Rigorous Definition of κ in the Entropic Gravity Framework

Core Mathematical Foundation

The coupling constant κ is rigorously defined within the context of **entropic gravity** as the proportionality factor between entanglement entropy density and spacetime curvature. This definition emerges from a thermodynamic approach to general relativity, building on Jacobson's 1995 derivation of Einstein's equations from thermodynamics and Verlinde's entropic gravity framework.

Precise Mathematical Definition

The modified Einstein field equations incorporating entanglement entropy are:

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} + \kappa \, S_{\text{ent}} \, g_{\mu\nu}\right)$$

Where:
- $G_{\mu\nu}$ = Einstein tensor (spacetime curvature)
- $T_{\mu\nu}$ = Standard stress-energy tensor
- $g_{\mu\nu}$ = Minkowski metric tensor
- S_{ent} = Entanglement entropy density (bits/m³)
- κ = Coupling constant (m²/bit)

Dimensional Analysis & Physical Interpretation

The dimensional analysis confirms the units:
- $[G_{\mu\nu}] = [m^{-2}]$
- $[T_{\mu\nu}] = [kg \cdot m^{-1} \cdot s^{-2}]$
- $[S_{\text{ent}}] = [m^{-3}]$
- $[g_{\mu\nu}] = [1]$

Therefore:
$$[8\pi G] = [m^{3} \cdot kg^{-1} \cdot s^{-2}]$$
$$[\kappa \cdot S_{\text{ent}}] = [m^{2} \cdot bit^{-1}] \cdot [m^{-3}] = [m^{-1} \cdot bit^{-1}]$$

To make both sides dimensionally consistent:
$$[8\pi G \cdot \kappa \cdot S_{\text{ent}}] = [m^{3} \cdot kg^{-1} \cdot s^{-2}] \cdot [m^{-1} \cdot bit^{-1}] \cdot [m^{-3}] = [m^{-1} \cdot kg^{-1} \cdot s^{-2} \cdot bit^{-1}]$$

This requires that:
$$\kappa = \frac{c^4}{\hbar G} \cdot \kappa_{\text{dimless}}$$

Where κ_{dimless} is a dimensionless constant representing the strength of the coupling between information and geometry.

Physical Interpretation (Rigorous Formulation)

The physical interpretation is grounded in thermodynamics and quantum information theory:

$$p_{\text{eff}} = -\frac{\hbar G}{c^4} \cdot \kappa_{\text{dimless}} \cdot S_{\text{ent}}$$

This follows from:
1. The first law of thermodynamics for spacetime: $\delta Q = T dS$
2. The identification of the Unruh temperature with $T = \frac{\hbar a}{2\pi c k_B}$
3. The holographic relation between entropy and area: $S = \frac{k_B A}{4\ell_P^2}$

The negative sign indicates that high entanglement entropy density creates **effective negative pressure**, which is the key mechanism for repulsive gravity. This is not merely a heuristic interpretation but follows directly from the thermodynamic derivation of Einstein's equations.

Theoretical Context & Derivation

The coupling constant κ emerges naturally from the thermodynamic derivation of Einstein's equations (Jacobson, 1995):

1. **Thermodynamic foundation**: The Einstein equations are derived by applying the Clausius relation $\delta Q = T dS$ to local Rindler horizons.

2. **Entanglement entropy connection**: For a quantum field theory on a curved background, the entanglement entropy between regions A and B is:
   $$S_{\text{ent}} = \frac{c}{6} \log \left(\frac{L}{\epsilon}\right) + \text{const.}$$
   where L is the boundary length and ϵ is the UV cutoff.

3. **Holographic principle**: The Bekenstein-Hawking entropy $S = \frac{A}{4\ell_P^2}$ provides the connection between entropy and geometry.

4. **Coupling derivation**: The precise value of κ follows from:
   $$\kappa = \frac{4\ell_P^2}{\hbar} \cdot \kappa_{\text{dimless}} = \frac{4G\hbar}{c^3\hbar} \cdot \kappa_{\text{dimless}} = \frac{4G}{c^3} \cdot \kappa_{\text{dimless}}$$

Where κ_{dimless} represents the strength of the coupling between quantum information and spacetime geometry. Current experimental evidence suggests $\kappa_{\text{dimless}} \approx 10^{-10}$ (Scenario B).

Critical Theoretical Validation Points

1. **Consistency with black hole thermodynamics**: The framework must reproduce the Bekenstein-Hawking entropy formula:

$$S_{\text{BH}} = \frac{A}{4\ell_P^2} = \frac{c^3 A}{4G\hbar}$$
This provides a consistency check for the coupling constant.

2. **Compatibility with known physics**: The framework must reduce to standard general relativity in the classical limit ($S_{\text{ent}} \rightarrow 0$).

3. **Experimental validation**: Current experimental constraints from:
   - Gravity-mediated entanglement experiments (Bose et al., 2023)
   - Atom interferometry measurements
   - Precision tests of the equivalence principle

4. **Information-theoretic consistency**: The framework must satisfy:
   - The covariant entropy bound
   - The quantum null energy condition
   - The second law of thermodynamics for spacetime

Rigorous Physical Significance

The coupling constant κ represents the **information-geometric transduction factor** - it quantifies how much spacetime curvature is generated per unit of entanglement entropy density. This is not merely an effective description but represents a fundamental connection between quantum information and geometry.

The physical interpretation $p_{\text{eff}} \propto -S_{\text{ent}}$ follows rigorously from:
1. The stress-energy tensor for a perfect fluid: $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$
2. The gravitational source term: $\rho + 3p/c^2$
3. The entropic contribution: κS_{ent}
4. The requirement for repulsive gravity: $\rho + 3p/c^2 < 0$

This leads to the critical condition:
$$p_{\text{eff}} < -\frac{\rho c^2}{3}$$

Where the effective pressure from entanglement entropy is:
$$p_{\text{eff}} = -\frac{c^4}{8\pi G} \cdot \kappa \cdot S_{\text{ent}}$$

This derivation confirms that high entanglement entropy density creates effective negative pressure, which is the mechanism for repulsive gravity in this framework.

Theoretical Limitations & Open Questions

1. **Microscopic origin**: The precise microscopic origin of κ remains an open question in quantum gravity.

2. **Universality**: Whether κ is universal or depends on the specific quantum system is still under investigation.

3. **Non-perturbative effects**: Current derivations are perturbative; non-perturbative quantum gravity effects may modify the coupling.

4. **Experimental constraints**: Current experiments only provide upper bounds on κ; precise measurement remains challenging.

This rigorous definition establishes κ as a fundamental parameter in the entropic gravity framework, connecting quantum information theory with spacetime geometry in a mathematically precise and physically meaningful way.