5, -7], \vec{n} = [0, 7]$
 c) $\vec{s} = [9, -3], \vec{t} = [3, -3]$
 d) $\vec{p} = [-6, 2], \vec{q} = [9, 1]$
 e) $\vec{a} = 2\vec{i} + 3\vec{j}, \vec{b} = 9\vec{i} - 7\vec{j}$
 f) $\vec{s} = 4\vec{i} + \vec{j}, \vec{t} = -\vec{i} - \vec{j}$

5. State whether each expression has any meaning. If not, explain why not.

- a) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$ b) $|\vec{u} \cdot \vec{v}|$ c) $\vec{u}(\vec{v} \cdot \vec{w})$
 d) $|\vec{u}|^2$ e) \vec{v}^2 f) $(\vec{u} \cdot \vec{v})^2$

B Connect and Apply

6. a) Calculate the dot product of the unit vectors \vec{i} and \vec{j} , first using geometric vectors and then using Cartesian vectors.

- b) Explain the results.

7. Let $\vec{u} = [3, -5]$, $\vec{v} = [-6, 1]$, and $\vec{w} = [4, 7]$. Evaluate each of the following, if possible. If it is not possible, explain why not.



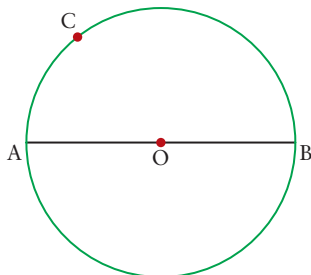
- a) $\vec{u} \cdot (\vec{v} + \vec{w})$ b) $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w}$
 c) $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v})$ d) $\vec{u} + \vec{v} \cdot \vec{w}$
 e) $-3\vec{v} \cdot (2\vec{w})$ f) $5\vec{u} \cdot (2\vec{v} - \vec{w})$
 g) $\vec{u} \cdot \vec{v} \cdot \vec{w}$ h) $(\vec{u} + 2\vec{v}) \cdot (3\vec{w} - \vec{u})$
 i) $\vec{u} \cdot \vec{u}$ j) $\vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$

8. a) Which of the following is a right-angled triangle? Identify the right angle in that triangle.
- $\triangle ABC$ for $A(3, 1)$, $B(-2, 3)$, and $C(5, 6)$
 - $\triangle STU$ for $S(4, 6)$, $T(-3, 7)$, and $U(-5, -4)$
- b) Describe another method for solving the problem in part a).

9. a) Compare the vectors $\vec{s} = [7, -3]$ and $\vec{t} = [3, 7]$.
 b) Make a hypothesis from your observation.
 c) Verify your hypothesis using algebra, construction, or logical argument, as appropriate.

10. Find a vector that is perpendicular to $\vec{u} = [9, 2]$. Verify that the vectors are perpendicular.
11. Determine the value of k so that $\vec{u} = [2, 5]$ and $\vec{v} = [k, 4]$ are perpendicular.
12. Determine the value of k so that $\vec{u} = [k, 3]$ and $\vec{v} = [k, 2k]$ are perpendicular.
13. Use an example to verify each property of the dot product for vectors \vec{a} and \vec{b} .
- a) If \vec{a} and \vec{b} are non-zero, $\vec{a} \cdot \vec{b} = 0$ if and only if \vec{a} is perpendicular to \vec{b} .
 b) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (the commutative property)
 c) If \vec{u} is a unit vector parallel to \vec{a} , then $\vec{a} \cdot \vec{u} = |\vec{a}|$.
14. Prove each property in question 13 using Cartesian vectors.

15. Find a counterexample to prove that it is not true that if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then $\vec{b} = \vec{c}$.
16. Points A, B, and C lie on the circumference of a circle so that AB is a diameter. Use vector methods to prove that $\angle ACB$ is a right angle.

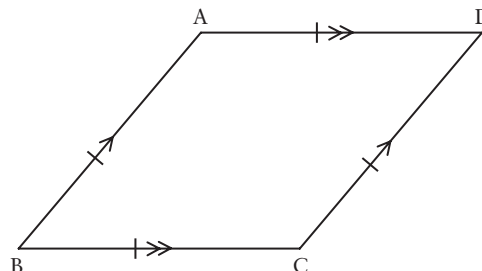


17. Use an example to verify each property of the dot product for any vectors \vec{u} , \vec{v} , and \vec{w} and scalar $k \in \mathbb{R}$.
- $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$
 - $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
 - $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

18. Prove each property of the dot product in question 17 using Cartesian vectors.

Achievement Check

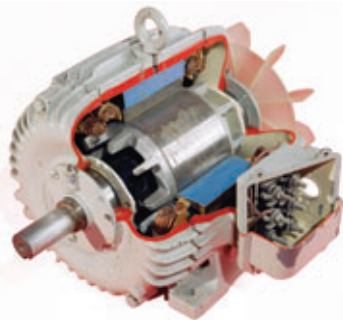
19. Consider rhombus ABCD.



- Find the resultants of $\vec{AB} + \vec{AD}$ and $\vec{AB} - \vec{AD}$.
- What will the value of $(\vec{AB} + \vec{AD}) \cdot (\vec{AB} - \vec{AD})$ always be? Explain.
- Is the value of $(\vec{AB} - \vec{AD}) \cdot (\vec{AB} + \vec{AD})$ the same as your answer to part b)? Explain.

C Extend and Challenge

20. Dot products can be used to calculate the average power requirements of appliances that run on alternating current, including many home appliances. The power, P , in watts, required to run an appliance is given by the formula $P = VI \cos \theta$, where V is the voltage, in volts; I is the current, in amperes; and θ is the phase angle between the voltage and the current.
- Write this formula as a dot product.
 - An electric motor is connected to a 120-V AC power supply and draws a current of 5 A with a phase angle of 15° . What is the average power rating of the motor?



21. Given the points $O(0, 0)$ and $P(5, 5)$, describe the set of points Q such that
- $\vec{OP} \cdot \vec{OQ} = 0$
 - $\vec{OP} \cdot \vec{OQ} = -\frac{\sqrt{2}}{2}$
 - $\vec{OP} \cdot \vec{OQ} = -\frac{\sqrt{3}}{2}$
22. **Math Contest** Because of rounding, some after-tax totals are not possible. For example, with a 13% tax, it is impossible to get a total of \$1.26 since $\$1.11 \times 1.13 \div \1.25 and $\$1.12 \times 1.13 \div \1.27 . How many other “impossible” totals less than \$5.00 are there after 13% tax? (Note: \$5.00 itself is an impossible total.)
23. **Math Contest** A transport truck 3 m wide and 4 m tall is attempting to pass under a parabolic bridge that is 6 m wide at the base and 5 m high at the centre. Can the truck make it under the bridge? If so, how much clearance will the truck have? If not, how much more clearance is needed?

7.3

Applications of the Dot Product

The dot product has many applications in mathematics and science. Finding the work done, determining the angle between two vectors, and finding the projection of one vector onto another are just three of these.



Example 1 Find the Work Done

Angela has entered the wheelchair division of a marathon race. While training, she races her wheelchair up a 300-m hill with a constant force of 500 N applied at an angle of 30° to the surface of the hill. Find the work done by Angela, to the nearest 100 J.

Solution

Let \vec{f} represent the force vector and \vec{s} represent the displacement vector. The work done, W , is given by

$$\begin{aligned} W &= \vec{f} \cdot \vec{s} \\ &= |\vec{f}| |\vec{s}| \cos \theta \\ &= (500)(300) \cos 30^\circ \\ &\doteq 129\,904 \end{aligned}$$

The work done by Angela is approximately 129 900 N·m or 129 900 J.

Example 2 Find the Angle Between Two Vectors

Determine the angle between the vectors in each pair.

- a) $\vec{g} = [5, 1]$ and $\vec{h} = [-3, 8]$ b) $\vec{s} = [-3, 6]$ and $\vec{t} = [4, 2]$

Solution

- a) From the definition of the dot product,

$$\begin{aligned} \vec{g} \cdot \vec{h} &= |\vec{g}| |\vec{h}| \cos \theta \\ \cos \theta &= \frac{\vec{g} \cdot \vec{h}}{|\vec{g}| |\vec{h}|} \\ \cos \theta &= \frac{5(-3) + 1(8)}{\sqrt{5^2 + 1^2} \sqrt{(-3)^2 + 8^2}} \quad \text{Use Cartesian vectors.} \\ \theta &= \cos^{-1} \left(\frac{-7}{\sqrt{26} \sqrt{73}} \right) \\ \theta &\doteq 99.246^\circ \end{aligned}$$

The angle between $\vec{g} = [5, 1]$ and $\vec{h} = [-3, 8]$ is approximately 99.2° .

$$\text{b) } \cos \theta = \frac{\vec{s} \cdot \vec{t}}{|\vec{s}| |\vec{t}|}$$

$$\cos \theta = \frac{-3(4) + 6(2)}{\sqrt{(-3)^2 + 6^2} \sqrt{4^2 + 2^2}}$$

$$\cos \theta = \frac{0}{\sqrt{45} \sqrt{20}}$$

$$\cos \theta = 0$$

$$\theta = 90^\circ$$

The angle between $\vec{s} = [-3, 6]$ and $\vec{t} = [4, 2]$ is 90° . These vectors are orthogonal.

To find the angle, θ , between two Cartesian vectors \vec{u} and \vec{v} , use the formula

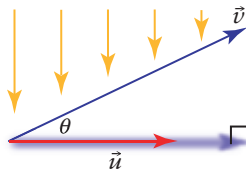
$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}.$$

Vector Projections

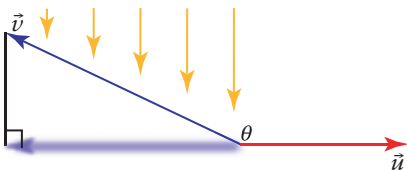
You can think of a vector projection like a shadow. Consider the following diagram, where the vertical arrows represent light from above.

Think of the **projection of \vec{v} on \vec{u}** as the shadow that \vec{v} casts on \vec{u} .

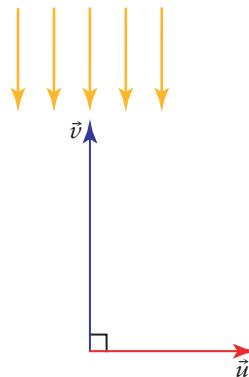
If the angle between \vec{v} and \vec{u} is less than 90° , then the projection of \vec{v} on \vec{u} , or $\text{proj}_{\vec{u}} \vec{v}$, is the vector component of \vec{v} in the direction of \vec{u} .



If the angle between \vec{v} and \vec{u} is between 90° and 180° , the direction of $\text{proj}_{\vec{u}} \vec{v}$ is opposite to the direction of \vec{u} .



If \vec{v} is perpendicular to \vec{u} , then \vec{v} casts no “shadow.” So, if $\theta = 90^\circ$, $\text{proj}_{\vec{u}} \vec{v} = \vec{0}$.

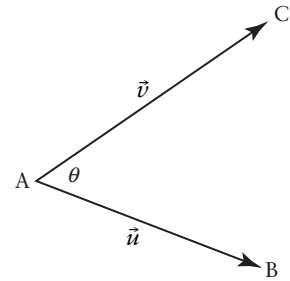
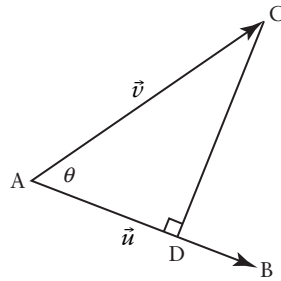


Example 3 Find the Projection of One Vector on Another

- a) Find the projection of \vec{v} on \vec{u} if $0 < \theta < 90^\circ$.
 b) Find $\text{proj}_{\vec{u}} \vec{v}$ if $90^\circ < \theta < 180^\circ$.

Solution

- a) Construct segment CD perpendicular to \overrightarrow{AB} , with D on \overrightarrow{AB} .



Then, $\overrightarrow{AD} = \text{proj}_{\vec{u}} \vec{v}$. From the triangle,

$$\frac{|\overrightarrow{AD}|}{|\vec{v}|} = \cos \theta$$

$$|\overrightarrow{AD}| = |\vec{v}| \cos \theta$$

$$|\text{proj}_{\vec{u}} \vec{v}| = |\vec{v}| \cos \theta$$

This gives the magnitude of the projection, which is also called the scalar component of \vec{v} on \vec{u} . The direction of $\text{proj}_{\vec{u}} \vec{v}$ is the same as the direction of \vec{u} . If we multiply the magnitude of $\text{proj}_{\vec{u}} \vec{v}$ by a unit vector in the direction of \vec{u} , we will get the complete form of $\text{proj}_{\vec{u}} \vec{v}$. Let $k\vec{u}$ be the unit vector in the direction of \vec{u} . Then,

$$|k\vec{u}| = 1$$

$$|k||\vec{u}| = 1$$

$$|k| = \frac{1}{|\vec{u}|}$$

Since we want the unit vector to be in the same direction as \vec{u} , k must be positive. Thus, $k = \frac{1}{|\vec{u}|}$.

So, a unit vector in the direction of \vec{u} is $\frac{1}{|\vec{u}|} \vec{u}$. Thus,

$$\text{proj}_{\vec{u}} \vec{v} = \underbrace{|\vec{v}| \cos \theta}_{\text{magnitude}} \underbrace{\left(\frac{1}{|\vec{u}|} \vec{u} \right)}_{\text{direction}}$$

CONNECTIONS

In question 13, you will show that an equivalent formula for the magnitude of the projection of \vec{v} on \vec{u} is $|\text{proj}_{\vec{u}} \vec{v}| = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|}$.

b) If $90^\circ < \theta < 180^\circ$, then, from the diagram,

$$\begin{aligned} |\text{proj}_{\vec{u}} \vec{v}| &= |\vec{v}| \cos(180^\circ - \theta) \\ &= |\vec{v}| (-\cos \theta) \\ &= -|\vec{v}| \cos \theta \end{aligned}$$

The direction of $\text{proj}_{\vec{u}} \vec{v}$ is opposite to the direction of \vec{u} .

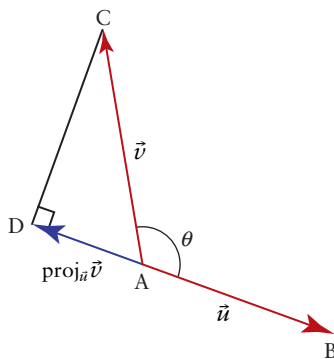
Thus, we want the negative value of k from part a).

$$k = -\frac{1}{|\vec{u}|}$$

So, a unit vector in the opposite direction to \vec{u} is $-\frac{1}{|\vec{u}|} \vec{u}$. Thus,

$$\begin{aligned} \text{proj}_{\vec{u}} \vec{v} &= \underbrace{-|\vec{v}| \cos \theta}_{\text{magnitude}} \underbrace{\left(-\frac{1}{|\vec{u}|} \vec{u}\right)}_{\text{direction}} \\ &= |\vec{v}| \cos \theta \left(\frac{1}{|\vec{u}|} \vec{u}\right) \end{aligned}$$

This is the same formula as in part a).



CONNECTIONS

In question 14, you will show that an equivalent formula for the projection of \vec{v} on \vec{u} is

$$\text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}.$$

The projection of \vec{v} on \vec{u} , where θ is the angle between \vec{v} and \vec{u} , is

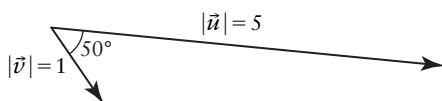
$$\text{proj}_{\vec{u}} \vec{v} = |\vec{v}| \cos \theta \left(\frac{1}{|\vec{u}|} \vec{u} \right) \text{ or } \text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}.$$

$$\text{If } 0^\circ < \theta < 90^\circ, |\text{proj}_{\vec{u}} \vec{v}| = |\vec{v}| \cos \theta \text{ or } |\text{proj}_{\vec{u}} \vec{v}| = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|}.$$

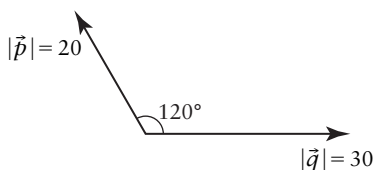
$$\text{If } 90^\circ < \theta < 180^\circ, \text{ then } |\text{proj}_{\vec{u}} \vec{v}| = -|\vec{v}| \cos \theta \text{ or } |\text{proj}_{\vec{u}} \vec{v}| = -\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|}.$$

Example 4 Find the Projection of One Vector on Another

a) Determine the projection of \vec{u} on \vec{v} .



b) Determine the projection of \vec{p} on \vec{q} .



- c) Determine the projection of $\vec{d} = [2, -3]$ on $\vec{c} = [1, 4]$.
- d) Illustrate the projections in parts a) to d) geometrically.

Solution

- a) Since $\theta < 90^\circ$, the magnitude of the projection is given by

$$\begin{aligned} |\text{proj}_{\vec{v}} \vec{u}| &= |\vec{u}| \cos \theta \\ &= 5 \cos 50^\circ \\ &\doteq 3.21 \end{aligned}$$

The direction of $\text{proj}_{\vec{v}} \vec{u}$ is the same as the direction of \vec{v} .

- b) Since $\theta > 90^\circ$, the magnitude of the projection is given by

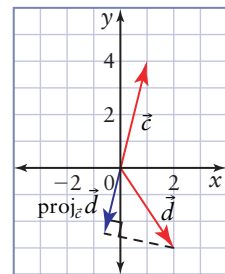
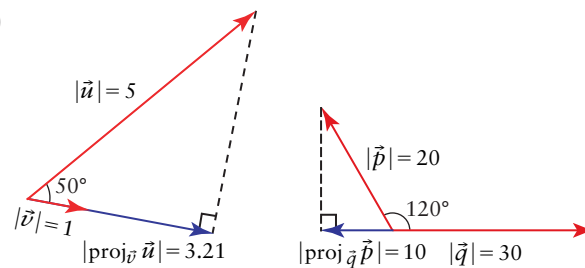
$$\begin{aligned} |\text{proj}_{\vec{q}} \vec{p}| &= -|\vec{p}| \cos \theta \\ &= -20 \cos 120^\circ \\ &= 10 \end{aligned}$$

The direction of $\text{proj}_{\vec{q}} \vec{p}$ is opposite to the direction of \vec{q} .

$$\begin{aligned} \text{c) } \text{proj}_{\vec{c}} \vec{d} &= \left(\frac{\vec{d} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \right) \vec{c} \\ &= \left(\frac{[2, -3] \cdot [1, 4]}{[1, 4] \cdot [1, 4]} \right) [1, 4] \\ &= \left(\frac{2(1) + (-3)(4)}{1^2 + 4^2} \right) [1, 4] \\ &= \frac{-10}{17} [1, 4] \\ &= \left[-\frac{10}{17}, -\frac{40}{17} \right] \end{aligned}$$

Note that the direction of $\text{proj}_{\vec{c}} \vec{d}$ is opposite to the direction of \vec{c} .

d)



Example 5 Dot Product in Sales

A shoe store sold 350 pairs of Excalibur shoes and 275 pairs of Camelot shoes in a year. Excalibur shoes sell for \$175 and Camelot shoes sell for \$250.

- Write a Cartesian vector, \vec{s} , to represent the numbers of pairs of shoes sold.
- Write a Cartesian vector, \vec{p} , to represent the prices of the shoes.
- Find the dot product $\vec{s} \cdot \vec{p}$. What does this dot product represent?

Solution

- $\vec{s} = [350, 275]$
- $\vec{p} = [175, 250]$
- $$\begin{aligned}\vec{s} \cdot \vec{p} &= [350, 275] \cdot [175, 250] \\ &= [350(175) + 275(250)] \\ &= 130\,000\end{aligned}$$

The dot product represents the revenue, \$130 000, from sales of the shoes.

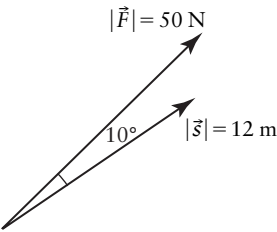
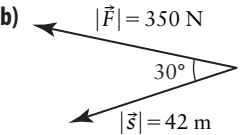
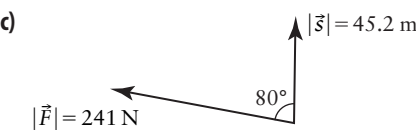
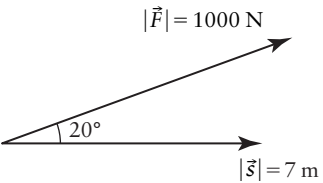
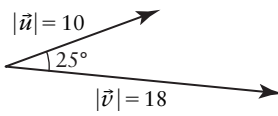
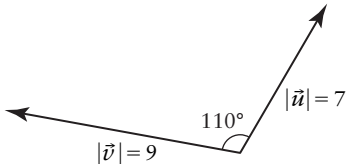
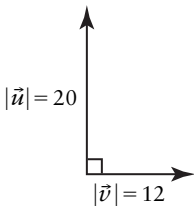
KEY CONCEPTS

- To find the angle, θ , between two Cartesian vectors \vec{u} and \vec{v} , use the formula $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$.
- For two vectors \vec{u} and \vec{v} with an angle of θ between them, the projection of \vec{v} on \vec{u} is the vector component of \vec{v} in the direction of \vec{u} :
 - $\text{proj}_{\vec{u}} \vec{v} = |\vec{v}| \cos \theta \left(\frac{1}{|\vec{u}|} \vec{u} \right)$ or $\text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$
 - $|\text{proj}_{\vec{u}} \vec{v}| = |\vec{v}| \cos \theta$ if $0^\circ < \theta < 90^\circ$
 - $|\text{proj}_{\vec{u}} \vec{v}| = -|\vec{v}| \cos \theta$ if $90^\circ < \theta < 180^\circ$
 - $|\text{proj}_{\vec{u}} \vec{v}| = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|}$
- If $0 < \theta < 90^\circ$, then $\text{proj}_{\vec{u}} \vec{v}$ is in the same direction as \vec{u} .
If $90^\circ < \theta < 180^\circ$, then $\text{proj}_{\vec{u}} \vec{v}$ is in the opposite direction as \vec{u} .
If $\theta = 90^\circ$, then $\text{proj}_{\vec{u}} \vec{v}$ is the zero vector, $\vec{0}$.

Communicate Your Understanding

- C1** There are two ways to calculate the magnitude of a projection. Describe when you would use each method.
- C2** What happens to the projection of \vec{v} on \vec{u} if \vec{v} is much longer than \vec{u} ?
- C3** What happens if the angle between \vec{u} and \vec{v} is close to, but less than, 180° ? What if it is equal to 180° ?
- C4** Is it possible for the angle between two vectors to be more than 180° ? How does the formula support this conclusion?
- C5** At the gym, Paul lifts an 80-kg barbell a distance of 1 m above the floor and then lowers the barbell to the floor. How much mechanical work has he done?

A Practise

- Determine the work done by each force, \vec{F} , in newtons, for an object moving along the vector, \vec{s} , in metres.
 - $\vec{F} = [5, 2]$, $\vec{s} = [7, 4]$
 - $\vec{F} = [100, 400]$, $\vec{s} = [12, 27]$
 - $\vec{F} = [67.8, 3.9]$, $\vec{s} = [4.7, 3.2]$
- Determine the work done by the force, \vec{F} , in the direction of the displacement, \vec{s} .
 - 
 - 
 - 
 - 
- Calculate the angle between the vectors in each pair. Illustrate geometrically.
 - $\vec{p} = [7, 8]$, $\vec{q} = [4, 3]$
 - $\vec{r} = [-2, -8]$, $\vec{s} = [6, -1]$
 - $\vec{t} = [-7, 2]$, $\vec{u} = [6, 11]$
 - $\vec{e} = [2, 3]$, $\vec{f} = [9, -6]$
- Determine the projection of \vec{u} on \vec{v} .
 - 
 - 
 - 
- In each case, determine the projection of the first vector on the second. Sketch each projection.
 - $\vec{a} = [6, -1]$, $\vec{b} = [11, 5]$
 - $\vec{c} = [2, 7]$, $\vec{d} = [-4, 3]$
 - $\vec{e} = [-2, -5]$, $\vec{f} = [-5, 1]$
 - $\vec{g} = [10, -3]$, $\vec{h} = [4, -4]$

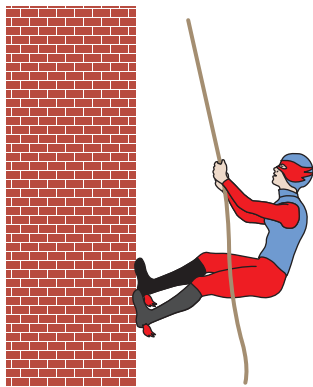
B Connect and Apply

6. A factory worker pushes a package along a broken conveyor belt from $(-4, 0)$ to $(4, 0)$ with a 50-N force at a 30° angle to the conveyor belt. How much mechanical work is done if the units of the conveyor belt are in metres?
7. A force, \vec{f} , of 25 N is acting in the direction of $\vec{a} = [6, 1]$.
- Find a unit vector in the direction of \vec{a} .
 - Find the Cartesian vector representing the force, \vec{f} , using your answer from part a).
 - The force \vec{f} is exerted on an object moving from point $(4, 0)$ to point $(15, 0)$, with distance in metres. Determine the mechanical work done.
8. Justin applies a force at 20° to the horizontal to move a football tackling dummy 8 m horizontally. He does 150 J of mechanical work. What is the magnitude of the force?
9. Determine the angles of $\triangle ABC$ for $A(5, 1)$, $B(4, -7)$, and $C(-1, -8)$.
10. Consider the parallelogram with vertices at $(0, 0)$, $(3, 0)$, $(5, 3)$, and $(2, 3)$. Find the angles at which the diagonals of the parallelogram intersect.
11. The points $P(-2, 1)$, $Q(-6, 4)$, and $R(4, 3)$ are three vertices of parallelogram PQRS.
- Find the coordinates of S.
 - Find the measures of the interior angles of the parallelogram, to the nearest degree.
 - Find the measures of the angles between the diagonals of the parallelogram, to the nearest degree.
12. Determine the angle between vector \overrightarrow{PQ} and the positive x -axis, given endpoints $P(4, 7)$ and $Q(8, 3)$.
13. Show that $|\text{proj}_{\vec{u}} \vec{v}| = |\vec{v}| \cos \theta$ can be written as $|\text{proj}_{\vec{u}} \vec{v}| = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|}$.
14. Show that $\text{proj}_{\vec{u}} \vec{v} = |\vec{v}| \cos \theta \left(\frac{1}{|\vec{u}|} \vec{u} \right)$ can be written as $\text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$.
15. A store sells digital music players and DVD players. Suppose 42 digital music players are sold at \$115 each and 23 DVD players are sold at \$95 each. The vector $\vec{a} = [42, 23]$ can be called the sales vector and $\vec{b} = [115, 95]$ the price vector. Find $\vec{a} \cdot \vec{b}$ and interpret its meaning.
16. A car enters a curve on a highway. If the highway is banked 10° to the horizontal in the curve, show that the vector $[\cos 10^\circ, \sin 10^\circ]$ is parallel to the road surface. The x -axis is horizontal but perpendicular to the road lanes, and the y -axis is vertical. If the car has a mass of 1000 kg, find the component of the force of gravity along the road vector. The projection of the force of gravity on the road vector provides a force that helps the car turn. (Hint: The force of gravity is equal to the mass times the acceleration due to gravity.)
17. **Chapter Problem** The town of Oceanside lies at sea level and the town of Seaview is at an altitude of 84 m, at the end of a straight, smooth road that is 2.5 km long. Following an automobile accident, a tow truck is pulling a car up the road using a force, in newtons, defined by the vector $\vec{F} = [30\,000, 18\,000]$.
- Find the force drawing the car up the hill and the force, perpendicular to the hill, tending to lift it.
 - What is the work done by the tow truck in pulling the car up the hill?
 - What is the work done in raising the altitude of the car?
 - Explain the differences in your answers to parts b) and c).
18. How much work is done against gravity by a worker who carries a 25-kg carton up a 6-m-long set of stairs, inclined at 30° ?

CONNECTIONS

A mass of 1 kg has a weight of about 9.8 N on Earth's surface.

19. A superhero pulls herself 15 m up the side of a wall with a force of 234 N, at an angle of 12° to the vertical. What is the work done?



20. A crate is dragged 3 m along a smooth level floor by a 30-N force, applied at 25° to the floor. Then, it is pulled 4 m up a ramp inclined at 20° to the horizontal, using the same force. Then, the crate is dragged a further 5 m along a level platform using the same force again. Determine the total work done in moving the crate.

21. A square is defined by the unit vectors \vec{i} and \vec{j} . Find the projections of \vec{i} and \vec{j} on each of the diagonals of the square.

22. The ramp to the loading dock at a car parts plant is inclined at 20° to the horizontal. A pallet of parts is moved 5 m up the ramp by a force of 5000 N, at an angle of 15° to the surface of the ramp.



- a) What information do you need to calculate the work done in moving the pallet along the ramp?

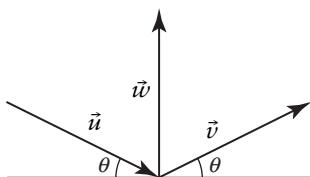
- b) Calculate the work done.

23. Draw a diagram to illustrate the meaning of each projection.

- a) $\text{proj}_{\text{proj}_{\vec{v}} \vec{u}} \vec{u}$
b) $\text{proj}_{\vec{u}} (\text{proj}_{\text{proj}_{\vec{v}} \vec{u}} \vec{u})$
c) $\text{proj}_{\vec{v}} (\text{proj}_{\vec{v}} \vec{u})$

C Extend and Challenge

24. In light reflection, the angle of incidence is equal to the angle of reflection. Let \vec{u} be the unit vector in the direction of incidence. Let \vec{v} be the unit vector in the direction of reflection. Let \vec{w} be the unit vector perpendicular to the face of the reflecting surface. Show that $\vec{v} = \vec{u} - 2(\vec{u} \cdot \vec{w})\vec{w}$.



25. a) Given vectors $\vec{a} = [6, 5]$ and $\vec{b} = [1, 3]$, find $\text{proj}_{\vec{b}} \vec{a}$.
b) Resolve \vec{a} into perpendicular components, one of which is in the direction of \vec{b} .

26. Consider your answer to question 25. Resolve $\vec{F} = [25, 18]$ into perpendicular components, one of which is in the direction of $\vec{u} = [2, 5]$.

27. Describe when each of the following is true, and illustrate with an example.

- a) $\text{proj}_{\vec{v}} \vec{u} = \text{proj}_{\vec{u}} \vec{v}$
b) $|\text{proj}_{\vec{v}} \vec{u}| = |\text{proj}_{\vec{u}} \vec{v}|$

28. **Math Contest** The side lengths of a right triangle are in the ratio 3:4:5. If the length of one of the three altitudes is 60 cm, what is the greatest possible area of this triangle?

29. **Math Contest** If $f(x) = \frac{1471}{n} \log_u x + \frac{538x}{u^n}$, where u and n are constants, determine $f(u^n)$.

7.4

Vectors in Three-Space

Force, velocity, and other vector quantities often involve a third dimension. How can you plot three-dimensional points and vectors in two dimensions? In this section, you will develop and use a Cartesian system to represent 3-D points and vectors. You will also extend the operations used on 2-D vectors to 3-D vectors.

CONNECTIONS

"2-D" means "two-dimensional" or "two dimensions." "Two-space" is short for "two-dimensional space." "3-D" means "three-dimensional" or "three dimensions." "Three-space" is short for "three-dimensional space."



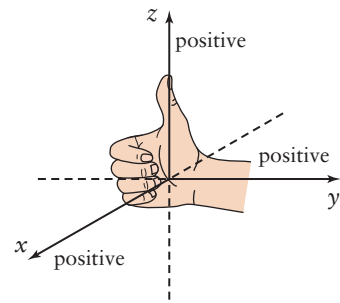
Investigate

How can you develop a 3-D coordinate system?

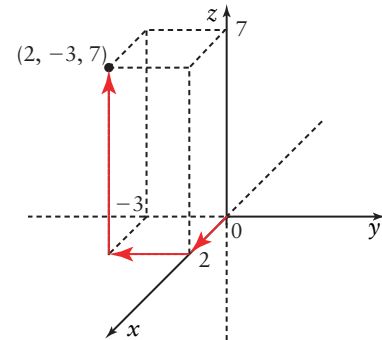
Consider a rectangular room as a three-dimensional coordinate system, where the points are defined by ordered triples (x, y, z) . If your classroom is rectangular, define the front left floor corner as the origin. Then, the x -axis is horizontal to your left, the y -axis is horizontal at the front of the room, and the z -axis is vertical. Use a 1-m scale for your coordinate system.

1. Describe the coordinates of all points on each axis.
2. Describe the location of the plane defined by the x -axis and the y -axis. This is called the **xy -plane**. Describe the set of ordered triples that represent points on this plane.
3. Describe the location of the xz -plane. Describe the set of ordered triples that represent points on this plane.
4. Describe the location of the yz -plane. Describe the set of ordered triples that represent points on this plane.
5. Describe all points above the floor. Describe all points below the floor. Describe all points behind the front wall.
6. Where is the point $(2, -1, 4)$? What about the point $(-4, -5, -2)$?
7.
 - a) Find the approximate coordinates of one of the feet of your desk.
 - b) Find the approximate coordinates of the top of your head when you are sitting at your desk.
8. **Reflect**
 - a) How can 2-D points such as $(2, 1)$ and $(-3, -4)$ be represented in a three-dimensional coordinate system?
 - b) Are there any points that do not have an ordered triple to represent them?
 - c) Are there any ordered triples that do not represent points?

It can be a challenge to draw a 3-D graph on a 2-D piece of paper. By convention, we set the x -, y -, and z -axes as shown. The negative axes are shown with broken lines. This is called a **right-handed system**. If you curl the fingers of your right hand from the positive x -axis to the positive y -axis, your thumb will point in the direction of the positive z -axis.



To plot the point $(2, -3, 7)$, start at the origin. Move 2 units along the positive x -axis, then 3 units parallel to the negative y -axis, and then 7 units parallel to the positive z -axis.



It is often a good idea to plot the coordinates of a 3-D point by drawing line segments illustrating their coordinate positions.

A 2-D Cartesian graph has four quadrants. A 3-D graph has eight octants. The first octant has points with three positive coordinates. There is no agreement on how the other seven octants should be named.

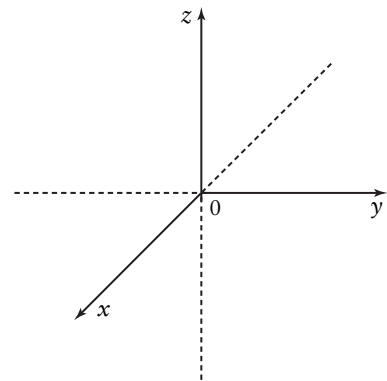
Example 1 Describe the Octants in a 3-D Graph

Describe the octant in which each point is located.

- a) $(1, 2, 3)$
- b) $(-3, 2, 1)$
- c) $(-3, -2, -1)$
- d) $(1, 2, -3)$

Solution

- a) Since all the coordinates are positive, $(1, 2, 3)$ is in the octant at the front right top of the 3-D grid.
- b) Since only the x -coordinate is negative, $(-3, 2, 1)$ is in the octant at the back right top of the 3-D grid.
- c) Since all the coordinates are negative, $(-3, -2, -1)$ is in the octant at the back left bottom of the 3-D grid.
- d) Since only the z -coordinate is negative, $(1, 2, -3)$ is in the octant at the front right bottom of the 3-D grid.

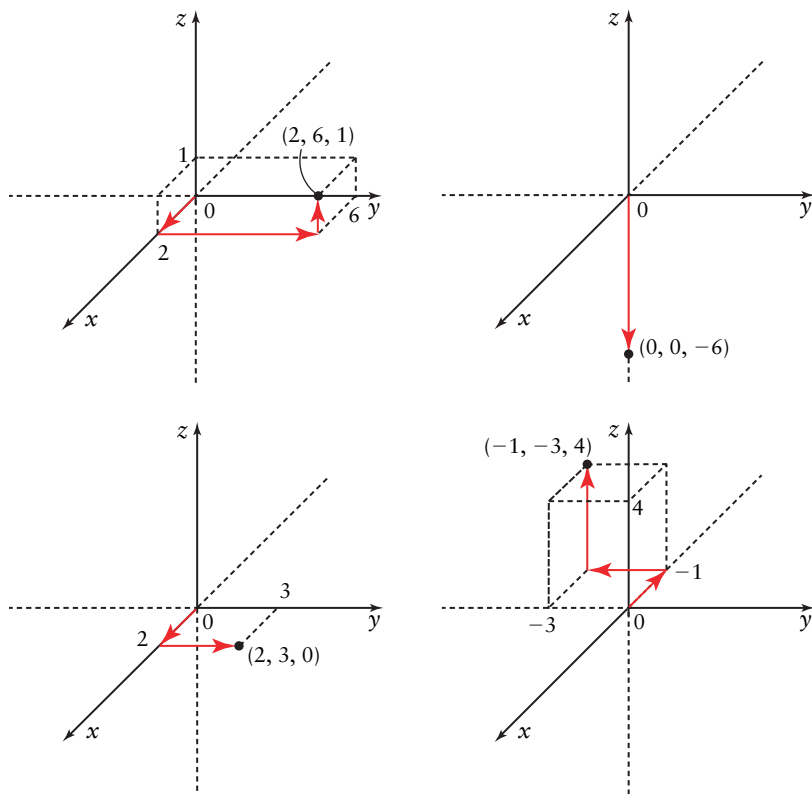


Example 2 Plot Points in 3-D

- a) Plot the points $A(2, 6, 1)$, $B(0, 0, -6)$, $C(2, 3, 0)$, and $D(-1, -3, 4)$.
b) Describe the location of each point.

Solution

a)



- b) Point A is in the octant that is to the front right top of the origin, where all three coordinates are positive.

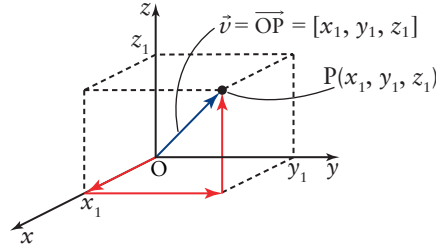
Point B is on the negative z -axis.

Point C is on the xy -plane.

Point D is in the octant that is to the back left top of the origin, where the x - and y -coordinates are negative and the z -coordinate is positive.

3-D Cartesian Vectors

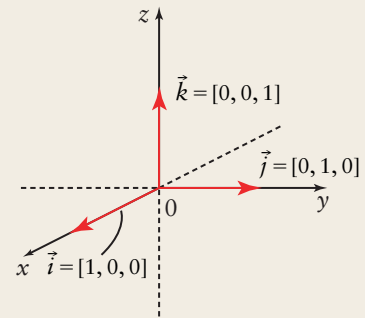
Let \vec{v} represent a vector in space. If \vec{v} is translated so that its tail is at the origin, O, then its tip will be at some point $P(x_1, y_1, z_1)$. Then, \vec{v} is the **position vector** of the point P, and $\vec{v} = \overrightarrow{OP} = [x_1, y_1, z_1]$.



$[x_1, y_1, z_1]$ also represents any vector that has the same magnitude and direction as \overrightarrow{OP} .

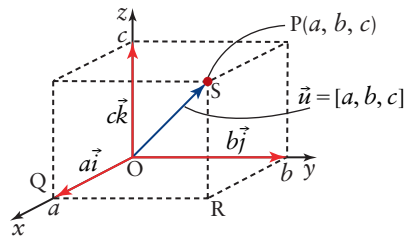
Unit Vectors

As in two dimensions, we define the unit vectors along the axes. The unit vector along the x -axis is $\vec{i} = [1, 0, 0]$, the unit vector along the y -axis is $\vec{j} = [0, 1, 0]$, and the unit vector along the z -axis is $\vec{k} = [0, 0, 1]$. From the diagram, and the definition of unit vectors, you can see that \vec{i} , \vec{j} , and \vec{k} all have magnitude, or length, one.



As with 2-D vectors, 3-D vectors can be written as the sum of multiples of \vec{i} , \vec{j} , and \vec{k} .

Consider a vector $\vec{u} = [a, b, c]$. Since the position vector \overrightarrow{OP} of \vec{u} , where the coordinates of P are (a, b, c) , has the same magnitude and direction as \vec{u} , we can use \overrightarrow{OP} to determine the characteristics of \vec{u} .



Vectors in 3-D are added in the same way as vectors in 2-D, by placing the tail of the second vector at the tip of the first vector. From the diagram,

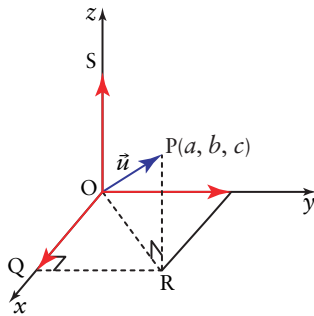
$$\begin{aligned}
 \vec{u} &= \overrightarrow{OP} \\
 &= \overrightarrow{OQ} + \overrightarrow{QR} + \overrightarrow{RS} \\
 &= a\vec{i} + b\vec{j} + c\vec{k} \\
 &= [a, 0, 0] + [0, b, 0] + [0, 0, c]
 \end{aligned}$$

From the previous page, $\vec{u} = [a, b, c]$. Thus,

$$[a, b, c] = [a, 0, 0] + [0, b, 0] + [0, 0, c] = a\vec{i} + b\vec{j} + c\vec{k}.$$

Magnitude of a Cartesian Vector

Consider the vector $\vec{u} = \overrightarrow{OP} = [a, b, c]$.



To find $|\vec{u}|$, use the Pythagorean theorem in $\triangle OQR$ to find $|\overrightarrow{OR}|^2$ first. Since $\angle OQR$ is a right angle,

$$\begin{aligned}
 |\overrightarrow{OR}|^2 &= |\overrightarrow{OQ}|^2 + |\overrightarrow{QR}|^2 \\
 &= a^2 + b^2
 \end{aligned}$$

Then, use the Pythagorean theorem in $\triangle ORP$ to find $|\overrightarrow{OP}|^2$. Since $\angle ORP$ is a right angle,

$$\begin{aligned}
 |\vec{u}|^2 &= |\overrightarrow{OP}|^2 \\
 &= |\overrightarrow{OR}|^2 + |\overrightarrow{RP}|^2 \\
 &= (a^2 + b^2) + c^2 \\
 &= a^2 + b^2 + c^2 \\
 |\vec{u}| &= \sqrt{a^2 + b^2 + c^2}
 \end{aligned}$$

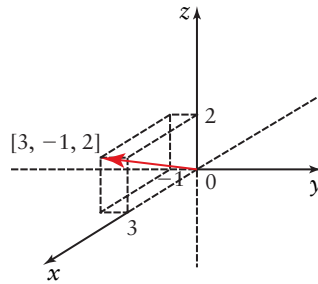
Example 3 Working With 3-D Vectors

For each vector below,

- i) sketch the position vector
 - ii) write the vector in the form $a\vec{i} + b\vec{j} + c\vec{k}$
 - iii) find the magnitude
- a) $\vec{u} = [3, -1, 2]$
- b) $\vec{v} = [-2, 0, 1]$

Solution

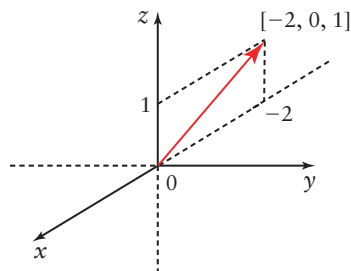
a) i)



$$\begin{aligned}\text{ii) } [3, -1, 2] &= 3\vec{i} + (-1)\vec{j} + 2\vec{k} \\ &= 3\vec{i} - \vec{j} + 2\vec{k}\end{aligned}$$

$$\begin{aligned}\text{iii) } |\vec{u}| &= |[3, -1, 2]| \\ &= \sqrt{3^2 + (-1)^2 + 2^2} \\ &= \sqrt{14}\end{aligned}$$

b) i)



$$\begin{aligned}\text{ii) } [-2, 0, 1] &= -2\vec{i} + 0\vec{j} + 1\vec{k} \\ &= -2\vec{i} + \vec{k}\end{aligned}$$

$$\begin{aligned}\text{iii) } |\vec{v}| &= |[-2, 0, 1]| \\ &= \sqrt{(-2)^2 + (0)^2 + 1^2} \\ &= \sqrt{5}\end{aligned}$$

3-D vectors can be combined to determine resultants in a similar way to 2-D vectors. For example, air traffic controllers need to consider ground speed, takeoff angle, and crosswinds while planning a takeoff.

Vectors in three-space have an additional component, z , but the properties of scalar multiplication, vector addition and subtraction, and dot product that were developed for two-space are valid in three-space.

Scalar Multiplication in 3-D

For any vector $\vec{u} = [u_1, u_2, u_3]$ and any scalar $k \in \mathbb{R}$, $k\vec{u} = [ku_1, ku_2, ku_3]$.

CONNECTIONS

You will prove this result in question 20.

As with 2-D vectors, $k\vec{u}$ is collinear with \vec{u} . This means that if you translate $k\vec{u}$ so that \vec{u} and $k\vec{u}$ are tail to tail, the two vectors lie on the same straight line.

Example 4 Find Collinear Vectors

- Find a such that $[1, 2, 3]$ and $[2, a, 6]$ are collinear.
- Find b and c such that $[-2, b, 7]$ and $[c, 6, 21]$ are collinear.

Solution

- For $[1, 2, 3]$ and $[2, a, 6]$ to be collinear, $[2, a, 6]$ must be a scalar multiple of $[1, 2, 3]$. Looking at the components of the two vectors, it appears that $[2, a, 6]$ is two times $[1, 2, 3]$. So, $a = 2(2)$ or 4.

- Let k represent the scalar. Then,

$$[c, 6, 21] = k[-2, b, 7]$$

$$[c, 6, 21] = [-2k, bk, 7k]$$

$$c = -2k \quad \text{①}$$

$$6 = bk \quad \text{②}$$

$$21 = 7k \quad \text{③}$$

From equation ③, $k = 3$.

Thus, $c = -6$ and $b = 2$.

Many of the tools we developed for 2-D vectors are easily modified for 3-D vectors.

CONNECTIONS

You will prove this result in question 21.

Vector Addition in 3-D

For two vectors $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$,

$$\vec{u} + \vec{v} = [u_1 + v_1, u_2 + v_2, u_3 + v_3]$$

Vector Subtraction in 3-D

If $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, then

$$\vec{u} - \vec{v} = [u_1 - v_1, u_2 - v_2, u_3 - v_3]$$

CONNECTIONS

You will prove this result in question 22.

Vector Between Two Points

The vector $\overrightarrow{P_1P_2}$ from point $P_1(x_1, y_1, z_1)$ to point $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1, z_2 - z_1].$$

Magnitude of a Vector Between Two Points

The magnitude of the vector $\overrightarrow{P_1P_2}$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 5 Interpret Cartesian Vectors

- Given the points $A(3, 6, -1)$ and $B(-1, 0, 5)$, express \overrightarrow{AB} as an ordered triple. Then, write \overrightarrow{AB} in terms of \vec{i} , \vec{j} , and \vec{k} .
- Determine the magnitude of \overrightarrow{AB} .
- Determine a unit vector, \vec{u} , in the direction of \overrightarrow{AB} .

Solution

$$\begin{aligned} \text{a) } \overrightarrow{AB} &= [-1 - 3, 0 - 6, 5 - (-1)] \\ &= [-4, -6, 6] \\ &= -4\vec{i} - 6\vec{j} + 6\vec{k} \end{aligned}$$

$$\begin{aligned} \text{b) } |\overrightarrow{AB}| &= \sqrt{(-4)^2 + (-6)^2 + 6^2} \\ &= \sqrt{88} \end{aligned}$$

- c) From Example 3 on page 380, a unit vector in the direction of \overrightarrow{AB} is $\frac{1}{|\overrightarrow{AB}|} \overrightarrow{AB}$. Let \vec{u} represent the unit vector in this case.

$$\begin{aligned}\vec{u} &= \frac{1}{|\overrightarrow{AB}|} \overrightarrow{AB} \\ &= \frac{1}{\sqrt{88}}[-4, -6, 6] \quad |\overrightarrow{AB}| = \sqrt{88} \text{ from part b).} \\ &= \frac{1}{2\sqrt{22}}[-4, -6, 6] \\ &= \left[-\frac{2}{\sqrt{22}}, -\frac{3}{\sqrt{22}}, \frac{3}{\sqrt{22}} \right]\end{aligned}$$

In Section 7.2, we defined the dot product of two vectors as $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$, where θ is the angle between \vec{u} and \vec{v} .

The Dot Product for 3-D Cartesian Vectors

For two 3-D Cartesian vectors, $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$,
 $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

CONNECTIONS

You will prove this relationship in question 28.

Example 6 Operations With Cartesian Vectors in 3-D

Given the vectors $\vec{u} = [2, 3, -5]$, $\vec{v} = [8, -4, 3]$, and $\vec{w} = [-6, -2, 0]$, simplify each vector expression.

- a) $-3\vec{v}$ b) $\vec{u} + \vec{v} + \vec{w}$ c) $|\vec{u} - \vec{v}|$ d) $\vec{u} \cdot \vec{v}$

Solution

- a) $-3\vec{v} = -3[8, -4, 3]$
 $= [-24, 12, -9]$
- b) $\vec{u} + \vec{v} + \vec{w} = [2, 3, -5] + [8, -4, 3] + [-6, -2, 0]$
 $= [2 + 8 + (-6), 3 + (-4) + (-2), -5 + 3 + 0]$
 $= [4, -3, -2]$
- c) $|\vec{u} - \vec{v}| = |[2, 3, -5] - [8, -4, 3]|$
 $= |[2 - 8, 3 - (-4), -5 - 3]|$
 $= |[-6, 7, -8]|$
 $= \sqrt{(-6)^2 + 7^2 + (-8)^2}$
 $= \sqrt{149}$
- d) $\vec{u} \cdot \vec{v} = [2, 3, -5] \cdot [8, -4, 3]$
 $= 2(8) + 3(-4) + (-5)(3)$
 $= -11$

CONNECTIONS

You could also check whether \vec{b} is a scalar multiple of \vec{a} to determine if the two vectors are collinear.

Example 7 Collinear Vectors and the Dot Product

Determine if the vectors $\vec{a} = [6, 2, 4]$ and $\vec{b} = [9, 3, 6]$ are collinear.

Solution

Find the angle between the vectors using the dot product definition.

$$\begin{aligned}\cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \\ &= \frac{[6, 2, 4] \cdot [9, 3, 6]}{|[6, 2, 4]| |[9, 3, 6]|} \\ &= \frac{6(9) + 2(3) + 4(6)}{\sqrt{6^2 + 2^2 + 4^2} \sqrt{9^2 + 3^2 + 6^2}} \\ &= \frac{84}{\sqrt{7056}} \\ &= 1 \\ \theta &= \cos^{-1}(1) \\ &= 0^\circ\end{aligned}$$

\vec{a} and \vec{b} are collinear.

Example 8 Find an Orthogonal Vector

- Prove that two non-zero vectors \vec{a} and \vec{b} are **orthogonal**, or perpendicular, if and only if $\vec{a} \cdot \vec{b} = 0$.
- Find a vector that is orthogonal to $[3, 4, 5]$.

Solution

- First, prove that if \vec{a} and \vec{b} are orthogonal, then $\vec{a} \cdot \vec{b} = 0$.

If \vec{a} and \vec{b} are orthogonal, then the angle between them is 90° . Thus,

$$\begin{aligned}\vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos 90^\circ \\ &= |\vec{a}| |\vec{b}| (0) \\ &= 0\end{aligned}$$

Next, prove that if $\vec{a} \cdot \vec{b} = 0$, then \vec{a} and \vec{b} are orthogonal.

Let $\vec{a} \cdot \vec{b} = 0$. Then,

$$|\vec{a}| |\vec{b}| \cos \theta = 0$$

Since $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$, we can divide both sides by $|\vec{a}| |\vec{b}|$.

$$\cos \theta = 0$$

$$\theta = 90^\circ$$

Thus, \vec{a} and \vec{b} are orthogonal.

Two non-zero vectors \vec{a} and \vec{b} are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

b) Let $[x, y, z]$ be a vector that is orthogonal to $[3, 4, 5]$.

$$[x, y, z] \cdot [3, 4, 5] = 0$$

$$3x + 4y + 5z = 0$$

An infinite number of vectors are orthogonal to $[3, 4, 5]$. Select any values that satisfy the equation.

Let $x = 2$ and $y = 1$.

$$3(2) + 4(1) + 5z = 0$$

$$10 + 5z = 0$$

$$5z = -10$$

$$z = -2$$

$[2, 1, -2]$ is orthogonal to $[3, 4, 5]$.

Check:

$$[2, 1, -2] \cdot [3, 4, 5] = 2(3) + 1(4) + (-2)(5) = 0$$

CONNECTIONS

The problem of finding lines and planes that are orthogonal to a given vector is significant in Chapter 8.

The properties of operations with Cartesian vectors in three-space are the same as those in two-space. You will use Cartesian vectors to prove some of these properties in questions 31 to 33. Consider non-zero vectors, \vec{a} , \vec{b} , and \vec{c} and scalar $k \in \mathbb{R}$.

Property	Addition and Scalar Multiplication	Dot Product
Commutative	$\vec{a} + \vec{b} = \vec{b} + \vec{a}$	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
Associative	$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$	$k(\vec{a} \cdot \vec{b}) = (k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b})$
Distributive	$k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$	$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
Addition Identity	$\vec{a} + \vec{0} = \vec{a}$	
Multiplication Identity	$1\vec{a} = \vec{a}$	$\vec{a} \cdot \vec{u} = \vec{u} \cdot \vec{a} = \vec{a} $ if \vec{u} is a unit vector in the same direction as \vec{a}

Example 9 Determine a Resultant Force

A crane lifts a steel beam upward with a force of 5000 N. At the same time, two workers push the beam with forces of 600 N toward the west and 650 N toward the north. Determine the magnitude of the resultant force acting on the beam.

Solution

The three forces are perpendicular to each other. Define the vertical vector as $[0, 0, 5000]$. The other two vectors are horizontal. Define the force to the west as $[-600, 0, 0]$ and the force to the north as $[0, 650, 0]$.

$$\vec{R} = [0, 0, 5000] + [-600, 0, 0] + [0, 650, 0]$$

$$= [-600, 650, 5000]$$

$$|\vec{R}| = \sqrt{(-600)^2 + 650^2 + 5000^2}$$

$$\doteq 5077.6 \text{ N}$$

The magnitude of the resultant force is about 5078 N.

KEY CONCEPTS

- A 3-D Cartesian system represents points as ordered triples (x, y, z) . To plot the point $P(x_1, y_1, z_1)$, move x_1 units along the x -axis, y_1 units parallel to the y -axis, and z_1 units parallel to the z -axis.
- Suppose a vector \vec{v} is translated so its tail is at the origin, O , and its tip is at the point $P(x_1, y_1, z_1)$. Then, $\vec{v} = \overrightarrow{OP} = [x_1, y_1, z_1]$ is the position vector of the point P . $[x_1, y_1, z_1]$ also represents any vector that has the same magnitude and direction as \overrightarrow{OP} .
- $\vec{i} = [1, 0, 0]$, $\vec{j} = [0, 1, 0]$, and $\vec{k} = [0, 0, 1]$ are unit vectors.
- Any 3-D Cartesian vector $\vec{v} = [a, b, c]$ can be written as the sum of scalar multiples of the unit vectors \vec{i} , \vec{j} , and \vec{k} . That is, $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k} = a[1, 0, 0] + b[0, 1, 0] + c[0, 0, 1]$.
- The magnitude of the vector $\vec{u} = [a, b, c]$ is $|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$.
- For vectors $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$ and scalar $k \in \mathbb{R}$,
 - $\vec{u} + \vec{v} = [u_1 + v_1, u_2 + v_2, u_3 + v_3]$
 - $k\vec{u} = [ku_1, ku_2, ku_3]$ and $k\vec{u}$ is collinear with \vec{u}
 - $\vec{u} - \vec{v} = [u_1 - v_1, u_2 - v_2, u_3 - v_3]$
- The coordinates of the vector $\overrightarrow{P_1P_2}$ between two points, $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, are $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$. The magnitude of $\overrightarrow{P_1P_2}$ is $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- The dot product of two vectors, \vec{u} and \vec{v} , with angle θ between them, is $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$. If $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, then $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.
- The properties of operations with Cartesian vectors in two-space hold in three-space.

Consider vectors \vec{a} , \vec{b} , and \vec{c} and scalar $k \in \mathbb{R}$.

Property	Addition and Scalar Multiplication	Dot Product
Commutative	$\vec{a} + \vec{b} = \vec{b} + \vec{a}$	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
Associative	$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$	$k(\vec{a} \cdot \vec{b}) = (k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b})$
Distributive	$k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$	$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
Addition Identity	$\vec{a} + \vec{0} = \vec{a}$	
Multiplication Identity	$1\vec{a} = \vec{a}$	$\vec{a} \cdot \vec{u} = \vec{u} \cdot \vec{a} = \vec{a} $ if \vec{u} is a unit vector in the same direction as \vec{a}

Communicate Your Understanding

- C1** Describe the method you would use to plot the point $(-3, 5, -7)$ on a piece of paper.
- C2** Describe the coordinates of all position vectors in three-space that are
 - a) parallel to the xy -plane
 - b) not parallel to the yz -plane
- C3** Which operations with 3-D vectors have vector results and which ones have scalar results? Explain.
- C4** Compare the vectors represented by $[a, b, c]$ and $[-a, -b, -c]$.
- C5** If the cosine of the angle between two vectors is -1 , what does this tell you? Explain.

A Practise

1. a) Plot the points $P(2, 3, -5)$, $Q(-4, 1, 3)$, and $R(6, -2, 1)$.
b) Describe the location of each point.
c) Which point lies closest to $S(1, 1, -1)$?
d) Which point lies closest to the xy -plane?
e) Which point lies closest to the z -axis?
2. a) Describe the form of the coordinates of all points that are equidistant from the x - and y -axes.
b) Describe the form of the coordinates of all points that are equidistant from the x -, y - and z -axes.
3. Draw each position vector.
 - a) $[-1, 5, -2]$
 - b) $[3, 3, 3]$
 - c) $[0, -2, -4]$
4. Find the exact magnitude of each position vector in question 3.
5. Express each vector as a sum of the vectors \vec{i} , \vec{j} , and \vec{k} .
 - a) $[3, -5, 2]$
 - b) $[-3, -6, 9]$
 - c) $[5, 0, -7]$
6. Express each vector in the form $[a, b, c]$.
 - a) $3\vec{i} + 8\vec{j}$
 - b) $-5\vec{i} - 2\vec{k}$
 - c) $7\vec{i} - 4\vec{j} + 9\vec{k}$
7. Are the vectors $\vec{u} = [6, -2, -5]$ and $\vec{v} = [-12, 4, 10]$ collinear? Explain.
8. Determine all unit vectors collinear with $[4, 1, -7]$.
9. Find a and b such that \vec{u} and \vec{v} are collinear.
 - a) $\vec{u} = [a, 3, 6]$, $\vec{v} = [-8, 12, b]$
 - b) $\vec{u} = a\vec{i} + 2\vec{j}$, $\vec{v} = -3\vec{i} - 6\vec{j} - b\vec{k}$
10. Draw the vector \overrightarrow{AB} joining each pair of points. Then, write the vector in the form $[x, y, z]$.
 - a) $A(2, 1, 3)$, $B(5, 7, 1)$
 - b) $A(-1, -7, 2)$, $B(-3, -2, 5)$
 - c) $A(0, 0, 6)$, $B(0, -5, 0)$
 - d) $A(3, -4, 1)$, $B(6, -1, 5)$
11. Determine the exact magnitude of each vector in question 10.
12. The initial point of vector $\overrightarrow{MN} = [2, 4, -7]$ is $M(-5, 0, 3)$. Determine the coordinates of the terminal point, N .
13. The terminal point of vector $\overrightarrow{DE} = [-4, 2, 6]$ is $E(3, 3, 1)$. Determine the coordinates of the initial point, D .
14. Write an ordered triple for each vector.
 - a) \overrightarrow{AB} with $A(0, -3, 2)$ and $B(0, 4, -4)$
 - b) \overrightarrow{CD} with $C(4, 5, 0)$ and $D(-3, -3, 5)$

15. Given the vectors $\vec{a} = [-4, 1, 7]$, $\vec{b} = [2, 0, -3]$, and $\vec{c} = [1, -1, 5]$, simplify each vector expression.

- a) $7\vec{a}$ b) $\vec{a} + \vec{b} + \vec{c}$
 c) $\vec{b} + \vec{c} + \vec{a}$ d) $\vec{c} - \vec{b}$
 e) $3\vec{a} - 2\vec{b} + 4\vec{c}$ f) $\vec{a} \cdot \vec{c}$
 g) $\vec{b} \cdot (\vec{a} + \vec{c})$ h) $\vec{b} \cdot \vec{c} - \vec{a} \cdot \vec{c}$
 i) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$

16. Determine the angle between the vectors $\vec{g} = [6, 1, 2]$ and $\vec{h} = [-5, 3, 6]$.
17. Determine two vectors that are orthogonal to each vector.

- a) $\vec{e} = [3, -1, 4]$ b) $\vec{f} = [-4, -9, 3]$

B Connect and Apply

18. Give a geometric interpretation to each set of vectors.

- a) $[0, 2, 0]$ $[0, 5, 0]$ $[0, -3, 0]$
 b) $[1, -3, 0]$ $[2, 1, 0]$ $[-3, -1, 0]$
 c) $[0, 2, 2]$ $[0, -4, -2]$ $[0, -1, -2]$

19. Find two unit vectors parallel to each vector.

- a) $\vec{a} = [5, -3, 2]$
 b) \overrightarrow{PQ} , given $P(-7, 8, 3)$ and $Q(5, -2, -2)$
 c) $\vec{u} = 5\vec{i} + 6\vec{j} - 3\vec{k}$
 d) $\vec{f} = -3\vec{i} - 2\vec{j} - 9\vec{k}$

20. Prove that $k\vec{u} = [ku_1, ku_2, ku_3]$ for any vector $\vec{u} = [u_1, u_2, u_3]$ and any scalar $k \in \mathbb{R}$.

21. Prove that $\vec{u} + \vec{v} = [u_1 + v_1, u_2 + v_2, u_3 + v_3]$ for any two vectors $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$. Use the method used to prove this property for 2-D vectors in Section 7.1.

22. Prove that the vector $\overrightarrow{P_1P_2}$ from point $P_1(x_1, y_1, z_1)$ to point $P_2(x_2, y_2, z_2)$ can be expressed as an ordered triple by subtracting the coordinates of P_1 from the coordinates of P_2 . That is, $\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$. Use the method used to prove this property for 2-D vectors in Section 7.1.

23. Identify the type of triangle with vertices $A(2, 3, -5)$, $B(-4, 8, 1)$, and $C(6, -4, 0)$.

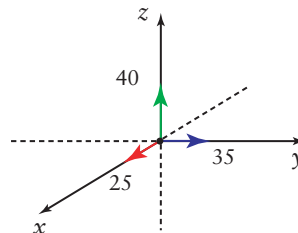
24. Prove that the magnitude of a vector can equal zero if and only if the vector is the zero vector, $\vec{0}$.

25. A triangle has vertices at the points $A(5, 0, 0)$, $B(0, 5, 0)$, and $C(0, 0, 5)$.

- a) What type of triangle is $\triangle ABC$? Explain.
 b) Identify the point in the interior of the triangle that is closest to the origin.

26. Three mutually perpendicular forces of 25 N, 35 N, and 40 N act on a body.

- a) Determine the magnitude of the resultant.
 b) Determine the angle the resultant makes with the 35-N force.



27. Let $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, $\vec{b} = 6\vec{i} + 3\vec{j} - \vec{k}$, and $\vec{c} = -5\vec{i} - \vec{j} + 7\vec{k}$. Simplify each vector expression.

- a) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$
- b) $\vec{a} \cdot (\vec{b} + \vec{c})$
- c) $(\vec{b} + \vec{c}) \cdot (\vec{b} + \vec{a})$
- d) $2\vec{c} \cdot (3\vec{a} - 2\vec{b})$

28. Prove that $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$ for any vectors $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$.

29. Determine the value(s) of k such that \vec{u} and \vec{v} are orthogonal.

- a) $\vec{u} = [4, 1, 3]$, $\vec{v} = [-1, 5, k]$
- b) $\vec{u} = [k, 3, 6]$, $\vec{v} = [5, k, 8]$
- c) $\vec{u} = [11, 3, 2k]$, $\vec{v} = [k, 4, k]$

30. Prove that \vec{i} , \vec{j} , and \vec{k} are mutually orthogonal using the dot product.

31. a) Use Cartesian vectors to prove that for any vectors \vec{u} and \vec{v} ,

- i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- ii) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

- b) What properties do the proofs in part a) depend on?

32. Use Cartesian vectors to prove that $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ for any vectors \vec{a} , \vec{b} , and \vec{c} .

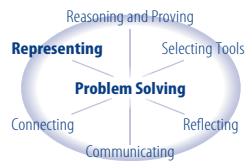
33. Use Cartesian vectors to prove that for any vectors \vec{a} , \vec{b} , and \vec{c} and scalar $k \in \mathbb{R}$,

- a) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
- b) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- c) $k(\vec{a} \cdot \vec{b}) = (k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b})$

34. Resolve $\vec{u} = [3, 4, 7]$ into rectangular components, one of which is collinear with $\vec{v} = [1, 2, 3]$.

35. An airplane takes off at a ground velocity of 200 km/h toward the west, climbing at an angle of 14° . A 20-km/h wind is blowing from the north. Determine the resultant air and ground velocities. Include a diagram in your solution.

36. The CN Tower is 553 m tall. A person is south of the tower, in a boat on Lake Ontario at a position that makes an angle of elevation of 10° to the top of the tower. A second person is sitting on a park bench, on a bearing of 060° relative to the tower, observing the top of the tower at an angle of elevation of 9° . Determine the displacement between the two people.



37. A projectile is launched from the origin at an angle of θ above the positive y -axis. Its position is affected by a crosswind and can be determined using the vector equation $[x, y, z] = [wt, vt \cos \theta, vt \sin \theta - gt^2]$, where v is the initial speed, in metres per second; t is the time, in seconds; w is the wind speed, in metres per second; and g is the acceleration due to gravity, in metres per square second.

- a) Determine the position at time t of a ball thrown from Earth for each scenario.

- i) initial speed of 30 m/s at an angle of 20° to the horizontal, with a crosswind blowing at 2 m/s
- ii) initial speed of 18 m/s at an angle of 35° to the horizontal, with a crosswind blowing at 6 m/s

- b) Calculate the position after 1 s for each scenario in part a).

- c) Calculate the position after 3 s for each scenario in part a).

- d) Determine the position of the ball at its maximum height for each scenario in part a).

38. The electron beam that forms the picture in an old-style cathode ray tube television is controlled by electric and magnetic forces in three dimensions.



Suppose that a forward force of 5.0 fN (femto-newtons) is exerted to accelerate an electron in a beam. The electronics engineer wants to divert the beam 30° left and then 20° upward.

CONNECTIONS

The prefix “femto” means 10^{-15} .

- Determine the magnitude of the component of the force to the left.
- Determine the magnitude of the upward component of the force.
- Determine the magnitude of the resultant force acting on the electron.

Achievement Check

39. Three vertices of a parallelogram are $D(2, 1, 3)$, $E(-4, 2, 0)$, and $F(6, -2, 4)$. Find all possible locations of the fourth vertex.

C Extend and Challenge

- Describe the coordinates of all points that are 10 units from both the x - and z -axes.
- We can visualize the vector $\vec{v} = [x, y, z]$. How might you describe the vector $\vec{v} = [w, x, y, z]$?

CONNECTIONS

Albert Einstein applied vectors in four dimensions in order to take time into account when developing his special theory of relativity.

- Find a vector that is orthogonal to both $\vec{a} = [2, 3, 1]$ and $\vec{b} = [4, 5, -2]$.
 - Is your answer to part a) unique? If so, prove it. If not, how many possible answers are there?
- For each set of vectors, determine the three angles that separate pairs of vectors.
 - $\vec{a} = [2, 1, 3]$, $\vec{b} = [-1, 2, 4]$, $\vec{c} = [3, 4, 10]$
 - $\vec{d} = [5, 1, 2]$, $\vec{e} = [-3, 1, 4]$, $\vec{f} = [6, -2, 3]$
- Prove, in three-space, that non-zero vectors \vec{u} and \vec{v} are orthogonal if and only if $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$.
- Suppose that \vec{a} and \vec{b} are position vectors. Prove that $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$ is the equation of a sphere, where \vec{r} is the position vector of any point on the sphere.
 - Where are \vec{a} and \vec{b} located relative to the sphere?

46. **Math Contest** Molly glances up at the digital clock on her DVD player and notices that the player’s remote control is blocking the bottoms of all the numbers (shown below). She also notices (remarkably!) that the tops of the numbers actually form the name of her cat, INOO. What is the probability of this event?



Hint: At 5:38 the clock looks like this:



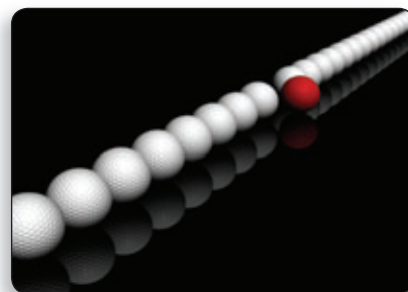
47. **Math Contest** The function $y = x^4 - 10x^3 + 24x^2 + 5x - 19$ has inflection points at $Q(1, 1)$ and $R(4, 1)$. The line through Q and R intersects the function again at points P and S . Show that $\frac{QR}{RS} = \frac{RQ}{QP} = \frac{1 + \sqrt{5}}{2}$ (the golden ratio).

7.5

The Cross Product and Its Properties

The dot product is a combination of multiplication and addition of vector components that provides a scalar result. In this section, we will develop a different vector product, called the cross product, that provides a vector result.

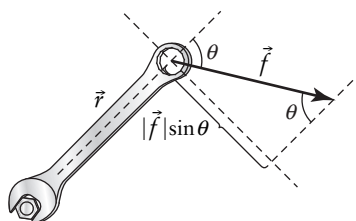
Geologists use cross products in vector calculus to analyse and predict seismic activity. In computer graphics and animation, programmers use the cross product to illustrate lighting relative to a given plane.



Consider using a wrench to tighten a typical bolt that is holding two pieces of wood together.

Suppose the force, \vec{f} , on the wrench turns it clockwise, as shown in the diagram.

The length of the handle of the wrench is represented by the vector \vec{r} . The effect of turning the wrench is called the moment, \vec{M} , of the force about the centre of the bolt.



The magnitude, $|\vec{M}|$, of the moment depends on the distance, $|\vec{r}|$, between the bolt and the point at which the force is applied, and on the magnitude of the force perpendicular to the wrench. From the diagram, the magnitude of the force is $|\vec{f}| \sin \theta$. Thus, $|\vec{M}| = |\vec{r}| \times (|\vec{f}| \sin \theta)$. In this case, the direction of the moment is into the wood, so the bolt is tightened. If the force was applied in the opposite direction, the direction of the moment would be out of the wood and the bolt would be loosened.

CONNECTIONS

Another word for moment is torque, which is denoted by the Greek letter τ (tau).

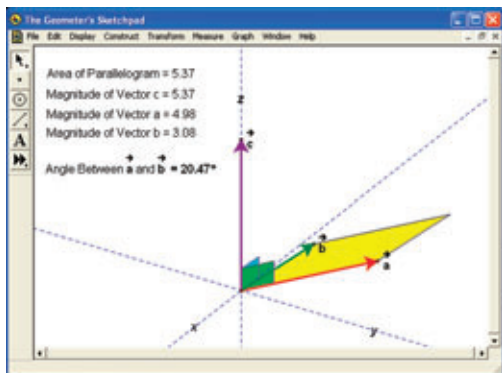
CONNECTIONS

This bolt has a standard right-handed thread. Some bolts have a left-handed thread, such as the ones that hold the pedals on a bicycle.

Investigate A new type of vector product

Go to www.mcgrawhill.ca/links/calculus12 and follow the links to 7.5. Download the file 7.5CrossProduct.gsp. Open the sketch.

You are given a parallelogram defined by vectors \vec{a} and \vec{b} , as well as a vector \vec{c} orthogonal to both \vec{a} and \vec{b} .



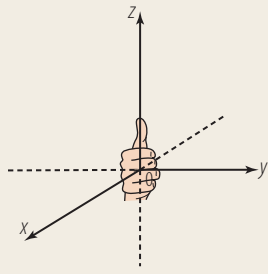
Tools

- computer with *The Geometer's Sketchpad*®
- 7.5CrossProduct.gsp

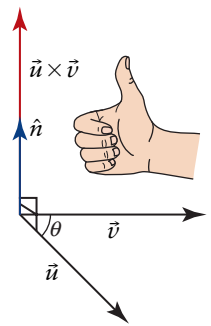
1. What happens if $|\vec{b}|$ changes?
 - a) What happens to the magnitude of \vec{c} ?
 - b) What happens to the direction of \vec{c} ?
2. Move \vec{b} closer to \vec{a} . How does this affect the magnitude of \vec{c} ? How does it affect the direction of \vec{c} ?
3. Slowly increase the angle between \vec{a} and \vec{b} . How does this affect the direction of \vec{c} ? What angle size maximizes the length of \vec{c} ?
4. How can you make the direction of \vec{c} change to its opposite?
5. **Reflect** Imagine spinning the plane about the z -axis. What is the relationship between \vec{c} and the parallelogram defined by \vec{a} and \vec{b} ?
6. **Reflect** How is this simulation related to the wrench scenario?

CONNECTIONS

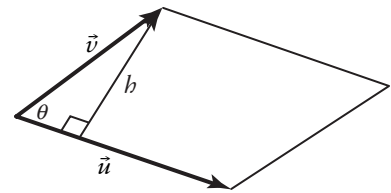
The right-hand rule is similar to the right-hand coordinate system. As the fingers on your right hand curl from the x -axis to the y -axis, your thumb points in the direction of the z -axis.



In the Investigate, you analysed what is known as the **cross product** of two vectors. In general, the cross product of two vectors \vec{u} and \vec{v} is defined as $\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin \theta)\hat{n}$, where θ is the angle between \vec{u} and \vec{v} and \hat{n} is a unit vector perpendicular to both \vec{u} and \vec{v} such that \vec{u} , \vec{v} , and \hat{n} form a right-handed system, as shown. As the fingers on your right hand curl from \vec{u} to \vec{v} , your thumb points in the direction of $\vec{u} \times \vec{v}$. This means that the order of the vectors \vec{u} and \vec{v} is important in determining which way the cross product vector points. Note that $\vec{u} \times \vec{v}$ is read as “vector u cross vector v ,” not “vector u times vector v .”



To determine the magnitude of $\vec{u} \times \vec{v}$, consider the parallelogram formed by \vec{u} and \vec{v} . The area of a parallelogram is given by the formula $\text{Area} = \text{base} \times \text{height}$.



$$\sin \theta = \frac{h}{|\vec{v}|}$$

$$h = |\vec{v}| \sin \theta$$

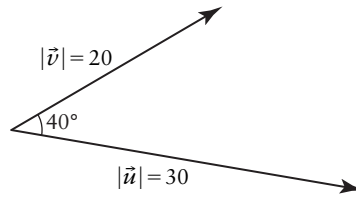
$$\begin{aligned} \text{Area} &= \text{base} \times \text{height} \\ &= |\vec{u}| |\vec{v}| \sin \theta \end{aligned}$$

Thus, the area of the parallelogram formed by \vec{u} and \vec{v} is equal to the magnitude of the cross product of \vec{u} and \vec{v} .

Example 1 Find a Cross Product

If $|\vec{u}| = 30$, $|\vec{v}| = 20$, the angle between \vec{u} and \vec{v} is 40° , and \vec{u} and \vec{v} are in the plane of the page, find

- a) $\vec{u} \times \vec{v}$ b) $\vec{v} \times \vec{u}$



Solution

- a) Using the right-hand rule, the direction of $\vec{u} \times \vec{v}$ is outward from the page. Let \hat{n} be the unit vector perpendicular to both \vec{u} and \vec{v} with direction outward from this page. Then,

$$\begin{aligned}\vec{u} \times \vec{v} &= (|\vec{u}||\vec{v}|\sin\theta)(\hat{n}) \\ &= [(30)(20)\sin 40^\circ](\hat{n}) \\ &\doteq 38.6\hat{n}\end{aligned}$$

- b) Using the right-hand rule, the direction of $\vec{v} \times \vec{u}$ is into the page. Thus,

$$\begin{aligned}\vec{v} \times \vec{u} &= (|\vec{v}||\vec{u}|\sin\theta)(-\hat{n}) \\ &= [(20)(30)\sin 40^\circ](-\hat{n}) \\ &\doteq -38.6\hat{n}\end{aligned}$$

From Example 1, $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ are *not* equal vectors, but they *are* opposite vectors. The cross product is not commutative.

$$\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}, \text{ but } \vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

The Cross Product in Cartesian Form

We want to develop a formula for $\vec{a} \times \vec{b}$ involving components.

Let $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$.

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$

$$= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$\begin{aligned}&= a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2 \\ &\quad - a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2 - 2a_1b_1a_2b_2 - 2a_1b_1a_3b_3 - 2a_2b_2a_3b_3\end{aligned}$$

$$= (a_2^2b_3^2 - 2a_2b_2a_3b_3 + a_3^2b_2^2) + (a_3^2b_1^2 - 2a_1b_1a_3b_3 + a_1^2b_3^2)$$

$$+ (a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2)$$

$$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= |[a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]|^2$$

$$\text{Thus, } |\vec{a} \times \vec{b}| = |[a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]|.$$

Magnitude of the cross product

Square both sides.

$$\cos^2\theta + \sin^2\theta = 1$$

$$|\vec{a}||\vec{b}|\cos\theta = (\vec{a} \cdot \vec{b})$$

Express \vec{a} , \vec{b} , and $\vec{a} \cdot \vec{b}$ using coordinates.

Rewrite, setting up perfect squares.

Let \vec{c} represent the vector on the right side of the last equation.

$$\vec{c} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$$

We want to check if \vec{c} has the same direction as $\vec{a} \times \vec{b}$, that is, if \vec{c} is orthogonal to both \vec{a} and \vec{b} .

\vec{c} is orthogonal to \vec{a} if $\vec{c} \cdot \vec{a} = 0$.

$$\begin{aligned}\vec{c} \cdot \vec{a} &= [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1] \cdot [a_1, a_2, a_3] \\ &= a_2b_3a_1 - a_3b_2a_1 + a_3b_1a_2 - a_1b_3a_2 + a_1b_2a_3 - a_2b_1a_3 \\ &= (a_2b_3a_1 - a_1b_3a_2) + (a_1b_2a_3 - a_3b_2a_1) + (a_3b_1a_2 - a_2b_1a_3) \quad \text{Collect like terms.} \\ &= 0\end{aligned}$$

Thus, \vec{c} is orthogonal to \vec{a} .

\vec{c} is orthogonal to \vec{b} if $\vec{c} \cdot \vec{b} = 0$.

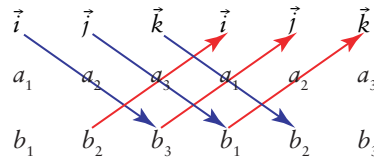
$$\begin{aligned}\vec{c} \cdot \vec{b} &= [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1] \cdot [b_1, b_2, b_3] \\ &= a_2b_3b_1 - a_3b_2b_1 + a_3b_1b_2 - a_1b_3b_2 + a_1b_2b_3 - a_2b_1b_3 \\ &= 0\end{aligned}$$

Thus, \vec{c} is orthogonal to \vec{b} .

Since \vec{c} is orthogonal to both \vec{a} and \vec{b} , and $|\vec{c}| = |\vec{a} \times \vec{b}|$, then $\vec{c} = \vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$$

Here is a visual way to remember how to perform these operations:



Example 2

Properties of the Cross Product of Cartesian Vectors

Consider the vectors $\vec{a} = [7, 1, -2]$, $\vec{b} = [4, 3, 6]$, and $\vec{c} = [-1, 2, 4]$.

- Find $\vec{a} \times \vec{b}$.
- Confirm that $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a} = [7, 1, -2]$ and $\vec{b} = [4, 3, 6]$.
- Determine $\vec{a} \times (\vec{b} + \vec{c})$ and $\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. What do you notice?
- Determine $(\vec{a} + \vec{b}) \times \vec{c}$ and $\vec{a} \times \vec{c} + \vec{b} \times \vec{c}$. What do you notice?

Solution

a)

$$\begin{aligned}\vec{a} \times \vec{b} &= [1(6) - 3(-2)]\vec{i} + [(-2)(4) - 6(7)]\vec{j} + [7(3) - 4(1)]\vec{k} \\ &= [12, -50, 17]\end{aligned}$$

b) $(\vec{a} \times \vec{b}) \cdot \vec{a} = [12, -50, 17] \cdot [7, 1, -2]$
 $= 12(7) + (-50)(1) + 17(-2)$
 $= 0$

Thus, $\vec{a} \times \vec{b}$ is orthogonal to \vec{a} .

$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot \vec{b} &= [12, -50, 17] \cdot [4, 3, 6] \\ &= 12(4) + (-50)(3) + 17(6) \\ &= 0\end{aligned}$$

Thus, $\vec{a} \times \vec{b}$ is orthogonal to \vec{b} .

c) $\vec{a} \times (\vec{b} + \vec{c}) = [7, 1, -2] \times ([4, 3, 6] + [-1, 2, 4])$
 $= [7, 1, -2] \times [3, 5, 10]$
 $= [1(10) - (-2)(5), -2(3) - 7(10), 7(5) - 1(3)]$
 $= [20, -76, 32]$

From part a), $\vec{a} \times \vec{b} = [12, -50, 17]$.

$$\begin{aligned}\vec{a} \times \vec{b} + \vec{a} \times \vec{c} &= [12, -50, 17] + [7, 1, -2] \times [-1, 2, 4] \\ &= [12, -50, 17] + [1(4) - (-2)(2), -2(-1) - 7(4), 7(2) - 1(-1)] \\ &= [12, -50, 17] + [8, -26, 15] \\ &= [20, -76, 32]\end{aligned}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

d) $(\vec{a} + \vec{b}) \times \vec{c} = ([7, 1, -2] + [4, 3, 6]) \times [-1, 2, 4]$
 $= [11, 4, 4] \times [-1, 2, 4]$
 $= [4(4) - 4(2), 4(-1) - 11(4), 11(2) - 4(-1)]$
 $= [8, -48, 26]$

From part b), $\vec{a} \times \vec{c} = [8, -26, 15]$.

$$\begin{aligned}\vec{a} \times \vec{c} + \vec{b} \times \vec{c} &= [8, -26, 15] + [4, 3, 6] \times [-1, 2, 4] \\ &= [8, -26, 15] + [3(4) - 6(2), 6(-1) - 4(4), 4(2) - 3(-1)] \\ &= [8, -26, 15] + [0, -22, 11] \\ &= [8, -48, 26]\end{aligned}$$

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

CONNECTIONS

You will prove this property in general in question 16.

CONNECTIONS

You will prove this property in general in question 16.

Example 3 Prove a Property of the Cross Product

Prove that, for two non-zero vectors \vec{u} and \vec{v} ,
 $\vec{u} \times \vec{v} = \vec{0}$ if and only if there is a scalar $k \in \mathbb{R}$ such that $\vec{u} = k\vec{v}$.

Solution

First, prove that if $\vec{u} \times \vec{v} = \vec{0}$, then $\vec{u} = k\vec{v}$.

Since $\vec{u} \times \vec{v} = \vec{0}$,

$$[u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1] = [0, 0, 0]$$

$$u_2v_3 - u_3v_2 = 0 \quad u_3v_1 - u_1v_3 = 0 \quad u_1v_2 - u_2v_1 = 0$$

$$u_2v_3 = u_3v_2 \quad u_3v_1 = u_1v_3 \quad u_1v_2 = u_2v_1$$

$$\frac{u_2}{v_2} = \frac{u_3}{v_3} \quad \frac{u_3}{v_3} = \frac{u_1}{v_1} \quad \frac{u_1}{v_1} = \frac{u_2}{v_2}$$

Thus,

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3}$$

Let the value of the fractions be k . Thus,

$$\frac{u_1}{v_1} = k \quad \frac{u_2}{v_2} = k \quad \frac{u_3}{v_3} = k$$

$$u_1 = kv_1 \quad u_2 = kv_2 \quad u_3 = kv_3$$

$$[u_1, u_2, u_3] = [kv_1, kv_2, kv_3]$$

$$\vec{u} = k\vec{v}$$

Next, prove that if $\vec{u} = k\vec{v}$, then $\vec{u} \times \vec{v} = \vec{0}$.

Since $\vec{u} = k\vec{v}$,

$$\begin{aligned} \vec{u} \times \vec{v} &= k\vec{v} \times \vec{v} \\ &= (kv_1, kv_2, kv_3) \times (v_1, v_2, v_3) \\ &= [kv_2v_3 - kv_3v_2, kv_3v_1 - kv_1v_3, kv_1v_2 - kv_2v_1] \\ &= [0, 0, 0] \\ &= \vec{0} \end{aligned}$$

Thus, for two non-zero vectors \vec{u} and \vec{v} ,

$\vec{u} \times \vec{v} = \vec{0}$ if and only if there is a scalar $k \in \mathbb{R}$ such that $\vec{u} = k\vec{v}$.

In other words, two non-zero vectors are parallel (collinear) if and only if their cross product is the zero vector, $\vec{0}$.

Examples 1 to 3 demonstrated four properties of the cross product. You will prove the fifth one in question 17.

Properties of the Cross Product

For any vectors \vec{u} , \vec{v} , and \vec{w} and any scalar $k \in \mathbb{R}$,

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- If \vec{u} and \vec{v} are non-zero, $\vec{u} \times \vec{v} = \vec{0}$ if and only if there is a scalar $m \in \mathbb{R}$ such that $\vec{u} = m\vec{v}$.
- $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$

Example 4 Apply the Cross Product

- a) Determine the area of the parallelogram defined by the vectors $\vec{u} = [4, 5, 2]$ and $\vec{v} = [3, 2, 7]$.
- b) Determine the angle between vectors \vec{u} and \vec{v} .

Solution

- a) We know that the magnitude of the cross product represents the area of the parallelogram formed by the two vectors.

$$\begin{aligned}
 A &= |\vec{u} \times \vec{v}| \\
 &= |[4, 5, 2] \times [3, 2, 7]| \\
 &= |[5(7) - 2(2), 2(3) - 7(4), 4(2) - 3(5)]| \\
 &= |[31, -22, -7]| \\
 &= \sqrt{31^2 + (-22)^2 + (-7)^2} \\
 &= \sqrt{1494} \\
 &\doteq 36.7
 \end{aligned}$$

- b) $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

$$\begin{aligned}
 \sin\theta &= \frac{|\vec{u} \times \vec{v}|}{|\vec{u}||\vec{v}|} \\
 &= \frac{3\sqrt{166}}{\sqrt{4^2 + 5^2 + 2^2}\sqrt{3^2 + 2^2 + 7^2}} \\
 &= \frac{3\sqrt{166}}{\sqrt{45}\sqrt{62}} \\
 \theta &= \sin^{-1}\left(\frac{3\sqrt{166}}{\sqrt{45}\sqrt{62}}\right) \\
 \theta &\doteq 47.0^\circ
 \end{aligned}$$

The angle between \vec{u} and \vec{v} is about 47° . Since $\sin(180^\circ - \theta) = \sin\theta$, another possible solution is $180^\circ - 47.0^\circ$ or 133.0° . A quick sketch will verify that the angle is 47° in this case. Also note that both of the corresponding position vectors lie in the first octant.

KEY CONCEPTS

- The cross product is defined as $\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}|\sin \theta)\hat{n}$, where \hat{n} is the unit vector orthogonal to both \vec{a} and \vec{b} , following the right-hand rule for direction, and θ is the angle between the vectors.
- If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$, then
 $\vec{a} \times \vec{b} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$.
- The magnitude of the cross product vector is $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin \theta$, which also calculates the area of the parallelogram defined by \vec{a} and \vec{b} .
- The angle, θ , between two vectors, \vec{a} and \vec{b} , is given by

$$\theta = \sin^{-1} \left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} \right).$$
- For any vectors \vec{u} , \vec{v} , and \vec{w} and scalar $k \in \mathbb{R}$,
 - $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
 - $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
 - $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
 - If \vec{u} and \vec{v} are non-zero, $\vec{u} \times \vec{v} = \vec{0}$ if and only if there is a scalar $m \in \mathbb{R}$ such that $\vec{u} = m\vec{v}$.
 - $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$

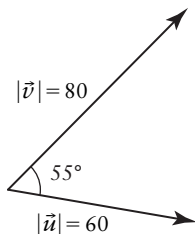
Communicate Your Understanding

- If $\vec{c} = \vec{a} \times \vec{b}$, what can be said about $\vec{c} \cdot \vec{a}$ and $\vec{c} \cdot \vec{b}$? Explain.
- Explain why we do not refer to the \times symbol in the cross product as “times.”
- How can the cross product be used to define the unit vector \vec{k} ? What about \vec{i} and \vec{j} ?

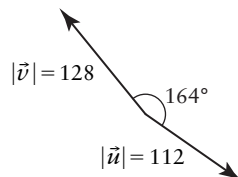
A Practise

- Determine $\vec{u} \times \vec{v}$.

a)



b)



- Determine $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ for each pair of vectors.

a) $\vec{a} = [3, -2, 9]$, $\vec{b} = [1, 1, 6]$

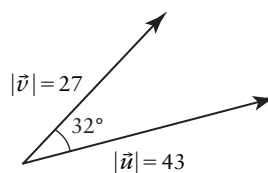
b) $\vec{a} = [6, 3, 2]$, $\vec{b} = [-5, 5, 9]$

c) $\vec{a} = [-8, 10, 3]$, $\vec{b} = [2, 0, 5]$

d) $\vec{a} = [4.3, 5.7, -0.2]$, $\vec{b} = [12.3, -4.9, 8.8]$

3. For each pair of vectors, confirm that $\vec{u} \times \vec{v}$ is orthogonal to \vec{u} and \vec{v} .
- a) $\vec{u} = [5, -3, 7], \vec{v} = [-1, 6, 2]$
 b) $\vec{u} = [-2, 1, 5], \vec{v} = [3, 2, 0]$
 c) $\vec{u} = [4, -6, 2], \vec{v} = [6, 8, -3]$

4. Determine the area of the parallelogram defined by each pair of vectors.
- a) $\vec{p} = [6, 3, 8], \vec{q} = [3, 3, 5]$
 b)



B Connect and Apply

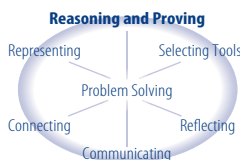
5. How could you use a wrench to explain why $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ for any vectors \vec{a} and \vec{b} ?



6. a) Use an example to verify that

$$|\vec{u} \times \vec{v}| = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}.$$

- b) Prove the identity in part a) for any vectors \vec{u} and \vec{v} .



7. Given $\vec{a} = [2, -6, 3]$, $\vec{b} = [-1, 5, 8]$, and $\vec{c} = [-4, 5, 6]$, evaluate each of the following.

- a) $\vec{a} \times (\vec{b} + \vec{c})$ b) $(\vec{b} + \vec{c}) \times \vec{a}$
 c) $\vec{a} \times \vec{b} - \vec{a} \times \vec{c}$ d) $\vec{a} \times (5\vec{a})$
 e) $|\vec{a} \times \vec{c}|$ f) $|\vec{b} \times (\vec{c} - \vec{a})|$

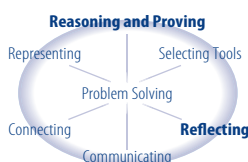
8. Determine two vectors that are orthogonal to both $\vec{c} = [4, 6, -1]$ and $\vec{d} = [-2, 10, 11]$.

9. Determine a unit vector that is orthogonal to both $\vec{u} = 3\vec{i} - 4\vec{j} + \vec{k}$ and $\vec{v} = 2\vec{i} + 3\vec{j} - 4\vec{k}$.

10. a) Show that the quadrilateral with vertices at P(0, 2, 5), Q(1, 6, 2), R(7, 4, 2), and S(6, 0, 5) is a parallelogram.

- b) Calculate its area.
 c) Is this parallelogram a rectangle? Explain.

11. Use the cross product to determine the angle between the vectors $\vec{g} = [4, 5, 2]$ and $\vec{h} = [-2, 6, 1]$. Verify using the dot product.



12. A parallelogram has area 85 cm². The side lengths are 10 cm and 9 cm. What are the measures of the interior angles?

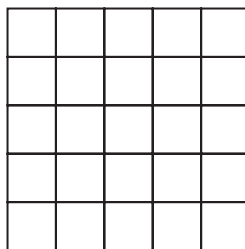
13. Given $\vec{u} = [3, 2, 9]$, $\vec{v} = [8, 0, 3]$, and $\vec{w} = [6, 2, 6]$, prove that the cross product is not associative, i.e., that $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$.

Achievement Check

14. a) If \vec{u} and \vec{v} are non-collinear vectors, show that \vec{u} , $\vec{u} \times \vec{v}$, and $(\vec{u} \times \vec{v}) \times \vec{u}$ are mutually orthogonal.
 b) Verify this property using vectors collinear with the unit vectors \vec{i} , \vec{j} , and \vec{k} .
 c) Use this property to determine a set of three mutually orthogonal vectors.
15. Use the cross product to describe when the area of a parallelogram will be zero.
16. a) Let $\vec{a} = [-2, 4, 3]$, $\vec{b} = [6, 1, 2]$, and $\vec{c} = [5, -3, -2]$. Use these vectors to verify that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
 b) Prove the property in part a) for any vectors \vec{a} , \vec{b} , and \vec{c} .
 c) Use the vectors in part a) to verify that $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$.
 d) Prove the property in part c) for any vectors \vec{a} , \vec{b} , and \vec{c} .
17. a) Let $\vec{u} = [-1, 4, 1]$, $\vec{v} = [3, -2, 4]$, and $k = 2$. Show that $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$.
 b) Prove the property in part a) for any vectors \vec{u} and \vec{v} and any scalar $k \in \mathbb{R}$.

C Extend and Challenge

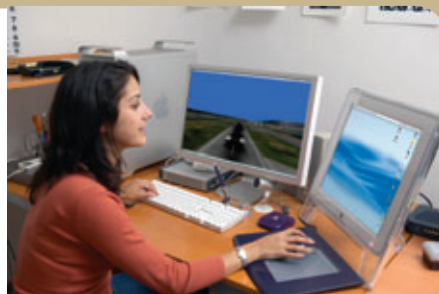
18. If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, does it follow that $\vec{b} = \vec{c}$? Justify your answer.
19. a) Devise an easy test to determine whether $|\vec{a} \times \vec{b}| < |\vec{a} \cdot \vec{b}|$, $|\vec{a} \times \vec{b}| > |\vec{a} \cdot \vec{b}|$, or $|\vec{a} \times \vec{b}| = |\vec{a} \cdot \vec{b}|$.
 b) Test your conjecture with $[2, 1, -1]$ and $[-1, -2, 1]$.
 c) Test your conjecture with $[2, 1, 1]$ and $[3, 1, 2]$.
 d) For randomly chosen vectors, which of these cases is most likely?
20. Use your result from question 6 to prove each of the following.
 a) $|\vec{u} \cdot \vec{v}| \leq |\vec{u}| |\vec{v}|$
 b) \vec{u} and \vec{v} are collinear if and only if $|\vec{u} \cdot \vec{v}| = |\vec{u}| |\vec{v}|$.
21. Let \vec{a} , \vec{b} , and \vec{c} be three vectors sharing a common starting point. The tips of the vectors form the vertices of a triangle. Prove that the area of the triangle is given by the magnitude of the vector $\frac{1}{2}(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$. This vector is known as the vector area of the triangle.
22. The expression $\vec{a} \times \vec{b} \times \vec{c}$ is called a triple vector product. Consider the special vectors \vec{i} , \vec{j} , and \vec{k} .
 a) Although it is not true in general, give two examples where $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$.
 b) Prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.
23. **Math Contest** Determine the area of the pentagon with vertices A(0, 5), B(2, 7), C(5, 6), D(6, 4), and E(1, 2).
24. **Math Contest** A wire screen made up of 20-mm squares and 1-mm-thick wire is shown. What is the probability that a marble 10 mm in diameter, thrown randomly at the screen, will pass through one of the holes cleanly (without touching any wire)?



CAREER CONNECTION

Lily, in her job as a computer programmer, helps her team to conceive, design, write, and test software programs for flight simulation. These programs must duplicate conditions on an airplane and are used for pilot training. Her work includes solving physical problems that involve motion in three dimensions. For every x , there is always a y and a z . Lily finds that using vectors and matrices simplifies the computer code tremendously, and using them also greatly reduces the chance of programming errors.

Lily trained for her programming career by taking a 4-year honours bachelor of computer science degree at the University of Waterloo.



7.6

Applications of the Dot Product and Cross Product

You can produce the same amount of turning force by applying a 6-N force 0.1 m from the pivot point as applying a 0.1-N force 6 m from the pivot point. This concept is used to design engines used in automobiles and train locomotives.

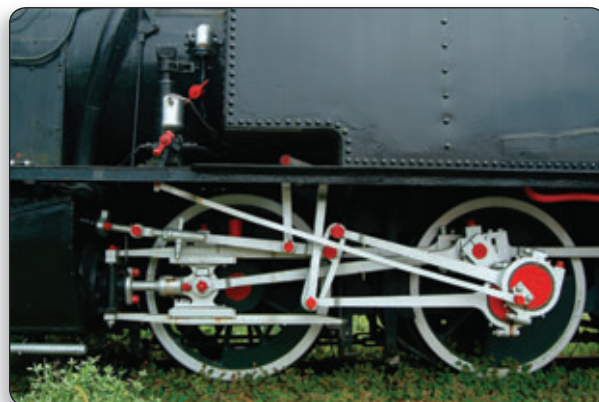
Torque, represented by the Greek letter tau (τ), is a measure of the force acting on an object that causes it to rotate. Torque is the cross product of the force and the torque arm, where the torque arm is the vector from the pivot point to the point where the force acts.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin \theta$$

In these equations, \vec{F} is the force acting on the object, \vec{r} represents the arm, and θ is the angle between \vec{r} and \vec{F} . We use the right-hand rule to determine the direction of the torque vector.

The unit of measure for torque is the newton-metre ($\text{N} \cdot \text{m}$). When we express the force in newtons and the displacement in metres, the units for the resulting torque are newton-metres.



CONNECTIONS

Torque is the same as moment, discussed in Section 7.5.

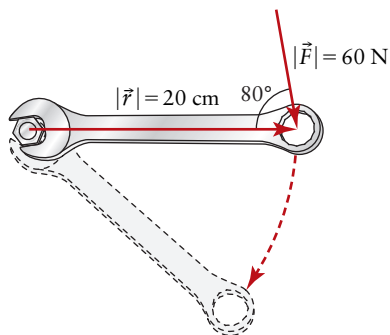
CONNECTIONS

Joules (J), which are used to describe energy, are equivalent to newton-metres. To avoid confusion, the newton-metre is the unit used for torque, a vector, while the joule is used for energy, a scalar.

Example 1 Find Torque

A wrench is used to tighten a bolt. A force of 60 N is applied in a clockwise direction at 80° to the handle, 20 cm from the centre of the bolt.

- Calculate the magnitude of the torque.
- In what direction does the torque vector point?



CONNECTIONS

Go to www.mcgrawhill.ca/links/calculus12 and follow the links for more information on torque.

Solution

- a) Convert the displacement to metres.

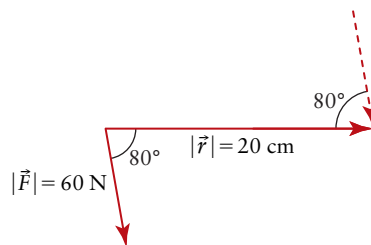
$$20 \text{ cm} = 0.2 \text{ m}$$

Translate \vec{F} so that it is tail to tail with \vec{r} .

The angle between \vec{F} and \vec{r} is 80° .

$$\begin{aligned} |\vec{\tau}| &= |\vec{r}| |\vec{F}| \sin \theta \\ &= (0.2)(60) \sin 80^\circ \\ &\doteq 11.8 \end{aligned}$$

The torque has a magnitude of about $11.8 \text{ N} \cdot \text{m}$.



- b) By the right-hand rule, the torque vector points downward into the material, which means that the bolt is being tightened.

Example 2 Projection

Determine the projection, and its magnitude, of $\vec{v} = [4, 2, 7]$ on $\vec{u} = [6, 3, 8]$.

Solution

Projections in three-space are similar to those in two-space, so you can use the same formula.

$$\begin{aligned} \text{proj}_{\vec{u}} \vec{v} &= \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u} \\ &= \left(\frac{[4, 2, 7] \cdot [6, 3, 8]}{[6, 3, 8] \cdot [6, 3, 8]} \right) [6, 3, 8] \\ &= \left(\frac{4(6) + 2(3) + 7(8)}{6^2 + 3^2 + 8^2} \right) [6, 3, 8] \\ &= \frac{86}{109} [6, 3, 8] \end{aligned}$$

$$\begin{aligned} |\text{proj}_{\vec{u}} \vec{v}| &= \left| \frac{86}{109} [6, 3, 8] \right| \\ &= \frac{86}{109} |[6, 3, 8]| \\ &= \frac{86}{109} \sqrt{6^2 + 3^2 + 8^2} \\ &= \frac{86}{109} \sqrt{109} \\ &\doteq 8.2 \end{aligned}$$

Example 3 Mechanical Work in Three-Space

A force with units in newtons and defined by $\vec{F} = [300, 700, 500]$ acts on an object with displacement, in metres, defined by $\vec{d} = [3, 1, 12]$.

- Determine the work done in the direction of travel.
- Determine the work done against gravity, which is a force in the direction of the negative z -axis.

Solution

- a) Work is equal to the dot product of the force, \vec{F} , and the displacement, \vec{s} .

$$\begin{aligned} W &= \vec{F} \cdot \vec{s} \\ &= |\vec{F}| |\vec{s}| \cos \theta \\ &= [300, 700, 500] \cdot [3, 1, 12] \\ &= 300(3) + 700(1) + 500(12) \\ &= 7600 \end{aligned}$$

The work performed in the direction of travel is 7600 J.

- b) Since the work is to be calculated against gravity, we must use only the vertical components of \vec{F} and \vec{s} , which are $[0, 0, 500]$ and $[0, 0, 12]$, respectively.

$$\begin{aligned} W &= \vec{F} \cdot \vec{s} \\ &= [0, 0, 500] \cdot [0, 0, 12] \\ &= 0(0) + 0(0) + 500(12) \\ &= 6000 \end{aligned}$$

The work performed against gravity is 6000 J.

The Triple Scalar Product

Certain situations require a combination of the dot and cross products. The **triple scalar product**, $\vec{a} \cdot \vec{b} \times \vec{c}$, is one such combination. Because the dot product is a scalar, this combination is only meaningful if the cross product is performed first.

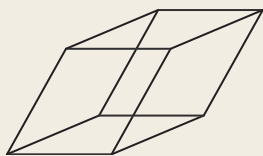
Example 4 Triple Scalar Product

Consider the vectors $\vec{u} = [4, 3, 1]$, $\vec{v} = [2, 5, 6]$, and $\vec{w} = [10, -3, -14]$.

- Evaluate the expression $\vec{u} \times \vec{v} \cdot \vec{w}$.
- Evaluate $\vec{w} \cdot \vec{u} \times \vec{v}$. Compare your answer to that in part a) and explain why this happens.
- Explain the geometric significance of the result.

CONNECTIONS

A rectangular prism has six orthogonal faces, each a rectangle. A parallelepiped is a six-faced solid with every face a parallelogram.



Solution

$$\begin{aligned}\text{a) } \vec{u} \times \vec{v} \cdot \vec{w} &= [4, 3, 1] \times [2, 5, 6] \cdot [10, -3, -14] \\ &= [3(6) - 1(5), 1(2) - 4(6), 4(5) - 3(2)] \cdot [10, -3, -14] \\ &= [13, -22, 14] \cdot [10, -3, -14] \\ &= 13(10) - 22(-3) + 14(-14) \\ &= 0\end{aligned}$$

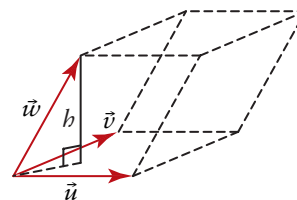
$$\begin{aligned}\text{b) } \vec{w} \cdot \vec{u} \times \vec{v} &= [10, -3, -14] \cdot [4, 3, 1] \times [2, 5, 6] \\ &= [10, -3, -14] \cdot [3(6) - 1(5), 1(2) - 4(6), 4(5) - 3(2)] \\ &= [10, -3, -14] \cdot [13, -22, 14] \\ &= 10(13) + (-3)(-22) + (-14)(14) \\ &= 0\end{aligned}$$

The answers in parts a) and b) are equal. This is because the dot product is commutative.

- c) $\vec{u} \times \vec{v}$ is orthogonal to both \vec{u} and \vec{v} . Since $\vec{u} \times \vec{v} \cdot \vec{w} = 0$, \vec{w} must be orthogonal to $\vec{u} \times \vec{v}$ and must lie in the same plane as \vec{u} and \vec{v} .

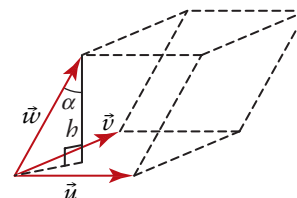
Example 5 Volume of a Parallelepiped

It can be shown that the volume, V , of a parallelepiped is given by the formula $V = A \times h$, where A is the area of the base and h is the height. Prove that the volume of the parallelepiped shown can be found using the triple scalar product $\vec{w} \cdot \vec{u} \times \vec{v}$.



Solution

In Section 7.5, you learned that the area of a parallelogram defined by vectors \vec{u} and \vec{v} is given by $|\vec{u} \times \vec{v}|$. The height is given by the component of \vec{w} that is perpendicular to the base. Let the angle between \vec{w} and the height be α .



Then,

$$\begin{aligned}\frac{h}{|\vec{w}|} &= \cos \alpha \\ h &= |\vec{w}| \cos \alpha \\ V &= Ah \\ &= |\vec{u} \times \vec{v}| |\vec{w}| \cos \alpha \\ &= |\vec{w}| |\vec{u} \times \vec{v}| \cos \alpha\end{aligned}$$

By parallel lines, α is also the angle between \vec{w} and the cross product $\vec{u} \times \vec{v}$, which is also perpendicular to the base.

Thus, by definition of the dot product, the volume is the dot product of \vec{w} and $\vec{u} \times \vec{v}$, or $V = \vec{w} \cdot \vec{u} \times \vec{v}$.

Volume of a Parallelepiped

In general, since the triple scalar product can sometimes result in a negative number, the volume of the parallelepiped defined by the vectors \vec{u} , \vec{v} , and \vec{w} is given by $V = |\vec{w} \cdot \vec{u} \times \vec{v}|$.

KEY CONCEPTS

- Torque is a measure of the force acting on an object causing it to rotate.

- $\vec{\tau} = \vec{r} \times \vec{F}$
- $|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin \theta$

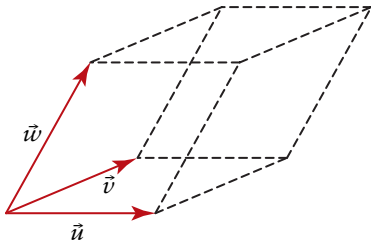
The direction of the torque vector follows the right-hand rule.

- The formulas for projection and work are the same in 3-D and 2-D:

- $\text{proj}_{\vec{u}} \vec{v} = |\vec{v}| \cos \theta \left(\frac{1}{|\vec{u}|} \vec{u} \right)$ or $\text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$
- $W = \vec{F} \cdot \vec{s}$

- The triple scalar product is defined as $\vec{a} \cdot \vec{b} \times \vec{c}$.

- The volume of the parallelepiped defined by \vec{u} , \vec{v} , and \vec{w} , as shown, is $V = |\vec{w} \cdot \vec{u} \times \vec{v}|$.

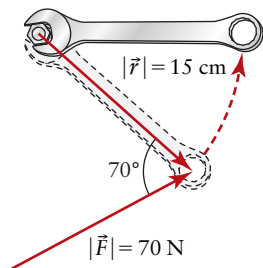


Communicate Your Understanding

- C1** Use a geometric interpretation to explain why $\vec{a} \times \vec{b} \cdot \vec{b} = 0$.
- C2** How can you check to see if your calculations in the cross product are correct? Explain.
- C3** At what angle, relative to the displacement vector, is the torque greatest? Why?
- C4** Does the expression $\vec{a} \times \vec{b} \cdot \vec{c} \times \vec{d}$ need brackets to indicate the order of operations? Explain.
- C5** Can you find the volume of a parallelepiped defined by \vec{u} , \vec{v} , and \vec{w} using the expression $\vec{v} \cdot \vec{u} \times \vec{w}$ or $\vec{u} \cdot \vec{v} \times \vec{w}$? Explain.

A Practise

1. A force of 90 N is applied to a wrench in a counterclockwise direction at 70° to the handle, 15 cm from the centre of the bolt.



- Calculate the magnitude of the torque.
 - In what direction does the bolt move?
2. Determine the projection, and its magnitude, of \vec{u} on \vec{v} .
- $\vec{u} = [3, 1, 4]$, $\vec{v} = [6, 2, 7]$
 - $\vec{u} = [5, -4, 8]$, $\vec{v} = [3, 7, 6]$

- $\vec{u} = -2\vec{i} - 7\vec{j} + 3\vec{k}$, $\vec{v} = 6\vec{i} + \vec{j} - 8\vec{k}$
- $\vec{u} = \vec{i} - \vec{k}$, $\vec{v} = 9\vec{i} + \vec{j}$

3. A force, $\vec{F} = [3, 5, 12]$, in newtons, is applied to lift a box, with displacement, \vec{s} , in metres as given. Calculate the work against gravity and compare it to the work in the direction of travel.

- $\vec{s} = [0, 0, 8]$
- $\vec{s} = [2, 0, 10]$
- $\vec{s} = [2, 1, 6]$

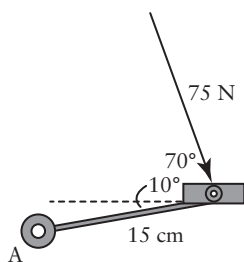
4. Given $\vec{a} = [-2, 3, 5]$, $\vec{b} = [4, 0, -1]$, and $\vec{c} = [2, -2, 3]$, evaluate each expression.

- $\vec{a} \times \vec{b} \cdot \vec{c}$
- $\vec{a} \cdot \vec{b} \times \vec{c}$
- $\vec{a} \times \vec{c} \cdot \vec{b}$
- $\vec{b} \cdot \vec{a} \times \vec{c}$

B Connect and Apply

5. Find the volume of each parallelepiped, defined by the vectors \vec{u} , \vec{v} , and \vec{w} .
- $\vec{u} = [1, 4, 3]$, $\vec{v} = [2, 5, 6]$, and $\vec{w} = [1, 2, 7]$
 - $\vec{u} = [-2, 5, 1]$, $\vec{v} = [3, -4, 2]$, and $\vec{w} = [1, 3, 5]$
 - $\vec{u} = [1, 1, 9]$, $\vec{v} = [0, 0, 4]$, and $\vec{w} = [-2, 0, 5]$
6. A triangle has vertices A(-2, 1, 3), B(7, 8, -4), and C(5, 0, 2). Determine the area of $\triangle ABC$.

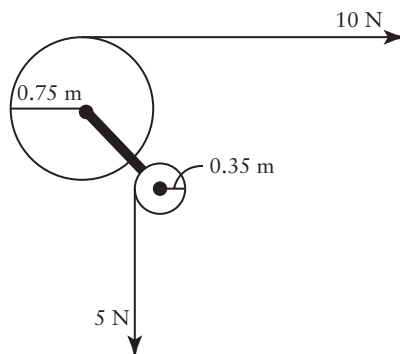
7. A bicycle pedal is pushed by a 75-N force, exerted as shown in the diagram. The shaft of the pedal is 15 cm long. Find the magnitude of the torque vector, in newton-metres, about point A.



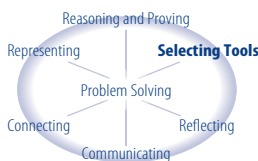
8. A 65-kg boy is sitting on a seesaw 0.6 m from the balance point. How far from the balance point should a 40-kg girl sit so that the seesaw remains balanced?



9. Given $\vec{u} = [2, 2, 3]$, $\vec{v} = [1, 3, 4]$, and $\vec{w} = [6, 2, 1]$, evaluate each expression.
- $|\vec{u} \times \vec{v}|^2 - (\vec{w} \cdot \vec{w})^2$
 - $|\vec{u} \times \vec{u}| + \vec{u} \cdot \vec{u}$
 - $\vec{u} \times \vec{v} \cdot \vec{v} \times \vec{w}$
 - $\vec{u} \times \vec{v} \cdot \vec{u} \times \vec{w}$
10. Consider two vectors \vec{a} and \vec{b} .
- In a single diagram, illustrate both $|\vec{a} \times \vec{b}|$ and $\vec{a} \cdot \vec{b}$.
 - Interpret $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2$ and illustrate it on your diagram.
 - Show that $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$.
11. An axle has two wheels of radii 0.75 m and 0.35 m attached to it. A 10-N force is applied horizontally to the edge of the larger wheel and a 5-N weight hangs from the edge of the smaller wheel. What is the net torque acting on the axle?

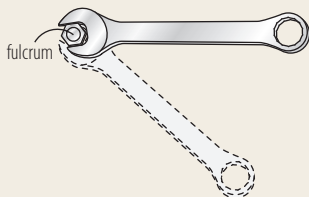


12. When a wrench is rotated, the magnitude of the torque is $10 \text{ N}\cdot\text{m}$. An 80-N force is applied 20 cm from the fulcrum. At what angle to the wrench is the force applied?



CONNECTIONS

The fulcrum of a lever, such as a wrench, is also known as the pivot point.



13. **Chapter Problem** The automotive industry often uses torque in its sales promotions when selling cars. As a result, engineers are regularly trying to increase torque production in car engines. Research the use of torque in car magazine articles or on the Internet, including
- the importance of torque as compared to horsepower
 - the units of measurement that are used
 - which variable(s) torque is plotted against on graphs

Find the torque and horsepower ratings for a current model of hybrid car. How does this compare with an all-electric car? a conventional gasoline-powered car?

14. Is the following statement true or false? “If $|\vec{e} \times \vec{f}| = 0$, then $|\vec{e} \cdot \vec{f}| = 0$.” Justify your response.
15. A wrench is rotated with torque of magnitude $100 \text{ N}\cdot\text{m}$. The force is applied 30 cm from the fulcrum, at an angle of 40° . What is the magnitude of the force, to one decimal place?

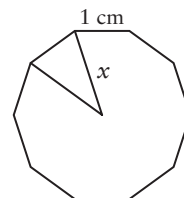
Achievement Check

16. Three edges of a right triangular prism are defined by the vectors $\vec{a} = [1, 3, 2]$, $\vec{b} = [2, 2, 4]$, and $\vec{c} = [12, 0, -6]$.
- Draw a diagram of the prism, identifying which edge of the prism is defined by each vector.
 - Determine the volume of the prism.
 - Explain how your method of solving this problem would change if the prism were not necessarily a right prism.

C Extend and Challenge

17. Given that $\vec{w} = k\vec{u} + m\vec{v}$, where $k, m \in \mathbb{R}$, prove algebraically that $\vec{u} \times \vec{v} \cdot \vec{w} = 0$.
18. Prove that $(\vec{a} \times \vec{b}) \times \vec{c}$ is in the same plane as \vec{a} and \vec{b} .
19. Let \vec{u} , \vec{v} , and \vec{w} be mutually orthogonal vectors. What can be said about $\vec{u} + \vec{v}$, $\vec{u} + \vec{w}$, and $\vec{v} + \vec{w}$?

20. **Math Contest** The figure shown is a regular decagon with side length 1 cm . Determine the exact value of x .



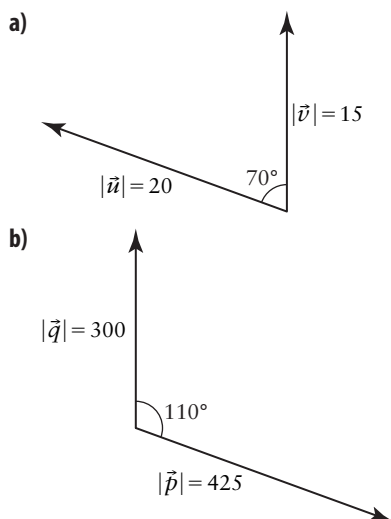
21. **Math Contest** Determine the next three terms of the sequence $333, 33, 23, 1213, \dots$

7.1 Cartesian Vectors

- Consider the vector $\vec{v} = [-6, 3]$.
 - Write \vec{v} in terms of \vec{i} and \vec{j}
 - State two unit vectors that are collinear with \vec{v} .
 - An equivalent vector \overrightarrow{AB} has initial point A(2, 9). Determine the coordinates of B.
- Given $\vec{u} = [5, -2]$ and $\vec{v} = [8, 5]$, evaluate each of the following.
 - $-5\vec{u}$
 - $\vec{u} + \vec{v}$
 - $4\vec{u} + 2\vec{v}$
 - $3\vec{u} - 7\vec{v}$
- An airplane is flying at an airspeed of 345 km/h on a heading of 040° . A wind is blowing at 18 km/h from a bearing of 087° . Determine the ground velocity of the airplane. Include a diagram in your solution.

7.2 Dot Product

- Calculate the dot product of each pair of vectors. Round your answers to two decimal places.



- Calculate the dot product of each pair of vectors.
 - $\vec{u} = [5, 2], \vec{v} = [-6, 7]$
 - $\vec{u} = -3\vec{i} + 2\vec{j}, \vec{v} = 3\vec{i} + 7\vec{j}$
 - $\vec{u} = [3, 2], \vec{v} = [4, -6]$

- Which vectors in question 5 are orthogonal? Explain.

7.3 Applications of the Dot Product

- Two vectors have magnitudes of 5.2 and 7.3. The dot product of the vectors is 20. What is the angle between the vectors? Round your answer to the nearest degree.
- Calculate the angle between the vectors in each pair. Illustrate geometrically.
 - $\vec{a} = [6, -5], \vec{b} = [7, 2]$
 - $\vec{p} = [-9, -4], \vec{q} = [7, -3]$
- Determine the projection of \vec{u} on \vec{v} .
 - $|\vec{u}| = 56, |\vec{v}| = 100$, angle θ between \vec{u} and \vec{v} is 125°
 - $\vec{u} = [7, 1], \vec{v} = [9, -3]$
- Determine the work done by each force, \vec{F} , in newtons, for an object moving along the vector \vec{d} , in metres.
 - $\vec{F} = [16, 12], \vec{d} = [3, 9]$
 - $\vec{F} = [200, 2000], \vec{d} = [3, 45]$
- An electronics store sells 40-GB digital music players for \$229 and 80-GB players for \$329. Last month, the store sold 125 of the 40-GB players and 70 of the 80-GB players.
 - Represent the total revenue from sales of the players using the dot product.
 - Find the total revenue in part a).

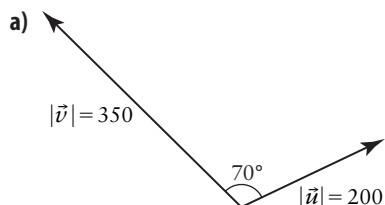
7.4 Vectors in Three-Space

- Determine the exact magnitude of each vector.
 - \overrightarrow{AB} , joining A(2, 7, 8) to B(-5, 9, -1)
 - \overrightarrow{PQ} , joining P(0, 3, 6) to Q(4, -9, 7)
- Given the vectors $\vec{a} = [3, -7, 8], \vec{b} = [-6, 3, 4]$, and $\vec{c} = [2, 5, 7]$, evaluate each expression.
 - $5\vec{a} - 4\vec{b} + 3\vec{c}$
 - $-5\vec{a} \cdot \vec{c}$
 - $\vec{b} \cdot (\vec{c} - \vec{a})$

14. If $\vec{u} = [6, 1, 8]$ is orthogonal to $\vec{v} = [k, -4, 5]$, determine the value(s) of k .

7.5 The Cross Product and Its Properties

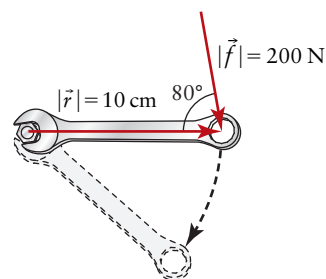
15. Determine $\vec{u} \times \vec{v}$ for each pair of vectors.



- b) $\vec{u} = [4, 1, -3]$, $\vec{v} = [3, 7, 8]$
16. Determine the area of the parallelogram defined by the vectors $\vec{u} = [6, 8, 9]$ and $\vec{v} = [3, -1, 2]$.
17. Use an example to verify that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ for all vectors \vec{a} , \vec{b} , and \vec{c} .

7.6 Applications of the Dot Product and Cross Product

18. A force of 200 N is applied to a wrench in a clockwise direction at 80° to the handle, 10 cm from the centre of the bolt.



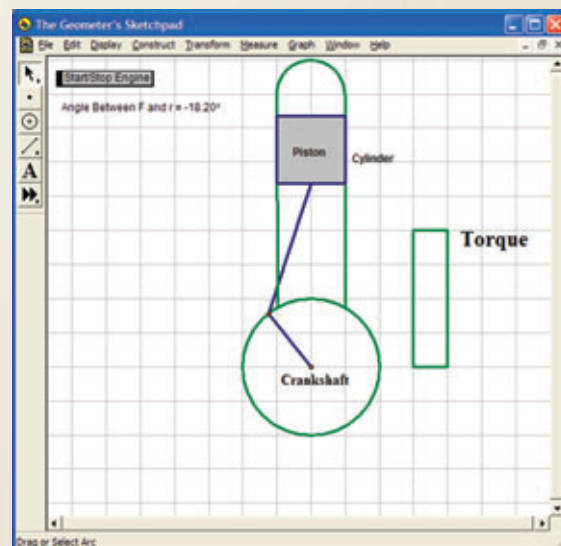
- a) Calculate the magnitude of the torque.
- b) In what direction does the torque vector point?
19. Determine the projection, and its magnitude, of $\vec{u} = [-2, 5, 3]$ on $\vec{v} = [4, -8, 9]$.

Go to www.mcgrawhill.ca/links/calculus12 and follow the links to 7review. Download the file **EngineTorque.gsp**, an applet that shows how a car engine creates torque with the combustion of air and gasoline pushing a piston down and turning the crankshaft. This is called the power stroke. Gases are released on the up, or exhaust, stroke, but no torque is produced.

This is a highly simplified model of an internal combustion engine. Engineers who design real engines must take more factors into account, such as the fuel being used, the way it burns in the cylinder and the transfer of force along the connecting rods.

Open the applet and click on the **Start/Stop Engine** button.

- a) As the crankshaft turns, at what point(s) would the torque be the greatest? Provide mathematical evidence.
- b) At what point(s) would the torque be the least? Provide mathematical evidence.
- c) Compare the rotation of the crankshaft to how you would use a wrench. Which method would be more efficient? Support your argument mathematically.



Chapter 7 PRACTICE TEST

For questions 1 to 4, choose the best answer.

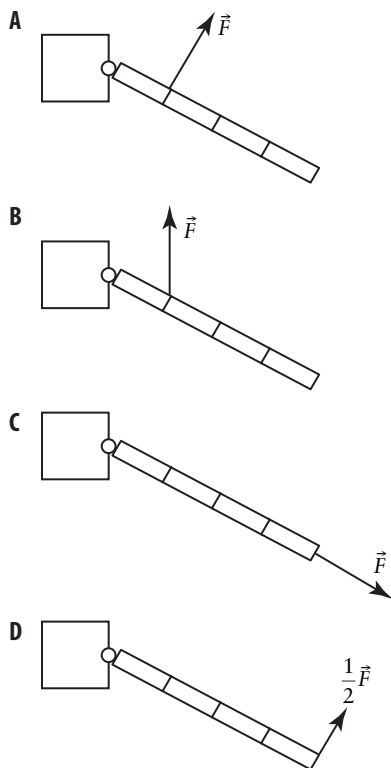
1. If $\vec{u} = [3, -4]$ and $\vec{v} = [6, k]$, and \vec{u} is orthogonal to \vec{v} , then the value of k is

A -8
 B 4.5
 C 8
 D -4.5

2. Which property of vectors is incorrect?

A $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
 B $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
 C $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
 D $\vec{a} \cdot \vec{b} = \vec{a} \times \vec{b}$

3. Each diagram is a top view of a door on hinges. Which configuration yields the greatest torque around the hinges of the door?



4. Which is the correct description of the coordinates of all vectors in three-space that are parallel to the yz -plane?

A x is a constant, $y \in \mathbb{R}$, $z \in \mathbb{R}$
 B $x \in \mathbb{R}$, $y = z$
 C $x \in \mathbb{R}$, y and z are constant
 D $x = 0$, $y \in \mathbb{R}$, $z \in \mathbb{R}$

5. Which vector is *not* orthogonal to $\vec{u} = [-6, 8]$?

A $[8, -6]$
 B $[8, 6]$
 C $[4, -3]$
 D $[-4, -3]$

6. Given $A(1, 3, -7)$ and $B(0, 2, -4)$, the exact magnitude of \overrightarrow{AB} is

A $\sqrt{147}$
 B 3
 C $\sqrt{11}$
 D 3.3

7. Consider the vectors $\vec{u} = [8, 3]$ and $\vec{v} = [2, 7]$.

a) Evaluate $\vec{u} \cdot \vec{v}$.
 b) What is the angle between \vec{u} and \vec{v} ?
 c) Determine $\text{proj}_{\vec{u}} \vec{v}$.
 d) Does $\text{proj}_{\vec{u}} \vec{v} = \text{proj}_{\vec{v}} \vec{u}$? Explain.

8. Use the dot product to determine the total revenue from sales of 100 video game players at \$399 and 240 DVD players at \$129.

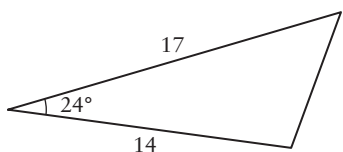
9. Determine $\vec{u} \times \vec{v}$ and $\vec{u} \cdot \vec{v}$ for each pair of vectors.

a) $\vec{u} = [5, 8, 2]$, $\vec{v} = [-7, 3, 6]$
 b) $\vec{u} = [1, 2, -5]$, $\vec{v} = [3, -4, 0]$
 c) $\vec{u} = [-3, 0, 0]$, $\vec{v} = [7, 0, 0]$
 d) $\vec{u} = [1, -9, 7]$, $\vec{v} = [9, 1, 0]$

10. Which vectors in question 9 are

a) collinear?
 b) orthogonal?

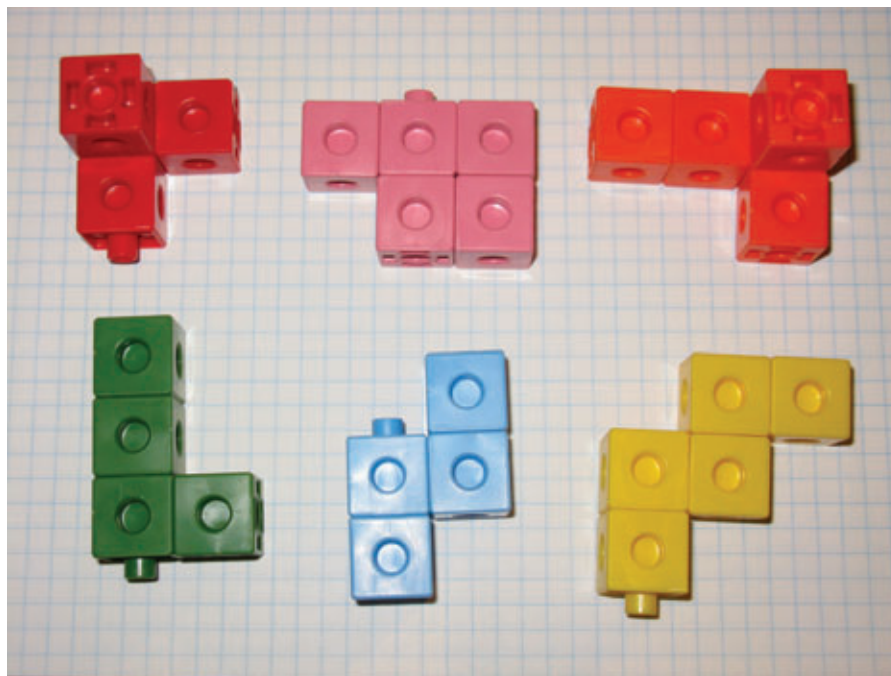
11. a) Describe the coordinates of all points that lie in the yz -plane. Include a sketch in your answer.
 b) Describe the coordinates of all points that are 10 units from the x -axis.
12. a) Determine all unit vectors that are collinear with $\vec{u} = [2, 4, -9]$.
 b) Determine two unit vectors that are orthogonal to $\vec{u} = [2, 4, -9]$.
 c) Explain why there is an infinite number of solutions to part b).
13. Calculate the area of the triangle shown.



14. Determine two unit vectors that are perpendicular to both $\vec{c} = [-5, 6, 2]$ and $\vec{d} = [3, 3, -8]$.
15. Find the area of the parallelogram defined by each pair of vectors.
 a) $\vec{a} = [1, -4]$, $\vec{b} = [3, 5]$
 b) $\vec{u} = [-3, 2, 0]$, $\vec{v} = [6, -4, 2]$
16. Find the volume of the parallelepiped defined by $\vec{u} = [1, 0, -4]$, $\vec{v} = [0, -3, 2]$, and $\vec{w} = [2, -2, 0]$.
17. Given $A(5, 2)$, $B(-3, 6)$, and $C(1, -3)$, determine if $\triangle ABC$ is a right triangle. If it is, identify the right angle.
18. A cart is pulled 5 m up a ramp under a constant force of 40 N. The ramp is inclined at an angle of 30° . How much work is done pulling the cart up the ramp?
19. A jet takes off at a ground velocity of 450 km/h toward the north, climbing at an angle of 12° . A 15-km/h wind is blowing from the east. Determine the resultant air and ground velocities. Include a labelled diagram in your solution.
20. A ramp is inclined at 12° to the horizontal. A cart is pulled 8 m up the ramp by a force of 120 N, at an angle of 10° to the surface of the ramp. Determine the mechanical work done against gravity. Would it be the same as the work done pulling the cart up the ramp? Explain.
21. A force of 70 N acts at an angle of 40° to the horizontal and a force of 125 N acts at an angle of 65° to the horizontal. Calculate the magnitude and direction of the resultant and the equilibrant. Include a diagram in your solution.
22. Resolve the vector $\vec{v} = [7, 9]$ into rectangular components, one of which is in the direction of $\vec{u} = [6, 2]$.
23. A parallelogram has an area of 200 cm^2 . The side lengths are 90 cm and 30 cm. What are the measures of the angles?
24. A wrench is rotated with torque of magnitude $175 \text{ N} \cdot \text{m}$. The force is applied 25 cm from the pivot point, at an angle of 78° . What is the magnitude of the force, to one decimal place?
25. Is the following statement true or false? Explain your decision. "If $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$, then \vec{u} is orthogonal to \vec{v} ."
26. A box weighing 20 N, resting on a ramp, is kept at equilibrium by a 4-N force at an angle of 20° to the ramp, together with a frictional force of 5 N, parallel to the surface of the ramp. Determine the angle of elevation of the ramp.
27. a) Given $\vec{u} = [4, -8, 12]$, $\vec{v} = [-11, 17, 12]$, and $\vec{w} = [15, 10, 8]$, evaluate each expression.
 i) $\vec{u} \times \vec{v} \cdot \vec{w}$ ii) $\vec{u} \cdot \vec{v} \times \vec{w}$
 b) Are your results in part a) equal? Explain.
28. Prove each property of vector addition using 2-D Cartesian vectors.
 a) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associative property)
 b) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, $k \in \mathbb{R}$ (distributive property)

TASK

The Cube Puzzle



Linking cubes can be used to build many 3-D mathematical shapes.

- Use 27 linking cubes to build the six puzzle pieces shown in the photograph. Keeping the puzzle pieces intact, fit them together to form a cube.
- Assign a set of vectors to represent each puzzle piece.
- Use your vector representation to prove that the pieces form a $3 \times 3 \times 3$ cube and that the faces are squares.
- Design a set of eight puzzle pieces that form a $4 \times 4 \times 4$ cube. Trade puzzles with another student. Use vectors to prove that these pieces form a cube.