ComS 331 ***Spring 2025*** ***Name:*** *Pranava Sai Maganti*

HW 0 **Due: 28 jan 2025**

1. **Answer**:

This is an inline equation: $x + y = 3$.
This is a displayed equation:

$$x + \frac{y}{z - \sqrt{3}} = 2.$$

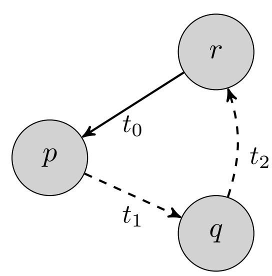
This is how you can define a piece-wise linear function:

$$f(x) = \begin{cases} 3x + 2 & \text{if } x < 0 \\ 7x + 2 & \text{if } x \geq 0 \text{ and } x < 10 \\ 5x + 22 & \text{otherwise.} \end{cases}$$

This is a matrix:

9	8	7	9
6	6	6	
3		3	3

This is a graph with two types (solid and dashed) of labeled edges:



2. **Answer**:

Given: A set of $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$ and a set of $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

To Prove: There exists a bijection $f : \mathbb{N} \to \mathbb{Z}$, demonstrating that $\mathbb{N}$ and $\mathbb{Z}$ are equinumerous.

Proof: We first start of by defining the piecewise functions as follows:

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{n+1}{2} & \text{if } n \mod 2 \neq 0 \\ \frac{-n}{2} & \text{if } n \mod 2 = 0 \end{cases}$$

Here the function $f(n)$ is the piecewise function, where $n \in \mathbb{N}$. Next, we will have prove that the defined function $f(n)$ is both **one-to-one** and **onto**.

(a) f **is One-to-One**
To prove f is one-to-one, let's assume $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{N}$. We consider two cases based on the parity (even or odd) of n_1 and n_2:

i. **Case 1: Both n_1 and n_2 are even.**
If $n_1 = 2k_1$ and $n_2 = 2k_2$, then:

$f(n_1) = \frac{-n_1}{2} = \frac{-2k_1}{2} = -k_1$

Similarly, $f(n_2) = \frac{-n_2}{2} = \frac{-2k_2}{2} = -k_2$

Since $f(n_1) = f(n_2)$, we have $-k_1 = -k_2 \implies k_1 = k_2$, which in-turn implies that $n_1 = n_2$.

ii. **Case 2: Both n_1 and n_2 are odd.**
If $n_1 = 2k_1 + 1$ and $n_2 = 2k_2 + 1$, then:

$f(n_1) = \frac{n_1+1}{2} = \frac{2k_1+2}{2} = \frac{2(k_1+1)}{2} = k_1$

Similarly, $f(n_2) = \frac{n_2+1}{2} = \frac{2k_2+2}{2} = \frac{2(k_2+1)}{2} = k_2$

Since $f(n_1) = f(n_2)$, we have $k_1 + 1 = k_2 + 1 \implies k_1 = k_2$ which implies $n_1 = n_2$.

iii. **Case 3: n_1 is even and n_2 is odd (or vice versa).**
If n_1 is even, $f(n_1) \leq 0$. If n_2 is odd, $f(n_2) > 0$. Therefore, $f(n_1) \neq f(n_2)$.

Since $f(n_1) = f(n_2)$ implies $n_1 = n_2$ in all cases, f is one-to-one.

(b) f **is Onto**
To prove f is onto, we show that for every $z \in \mathbb{Z}$, there exists $n \in \mathbb{N}$ such that $f(n) = z$.

i. **Case 1: $z \leq 0$.**
Let $n = -2z$. Since $z \leq 0$, $-2z \geq 0$, so $n \in \mathbb{N}$. For this n,

$$f(n) = -\frac{n}{2} = -\frac{-2z}{2} = z.$$

ii. **Case 2: $z > 0$.**
Let $n = 2z - 1$. Since $z > 0$, $n = 2z - 1 \geq 0$, so $n \in \mathbb{N}$. For this n,

$$f(n) = \frac{n+1}{2} = \frac{(2z-1)+1}{2} = z.$$

Thus, for every $z \in \mathbb{Z}$, we can find $n \in \mathbb{N}$ such that $f(n) = z$. Therefore, f is onto.

Since f is both one-to-one and onto, it is a bijection.

3. **Answer**:
Given: A relation R on $\mathbb{N}$ defined by:
$\forall m, n \in \mathbb{N}, (m, n) \in R \iff (m - n) \mod 3 = 0$

To Prove:

(a) R is an equivalence relation (i.e., R is reflexive, symmetric, and transitive).

(b) Describe the equivalence classes of R.

Proof:

(a) **Reflexive:** For any $m \in \mathbb{N}$, $m - m = 0$, and $0 \mod 3 = 0$.
Hence, $(m, m) \in R$.
Therefore, R is reflexive.

(b) **Symmetric:** Let $(m, n) \in R$. This means $(m - n) \mod 3 = 0$.
Then, $m - n = 3k$ for some $k \in \mathbb{Z}$.
Rearranging$n - m = -3k$, and since $-3k \mod 3 = 0$, we have $(n, m) \in R$.
Therefore, R is symmetric.

(c) **Transitive:** Let $(m, n) \in R$ and $(n, p) \in R$. This means $(m-n) \mod 3 = 0$ and $(n-p) \mod 3 = 0$. Then, $m - n = 3k$ and $n - p = 3l$ for some $k, l \in \mathbb{Z}$. Adding these equations, $m - p = 3k + 3l = 3(k + l)$. Hence, $(m - p) \mod 3 = 0$, so $(m, p) \in R$. Therefore, R is transitive.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation.

Equivalence Classes: Two numbers $m, n \in \mathbb{N}$ are equivalent under R if $(m-n) \mod 3 = 0$. Thus, the equivalence classes are determined by the remainder when dividing a number by 3:

$$[0] = \{n \in \mathbb{N} \mid n \mod 3 = 0\}, \quad \text{i.e., } \{0, 3, 6, 9, \cdots\}$$

$$[1] = \{n \in \mathbb{N} \mid n \mod 3 = 1\}, \quad \text{i.e., } \{1, 4, 7, 10, \cdots\}$$

$$[2] = \{n \in \mathbb{N} \mid n \mod 3 = 2\}, \quad \text{i.e., } \{2, 5, 8, 11, \cdots\}$$

Therefore, the equivalence classes are: $[0], [1], [2]$

4. **Answer**:
   To Prove:

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$$

Proof:

(a) **Base Case**:
For $n = 1$ on LHS:

$$\sum_{i=1}^{n} i^2 = 1$$

For $n = 1$ on RHS:

$$\frac{(2(1)+1)((1)+1)(1)}{6} = \frac{(2+1)(1+1)(1)}{6} = \frac{3*2*1}{6} = \frac{6}{6} = 1$$

Since LHS = RHS, base case holds true.

(b) **Inductive Hypothesis**: Assuming that the formula holds true for $n = k$:

$$\sum_{i=1}^{k} i^2 = \frac{(2k+1)(k+1)k}{6}$$

(c) **Inductive Step**: For the inductive step, we need to prove that the formula holds true for $n = k+1$ also. For $n = k+1$:

$$\sum_{i=1}^{k+1} i^2 = \frac{(2(k+1)+1)((k+1)+1)(k+1)}{6}$$

Simplifying on the LHS:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

From the inductive hypothesis, we know that:

$$\sum_{i=1}^{k} i^2 = \frac{(2k+1)(k+1)k}{6}$$

Substituting this in the inductive step, we get:

$$\sum_{i=1}^{k+1} i^2 = \frac{(2k+1)(k+1)k}{6} + (k+1)^2 = \frac{k+1}{6}[(2k+1)k+6(k+1)] = \frac{k+1}{6}[2k^2+k+6k+6]$$

$$= \frac{k+1}{6} \cdot (2k+3)(k+2)$$

Now, simplifying on the RHS, we get:

$$\frac{(2(k+1)+1)((k+1)+1)(k+1)}{6} = \frac{(2k+3)(k+2)(k+1)}{6} = \frac{k+1}{6} \cdot (2k+3)(k+2)$$

Since, LHS = RHS, this proves that the formula holds true for $n = k+1$.
Hence Proved.

5. **Answer**:
   To Prove:
   For $n \geq 1$:

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

   Proof:

   (a) **Base Case**:
   For $n = 1$ on LHS:

$$\sum_{i=1}^{1} \frac{1}{i^2} = \frac{1}{1^2} = 1$$

   For $n = 1$ on RHS:

$$2 - \frac{1}{n} = 2 - \frac{1}{1} = 2 - 1 = 1$$

   Since, LHS = RHS, the base case holds true.

   (b) **Inductive Hypothesis**:
   Assuming the inequality holds true for $n = k$:

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

(c) **Inductive Step**:
For the inductive step, we need to prove that the inequality holds true for $n = k+1$ also. For $n = k+1$:

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$$

Expanding the terms for simplifying on LHS we get:

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$$

From the inductive hypothesis we know that:

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

Substituting this into the equation, we get:

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Now simplifying the RHS:

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

Solving this equation we get:

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1} = \frac{1}{k(k+1)}$$

$$\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$$

Since $k+1 > k$, we know that $k(k+1) > (k+1)^2$, and so:

$$\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$$

This proves the inequality for $n = k+1$. Hence Proved.

6. **Answer**:
<u>To Prove</u>:

$$\forall a \in \mathbb{R}, a \neq 1, \sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1-a}$$

<u>Proof</u>:

(a) **Base Case**:
For $n = 0$, on LHS:

$$\sum_{i=0}^{0} a^i = a^0 = 1$$

For $n = 0$, on RHS:

$$\frac{1 - a^{n+1}}{1 - a} = \frac{1 - a^{0+1}}{1 - a} = \frac{1 - a}{1 - a} = 1$$

Since LHS = RHS, the base case holds true.

(b) **Inductive Hypothesis**:
Assuming that the formula holds true for $n = k$:

$$\sum_{i=0}^{k} a^i = \frac{1 - a^{k+1}}{1 - a}$$

(c) **Inductive Step**: For the inductive step, we need to prove that the formula holds true for $n = k + 1$ also. For $n = k + 1$:

$$\sum_{i=0}^{k+1} a^i = \frac{1 - a^{(k+1)+1}}{1 - a}$$

Simplifying on the LHS:

$$\sum_{i=0}^{k+1} a^i = \sum_{i=0}^{k} a^i + a^{k+1}$$

From the inductive hypothesis, we know that:

$$\sum_{i=0}^{k} a^i = \frac{1 - a^{k+1}}{1 - a}$$

Substituting this in the inductive step:

$$\sum_{i=0}^{k+1} a^i = \frac{1 - a^{k+1}}{1 - a} + a^{k+1}$$

Simplifying it further we get:

$$\sum_{i=0}^{k+1} a^i = \frac{1 - a^{k+1} + a^{k+1}(1 - a)}{1 - a} = \frac{1 - a^{k+2}}{1 - a}$$

Simplifying on the RHS:

$$\frac{1 - a^{(k+1)+1}}{1 - a} = \frac{1 - a^{k+2}}{1 - a}$$

Since the LHS = RHS, the formula holds true for $n = k + 1$.
Using the result, to find the sum of the first 30 powers of 2 starting from 2^0:

$$\sum_{i=0}^{29} 2^i$$

Here a = 2 and n = 30, because a if 0, would be 0 always and the condition that $a \neq 1$, substituting this into the formula:

$$\sum_{i=0}^{30} 2^i = \frac{1 - 2^{30+1}}{1 - 2} = \frac{1 - 2^{31}}{-1} = 2^{31} - 1 = 2147483648 - 1 = 2147483647$$

Therefore, the sum of the first 30 powers of 2 (starting from 2^0) is 2147483647.