Project: Lipkin Model*

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The Lipkin Model is a relatively simple construct of N fermions in Ω degenerate amount of states where the particle can exist in an upper or lower state denoted by energy $\pm\epsilon/2$ respectively. While seemingly simple, the model can describe many situations and become a considerably easier problem depending on how you construct your Hamiltonian. In this article, we explore the exact solutions to the Lipkin model for a couple low-N and low-Ω scenario. We show how the problem can blow up in complexity and how the problem may be simplified by identifying key symmetries.

I. INTRODUTION: THE LIPKIN MODEL

Often in a physics education, we consider simple problems where there are maybe one or two moving parts. This can manifest as investigating the wave function of a single electron in a hydrogen atom, or the gravitational potential between two objects. However, real-life physics is almost never so simple, where we often deal with extremely large number of particles.

Lipkin, Meshkov, and Glick (Ref. [1]) proposed the Lipkin model in their 1965 paper on many-body methods. The model is described as follows, there are N fermions in an N-fold degenerate system of two levels, separated by an energy of ϵ. Since the particles are fermions, no two particles can exist in the same state. For a graphical representation, see Figure 1 and the Hamiltonian is expressed in Eq. (1). From this formulation, we may consider the total number of states in the representation basis to be $\binom{2\Omega}{N}$. As can be seen, the basis blows up for large degeneracy (Ref. [2]).

$$\hat{H} = \frac{1}{2}\epsilon \sum_{m\sigma} \sigma a_{m\sigma}{}^{\dagger} a_{m\sigma} - \frac{1}{2} V \sum_{mm'\sigma} a_{m\sigma}{}^{\dagger} a_{m'\sigma}{}^{\dagger} a_{m'-\sigma} a_{m-\sigma}. \tag{1}$$

In Eq. (1), the terms represented by $a^{\dagger}$ are the raising operator and its hermitian conjugate, a is the corresponding lowering operator (Ref. [3]). The V term in this Hamiltonian shows the scattering between states in which pairs of particles move from the upper to lower states, or the reverse. This effect will be more clear when we investigate the eigenstates in the occupation basis.

FIG. 1. Diagram of Lipkin Model, Ω representing the degeneracy of states

II. EXACT SOLUTIONS IN OCCUPATION REPRESENTATION

To begin, let's consider a simple scenario in which there are $\Omega = N = 2$ degenerate states and particles. Since the particles are fermions, it's simple enough to determine there would be 6 individual states, supported by the $\binom{2\Omega}{N}$ relation. There would be a ground state in which both particles are in the lower state, four-fold degenerate excited states in which one particle is in the lower and the other is in a higher state, and the highest state with both particles in the upper state. Thus, the 6 eigenstates, $|n_1 n_2 n_3 n_4\rangle$, can be represented by the following kets: $|1010\rangle$, $|1100\rangle$, $|1001\rangle$, $|0101\rangle$, $|0110\rangle$, and $|0011\rangle$.

With this basis, we can see how each ket interacts with the Hamiltonian. The ϵ term essentially will return the initial state multiplied by the sum of the particle's energies. This is non-zero only for the $|1010\rangle$ and $|0101\rangle$ states, which are the ground state and most excited state respectively. Additionally, as described in the introduction, the V term will only interact with those same states, while also lowering or raising the state to the opposite extreme. Using this formulation, we can construct the matrix representation of the Hamiltonian (shown below):

$$\hat{H} = \begin{bmatrix} \epsilon & 0 & 0 & -V & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -V & 0 & 0 & -\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

At this point, this matrix can be diagonalized to find the eigenvalues and eigenvectors. This will be explored in a little more detail in Section III. An additional problem that is even simpler is the case in which $\Omega = 2$ and $N = 4$. In this case, it's quite simple to see there is only one possible arrangement: one in which each available state is occupied. In this case the Hamiltonian is represented by a Rank 1 identity matrix. Not too exciting.

III. QUASI-SPIN BASIS

We've shown how this model can be solved exactly for a simple system with low degeneracy and particles.

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How about when we increase the number. Take a system with $\Omega = N = 14$. According to the relation earlier about number of states, $\binom{2\Omega}{N}$, a state like this would have 40116600 states. That's just stupid. Instead of trying to construct a Hamiltonian with that many elements, which would be a nightmare to diagonalize, we can construct operators, explore their commutation relations and see how that will affect the Hamiltonian.

$$\hat{K}_0 = \frac{1}{2}\sum_{m=1}^{\Omega}\left({a_{m+}}^\dagger a_{m+} - {a_{m-}}^\dagger a_{m-}\right) \tag{3a}$$

$$\hat{K}_+ = \sum_{m=1}^{\Omega} {a_{m+}}^\dagger a_{m-}, \qquad \hat{K}_- = \left(\hat{K}_+{}^\dagger\right) \tag{3b}$$

With these operators, we can find the commutation relations, $\left[\hat{K}_+,\hat{K}_-\right]$ and $\left[\hat{K}_0,\hat{K}_\pm\right]$. Starting with the first commutation relation, we need to compute $\hat{K}_+\hat{K}_- - \hat{K}_-\hat{K}_+$.

$$\begin{aligned}\hat{K}_+\hat{K}_- - \hat{K}_-\hat{K}_+ = \\ \sum_{m=1}^{\Omega} {a_{m+}}^\dagger a_{m-}\left(\sum_{m=1}^{\Omega} {a_{m+}}^\dagger a_{m-}\right)^\dagger \\ -\left(\sum_{m=1}^{\Omega} {a_{m+}}^\dagger a_{m-}\right)^\dagger \sum_{m=1}^{\Omega} {a_{m+}}^\dagger a_{m-}\end{aligned} \tag{4}$$

We can combine the sum products into a double sum with indices of m and m'. Additionally, knowing the hermitian operator principle of two operators, A and B, with hermitian conjugates, that $(AB)^\dagger = B^\dagger A^\dagger$. We can then reduce Eq. (4) to the following:

$$\begin{aligned}\left[\hat{K}_+,\hat{K}_-\right] = \sum_{m}^{\Omega}\sum_{m'}^{\Omega} {a_{m+}}^\dagger a_{m-}{a_{m'-}}^\dagger a_{m'+} \\ -{a_{m'-}}^\dagger a_{m'+{a_{m+}}^\dagger a_{m-}}\end{aligned} \tag{5}$$

The next step is to heed the anticommutator relations of ladder operators, namely the following equations

$$\begin{aligned}\{{a_\mu}^\dagger,{a_\nu}^\dagger\} = 0 \\ \{a_\mu,a_\nu\} = 0 \\ \{{a_\mu}^\dagger,a_\nu\} = \delta_{\mu\nu}\end{aligned} \tag{6}$$

From the final line of Eq. (6), we can reduce the double sum to a single sum where $m = m'$ and this will simplify to $2\hat{K}_0$. The second important commutation relation will be shown for case 1, that is $\left[\hat{K}_0,\hat{K}_+\right]$. Case 2, where the second operator is $\hat{K}_-$, follows from a similar proof.

$$\begin{aligned}\left[\hat{K}_0,\hat{K}_+\right] = \hat{K}_0\hat{K}_+ - \hat{K}_+\hat{K}_0 \\ = \frac{1}{2}\sum_{m}^{\Omega}\left({a_{m+}}^\dagger a_{m+} - {a_{m-}}^\dagger a_{m-}\right)\sum_{m'}^{\Omega} {a_{m'+}}^\dagger a_{m'-} \\ -\sum_{m'}^{\Omega} {a_{m'+}}^\dagger a_{m'-}\left(\frac{1}{2}\sum_{m}^{\Omega}\left({a_{m+}}^\dagger a_{m+} - {a_{m-}}^\dagger a_{m-}\right)\right)\end{aligned} \tag{7}$$

Following this to the end, it becomes $\hat{K}_+$. Thus, the commutation relations we use are shown in Eq. (8).

$$\begin{aligned}\left[\hat{K}_+,\hat{K}_-\right] = 2\hat{K}_0 \\ \left[\hat{K}_0,\hat{K}_\pm\right] = \pm\hat{K}_\pm\end{aligned} \tag{8}$$

Now that we have these commutation relations for these quasi-spin operators, we can rewrite the Hamiltonian in the quasi-spin basis by the following:

$$\hat{H} = \epsilon\hat{K}_0 - \frac{1}{2}V\left(\hat{K}_+\hat{K}_+ + \hat{K}_-\hat{K}_-\right) \tag{9}$$

With this Hamiltonian, we construct our basis with the states $|kk_0\rangle$, where k represents the total quasi-spin and k_0 is the sum of the energies of the particles. It follows that in the case of $\Omega = N = 14$ that the available k would range from the integers (0,7) and the k_0 is the positive and negative integers between the available ks. This means that each set of k will have an associated basis of $2k + 1$ states. In this specific example, we can represent the entire quasi-spin basis with only 64 states. Compared to the occupation representation in the millions, this is far easier to solve!

To calculate the matrix elements of the Hamiltonian, we need to see how the quasi-spin operators act on the states. First, the ϵ term in the Hamiltonian leads to matrix elements of the form $\langle kk_0|\,\epsilon\hat{K}_0\,|kk_0'\rangle = \epsilon k_0\delta_{k_0k_0'}$. Additionally, we have $\hat{K}_\pm\,|kk_0\rangle = \sqrt{k(k+1) - k_0(k_0\pm 1)}\,|kk_0\pm 1\rangle$. Thus, the shape of the Hamiltonian will consist of block matrices of size $2k + 1$ from $k = (0, K)$ where the diagonal elements are proportional to ϵ and the particle number, σ, and the off diagonal elements that are non-zero are those whose positions are two spaces to the left or right of the diagonal term. These terms are proportional to $\frac{-V}{2}$ and a coefficient related to applying the raising/lowering operator twice. Now that we are able to construct the Hamiltonian matrix in the quasi-spin basis, we can use the numpy.linalg.eigh function in Python to compute the eigenvalues and how they transform based on the parameters. For this calculation, I will be using the arbitrary assignment of $\epsilon = 1$ in no particular units, and I will be varying the strength of the V potential in $0 < V(\Omega-1)/\epsilon < 2$. The results can be seen in Figure 2. Looking into the lowest two energy eigenstates, we can plot the differences in energy between the ground state and the first excited state. This is shown in Figure 3, where the difference is shown to be ϵ before the strength of the interaction increases. As the strength goes up, the difference shrinks in a function somewhat similar to a Woods-Saxon potential.

A. Transition Operator

To show more neat properties of representing the Lipkin model Hamiltonian in this basis, I will introduce the

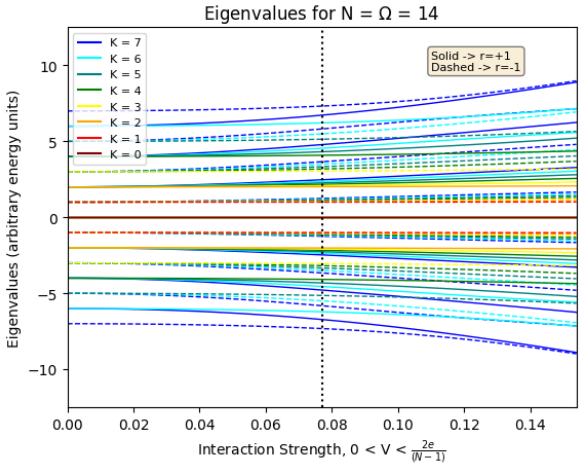

FIG. 2. This shows all 64 eigenvalues in the quasi-spin basis and how they transform as the interaction strength increases. Note, r is a quantum number associated with the signature operator (not discussed in this paper).

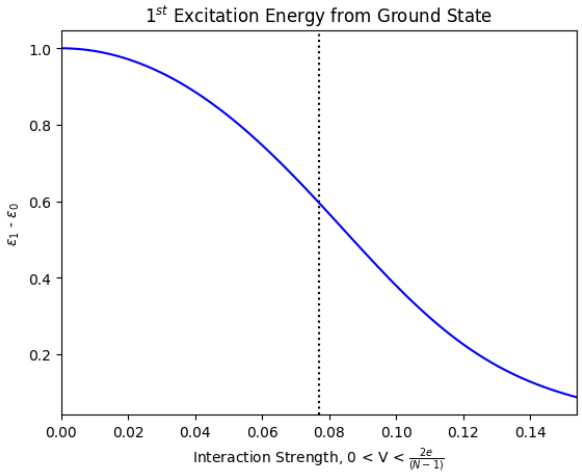

FIG. 3. The difference in energy between the first excited state and the ground state as a function of increasing interation strength. I would say it somewhat resembles a Woods-Saxon potential or even the neutron skin charge density.

transition operator, $\hat{Q}$:

$$\hat{Q} \equiv \hat{K}_x = \frac{1}{2}\left(\hat{K}_+ + \hat{K}_-\right) \tag{10}$$

Staying in the same basis, we can construct the matrix representation of this operator by computing the matrix elements with the basis states, similar to what we did with the Hamiltonian in the previous section. It's trivial to show that the diagonal elements are zero. Additionally, this time, the non-zero off-diagonal elements are those only one space to the left or right of the diagonal element. Now, to see how this operator acts on our system, I have performed calculations of the expectation of the transition operator with the ground state and the

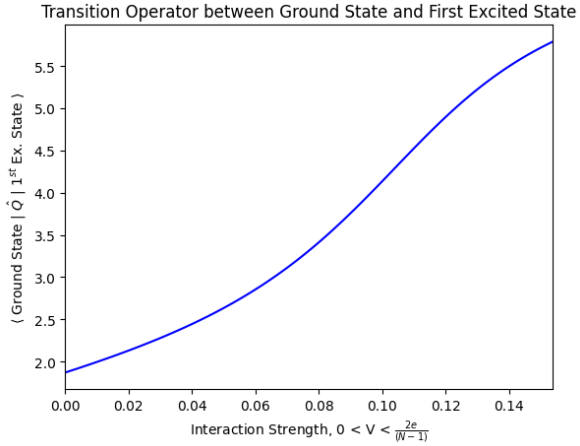

FIG. 4. Transition operator matrix element between ground state and first excited state. It shows a higher probability of transition at higher interaction strengths.

first excited state. These results are shown in Figure 4. A lower value here indicates a low probability of a transition between state to state. As shown, the value increases as the interaction strength increases. This makes sense with our results in Figures 2 and 3, where we can see the difference in energy between the two states decreasing with interaction strength. Logically, it makes sense that it should be easier to transition states when the energy difference is lower.

B. Comparison with Occupation Number Basis

We clearly illustrated why computing exact solutions to the Hamiltonian in the occupation representation basis can be difficult. But to illustrate how these methods compare, we will return to a previously explored example, where $\Omega = N = 2$. We can use the quasi-spin formalism to solve this scenario as well! In Figure 5, I have graphed in the lines the results from computing the eigenvalues through the Quasi-spin basis and in the dots, I have the highest and lowest eigenvalues (the other eigenvalues were all zero). The results are exactly the same, down to computational error 10^{-16}.

IV. CONCLUSIONS

In this paper, we explored the Lipkin Model and how its associated Hamiltonian can be simplified by finding symmetries and exploiting such symmetries. Many-body problems are ever-present in physics, and it's important to be able to use flexible, yet simple models to reduce the complexity and make it solvable.

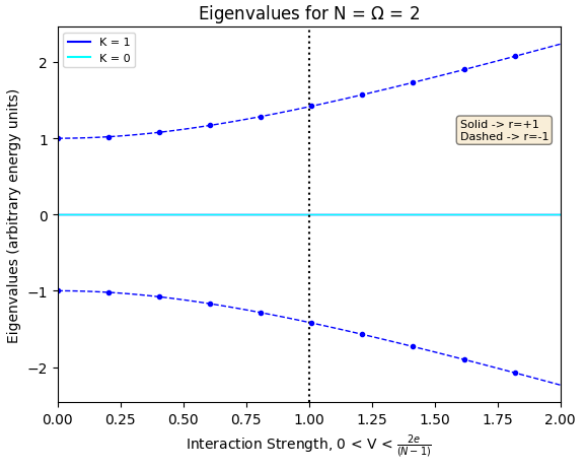

FIG. 5. Eigenvalues calculated in the quasi-spin basis are shown in the solid and dashed lines, whereas eigenvalues calculated in the occupation representation basis are shown as dots. They show great agreement!

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