Chapter 2

Derivatives

In Chapter 1, you learned that instantaneous rate of change is represented by the slope of the tangent at a point on a curve. You also learned that you can determine this value by taking the derivative of the function using the first principles definition of the derivative. However, mathematicians have derived a set of rules for calculating derivatives that make this process more efficient. You will learn to use these rules to quickly determine instantaneous rate of change.

By the end of this chapter you will

- verify the power rule for functions of the form $f(x) = x^n$, where n is a natural number
- verify the constant, constant multiple, sum, and difference rules graphically and numerically, and read and interpret proofs involving $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ of the constant, constant, power, and product rules
- determine algebraically the derivatives of polynomial functions, and use these derivatives to determine the instantaneous rate of change at a point and to determine point(s) at which a given rate of change occurs
- verify that the power rule applies to functions of the form $f(x) = x^n$, where n is a rational number, and verify algebraically the chain rule using monomial functions and the product rule using polynomial functions
- solve problems, using the product and chain rules, involving the derivatives of polynomial functions, rational functions, radical functions, and other simple combinations of functions
- make connections between the concept of motion and the concept of the derivative in a variety of ways
- make connections between the graphical or algebraic representations of derivatives and real-world applications
- solve problems, using the derivative, that involve instantaneous rate of change, including problems arising from real-world applications, given the equation of a function

Prerequisite Skills

Identifying Types of Functions

1. Identify the type of function (polynomial, rational, logarithmic, etc.) represented by each of the following. Justify your response.

a) $f(x) = 5x^3 + 2x - 4$

b) $y = \sin x$

c) $g(x) = -2x^2 + 7x + 1$

d) $f(x) = \sqrt{x}$

e) $h(x) = 5^x$

f) $q(x) = \dfrac{x^2 + 1}{3x - 2}$

g) $y = \log_3 x$

h) $y = (4x + 5)(x^2 - 2)$

Determining Slopes of Perpendicular Lines

2. For each function, state the slope of a line that is perpendicular to it.

a) $y = 2x + 9$ **b)** $y = -5x - 3$

c) $\dfrac{2}{3}x - y + 3 = 9$ **d)** $y = 26$

e) $y = x$ **f)** $x = -3$

Using the Exponent Laws

3. Express each radical as a power.

a) $\sqrt{x}$ **b)** $\sqrt[3]{x}$ **c)** $\left(\sqrt[4]{x}\right)^3$ **d)** $\sqrt[5]{x^2}$

4. Express each term as a power with a negative exponent.

a) $\dfrac{1}{x}$ **b)** $-\dfrac{2}{x^4}$ **c)** $\dfrac{1}{\sqrt{x}}$ **d)** $\dfrac{1}{\left(\sqrt[3]{x}\right)^2}$

5. Express each quotient as a product by using negative exponents.

a) $\dfrac{x^3 - 1}{5x + 2}$ **b)** $\dfrac{3x^4}{\sqrt{5x + 6}}$

c) $\dfrac{(9 - x^2)^3}{(2x + 1)^4}$ **d)** $\dfrac{(x + 3)^2}{\sqrt[3]{1 - 7x^2}}$

Simplify Expressions with Negative Exponents

6. Simplify. Express answers using positive exponents.

a) $(x^2)^{-3}$ **b)** $\dfrac{2x^3 - x^2 + 3x}{x^3}$

c) $\dfrac{x^5}{x^8}$ **d)** $x^{-\frac{1}{2}}(x - 1)$

e) $\dfrac{c^6}{c^{-3}}$ **f)** $(x^2 + 3)^{-\frac{3}{2}}(4x - 3)^2$

Analysing Polynomial Graphs

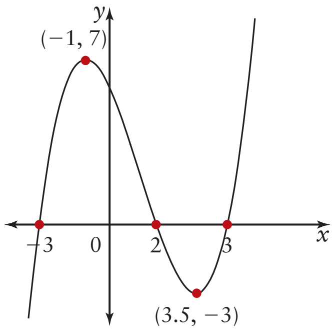
7. Maximum and minimum points and x-intercepts are indicated on each graph. Determine the intervals, or values of x, over which

i) the function is increasing and decreasing

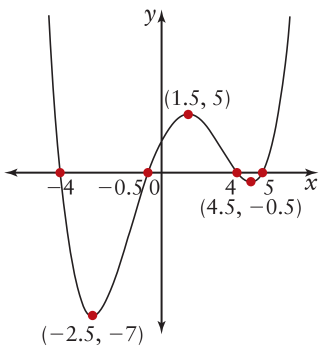
ii) the function is positive and negative

iii) the curve has zero slope, positive slope, and negative slope

a)



b)



Solving Equations

8. Solve.

a) $x^2 - 8x + 12 = 0$ **b)** $4x^2 - 16x - 84 = 0$

c) $5x^2 - 14x + 8 = 0$ **d)** $6x^2 - 5x - 6 = 0$

e) $x^2 + 5x - 4 = 0$ **f)** $2x^2 + 13x - 6 = 0$

g) $4x^2 = 9x - 3$ **h)** $-x^2 + 7x = 1$

Factoring Polynomials

9. Solve using the factor theorem.

a) $x^3 + 3x^2 - 6x - 8 = 0$

b) $2x^3 - x^2 - 5x - 2 = 0$

c) $3x^3 + 4x^2 - 35x - 12 = 0$

d) $5x^3 + 11x^2 - 13x - 3 = 0$

e) $3x^3 + 2x^2 - 7x + 2 = 0$

f) $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$

Simplify Expressions

10. Expand and simplify.

a) $(x^2 + 4)(5) + 2x(5x - 7)$

b) $(9 - 5x^3)(14x) + (-20x^3)(7x^2 + 2)$

c) $(3x^4 - 6x)(6x^2 + 5) + (12x^3 - 6)(2x^3 + 5x)$

11. Factor first and then simplify.

a) $8(x^3 - 1)^5(2x + 7)^3 + 15x^2(x^3 - 1)^4(2x + 7)^4$

b) $6(x^3 + 4)^{-1} - 3x^2(6x - 5)(x^3 + 4)^{-2}$

c) $2x^{\frac{7}{2}} - 2x^{\frac{1}{2}}$

d) $1 + 2x^{-1} + x^{-2}$

12. Determine the value of y when $x = 4$.

a) $y = 6u^2 - 1,\ u = \sqrt{x}$

b) $y = -\dfrac{5}{u^3},\ u = 9 - 2x$

c) $y = -u^2 + 3u + 1,\ u = 5x - 18$

Creating Composite Functions

13. Given $f(x) = x^3 + 1,\ g(x) = \dfrac{1}{x - 2}$, and $h(x) = \sqrt{1 - x^2}$, determine

a) $f \circ g(x)$ **b)** $g \circ h(x)$

c) $h[f(x)]$ **d)** $g[f(x)]$

14. Express each function $h(x)$ as a composition of two simpler functions $f(x)$ and $g(x)$.

a) $h(x) = (2x - 3)^2$ **b)** $h(x) = \sqrt{2 + 4x}$

c) $h(x) = \dfrac{1}{3x^2 - 7x}$ **d)** $h(x) = \dfrac{1}{(x^3 - 4)^2}$

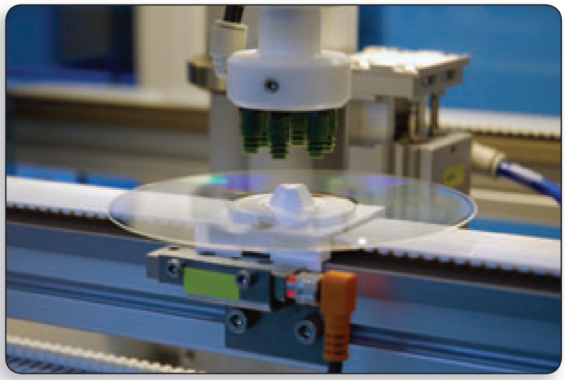
CHAPTER PROBLEM

Five friends in Ottawa have decided to start a fresh juice company with a Canadian flavour. They call their new enterprise Mooses, Gooses, and Juices. The company specializes in making and selling a variety of fresh fruit drinks, smoothies, frozen fruit yogurt, and other fruit snacks. The increased demand for these healthy products has had a positive influence on sales, and business is expanding. How can the young entrepreneurs use derivatives to analyse their costs, revenues, profits, and employee productivity, thereby increasing their chance for success?

2.1 Derivative of a Polynomial Function

There are countless real-world situations that can be modelled by polynomial functions. Consider the following:

- A recording studio determines that the cost, in dollars, of producing x music CDs is modelled by the function $C(x) = 85\ 000 + 25x + 0.015x^2$.
- The amount of fuel required to travel a distance of x kilometres by a vehicle that consumes 8.5 L/100 km is represented by the function $f(x) = 0.085x$.
- The price of a stock, p, in dollars, t years after it began trading on the stock exchange is modelled by the function $p(t) = 0.5t^3 - 5.7t^2 + 12t$.



Whether you are trying to determine the costs and price points that will maximize profits, calculating optimum fuel efficiency, or deciding the best time to buy or sell stocks, calculating instantaneous rates of change heighten the chances of making good decisions.

Instantaneous rate of change in each of the above situations can be found by differentiating the polynomial function. In this section, you will examine five rules for finding the derivative of functions: the constant rule, the power rule, the constant multiple rule, and the sum and difference rules.

Investigate What derivative rules apply to polynomial functions?

Tools

- graphing calculator

As you work through this Investigate, you will explore five rules for finding derivatives. Create a table similar to the one below, and record the findings from your work.

	Original Function	Derivative Function
Constant Rule	$y = c$	
Power Rule	$y = x^n$	
Sum Rule	$y = f(x) + g(x)$	
Difference Rule	$y = f(x) - g(x)$	
Constant Multiple Rule	$y = cf(x)$	

A: The Constant Rule

1. a) Graph the following functions on a graphing calculator. What is the slope of each function at any point on its graph?

i) $y = 2$ **ii)** $y = -3$ **iii)** $y = 0.5$

b) **Reflect** Would the slope of the function $y = c$ be different for any value of $c \in \mathbb{R}$? Explain.

c) **Reflect** Write a rule for the derivative of a constant function $y = c$ for any $c \in \mathbb{R}$.

B: The Power Rule

1. a) Use the nDeriv function to graph the derivative of $y = x$ as follows:

- Enter **Y1** $= x$.

  Move the cursor to **Y2**.

- Press MATH.

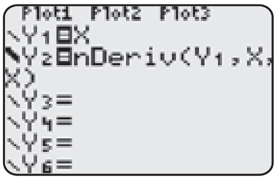
  Select **8:nDeriv(**.

- Press VARS to display the **Y-VARS** menu.

  Select **1: Function**, and then select **1:Y1**.

- Press , X, T, θ, n , X,T,θ,n).

- Select a thick line for **Y2.** Press GRAPH.



b) Write the equation that corresponds to the graph of the derivative.

c) Press 2ND GRAPH on your calculator to display a table of values for the function and its derivative. How does the table of values confirm the equation in part b)?

2. Repeat step 1 for each function.

a) $\mathbf{Y1} = x^2$ **b)** $\mathbf{Y1} = x^3$ **c)** $\mathbf{Y1} = x^4$

3. Complete the following chart based on your findings in steps 1 and 2.

$f(x)$	$f'(x)$
x	
x^2	
x^3	
x^4	

4. a) **Reflect** Is there a pattern in the graphs and the derivatives in step 3 that could be used to formulate a rule for determining the derivative of a power, $f(x) = x^n$?

b) Apply your rule to predict the derivative of $f(x) = x^5$. Verify the accuracy of your rule by repeating step 1 for $\mathbf{Y1} = x^5$.

Technology Tip

The **nDeriv** (numerical derivative) on a graphing calculator can be used to graph the derivative of a given function.

C: The Constant Multiple Rule

1. a) Graph the following functions on the same set of axes.

$\mathbf{Y1} = x \qquad \mathbf{Y2} = 2x \qquad \mathbf{Y3} = 3x \qquad \mathbf{Y4} = 4x$

b) How are the graphs of **Y2, Y3,** and **Y4** related to the graph of **Y1**?

c) State the slope and the equation of the derivative of each function in part a). How are the derivatives related?

d) **Reflect** Predict a rule for the derivative of $y = cx$, for any constant $c \in \mathbb{R}$. Verify your prediction for other values of c.

2. a) Graph the following functions on the same set of axes.

$\mathbf{Y1} = x^2 \qquad \mathbf{Y2} = 2x^2 \qquad \mathbf{Y3} = 3x^2 \qquad \mathbf{Y4} = 4x^2$

b) How are the graphs of **Y2, Y3,** and **Y4** related to the graph of **Y1**?

c) Use what you learned about the power rule to predict the derivative of each function in part a). Confirm your prediction using the **nDeriv** function.

d) **Reflect** Predict a rule for the derivative of $y = cx^2$ for any constant $c \in \mathbb{R}$. Verify your prediction for other values of c.

3. **Reflect** Predict a general rule for the derivative of $f(x) = cx^n$, where c is any real number. Verify your prediction using a graphing calculator.

D: The Sum and Difference Rules

1. a) Given $f(x) = 4x$ and $g(x) = 7x$, determine

i) $f'(x)$, $g'(x)$, and $f'(x) + g'(x)$

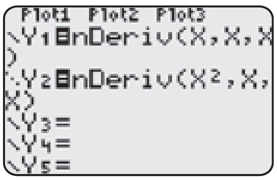
ii) $h(x) = f(x) + g(x)$ and $h'(x)$

b) **Reflect** Compare your results in part a), i) and ii). Use these results to predict the derivative of $h(x) = x + x^2$.

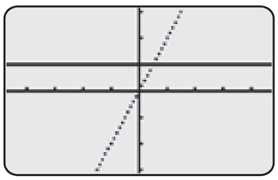
c) Verify the accuracy of your prediction using a graphing calculator.

- Enter the two functions as shown in Screen 1 below.
- Press ZOOM and select **4:ZDecimal** to view Screen 2.
- Press 2ND GRAPH to see the table of values in Screen 3.

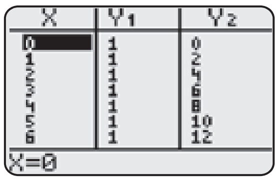
Screen 1



Screen 2



Screen 3

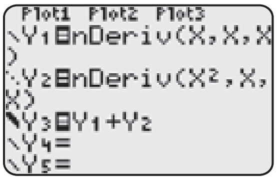


Window variables:
$x \in [-4.7, 4.7]$, $y \in [-3.1, 3.1]$

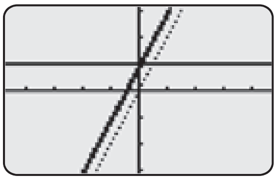
- Enter **Y1** + **Y2** as shown in Screen 4. Select a thick line to graph **Y3**.

View the graph and the table of values, shown in Screens 5 and 6. You will have to scroll to the right to see the **Y3** column.

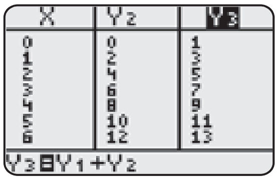
Screen 4



Screen 5



Screen 6



Window variables:
$x \in [-4.7, 4.7]$, $y \in [-3.1, 3.1]$

d) **Reflect** What do the table values in Screen 6 represent?

e) What is the relationship between **Y1**, **Y2**, and **Y3**?

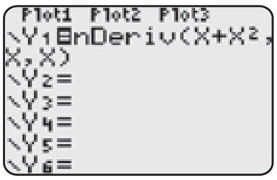
2. Determine the derivative of the sum of the two functions by entering the function shown in Screen 7. View the graph and the table of values shown in Screens 8 and 9.

a) **Reflect** Compare Screens 6 and 9. What do the table values suggest to you?

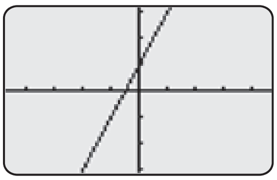
b) Compare Screens 5 and 8. Describe how the graphs are related.

c) How do your results in parts a) and b) compare to the prediction you made in step 1 part b)?

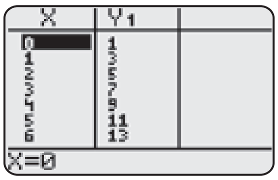
Screen 7



Screen 8



Screen 9



Window variables:
$x \in [-4.7, 4.7]$, $y \in [-3.1, 3.1]$

3. **Reflect** Predict a general rule about the derivative of the sum of two functions.

4. Do you think there is a similar rule for the derivative of the difference of two functions? Describe how you could confirm your prediction.

Derivative Rules

Rule	Lagrange Notation	Leibniz Notation
Constant Rule If $f(x) = c$, where c is a constant, then	$f'(x) = 0$	$\frac{d}{dx}(c) = 0$
Power Rule If $f(x) = x^n$, where n is a positive integer, then	$f'(x) = nx^{n-1}$	$\frac{d}{dx}x^n = nx^{n-1}$
Constant Multiple Rule If $f(x) = cg(x)$, for any constant c, then	$f'(x) = cg'(x)$	$\frac{d}{dx}cg(x) = c\frac{d}{dx}g(x)$
Sum Rule If functions $f(x)$ and $g(x)$ are differentiable, and $h(x) = f(x) + g(x)$, then	$h'(x) = f'(x) + g'(x)$	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
Difference Rule If functions $f(x)$ and $g(x)$ are differentiable, and $h(x) = f(x) - g(x)$, then	$h'(x) = f'(x) - g'(x)$	$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

The five differentiation rules you have explored can be proved using the first principles definition for the derivative. These proofs confirm that the patterns that you observed will apply to all functions. The following proofs of the constant rule and the power rule show how these proofs work. The constant multiple rule, sum, and difference rules can be similarly proved.

The Constant Rule

If $f(x) = c$, where c is a constant, then $f'(x) = 0$.

Proof:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{c - c}{h} && f(x) \text{ is constant, so } f(x+h) \text{ is also equal to } c. \\
&= \lim_{h \to 0} \frac{0}{h} \\
&= \lim_{h \to 0} 0 \\
&= 0
\end{aligned}
$$

To reflect on how the preceding proof works, consider the following:

- Why are $f(x+h)$ and $f(x)$ equal? To support your answer, substitute a particular value for c and work through the steps of the proof again.
- What would happen if you let $h \to 0$ before simplifying the expression?
- Describe how this proof can be verified graphically.

The Power Rule

If $f(x) = x^n$, where n is a natural number, then $f'(x) = nx^{n-1}$.

Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Factor the numerator. Use $a^n - b^n$ with $a = (x+h)$ and $b = x$.

$$= \lim_{h \to 0} \frac{[(x+h) - x][(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}]}{h}$$
$$= \lim_{h \to 0} \frac{[h][(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}]}{h}$$
$$= \lim_{h \to 0}[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}]$$
$$= x^{n-1} + x^{n-2}x + \ldots + xx^{n-2} + x^{n-1}$$
$$= x^{n-1} + x^{n-1} + \ldots + x^{n-1} + x^{n-1}$$ Simplify using laws of exponents.
$$= nx^{n-1}$$ There are n terms.

This proves the power rule for any exponent $n \in \mathbb{N}$. A generalized version of the power rule is also valid. If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$. This proof is explored in later calculus courses.

Reflect on the steps in the preceding proof by considering these questions:

- How does factoring help to prove the power rule?
- How are the laws of exponents applied in the proof?
- Why is it important to first simplify and reduce the expression before letting $h \to 0$?
- Why is it important to state that there are n terms?

Example 1 Justify the Power Rule for Rational Exponents Graphically and Numerically

Tools

- graphing calculator

a) Use the power rule with $n = \frac{1}{2}$ to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

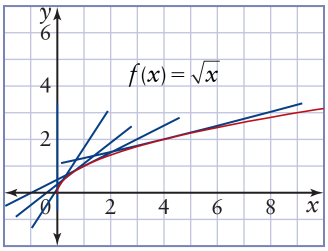
b) Verify this derivative graphically and numerically.

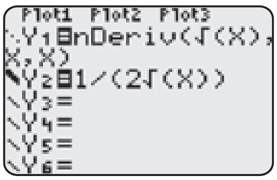
Solution

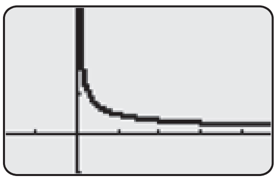
a)
$$
\begin{aligned}
f(x) &= \sqrt{x} \\
&= x^{\frac{1}{2}} \\
f'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} \qquad \text{Apply the power rule.} \\
&= \frac{1}{2}x^{-\frac{1}{2}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

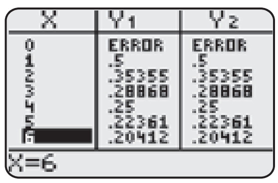
b) Sketch the graph of the function $f(x) = \sqrt{x}$. The graph's domain is restricted because $x \geq 0$. At $x = 0$, the tangent is undefined. For values close to zero, the slope of the tangent is very large. As the x-values increase, the slope of the tangent becomes smaller, approaching zero.

You can verify these results using a graphing calculator. Enter the equations as shown and graph the functions. The graph of the derivative confirms that its value is very large when x is close to zero, and gets smaller, approaching zero, as the x-value increases. Also, the fact that the two functions have identical graphs proves that the functions are equal. This is also confirmed by the table of values.









X	Y1	Y2
0	ERROR	ERROR
1	.5	.5
2	.35355	.35355
3	.28868	.28868
4	.25	.25
5	.22361	.22361
6	.20412	.20412

X=6

Window variables:
$x \in [-1, 4.7]$, $y \in [-1, 3.1]$

Example 2 Rational Exponents and the Power Rule

Determine $\frac{dy}{dx}$ for each function. Express your answers using positive exponents.

a) $y = \sqrt[3]{x}$ **b)** $y = \frac{1}{x}$ **c)** $y = -\frac{1}{x^5}$

Solution

First express the function in the form $y = x^n$, and then differentiate.

a)
$$\begin{aligned} y &= \sqrt[3]{x} \\ &= x^{\frac{1}{3}} \\ \frac{dy}{dx} &= \frac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{3x^{\frac{2}{3}}} \end{aligned}$$

b)
$$\begin{aligned} y &= \frac{1}{x} \\ &= x^{-1} \\ \frac{dy}{dx} &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

c)
$$\begin{aligned} y &= -\frac{1}{x^5} \\ &= (-1)x^{-5} \\ \frac{dy}{dx} &= 5x^{-6} \\ &= \frac{5}{x^6} \end{aligned}$$

Example 3 Apply Strategies to Differentiate Polynomial Functions

Differentiate each function, naming the derivative rule(s) that are being used.

a) $y = 5x^6 - 4x^3 + 6$ **b)** $f(x) = -3x^5 + 8\sqrt{x} - 9.3$

c) $g(x) = (2x - 3)(x + 1)$ **d)** $h(x) = \frac{-8x^6 + 8x^2}{4x^5}$

Solution

a)
$$\begin{aligned} y &= 5x^6 - 4x^3 \\ y' &= 5(6x^5) - 4(3x^2) \\ &= 30x^5 - 12x^2 \end{aligned}$$

Use the difference, constant multiple, and power rules.

b)
$$\begin{aligned} f(x) &= -3x^5 + 8\sqrt{x} - 9.3 \\ &= -3x^5 + 8x^{\frac{1}{2}} - 9.3 \end{aligned}$$

Express the root as a rational exponent.

$$f'(x) = -3(5x^4) + 8\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

Use the sum, constant multiple, power, and constant rules.

$$= -3(5x^4) + 8\left(\frac{1}{2\sqrt{x}}\right)$$

Express as a positive exponent.

$$= -15x^4 + \frac{4}{\sqrt{x}}$$

c) $g(x) = (2x - 3)(x + 1)$
$$= 2x^2 + 2x - 3x - 3$$
$$= 2x^2 - x - 3$$
$g'(x) = 4x - 1$ Use the difference, constant multiple, power, and constant rules.

d) $h(x) = \dfrac{-8x^6 + 8x^2}{4x^5}$
$$= \frac{-8x^6}{4x^5} + \frac{8x^2}{4x^5}$$
$$= -2x + \frac{2}{x^3}$$
$$= -2x + 2x^{-3}$$
$h'(x) = -2 - 6x^{-4}$ Use the difference, constant multiple, and power rules.
$$= -2 - \frac{6}{x^4}$$

Example 4 Apply Derivative Rules to Determine the Equation of a Tangent

Determine the equation of the tangent to the curve $f(x) = 4x^3 + 3x^2 - 5$ at $x = -1$.

Solution

Method 1: ***Use Paper and Pencil***

The derivative represents the slope of the tangent at any value x.

$f(x) = 4x^3 + 3x^2 - 5$
$f'(x) = 12x^2 + 6x - 0$ Use the sum and difference, constant multiple, power, and constant rules.
$$= 12x^2 + 6x$$

Substitute $x = -1$ into the derivative function.

$$f'(-1) = 12(-1)^2 + 6(-1)$$
$$= 6$$

When $x = -1$, the derivative, or slope of the tangent, is 6.

To find the point on the curve corresponding to $x = -1$, substitute $x = -1$ into the original function.

$$f(-1) = 4(-1)^3 + 3(-1)^2 - 5$$
$$= -4 + 3 - 5$$
$$= -6$$

The tangent point is $(-1, -6)$.

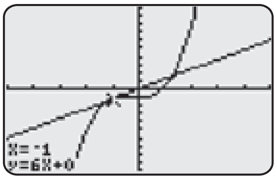
To find the equation of the tangent line, use the point-slope form for the equation of a line, $y - y_1 = m(x - x_1)$, substituting $m = 6$, $x_1 = -1$, and $y_1 = -6$.

$$\begin{aligned} y - (-6) &= 6(x - (-1)) \\ y &= 6x + 6 - 6 \\ y &= 6x \end{aligned}$$

The equation of the tangent line is $y = 6x$.

Method 2: ***Use Technology***

Use the Tangent operation on a graphing calculator to graph the tangent to the function $\mathbf{Y1} = 4x^3 + 3x^2 - 5$ at $x = -1$.



Window variables:
$x \in [-5, 5]$, $y \in [-50, 50]$, Yscl = 5

Tools

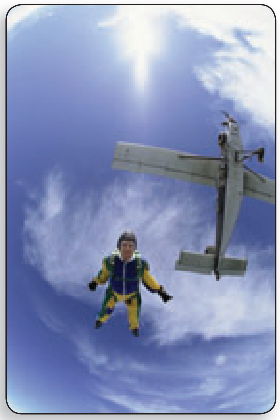
- graphing calculator

Technology Tip

To get the result shown in this example, set the number of decimal places to 1, using the MODE menu on your graphing calculator.

Example 5 Apply Derivative Rules to Solve an Instantaneous Rate of Change Problem

A skydiver jumps out of a plane from a height of 2200 m. The skydiver's height above the ground, in metres, after t seconds is represented by the function $h(t) = 2200 - 4.9t^2$ (assuming air resistance is not a factor). How fast is the skydiver falling after 4 s?



Solution

The instantaneous rate of change of the height of the skydiver at any point in time is represented by the derivative of the height function.

$$\begin{aligned} h(t) &= 2200 - 4.9t^2 \\ h'(t) &= 0 - 4.9(2t) \\ &= -9.8t \end{aligned}$$

Substitute $t = 4$ into the derivative function to find the instantaneous rate of change at 4 s.

$$\begin{aligned} h'(4) &= -9.8(4) \\ &= -39.2 \end{aligned}$$

After 4 s, the skydiver is falling at a rate of 39.2 m/s.

CONNECTIONS

Earth's atmosphere is not empty space. It is filled with a mixture of gases, especially oxygen, that is commonly referred to simply as air. Falling objects are slowed by friction with air molecules. This resisting force is "air resistance." The height function used in this text ignores the effects of air resistance. Taking its effects into account would make these types of problems considerably more difficult.

Example 6 Apply a Strategy to Determine Tangent Points for a Given Slope

Determine the point(s) on the graph of $y = x^2(x + 3)$ where the slope of the tangent is 24.

Solution

Expand the function to put it in the form of a polynomial, and then differentiate.

$$\begin{aligned} y &= x^2(x + 3) \\ &= x^3 + 3x^2 \\ y' &= 3x^2 + 6x \end{aligned}$$

Since the derivative is the slope of the tangent, substitute $y' = 24$ and solve.

$$\begin{aligned} 24 &= 3x^2 + 6x \\ 0 &= 3x^2 + 6x - 24 \\ 0 &= 3(x^2 + 2x - 8) \\ 0 &= 3(x - 2)(x + 4) \end{aligned}$$

The equation is true when $x = 2$ or $x = -4$.

Determine y by substituting the x-values into the original function.

For $x = 2$

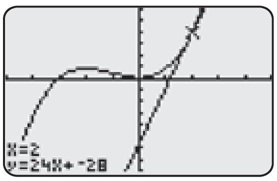
$$\begin{aligned} y &= 2^3 + 3(2)^2 \\ &= 8 + 12 \\ &= 20 \end{aligned}$$

For $x = -4$,

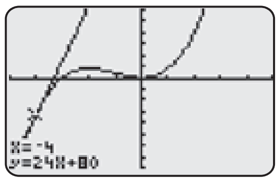
$$\begin{aligned} y &= (-4)^3 + 3(-4)^2 \\ &= -64 + 48 \\ &= -16 \end{aligned}$$

The two tangent points at which the slope is 24 are (2, 20) and $(-4, -16)$.

Use the Tangent function on a graphing calculator to confirm these points. Enter the two x-values, noting the slope in the equation of the tangent at the bottom of the screens.



Window variables:
$x \in [-5, 5]$, $y \in [-30, 20]$, Yscl = 4



Window variables:
$x \in [-5, 5]$, $y \in [-40, 30]$, Yscl = 5

KEY CONCEPTS

- Derivative rules simplify the process of differentiating polynomial functions.
- To differentiate a radical, first express it as a power with a rational exponent (e.g., $\sqrt[3]{x} = x^{\frac{1}{3}}$).
- To differentiate a power of x that is in the denominator, first express it as a power with a negative exponent (e.g., $\frac{1}{x^2} = x^{-2}$).

Communicate Your Understanding

C1 How can you use slopes to prove that the derivative of a constant is zero?

C2 How can the sum and difference rules help differentiate polynomial functions?

C3 Why can you extend the sum and difference rules to three or more functions that are added and subtracted? Describe an example to support your answer.

C4 What is the difference between proving a derivative rule and showing that it works for certain functions?

A Practise

1. Which of the following functions have a derivative of zero?

a) $y = 9.8$ **b)** $y = 11$

c) $y = -4 + x$ **d)** $y = \frac{5}{9}x$

e) $y = \sqrt{7}$ **f)** $y = x$

g) $y = \frac{3}{4}$ **h)** $y = -2.8\pi$

2. For each function, determine $\frac{dy}{dx}$.

a) $y = x$ **b)** $y = \frac{1}{4}x^2$

c) $y = x^5$ **d)** $y = -3x^4$

e) $y = 1.5x^3$ **f)** $y = \sqrt[5]{x^3}$

g) $y = \frac{5}{x}$ **h)** $y = \frac{4}{\sqrt{x}}$

3. Determine the slope of the tangent to the graph of each function at the indicated value.

a) $y = 6$, $x = 12$

b) $f(x) = 2x^5$, $x = \sqrt{3}$

c) $g(x) = -\frac{3}{\sqrt{x}}$, $x = 4$

d) $h(t) = -4.9t^2$, $t = 3.5$

e) $A(r) = \pi r^2$, $r = \frac{3}{4}$

f) $y = \frac{1}{3x}$, $x = -2$

4. Determine the derivative of each function. State the derivative rules used.

a) $f(x) = 2x^2 + x^3$ **b)** $y = \frac{4}{5}x^5 - 3x$

c) $h(t) = -1.1t^4 + 78$ **d)** $V(r) = \frac{4}{3}\pi r^3$

e) $p(a) = \frac{a^5}{15} - 2\sqrt{a}$ **f)** $k(s) = -\frac{1}{s^2} + 7s^4$

B Connect and Apply

5. a) Determine the point at which the slope of the tangent to each parabola is zero.

i) $y = 6x^2 - 3x + 4$

ii) $y = -x^2 + 5x - 1$

iii) $y = \frac{3}{4}x^2 + 2x + 3x^2 + 2x + 3$

b) **Use Technology** Graph each parabola in part a). What does the point found in part a) correspond to on each of these graphs?

6. Simplify, and then differentiate.

a) $f(x) = \frac{10x^4 - 6x^3}{2x^2}$

b) $g(x) = (3x + 4)(2x - 1)$

c) $p(x) = \frac{x^8 - 4x^6 + 2x^3}{4x^3}$

d) $f(x) = (5x + 2)^2$

7. a) Describe the steps you would follow to determine the equation of a tangent to a curve at a given x-value.

b) How is the derivative used to determine the tangent point when the slope of the tangent is known?

8. Consider the function $f(x) = (2x - 1)^2(x + 3)$.

a) Explain why the derivative rules from this section cannot be used to differentiate the function in the form in which it appears here.

b) Describe what needs to be done to $f(x)$ before it can be differentiated.

c) Apply the method you described in part b), and then differentiate $f(x)$.

9. **Use Technology** Verify algebraically, numerically, and graphically that the derivative of $y = \frac{1}{x}$ is $\frac{dy}{dx} = -\frac{1}{x^2}$.

10. The amount of water flowing into two barrels is represented by the functions $f(t)$ and $g(t)$. Explain what $f'(t)$, $g'(t)$, $f'(t) + g'(t)$, and $(f + g)'(t)$ represent. Explain how you can use this context to verify the sum rule.

11. A skydiver jumps out of a plane that is flying 2500 m above the ground. The skydiver's height above the ground, in metres, after t seconds is $h(t) = 2500 - 4.9t^2$.

a) Determine the rate of change of the height of the skydiver after 5 s.

b) The skydiver's parachute opens at 1000 m above the ground. After how many seconds does this happen?

c) What is the rate of change of the height of the skydiver at the time found in part b)?

12. The following chart lists the acceleration due to gravity on several planets.

Planet	Acceleration Due to Gravity (m/s^2)
Earth	9.8
Venus	8.9
Mars	3.7
Saturn	10.5
Neptune	11.2

The height of a free-falling object on any planet is represented by the function $h(t) = -0.5gt^2 + k$, where h is the height, in metres, t is time, in seconds, $t \geq 0$, g is the planet's acceleration due to gravity, in metres per second squared, and k is the height, in metres, from which the object is dropped. Suppose a rock is dropped from a height of 250 m on each planet listed in the table. Use derivatives to determine the instantaneous rate of change of the height of the rock on each planet after 4 s.

13. a) Determine the slope of the tangent to the graph of $y = -6x^4 + 2x^3 + 5$ at the point $(-1, -3)$.

b) Determine the equation of the tangent at this point.

c) **Use Technology** Confirm your equation using a graphing calculator.

14. a) Determine the slope of the tangent to the graph of $y = -1.5x^3 + 3x - 2$ at the point $(2, -8)$.

b) Determine the equation of the tangent at this point.

c) Use Technology Confirm your equation using a graphing calculator.

15. A flaming arrow is shot into the air to mark the beginning of a festival. Its height, in metres, after t seconds is modelled by the function $h(t) = -4.9t^2 + 24.5t + 2$.

a) Determine the height of the arrow at 2 s.

b) Determine the rate of change of the height of the arrow at 1, 2, 4, and 5 s.

c) What happens at 5 s?

d) How long does it take the arrow to return to the ground?

e) How fast is the arrow travelling when it hits the ground? Explain how you arrived at your answer.

f) Graph the function. Use the graph to confirm your answers in parts a) to e).

16. a) Determine the coordinates of the point(s) on the graph of $f(x) = x^3 - 7x$ where the slope of the tangent is 5.

b) Determine the equation(s) of the tangent(s) to the graph of $f(x)$ at the point(s) found in part a).

c) Use Technology Confirm your equation(s) using a graphing calculator.

17. a) Find the values of x at which the tangents to the graphs of $f(x) = 2x^2$ and $g(x) = x^3$ have the same slope.

b) Determine the equations of the tangent lines to each curve at the points found in part a).

c) Use Technology Confirm your equations using a graphing calculator.

18. Use the first principles definition of the derivative and the properties of limits to prove the sum rule: $h'(x) = f'(x) + g'(x)$.

19. Chapter Problem The cost, in dollars, of producing x frozen fruit yogurt bars can be modelled by the function $C(x) = 3450 + 1.5x - 0.0001x^2$, $0 \le x \le 5000$. The revenue from selling x yogurt bars is $R(x) = 3.25x$.

a) Determine the cost of producing 1000 frozen fruit yogurt bars. What is the revenue generated from selling this many bars?

b) Compare the values for $C'(1000)$ and $C'(3000)$. What information do these values provide?

c) When is $C'(x) = 0$? Explain why this is impossible.

d) Determine $R'(x)$. What does this value represent?

e) The profit function, $P(x)$, is the difference of the revenue and cost functions. Determine the equation for $P(x)$.

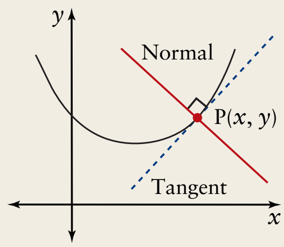
f) When is the profit function positive? When is it negative? What important information does this provide the owners?

20. a) Determine the slope of the tangent to the curve $f(x) = -2x^3 + 5x^2 - x + 3$ at $x = 2$.

b) Determine the equation of the normal to $f(x)$ at $x = 2$.

CONNECTIONS

The normal to a curve at a point (x, y) is the line perpendicular to, and intersecting, the curve's tangent at that point.



21. a) Determine the slope of the tangent to the curve $f(x) = -4x^3 + \dfrac{3}{x} + \sqrt{x} - 2$ at $x = 1$.

b) Determine the equation of the normal to $f(x)$ at $x = 1$.

22. a) Determine the equation of the tangent to the graph of $f(x) = \left(2 - \sqrt{x}\right)^2$ at $x = 9$.

b) Use Technology Confirm your equation using a graphing calculator.

23. a) Determine the equation of the tangent to the graph of $g(x) = \left(\frac{4}{x^3} + 1\right)(x - 3)$ at $x = -1$.

b) Use Technology Confirm your equation using a graphing calculator.

24. a) Determine the equations of the tangents to the points on the curve $y = -x^4 + 8x^2$ such that the tangent lines are perpendicular to the line $x = 1$.

b) Use Technology Verify your solution using a graphing calculator.

25. a) Show that there are no tangents to the curve $y = 6x^3 + 2x^2$ that have a slope of -5.

b) Use Technology Verify your solution using a graphing calculator.

Achievement Check

26. The population of a bacteria colony is modelled by the function $p(t) = 200 + 20t - t^2$, where t is time, in hours, $t \geq 0$, and p is the number of bacteria, in thousands.

a) Determine the growth rate of the bacteria population at each of the following times.

i) 3 h **ii)** 8 h **iii)** 13 h **iv)** 18 h

b) What are the implications of the growth rates in part a)?

c) Determine the equation of the tangent to $p(t)$ at the point corresponding to $t = 8$.

d) When does the bacteria population stop growing? What is the population at this time?

e) Graph the growth function and its derivative. Describe how each graph reflects the rate of change of the bacteria population.

f) Determine the time interval over which the bacteria population

i) increases **ii)** decreases

C Extend and Challenge

27. Determine the values of a and b for the function $f(x) = ax^3 + bx^2 + 3x - 2$ given that $f(2) = 10$ and $f'(-1) = 14$.

28. Determine the equations of two lines that pass through the point $(1, -5)$ and are tangent to the graph of $y = x^2 - 2$.

29. a) Determine the equations of the tangents to the cubic function $y = 2x^3 - 3x^2 - 11x + 8$ at the points where $y = 2$.

b) Use Technology Verify your solution using a graphing calculator.

30. Show that there is no polynomial function that has a derivative of x^{-1}.

31. Math Contest Consider the polynomials $p(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$ and $q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$, where $a_m \neq 0 \neq b_n$ and $m, n \geq 1$. If the equation $p(x) - q(x) = 0$ has $m + n$ real roots, which of the following must be true?

i) $p(x)$ and $q(x)$ have no common factors.

ii) The equation $p'(x) - q'(x) = 0$ has exactly $m + n - 1$ real roots.

iii) The equation $p(x) - q(x) = 0$ has infinitely many real roots.

A i) only **B** ii) only

C i) and ii) only **D** iii) only

E i) and iii) only

32. Math Contest If p and q are two polynomials such that $p'(x) = q'(x)$ for all real x with $p(0) = 1$ and $q(0) = 2$, then which of the following best describes the intersection of the graphs of $y = p(x)$ and $y = q(x)$?

A They intersect at exactly one point.

B They intersect at at least one point.

C They intersect at at most one point.

D They intersect at more than one point.

E They do not intersect.

Extension Problem Solving with a Computer Algebra System

You can use a Computer Algebra System (CAS) to solve exercise 16 from Section 2.1.

16. a) Determine the coordinates of the point(s) on the graph of $f(x) = x^3 - 7x$ where the slope of the tangent is 5.

b) Determine the equation(s) of the tangent(s) to the graph of $f(x)$ at the point(s) found in part a).

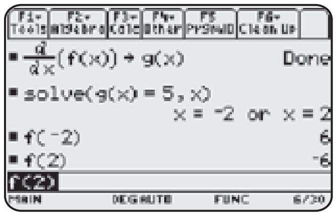
Solution:

a) Turn on the CAS. If necessary press HOME to display the HOME screen. Ensure that the following parameters are correctly set.

- Clear the CAS variables by pressing 2ND F1 to access the F6 menu. Select **2:NewProb**. Press ENTER.
- Press MODE. Scroll down to **Exact/Approx**, and ensure that **AUTO** is selected. Enter the function and store it as $f(x)$.
- From the F4 menu, select **1:Define**. Enter the function $f(x) = x^3 - 7x$ and press ENTER.

Determine the derivative of the function, and the value(s) of x when the derivative is 5.

- From the F3 menu, select **1:*d*(differentiate**. Enter $f(x)$, x).
- Press STO, and enter $g(x)$. Press ENTER.
- From the F2 menu, select **1:solve(**. Enter $g(x) = 5$, x), and press ENTER.



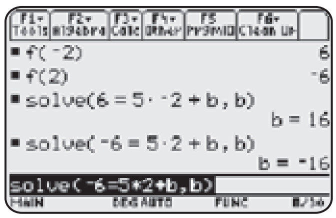
The slope of the tangent is equal to 5 when $x = -2$ and $x = 2$.

Determine $f(-2)$ and $f(2)$.

- Enter $f(-2)$, and press ENTER. Similarly, enter $f(2)$, and press ENTER.

The slope of the tangent is equal to 5 at the tangent points $(-2, 6)$ and $(2, -6)$.

b) The equation of a straight line is $y = mx + b$. Substitute $m = 5$, $x = -2$, and $y = 6$, and solve for b. When this is entered, $b = 16$. The equation of the tangent to the curve at $(-2, 6)$ is $y = 5x + 16$.



In a similar manner, substitute $m = 5$, $x = 2$, and $y = -6$, and solve for b. When this is entered, $b = -16$. The equation of the tangent to the curve at $(2, -6)$ is $y = 5x - 16$.

To view the form of the derivative, enter $g(x)$, and press ENTER.

2.2 The Product Rule

The manager of a miniature golf course is planning to raise the ticket price per game. At the current price of \$6.50, an average of 81 rounds is played each day. The manager's research suggests that for every \$0.50 increase in price, an average of four fewer games will be played each day. Based on this information, the revenue from sales is represented by the function $R(n) = (81 - 4n)(6.50 + 0.50n)$, where n represents the number of \$0.50 increases in the price.

The manager can use derivatives to determine the price that will provide maximum revenue. To do so, he could use the differentiation rules explored in Section 2.1, but only after expanding and simplifying the expression. In this section, you will explore the product rule, which is a more efficient method for differentiating a function like the one described above. Not only is this method more efficient in these cases, but it can also be used to differentiate functions that cannot be expanded and simplified.

Investigate Does the derivative of a product of two functions equal the product of their derivatives?

Complete each step for the functions $f(x) = x^3$ and $g(x) = x^4$.

1. Determine the following:

a) $f'(x)$ and $g'(x)$

b) $f'(x) \times g'(x)$, in simplified form

2. Determine the following:

a) $f(x) \times g(x)$, in simplified form

b) $\frac{d}{dx}[f(x) \times g(x)]$, in simplified form

3. **Reflect** Does the above result verify that $f'(x) \times g'(x)$ does not equal $\frac{d}{dx}[f(x)g(x)]$.

4. **Reflect** You know that the derivative of x^7 is $7x^6$. Can you combine $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ in an expression that will give the result $7x^6$? Do you think this will always work?

The Product Rule

If $P(x) = f(x)g(x)$, where $f(x)$ and $g(x)$ are differentiable functions, then $P'(x) = f'(x)g(x) + f(x)g'(x)$.

In Leibniz notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

CONNECTIONS

A simple way to remember the product rule is to express the product, P, in terms of f (first term) and s (second term): $P = fs$.
The product rule then becomes $P' = f's + fs'$.
In words: "the derivative of the first times the second, plus the first times the derivative of the second."

Proof:

Use the first principles definition for $P(x) = f(x)g(x)$.

$$P'(x) = \lim_{h\to 0}\frac{P(x+h) - P(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$P(x+h) = f(x+h)g(x+h)$.

Reorganize this expression as two fractions, one involving $f'(x)$ and one involving $g'(x)$, by adding and subtracting $f(x+h)g(x)$ in the numerator.

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{[f(x+h)g(x+h) - f(x+h)g(x)]}{h} + \lim_{h\to 0}\frac{[f(x+h)g(x) - f(x)g(x)]}{h}$$

Separate the limit into two fractions.

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0}\frac{g(x)[f(x+h) - f(x)]}{h}$$

Common factor each numerator.

$$= \lim_{h\to 0}f(x+h)\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0}g(x)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Apply limit rules.

$$= f(x)g'(x) + g(x)f'(x)$$

$$= f'(x)g(x) + f(x)g'(x)$$

Recognize the expressions for $g'(x)$ and $f'(x)$ and rearrange.

Notice that $\lim_{h\to 0} g(x) = g(x)$, since $g(x)$ is a constant with respect to h, and that $\lim_{h\to 0} f(x+h) = f(x)$, since $f(x)$ is continuous.

Reflect on the steps in the preceding proof by considering these questions:

- What special form of zero is added to the numerator in the proof?
- Why is it necessary to add this form of zero?
- How is factoring used in the proof?
- Why is it important to separate the expression into individual limits?

Example 1 Apply the Product Rule

Use the product rule to differentiate each function.

a) $p(x) = (3x - 5)(x^2 + 1)$

Check your result algebraically.

b) $y = (2x + 3)(1 - x)$

Check your result graphically and numerically using technology.

Solution

a) $p(x) = (3x - 5)(x^2 + 1)$

Let $f(x) = 3x - 5$ and $g(x) = x^2 + 1$.

Then $f'(x) = 3$ and $g'(x) = 2x$.

Apply the product rule, $p'(x) = f'(x)g(x) + f(x)g'(x)$.

$$\begin{aligned} p'(x) &= 3(x^2 + 1) + (3x - 5)(2x) \\ &= 3x^2 + 3 + 6x^2 - 10x \\ &= 9x^2 - 10x + 3 \end{aligned}$$

Checking the solution algebraically,

$$\begin{aligned} p(x) &= (3x - 5)(x^2 + 1) \\ &= 3x^3 + 3x - 5x^2 - 5 \\ &= 3x^3 - 5x^2 + 3x - 5 \\ p'(x) &= 9x^2 - 10x + 3 \end{aligned}$$

b) $y = (2x + 3)(1 - x)$

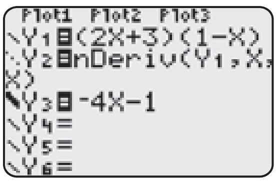
Let $f(x) = 2x + 3$ and $g(x) = 1 - x$.

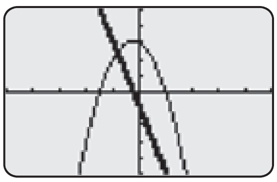
Then $f'(x) = 2$ and $g'(x) = -1$.

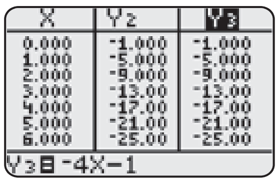
Apply the product rule, $y' = f'(x)g(x) + f(x)g'(x)$

$$\begin{aligned} y' &= 2(1 - x) + (2x + 3)(-1) \\ &= 2 - 2x - 2x - 3 \\ &= -4x - 1 \end{aligned}$$

To check the solution using a graphing calculator, enter the functions as shown and press GRAPH. The straight line represents both **Y2** and **Y3,** confirming that $-4x - 1$ is equal to the derivative. The table of values confirms this result numerically.







Example 2 Find Equations of Tangents Using the Product Rule

Determine the equation of the tangent to the curve $y = (x^2 - 1)(x^2 - 2x + 1)$ at $x = 2$. Use technology to confirm your solution.

Solution

Apply the product rule.

$$\frac{dy}{dx} = \left[\frac{d}{dx}(x^2 - 1)\right](x^2 - 2x + 1) + (x^2 - 1)\left[\frac{d}{dx}(x^2 - 2x + 1)\right]$$

$$\frac{dy}{dx} = (2x)(x^2 - 2x + 1) + (x^2 - 1)(2x - 2)$$

There is no need to simplify the expression to calculate the slope. It is quicker to simply substitute $x = 2$ and then simplify the expression.

$$\begin{aligned} m &= \left.\frac{dy}{dx}\right|_{x=2} \\ &= 2(2)[2^2 - 2(2) + 1] + (2^2 - 1)[2(2) - 2] \\ &= 10 \end{aligned}$$

Determine the y-coordinate of the tangent point by substituting the x-value into the original function.

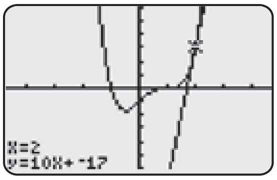
$$\begin{aligned} y &= f(2) \\ &= (2^2 - 1)(2^2 - 2(2) + 1) \\ &= 3 \end{aligned}$$

The tangent point is (2, 3).

The equation of the line through (2, 3) with slope 10 is

$$\begin{aligned} y - 3 &= 10(x - 2) \\ y &= 10x - 17 \end{aligned}$$

This answer can be confirmed using the Tangent function on a graphing calculator.



Example 3 Apply Mathematical Modelling to Develop a Revenue Function

The student council is organizing its annual trip to an out-of-town concert. For the past three years, the cost of the trip was \$140 per person. At this price, all 200 seats on the train were filled. This year, the student council plans to increase the price of the trip. Based on a student survey, the council estimates that for every \$10 increase in price, 5 fewer students will attend the concert.

a) Write an equation to represent revenue, R, as a function of the number of \$10 increases, n.

b) Determine an expression, in simplified form, for $\frac{dR}{dn}$ and interpret its meaning for this situation.

c) What is the rate of change in revenue when the price of the trip is \$200? How many students will attend the concert at this price?

Solution

a) Revenue, R, is the product of the price per student and the number of students attending. So, $R = \text{price} \times \text{number of students}$. The following table illustrates how each \$10 increase will affect the price, the number of students attending, and the revenue.

Number of \$10 Increases, *n*	Cost/Student (\$)	Number of Students	Revenue (\$)
0	140	200	140×200
1	$140 + 10(1)$	$200 - 5(1)$	$(140 + 10(1))(200 - 5(1))$
2	$140 + 10(2)$	$200 - 5(2)$	$(140 + 10(2))(200 - 5(2))$
3	$140 + 10(3)$	$200 - 5(3)$	$(140 + 10(3))(200 - 5(3))$
⋮	⋮	⋮	⋮
n	$140 + 10n$	$200 - 5n$	$(140 + 10n)(200 - 5n)$

The revenue function is $R(n) = (140 + 10n)(200 - 5n)$.

b) Apply the product rule.

$$\begin{aligned}\frac{dR}{dn} &= \left[\frac{d}{dn}(140 + 10n)\right](200 - 5n) + (140 + 10n)\left[\frac{d}{dn}(200 - 5n)\right]\\ &= (10)\,(200 - 5n) + (140 + 10n)(-5)\\ &= 2000 - 50n - 700 - 50n\\ &= 1300 - 100n\end{aligned}$$

$\frac{dR}{dn} = 1300 - 100n$ represents the rate of change in revenue for each \$10 increase.

c) The price of the trip is \$200 when there are 6 increases of \$10. Evaluate the derivative for $n = 6$.

$$\begin{aligned}R'(6) &= 1300 - 100(6)\\ &= 700\end{aligned}$$

When the price of the trip is \$200, the rate of increase in revenue is \$700 per price increase. The number of students attending when $n = 6$ is $200 - 5(6) = 170$. Thus, 170 students will attend the concert when the price is \$200.

KEY CONCEPTS

- The Product Rule

  If $h(x) = f(x)g(x)$, where $f(x)$ and $g(x)$ are differentiable functions, then $h'(x) = f'(x)g(x) + f(x)g'(x)$.

  In Leibniz notation,

  $$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

Communicate Your Understanding

C1 Why is the following statement false? "The product of the derivatives of two functions is equal to the derivative of the product of the two functions." Support your answer with an example.

C2 What response would you give to a classmate who makes the statement, "I don't need to learn the product rule because I can always expand each product and then differentiate"?

C3 Why is it unnecessary to simplify the derivative before substituting the value of x to calculate the slope of a tangent?

C4 How can the product rule be used to differentiate $(2x-5)^3$?

A Practise

1. Differentiate each function using the two different methods presented below. Compare your answers in each case.

i) Expand and simplify each binomial product and then differentiate.

ii) Apply the product rule and then simplify.

a) $f(x) = (x+4)(2x-1)$

b) $h(x) = (5x-3)(1-2x)$

c) $h(x) = (-x+1)(3x+8)$

d) $g(x) = (2x-1)(4-3x)$

2. Use the product rule to differentiate each of the following functions.

a) $f(x) = (5x+2)(8x-6)$

b) $h(t) = (-t+4)(2t+1)$

c) $p(x) = (-2x+3)(x-9)$

d) $g(x) = (x^2+2)(4x-5)$

e) $f(x) = (1-x)(x^2-5)$

f) $h(t) = (t^2+3)(3t^2-7)$

3. Differentiate.

a) $M(u) = (1-4u^2)(u+2)$

b) $g(x) = (-x+3)(x-10)$

c) $p(n) = (5n+1)(-n^2+3)$

d) $A(r) = (1+2r)(2r^2-6)$

e) $b(k) = (-0.2k+4)(2-k)$

4. The derivative of the function $h(x) = f(x)g(x)$ is given in the form $h'(x) = f'(x)g(x) + f(x)g'(x)$. Determine $f(x)$ and $g(x)$ for each derivative.

a) $h'(x) = (10x)(21-3x) + (5x^2+7)(-3)$

b) $h'(x) = (-12x^2+8)(2x^2-4x) + (-4x^3+8x)(4x-4)$

c) $h'(x) = (6x^2-1)(0.5x^2+x) + (2x^3-x)(x+1)$

d) $h'(x) = (-3x^3+6)\left(7x-\frac{2}{3}x^2\right) + \left(-\frac{3}{4}x^4+6x\right)\left(7-\frac{4}{3}x\right)$

B Connect and Apply

5. Determine $f'(-2)$ for each function.

a) $f(x) = (x^2 - 2x)(3x + 1)$

b) $f(x) = (1 - x^3)(-x^2 + 2)$

c) $f(x) = (3x - 1)(2x + 5)$

d) $f(x) = (-x^2 + x)(5x^2 - 1)$

e) $f(x) = (2x - x^2)(7x + 4)$

f) $f(x) = (-5x^3 + x)(-x + 2)$

6. Determine the equation of the tangent to each curve at the indicated value.

a) $f(x) = (x^2 - 3)(x^2 + 1)$, $x = -4$

b) $g(x) = (2x^2 - 1)(-x^2 + 3)$, $x = 2$

c) $h(x) = (x^4 + 4)(2x^2 - 6)$, $x = -1$

d) $p(x) = (-x^3 + 2)(4x^2 - 3)$, $x = 3$

7. Determine the point(s) on each curve that correspond to the given slope of the tangent.

a) $y = (-4x + 3)(x + 3)$, $m = 0$

b) $y = (5x + 7)(2x - 9)$, $m = \frac{2}{5}$

c) $y = (2x - 1)(-4 + x^2)$, $m = 3$

d) $y = (x^2 - 2)(2x + 1)$, $m = -2$

8. Differentiate.

a) $y = (5x^2 - x + 1)(x + 2)$

b) $y = (1 - 2x^3 + x^2)\left(\frac{1}{x^3} + 1\right)$

c) $y = -x^2(4x - 1)(x^3 + 2x + 3)$

d) $y = \left(2x^2 - \sqrt{x}\right)^2$

e) $y = (-3x^2 + x + 1)^2$

9. Recall the problem introduced at the start of this section: The manager of a miniature golf course is planning to raise the ticket price per game. At the current price of \$6.50, an average of 81 rounds is played each day. The manager's research suggests that for every \$0.50 increase in price, an average of 4 fewer games will be played each day. The revenue from sales is represented by the function $R(n) = (81 - 4n)(6.50 + 0.50n)$, where n represents the number of \$0.50 increases in the price. By finding the derivative, the manager can determine the price that will provide the maximum revenue.

a) Describe two methods that could be used to determine $R'(n)$. Apply your methods and then compare the answers. Are they the same?

b) Evaluate $R'(4)$. What information does this value give the manager?

c) Determine when $R'(n) = 0$. What information does this give the manager?

d) Sketch the graph of $R(n)$. Determine the maximum revenue from the graph. Compare this value to your answer in part c). What do you notice?

e) Describe how the derivative could be used to find the value in part d).

10. Some years ago, an orchard owner began planting ten saplings each year. The saplings have begun to mature, and the orchard is expanding at a rate of ten fruit-producing trees per year. There are currently 120 trees in the apple orchard, producing an average yield of 280 apples per tree. Also, because of improved soil conditions, the average annual yield has been increasing at a rate of 15 apples per tree.

a) Write an equation to represent the annual yield, Y, as a function of t years from now.

b) Determine $Y'(2)$ and interpret its meaning for this situation.

c) Evaluate $Y'(6)$. Explain what this value represents.

11. The owner of a local hair salon is planning to raise the price for a haircut and blow dry. The current rate is \$30 for this service, with the salon averaging 550 clients a month. A survey indicates that the salon will lose 5 clients for every incremental price increase of \$2.50.

a) Write an equation that models the salon's monthly revenue, R, as a function of x, where x represents the number of \$2.50 increases in the price.

b) Use the product rule to determine $R'(x)$.

c) Evaluate $R'(3)$ and interpret its meaning for this situation.

d) Solve $R'(x) = 0$.

e) Explain how the owner can use the result of part d). Justify your answer graphically.

12. a) Determine the equation of the tangent to the graph of $f(x) = 2x^2(x^2 + 2x)(x - 1)$ at the point $(-1, 4)$.

b) **Use Technology** Use a graphing calculator to sketch a graph of the function and the tangent.

13. a) Determine the points on the graph of $f(x) = (3x - 2x^2)^2$ where the tangent line is parallel to the x-axis.

b) **Use Technology** Use a graphing calculator to sketch a graph of the function and the tangents.

14. The gas tank of a parked pickup truck develops a leak. The amount of gas, in litres, remaining in the tank after t hours is represented by the function

$$V(t) = 90\left(1 - \frac{t}{18}\right)^2, 0 \le t \le 18.$$

a) How much gas was in the tank when the leak developed?

b) How fast is the gas leaking from the tank after 12 h?

c) How fast is the gas leaking from the tank when there are 40 L of gas in the tank?

15. The fish population in a lake can be modelled by the function $p(t) = 15(t^2 + 30)(t + 8)$, where t is time, in years, from now.

a) What is the current fish population?

b) Determine the rate of change of the fish population in 3 years.

c) Determine the rate of change of the fish population when there are 5000 fish in the lake.

d) When will the fish population double from its current level? What is the rate of change in the fish population at this time?

16. **Chapter Problem** The owners of Mooses, Gooses, and Juices are considering an increase in the price of their frozen fruit smoothies. At the current price of \$1.75 they sell on average 150 smoothies a day. Their research shows that every \$0.25 increase in the price of a smoothie will result in a decrease of 10 sales per day.

a) Write an equation to represent the revenue, R, as a function of n, the number of \$0.25 price increases.

b) Compare the rate of change of revenue when the price increases by \$0.25, \$0.75, \$1.00, \$1.25, and \$1.50.

c) When is $R'(n) = 0$? Interpret the meaning of this value for this situation.

d) If it costs \$0.75 to make one smoothie, what will be the rate of change in profit for each price increase indicated in part b)?

e) What price will result in a maximum profit? Justify your answer. How can this be confirmed using derivatives?

f) Compare your answers in part b) and part d). Give reasons for any similarities or differences.

C Extend and Challenge

17. a) Use the product rule to differentiate each function. Do not simplify your final answer.

i) $y = (x^2 - 3x)^2$

ii) $y = (2x^3 + x)^2$

iii) $y = (-x^4 + 5x^2)^2$

What do you notice about the two parts that make up the derivative?

b) Make a conjecture about $\frac{d}{dx}[f(x)]^2$.

c) Verify whether your conjecture is true by replacing $g(x)$ with $f(x)$ in the product rule and then comparing the result to your conjecture.

d) Use your result in part c) to differentiate the functions in part a). Compare your derivatives to those found in part a). What do you notice?

18. a) Use the product rule to show that $(fgh)' = f'gh + fg'h + fgh'$.

b) Apply the above result to differentiate $f(x) = (x^2 + 4)(3x^4 - 2)(5x + 1)$.

c) Describe another method for finding the derivative in part b). Apply the method you have described.

d) Verify that the derivatives in parts b) and c) are the same.

19. a) Make a conjecture about $\frac{d}{dx}[f(x)]^3$.

b) Use the results of question 18 part a), replacing both $g(x)$ and $h(x)$ with $f(x)$, to verify whether your conjecture is true.

c) Use the results of part b) to differentiate each function.

i) $y = (4x^2 - x)^3$

ii) $y = (x^3 + x)^3$

iii) $y = (-2x^4 + x^2)^3$

20. Determine expressions for each derivative, given that $f(x)$ and $g(x)$ are differentiable functions.

a) $h(x) = x^3 f(x)$

b) $p(x) = g(x)(x^4 - 3x^2)$

c) $q(x) = (-3x^4 - 8x^2 + 5x + 6)f(x)$

d) $r(x) = f(x)(2x^3 + 5x^2)^2$

21. a) Use the results of questions 17 and 19 to make a conjecture about $\frac{d}{dx}[f(x)]^n$.

b) Test your conjecture by differentiating $y = (2x^3 + x^2)^n$, for $n = 4$, 5, and 6.

22. Math Contest Let f be a function such that $\frac{d}{dx}[(x^2 + 1)f(x)] = 2xf(x) + 3x^4 + 3x^2$. Which of the following could $f(x)$ be?

A $6x$

B $3x^2$

C x^3

D $12x^3$

E $3x^4$

23. Math Contest Let p be a polynomial function with $p(a) = 0 = p'(a)$ for some real a. Which of the following *must* be true?

A $p(x)$ is divisible by $x + a$

B $p(x)$ is divisible by $x^2 + a^2$

C $p(x)$ is divisible by $x^2 - a^2$

D $p(x)$ is divisible by $x^2 + 2ax + a^2$

E $p(x)$ is divisible by $x^2 - 2ax + a^2$

2.3 Velocity, Acceleration, and Second Derivatives

When Sir Isaac Newton was working on his "method of fluxions," he recognized how these concepts could be applied to the study of objects in motion. This section will explore the use of derivatives to analyse the motion of objects travelling in a straight line. Three related concepts will be considered in relation to this type of motion:

- **Displacement** – the distance and direction an object has moved from an origin over a period of time
- **Velocity** – the rate of change of displacement of an object with respect to time
- **Acceleration** – the rate of change of velocity with respect to time

Investigate A What is the derivative of a derivative?

Method 1: ***Use Paper and Pencil***

1. a) Determine the derivative of $y = x^3$.

b) Determine the derivative of the derivative you found in part a).

c) **Reflect** How is the result in part b) related to the original function?

d) **Reflect** Why does it make sense to call your result in part b) a **second derivative**?

2. a) Sketch the graphs of y, y', and y''.

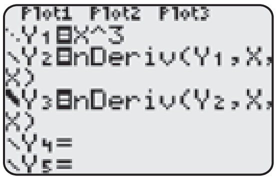
b) **Reflect** Describe how the graphs show the relationships among the three functions.

CONNECTIONS

There are several notations for the second derivative, including y'', $f''(x)$, $\frac{d^2y}{dx^2}$, $D^2f(x)$, and $D_x^2f(x)$.

Method 2: ***Use Technology***

1. a) Consider the function $y = x^3$. Use a graphing calculator to determine the derivative of the derivative of this function. Enter the information as shown, and then change the window variables to $x \in [-4, 4]$, $y \in [-20, 20]$, Yscl = 2. Before pressing GRAPH, draw a sketch to predict the shape of the graphs of Y1, Y2, and Y3.



b) Press GRAPH. Are the graphs you predicted accurate?

c) **Reflect** What is the relationship between the three graphs?

2. a) Determine the equations of the graphs of **Y2** and **Y3**.

b) **Reflect** Is it possible to differentiate a derivative? Why does it make sense to call **Y3** a second derivative?

Tools

- graphing calculator
- graphing software

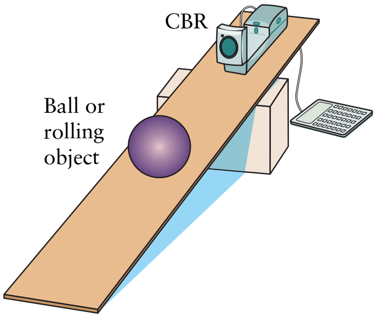
Investigate B What is the relationship between displacement, velocity, and acceleration?

In this activity, you will use a motion sensor to gather displacement and time data for a ball rolling up and down a ramp. You will investigate the displacement-time, velocity-time, and acceleration-time graphs, and form connections with derivatives.

Tools

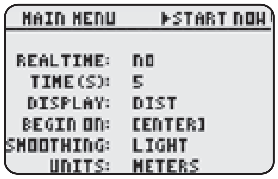
- graphing calculator
- Calculator-Based Ranger (CBR™)
- calculator-to-CBR™ cable
- ramp at least 3 m long
- large ball, such as a basketball

1. Set up the ramp as shown.



2. Prepare the CBR™ and calculator to collect data.

- Connect the CBR™ to the calculator using the calculator-to-CBR™ cable. Ensure that both ends of the cable are firmly in place.
- Press APPS. Select **2:CBL/CBR**. Press ENTER.
- To access the programs available, select **3:RANGER**.
- When the **RANGER** menu is displayed, press ENTER.
- From the **MAIN MENU** screen, select **1:SETUP/SAMPLE**.



- To change the **TIME (S)** setting, move the cursor down to **TIME (S)** and enter 5. Press ENTER.
- Move the cursor up to **START NOW** at the top of the screen, and press ENTER.

3. Collect the data.

- Align the CBR™ on the ramp, as shown in step 1.
- Press ENTER, and roll the ball.

Technology Tip

All settings, except **TIME (S)**, can be changed by using the cursor keys to position the ▶ beside the current option and pressing ENTER to cycle through the choices.

Technology Tip

The CBR™ is most accurate in an interval from about 1 m to 3 m. Practise rolling the ball such that it stays within this interval on the ramp.

4. The displacement-time graph will be displayed. If necessary, the graph can be redrawn to display the part that represents the motion in more detail.

   Press ENTER to display the **PLOT MENU** screen, and choose **4:PLOT TOOLS**. Select **1:SELECT DOMAIN**. Move the cursor to the point where the motion begins and press ENTER. Move the cursor to the point where the motion ends and press ENTER.

5. Use the **TRACE** feature to investigate displacement and time along the curve.

   a) Determine the time when the ball was closest to the CBR™. Describe what point this represents on the curve.

   b) For what time interval was the displacement increasing? For what time interval was it decreasing?

   c) **Reflect** Which direction was the ball rolling during the intervals in part b)?

6. **a)** Sketch a graph of your prediction for the corresponding velocity-time graph. Give reasons for your prediction.

   b) Return to the **PLOT MENU** screen. Select **2:VEL-TIME** to display the velocity-time graph. How does your sketch from part a) compare to the actual graph?

   c) **Reflect** Explain the significance of the time you found in step 5 part a), in regard to the velocity-time graph.

   d) **Reflect** How are the intervals you found in step 5 part b) reflected on the velocity-time graph? Explain.

   e) **Reflect** Think about rates of change and derivatives. What is the relationship between the distance-time graph and the velocity-time graph? Justify your answer.

7. **a)** Sketch a graph of your prediction for the corresponding acceleration-time graph. Give reasons for your prediction.

   b) Return to the **PLOT MENU** screen. Select **3:ACCEL-TIME** to display the acceleration-time graph. How does your sketch from part a) compare to the actual graph?

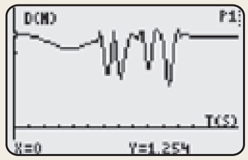
   c) For what time interval was the acceleration positive? For what interval was it negative? How do these intervals reflect the motion of the ball?

   d) **Reflect** Think about rates of change and derivatives. What is the relationship between the velocity-time graph and the acceleration-time graph? Justify your answer.

8. **Reflect** How can derivatives be used to determine the relationships between displacement-time, velocity-time, and acceleration-time graphs? Justify your reasoning.

Technology Tip

If you see any spikes, like those shown here, or other "artifacts" on the graph, it means that the CBR™ is intermittently losing the signal. Ensure that it is aimed properly, and that the ball you are using is big enough to reflect the sound waves. If necessary, repeat the data collection step until you have a graph free of artifacts.



Technology Tip

The data for t, d, v, and a are stored in L1, L2, L3, and L4 respectively.

Example 1 Apply Derivative Rules to Determine the Value of a Second Derivative

Determine $f''(2)$ for the function $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$.

Solution

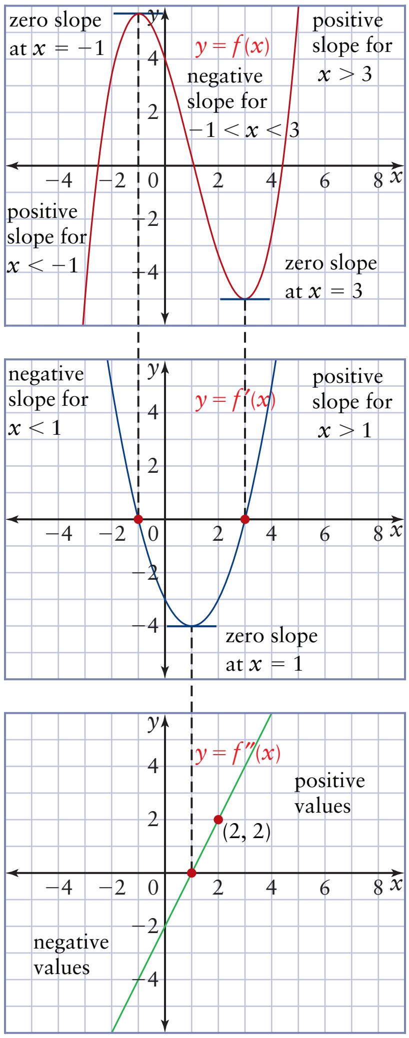
$$f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$$

$$\begin{aligned} f'(x) &= \frac{1}{3}(3x^2) - 2x - 3 \\ &= x^2 - 2x - 3 \end{aligned}$$

$$f''(x) = 2x - 2 \qquad \text{Differentiate } f'(x)$$

$$f''(2) = 2(2) - 2 = 2$$

The following graphs show the relationship between the original function, $f(x)$, the derivative, $f'(x)$, and the second derivative, $f''(x)$. The point (2, 2) on the graph of $f''(x)$ corresponds to $f''(2) = 2$.



Negative y-values for $-1 < x < 3$ on the derivative graph correspond to the negative slopes on the original graph. Positive y-values for $x < -1$ and $x > 3$ on the derivative graph correspond to positive slopes on the original graph. The x-intercepts, $x = -1$ and 3, of the derivative graph correspond to points on $f(x)$ that have zero slope.

Negative y-values for $x < 1$ on the second derivative graph correspond to the negative slopes on the derivative graph. Positive y-values for $x > 1$ on the second derivative graph correspond to positive slopes on the derivative graph. The x-intercept, $x = 1$, of the second derivative graph corresponds to the point on $f'(x)$ that has zero slope.

Displacement, Velocity, and Acceleration

	Displacement (s)	Velocity (v)	Acceleration (a)
Definition in Words	Distance an object has moved from the origin over a period of time (t)	The rate of change of displacement (s) with respect to time (t)	The rate of change of velocity (v) with respect to time (t)
Relationship	$s(t)$	$s'(t) = v(t)$	$v'(t) = a(t)$ and $s''(t) = a(t)$
Typical Units	m	m/s	m/s^2

When describing motion, there are two terms that are sometimes misused in everyday speech: speed and velocity. These are sometimes used interchangeably, but they are, in fact, different.

Speed is a scalar quantity. It describes the magnitude of motion, but does not describe direction. **Velocity**, on the other hand, is a vector quantity. It has both magnitude and direction. The answer in a velocity problem will be either a negative or positive value. The sign indicates the direction the object is travelling. That is, the original position of the object is considered the origin. One direction from the origin is assigned positive values, and the opposite direction is assigned negative values, depending on what makes sense for the question.

Example 2 Solve a Velocity and Acceleration Problem Involving a Falling Object

A construction worker accidently drops a hammer from a height of 90 m while working on the roof of a new apartment building. The height of the hammer, s, in metres, after t seconds is modelled by the function $s(t) = 90 - 4.9t^2$, $t \geq 0$.

a) Determine the average velocity of the hammer between 1 s and 4 s.

b) Explain the significance of the sign of your result in part a).

c) Determine the velocity of the hammer at 1 s and at 4 s.

d) When will the hammer hit the ground?

e) Determine the impact velocity of the hammer.

f) Determine the acceleration function. What do you notice? Interpret its meaning for this situation.

Solution

a) Average velocity $= \dfrac{\Delta s}{\Delta t}$

$$= \frac{s(4) - s(1)}{4 - 1}$$

$$= \frac{[90 - 4.9(4)^2] - [90 - 4.9(1)^2]}{4 - 1}$$

$$= -\frac{73.5}{3}$$

$$= -24.5$$

The average velocity of the hammer between 1 s and 4 s is -24.5 m/s.

b) In this type of problem, the movement in the upward direction is commonly assigned positive values. Therefore, the negative answer indicates that the motion of the hammer is downward. (Note that the speed of the hammer is 24.5 m/s.)

c) $v(t) = s'(t)$

$$= \frac{d}{dt}(90 - 4.9t^2)$$

$$= -9.8t$$

Substitute $t = 1$ and $t = 4$.

$v(1) = -9.8(1)$
$= -9.8$
$v(4) = -9.8(4)$
$= -39.2$

The velocity of the hammer at 1 s is -9.8 m/s, and at 4 s it is -39.2 m/s. Once again, the negative answers indicate downward movement.

d) The hammer hits the ground when the displacement is zero.
Solve $s(t) = 0$.

$$90 - 4.9t^2 = 0$$

$$t^2 = \frac{90}{4.9}$$

$$\doteq 18.37$$

$$t \doteq \pm 4.29$$

Since $t \geq 0$, $t = 4.29$.
The hammer takes approximately 4.3 s to hit the ground.

e) The impact velocity is the velocity of the hammer when it hits the ground.

$v(4.3) = -9.8(4.3)$
$= -42.14$

The impact velocity of the hammer is about 42 m/s.

CONNECTIONS

Earlier in the chapter, air resistance was defined as a force that counters the effects of gravity as falling objects encounter friction with air molecules. When this force, also called atmospheric drag, becomes equal to the force of gravity, a falling object will accelerate no further. Its velocity remains constant after this point. This maximum velocity is called terminal velocity.

f) The acceleration function is the derivative of the velocity function.

$$\begin{aligned} a(t) &= v'(t) \\ &= \frac{d}{dt}(-9.8t) \\ &= -9.8 \end{aligned}$$

The hammer falls at a constant acceleration of -9.8 m/s^2. This value is the acceleration due to gravity for any falling object on Earth (when air resistance is ignored).

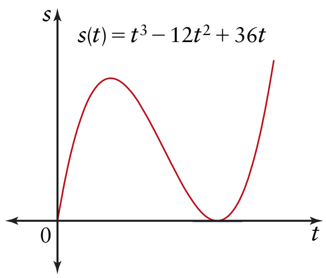
In general, if the acceleration and velocity of an object have the same sign at a particular time, then the object is being pushed in the direction of the motion, and the object is speeding up. If the acceleration and velocity have opposite signs at a particular time, the object is being pushed in the opposite direction to its motion, and it is slowing down.

An object is speeding up, at time t if $v(t) \times a(t) > 0$.

An object is slowing down, at time t if $v(t) \times a(t) < 0$.

Example 3 Relate Velocity and Acceleration

The position of a particle moving along a straight line is represented by the function $s(t) = t^3 - 12t^2 + 36t$, where distance, s, is in metres, time, t, is in seconds, and $t \geq 0$.

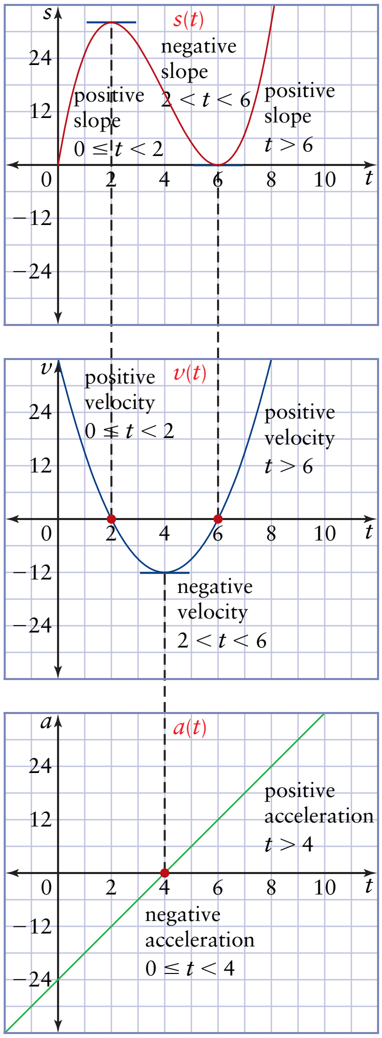


a) The graph of the position function is given. Sketch the graph of the velocity and acceleration functions.

b) Determine when the particle is speeding up and slowing down. How does this relate to the slope of the position function?

Solution

a)



Begin with $t = 2$ and $t = 6$ on the graph of $s(t)$, where the slope of the tangent is zero. Mark these as the t-intercepts of the derivative graph, $v(t)$.

The graph of $s(t)$ has positive slope over the intervals $[0, 2)$ and $(6, 8)$, so the graph of $v(t)$ is positive (lies above the t-axis) over these intervals. The graph of $s(t)$ has negative slope over the interval $(2, 6)$, so the graph of $v(t)$ is negative (lies below the t-axis) over this interval.

Begin with $t = 4$ on the graph of $v(t)$, where the slope of the tangent is zero. Mark this as the t-intercept of the derivative graph, $a(t)$.

The graph of $v(t)$ has negative slope over the interval $[0, 4)$, so the graph of $a(t)$ is negative over this interval. The graph of $v(t)$ has positive slope over the interval $(4, 8)$, so the graph of $a(t)$ is positive over this interval.

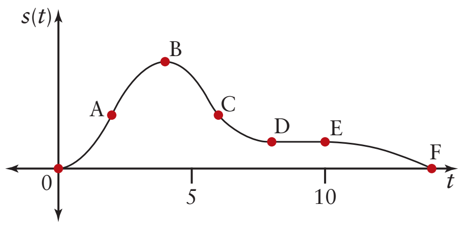
b) The graph of $v(t)$ changes sign at the intercepts $t = 2$ and $t = 6$. The graph of $a(t)$ changes sign at the intercept $t = 4$. The signs of $v(t)$ and $a(t)$ are easily observed by determining whether the respective graph lies below or above the t-axis. Consider the following four time intervals: $[0, 2)$, $(2, 4)$, $(4, 6)$, $(6,8)$. The following chart summarizes this information.

Interval	$v(t)$	$a(t)$	$v(t) \times a(t)$	Motion of Particle	Description of slope of $s(t)$
[0, 2)	+	−	−	slowing down and moving forward	positive slope that is decreasing
(2, 4)	−	−	+	speeding up and moving in reverse	negative slope that is decreasing
(4, 6)	−	+	−	slowing down and moving in reverse	negative slope that is increasing
(6, 8)	+	+	+	speeding up and moving forward	positive slope that is increasing

Therefore, the particle is slowing down between 0 s and 2 s and again between 4 s and 6 s. The particle is speeding up between 2 s and 4 s and after 6 s.

Example 4 Analyse and Interpret a Position-Time Graph

The graph shows the position function of a motorcycle. Describe the slope of the graph, in terms of being positive, negative, increasing, or decreasing, over the interval between consecutive pairs of points, beginning at the origin. For each interval, determine the sign of the velocity and acceleration by considering the slope of the graph.



Solution

The analysis of this situation is organized in the following chart.

Interval	Slope of Graph	Velocity	Acceleration
0 to A	positive slope that is increasing	+	+
A to B	positive slope that is decreasing	+	−
B to C	negative slope that is decreasing	−	−
C to D	negative slope that is increasing	−	+
D to E	slope = zero, horizontal segment	0	0
E to F	negative slope that is decreasing	−	−

KEY CONCEPTS

- The second derivative of a function is determined by differentiating the first derivative of the function.
- For a given position function $s(t)$, its velocity function is $v(t)$, or $s'(t)$, and its acceleration function is $a(t)$, $v'(t)$, or $s''(t)$.
- When $v(t) = 0$, the object is at rest, or stationary. There are many instances where an object will be momentarily at rest when changing directions. For example, a ball thrown straight upward will be momentarily at rest at its highest point, and will then begin to descend.
- When $v(t) > 0$, the object is moving in the positive direction.
- When $v(t) < 0$, the object is moving in the negative direction.
- When $a(t) > 0$, the velocity of an object is increasing (i.e., the object is accelerating).
- When $a(t) < 0$, the velocity of an object is decreasing (i.e., the object is decelerating).
- An object is speeding up if $v(t) \times a(t) > 0$ and slowing down if $v(t) \times a(t) < 0$.

Communicate Your Understanding

C1 Under what conditions is an object speeding up? Under what conditions is it slowing down? Support your answers with examples.

C2 Give the graphical interpretation of positive velocity and negative velocity.

C3 How are speed and velocity similar? How are they different?

C4 What is the relationship between the degrees of $s(t)$, $v(t)$, and $a(t)$, if $s(t)$ is a polynomial function?

A Practise

1. Determine the second derivative of each function.

a) $y = 2x^3 + 21$

b) $s(t) = -t^4 + 5t^3 - 2t^2 + t$

c) $h(x) = \frac{1}{6}x^6 - \frac{1}{5}x^5$

d) $f(x) = \frac{1}{4}x^3 - 2x^2 + 8$

e) $g(x) = x^5 + 3x^4 - 2x^3$

f) $h(t) = -4.9t^2 + 25t + 4$

2. Determine $f''(3)$ for each function.

a) $f(x) = 2x^4 - 3x^3 + 6x^2 + 5$

b) $f(x) = 4x^3 - 5x + 6$

c) $f(x) = -\frac{2}{5}x^5 - x^3 + 0.5$

d) $f(x) = (3x^2 + 2)(1 - x)$

e) $f(x) = (6x - 5)(x^2 + 4)$

f) $f(x) = 4x^5 - \frac{1}{2}x^4 - 3x^2$

3. Determine the velocity and acceleration functions for each position function $s(t)$. Where possible, simplify the functions before differentiating.

a) $s(t) = 5 + 7t - 8t^3$

b) $s(t) = (2t + 3)(4 - 5t)$

c) $s(t) = -(t + 2)(3t^2 - t + 5)$

d) $s(t) = \dfrac{-2t^4 - t^3 + 8t^2}{4t^2}$

4. Determine the velocity and acceleration at $t = 2$ for each position function $s(t)$, where s is in metres and t is in seconds.

a) $s(t) = t^3 - 3t^2 + t - 1$

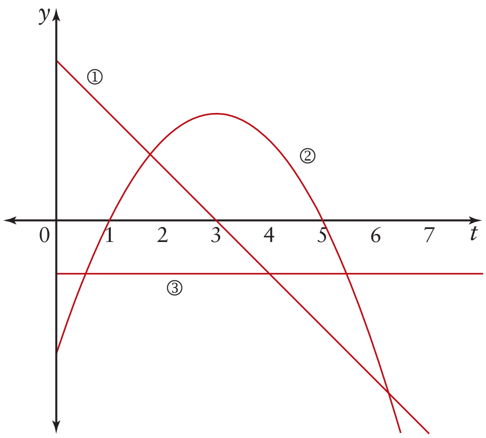
b) $s(t) = -4.9t^2 + 15t + 1$

c) $s(t) = t(3t + 5)(1 - 2t)$

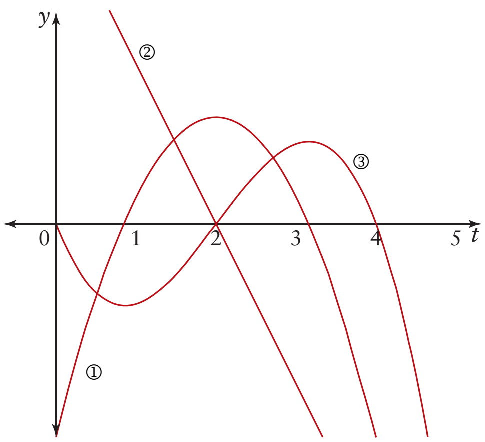
d) $s(t) = (t^2 - 2)(t^2 + 2)$

5. In each graph, identify which curve or line represents $y = s(t)$, $y = v(t)$, and $y = a(t)$. Justify your choices.

a)



b)



6. Copy and complete the chart for each graph in question 5.

a)

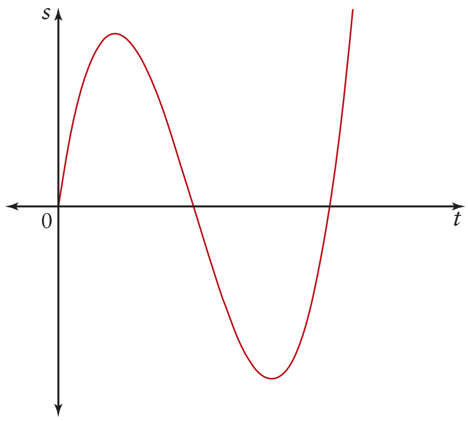
Interval	$v(t)$	$a(t)$	$v(t) \times a(t)$	Motion of Object	Description of slope of $s(t)$

b)

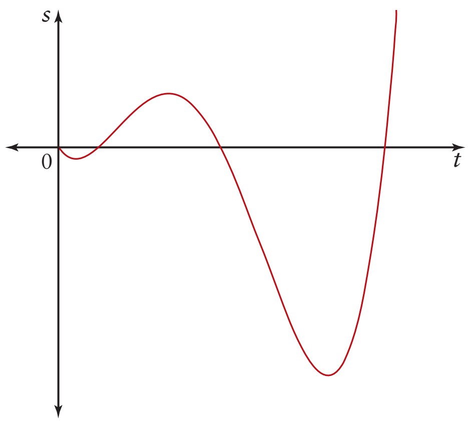
Interval	$v(t)$	$a(t)$	$v(t) \times a(t)$	Motion of Object	Description of slope of $s(t)$

7. For each position function $y = s(t)$ given, sketch the graphs of $y = v(t)$ and $y = a(t)$.

a)



b)

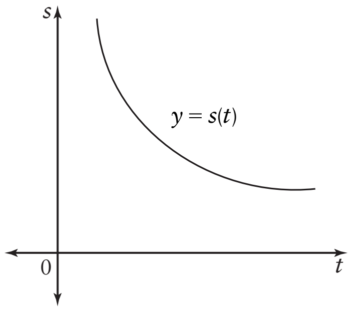


8. Answer the following for each graph of $y = s(t)$. Explain your reasoning.

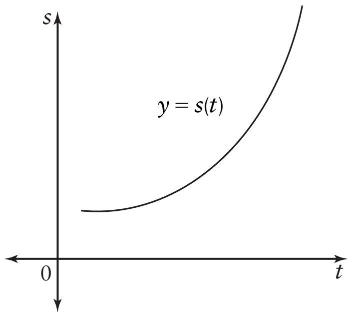
i) Is velocity increasing, decreasing, or constant?

ii) Is acceleration positive, negative, or zero?

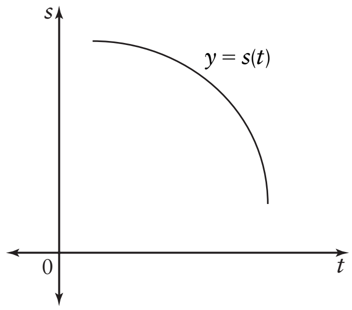
a)



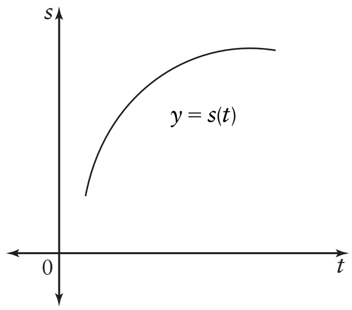
b)



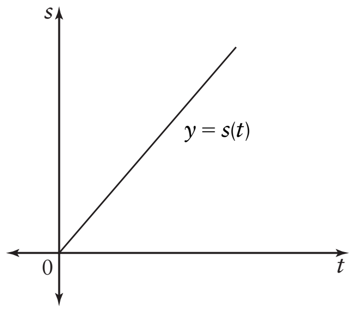
c)



d)

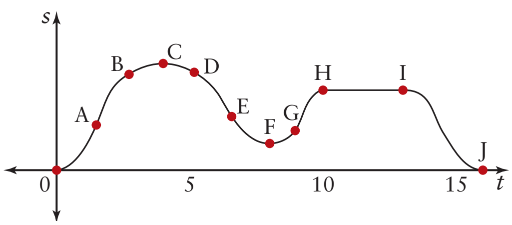


e)



B Connect and Apply

9. The following graph shows the position function of a bus during a 15-min trip.



a) What is the initial velocity of the bus?

b) What is the bus's velocity at C and at F?

c) Is the bus going faster at A or at B? Explain.

d) What happens to the motion of the bus between H and I?

e) Is the bus speeding up or slowing down at A, B, and D?

f) What happens at J?

10. Refer to the graph in question 9. Is the acceleration positive, zero, or negative during the following intervals?

a) 0 to A

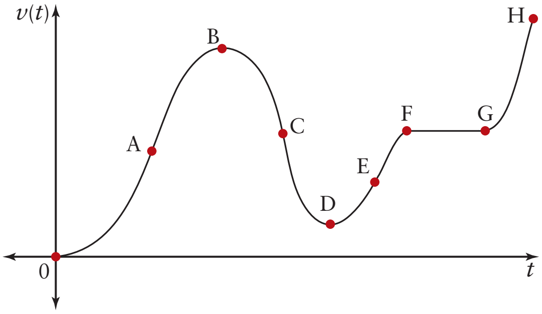
b) C to D

c) E to F

d) G to H

e) F to G

11. The following is the graph of a velocity function.



a) State whether the acceleration is positive, negative, or zero for the following intervals or points.

i) 0 to B

ii) B to D

iii) D to F

iv) F to G

v) G to H

vi) at A and at D

b) Describe similarities and differences in the acceleration for the given intervals. Justify your response.

i) 0 to A, D to E, and G to H

ii) A to B and E to F

iii) B to C and C to D

12. A water bottle rolls off a rooftop patio from a height of 80 m. The distance, s, in metres, the bottle is above the ground after t seconds is modelled by the function $s(t) = 80 - 4.9t^2$, $t \geq 0$.

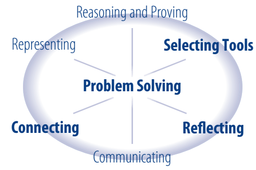
a) Determine the average velocity of the bottle between 1 s and 3 s.

b) Determine the velocity of the bottle at 3 s.

c) When will the bottle hit the ground?

d) Determine the impact velocity of the bottle.

13. During a fireworks display, a starburst rocket is shot upward with an initial velocity of 34.5 m/s from a platform 3.2 m high. The height, in metres, of the rocket after t seconds is represented by the function $h(t) = -4.9t^2 + 34.5t + 3.2$.



a) Determine the velocity and acceleration of the rocket at 3 s.

b) When the rocket reaches its maximum height, it explodes to create a starburst display. How long does it take for the rocket to reach its maximum height?

c) At what height does the starburst display occur?

d) Sometimes the rockets malfunction and do not explode. How long would it take for an unexploded rocket to return to the ground?

e) At what velocity would it hit the ground?

14. Consider the motion of a truck that is braking while moving forward. Justify your answer to each of the following.

a) Is the velocity positive or negative?

b) Is the velocity increasing or decreasing?

c) Is the acceleration positive or negative?

C Extend and Challenge

15. A bald eagle flying horizontally at 48 km/h drops its prey from a height of 50 m.

a) State the equation representing the horizontal displacement of the prey while held by the eagle.

b) Determine the velocity and acceleration functions for the prey's horizontal displacement.

c) State the equation representing the vertical displacement of the prey as it falls. (Assume the acceleration due to gravity is -9.8 m/s^2.)

d) Determine the velocity and acceleration functions for the prey's vertical displacement.

e) What is the prey's vertical velocity when it hits the ground?

f) When is the vertical speed greater than the horizontal speed?

g) Develop an equation to represent the total velocity. Determine the velocity.

h) Develop an equation to represent the total acceleration.

i) Determine the prey's acceleration 4 s after being dropped.

16. The position function of an object moving along a straight line is represented by the function $s(t) = 2t^3 - 15t^2 + 36t + 10$, where t is in seconds and s in metres.

a) What is the velocity of the object after 1 s and after 4 s? What is the object's acceleration at these times?

b) When is the object momentarily at rest? What is the object's position when stopped?

c) When is the object moving in a positive direction? When is it moving in a negative direction?

d) Determine the total distance travelled by the object during the first 7 s.

e) Sketch a diagram to illustrate the motion of the object.

17. The height of any object that is shot into the air is given by the position function $h(t) = 0.5gt^2 + v_0t + s_0$, where h is in metres and t is in seconds, s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity ($g = -9.8\ \text{m/s}^2$).

a) Determine the velocity function and the acceleration function for $h(t)$.

b) An arrow is shot upward at 17.5 m/s from a position in a tree 4 m above the ground. State the position, velocity, and acceleration functions for this situation.

c) Suppose a flare is shot upward, and after 2 s its velocity is 10.4 m/s and its height is 42.4 m. Determine the position, velocity, and acceleration functions for this situation.

18. The position function $s(t) = 0.5gt^2 + v_0t + s_0$ is also used to represent the motion of an object moving along a straight line, where s is in metres and t is in seconds, and where s_0 is the initial position of the object, v_0 is the initial velocity of the object, and g is the acceleration. The driver of a pickup truck travelling at 86.4 km/h suddenly notices a stop sign and applies the brakes, resulting in a constant deceleration of 12 m/s^2.

a) Determine the position, velocity, and acceleration function for this situation.

b) How long does it take for the truck to stop?

19. Math Contest If p and q are two polynomials such that $p''(x) = q''(x)$ for $x \in \mathbb{R}$, which of the following *must* be true?

A $p(x) = q(x)$ for $x \in \mathbb{R}$

B $p'(x) = q'(x)$ for $x \in \mathbb{R}$

C $p(0) - q(0) = 0$

D The graph of $y = p(x) - q(x)$ is a horizontal line.

E The graph of $y = p'(x) - q'(x)$ is a horizontal line.

20. Math Contest

$f^{(n)}(x) = \dfrac{d}{dx}\left(\dfrac{d}{dx}\left(\dfrac{d}{dx}\left(\ldots\dfrac{d}{dx}(f(x)\ldots)\right)\right)\right)$ denotes the nth derivative of the function $f(x)$. If $f'(x) = g(x)$ and $g'(x) = -f(x)$, then $f^{(n)}(x)$ is equal to

A $\dfrac{1}{2}[1 + (-1)^n]f(x) + \dfrac{1}{2}[1 + (-1)^{n-1}]g(x)$

B $\dfrac{(-1)^n}{2}[1 + (-1)^n]f(x) + \dfrac{(-1)^{n-1}}{2}[1 + (-1)^{n-1}]g(x)$

C $\dfrac{(-1)^{n-1}}{2}[1 + (-1)^n]f(x) + \dfrac{(-1)^n}{2}[1 + (-1)^{n-1}]g(x)$

D $\dfrac{i^n}{2}[1 + (-1)^n]f(x) + \dfrac{i^{n-1}}{2}[1 + (-1)^{n-1}]g(x)$, where $i = \sqrt{-1}$

E $\dfrac{i^{n-1}}{2}[1 + (-1)^{n-1}]f(x) + \dfrac{i^n}{2}[1 + (-1)^n]g(x)$, where $i = \sqrt{-1}$

2.4 The Chain Rule

Andrew and David are both training to run a marathon, a long-distance running event that covers a distance of 42.195 km (26.22 mi). They both go for a run on Sunday mornings at precisely 7 A.M. Andrew's house is 22 km south of David's house. One Sunday morning, Andrew leaves his house and runs north at 9 km/h. At the same time, David leaves his house and runs west at 7 km/h. The distance between the two runners can be modelled by the function $s(t) = \sqrt{130t^2 - 396t + 484}$, where s is in kilometres and t is in hours. You can use differentiation to determine the rate at which the distance between the two runners is changing. This rate of change is given by $s'(t)$.

To this point, you have not learned a differentiation rule that will help you find the derivative of this type of function. You could use the first principles limit definition, but that would be complicated. Notice, however, that $s(t)$ is the composition of a root function and a polynomial function: $s(t) = f \circ g(t) = f(g(t))$, where $f(t) = \sqrt{t}$, and $g(t) = 130t^2 - 396t + 484$. Both $f'(t)$ and $g'(t)$ are easily computed using the derivative rules you already know. This section will develop a general rule that can be used to find the derivative of composite functions. This rule is called the **chain rule**.

Investigate — How are composite functions differentiated?

1. Consider the function $f(x) = (8x^3)^{\frac{1}{3}}$.

a) Simplify $f(x)$, using the laws of exponents.

b) Use your result from part a) to determine $f'(x)$.

2. a) Let $g(x) = x^{\frac{1}{3}}$. Determine $g'(x)$.

b) Let $h(x) = 8x^3$. Replace x with $8x^3$ in your expression for $g'(x)$ from part a). This will give an expression for $g'[h(x)]$. Do not simplify.

c) **Reflect** Why is it appropriate to refer to the expression $g'[h(x)]$ as a composite function?

d) Determine $h'(x)$.

3. a) Use the results of step 2 parts b) and d) to write an expression for the product $g'[h(x)] \times h'(x)$.

b) Simplify your answer from part a).

c) Compare the derivative result from step 1 part b) with the derivative result from step 3 part b). What do you notice?

4. **a)** **Reflect** Use the above results to write a rule for differentiating a composite function $f(x) = g \circ h(x)$. This rule is called the chain rule. What operation forms the "chain"? Explain.

b) Write the rule from part a) in terms of the "outer function" and the "inner function."

c) Use your rule to differentiate $f(x) = (2x^3 - 5)^2$.

d) Verify the accuracy of your rule by differentiating $f(x) = (2x^3 - 5)^2$ using the product rule.

The Chain Rule

Given two differentiable functions $g(x)$ and $h(x)$, the derivative of the composite function $f(x) = g[h(x)]$ is $f'(x) = g'[h(x)] \times h'(x)$.

A composite function $f(x) = (g \circ h)(x) = g[h(x)]$ consists of an outer function, $g(x)$, and an inner function, $h(x)$. The chain rule is an efficient way of differentiating a composite function by first differentiating the outer function with respect to the inner function, and then multiplying by the derivative of the inner function.

Example 1 Apply the Chain Rule

Differentiate each function using the chain rule.

a) $f(x) = (3x - 5)^4$

b) $f(x) = \sqrt{4 - x^2}$

Solution

a) $f(x) = (3x - 5)^4$ is a composite function, $f(x) = g[h(x)]$, with $g(x) = x^4$ and $h(x) = 3x - 5$. Then $g'(x) = 4x^3$, $g'[h(x)] = 4(3x - 5)^3$, and $h'(x) = 3$.

$$\begin{aligned} f'(x) &= g'[h(x)]h'(x) && \text{Apply the chain rule.} \\ &= 4(3x-5)^3(3) \\ &= 12(3x-5)^3 \end{aligned}$$

b) $f(x) = \sqrt{4 - x^2}$ is a composite function, $f(x) = g[h(x)]$, with $g(x) = x^{\frac{1}{2}}$ and $h(x) = 4 - x^2$. Then $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $g'[h(x)] = \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}$, and $h'(x) = -2x$.

$$\begin{aligned} f'(x) &= g'[h(x)]h'(x) && \text{Apply the chain rule.} \\ &= \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x) \\ &= -x(4 - x^2)^{-\frac{1}{2}} \\ &= \frac{-x}{\sqrt{4 - x^2}} \end{aligned}$$

Alternate Form of the Chain Rule

Consider the function in Example 1 part b): $f(x) = \sqrt{4 - x^2}$.

Let $y = \sqrt{u}$. Let $u = 4 - x^2$.

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ $\frac{du}{dx} = -2x$

The product of these two derivatives, $\frac{dy}{du} \times \frac{du}{dx}$, results in the derivative of y with respect to x, $\frac{dy}{dx}$.

Therefore, $\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(-2x)$. Replacing u with $4 - x^2$, the result is $\frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$.

Leibniz Form of the Chain Rule

If $y = f(u)$ and $u = g(x)$ are differentiable functions, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

The Leibniz form of the chain rule can be expressed in words as follows: If y is a function in u, and u is a function in x, then the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x.

This form of the chain rule is easily remembered if you think of each term as a fraction. You can then cancel du. Keep in mind, however, that this is only a memory device, and does not reflect the mathematical reality. These terms are not really fractions because du has not been defined.

Example 2 Represent the Chain Rule in Leibniz Notation

a) If $y = -\sqrt{u}$ and $u = 4x^3 - 3x^2 + 1$, determine $\dfrac{dy}{dx}$.

b) If $y = u^{-3}$ and $u = 2x - x^3$, determine $\left.\dfrac{dy}{dx}\right|_{x=2}$.

Solution

a)

$$y = -\sqrt{u} \text{ and } u = 4x^3 - 3x^2 + 1$$

$$y = -u^{\frac{1}{2}}$$

$$\frac{dy}{du} = -\frac{1}{2}u^{-\frac{1}{2}} \text{ and } \frac{du}{dx} = 12x^2 - 6x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= -\frac{1}{2}u^{-\frac{1}{2}}(12x^2 - 6x)\\ &= -\frac{1}{2}(4x^3 - 3x^2 + 1)^{-\frac{1}{2}}(12x^2 - 6x)\\ &= \frac{-12x^2 + 6x}{2\sqrt{4x^3 - 3x^2 + 1}}\end{aligned}$$

Substitute $u = 4x^3 - 3x^2 + 1$ to express the answer in terms of x.

b) For $y = u^{-3}$, $\dfrac{dy}{du} = -3u^{-4}$.

For $u = 2x - x^3$, $\dfrac{du}{dx} = 2 - 3x^2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= -3u^{-4}(2 - 3x^2)\\ &= \frac{-3(2 - 3x^2)}{u^4}\end{aligned}$$

Notice that it was not necessary to write the expression entirely in terms of x, because you can determine the value of u when $x = 2$ and then substitute as shown below.

When $x = 2$, $u = 2(2) - 2^3 = -4$.

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=2} &= \frac{-3(2 - 3(2)^2)}{(-4)^4}\\ &= \frac{-3(2 - 12)}{256}\\ &= \frac{15}{128}\end{aligned}$$

Example 2 illustrates a special case of the chain rule that occurs when the outer function is a power function, such as $y = u^n$ or $[g(x)]^n$. The derivative consists of using the power rule first.

Power of a Function Rule

If $y = u^n$ and $u = g(x)$, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \text{ or } \frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x).$$

Example 3 Combine the Chain Rule and the Product Rule

Determine the equation of the tangent to $f(x) = 3x(1 - x)^2$ at $x = 0.5$.

Solution

Differentiate the function.

$$f(x) = 3x(1 - x)^2$$

$$f'(x) = (1 - x^2)\frac{d}{dx}(3x) + (3x)\frac{d}{dx}(1 - x)^2 \qquad \text{Apply the product rule first.}$$

$$= 3(1 - x)^2 + 3x[2(1 - x)(-1)] \qquad \text{Apply the power of a function rule.}$$

Determine the slope at $x = 0.5$ by substituting into the derivative function.

$$\begin{aligned} f'(0.5) &= 3(1 - 0.5)^2 + 3(0.5)[2(1 - 0.5)(-1)] \\ &= 0.75 - 1.5 \\ &= -0.75 \end{aligned}$$

The slope of the tangent at $x = 0.5$ is -0.75.

Calculate the tangent point by substituting the x-value into the original function.

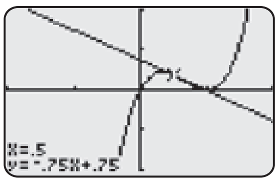
$$\begin{aligned} f(0.5) &= 3(0.5)(1 - 0.5)^2 \\ &= 1.5(0.25) \\ &= 0.375 \end{aligned}$$

The tangent point is (0.5, 0.375).

Substitute $m = -0.75$ and (0.5, 0.375) into $y - y_1 = m(x - x_1)$.

$$\begin{aligned} y - 0.375 &= -0.75(x - 0.5) \\ y &= -0.75x + 0.375 + 0.375 \\ y &= -0.75x + 0.75 \end{aligned}$$

You can check your answer using the Tangent function on a graphing calculator.

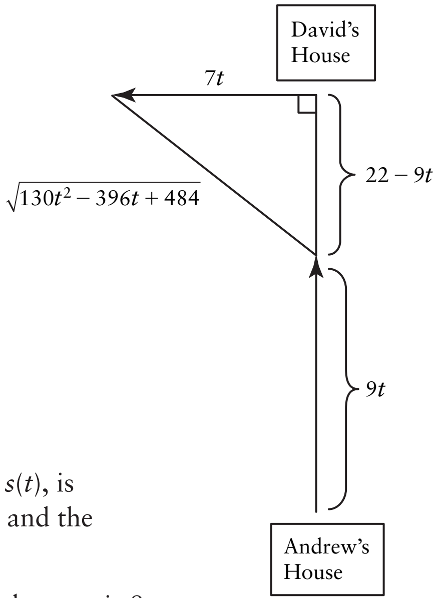


Technology Tip

To get the screen shown here, you will have to change the number of decimal places to 2.

Example 4 Apply the Chain Rule to Solve a Rate of Change Problem

The chain rule can be used to solve the problem presented at the beginning of this section. Andrew and David both leave their houses at 7 A.M. for their Sunday run. Andrew's house is 22 km south of David's house. Andrew runs north at 9 km/h, while David runs west at 7 km/h. Determine the rate of change of the distance between the two runners after 1 h.



Solution

The distance between the two runners, $s(t)$, is obtained by using two formulas: $d = vt$ and the Pythagorean theorem.

Andrew runs at 9 km/h, so the distance he runs is $9t$, with t measured in hours. Since his house is 22 km from David's, the distance he is from David's house as he runs is represented by $22 - 9t$. David runs at 7 km/h, so the distance he runs is represented by $7t$.

$$s(t) = \sqrt{(22 - 9t)^2 + (7t)^2} \qquad \text{Pythagorean theorem.}$$

$$= \sqrt{130t^2 - 396t + 484}$$

$$= (130t^2 - 396t + 484)^{\frac{1}{2}}$$

$$s'(t) = \frac{1}{2}(130t^2 - 396t + 484)^{-\frac{1}{2}} \frac{d}{dt}(130t^2 - 396t + 484) \qquad \text{Power of a function rule.}$$

$$= \frac{1}{2\sqrt{130t^2 - 396t + 484}}(260t - 396)$$

$$s'(1) = \frac{260(1) - 396}{2\sqrt{130(1)^2 - 396(1) + 484}}$$

$$= \frac{-136}{2\sqrt{218}}$$

$$\doteq -4.6$$

After 1 h, the distance between Andrew and David is decreasing at 4.6 km/h.

KEY CONCEPTS

- **The Chain Rule**

  The chain rule is used to differentiate composite functions, $f = g \circ h$. Given two differentiable functions $g(x)$ and $h(x)$, the derivative of the composite function $f(x) = g[h(x)]$ is $f'(x) = g'[h(x)] \times h'(x)$.

- **The Chain Rule in Leibniz Notation**

  If $y = f(u)$ and $u = g(x)$ are differentiable functions, then

  $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

- **The Power of a Function Rule**

  If $y = u^n$ and $u = g(x)$, then

  $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \text{ or } \frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x).$$

Communicate Your Understanding

C1 Use an example to explain the meaning of the statement, "The derivative of a composite function is equal to the derivative of the outer function multiplied by the derivative of the inner function."

C2 Can the product rule be used to verify the chain rule? Support your answer with an example.

C3 Would it be true to say, "The power of a function rule is separate and distinct from the chain rule"? Use an example to justify your response.

C4 What operation creates the "chain" in the chain rule?

A Practise

1. Determine the derivative of each function by using the following methods.

i) Use the chain rule, and then simplify.

ii) Simplify first, and then differentiate.

a) $f(x) = (2x)^3$

b) $g(x) = (-4x^2)^2$

c) $p(x) = \sqrt{9x^2}$

d) $f(x) = (-16x^2)^{\frac{3}{4}}$

e) $q(x) = (8x)^{\frac{2}{3}}$

2. Copy and complete the following table.

$f(x) = g[h(x)]$	$g(x)$	$h(x)$	$h'(x)$	$g'[h(x)]$	$f'(x)$
a) $(6x - 1)^2$					
b) $(x^2 + 3)^3$					
c) $(2 - x^3)^4$					
d) $(-3x + 4)^{-1}$					
e) $(7 + x^2)^{-2}$					
f) $\sqrt{x^4 - 3x^2}$					

3. Differentiate, expressing each answer using positive exponents.

a) $y = (4x + 1)^2$ **b)** $y = (3x^2 - 2)^3$

c) $y = (x^3 - x)^{-3}$ **d)** $y = (4x^2 + 3x)^{-2}$

4. Express each function as a power with a rational exponent, and then differentiate. Express each answer using positive exponents.

a) $y = \sqrt{2x - 3x^5}$ **b)** $y = \sqrt{-x^3 + 9}$

c) $y = \sqrt[3]{x - x^4}$ **d)** $y = \sqrt[5]{2 + 3x^2 - x^3}$

5. Express each of the following as a power with a negative exponent, and then differentiate. Express each answer using positive exponents.

a) $y = \dfrac{1}{(-x^3 + 1)^2}$ **b)** $y = \dfrac{1}{(3x^2 - 2)}$

c) $y = \dfrac{1}{\sqrt{x^2 + 4x}}$ **d)** $y = \dfrac{1}{\sqrt[3]{x - 7x^2}}$

6. **a)** Use two different methods to differentiate $f(x) = \sqrt{25x^4}$.

b) **Reflect** Explain why you prefer one of the methods in part a) over the other.

c) **Reflect** Can both methods that you described in part a) be used to differentiate $f(x) = \sqrt{25x^4 - 3}$? Explain.

B Connect and Apply

7. Determine $f'(1)$.

a) $f(x) = (4x^2 - x + 1)^2$

b) $f(x) = (3 - x + x^2)^{-2}$

c) $f(x) = \sqrt{4x^2 + 1}$

d) $f(x) = \dfrac{5}{\sqrt[3]{2x - x^2}}$

8. Using Leibniz notation, apply the chain rule to determine $\dfrac{dy}{dx}$ at the indicated value of x.

a) $y = u^2 + 3u,\ u = \sqrt{x},\ x = 4$

b) $y = \sqrt{u},\ u = 2x^2 + 3x + 4,\ x = -3$

c) $y = \dfrac{1}{u^2},\ u = x^3 - 5x,\ x = -2$

d) $y = u(2 - u^2),\ u = \dfrac{1}{x},\ x = 2$

9. Determine the equation of the tangent to the curve $y = (x^3 - 4x^2)^3$ at $x = 3$.

10. Determine the equation of the tangent to the curve $y = \dfrac{1}{\sqrt[5]{5x^3 - 2x^2}}$ at $x = 2$.

11. The position function of a moving particle is given by $s(t) = \sqrt[3]{t^5 - 750t^2}$, where s is in metres and t is in seconds. Determine the velocity of the particle at 5 s.

12. Determine the point(s) on the curve $y = x^2(x^3 - x)^2$ where the tangent line is horizontal.

13. **Chapter Problem** The owners of Mooses, Gooses, and Juices are interested in analysing the productivity of their staff. The function $N(t) = 150 - \dfrac{600}{\sqrt{16 + 3t^2}}$ models the total number of customers, N, served by the staff after t hours during an 8-h workday $(0 \le t \le 8)$.

a) Determine $N'(t)$. What does the derivative represent for this situation?

b) Determine $N(4)$ and $N'(4)$. Interpret the meaning of each of these values for this situation.

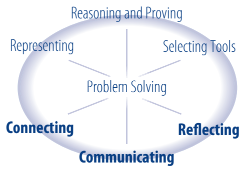
c) Solve $N(t) = 103$. Interpret the meaning of your answer for this situation.

d) Determine $N'(t)$ for the value you found in part c). Compare this value with $N'(4)$. What conclusion, if any, can be made from comparing these two values?

14. The population of a small town is modelled by the function $P(t) = \dfrac{1250}{1 + 0.01t}$, where P is the number of people, and t is time, in years, $t \ge 0$.

Determine the instantaneous rate of change of the population after 2 years, 4 years, and 7 years.

15. The formula for the volume of a cube in terms of its side length, s, is $V(s) = s^3$. If the side length is expressed in terms of a variable, x, measured in metres, such that $s = 3x^2 - 7x + 1$, determine $\left.\dfrac{dv}{dx}\right|_{x=3}$. Interpret the meaning of this value for this situation.



16. Express $y = \dfrac{4x - x^3}{(3x^2 + 2)^2}$ as a product and then differentiate. Simplify your answer using positive exponents.

Achievement Check

17. The red squirre l population in a neighbourhood park can be modelled by the function $p(t) = \sqrt{210t + 44t^2}$, where p is the number of red squirrels, and t is time, in years.

a) Determine the rate of growth of the squirrel population after 2 years.

b) When will the population reach 60 squirrels?

c) What is the rate of change of the population at the time in part b)?

d) When is the rate of change of the squirrel population approximately 7 squirrels per year?

C Extend and Challenge

18. Determine the equations of the tangents to the curve $y = x^3\sqrt{8x^2 + 1}$ at the points where $x = 1$ and $x = -1$. How are the tangent lines related? Explain why this relationship is true at all points with corresponding positive and negative values $x = a$ and $x = -a$.

19. Determine $f'(2)$ for $f(x) = g[h(x)]$, given $g(2) = 5$, $g'(2) = -3$, $g'(-6) = -3$, $h(2) = -6$, and $h'(2) = 4$.

20. Determine $\dfrac{d^2y}{dx^2}$ for the function $y = \sqrt{2x + 1}$.

21. Consider the statement $\dfrac{d}{dx} f \circ g(x) = \dfrac{d}{dx} g \circ f(x)$.

Determine examples of two differentiable functions $f(x)$ and $g(x)$ to show

a) when the statement *is not* true

b) when the statement *is* true

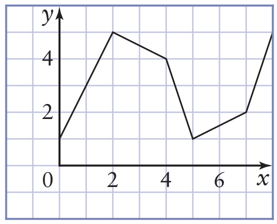
22. If $f(x) = x^2$, $g(x) = \dfrac{1}{x}$, and $h(x) = \sqrt{x^2 + 2x}$, determine the derivative of each composite function.

a) $y = f \circ g \circ h(x)$ **b)** $y = g \circ f \circ h(x)$

c) $y = g \circ h \circ f(x)$ **d)** $y = h \circ g \circ f(x)$

23. Determine a rule to differentiate a composite function of the form $y = f \circ g \circ h(x)$ given that f, g, and h are all differentiable functions.

24. **Math Contest** This figure shows the graph of $y = f(x)$.



If the function F is defined by $F(x) = f[f(x)]$, then $F(1)$ equals

A -1 **B** 2 **C** 4

D 4.5 **E** undefined

25. **Math Contest** Let f be a function such that $f'(x) = \dfrac{1}{x}$. What is the derivative $(f^{-1})'(x)$ of its inverse? Hint: If $y = f^{-1}(x)$, you can write $f(y) = x$.

A 0 **B** x **C** $-\dfrac{1}{x^2}$ **D** $f(x)$ **E** $f^{-1}(x)$

2.5 Derivatives of Quotients

Suppose the function $V(t) = \dfrac{50\,000 + 6t}{1 + 0.4t}$ represents the value, in dollars, of a new car t years after it is purchased. You want to calculate the rate of change in the value of the car at 2 years, 5 years, and 7 years to determine at what rate the car is depreciating.

This problem is similar to the one considered in Section 1.1, though at that time you were only considering data and graphs. In this section, you will explore strategies for differentiating a function that has the form $q(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$.

Example 1 Differentiate a Quotient When the Denominator's Exponent is 1

Differentiate $q(x) = \dfrac{6x - 5}{x^3 + 4}$. State the domain of $q(x)$ and $q'(x)$.

Solution

$$q(x) = (6x - 5)(x^3 + 4)^{-1}$$

Express as a product.

$$q'(x) = \left[\frac{d}{dx}(6x - 5)\right](x^3 + 4)^{-1} + (6x - 5)\left[\frac{d}{dx}(x^3 + 4)^{-1}\right]$$

Apply the product rule.

$$= (6)(x^3 + 4)^{-1} + (6x - 5)(-1)(x^3 + 4)^{-2}(3x^2)$$

Apply the chain rule.

$$= 3(x^3 + 4)^{-2}[2(x^3 + 4)] + (6x - 5)(-1)x^2]$$

Common factor $3(x^3 + 4)^{-2}$.

$$= 3(x^3 + 4)^{-2}(2x^3 + 8 - 6x^3 + 5x^2)$$

$$= \frac{3(-4x^3 + 5x^2 + 8)}{(x^3 + 4)^2}$$

The denominator cannot be zero, so

$$x^3 + 4 \neq 0$$
$$x^3 \neq -4$$
$$x \neq \sqrt[3]{-4}$$

The domain of $q(x)$ and $q'(x)$ is $\{x \in \mathbb{R} \mid x \neq \sqrt[3]{-4}\}$.

Differentiating a Simple Quotient Function

If $q(x) = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are differentiable functions, and $g'(x) \neq 0$, then $q(x)$ can be expressed as $q(x) = f(x)[g(x)]^{-1}$.

Then $q'(x) = f(x)(-1)[g(x)]^{-2}\, g'(x) + f'(x)[g(x)]^{-1}$.

Example 2 Differentiate a Quotient When the Denominator's Exponent is *n*

Differentiate $q(x) = \dfrac{x+3}{\sqrt{x^2-1}}$. State the domain of $q(x)$ and $q'(x)$.

Solution

$q(x) = (x+3)(x^2-1)^{-\frac{1}{2}}$ Express as a product.

$q'(x) = \left[\dfrac{d}{dx}(x+3)\right](x^2-1)^{-\frac{1}{2}} + (x+3)\left[\dfrac{d}{dx}(x^2-1)^{-\frac{1}{2}}\right]$ Apply the product rule.

$= 1(x^2-1)^{-\frac{1}{2}} + (x+3)\left(-\dfrac{1}{2}\right)(x^2-1)^{-\frac{3}{2}}(2x)$ Apply the chain rule.

$= (x^2-1)^{-\frac{3}{2}}[(x^2-1) - x(x+3)]$ Common factor $(x^2-1)^{-\frac{3}{2}}$

$= (x^2-1)^{-\frac{3}{2}}(x^2-1-x^2-3x)$

$= \dfrac{-3x-1}{\left(\sqrt{x^2-1}\right)^3}$

The denominator cannot be zero. Also, x^2-1 must be positive. So,

$x^2-1>0$
$x^2>1$
$x<-1$ or $x>1$

The domain of $q(x)$ and $q'(x)$ is $\{x \in \mathbb{R} \mid x>1 \text{ or } x<-1\}$.

Example 3 Find the Equation of the Tangent

Determine the equation of the tangent to the curve $y = \dfrac{x^2 - 3}{5 - x}$ at $x = 2$.

Solution

$$y = (x^2 - 3)(5 - x)^{-1}$$

$$\begin{aligned} y' &= \left[\frac{d}{dx}(x^2 - 3)\right](5 - x)^{-1} + (x^2 - 3)\left[\frac{d}{dx}(5 - x)^{-1}\right] \\ &= (2x)(5 - x)^{-1} + (x^2 - 3)(-1)(5 - x)^{-2}(-1) \end{aligned}$$

Use the unsimplified form to determine the slope of the tangent.

$$\begin{aligned} f'(2) &= 2(2)(5 - 2)^{-1} + (2^2 - 3)(-1)(5 - 2)^{-2}(-1) \\ &= 4(3)^{-1} + (1)(3)^{-2} \\ &= \frac{4}{3} + \frac{1}{9} \\ &= \frac{13}{9} \end{aligned}$$

Determine the y-coordinate for the function at $x = 2$ by substituting into the original function.

$$\begin{aligned} y &= \frac{2^2 - 3}{5 - 2} \\ &= \frac{1}{3} \end{aligned}$$

The tangent point is $\left(2, \frac{1}{3}\right)$.

Substitute $m = \frac{13}{9}$ and $(x_1, y_1) = \left(2, \frac{1}{3}\right)$ into $y - y_1 = m(x - x_1)$.

$$\begin{aligned} y - \frac{1}{3} &= \frac{13}{9}(x - 2) \\ y &= \frac{13}{9}(x - 2) + \frac{1}{3} \\ &= \frac{13}{9}x - \frac{26}{9} + \frac{3}{9} \\ &= \frac{13}{9}x - \frac{23}{9} \end{aligned}$$

The equation of the tangent is $y = \frac{13}{9}x - \frac{23}{9}$.

Example 4 Solve a Rate of Change Problem Involving a Quotient

Recall the problem introduced at the beginning of this section:

Suppose the function $V(t) = \dfrac{50\,000 + 6t}{1 + 0.4t}$ represents the value, in dollars, of a new car t years after it is purchased.

a) What is the rate of change of the value of the car at 2 years, 5 years, and 7 years?

b) What was the initial value of the car?

c) Explain how the values in part a) can be used to support an argument in favour of purchasing a used car, rather than a new one.

Solution

a) $V(t) = (50\,000 + 6t)(1 + 0.4t)^{-1}$

$$\begin{aligned}
V'(t) &= (50\,000 + 6t)\frac{d}{dt}\left[(1 + 0.4t)^{-1}\right] + (1 + 0.4t)^{-1}\frac{d}{dt}\left[50\,000 + 6t\right] \\
&= (50\,000 + 6t)[-(1 + 0.4t)^{-2}(0.4)] + (6)(1 + 0.4t)^{-1} \\
&= (1 + 0.4t)^{-2}[-0.4(50\,000 + 6t) + 6(1 + 0.4t)] \qquad \text{Common factor } (1 + 0.4t)^{-2}. \\
&= (1 + 0.4t)^{-2}(-20\,000 - 2.4t + 6 + 2.4t) \\
&= (1 + 0.4t)^{-2}(-19\,994) \\
&= \frac{-19\,994}{(1 + 0.4t)^2}
\end{aligned}$$

When $t = 2$, $V'(2) = \dfrac{-19\,994}{[1 + 0.4(2)]^2}$
$= -6170.99$

After 2 years, the value of the car is decreasing at a rate of \$6170.99/year.

When $t = 5$, $V'(5) = \dfrac{-19\,994}{[1 + 0.4(5)]^2}$
$= -2221.56$

After 5 years, the value of the car is decreasing at a rate of \$2221.56/year.

When $t = 7$, $V'(7) = \dfrac{-19\,994}{[1 + 0.4(7)]^2}$
$= -1384.63$

After 7 years, the value of the car is decreasing at a rate of \$1384.63/year.

b) The initial value of the car occurs when $t = 0$.

$V(0) = 50\,000$

The initial value of the car was \$50 000.

c) The value of the car decreases at a slower rate as the vehicle ages. This would suggest that it makes less sense to purchase a new vehicle because it depreciates rapidly in the first few years.

KEY CONCEPTS

- To find the derivative of a quotient $q(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$,
  i) Express $q(x)$ as a product.
  ii) Differentiate the resulting expression using the product and chain rules.
  iii) Simplify the result to involve only positive exponents, if appropriate.

Communicate Your Understanding

C1 Describe the similarities and differences between the derivatives of $y = \frac{1}{x^2+1}$ and $y = \frac{x}{x^2+1}$.

C2 Predict a rule that could be used to determine the derivative of quotients of the form $y = \frac{1}{g(x)}$, $g(x) \neq 0$.

C3 "The derivative of $q(x) = \frac{f(x)}{g(x)}$ is $q'(x) = \frac{f'(x)}{g'(x)}$." Is this statement true or false? Use an example to explain your answer.

C4 Why is it important to state that $f(x)$ and $g(x)$ are two differentiable functions, and that $g(x) \neq 0$ when differentiating $q(x) = \frac{f(x)}{g(x)}$?

A Practise

1. Express each quotient as a product, and state the domain of x.

a) $q(x) = \frac{1}{3x+5}$

b) $f(x) = \frac{-2}{x-4}$

c) $g(x) = \frac{6}{7x^2+1}$

d) $r(x) = \frac{-2}{x^3-27}$

2. Differentiate each function in question 1. Do not simplify your answers.

3. Express each quotient as a product, and state the domain of x.

a) $q(x) = \frac{3x}{x+1}$

b) $f(x) = \frac{-x}{2x+3}$

c) $g(x) = \frac{x^2}{5x-4}$

d) $r(x) = \frac{8x^2}{x^2-9}$

4. Differentiate each function in question 3. Do not simplify.

B Connect and Apply

5. Differentiate.

a) $y = \dfrac{-x + 3}{2x^2 + 5}$

b) $y = \dfrac{4x + 1}{x^3 - 2}$

c) $y = \dfrac{9x^2 - 1}{1 + 3x}$

d) $y = \dfrac{x^4}{x^2 - x + 1}$

6. Determine the slope of the tangent to each curve at the indicated value of x.

a) $y = \dfrac{x^2}{6x + 2}$, $x = -2$

b) $y = \dfrac{\sqrt{x}}{3x^2 - 1}$, $x = 1$

c) $y = \dfrac{4x + 1}{x^2 - 1}$, $x = -3$

d) $y = \dfrac{2x}{x^2 - x + 1}$, $x = -1$

e) $y = \dfrac{x^3 - 3}{x^2 + x - 1}$, $x = 2$

7. a) Describe two different methods that can be used to differentiate $q(x) = \dfrac{-4x^3 + 5x^2 - 2x + 6}{x^3}$. Differentiate using both methods and explain why you prefer one method over the other.

b) Can both methods that you described in part a) be used to differentiate $q(x) = \dfrac{-4x^3 + 5x^2 - 2x + 6}{x^3 + 1}$? Explain.

8. Determine the points on the curve $y = \dfrac{x^2}{x + 2}$ where the slope of the tangent line is -3.

9. Determine the equation of the tangent to the curve $y = \left(\dfrac{x^3 - 1}{x + 2}\right)^2$ at the point where $x = -1$.

10. Alison has let her hamster run loose in her living room. The position function of the hamster is $s(t) = \dfrac{5t}{t^2 + 4}$, $t \geq 0$, where s is in metres, and t is in seconds.

a) How fast is the hamster moving after 1 s?

b) When does the hamster change direction?

11. The number of clients investing in a new mutual fund w weeks after it is introduced into the market is modelled by the function $C(w) = \dfrac{800w^2}{200 + w^3}$, where C is the number of clients, and $w \geq 0$.

a) Determine $C'(1)$, $C'(3)$, $C'(5)$, and $C'(8)$. Interpret the meaning of these values for this situation.

b) **Use Technology** Use a graphing calculator to sketch the graph of $C(w)$. Explain how this graph can be used to determine when $C'(w)$ is positive, zero, and negative.

c) **Use Technology** Use a graphing calculator to sketch the graph of $C'(w)$. Use this graph to confirm your answers to part b).

d) Confirm your solution to part b) algebraically.

e) Interpret your results from part b).

12. **Chapter Problem** Mooses, Gooses, and Juices has launched a television and Internet advertising campaign to attract new customers. The predicted number of new customers, N, can be modelled by the function $N(x) = \dfrac{500x^2}{\sqrt{280 + x^2}} + 10x$, where x is the number of weeks after the launch of the advertising campaign.

a) Determine the predicted number of new customers 8 weeks after the campaign launch.

b) Determine the predicted average number of new customers per week between 1 and 6 weeks.

c) Determine the rate of change of the predicted number of new customers at week 1 and at week 6.

d) Is the rate of change of new customers ever negative? Explain what this implies.

13. The value of an original painting t years after it is purchased is modelled by the function $V(t) = \frac{(2500 + 0.2t)(1 + t)}{\sqrt{0.5t + 2}}$, where V is in dollars, and $t \geq 0$.

a) What was the purchase price of the painting?

b) Determine the rate of change of the value of the painting after t years.

c) Is the value of the painting increasing or decreasing? Justify your answer.

d) Compare $V'(2)$ and $V'(22)$. Interpret the meaning of these values.

C Extend and Challenge

14. **a)** Determine a pattern for the nth derivative of $f(x) = \frac{1}{ax + b}$. State the restriction on the denominator.

b) Use your pattern from part a) to determine the fourth derivative of $f(x) = \frac{1}{2x - 3}$.

15. Consider the function $p(x) = \frac{x^2 - 4}{x^2 + 4}$.

a) Determine the points on the graph of $p(x)$ that correspond to $p'(x) = 0$.

b) Determine the points on the graph of $p(x)$ that correspond to $p''(x) = 0$.

c) **Use Technology** Use graphing technology to sketch $p(x)$. What do the points found in parts a) and b) represent on the graph of $p(x)$? Explain.

d) **Use Technology** Use graphing technology to sketch $p''(x)$. What do the points found in part b) represent on the graph of $P'(x)$? Explain.

16. Given $f(x) = \frac{x}{\sqrt{x - 1}}$ and $g(x) = \frac{1}{x} + x$, determine the derivative of each composite function and state its domain.

a) $y = f \circ g(x)$ **b)** $y = g \circ f(x)$

17. **Math Contest** Consider the function $f(x) = \frac{ax + b}{cx + d}$. Depending on the choice of the real constants a, b, c, and d, which of the following are possible for the graph of $y = f(x)$?

i) The graph has no points with horizontal tangents.

ii) The graph has exactly one point with a horizontal tangent.

iii) The graph has infinitely many points with horizontal tangents.

A i) only

B ii) only

C iii) only

D i) and iii) only

E i), ii), and iii)

18. **Math Contest** Let p be a quadratic polynomial such that $p(0) = 0$. Consider the function $F(x) = \frac{p(x)}{x^2 + 1}$. If $F'(0) = 0$ and $F'(1) = 1$, then $p(x)$ equals

A $\frac{2x^2}{3}$

B $\frac{4x^2}{3}$

C $\frac{4x^2}{5}$

D $2x^2$

E $-2x^2$

Extension The Quotient Rule

In the previous section, you differentiated rational functions by expressing the denominator in terms of a negative exponent and then used the product and chain rules to determine the derivative. Another method that can be used to differentiate functions of the form $q(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$, is the quotient rule.

Investigate What is the quotient rule for derivatives?

A: Develop the Quotient Rule From the Product Rule

Consider the quotient $Q(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are two differentiable functions, and $g(x) \neq 0$.

1. Multiply each side of the above equation by $g(x)$.
2. Differentiate both sides of the equation, using the product rule where needed.
3. Isolate $Q'(x)\,g(x)$ in the equation in step 2.
4. Substitute $Q(x) = \frac{f(x)}{g(x)}$ in the result of step 3.
5. Express the right side of the result of step 4 in terms of a common denominator.
6. Isolate $Q'(x)$ in the result of step 5.
7. Now look at another way to develop a derivative for $Q(x)$.

   $Q(x) = \frac{f(x)}{g(x)}$ can also be written as $Q(x) = f(x)g^{-1}(x)$.

   Differentiating this expression using the product and chain rules gives $Q'(x) = f(x)(-1)[g(x)]^{-2}\,g'(x) + f'(x)[g(x)]^{-1}$.

   Common factor $g(x)^{-2}$ from this expression. Simplify the resulting expression using only positive exponents.
8. **Reflect** Compare the results of steps 6 and 7.

B: Verify the Quotient Rule

Consider the function $Q(x) = \frac{x^3 - 4x^2}{3x^2 + x}$.

1. Differentiate $Q(x)$ by changing it to a product and differentiating the result, as was done in Section 2.5. Simplify your final answer using positive exponents.
2. Differentiate $Q(x)$ using the result from step 6 of Part A of this Investigate. Simplify your answer.
3. a) **Reflect** Compare your answers in steps 1 and 2. Describe the results.

   b) Repeat steps 1 and 2 for a quotient of your choice. Are the answers the same? Which method do you prefer? Why?

Quotient Rule

If $q(x) = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are differentiable functions and $g'(x) \neq 0$,

then $q'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

Example 1 Apply the Quotient Rule

Differentiate $q(x) = \dfrac{4x^3 - 7}{2x^2 + 3}$.

Solution

Apply the quotient rule.

$$\begin{aligned} q'(x) &= \frac{(2x^2+3)\frac{d}{dx}(4x^3-7) - (4x^3-7)\frac{d}{dx}(2x^2+3)}{(2x^2+3)^2} \\ &= \frac{(2x^2+3)(12x^2) - (4x^3-7)(4x)}{(2x^2+3)^2} \\ &= \frac{(24x^4+36x^2) - (16x^4-28x)}{(2x^2+3)^2} \\ &= \frac{8x^4+36x^2+28x}{(2x^2+3)^2} \end{aligned}$$

Example 2 Combine the Chain Rule and the Quotient Rule

Differentiate $g(x) = \dfrac{4x+1}{\sqrt{1-x}}$.

Solution

$$g'(x) = \frac{(1-x)^{\frac{1}{2}}\frac{d}{dx}(4x+1) - (4x+1)\frac{d}{dx}(1-x)^{\frac{1}{2}}}{(1-x)}$$ Apply the quotient rule.

$$= \frac{(1-x)^{\frac{1}{2}}(4) - (4x+1)\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)}{(1-x)}$$ Apply the chain rule.

$$= \frac{\frac{1}{2}(1-x)^{-\frac{1}{2}}[(1-x)(8) + (4x+1)(-1)]}{(1-x)}$$ Common factor.

$$= \frac{8-8x-4x-1}{2(1-x)^{\frac{3}{2}}}$$

$$= \frac{7-12x}{2(1-x)^{\frac{3}{2}}}$$

A Practise

In the first three questions, you will revisit questions 4, 5, and 6 of section 2.5, using the quotient rule to differentiate.

1. Differentiate using the quotient rule.

a) $q(x) = \dfrac{3x}{x+1}$ **b)** $f(x) = \dfrac{-x}{2x+3}$

c) $g(x) = \dfrac{x^2}{5x-4}$ **d)** $r(x) = \dfrac{8x^2}{x^2-9}$

2. Differentiate using the quotient rule.

a) $y = \dfrac{-x+3}{2x^2+5}$ **b)** $y = \dfrac{4x+1}{x^3-2}$

c) $y = \dfrac{9x^2-1}{1+3x}$ **d)** $y = \dfrac{x^4}{x^2-x+1}$

3. Use the quotient rule to determine the slope of the tangent to each curve at the indicated value of x.

a) $y = \dfrac{x^2}{6x+2}$, $x = -2$

b) $y = \dfrac{\sqrt{x}}{3x^2-1}$, $x = 1$

c) $y = \dfrac{4x+1}{x^2-1}$, $x = -3$

d) $y = \dfrac{2x}{x^2-x+1}$, $x = -1$

e) $y = \dfrac{x^3-3}{x^2+x-1}$, $x = 2$

4. a) Given $y = \dfrac{1}{(x^2+3x)^5}$, determine the derivative by using the following two methods.

i) Use the quotient rule.

ii) First express the quotient using a negative exponent and then use the power of a function rule.

b) Which method in part a) is more efficient? Justify your answer.

c) Which method in part a) do you prefer? Explain.

5. Use the chain rule to determine

$$\left.\frac{dy}{dx}\right|_{x=2} \text{ for } y = \frac{u^3}{u^2+1},\ u = 3x - x^2.$$

6. Determine the points on the curve $y = \dfrac{x^2}{2x+5}$ where the tangent line is horizontal.

7. Given $f(x) = 15x^5 - 9x^3$ and $g(x) = 3x^2$, is the following true? Justify your answer.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \left[\frac{d}{dx}f(x)\right] \div \left[\frac{d}{dx}g(x)\right]$$

8. The number, n, of new FAST cars sold by RACE dealership w weeks after going on the market is represented by the function $n(w) = \dfrac{300w^2}{1+w^2}$, where $0 \le w \le 10$.

a) At what rate is the number of sales changing after 1 week? At what rate are the sales changing after 5 weeks?

b) Does the number of sales per week decrease at any time during this 10-week period? Justify your answer.

9. The concentration of an antibiotic in the blood t hours after it is taken is represented by the function $c(t) = \dfrac{4t}{3t^2+4}$. Determine $c'(3)$ and interpret its meaning for this situation.

10. A chemical cleaner from a factory is accidently spilled into a nearby lake. The concentration of cleaner in the water t days after it is spilled is represented by the function $c(t) = \dfrac{6t}{2t^2+9}$, where c is in grams per litre.

a) At what rate is the concentration of the chemical cleaner changing after 1 day, 4 days, and 1 week?

b) When is the rate of change of concentration zero? When is it positive? When is it negative?

c) **Use Technology** Confirm your answers to part b) using the graphs of $c(t)$ and $c'(t)$.

d) Interpret the meaning of your answers to part b) for this situation.

e) Determine $c''(4)$ and interpret its meaning for this situation.

2.6 Rate of Change Problems

Earlier in this chapter, the connection between calculus and physics was examined in relation to velocity and acceleration. There are many other applications of calculus to physics, such as the analysis of the change in the density of materials and the rate of flow of an electrical current. But calculus is applied far beyond the realm of physics. In biology, derivatives are used to determine growth rates of populations or the rate of concentration of a drug in the bloodstream. In chemistry, derivatives are used to analyse the rate of reaction of chemicals. In the world of business and economics, rates of change pertaining to profit, revenue, cost, price, and demand are measured in terms of the number of items sold or produced. This section will focus on applying derivatives to solve problems involving rates of change in the social and physical sciences.

Rates of Change in Business and Economics

The primary goal of most businesses is to generate profits. To achieve this goal, many different aspects of the business need to be considered and measured. For instance, a business has to determine the price for its products that will maximize profits. If the price of a product or service is set too high, fewer consumers are willing to buy it. This often results in lower revenues and lower profits. If the price is set too low, the cost of producing large quantities of the item may also result in reduced profits. A delicate balance often exists between the cost, revenue, profit, and demand functions.

Functions Pertaining to Business

- The **demand function**, or **price function**, is $p(x)$, where p is the number of units of a product or service that can be sold at a particular price, x.
- The **revenue function**, is $R(x) = xp(x)$, where x is the number of units of a product or service sold at a price per unit of $p(x)$.
- The **cost function**, $C(x)$, is the total cost of producing x units of a product or service.
- The **profit function**, $P(x)$, is the profit from the sale of x units of a product or service. The profit function is the difference between the revenue function and the cost function: $P(x) = R(x) - C(x)$.

Derivatives of Business Functions

Economists use the word *marginal* to indicate the derivative of a business function.

- $C'(x)$ or $\dfrac{dC}{dx}$ is the **marginal cost function** and refers to the instantaneous rate of change of total cost with respect to the number of items produced.
- $R'(x)$ or $\dfrac{dR}{dx}$ is the **marginal revenue function** and refers to the instantaneous rate of change of total revenue with respect to the number of items sold.
- $P'(x)$ or $\dfrac{dP}{dx}$ is the **marginal profit function** and refers to the instantaneous rate of change of total profit with respect to the number of items sold.

Example 1 Apply Mathematical Modelling to Determine the Demand Function

A company sells 1500 movie DVDs per month at \$10 each. Market research has shown that sales will decrease by 125 DVDs per month for each \$0.25 increase in price.

a) Determine the demand, or price, function.

b) Determine the marginal revenue when sales are 1000 DVDs per month.

c) The cost of producing x DVDs is $C(x) = -0.004x^2 + 9.2x + 5000$. Determine the marginal cost when production is 1000 DVDs per month.

d) Determine the actual cost of producing the 1001st DVD.

e) Determine the profit and marginal profit from the monthly sales of 1000 DVDs.

Solution

a) Let p be the price of one movie DVD.
Let x be the number of DVDs sold per month.
Let n be the number of \$0.25 increases in price.

Two equations can be derived from this information:

① $x = 1500 - 125n$ and

② $p = 10 + 0.25n$

You want to express p in terms of x.

From ①, you have $n = \dfrac{1500 - x}{125}$.

Substitute this expression into ②.

$$\begin{aligned} p &= 10 + 0.25\left(\frac{1500 - x}{125}\right) \\ &= 10 + 0.002(1500 - x) \\ &= 10 + 3 - 0.002x \\ &= 13 - 0.002x \end{aligned}$$

The demand function is $p(x) = 13 - 0.002x$. This function gives the price for one DVD when x of them are being sold.

b) The revenue function is

$$\begin{aligned} R(x) &= xp(x) \\ &= x(13 - 0.002x) \\ &= 13x - 0.002x^2 \end{aligned}$$

Take the derivative to determine the marginal revenue function for this situation.

$$\begin{aligned} R'(x) &= 13 - 0.004x \\ R'(1000) &= 13 - 0.004(1000) \\ &= 9 \end{aligned}$$

When sales are at 1000 DVDs per month, revenue is increasing at the rate of \$9.00 per additional DVD.

c)

$$\begin{aligned} C(x) &= -0.004x^2 + 9.2x + 5000 \\ C'(x) &= -0.008x + 9.2 \\ C'(1000) &= -0.008(1000) + 9.2 \\ &= 1.20 \end{aligned}$$

When production is at 1000 DVDs per month, the marginal cost is \$1.20.

d) The cost of producing the 1001st DVD is

$$\begin{aligned} C(1001) - C(1000) &= [-0.004(1001)^2 + 9.2(1001) + 5000] \\ &\quad - [-0.004(1000)^2 + 9.2(1000) + 5000] \\ &= 10\,201.196 - 10\,200.00 \\ &= 1.196 \end{aligned}$$

The actual cost of producing the 1001st DVD is \$1.196. Notice the similarity between the marginal cost of the 1000th DVD and the actual cost of producing the 1001st DVD. For large values of x, the marginal cost when producing x items is approximately equal to the cost of producing one more item, the $(x + 1)$th item.

e) The profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 13x - 0.002x^2 - (-0.004x^2 + 9.2x + 5000) \\ &= 0.002x^2 + 3.8x - 5000 \\ P(1000) &= 0.002(1000)^2 + 3.8(1000) - 5000 \\ &= 800 \\ P'(x) &= 0.004x + 3.8 \\ P'(1000) &= 0.004(1000) + 3.8 \\ &= 7.80 \end{aligned}$$

When 1000 DVDs per month are sold, the total profit is \$800, and the marginal profit is \$7.80 per DVD.

Example 2 Apply Mathematical Modelling to Determine the Revenue Function

An ice cream shop sells 150 cookies 'n' cream ice cream cakes per month at a price of \$40 each. A customer survey indicates that for each \$1 decrease in price, sales will increase by 5 cakes.

a) Determine a revenue function based on the number of price decreases.

b) Determine the marginal revenue for the revenue function developed in part a).

c) When is this marginal revenue function equal to zero? What is the total revenue at this time? How can the owners use this information?

Solution

a) It is not always necessary to involve the demand function as in Example 1.
Revenue = price × sales.
Let n represent the number of \$1 decreases in the cake price.
The price is $p = 40 - n$.
For each decrease in price, cake sales increase by 5, so sales = $150 + 5n$.
The revenue function is $R(n) = (40 - n)(150 + 5n)$.

b)
$$\begin{aligned} R'(n) &= (-1)(150 + 5n) + (40 - n)(5) \\ &= -150 - 5n + 200 - 5n \\ &= -10n + 50 \end{aligned}$$

c) Solve $R'(n) = 0$.

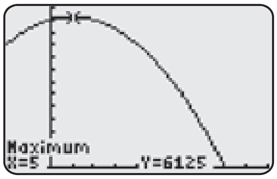
$$\begin{aligned} -10n + 50 &= 0 \\ -10n &= -50 \\ n &= 5 \end{aligned}$$

The marginal revenue is zero when there are five \$1 decreases in the price of the cakes. Cakes then sell for \$35.

$$\begin{aligned} R(5) &= (40 - 5)(150 + 5(5)) \\ &= (35)(175) \\ &= 6125 \end{aligned}$$

When the price is \$35, total revenue is \$6125.

As shown below, you can use a graphing calculator to verify that the maximum point on the graph of $R(n)$ occurs at $x = 5$.



Window variables:
$x \in [-10, 50]$, Xscl = 5,
$y \in [0, 6500]$, Yscl = 500

The owners of the ice cream shop should realize that decreasing the price further will lead to increased sales, but decreased total revenue.

Example 3 Apply Derivatives to Kinetic Energy

Kinetic energy, K, is the energy due to motion. When an object is moving, its kinetic energy is determined by the formula $K(v) = 0.5mv^2$, where K is in joules, m is the mass of the object, in kilograms, and v is the velocity of the object, in metres per second.

Suppose a ball with a mass of 350 g is thrown vertically upward with an initial velocity of 40 m/s. Its velocity function is $v(t) = 40 - 9.8t$, where t is time, in seconds.

a) Express the kinetic energy of the ball as a function of time.

b) Determine the rate of change of the kinetic energy of the ball at 3 s.

Solution

a) Substitute $m = 0.350$ kg and $v(t) = 40 - 9.8t$ into $K(v) = 0.5mv^2$.

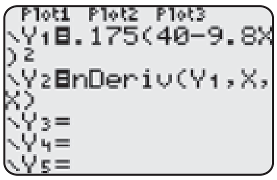
$$K[v(t)] = 0.5(0.350)(40 - 9.8t)^2$$
$$K(t) = 0.175(40 - 9.8t)^2$$

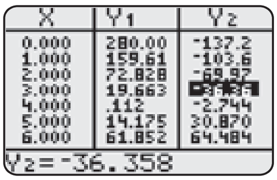
b) Differentiate.

$$\begin{aligned} K'(t) &= 0.175(2)(40 - 9.8t)(-9.8) \\ &= -3.43(40 - 9.8t) \\ K'(3) &= -3.43(40 - 9.8(3)) \\ &= -36.358 \end{aligned}$$

At 3 s, the rate of change of the kinetic energy of the ball is -36.358 J/s. The negative value indicates that the kinetic energy is decreasing.

Confirm this value using a graphing calculator as shown below.





Example 4 Apply Derivatives to Electrical Currents

In a certain electrical circuit, the resistance, R, in ohms, is represented by the function $R = \frac{150}{I}$, where I is the current, in amperes (A). Determine the rate of change of the resistance with respect to the current when the current is 10 A.

Solution

$R = 150I^{-1}$ Express $R = \frac{150}{I}$ as a power with a negative exponent.

$$\frac{dR}{dI} = (-1)150I^{-2}$$

$$\frac{dR}{dI} = -\frac{150}{I^2}$$

$$\left.\frac{dR}{dI}\right|_{I=10} = -\frac{150}{10^2}$$

$$= -1.5$$

When the current is 10 A, the rate of decrease of the resistance is 1.5 Ω /A.

CONNECTIONS

Ohm's law is $R = \frac{V}{I}$. It is named after its discoverer, Georg Ohm, who published it in 1827.

Derivatives and Linear Density

Derivatives can be used in the analysis of different types of densities. For instance, population density refers to the number of people per unit area. Colour density, used in the study of radiographs, refers to the depth of colour per unit area. Linear density refers to the mass of an object per unit length: linear density $= \frac{\text{mass}}{\text{length}}$.

Consider a linear object, such as a rod or wire. If it is made of the exact same material along its entire length, it is said to be made out of homogenous material (*homogeneous* means the same or similar). In cases like this, the linear density of the object is constant at every point. This would not be true of an object made of nonhomogenous materials, in which case the linear density would vary along the object's length.

Suppose the function $f(x)$ gives the mass, in kilograms, of the first x metres of the object. For the part of the object that lies between $x = x_1$ and $x = x_2$, the average linear density (or mass per unit length) is defined as $\frac{f(x_1) - f(x_2)}{x_1 - x_2}$. The corresponding derivative function $\rho(x) = f'(x)$ is the linear density, the rate of change of density at a particular length x.

CONNECTIONS

ρ is the Greek letter rho.

Example 5 Represent Linear Density as a Rate of Change

The mass, in kilograms, of the first x metres of a wire is represented by the function $f(x) = \sqrt{3x + 1}$.

a) Determine the average linear density of the part of the wire from $x = 5$ to $x = 8$.

b) Determine the linear density at $x = 5$ and at $x = 8$. Compare the densities at the two points. What do these values confirm about the wire?

Solution

a)

$$\begin{aligned} f(x) &= \sqrt{3x+1} \\ \text{average linear density} &= \frac{f(x_1) - f(x_2)}{x_1 - x_2} \\ &= \frac{f(8) - f(5)}{8 - 5} \\ &= \frac{\sqrt{3(8)+1} - \sqrt{3(5)+1}}{8 - 5} \\ &= 0.33 \end{aligned}$$

The average linear density for this part of the wire is approximately 0.33 kg/m.

b)

$$\begin{aligned} f(x) &= \sqrt{3x+1} \\ &= (3x+1)^{\frac{1}{2}} \\ \rho(x) &= f'(x) \\ &= \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3) \\ &= \frac{3}{2\sqrt{3x+1}} \\ \rho(5) &= \frac{3}{2\sqrt{3(5)+1}} \\ &= \frac{3}{2(4)} \\ &= 0.375 \\ \rho(8) &= \frac{3}{2\sqrt{3(8)+1}} \\ &= 0.3 \end{aligned}$$

The linear density at $x = 5$ is 0.375 kg/m, and at $x = 8$ it is 0.3 kg/m. Since the two density values are different, they confirm that the material of which the wire is composed is nonhomogenous.

KEY CONCEPTS

- The cost function, $C(x)$, is the total cost of producing x units of a product or service.
- The revenue function, $R(x)$, is the total revenue (income) from the sale of x units of a product or service. The revenue function is the product of the demand function, $p(x)$, and the number of items sold: $R(x) = xp(x)$.
- The profit function, $P(x)$, is the total profit from the sales of x units of a product or service. The profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.
- The demand, or price, function, $p(x)$, is the price at which x units of a product or service can be sold.
- $C'(x)$ is the marginal cost function.
- $R'(x)$ is the marginal revenue function.
- $P'(x)$ is the marginal profit function.

Communicate Your Understanding

C1 What does the word *marginal* refer to in economics and business problems?

C2 The demand function is also referred to as the price function. Explain why this is appropriate.

C3 What is the difference between negative marginal revenue and positive marginal revenue?

C4 Why is it true to say that for certain items the actual cost of producing the 1001st item is the same as the marginal cost of producing 1000 items? Explain your answer.

A Practice

1. The demand function for a DVD player is $p(x) = \dfrac{575}{\sqrt{x}} - 3$, where x is the number of DVD players sold and p is the price, in dollars. Determine the following:

a) the revenue function

b) the marginal revenue function

c) the marginal revenue when 200 DVD players are sold

2. Refer to question 1. If the cost, in dollars, of producing x DVD players is $C(x) = 2000 + 150x - 0.002x^2$, determine the following:

a) the profit function

b) the marginal profit function

c) the marginal profit for the sale of 500 DVD players

3. The cost, in dollars, of making x large combo pizzas at a local pizzeria is modelled by the function $C(x) = -0.001x^3 + 0.025x^2 + 4x$, and the price per large combo pizza is \$17.50. Determine the following:

   a) the revenue function

   b) the marginal revenue function

   c) the profit function

   d) the marginal profit function

   e) the marginal revenue and marginal profit for the sale of 300 large combo pizzas

B Connect and Apply

4. The mass, in grams, of the first x metres of a wire is represented by the function $f(x) = \sqrt{2x - 1}$.

   a) Determine the average linear density of the part of the wire from $x = 1$ to $x = 8$.

   b) Determine the linear density at $x = 5$ and at $x = 8$, and compare the densities at the two points. What do these values confirm about the wire?

5. A paint store sells 270 pails of paint per month at a price of \$32 each. A customer survey indicates that for each \$1.20 decrease in price, sales will increase by 6 pails of paint.

   a) Determine the demand, or price, function.

   b) Determine the revenue function.

   c) Determine the marginal revenue function.

   d) Solve $R'(x) = 0$. Interpret the meaning of this value for this situation.

   e) What is the price that corresponds to the value found in part d)? How can the paint store use this information?

6. The mass, in kilograms, of the first x metres of a metal rod is represented by the function $f(x) = (x - 0.5)^3 + 5x$.

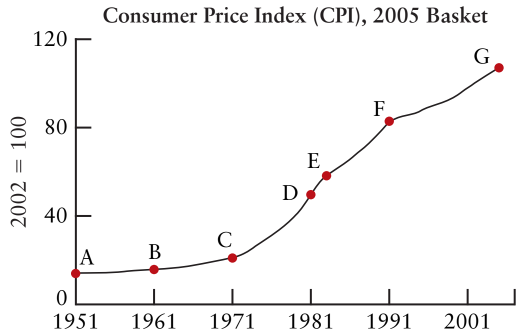
   a) Determine the average linear density of the part of the rod from $x = 1$ to $x = 3$.

   b) Determine the linear density at $x = 2$.

CONNECTIONS

Find out more about the consumer price index online. The Statistics Canada web site might be a good place to begin your search.

7. The following graph represents Canada's consumer price index (CPI) between 1951 and 2007. The CPI is an index number measuring the average price of consumer goods and services purchased by households. The percent change in the CPI is a measure of inflation.



Source: Statistics Canada. Table 326-0021 - Consumer price index (CPI), 2005 basket, annual (2002 = 100 unless otherwise noted), CANSIM (database).

   a) Consider the interval from A to G.

      i) Is the CPI increasing or decreasing over this interval? Justify your answer.

      ii) Is the rate of growth positive or negative during this period? Explain.

   b) Place each interval in order from the lowest rate of growth to the highest rate of growth. Explain your reasoning. State the years for each interval.

      i) A to B ii) B to C iii) C to D

      iv) D to E v) E to F vi) F to G

   c) Compare the rate of inflation from 1951 to 1975 with the rate of inflation after 1975. What conclusions can be made? Explain.

   d) Has the rate of inflation been increasing or decreasing since 1981? Justify your answer.

8. A yogurt company estimates that the revenue from selling x containers of yogurt is $4.5x$. Its cost for producing this number of containers of yogurt is $C(x) = 0.0001x^2 + 2x + 3200$.

a) Determine the marginal cost of producing 4000 containers of yogurt.

b) Determine the marginal profit from selling 4000 containers of yogurt.

c) What is the selling price of a container of yogurt? Explain.

9. The total cost, C, in dollars, of operating a factory that produces kitchen utensils is $C(x) = 0.5x^2 + 40x + 8000$, where x is the number of items produced, in thousands.

a) Determine the marginal cost of producing 5000 items and compare this with the actual cost of producing the 5001st item.

b) The average cost is found by dividing the total cost by the number of items produced. Determine the average cost of producing 5000 items. Compare this value to those found in part a). What do you notice?

c) Determine the rate of change of the average cost of producing 5000 items. Interpret the meaning of this value.

10. A demographer develops the function $P(x) = 12\ 500 + 320x - 0.25x^3$ to represent the population of the town of Calcville x years from today.

a) Determine the present population of the town.

b) Predict the rate of change of the population in 3 years and in 8 years.

c) When will the population reach 16 294?

d) When will the rate of growth of the population be 245 people per year?

e) Is the rate of change of the population increasing or decreasing? Explain.

11. The cost, in dollars, of producing x hot tubs is represented by the function $C(x) = 3450x - 1.02x^2$, $0 \le x \le 1500$.

a) Determine the marginal cost at a production level of 750 hot tubs. Explain what this means to the manufacturer.

b) Find the cost of producing the 751st hot tub.

c) Compare and comment on the values you found in parts a) and b).

d) Each hot tub is sold for \$9200. Write an expression that represents the total revenue from the sale of x hot tubs.

e) Determine the rate of change of profit for the sale of 750 hot tubs.

12. A certain electrical current, I, in amperes, can be modelled by the function $I = \dfrac{120}{R}$, where R is the resistance, in ohms. Determine the rate of change of the current with respect to the resistance when the resistance is 18 Ω.

13. An iron bar with an initial temperature of 10°C is heated such that its temperature increases at a rate of 4°C/min. The temperature, C, in degrees Celsius, at any time, t, in minutes, after heat is applied is given by the function $C = 10 + 4t$. The equation $F = 1.8C + 32$ is used to convert from C degrees Celsius to F degrees Fahrenheit. Determine the rate of change of the temperature of the bar with respect to time, in degrees Fahrenheit, after 4 min.

14. The size of the pupil of a certain animal's eye, in millimetres, is given by the function $f(x) = \dfrac{155x^{-0.5} + 85}{3x^{-0.5} + 18}$, where x is the intensity of light the pupil is exposed to. Is the rate of change of the size of the animal's pupil positive or negative? Interpret this result in terms of the response of the pupil to light.

15. A company's revenue for selling x items of a commodity, in thousands of dollars, is represented by the function $R(x) = \dfrac{15x - x^2}{x^2 + 15}$.

a) Determine the rate of change of revenue for the sale of 1000 items and of 5000 items.

b) Compare the values found in part a). Explain their meaning.

c) Determine the number of items that must be sold to obtain a $0 rate of change in revenue.

d) Determine the revenue for the value found in part c). Explain the significance of this value.

16. When a person coughs, the airflow to the lungs is increased because the cough dislodges particles that may be blocking the windpipe, thereby increasing the radius of the windpipe. Suppose the radius of a windpipe, when there is no cough, is 2.5 cm. The velocity of air moving through the windpipe at radius r can be modelled by the function $V(r) = cr^2(2.5 - r)$ for some constant, c.

a) Determine the rate of change of the velocity of air through the windpipe with respect to r when $r = 2.75$ cm.

b) Determine the value of r that results in $V'(r) = 0$. Interpret the meaning of this value for this situation.

17. A coffee shop sells 500 mocha lattes a month at $4.75 each. The results of a customer survey indicate that sales of mocha lattes would increase by 125 per month for each $0.25 decrease in price.

a) Determine the demand, or price, function.

b) Determine the revenue and marginal revenue from the monthly sales of 350 mocha lattes.

c) The cost of producing x mocha lattes is $C(x) = -0.0005x^2 + 3.5x + 400$. Determine the marginal cost of producing the 350 mocha lattes.

d) Determine the actual cost of producing the 351st mocha latte.

e) Determine the profit and marginal profit from the monthly sales of 350 mocha lattes.

f) Determine the average revenue and average profit for the sale of 360 mocha lattes. Compare these values with the results of parts b) and e). Explain any similarities, or give reasons for differences.

18. The mass, in grams, of a compound being formed during a chemical reaction is modelled by the function $M(t) = \dfrac{6.3t}{t + 2.2}$, where t is the time after the start of the reaction, in seconds.

a) Determine the rate of change of the mass after 6 s.

b) Is the rate of change of the mass ever negative? Explain.

C Extend and Challenge

19. The wholesale demand function of a personal digital assistant (PDA) is $p(x) = \dfrac{650}{\sqrt{x}} - 4.5$, where x is the number of PDAs sold, and p is the wholesale price, in dollars.

a) Determine the revenue function.

b) Determine the marginal revenue for the sale of 500 PDAs.

c) If it costs $125 to produce each PDA, determine the profit function.

d) Determine the marginal profit for the sale of 500 PDAs.

20. A patient's reaction to an antibacterial drug is represented by the function $r = \dfrac{m^2}{a}\left(\dfrac{1}{b} - \dfrac{m}{c}\right)$, where r is the time it takes for the body to react, in minutes, m is the amount of drug absorbed by the blood, in millilitres, and a, b, and c are positive constants. Determine $\dfrac{dr}{dm}$, the sensitivity of the patient to the drug, when 15 mL of the drug is administered.

21. Many sports involve hitting a ball with a striking object, such as a racquet, club, or bat. The velocity of the ball after being hit is represented by the function $u(W) = \dfrac{WV(1+c) + v(cW - w)}{W + w}$, where w is the weight of the ball (in grams), v is the velocity of the ball before it is hit (in metres per second), W is the weight of the striking object (in grams), $-V$ is the velocity of the striking object (in metres per second) before the collision (the negative value indicates that the striking object is moving in the opposite direction), and c is the coefficient of restitution, or bounciness, of the ball.

a) Show that $\dfrac{du}{dW} = \dfrac{V(1+c)w + cvw + vw}{(W+w)^2}$.

b) A baseball with mass 0.15 kg, coefficient of restitution of 0.575, and speed of 40 m/s is struck with a bat of mass m and speed 35 m/s (in the opposite direction to the ball's motion).

i) Determine the velocity of the ball after being hit, in terms of m.

ii) Determine the rate of change of the velocity of the ball when $m = 1.05$ kg.

22. **Math Contest** A water tank that holds V_0 litres of water drains in T minutes. The volume of water remaining in the tank after t minutes is given by the function $V = V_0\left(1 - \dfrac{t}{T}\right)^2$. The rate at which water is draining from the tank is the slowest when t equals

A 0 **B** $\dfrac{T}{2}$ **C** $\dfrac{(\sqrt{2}-1)T}{\sqrt{2}}$ **D** T **E** ∞

23. **Math Contest** In a certain chemical reaction, $X + Y \rightarrow Z$, the concentration of the product Z at time t is given by the function $z = \dfrac{c^2kt}{ckt+1}$, where c and k are positive constants. The rate of reaction $\dfrac{dz}{dt}$ can be written as

A $k(c-z)$

B $\dfrac{k}{c-z}$

C $k(c-z)^2$

D $\dfrac{k}{(c-z)^2}$

E none of the above

CAREER CONNECTION

Prakesh took a 4-year Bachelor of Science degree at the University of Guelph, specializing in microbiology. Since graduating, Prakesh has worked as a public health microbiologist. He detects and identifies micro-organisms, such as bacteria, fungi, viruses, and parasites, that are associated with infectious and communicable diseases. Prakesh uses derivatives in his work to help determine the growth rate of a bacterial culture when variables, such as temperature or food source are changed. He can then help the culture to increase its rate of growth of beneficial bacteria, or decrease the rate of growth of harmful bacteria.

Chapter 2 REVIEW

2.1 Derivative of a Polynomial Function

1. Differentiate each function. State the derivative rules used.

a) $h(t) = t^3 - 2t^2 + \frac{1}{t^2}$

b) $p(n) = -n^5 + 5n^3 + \sqrt[3]{n^2}$

c) $p(r) = r^6 - \frac{2}{5\sqrt{r}} + r - 1$

2. Air is being pumped into a spherical balloon. The volume of the balloon is $V = \frac{4}{3}\pi r^3$, where the radius, r, is in centimetres.

a) Determine the instantaneous rate of change of the volume of the balloon when its radius is 1.5 cm, 6 cm, and 9 cm.

b) Sketch a graph of the curve and the tangents corresponding to each radius in part a).

c) State the equations of the tangent lines.

2.2 The Product Rule

3. Differentiate using the power rule.

a) $f(x) = (5x + 3)(2x - 11)$

b) $h(t) = (2t^2 + \sqrt[3]{t})(4t - 5)$

c) $g(x) = (-1.5x^6 + 1)(3 - 8x)$

d) $p(n) = (11n + 2)(-5 + 3n^2)$

4. Determine the equation of the tangent to the graph of each curve at the point that corresponds to each value of x.

a) $y = (6x - 3)(-x^2 + 2)$, $x = 1$

b) $y = (-3x + 8)(x^3 - 7)$, $x = 2$

2.3 Velocity, Acceleration, and Second Derivatives

5. Determine $f''(-2)$ for $f(x) = (4 - x^2)(3x + 1)$.

6. A toy missile is shot into the air. Its height, in metres, after t seconds is given by the function $h(t) = -4.9t^2 + 15t + 0.4$, $t \geq 0$.

a) Determine the height of the missile after 2 s.

b) Determine the rate of change of the height of the missile at 1 s and at 4 s.

c) How long does it take the missile to return to the ground?

d) How fast was the missile travelling when it hit the ground? Explain your reasoning.

e) **Use Technology** Graph $h(t)$ and $v(t)$.

i) When does the toy missile reach its maximum height?

ii) What is the maximum height of the toy missile?

iii) What is the velocity of the missile when it reaches its maximum height? How can you tell this from the graph of the velocity function?

2.4 The Chain Rule

7. The population of a certain type of berry bush in a conservation area is represented by the function $p(t) = \sqrt[3]{16t + 50t^3}$, where p is the number of berry bushes and t is time, in years.

a) Determine the rate of change of the number of berry bushes after 5 years.

b) When will there be 40 berry bushes?

c) What is the rate of change of the berry bush population at the time found in part b)?

8. Apply the chain rule, in Leibniz notation, to determine $\frac{dy}{dx}$ at the indicated value of x.

a) $y = u^2 + 3u$, $u = \sqrt{x - 1}$, $x = 5$

b) $y = \sqrt{2u}$, $u = 6 - x$, $x = -3$

c) $y = 8u(1 - u)$, $u = \frac{1}{x}$, $x = 4$

2.5 Derivatives of Quotients

9. Determine the slope of the tangent to each.

a) $y = \frac{2x^2}{x + 1}$ at $x = 2$

b) $y = \frac{\sqrt{3x}}{x^2 - 4}$ at $x = 3$

c) $y = \frac{5x + 3}{x^3 + 1}$ at $x = -2$

d) $y = \frac{-4x + 2}{3x^2 - 7x - 1}$ at $x = 1$

10. Differentiate each function.

a) $q(x) = \dfrac{-7x + 2}{(4x^2 - 3)^3}$ **b)** $y = \dfrac{8x^3}{\sqrt{3x - 2}}$

c) $m(x) = \dfrac{(-x + 2)^2}{(3 + 5x)^4}$ **d)** $y = \dfrac{(x^2 - 3)^2}{\sqrt{4x + 5}}$

e) $y = \dfrac{(2\sqrt{x} + 7)^3}{(x^3 - 3x^2 + 1)^7}$

11. Determine the equation of the tangent to the curve $y = \left(\dfrac{x^2 - 1}{4x + 7}\right)^3$ at the point where $x = -2$.

2.6 Rate of Change Problems

12. A music store sells an average of 120 music CDs per week at \$24 each. A market survey indicates that for each \$0.75 decrease in price, 5 additional CDs will be sold per week.

a) Determine the demand, or price, function.

b) Determine the marginal revenue from the weekly sales of 150 music CDs.

c) The cost of producing x music CDs is $C(x) = -0.003x^2 + 4.2x + 3000$. Determine the marginal cost of producing 150 CDs.

d) Determine the marginal profit from the weekly sales of 150 music CDs.

13. The voltage across a resistor in an electrical circuit is $V = IR$, where $I = 4.85 - 0.01t^2$ is the current through the resistor, in amperes, $R = 15.0 + 0.11t$ is the resistance, in ohms, and t is time, in seconds.

a) Write an equation for V in terms of t.

b) Determine $V'(t)$ and interpret its meaning for this situation.

c) Determine the rate of change of voltage after 2 s.

d) What is the rate of change of current after 2 s?

e) What is the rate of change of resistance after 2 s?

f) Is the product of the values in parts d) and e) equal to the value in part b)? Give reasons why or why not.

CHAPTER

PROBLEM WRAP-UP

The owners of Mooses, Gooses, and Juices hired a research firm to perform a market survey on their products. They discovered that the yearly demand for their Brain Boost BlueBerry frozen smoothie, also know as the B^4, is represented by the function $p(x) = \dfrac{45\,000 - x}{10\,000}$, where p is the price, in dollars, and x is the number of B^4s ordered each year.

a) Graph the demand function.

b) Would you use a graph or an equation to determine the quantity of B^4s ordered when the price is \$0.50 and \$3.00? Explain your choice and determine each quantity.

c) Would you use a graph or an equation to determine the quantity of B^4s ordered when the price is \$2.75 and \$3.90? Explain your choice and determine each quantity.

d) Determine the marginal revenue when 20 000 B^4s are made each year. Explain the significance of this value.

e) Research shows that the cost, in dollars, of producing x number of B^4s is modelled by the function $C(x) = 10\,000 + 0.75x$. Compare the profit and marginal profit when 15 000 B^4s are sold each year, versus 30 000 B^4s. Explain the meaning of the marginal profit for these two quantities.

Chapter 2 PRACTICE TEST

For questions 1 to 3, choose the best answer.

1. Which of the following is not a derivative rule? Justify your answer with an example.

A $\dfrac{d}{dx}[f(x)+g(x)] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$

B $\dfrac{d}{dx}f[g(x)] = \dfrac{d}{dx}f(x)\dfrac{d}{dx}g(x)$

C $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = f(x)\dfrac{d}{dx}[g(x)]^{-1} + [g(x)]^{-1}\dfrac{d}{dx}f(x)$

D $\dfrac{d}{dx}cf(x) = c\dfrac{d}{dx}f(x)$

2. Which statement is always true for an object moving along a vertical straight line? Explain why each of the other statements is not true.

A The object is speeding up when $v(t)a(t)$ is negative.

B The object is slowing down when $v(t)a(t)$ is positive.

C The object is moving upward when $v(t)$ is positive.

D The object is at rest when the acceleration is zero.

3. Which of the following are incorrect derivatives for $y = \dfrac{-4x}{x^2+1}$? Justify your answers.

A $y' = \dfrac{-4}{2x}$

B $y' = \dfrac{(x^2+1)(-4) - 4x(2x)}{(x^2+1)^2}$

C $y' = -4(x^2+1)^{-1} + 8x^2(x^2+1)^{-2}$

D $y' = \dfrac{(x^2+1)(-4) + 4x(2x)}{(x^2+1)^2}$

4. Determine $f''(3)$ for the function $f(x) = (5x^2 - 3x)^2$.

5. Describe two different methods that can be used to differentiate each of the following. Differentiate each function using the methods you described.

a) $y = (3x^6)^{\frac{1}{3}}$ **b)** $y = (x^2 - 4)(2x + 1)$

6. Differentiate each function.

a) $y = -5x^3 + \dfrac{4}{x^5} + 1.7\pi$

b) $g(x) = (8x^2 - 3x)^3$

c) $m(x) = \sqrt{9-2x}\left(x^2 + \dfrac{2}{x^3}\right)$

d) $f(x) = \dfrac{3x-2}{\sqrt{1-x^2}}$

7. Mia shoots an arrow upward with an initial vertical velocity of 11 m/s from a platform that is 2 m high. The height, h, in metres, of the arrow after t seconds is modelled by the equation $h(t) = -4.9t^2 + 11t + 2$, $t \geq 0$.

a) Determine the velocity and acceleration of the arrow after 3 s.

b) When is the arrow moving upward? When is it moving downward? Justify your answer.

c) When is the arrow momentarily at rest?

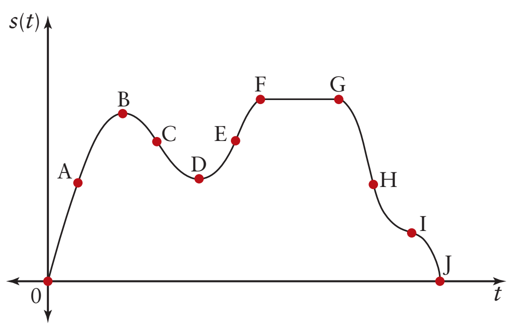
d) What is the height of the arrow for the time found in part c)? What is the significance of this value?

e) When does the arrow hit the ground? With what velocity does it hit the ground?

8. Determine the equation of the tangent to the curve $y = \dfrac{-x}{(3x+2)^3}$ at the point where $x = -1$.

9. Determine the coordinates of the point on the graph of $f(x) = \sqrt{2x+1}$ where the tangent line is perpendicular to the line $3x + y + 4 = 0$.

10. The graph below shows the position function of a vehicle.



a) Is the vehicle going faster at A or at E? Is it going faster at C or at H?

b) What is the velocity of the vehicle at B and at D?

c) What happens between F and G?

d) Is the vehicle speeding up or slowing down at C and I?

e) What happens at J?

f) State whether the acceleration is positive, negative, or zero over each interval.

i) 0 to A ii) B to C

iii) D to E iv) F to G

v) I to J

11. The student council normally sells 1500 school T-shirts for \$12 each. This year they plan to decrease the price of the T-shirts. Based on student feedback, they know that for every \$0.50 decrease in price, 20 more T-shirts will be sold.

a) Determine the demand, or price, function.

b) Determine the marginal revenue from the sales of 1800 T-shirts.

c) The cost of producing x T-shirts is $C(x) = -0.0005x^2 + 7.5x + 200$. Determine the marginal cost of producing 1800 T-shirts.

d) Determine the actual cost of producing the 1801st T-shirt.

e) Determine the profit and marginal profit from the sale of 1800 T-shirts.

12. Suppose the function $V(t) = \dfrac{100\,000 + 5t}{1 + 0.02t}$ represents the value, in dollars, of a new motorboat t years after it is purchased.

a) What is the rate of change of the value of the motorboat at 1, 3, and 6 years?

b) What was the initial value of the motorboat?

c) Do the values in part a) support the purchase of a new motorboat or a used one? Explain your reasoning.

13. The cost, C, in dollars, of manufacturing x MP3 players per day is represented by the function $C(x) = 0.01x^2 + 42x + 300$, $0 \le x \le 300$. The demand function is $p(x) = 130 - 0.4x$.

a) Determine the marginal cost at a production level of 250 players.

b) Determine the actual cost of producing the 251st player.

c) Compare and describe your results from parts a) and b).

d) Determine the revenue function and the profit function.

e) Determine the marginal revenue and marginal profit for the sale of 250 players.

f) Interpret the meaning of the values in part e) for this situation.

14. The value of an antique solid wood dining set t years after it is purchased is modelled by the function $V(t) = \dfrac{(5500 + 6t^3)}{\sqrt{0.002t^2 + 1}}$, where V is in dollars, and $t \ge 0$.

a) What was the price of the dining set when it was purchased?

b) Determine the rate of change of the value of the dining set after t years.

c) Is the value of the dining set increasing or decreasing? Justify your answer.

d) What is the dining set worth after 3 years and after 10 years?

e) Compare $V'(3)$ and $V'(10)$. Interpret the meaning of these values for this situation.

f) **Use Technology** When will the dining set be worth about \$10 500? What is the rate of change of the value of the dining set at this time?

TASK

The Disappearing Lollipop

You have learned how to apply the chain rule and to solve problems involving rates of change. In this task, you will solve a problem involving a combination of these ideas.

Hypothesis: The rate of change of the volume of a sphere is proportional to its surface area.

Experiment:

a) Obtain a spherical lollipop on a stick. Assume that it is a perfect sphere. Measure and record the initial radius of this sphere. (Hint: Remember the relationship between radius and circumference).

b) Place the lollipop in your mouth and carefully consume it (uniformly) for 30 s. Measure and record the new radius in a table similar to the one here. Repeat until you have at least ten measurements.

Time (s)	0	30	60	90	120	150	180	210	240	270	300
Radius of Lollipop (cm)											

c) Use your data to determine the rate of change of radius with respect to time. Write an equation to model the radius as a function of time. Justify your choice of models.

d) Write the formula to model the volume as a function of the radius. Use your equation from part c) to model the volume as a function of time.

e) Use your model from part c) to calculate the rate of change of volume with respect to time when the radius has reached half its original value.

f) Explain why you should not expect the rate of change of volume with respect to time to be constant in this situation.

g) Use your model to estimate how much time is required to completely consume the lollipop.

h) Does this experiment confirm the hypothesis? Explain.

i) How would the time to consume the lollipop change if the initial radius were multiplied by a constant k? Justify your response.