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# Chapter 1

## Introduction

### 1.1 Motivation

The goal of this thesis is...

### 1.2 Traffic flow modelling

Analyzing traffic flow has been an interdisciplinary research field of both mathematicians and civil engineers since the early 1950s. Over time different approaches for modelling traffic have been introduced and applied in extensive studies.

- On the **microscopic** scale every vehicle is considered as an individual agent whose dynamics are determined by the solution of an ordinary differential equation (ODE). The interaction between neighboring vehicles are usually based on simple equations and determine the behaviour of the collective. The most famous models of this approach are those of the *follow-the-leader* kind [17]. In those models the entire behaviour of the collective cars is fully determined by the one of the first car, namely the leader. Microscopic models can not only illustrate collective phenomena like traffic jams, but it can also be shown that under certain conditions the microscopic solution converges to the macroscopic solution as the number of vehicles approaches infinity [7].
- **Mesoscopic** - or kinetic - models analyze transportation elements in small homogenous groups, where the probability of each group to be at time  $t$  at location  $x$  with velocity  $v$  can be described by a function  $f(t, x, v)$ . This modelling approach requires the use of methods from statistical mechanics.
- **Macroscopic** models utilize the similarities of traffic flow and fluid dynamics. In those models the change of averaged quantities, like density, velocity etc., is described by means of partial differential equations (PDE). The oldest representative of macroscopic modelling are the LWR-models [18], named after their inventors M.

J. Lighthill, G. B. Whitham and P. I. Richards. Others include the Aw-Rascle model and the Payne-Whitham approach [1, 16]. For the rest of this thesis, this is the modelling approach of choice.

In macroscopic models the target is to analyze the change of measurable quantities. A typical quantity used in the analysis of traffic flow, analogous to fluid dynamics, is the density of mass. Let us denote the traffic density on a space interval  $[x_1, x_2]$  at some time  $t \in \mathbb{R}$  as  $\rho(x, t)$ , then we can write the total mass on this interval as

$$\text{amount of cars} = \int_{x_1}^{x_2} \rho(x, t) dx \quad (1.2.1)$$

For conserved quantities no mass is neither destroyed nor created. Therefore the mass on an interval can only change by inflowing and outflowing quantities. The quantity mass crossing the point  $x$  at a time  $t$  is given by the **traffic flux**  $f : [0, \rho_{max}] \rightarrow [0, f^{max}]$ .

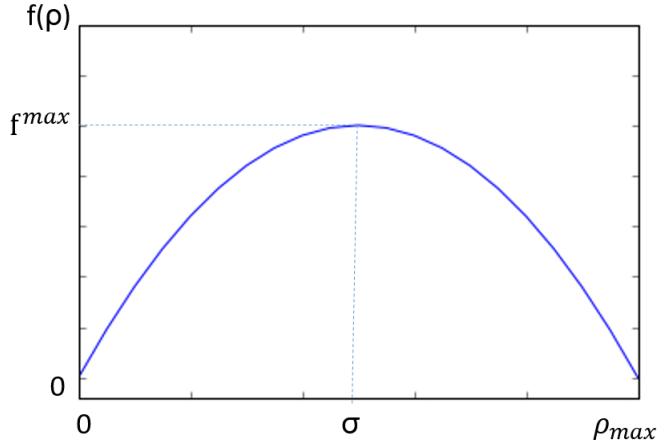


Figure 1.1: Typical shape of the fundamental diagram of traffic flow

The traffic flux, or also referred to as *fundamental diagram*, is typically assumed to follow the rule

$$f'(\rho)(\sigma - \rho) > 0$$

where

$$\sigma := \arg \max_{\rho} f(\rho)$$

denotes the density where the maximum flux

$$f^{max} := f(\sigma)$$

is attained.

Since the total mass on the global space interval is preserved, the rate of change of the

mass on the interval  $[x_1, x_2]$  is defined by the difference of the in- and outfluxes of the interval, e.g.

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho(x, t) dx = f(\rho(x_1, t)) - f(\rho(x_2, t))$$

Rewriting this assuming differentiability of both functions  $u$  and  $f$  and integration over the time interval  $[t_1, t_2]$  leads to:

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{d}{dt} \rho(x, t) dx dt = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{d}{dx} \{-f(\rho)\} dx dt \quad (1.2.2)$$

Assuming that  $f$  is smooth, this leads the differential form of the **conservation law**:

$$\rho_t + (f(\rho))_x = 0 \quad (1.2.3)$$

The LWR-model, as it will be the basis for the remaining thesis, assumes that the flux  $f(\rho) = \rho(x, t)v(x, t)$  is linearly dependent on the velocity  $v(x, t)$  of the wave at point  $x$ . In particular the flux used for the LWR-model can be written as

$$f(\rho(x, t)) = \rho(x, t)v_{max} \left(1 - \frac{\rho(x, t)}{\rho_{max}}\right) \quad (1.2.4)$$

where  $u_{max}$  is the maximum density permitted by the road, and  $v_{max}$  the maximum velocity obtainable by the wave.

Other approaches like the Greenberg and Payne-Whitham model use highly nonlinear functions for their wave velocities, see [14, 16].

Extending the conservation law 1.2.3 with suitable initial conditions, we can formulate the full **Lighthill-Whitham-Richards model** as

$$\begin{aligned} \rho_t + \left[ v_{max} \rho \left(1 - \frac{\rho}{\rho_{max}}\right) \right]_x &= 0 & t > 0 \\ \rho(x, 0) &= \rho_0(x) \end{aligned} \quad (1.2.5)$$

The remaining thesis will be structured as follows: In chapter 2 we will extend the given LWR-model on a network of roads in order to be able to analyze urban traffic situations. We will also introduce the concept of buffers to guarantee the existence of unique solutions for the model. In chapter 3 we will discuss the numerical procedures in order to qualitatively simulate traffic flow computationally. Chapter 4 will deal with introduce the concept of traffic lights on the network. We will then propose two different optimization methods to control the light signals in order to maximize the flux on the network. The two approaches consist of a rather brute-force approach and a concept called model predictive control (MPC). The last chapter will then expand the model by the concept of pollution.

# Chapter 2

## The LWR model on networks

So far we have only solved traffic flow problems on the real line. In order to analyze traffic flow more sophisticated problems under realistic scenarios, in particular in urban environments, it is necessary to extend the mathematical theory and the representation of traffic networks. The method of choice is hereby to represent traffic networks via directed graphs. This directed graph consists of a set of edges, representing the roads, and a set of nodes representing junctions. Every junction is thereby characterized by a finite number of incoming and a finite number of outgoing roads.

### 2.1 Related literature

Traffic flow on road networks has been a widely discussed topic over the past 70 years with an increasing interest in the past century.

**TODO:**

- summary of cumulative number pde, multi-path, garavello approach...

A much larger and well presented genealogy of the history of traffic flow modelling can be found in [19].

### 2.2 Basic principles of networks

Rigorously we can define a network as follows, cf. [8]:

**Definition 2.2.1** *A network is a tuple  $(\mathcal{N}, \mathcal{E})$ , where  $\mathcal{N}$  is a finite collection of vertices, and  $\mathcal{E}$  a finite collection of  $n_R$  edges, where every node represents a junction and every edge represents a unidirectional road. Each road  $e_i = [a_i, b_i] \in \mathbb{R}$ ,  $i = 1 \dots n_R$  is defined over a real interval.*

We further require the following properties:

- 1) Every junction  $J \in \mathcal{N}$  defines a union of two non-empty sets  $\delta^{in}(J)$  and  $\delta^{out}(J)$  where  $\delta^{in}(J), \delta^{out}(J) \subset \mathcal{E}$ , representing incoming and outgoing roads of the junction  $J$ , respectively.
- 2) For junctions  $I, J \in \mathcal{N}, I \neq J$  we require  $\delta^{in}(I) \cap \delta^{in}(J) = \emptyset$  and  $\delta^{out}(I) \cap \delta^{out}(J) = \emptyset$
- 3) If  $e_i \notin \bigcup_{J \in \mathcal{N}} \delta^{in}(J)$  then  $b_i = \infty$ . In this case  $e_i \in \mathcal{E}^{out}$  where  $\mathcal{E}^{out}$  denotes the set of all outgoing roads for the network. Furthermore if  $e_i \notin \bigcup_{J \in \mathcal{N}} \delta^{out}(J)$  then  $a_i = -\infty$ . In this case  $e_i \in \mathcal{E}^{in}$  where  $\mathcal{E}^{in}$  denotes the set of all incoming roads for the network.

These assumptions guarantee that the resulting network is indeed a valid graph. Requirement 1) implies that every road starts and ends in at most one junction. Requirement 2) then says that if a road does not start (end) at a junction then it is considered as an inflow (outflow) of the network.

Another method to represent a road network is via so-called **adjescency matrices** [2].

**Definition 2.2.2** Let  $\mathcal{N}$  be a set of edges with  $|\mathcal{N}| = n_R$ . Then the adjescency matrix is a square  $n_R \times n_R$ -matrix  $A$  with

$$A_{ij} = \begin{cases} 1 & \text{if there exists a junction between road } e_j \text{ and } e_i \\ 0 & \text{else} \end{cases}$$

The adjescency matrix stores information if there is a direct connection between road  $e_j$  and  $e_i$ .

**Remark:**

Having given the network as in definition 2.2.1 one can easily derive the corresponding adjescency matrix as in definition 2.2.2, and vice versa. **BEING MORE SPECIFIC???** also about problem of non-unique indexing of junctions?

**Example 1** Let us consider network displayed in figure 2.1. Then the corresponding adjescency matrix can be written as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

## 2.3 Modelling on networks - the buffer model

**The setting:**

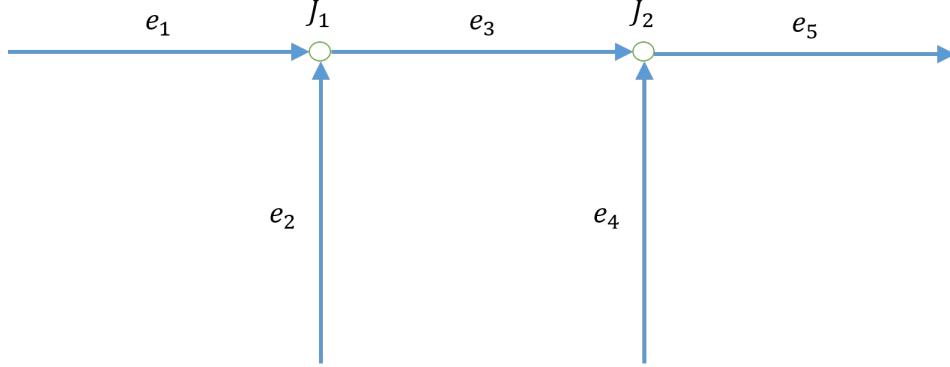


Figure 2.1: Example of a network

Consider a family of  $n + m$  roads, all joining at a single junction. The indices  $i \in \{1, \dots, m\} =: \mathcal{I}$  hereby denote *incoming roads* and indices  $j \in \{m + 1, \dots, m + n\} =: \mathcal{O}$  denote *outgoing roads*. Then the evolution of the density of cars  $\rho_k(x, t)$  on the  $k - \text{th}$  road can be described by the **conservation law**

$$(\rho_k)_t + f(\rho_k)_x = 0. \quad (2.3.1a)$$

accompanied by intial densities  $\rho_k(x, 0) = \rho_{k,0}(x)$  and external inflows into the network defined by

$$f(\rho_k(a_k, t)) = q_k(t) \quad \forall e_k \in \mathcal{E}^{in} \quad (2.3.1b)$$

Furthermore the corresponding inflows and outflows of the junctions are given as

$$f(\rho_k(b_k, t)) = \bar{f}_k(t) \quad \forall e_k \in \mathcal{E} \setminus \mathcal{E}^{out} \quad (2.3.1c)$$

$$f(\rho_k(a_k, t)) = \hat{f}_k(t) \quad \forall e_k \in \mathcal{E} \setminus \mathcal{E}^{in} \quad (2.3.1d)$$

where the junctions fluxes  $\bar{f}_k$  and  $\hat{f}_k$  are defined later.

For the roads leaving the network we also impose Dirichlet boundary conditions.

### The model

In order to be useful for the analysis of global optimization, the used traffic flow model at junctions should provide two crucial properties:

- Well posedness for  $\mathcal{L}^\infty$  data
- Continuous solution w.r.t. weak convergence

Due to the ill-posedness of the general junction model in [8] for certain input data, we need to come up with a different approach.

In [3] a model is proposed where each intersection in the model includes a **buffer** with

limited capacity. The current filling level of the buffer in front of the outgoing road  $j \in \mathcal{O}$  may be denoted as the *queue length*  $q_j(t)$ . The rate of flux at which cars from incoming roads enter the intersection is controlled by the current length of the queues. The outgoing fluxes are governed by the queue length and the personal preference destination of the individual drivers, as well as the maximum permitted fluxes on the designated roads.

The main results of their analysis can be stated as:

- I. If the queue lengths  $q_j$  for every outgoing road are given, the initial boundary value problems on each road become decoupled and can be solved individually, first on every incoming road, and secondly on every outgoing road. The densities  $\rho_k(t, x)$  on every road  $k = 1 \dots m + n$  can then be explicitly computed via a Lax type formula.
- II. Given the densities, the lengths  $q_j$  of the queues can be determined by balancing the influxes and outfluxes of the intersection. The queue lengths can finally be obtained as a fixpoint of a contractive transformation  $q \rightarrow \Delta(q)$  where  $q$  needs to be Lipschitz continuous.
- III. The buffer model is thus well-posed at intersections for general  $\mathcal{L}^\infty$  data. It is also shown that the traffic flow model is continuous w.r.t. weak convergence.

The interested reader is referred to [3] for the proofs.

Let the general setting be as earlier in this section. Further we include two realistic assumptions for the boundary values at the entrance and exit points of junctions:

- i. **Driver's preferences**  $\Theta_{i,j}$  define the fraction of cars on road  $i$  that want to exit the junction in direction of road  $j$ .

$$\Theta_{i,j} \in [0, 1]$$

- ii. **Relative priorities**  $\eta_i$  given to incoming roads e.g. external effects like traffic lights

For every junction  $J$  we can define the driver's preferences  $\Theta$ , i.e. the percentage of drivers going from one incoming to out outgoing road, as followed:

**Definition 2.3.1** *Given a junction  $J$  with  $n := |\delta^{in}(J)|$  incoming roads, say  $e_1, \dots, e_n$ , and  $m := |\delta^{out}(J)|$  outgoing roads, say  $e_{n+1}, \dots, e_{n+m}$ . Then the traffic distribution matrix  $\Theta$  is given by*

$$\Theta = \begin{pmatrix} \Theta_{n+1, 1} & \cdots & \Theta_{n+1, n} \\ \vdots & \ddots & \vdots \\ \Theta_{n+m, 1} & \cdots & \Theta_{n+m, n} \end{pmatrix} \quad (2.3.2)$$

where  $0 \leq \Theta_{i,j} \leq 1$  for every  $i = 1, \dots, n$  and  $j = n+1, \dots, n+m$ . Furthermore we impose the validity condition

$$\Theta_{i,j} \in [0, 1], \quad \sum_{j \in \delta^{out}(J)} \Theta_{i,j} = 1$$

In general,  $\Theta_{i,j} = \Theta_{i,j}(t, x)$  needs not to be necessarily constant, but can be time- and location-dependent. In the following it is assumed that the drivers' preferences are known in advance and that they do not change their itinerary throughout the network. Then the conservation law reads

$$(\Theta_{i,j}\rho_i)_t + [\rho_i\Theta_{i,j}v(\rho_i)]_x = 0$$

Using product rule and reordering yields

$$\rho_i[(\Theta_{i,j})_t + v(\rho_i)(\Theta_{i,j})_x] + \Theta_{i,j}[(\rho_i)_t + (\rho_i v(\rho_i))_x] = 0$$

The second term vanishes as the general conservation law still needs to be fulfilled. For zero density on the road this is fulfilled trivially. For  $\rho_i \neq 0$  we obtain the passive scalar transport equation along the flux:

$$(\Theta_{i,j})_t + v(\rho_i)(\Theta_{i,j})_x = 0 \tag{2.3.3}$$

A similar approach has been pursued by [4]. In their intersection model the capacity of the queues is arbitrarily big. Cars wanting to enter road  $j$  but exceed the maximum outflux of the intersection are instead stored in the queue. As a consequence there is no backward propagation of queues and therefore no emergence of shocks on incoming roads.

The model by [3] extends this model. Consider a single junction and let  $M > 0$  be the maximum capacity of the queue. Then the incoming fluxes into the junction depend on the current degree of occupancy of the buffer, which is defined by

$$q = (q_j)_{j \in \mathcal{O}}, \quad q \in \mathbb{R}^n$$

The Cauchy problem for traffic flow on a network can thus be formulated as

$$\begin{aligned} (\rho_i)_t + f_i(\rho_i)_x &= 0 \\ (\Theta_{ij})_t + v(\rho_i)(\Theta_{ij})_x &= 0 \end{aligned}$$

supplemented by suitable initial conditions.

The **maximum fluxes** that can enter and exit the intersection depend on the current filling level on the incoming and outgoing road, respectively. In particular they are given by the following equations.

- Maximum fluxes on incoming roads:

$$\omega_i(\rho) := \begin{cases} f(\rho) & \rho \leq \sigma \\ f(\sigma) & \rho > \sigma \end{cases}$$

- Maximum fluxes on outgoing roads:

$$\omega_j(\rho) := \begin{cases} f(\sigma) & \rho \leq \sigma \\ f(\rho) & \rho > \sigma \end{cases}$$

This property ensures that Riemann problems on incoming roads are solved by waves with negative speed, and Riemann problems on outgoing roads are solved by waves with positive speed.

Based on the junctions fluxes, we can also derive the rate of change of the size of the junction buffer. In detail, conservation of mass yields the additional differential equation for the **evolution of the queue size**

$$\dot{q}_j = \sum_{i \in \mathcal{I}} \bar{\Theta}_{i,j} \bar{f}_i - \bar{f}_j \quad (2.3.4)$$

where  $\bar{f}_j$  denote the boundary fluxes at the junction,  $k \in \mathcal{I} \cup \mathcal{O}$ .

We are now ready to state two sets of equations regarding the incoming and outgoing junction fluxes depending on the drivers' choices  $\Theta_{ij}$  and the queue lengths  $q_j$ .

The first model provides a shared buffer of capacity  $M$  for every outgoing junction. Incoming cars can cross the intersection governed by the amount of free space left in the queue, regardless of the car's destination. Once within the junction, cars leave at the maximum rate allowed by the outgoing road of their choice.

**Single Buffer Junction model (SBJ):** Consider a constant  $M > 0$ , describing the maximum capacity of the junction at any given time, and constants  $c_i > 0, i \in \mathcal{I}$  accounting for priorities given to the different incoming roads.

We then require that the incoming fluxes  $\bar{f}_i$  satisfy

$$\bar{f}_i = \min\{\omega_i, c_i(M - \sum_{j \in \mathcal{O}} q_j)\}, \quad i \in \mathcal{I} \quad (2.3.5a)$$

Furthermore, the outgoing fluxes  $\hat{f}_j$  should satisfy

$$\hat{f}_j = \begin{cases} \omega_j & q_j > 0 \\ \min\{\omega_j, \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\Theta}_{i,j}\} & q_j = 0 \end{cases} \quad (2.3.5b)$$

As seen, the outgoing fluxes are uniquely defined once the incoming fluxes are known in addition to the current status of the queue.

The second model uses  $n$  different buffers, one for every outgoing road. Once having entered the junction, cars are admitted to their desired road of destination depending on

the length of the queue in front of the desired road.

**Multiple Buffer Junction model (MBJ):** Consider constants  $M_j > 0, j \in \mathcal{O}$ , describing the maximum capacities of the buffers in front of the  $n$  outgoing roads of the junction at any given time, and constants  $c_i > 0, i \in \mathcal{I}$  accounting for priorities given to the different incoming roads.

We then require that the incoming fluxes  $\bar{f}_i$  satisfy

$$\bar{f}_i = \min\{\omega_i, \frac{c_i(M_j - q_j)}{\Theta_{i,j}}\}, j \in \mathcal{O}\}, \quad i \in \mathcal{I}$$

Furthermore, the outgoing fluxes  $\hat{f}_j$  should satisfy

$$\hat{f}_j = \begin{cases} \omega_j & q_j > 0 \\ \min\{\omega_j, \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\Theta}_{i,j}\} & q_j = 0 \end{cases}$$

Now we can define the full **buffer model** by Bressan et al. [3]

$$\begin{aligned} (\rho_i)_t + f_i(\rho_i)_x &= 0 & e_i \in \mathcal{E} \\ \dot{q}_j &= \sum_{i \in \mathcal{I}} \bar{\Theta}_{i,j} \bar{f}_i - \bar{f}_j \\ f(\rho_k(a_k, t)) &= q_k(t) & \forall e_k \in \mathcal{E}^{in} \\ f(\rho_k(b_k, t)) &= \bar{f}_k(t) & \forall e_k \in \mathcal{E} \setminus \mathcal{E}^{out} \\ f(\rho_k(a_k, t)) &= \hat{f}_k(t) & \forall e_k \in \mathcal{E} \setminus \mathcal{E}^{in} \end{aligned} \tag{2.3.6}$$

with external inflows  $q_k(t)$  and junctions fluxes  $\bar{f}_k(t)$  and  $\hat{f}_k(t)$  as defined in (??)

### Limitations of this approach:

- This approach focuses on maximizing the flux of mass through the junction. A counter-intuitive effect of this approach is the following:
- Consider a trivial junction at  $x = 0$  with one incoming and one outgoing road and initial data  $\rho_{in} > \sigma$  and  $\rho_{out} = 0$ . Then the result won't be continuous at  $x = 0$ . In particular the outgoing flux will be  $f^{max} = f(\sigma)$ , hence  $\rho(0^+, t) = \sigma$  whereas  $\rho(0^-, t) > \sigma$  by assumption.

# Chapter 3

## Numerical approximations

In order to solve conservation laws of the form (1.2.3) we usually assume smooth initial conditions  $\rho(x, 0)$ . We are naturally interested in the difficulties caused by discontinuities in the solution. Straight-forward numerical methods like the finite difference approximation usually have difficulties near the discontinuity. Consider for instance the scalar (linear) advection equation

$$u_t + Au_x = 0, \quad -\infty < x < \infty, t \geq 0$$

$$u(x, 0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \quad (3.0.1)$$

Obviously the analytical solution obtained by the method of characteristics is given by a travelling shock wave  $u(x, t) = u_0(x - At)$  with wave speed  $A$ . **ADD Lax Equivalence Theorem??** Unfortunately the finite difference approach is inconsistent, as the finite difference approximation to  $u_x$  at the discontinuity  $x = 0$  will not approach 0, e.g.

$$u_x \sim \frac{u_0(At + h) - u_0(At - h)}{2h} = \frac{0 - 1}{2h} \rightarrow -\infty, \text{ as } h \rightarrow 0$$

In particular, the *local truncation error*

$$L_k(0, t) := \frac{1}{k} [u(0, t + k) - u(0, t)] - a \frac{u_0(At + h) - u_0(At - h)}{2h} \quad (3.0.2)$$

for the finite difference scheme does not vanish for  $h, k \rightarrow 0$  (see also [13]). A numerical method of **order p** fulfills that  $L_k(x, t) = \mathcal{O}(k^p)$ . This means that the finite difference scheme for discontinuous data does neither provide local nor global convergence towards the analytical solution.

For linear systems of the form (3.0.1) a wide variety of convergent numerical methods using adjusted finite difference schemes are given. Some possibilities are stated in Table 3.1. First order method (Upwind, Lax-Friedrich) typically produce a flattened solutions compared to the original solutions, whereas second order methods (Lax-Wendroff and Beam-Warming) give oscillations. These phenomena are displayed in Figure 3.1.

Name	Difference equation	Order
Lax-Friedrich	$\begin{aligned} u(x, t+k) &= \frac{1}{2}(u(x+h, t) + u(x-h, t)) \\ &\quad - \frac{k}{2h}A(u(x+h, t) - u(x-h, t)) \end{aligned}$	1
Upwind ( $A > 0$ )	$u(x, t+k) = u(x, t) - \frac{k}{2h}A(u(x, t) - u(x-h, t))$	1
Lax-Wendroff	$\begin{aligned} u(x, t+k) &= u(x, t) - \frac{k}{2h}A(u(x+h, t) - u(x-h, t)) \\ &\quad + \frac{k^2}{2h^2}A^2(u(x+h, t) - 2u(x, t) + u(x-h, t)) \end{aligned}$	2
Beam-Warming	$\begin{aligned} u(x, t+k) &= u(x, t) - \frac{k}{2h}A(3u(xh, t) - 4u(x-h, t) \\ &\quad + u(x-2h, t)) + \frac{k^2}{2h^2}A^2((u(xh, t) \\ &\quad - 2u(x-h, t) + u(x-2h, t))) \end{aligned}$	2

Table 3.1: Finite difference schemes for the linear advection problem (3.0.1).

### 3.1 The Godunov scheme for nonlinear conservation laws

Finite difference schemes as discussed in the introductory part of this chapter provide *smooth* solutions, even if the initial datum of the conservation law contains discontinuities and jumps. Even to nonlinear problems these PDEs can often be linearized and therefore results from the linear FD methods applied in order to obtain convergence results for nonlinear problems [? ]. There are several limitations to this approach:

Firstly it is not guaranteed that the obtained numerical solution really converges towards the analytical discontinuous solution (see e.g. Burger's equation). Secondly the modelling of travelling waves, in particular the evolution of traffic jams and shocks often emerge from initial discontinuities discontinuities. Thus a different approach has to be pursued.

#### 3.1.1 Theory of conservative numerical schemes

Let us consider the x-t-plane as the operating space. We can discretize this space by choosing a mesh with step size  $h := \Delta x$  and time step  $k := \Delta t$ ,  $\frac{h}{k} = C > 0$  fixed. In particular the mesh points are given by

$$\begin{aligned} x_j &= jh, & j &= \dots, -1, 0, 1, \dots \\ t_n &= nk, & n &= 0, 1, \dots \end{aligned}$$

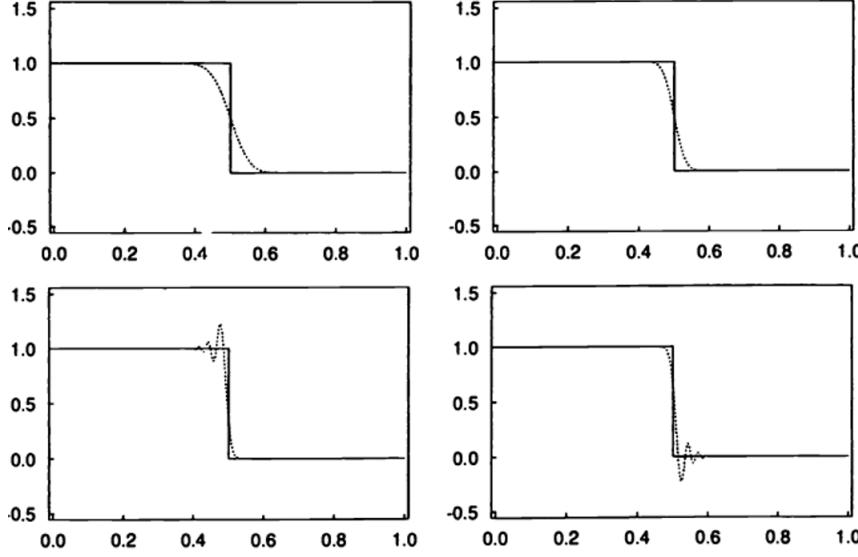


Figure 3.1: Numerical and exact solution of conservation law (3.0.1) at  $t = 0.5$  and step size  $h = 0.0025$  using the following methods (top left to bottom right) (a) Lax-Friedrich, (b) Upwind, (c) Lax-Wendroff, (d) Beam-Warming

We also define the midpoints in space

$$x_{j \pm \frac{1}{2}} := x_j \pm \frac{h}{2}$$

We can now define the **cell averages** of the density  $\rho(x, t)$  on the mesh as

$$u_j^n := \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho(x, t_n) dx \quad (3.1.1)$$

The idea of the **Godunov scheme** is to approximate the solution  $\rho(x, t_n)$  to conservation law (1.2.2) by a piecewise constant function

$$u^n(x, t_n) = u_j^n \quad \text{if } x_{j-\frac{1}{2}} \leq x < x_{j+\frac{1}{2}} \quad (3.1.2)$$

and by solving the Riemann problems caused by the space discretization on the time interval  $[t_n, t_{n+1}]$ . Following from the integral form of conservation law (1.2.2), namely

$$\begin{aligned} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho(x, t_{n+1}) dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho(x, t_n) dx \\ &\quad - \left[ \int_{t_n}^{t_{n+1}} f(\rho(x_{j+\frac{1}{2}}, t)) dt - \int_{t_n}^{t_{n+1}} f(\rho(x_{j-\frac{1}{2}}, t)) dt \right] \end{aligned} \quad (3.1.3)$$

we obtain

$$u_j^{n+1} = u_j^n - \frac{1}{h} \left[ \int_{t_n}^{t_{n+1}} f(\rho(x_{j+\frac{1}{2}}, t)) dt - \int_{t_n}^{t_{n+1}} f(\rho(x_{j-\frac{1}{2}}, t)) dt \right] \quad (3.1.4)$$

with the **numerical flux function**

$$F(u_j^n, u_{j+1}^n) \sim \frac{1}{k} \int_{t_n}^{t_{n+1}} f(\rho(x_{j+\frac{1}{2}}, t)) dt \quad (3.1.5)$$

which plays the role of the average flux through  $x_{j+\frac{1}{2}}$  over the time interval  $[t_n, t_{n+1}]$ . Numerical methods of the form (3.1.4) are called *in conservative form* as they still fulfill the conservation law.

We also note that in practise equation 3.1.5 simplifies since  $u_j^n$  at point  $x_{j+\frac{1}{2}}$  is constant along the line  $[t_n, t_{n+1}]$  (also see picture ADD picture like LeVeque p.139). Therefore the numerical flux at point  $x_{j+\frac{1}{2}}$  only depends on the neighboring densities  $u_j^n$  and  $u_{j+1}^n$ . Denoting this value by  $u^*(u_j^n, u_{j+1}^n)$  the numerical flux reduces to

$$F(u_j^n, u_{j+1}^n) = f(u^*(u_j^n, u_{j+1}^n)) \quad (3.1.6)$$

Obviously the scheme is consistent, since  $F(u_j^n, u_j^n) = f(u_j^n)$ .

We mentioned earlier that  $u_j^n$  at  $x_{j+\frac{1}{2}}$  is constant along the line  $[t_n, t_{n+1}]$ . This only holds if the cells are "small enough" such that two shocks resulting from neighboring Riemann problems do not interact between two time steps. As the wave speed is bounded by the eigenvalue of  $f'(u)$  we require that

$$c_C := \frac{k}{h} |\sup(f'(u))| \leq 1 \quad (3.1.7)$$

$c_C$  is also called the Courant-number and serves as the one-dimensional CFL-condition [6].

The **Godunov method** now consists of separately solving every Riemann problem between every two neighboring cells. Let  $u_l$  and  $u_r$  be the densities on two neighboring cells. Then the flux  $f(u^*(u_l, u_r))$  is given by

$$f(u^*(u_l, u_r)) = \begin{cases} \min[f(u_l), f(u_r)] & \text{if } u_l \leq u_r \\ f(\sigma) & \text{if } u_r \leq \sigma \leq u_l \\ f(u_r) & \text{if } \sigma \leq u_r \leq u_l \text{ or } u_r \leq u_l \leq \sigma \end{cases} \quad (3.1.8)$$

ADD picture of different shocks, rarefaction waves

In the following we want to apply the Godunov scheme on the buffer model defined in section ??

### 3.1.2 The Godunov scheme on networks

Let us consider an arbitrary connected road network  $(\mathcal{N}, \mathcal{E})$  with  $n_R$  roads and  $n_J$  junctions. On every road  $e_i = [a_i, b_i]$  we apply the discretization  $x_{i,j} = a_{i,jh}, j = 0, \dots, D_i, x_{i,D_i h} = b_i$ , with  $D_i = \frac{b_i - a_i}{h} - 1$  the number of cells on road  $e_i$ . Let us then

denote  $u_{i,j}^n$  the density on the  $j$ -th cell of road  $e_i$  at time  $t = nk$ .

We can now define on all *inner cells* of every road  $e_i$  the Godunov scheme

$$u_{i,j}^{n+1} = u_{i,j}^n - \frac{k}{h} [f(u^*(u_{i,j}^n, u_{i,j+1}^n)) - f(u^*(u_{i,j-1}^n, u_{i,j}^n))], \quad e_i \in \mathcal{E}, j = 1, \dots, D_i - 1 \quad (3.1.9)$$

We incorporate the *externally inflowing densities*  $q_i(t), e_i \in \mathcal{E}^{in}$  by introducing a ghost cell at the beginning of each road  $e_i$ . Then we define

$$u_{i,0}^{n+1} = u_{i,0}^n - \frac{k}{h} [f(u^*(u_{i,0}^n, u_{i,1}^n)) - f(u^*(v_{i,1}^n, u_{i,0}^n))], \quad i \in \mathcal{E}^{in} \quad (3.1.10)$$

with the inflowing density  $v_{i,1}^n := \int_{t_n}^{t_{n+1}} q_i(t) dt$ .

The outfluxes leaving the network can be treated analogously.

For junctions we recall Definiton ?? for the junction fluxes in the single buffer model. Accordingly, we define for roads coming into junction  $J$

$$u_{i,D_i}^{n+1} = u_{i,D_i}^n - \frac{k}{h} [\bar{f}_i - f(u^*(u_{i,D_i-1}^n, u_{i,D_i}^n))], \quad i \in \delta^{in}(J) \quad (3.1.11)$$

For outgoing roads of junction  $J$  we analogously define the scheme

$$u_{i,0}^{n+1} = u_{i,0}^n - \frac{k}{h} [f(u^*(u_{i,0}^n, u_{i,1}^n)) - \hat{f}_i], \quad i \in \delta^{out}(J) \quad (3.1.12)$$

## 3.2 Examples

# Chapter 4

## Traffic light optimization

As discussed in section 2.3 we can model external effects, especially traffic lights, by assigning relative priorities

$$\eta_i \in [0, 1], \quad e_i \in \mathcal{E} \setminus \mathcal{E}^{out}$$

to roads prior to intersections. These values have the effect that only a fraction of the flux  $\bar{f}_i$  as defined in the buffer model can enter the junction and pass it towards outgoing roads. In particular the binary values  $\eta_i = 1$  and  $\eta_i = 0$  for road  $e_i$  correspond to green and red phases, respectively, for the particular incoming road.

For every junction  $J$  the **feasibility condition** must hold. It is given by

$$\sum_{e_i \in \delta^{in}(J)} \eta_i(t) = 1 \tag{4.0.1}$$

assuring that cars from only one road can cross the junction at any time  $t$ .

This leads to a necessary update of the junction fluxes. In detail we define the **influx under applied traffic control** as

$$\bar{\bar{f}}_k(t, \eta_k) := \min\{\omega_k, c_k(M - \sum_{j \in \mathcal{O}} q_j)\} \cdot \eta_k(t) \tag{4.0.2}$$

The corresponding **outflux under applied traffic control** is given by

$$\hat{\bar{f}}_j(t, \eta) := \begin{cases} \omega_j & q_j > 0 \\ \min\{\omega_j, \sum_{i \in \mathcal{I}} \bar{\bar{f}}_i \bar{\Theta}_{ij}\} & q_j = 0 \end{cases} \tag{4.0.3}$$

For the conservation law

$$(\rho_k)_t + f(\rho_k)_x = 0 \quad \forall e_k \in \mathcal{E} \tag{4.0.4a}$$

$$f(\rho_k(b_k, t)) = \bar{\bar{f}}_k(t) \quad \forall e_k \in \mathcal{E} \setminus \mathcal{E}^{out} \tag{4.0.4b}$$

$$f(\rho_k(a_k, t)) = \hat{\bar{f}}_k(t) \quad \forall e_k \in \mathcal{E} \setminus \mathcal{E}^{in} \tag{4.0.4c}$$

with junction fluxes  $\bar{f}_k$  and  $\hat{f}_k$  on a given network  $(\mathcal{N}, \mathcal{E})$  the goal is to optimize the flow on every single road and through their intersections. The considered objective function - or also referred to as *cost functional* - can hereby vary depending on the goals of the simulation and the specific definition of the targeted problem. Typical **objective functions** for the optimization are introduced in the following.

### a) Mean travel time

From driver's point of view, the key quantity to determine the quality - and therefore the optimal value - of traffic is related to the time needed to reach the desired location. Taking into account the sum of every personal preference hence leads to the definition of the mean arrival time or **mean travel time** (cf. [5]).

Let  $x = 0$  be a point on the network. Then the mean travel time needed to reach point  $x = \bar{x} > 0$  can be described by

$$T(\bar{x}) := \frac{1}{Q_{in}} \int_{t_0}^{\infty} t f(\rho(\bar{x}, t)) dt$$

where  $Q_{in} = \int_{t_0}^{t_{end}} q_0(t) dt$  is the accumulated influx  $q_0(t)$  into point  $x = 0$  on a compact time interval  $[t_0, t_{end}]$ .

#### Remarks:

- This approach expects compactly supported inflow into  $x = 0$ .
- On complex networks drivers can have distinct preference of their respective arrival points and favoured routes. This means that cars, despite also crossing the point  $x = 0$ , might never reach the reference point  $x = \bar{x}$ , which complicates the computation of the average travel time between two points on a network.

### b) Cumulative traffic flux

From the traffic planner's perspective a more relevant quantity might be the overall flux on the entire network. Therefore the desired goal is to maximize the total number of cars travelling through the network over a certain time interval.

Following equation 4.0.4 we denote by  $\bar{f}$  and  $\hat{f}$  incoming fluxes into and outgoing fluxes out of junctions, respectively. Then we can formulate the **cumulated traffic flux** on the network during time  $t = [t_0, T]$  as

$$\begin{aligned} F_T(\eta) := & \sum_{i \in \mathcal{E}} \int_0^T \int_{a_i}^{b_i} f(\rho_i(x, t)) dx dt + \sum_{i \in \mathcal{E} \setminus \mathcal{E}^{out}} \int_0^T \bar{f}_i(t, \eta_i) dt \\ & + \sum_{j \in \mathcal{E} \setminus \mathcal{E}^{in}} \int_0^T \hat{f}_j(t, \eta_j) dt \end{aligned} \quad (4.0.5)$$

where  $\bar{f}$  and  $\hat{f}$  are as defined as in equations 4.0.2 and 4.0.3 in the buffer model.

**Remark:**

During the following optimization studies the cumulative traffic flux is the cost functional of choice.

## 4.1 Traffic light coordination

Optimally tuned traffic lights settings provide a setting where drivers encounter a green wave, in particular a sequence of consecutive green lights. The distinction between synchronized and coordinated traffic lights is important. Synchronized traffic signals all switch at the same time and are hardly used in practice. On the other hand coordinated signals are controlled by a master controller and are set up such that they progress (switch) in sequence in order to generate a green wave for crossing vehicles. **ADD weiter ausführlich**

### The model

Consider a sequence of two intersections with two incoming and one outgoing road (cf. figure 4.1) with inflows  $q_k(t)$  for  $e_k \in \mathcal{E}^{in}$ . Then we can refine the conservation law of

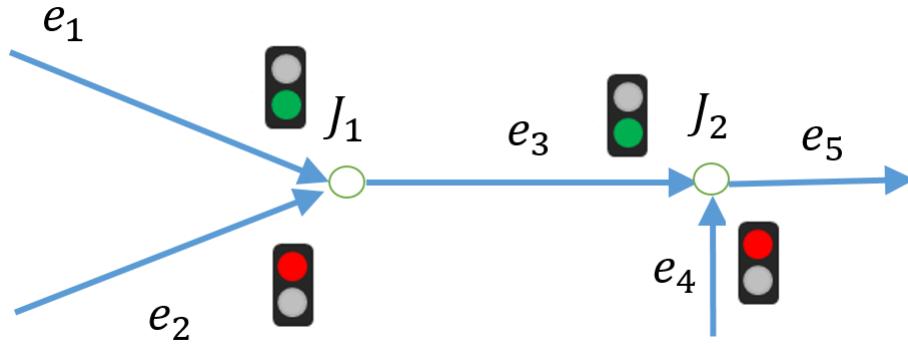


Figure 4.1: Example network consisting of two junctions and four controls.

1.2.3 to

$$(\rho_k)_t + f(\rho_k)_x = 0 \quad \forall e_k, k = 0, \dots, 4 \quad (4.1.1a)$$

$$f(\rho_k(b_k, t)) = \bar{f}_k(t, \eta_k(t)) \quad \forall e_k, k = 0, 1, 2, 3 \quad (4.1.1b)$$

$$f(\rho_k(a_k, t)) = \hat{f}_k(t, \eta(t)) \quad \forall e_k, k = 2, 4 \quad (4.1.1c)$$

accounting the restrictions on the fluxes induced by the traffic configuration  $\eta$ , where  $\eta(t) = (\eta_0(t), \dots, \eta_3(t)) \in [0, 1]^4$  is the vector containing all control values.

Impose now that the two traffic lights  $\eta_0, \eta_2$  have the same fixed frequency of red/green light - say one time unit -, only set apart by a delay  $\tau$ , and recall the feasibility condition 4.0.1. Then the controls satisfy

$$\begin{aligned}\eta_0(t) &= \chi_{[0,1] \cap [2,3] \cap \dots} =: \eta_C(t) \\ \eta_1(t) &= 1 - \eta_C(t) \\ \eta_2(t) &= \eta_C(t - \tau) \\ \eta_3(t) &= 1 - \eta_C(t - \tau)\end{aligned}$$

We fix  $\eta_C(t)$ . The goal now is to find the optimal delay  $\tau$  in order to obtain the best value for  $F_T$ .

In particular, the optimization problem can be formulated as

$$\begin{aligned}F_T^* := \max_{\tau \in \mathbb{R}} \quad & \sum_{i \in \mathcal{E}} \int_0^T \int_{a_i}^{b_i} f(\rho_i(x, t)) dx dt + \sum_{i \in \mathcal{E} \setminus \mathcal{E}^{out}} \int_0^T \bar{f}_i(t) \cdot \eta_i(t) dt \\ & + \sum_{j \in \mathcal{E} \setminus \mathcal{E}^{in}} \int_0^T \hat{f}_j(t, \eta) dt\end{aligned}\tag{4.1.2}$$

and the optimal delay is given by

$$\begin{aligned}\tau^* := \quad & \arg \max_{\tau \in \mathbb{R}} \sum_{i \in \mathcal{E}} \int_0^T \int_{a_i}^{b_i} f(\rho_i(x, t)) dx dt + \sum_{i \in \mathcal{E} \setminus \mathcal{E}^{out}} \int_0^T \bar{f}_i(t) \cdot \eta_i(t) dt \\ & + \sum_{j \in \mathcal{E} \setminus \mathcal{E}^{in}} \int_0^T \hat{f}_j(t, \eta) dt\end{aligned}\tag{4.1.3}$$

### Example

Consider the same network as provided in figure 4.1 with the following properties:

road	length	initial density $\rho_0$	inflow $q_i(t)$ on $[0, 200]$
$e_0$	50	0.2	$f(0.4)$
$e_1$	50	0.2	$f(0.2)$
$e_2$	100	0.2	-
$e_3$	50	0.2	$f(0.2)$
$e_4$	50	0.2	-

Table 4.1: Setup for the network of example

Furthermore let the external inflows be compactly supported on  $t \in [0s, 200s]$  and let the fixed green/red phase duration be set to 60s. Then the evaluation of our cost functional with respect to the delay  $\tau \in [0, 60]$  up to final time  $T = 500$ , according to equation 4.1.2, leads to an optimal delay of  $\tau^* = 31s$  with an optimal cumulated traffic flux  $F_T^* \sim 83174.7$ .

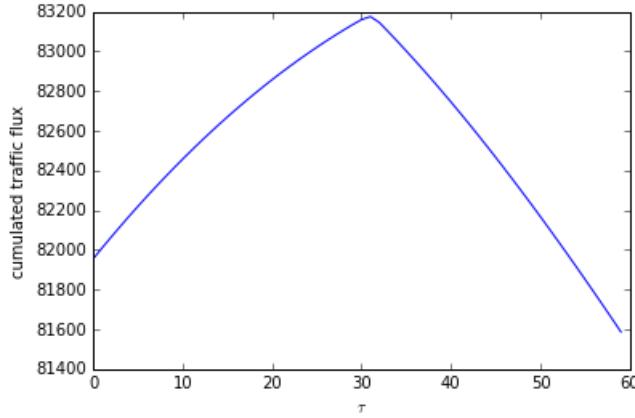


Figure 4.2: Dependency of the cumulated traffic flux  $F_T$  on the value for the delay. Obviously the objective function attains its maximum  $F_T^*$  at  $\tau^* \sim 83174.7$ .

### TODO:

- Discussion
- add sequence of plots of road densities and/or  $\eta$

## 4.2 Optimization via Model Predictive Control

In contrary to chapter 4.1 the optimal master controller should not work on fixed green and red light periods but be able to adjust traffic lights based on the current - and ideally even future - traffic situation in front of the traffic lights. Based on the possible fluxes the master controller would assign traffic light configurations such that the total flow on the network is maximized over the full time horizon. Unfortunately in real networks complete knowledge of the future volume of traffic over the whole time horizon is usually not given (at best educated guesses about the volume of traffic and the occupation of the network can be made at certain times throughout the day). This is where **model predictive control** comes into play.

Model Predictive Control (MPC) is a method to control complex dynamic processes and has a wide range of applications. MPC algorithms use a model of the underlying system under consideration in order to find optimal control signals, taking the future behaviour of the system into account. MPC methods are suitable to control systems in which prediction is a key aspect.

The main advantage of MPC over non-predictive control is, that MPC methods inherently make a trade-off between immediate performance and future outputs [10].

MPC is based on an iterative, finite-horizon optimization of an objective function whose state variables are derived from the solution of a system of PDEs.

The finite time interval used for MPC optimization is called **predictive horizon**  $n_p$ . The resulting optimal control sequence obtained after one optimization step is then applied as input for the light signals up to a **control horizon**  $n_c$ , when the optimization step repeats on the new time interval. As the predictive horizon keeps being shifted forward MPC is also referred to as *receding horizon control*.

In the following we provide a high level description of the MPC method [11]:

1. **Prediction.** The current state  $x(t_n)$  of the system, expected external influences and a planned control signal are used to predict the behavior of the considered system in the the predictive horizon  $[t_n, t_n + n_P \Delta t]$ . For traffic flow, this involves the evaluation of a model to predict the future road conditions.
2. **Performance evaluation.** An objective function is used to evaluate the performance of the system under the planned control signal from the prediction phase. Typical functionals consist of the average travel time or the cummulated traffic flux on the network.
3. **Optimization.** In an optimization step, the optimal control signal is found. This control signal optimizes the objective function for the chosen time horizon of the prediction phase. Typical methods for this step use gradient-descent methods, LP solvers or least-squared methods.

**4. Model dynamics.** Given the optimal control signal from the previous step, the next control action is taken from the optimal control signal and subsequently applied to the system on the control horizon  $[t_n, t_n + n_C \Delta t]$ . Recalculating optimal control signals proceeds using a receding horizon scheme.

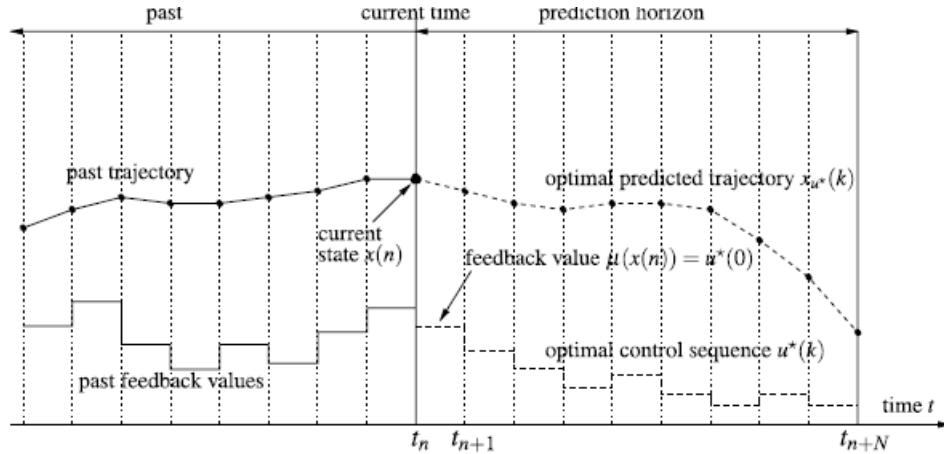


Figure 4.3: Illustration of an MPC step at time  $t_n$  ADD better picture

In the following the theory on MPC will be provided, supplemented by numerical examples. Also a comparison with the results from the delay modelling in 4.1 will be given.

## Theory

relaxation of binary values for the light signals on  $[0, 1]$ .

Let  $K_i \in \{0, 1\}^{n_J}$  be a feasible discrete configuration for all traffic lights such that the feasibility condition 4.0.1 is met, where  $K_i^j \in \{0, 1\}$  denotes the individual configuration of the  $j$ -th traffic light. Then we can define  $\Omega := \{K_1, \dots, K_{n_\Omega}\}$  as the set of all **feasible discrete traffic light configurations**. ADD different repr of K

**Example 2** For a single junction with 3 incoming and a single outgoing road  $\Omega$  would be defined in a straight forward way as

$$\Omega := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

supposing that only one signal can be green at every instant.

The flux functional  $F_{MPC}$  can then be defined as

$$\begin{aligned}
F_{MPC}(t_n, \eta) &:= \sum_{i \in \mathcal{E}} \int_{t_n}^{t_n + n_p \Delta t} \int_{a_i}^{b_i} f(\rho_i(x, t)) dx dt + \sum_{i \in \mathcal{E} \setminus \mathcal{E}^{out}} \int_{t_n}^{t_n + n_p \Delta t} \bar{f}_i(t, \eta_i) dt \\
&+ \sum_{j \in \mathcal{E} \setminus \mathcal{E}^{in}} \int_{t_n}^{t_n + n_p \Delta t} \hat{f}_j(t, \eta) dt + \gamma_1 \int_{t_n}^{t_n + n_p \Delta t} \sum_{i=1}^{n_J} W_i(t, \eta_i) dt \\
&+ \gamma_2 \int_{t_n}^{t_n + n_p \Delta t} \|\dot{\eta}\|^2 dt \\
&= F_{n_C}(t_n, \eta) + R(t_n, \eta)
\end{aligned}$$

with  $\gamma_1, \gamma_2 > 0$  and where

- $F_{n_C}(t_n, \eta)$  refers to the cummulated flux travelled through the network during  $t \in [t_n, t_n + n_C \Delta t]$  and
  - $R$  marks the influencing terms ??????
- $W_i(t) := \|\prod_j (\eta_j(t) - K_i^j)\|^2$  is a **multi-well function**, as it is also used in the Modica-Mortola-functional [15]. It works as a de-relaxation term meaning that the resulting controls  $\eta_i$  stay close to the binary values 0 and 1
  - and the term  $\|\dot{\eta}\|^2$  which limits the frequency with which traffic lights can switch.

Then by substituting the time integrals by finite sums and discretizing the derivative via the backward difference quotient

$$\int_{t_n}^{t_n + n_p \Delta t} \|\dot{\eta}\|^2 dt \sim \gamma_2 \sum_{j=n+1}^{n+n_p} \frac{\Delta t}{\Delta t} \sum_{i=1}^{n_J} (\eta_i(t_j) - \eta_i(t_{j-1}))^2 = \gamma_2 \sum_{j=n+1}^{n+n_p} \sum_{i=1}^{n_J} (\eta_i(t_j) - \eta_i(t_{j-1}))^2$$

we can define the **discrete optimal control problem** (OCP) over the finite horizon  $[t_n, t_n + n_p \Delta t]$  as

The resulting flux  $F_{n_C}(t_{n \cdot n_C}, u^*)$  is then (locally) optimal on the interval  $[t_n, t_n + n_C \Delta t]$ . The **total flux** on the network over the entire time horizon  $[0, T]$  can finally be computed by summing up the local optima, i.e.

$$F_{MPC,T}(u^*) = \sum_{n=0}^{n_{MPC}} F_{n_C}(t_{n \cdot n_C}, u^*|_{n_C}(t_n)) \tag{4.2.2}$$

where  $n_{MPC}$  denotes the number of MPC-steps computed.

A similar variation of algorithm 1 can be found in [9]. The solution of this algorithm then converges to a local optimum and for appropriate choices of  $\gamma_1, \gamma_2$  the elements of

**OCP**

$$F_{n_C}(t_{n \cdot n_C}, u^*) := \max_{\eta_i \in [0,1]} F_{MPC,discrete}(t_n, \eta) \quad (4.2.1a)$$

s.t.

$$\text{valid TL configs} \quad \sum_{e_i \in \delta^{in}(J)} \eta_i(t_k) = 1, \quad k = n \dots n + n_p, J \in \mathcal{N} \quad (4.2.1b)$$

$$\text{Godunov scheme} \quad \rho_j^{n+1} = \rho_j^n - \frac{k}{h} [f(u^*(\rho_j^n, \rho_{j+1}^n)) - f(u^*(\rho_{j-1}^n, \rho_j^n))] \quad (4.2.1c)$$

$$\text{maximum fluxes} \quad \omega_i, \omega_j, \text{cf.2.3} \quad (4.2.1d)$$

$$\text{junction influxes} \quad \bar{\bar{f}}_i = \min\{\omega_i, c_i(M - \sum_{j \in \mathcal{O}} q_j)\} \quad (4.2.1e)$$

$$\text{junction outfluxes} \quad \hat{\hat{f}}_j = \begin{cases} \omega_j & q_j > 0 \\ \min\{\omega_j, \sum_{i \in \mathcal{I}} \hat{\hat{f}}_i \bar{\Theta}_{ij}\} & q_j = 0 \end{cases} \quad (4.2.1f)$$

$$\text{queue} \quad q_j^{n+1} = q_j^n + \Delta t \left( \sum_{i \in \mathcal{I}} \bar{\Theta}_{ij} \bar{\bar{f}}_i - \bar{\bar{f}}_j \right) \quad (4.2.1g)$$

$$\text{initial values} \quad \rho_i^n, q_j^n, \eta^n \quad (4.2.1h)$$

$$\text{external inflow} \quad \rho_i(0, t_m) \forall m = 0, 1 \dots n_T \forall i \in \mathcal{E}^{in} \quad (4.2.1i)$$

Figure 4.4: The MPC optimal control problem.

the optimal control sequences are close to binary.

**Justification of the use of  $W(t)$  and  $\|\dot{\eta}\|$ :**

Consider a simple network consisting of a single junction with two incoming roads and one outgoing road. Initially the network is empty, so  $\rho_i(x, 0) = 0 \forall i = 1 \dots 3$ . In the given network we have two traffic lights (controls)  $\eta_1$  and  $\eta_2$  referring to the two incoming roads. Furthermore at every time step  $t$  the feasibility condition 4.0.1 is satisfied.

Using this basic network we want to evaluate the effect of the two additional terms  $W(t)$  and  $\|\dot{\eta}\|$ .

	<b>fluxes</b>	<b>computation time [s]</b>
a)	48517.24	518.28
b)	48120.96	417.30
c)	48503.42	295.84
d)	48398.24	396.31

Table 4.2: Resulting optimal fluxes and needed computation time for the different functionals

**Algorithm 1** (*MPC*)

For every sampling time  $t_n, n = 0, n_c, 2n_c \dots$  do the following steps 1.-4.:

1. Measure the densities  $\rho_i(x, t_n)$  on every road  $e_i$  of the network
2. Solve the OCP 4.2.1 and denote the resulting optimal control sequence for TL  $i$  by  $\eta_i^* := \{\eta_i(t_n), \eta_i(t_{n+1}), \dots, \eta_i(t_{n+n_p})\}$
3. Apply the rounding function

$$P(\eta) := \begin{cases} 1 & \eta \geq 0.5 \\ 0 & \eta < 0.5 \end{cases} \quad (4.2.3)$$

on the optimal control sequence  $u_i^*|_{n_c} := \{\eta_i(t_n), \eta_i(t_{n+1}), \dots, \eta_i(t_{n+n_c})\}$  in order to obtain a (sub-)optimal binary control sequence  $u_{i,bin}^*|_{n_c}$ .

4. Take  $u_{bin}^*|_{n_c} := \{u_{0,bin}^*|_{n_c}, u_{1,bin}^*|_{n_c}, \dots, u_{n_J,bin}^*|_{n_c}\}$  as an input for the Godunov scheme to obtain densities  $\rho_i(x, t_{n+n_c})$  and to compute the optimal flux  $F_{n_C}(t_n, u_{bin}^*|_{n_c})$ . Proceed with 1.
5. Compute the total flux  $F_T(u_{bin}^*)$  based on equation 4.2.2.

Figure 4.5: The MPC algorithm.

### Examples

To solve the optimization problems (4.2.1) we use a sequential least squares programming optimizer (SLSQP) implemented as part of the SciPy module<sup>1</sup> in Python 2.7<sup>2</sup>. The SLSQP uses the Han-Powell quasi-Newton method with a Broyden-Fletcher-Goldfarb-Shanno (BFGS) update [12]. In particular we choose the convergence accuracy  $\epsilon = 10^{-6}$  for the optimization.

a) single junction: two incoming, one outgoing road

$n_p$	$n_c$	$F_T$	computation time [s]
1	1	21541.159	24
2	1	21535.535	36
10	1	21539.150	903
10	10	22046.865	236
optimal control value $\eta_0$			

<sup>1</sup>Jones E, Oliphant E, Peterson P, et al. SciPy: Open Source Scientific Tools for Python, 2001-, <http://www.scipy.org/>

<sup>2</sup>Python Software Foundation. Python Language Reference, version 2.7. Available at <http://www.python.org>

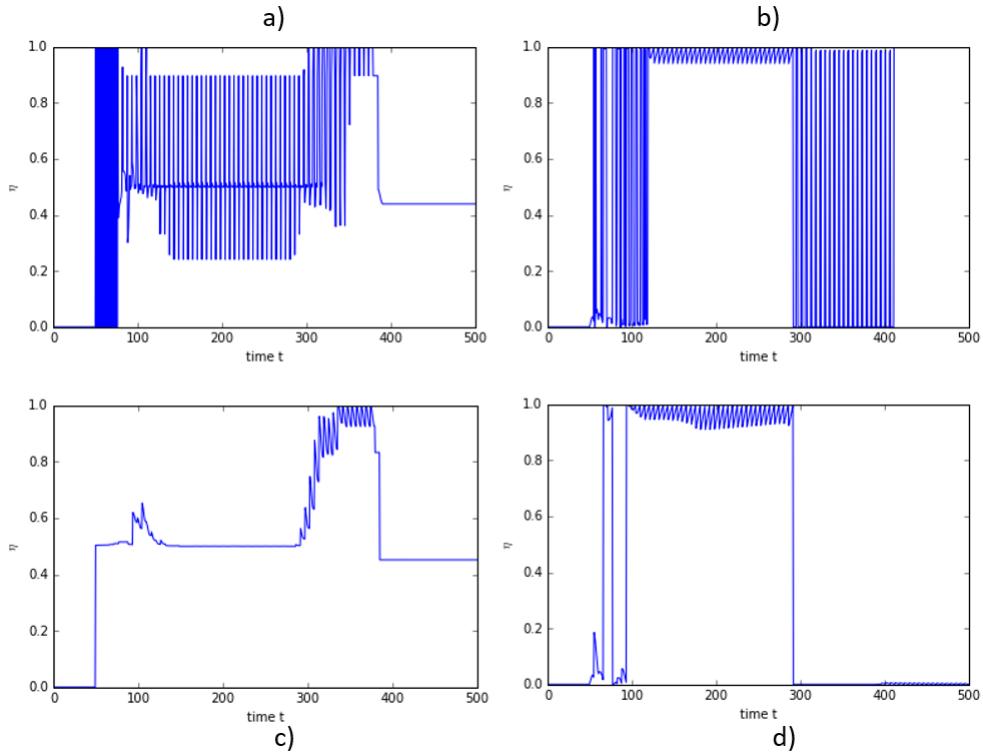


Figure 4.6: Plots of control  $\eta_1(t)$  using different functionals. (a) The flux functional  $\tilde{F}$  without considering terms  $\tilde{W}$  and  $\tilde{D}$ . We notice high frequencies between 0-(red-)phases and 1-(green-)phases and the existence of clearly non-binary states  $\eta_1(t) \in (0, 1)$ . (b)  $\gamma_1 = 0, \gamma_2 = 10$ . Application of the de-relaxation term  $\tilde{W}$ . The rate of occurrence for non-binary controls is reduced. Nevertheless we still observe high frequencies of switching. (c)  $\gamma_1 = 10, \gamma_2 = 0$ . Application of switching cost  $\|\dot{\eta}\|$ . Switching frequencies are decreased but mainly non-binary control states occur. (d)  $\gamma_1 = 10, \gamma_2 = 10$ . Both de-relaxation and switching cost are applied. Low frequency and mainly binary states occur.

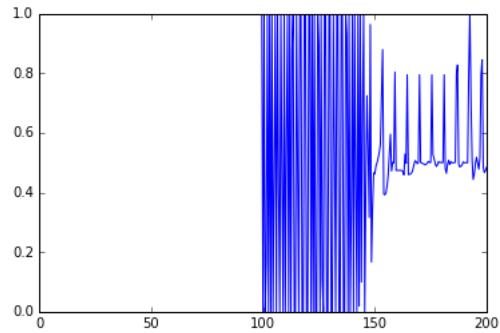


Figure 4.7: evolution of the control  $\eta_0$  over time. it starts only at  $t=100$ s because the network is initially empty and therefore the controls stay at the initial value until the first cars reach the junction. controls change with high frequency until they level out eventually at around 0.5

b) network with 2 junctions - same as 4.1

same input values as in 4.1

$n_p$	$n_c$	$F_T$	computation time [s]
10	10	89723.626	518.85

comparing this to the optimal delay value in 4.1:  $F_T(\tau^*) \sim 83174.7$

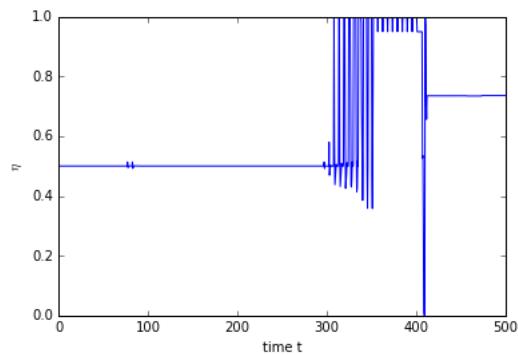


Figure 4.8: abb

c) network with 2 junctions: initially empty

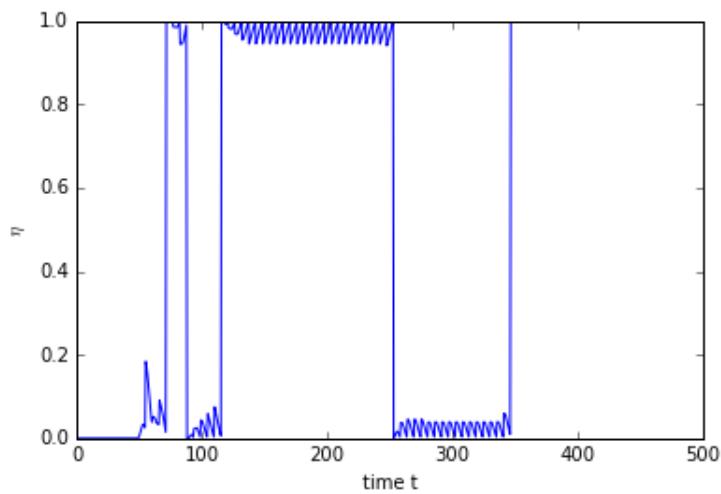


Figure 4.9: Plot of control  $\eta_1(t)$  for a 2-junction network with 0 initial data

### 4.3 Conclusion and possible extensions

- Instead of MPC one could use compass search in order to determine the time points at which traffic lights switch...ref to handwritten papers

## Abbreviations

LWR	Lighthill-Whitham-Richards
MPC	model predictive control
ODE	ordinary differential equation
OCP	optimal control problem
PDE	partial differential equation
TL	traffic light

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