CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 3

Sezgin, Mustafa
e2380863@ceng.metu.edu.tr

May 13, 2022

---

1. (a) $\omega_0 = GCD(\frac{\pi}{5}, \frac{\pi}{4}) = \frac{\pi}{20} \implies T_0 = \frac{2\pi}{\pi/20} = 40$

$$\begin{aligned} x(t) &= \sin\left(\frac{\pi}{5}t\right) + \cos\left(\frac{\pi}{4}t\right) \\ &= \frac{1}{2j}\left(e^{j\frac{\pi}{5}t} - e^{j\frac{-\pi}{5}t}\right) + \frac{1}{2}\left(e^{j\frac{\pi}{4}t} + e^{j\frac{-\pi}{4}t}\right) \\ &= \frac{1}{2j}e^{j4\frac{\pi}{20}t} - \frac{1}{2j}e^{j(-4)\frac{\pi}{20}t} + \frac{1}{2}e^{j5\frac{\pi}{20}t} + \frac{1}{2}e^{j(-5)\frac{\pi}{20}t} \end{aligned}$$

$$a_4 = \frac{1}{2j},\ a_{-4} = -\frac{1}{2j},\ a_5 = \frac{1}{2},\ a_{-5} = \frac{1}{2}$$

(b)

$$\begin{aligned} \omega = GCD(4\pi, 2\pi) = 2\pi &\implies N = p\frac{2\pi}{2\pi} = p,\ p \in \mathbb{Z} \\ e^{j\pi N} = 1 = e^{jq2\pi} &\implies N = 2q,\ q \in \mathbb{Z} \\ N_0 &= 2 \end{aligned}$$

$$\begin{aligned} x[n] &= \frac{1}{2} + e^{j\pi n} + \sin[4\pi n] + \cos[2\pi n] \\ &= \frac{1}{2} + e^{j\pi n} + \frac{1}{2j}\left(e^{j4\pi n} - e^{j(-4)\pi n}\right) + \frac{1}{2}\left(e^{j2\pi n} + e^{j(-2)\pi n}\right) \end{aligned}$$

$$a_0 = \frac{1}{2},\ a_1 = 1,\ a_4 = \frac{1}{2j},\ a_{-4} = -\frac{1}{2j},\ a_2 = \frac{1}{2},\ a_{-2} = \frac{1}{2}$$

2. $N = 7 \implies \omega_0 = \frac{2\pi}{7}$

$$x[n] = A_0 + \sum_{k=1}^{\infty}\left(B_k \cos\left[k\frac{2\pi}{7}n\right] + C_k \sin\left[k\frac{2\pi}{7}n\right]\right),$$

where $B_k = a_k + a_{-k}$ and $C_k = j(a_k - a_{-k})$.

$$\begin{aligned} B_1 &= 2j + (-2j) = 0 \\ C_1 &= j(2j - (-2j)) = -4 \\ B_2 &= 2 + 2 = 4 \\ C_2 &= j(2 - 2) = 0 \\ B_3 &= 2j + (-2j) = 0 \\ C_3 &= j(2j - (-2j)) = -4 \\ A_0 &= B_4 = C_4 = B_5 = C_5 = \cdots = 0 \end{aligned}$$

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin\left[k\frac{2\pi}{7}n + \phi_k\right],$$

where $A_k = \sqrt{B_k^2 + C_k^2}$ and $\phi_k = \arctan(\frac{B_k}{C_k})$.

$$
\begin{aligned}
A_1 &= \sqrt{0^2 + (-4)^2} = 4 \\
\phi_1 &= \arctan\left(\frac{0}{-4}\right) = \pi \\
A_2 &= \sqrt{4^2 + 0^2} = 4 \\
\phi_2 &= \arctan\left(\frac{4}{0}\right) = \frac{\pi}{2} \\
A_3 &= \sqrt{0^2 + (-4)^2} = 4 \\
\phi_3 &= \arctan\left(\frac{0}{-4}\right) = \pi
\end{aligned}
$$

$$
\begin{aligned}
x[n] &= 4\sin\left[\frac{2\pi}{7}n + \pi\right] + 4\sin\left[\frac{4\pi}{7}n + \frac{\pi}{2}\right] + 4\sin\left[\frac{6\pi}{7}n + \pi\right] \\
&= -4\sin\left[\frac{2\pi}{7}n\right] + 4\sin\left[\frac{4\pi}{7}n + \frac{\pi}{2}\right] - 4\sin\left[\frac{6\pi}{7}n\right]
\end{aligned}
$$

3. (a) $\omega_0 = \frac{\pi}{8} \implies T_0 = \frac{2\pi}{\pi/8} = 16$

$$x(t) = \sin\left(\frac{\pi}{8}t\right) = \frac{1}{2j}\left(e^{j\frac{\pi}{8}t} - e^{j\frac{-\pi}{8}t}\right)$$

$$a_1 = \frac{1}{2j},\ a_{-1} = -\frac{1}{2j}$$

(b) $\omega_0 = \frac{\pi}{8} \implies T_0 = \frac{2\pi}{\pi/8} = 16$

$$x(t) = \cos\left(\frac{\pi}{8}t\right) = \frac{1}{2}\left(e^{j\frac{\pi}{8}t} + e^{j\frac{-\pi}{8}t}\right)$$

$$b_1 = \frac{1}{2},\ b_{-1} = \frac{1}{2}$$

(c)

$$
\begin{aligned}
c_k &= \sum_{\forall l} a_l b_{k-l} \\
&= \cdots + a_{-2}b_{k+2} + a_{-1}b_{k+1} + a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \cdots \\
&= \cdots + 0 - \frac{1}{2j}b_{k+1} + 0 + \frac{1}{2j}b_{k-1} + 0 + \cdots \\
&= \frac{1}{2j}(b_{k-1} - b_{k+1})
\end{aligned}
$$

$$
\begin{aligned}
c_{-2} &= \frac{1}{2j}(b_{-3} - b_{-1}) = \frac{1}{2j}\left(0 - \frac{1}{2}\right) = -\frac{1}{4j} \\
c_0 &= \frac{1}{2j}(b_{-1} - b_1) = \frac{1}{2j}\left(\frac{1}{2} - \frac{1}{2}\right) = 0 \\
c_2 &= \frac{1}{2j}(b_1 - b_3) = \frac{1}{2j}\left(\frac{1}{2} - 0\right) = \frac{1}{4j} \\
c_k &= 0,\ \forall k \in \mathbb{Z} - \{-2, 0, 2\}
\end{aligned}
$$

4. $T = 4 \implies \omega = \frac{2\pi}{4} = \frac{\pi}{2}$
Since $x(t)$ is an odd function,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{2}t} = -\sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{\pi}{2}t} = -x(-t)$$

implies that $a_0 = 0$ and $a_{-k} = -a_k$. Therefore

$$\begin{aligned} x(t) &= a_1\left(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}\right) + a_2\left(e^{j\pi t} - e^{-j\pi t}\right) \\ &= a_1(2j)\sin\left(\frac{\pi}{2}t\right) + a_2(2j)\sin(\pi t) \\ &= a_1(2j)\sin\left(\frac{\pi}{2}t\right) - 6\sin(\pi t)\,. \end{aligned}$$

$$\begin{aligned} \frac{1}{4}\int_0^4 |x(t)|^2 dt &= \frac{1}{4}\int_0^4 \left|a_1(2j)\sin\left(\frac{\pi}{2}t\right) - 6\sin(\pi t)\right|^2 dt \\ &= -a_1^2\int_0^4 \sin^2\left(\frac{\pi}{2}t\right) dt + 9\int_0^4 \sin^2(\pi t) - a_1(4j)\int_0^4 \sin\left(\frac{\pi}{2}t\right)\sin(\pi t)\, dt \\ &= -2a_1^2 + 18 + 0 \\ &= 18 \end{aligned}$$

This implies that $a_1 = 0$ and $x(t) = -6\sin(\pi t)$.

5. (a) $N = 9 \implies \omega_0 = \frac{2\pi}{9}$

$$\begin{aligned} a_0 &= \frac{1}{9}\sum_{n=0}^{8} x[n] = \frac{5}{9} \\ a_k &= \frac{1}{9}\sum_{n=0}^{8} x[n]e^{-jk\frac{2\pi}{9}n} \\ &= \frac{1}{9}\sum_{n=0}^{4} e^{-jk\frac{2\pi}{9}n} \\ &= \frac{1}{9}\cdot\frac{1 - e^{-jk\frac{10\pi}{9}}}{1 - e^{-jk\frac{2\pi}{9}}} \end{aligned}$$

(b) $N = 9 \implies \omega_0 = \frac{2\pi}{9}$

$$\begin{aligned} a_0 &= \frac{1}{9}\sum_{n=0}^{8} y[n] = \frac{4}{9} \\ a_k &= \frac{1}{9}\sum_{n=0}^{8} y[n]e^{-jk\frac{2\pi}{9}n} \\ &= \frac{1}{9}\sum_{n=0}^{3} e^{-jk\frac{2\pi}{9}n} \\ &= \frac{1}{9}\cdot\frac{1 - e^{-jk\frac{8\pi}{9}}}{1 - e^{-jk\frac{2\pi}{9}}} \end{aligned}$$

(c)

$$\begin{aligned} x[n] &= \sum_{k=-\infty}^{\infty} (u[n-9k] - u[n-9k-5]) \\ &= \cdots + u[n] - u[n-5] + u[n-9] - u[n-14] + \cdots \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} (u[n-9k] - u[n-9k-4]) \\ &= \cdots + u[n] - u[n-4] + u[n-9] - u[n-13] + \cdots \\ &= -x[n-4] \end{aligned}$$

$$
\begin{aligned}
x[n] &= \frac{1}{9}\sum_{k=-\infty}^{\infty}\frac{1-e^{-jk\frac{10\pi}{9}}}{1-e^{-jk\frac{2\pi}{9}}}e^{jk\frac{2\pi}{9}n} \\
y[n] &= -\frac{1}{9}\sum_{k=-\infty}^{\infty}\frac{1-e^{-jk\frac{10\pi}{9}}}{1-e^{-jk\frac{2\pi}{9}}}e^{jk\frac{2\pi}{9}(n-4)} \\
&= -\frac{1}{9}\sum_{k=-\infty}^{\infty}\frac{e^{-jk\frac{8\pi}{9}}-e^{-jk\frac{18\pi}{9}}}{1-e^{-jk\frac{2\pi}{9}}}e^{jk\frac{2\pi}{9}n} \\
&= \frac{1}{9}\sum_{k=-\infty}^{\infty}\frac{1-e^{-jk\frac{8\pi}{9}}}{1-e^{-jk\frac{2\pi}{9}}}e^{jk\frac{2\pi}{9}n}
\end{aligned}
$$