# Assignment #1

**Due date**: Sep 20, beginning of the class.

- Plesae write your name here
- Please write your UID here
- Please write your solutions in the free space after problem statements. Hand in the hard copy of your solutions on Sep 20, before the class begins.
- Your grades depend on the correctness and clarity of your answers.
- Write your answers with enough detail about your approach and concepts used, so that the grader will be able to understand it easily. You should ALWAYS prove the correctness of your algorithms either directly or by referring to a proof in the book.
- Write your answers in the spaces provided. If needed, attach other pages.
- You should scan and submit your solutions one week before the deadline to show your progress. Send your .pdf file to saeedreza.seddighin@gmail.com with title "HW1-so-far".

(1) A standard 52-card deck includes thirteen *ranks* of $\{2, 3, \ldots, 10, J, Q, K, A\}$ in each of the four *suits* $\{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$. In poker, any subset of 5 cards is called a *hand*. A hand is called a *full-house*, if it includes exactly three cards of the same rank and two cards of another rank.

(a) Suppose that we draw 5 cards out of the 52 cards randomly. What is the probability that these 5 cards form a full-house?

(b) Suppose that we draw 7 cards out of the 52 cards randomly. What is the probability that there exists at least one full-house hand within these 7 cards?

(a)

Let $x$ be the rank from which we have 3 cards and $y$ be the rank from which we have 2 cards in a full house. There are $2\binom{13}{2}$ choices for $x$ and $y$. Once we fix $x$ and $y$, there are $\binom{4}{3}$ choices for the three cards of rank $x$, and $\binom{4}{2}$ choices for the two cards of rank $y$. Multiplying these together, there are

$$2\binom{13}{2}\binom{4}{3}\binom{4}{2} = 3744$$

possible full house hands. On the other hand, the total number of hands is $\binom{52}{5} = 2598960$, thus probability that we draw one full-house would be $\frac{3744}{2598960} = 0.00144058$.

(b)

It is important to ensure that a set of 7 cards with multiple full-houses is counted only once. We do this by considering different types of hands that include at least one full-house. Note that a hand with a full-house belongs to exactly one of the following hands:

**Case 1:** hands of form AAA-BB-CC

There are $\binom{13}{3}$ ways to choose three ranks, and 3 ways to pick one to be $A$, leaving the other 2 for $B$ and $C$. Furthermore, once we fix ranks $A$, $B$, and $C$, there are $\binom{4}{3} \times \binom{4}{2} \times \binom{4}{2}$ ways to choose the actual cards of these ranks. Overall, the total number of hands of this form is

$$\binom{13}{3} \times 3 \times \binom{4}{3} \times \binom{4}{2} \times \binom{4}{2} = 123552$$

**Case 2:** hands of form AAA-BB-CD

$$\binom{13}{4} \times 4 \times 3 \times \binom{4}{3} \times \binom{4}{2} \times \binom{4}{1} \times \binom{4}{1} = 3294720$$

**Case 3:** hands of form AAA-BBB-C

$$\binom{13}{3} \times 3 \times \binom{4}{3} \times \binom{4}{3} \times \binom{4}{1} = 54912$$

**Case 4:** hands of form AAAA-BB-C

$$\binom{13}{3} \times 3 \times 2 \times \binom{4}{4} \times \binom{4}{2} \times \binom{4}{1} = 41184$$

**Case 5:** hands of form AAAA-BBB

$$\binom{13}{2} \times 2 \times \binom{4}{4} \times \binom{4}{3} = 624$$

Therefore, the total number of 7 card hands with at least one full-house in them is $123552 + 3294720 + 54912 + 41184 + 624 = 3514992$. Thus the probability of observing one is $\frac{3514992}{\binom{52}{7}} = 0.0262735$.

(2) Suppose that you play the following game against a house in Las Vegas. You pick a number between one and six, and then the house rolls three dices. The house pays you \$1,500 if your number comes up on one dice, \$2,000 if your number comes up on two dices, and \$2,500 if your number comes up on all three dices. You have to pay the house \$1,000 if your number does not show up at all. How much can you expect to win (or lose)?

Let $X$ denote the payoff you get. The expected payoff you get is:

$$\begin{aligned}\mathbb{E}[X] = 1500 \cdot \mathsf{Pr}[&\text{your number comes up on one die}]\\ &+ 2000.\mathsf{Pr}[\text{your number comes up on two dice}]\\ &+ 2500.\mathsf{Pr}[\text{your number comes up on all three dice}]\\ &- 1000.\mathsf{Pr}[\text{your number does not show up at all}].\end{aligned}$$

The probability of rolling $n$ dice, where a given number comes up $k$ times is $\binom{n}{k}(1/6)^k(5/6)^{n-k}$. Thus,

$$\begin{aligned}\mathbb{E}[X] &= 1500 \times 3 \times (1/6) \times (5/6)^2 + 2000 \times 3 \times (1/6)^2 \times (5/6) + 2500 \times 1 \times (1/6)^3 - 1000(5/6)^3\\ &= 92.5926.\end{aligned}$$

(3) You are looking for your hat in one of six drawers. There is a 10% chance that it is not in the drawers at all, but if it is in a drawer, it is equally likely to be in each. Suppose that we have opened the first two drawers and noticed that the hat is not in them.

(a) What is the probability that the hat is in the third drawer?

(b) What is the probability that the hat is not in any of the drawers?

Let $A$ be the event that the hat is not in the drawer and $B_i$ for $1 \leq i \leq 6$ denote the event that it is in the $i$'th drawer. Therefore, $\Pr[A] = 0.1$ and also $\Pr[B_i] = 0.15$ follows from the fact that for each $i$ and $j$ we have $\Pr[B_i] = \Pr[B_j]$. Now, the probability that we find the hat in the third drawer given that it is not in the first two is:

$$\Pr[B_3|\bar{B}_1 \wedge \bar{B}_2] = \frac{\Pr[B_3 \wedge \bar{B}_1 \wedge \bar{B}_2]}{\Pr[\bar{B}_1 \wedge \bar{B}_2]} = \frac{\Pr[B_3]}{1 - 0.15 - 0.15} = \frac{0.15}{0.7} = 0.214286$$

Moreover, the probability that the hat is not in any of the drawers would be

$$\Pr[A|\bar{B}_1 \wedge \bar{B}_2] = \frac{\Pr[A \wedge \bar{B}_1 \wedge \bar{B}_2]}{\Pr[\bar{B}_1 \wedge \bar{B}_2]} = \frac{\Pr[A]}{0.7} = \frac{0.1}{0.7} = 0.142857$$

(4) Let $S$ be the set of all sequences of three rolls of a dice.

- Let $X$ be the sum of the number of dots on the three rolls. What is $E(X)$?

  Let $X_1, X_2$, and $X_3$ denote the expected number of dots of the first, second, and the third dice, respectively. Hence $\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + X_3]$. According to the linearity of expectation, $\mathbb{E}[X_1 + X_2 + X_3] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] = 3 \cdot 3.5 = 10.5$.

- Let $Y$ be the product of the number of dots on the three rolls. What is $E(Y)$?

  Let $X_1, X_2$, and $X_3$ denote the expected number of dots of the first, second, and the third dice, respectively. By the definition of the expectation we have

$$\mathbb{E}[X_1 \cdot X_2 \cdot X_3] = \sum_{i=1}^{6}\sum_{j=1}^{6}\sum_{k=1}^{6} i \cdot j \cdot k \cdot \mathsf{Pr}[X_1 = i \wedge X_2 = j \wedge X_3 = k].$$

  Note that in this equation we consider every possible situation, where each of die is an integer between 1 and 6, and compute the value of $X_1$,$X_2$,$X_3$ times its probability. Since the $X_1$, $X_2$, and $X_3$ are independent

$$\mathsf{Pr}[X_1 = i \wedge X_2 = j \wedge X_3 = k] = \mathsf{Pr}[X_1 = i] \cdot \mathsf{Pr}[X_2 = j] \cdot \mathsf{Pr}[X_3 = k] = 1/6^3$$

  and thus,

$$\begin{aligned}\mathbb{E}[X_1 \cdot X_2 \cdot X_3] &= \sum_{i=1}^{6}\sum_{j=1}^{6}\sum_{k=1}^{6} i \cdot j \cdot k \cdot 1/6^3 \\ &= 1/6^3 \sum_{i=1}^{6} i \sum_{j=1}^{6} j \sum_{k=1}^{6} k.\end{aligned}$$

  Therefore, $\mathbb{E}[X_1 \cdot X_2 \cdot X_3] = (3.5)^3 = 42.875$.

(7) Find $x$, $y$, and $z$ such that

- $x \leq 10$
- $x + y \leq 17$
- $2x + 3z \leq 25$
- $y + z \geq 11$
- $5x + 2y + z$ is maximized

Your soulution:

- $x$ : [ 43/5 ]
- $y$ : [ 42/5 ]
- $z$ : [ 13/5 ]
- $5x + 2y + z$: [ 148.4 ]

We will first solve for x and y by eliminating z.

$$2x + 3z \leq 25$$

$$3z \leq 25 - 2x$$

$$z \leq \frac{25 - 2x}{3}$$

$$y + z \geq 11$$

$$z \geq 11 - y$$

$$-z \leq -11 + y$$

We will combine the two inequalities to remove z

$$z \leq \frac{25 - 2x}{3}$$

$$-z \leq -11 + y$$

$$0 \leq \frac{25 - 2x}{3} + (-11 + y)$$

$$0 \leq \frac{-2x + 3y - 8}{3}$$

$$0 \leq -2x + 3y - 8$$

$$2x - 3y \leq -8$$

We can now solve for x and y using:

$$x + y \leq 17$$

$$2x - 3y \leq -8$$

$$3(x+y) \leq 3(17)$$
$$2x - 3y \leq -8f$$

$$3x + 3y \leq 51$$
$$2x - 3y \leq -8f$$

We can combine two inequalities and we will get:

$$5x \leq 43$$
$$x \leq \frac{43}{5}$$

Substituting to original inequality:

$$x + y \leq 17$$
$$\frac{43}{5} + y \leq 17$$
$$y \leq 17 - \frac{43}{5}$$
$$y \leq \frac{42}{5}$$

Substituting x to inequality with z to solve for z:

$$2x + 3z \leq 25$$
$$2(\frac{43}{5}) + 3z \leq 25$$
$$\frac{86}{5} + 3z \leq 25$$
$$3z \leq 25 - \frac{86}{5}$$
$$3z \leq \frac{125 - 86}{5}$$
$$3z \leq \frac{39}{5}$$
$$z \leq \frac{39}{15}$$
$$z \leq \frac{13}{5}$$

Since our goal is to maximize $15x + 2y + z$, we take the greatest value of x,y, and z, which are:

$$x = \frac{43}{5}, y = \frac{42}{5}, z = \frac{13}{5}$$

$$15x + 2y + z$$

$$= 15(\frac{43}{5}) + 2(\frac{42}{5}) + \frac{13}{5}$$

$$= \frac{645 + 84 + 13}{5}$$

$$= \frac{742}{5}$$

$$148.4$$

(8) We are given a graph $G$ with vertex set $V(G)$ and edge set $E(G)$. Every edge $e$ has a weight $w_e$. Write an LP whose optimal solution is equal to the length of the shortest path from a vertex $s$ to a vertex $t$ in this graph. You may write some constraints over the edge set or vertex set of the graph. For instance you can write

$$\forall (u,v) \in E(G) \qquad\qquad w_u f_u + w_v f_v \leq 10.$$

In the above example, $f_i$'s are the variables of your LP.

$$\begin{array}{ll} \max & d_t \\ s.t. & d_s = 0 \\ & d_v - d_u \leq w_e \quad \forall e = \langle u,v \rangle \in E(G) \end{array}$$

The variables of the above LP are $d_1, d_2, \ldots, d_n$. Every $d_i$ denotes the distance of vertex $i$ from vertex $s$. By definition, we have $d_s = 0$. Moreover, for every edge $e = \langle u,v \rangle$ we have $d_v - d_u \leq w_e$ since one way to get to vertex $v$ is through vertex $u$ which costs $d_u + w_e$.

For example at $d_t$ where v is neighbors of t:

$$d_t \leq d_v + w_e \quad e = \langle t,v \rangle \in E(G)$$

$d_t$ is the shortest path and thus should be less than equal to $d_v + w_e$ from all its neighbors By maximizing $d_t$, $d_t$ will equal to smallest $d_v + w_e$.

## References

(4) Prove that the expected value of the binomial distribution with $n$ draws and probability $p$ is equal to $np$ and its variance is $np(1-p)$.

We denote the binomial distribution with random variable $x = \sum_{i=1}^{n} x_i$ where $x_i$ denotes the $i$th realization. Therefore, we have

$$\begin{aligned}
\mathbb{E}[x] &= \mathbb{E}[sum_{i=1}^{n} x_i] \\
&= \sum_{i=1}^{n} \mathbb{E}[x_i] \\
&= \sum_{i=1}^{n} p \\
&= np
\end{aligned}$$

$$\begin{aligned}
\mathsf{Var}[x] &= \mathbb{E}[x^2] - \mu^2 \\
&= \mathbb{E}[x^2] - n^2p^2 \\
&= \mathbb{E}[(\sum x_i)(\sum x_i)] - n^2p^2 \\
&= (\sum_{i=1}^{n} \sum j = i + 1^n \mathbb{E}[2x_i x_j]) + (\sum_{i=1}^{n} \mathbb{E}[x_i^2]) - n^2p^2 \\
&= (n^2 - n)p^2 + (\sum_{i=1}^{n} \mathbb{E}[x_i^2]) - n^2p^2 \\
&= (n^2 - n)p^2 + np - n^2p^2 \\
&= np(1-p)
\end{aligned}$$

(6) Assume there is a set $J$ of jobs and a set $C$ of cpu cores such that $|C| = |J|$. The time required for a job $j \in J$ to be processed on cpu $c \in C$ is $t_{j,c}$.

(a) Write an integer program to assign each job to a cpu which minimizes the total amount of time required to finish all the jobs. Note that each cpu has to receive exactly one job.

(b) Give an example with two jobs and two cpu cores for which the LP relaxation of your program has a lower objective value than the optimal integer one.

(a) Let variable $x_{j,c} \in \{0,1\}$ for each $j \in J, c \in C$ denote if we assign job $j$ to cpu $c$. In other words if $x_{j,c} = 1$, it means that we assign job $j$ to cpu $c$ and if $x_{j,c} = 0$ it means we do not assign. The IP is the following.

| | | |
|---|---|---|
| $\min \sum_{j \in J} \sum_{c \in C} x_{jc}\, t_{jc}$ | | Minimizing the sum of times. |
| $\sum_{c \in C} x_{jc} \geq 1$ | $\forall j \in J$ | Making sure that each job is assigned to at least one cpu. |
| $\sum_{j \in J} x_{jc} = 1$ | $\forall c \in C$ | Making sure that no cpu is assigned to more than one job. |
| $x_{ij} \in \{0,1\}$ | $\forall j \in J, \forall c \in C$ | |

(b) Consider the case when the assignment times are all one. In this case the optimal solution of LP has total time 2. Now a candidate optimal solution can be the following: $x_{1,1} = x_{1,2} = x_{2,1} = x_{2,2} = 1/2$. Which says that half of each job is assigned to each cpu. You can check that this solution satisfies all the constraints of the LP and has the optimal total time which is 2.

(7) The bisection [1] of a graph is defined as the smallest number of crossing edges when dividing the vertices of the graph into two sets of equal size (there is no connectivity requierment for the sets). Write an integer program that computes the bisection of a graph with even number of vertices.

We define an integer variable $x_i$ for every vertex $i$ where $x_i = 0$ means that veretex $i$ belongs to the first set and $x_i = 1$ means otherwise. Moreover, for every edge $e = \langle i, j \rangle$ we have a varibale $y_e$ where $y_e \geq 1$ whenever the end points of $e$ are not in the same set $(x_i \neq x_j)$. Therefore the program is as follows:

$$
\begin{array}{lll}
\min & \sum_e y_e & \\
s.t. & y_e \geq x_i - x_j & \forall e = \langle i, j \rangle \\
 & y_e \geq x_j - x_i & \forall e = \langle i, j \rangle \\
 & \sum x_i = n/2 & \\
 & x_i \in \{0, 1\} & \forall 1 \leq i \leq n
\end{array}
$$