Fluctuations of Kendall random walks

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Generalized convolution of the Kendall type

Definition 1 (Kendall convolution)

1. $\delta_a \vartriangle_\alpha \delta_b = T_{max\{|a|,|b|\}} \left((1 - z^\alpha)\widetilde{\delta}_1 + z^\alpha \widetilde{\pi}_{2\alpha} \right),$
   where $z = \frac{min\{|a|,|b|\}}{max\{|a|,|b|\}}$, $\alpha \in (0, 1]$, $\widetilde{\pi}_{2\alpha}(dx) = \frac{\alpha}{|x|^{2\alpha+1}} \mathbf{1}_{(1,\infty)}(|x|)dx.$
2. $\nu_1 \vartriangle_\alpha \nu_2(A) = \int_{\mathbb{R}^2} (\delta_x \vartriangle_\alpha \delta_y)(A)\nu_1(dx)\nu_2(dy)$

Kendall random walk

Definition 2 (M. Borowiecka-Olszewska, B.H. Jasiulis-Gołdyn, J.K. Misiewicz, J. Rosiński)

The Kendall random walk is a Markov process $\{X_n : n \in \mathbb{N}_0\}$ *with* $X_0 = 0$ *and the transition probabilities*

$$P_n(x, A) = P(X_{n+k} \in A | X_k = x) = \delta_x \vartriangle_\alpha \nu^{\vartriangle_\alpha n}(A)$$

where measure $\nu \in \mathcal{P}_s$ *is called the step distribution.*

Kendall random walk representation

Theorem 1 (Jasiulis-Gołdyn)

The Markov process $\{X_n : n \in \mathbb{N}_0\}$ with step distribution $Y_i \sim \nu$ has the following properties: $X_0 = 0$, $X_1 = Y_1$,

$$X_{n+1} = v(|X_n|, |Y_{n+1}|)u(X_n, Y_{n+1})\theta_n^{Q_n} a.e.,$$

where θ_n is independent of $v(|X_n|, |Y_{n+1}|)u(X_n, Y_{n+1})$, $v(x, y) = x \vee y$, $u(x, y) = \frac{x \wedge y}{x \vee y}$,

$$u(x, y) = \begin{cases} sgn(x), & |x| \geq |y|, \\ sgn(y), & |x| < |y|, \end{cases}$$

and

$$\mathbb{P}(Q_n = k | X_n, Y_{n+1}) = \begin{cases} (z(|X_n|, |Y_{n+1}|))^{\alpha}, & k = 1, \\ 1 - (z(|X_n|, |Y_{n+1}|))^{\alpha}, & k = 0. \end{cases}$$

Williamson transform

Definition 3

Williamson transform Operation $\nu \to \widehat{\nu}$ given by

$$\widehat{\nu}(t) = \int_{\mathbb{R}} (1 - |xt|^{\alpha})_+ \, \nu(dx), \quad \nu \in \mathcal{P}_s,$$

where $a_+ = a$ if $a \geqslant 0$ and $a_+ = 0$ otherwise.

Theorem 2

Let $\nu_1, \nu_2 \in \mathcal{P}_s$ be probability measures with Williamson transforms $\widehat{\nu_1}, \widehat{\nu_2}$. Then

$$\int_{\mathbb{R}} (1 - |xt|^{\alpha})_+ \big(\nu_1 \vartriangle_{\alpha} \nu_2\big)(dx) = \widehat{\nu_1}(t)\widehat{\nu_2}(t).$$

Inverse Williamson transform

Theorem 3 (Jasiulis-Gołdyn, Misiewicz, 2017)

The correspondence between measure $\nu \in \mathcal{P}_s$ *and its Williamson transform is* $1-1$*. Moreover, denoting by* F *the cumulative distribution function of* ν*,* $\nu(\{0\}) = 0$ *and* $G(t) = \hat{\nu}(\frac{1}{t})$*, we have*

$$F(t) = \begin{cases} \frac{1}{2\alpha}\left[\alpha(G(t)+1) + tG'(t)\right] & \text{if} \quad t > 0; \\ 1 - F(-t) & \text{if} \quad t < 0. \end{cases}$$

except for the countable many $t \in \mathbb{R}$*.*

Definitions

- $\tau_a^+ = \inf\{i \geq 0 : X_i > a\}$
- $\tau_a^- = \inf\{i \geq 0 : X_i < a\}$
- Convention: $\min \emptyset = \infty$
- Notation: $H(x) = 2F(x) - 1 - G(x)$
- Notation: $\Psi(t) = (1 - |t|^\alpha)_+$

Useful lemmas

Lemma 4 (Jasiulis-Gołdyn, Misiewicz)

$$\begin{aligned}\delta_x \vartriangle_\alpha \delta_y(0,t) &:= \frac{1}{2}\left(1 - \left|\frac{xy}{t^2}\right|^\alpha\right)\mathbf{1}_{\{|x|<t,|y|<t\}} \\ &= \frac{1}{2}\left[\Psi\left(\frac{x}{t}\right) + \Psi\left(\frac{y}{t}\right) - \Psi\left(\frac{x}{t}\right)\Psi\left(\frac{y}{t}\right)\right]\mathbf{1}_{\{|x|<t,|y|<t\}}\end{aligned}$$

Lemma 5 (Jasiulis-Gołdyn, Misiewicz)

$$\begin{aligned}&\delta_x \vartriangle_\alpha \nu\,(0,t) = P_1(x,[0,t)) \\ &= \left[\Psi\left(\frac{x}{t}\right)\left(F(t) - \frac{1}{2}\right) + \frac{1}{2}G(t) - \frac{1}{2}\Psi\left(\frac{x}{t}\right)G(t)\right]\mathbf{1}_{\{|x|<t\}} \\ &= \frac{1}{2}\left[\Psi\left(\frac{x}{t}\right)H(t) + G(t)\right]\mathbf{1}_{\{|x|<t\}}\end{aligned}$$

Useful lemmas

Lemma 6 (Jasiulis-Gołdyn, Staniak)

Under the notation $M(a,t) = H(a)\Psi\left(\frac{a}{t}\right) + \left(1 - \Psi\left(\frac{a}{t}\right)\right)G(a)$ the following equality holds

$$\begin{aligned}
& \int_{-\infty}^{a} \Psi\left(\frac{x}{t}\right)(\delta_y \vartriangle_\alpha \nu)(dx) \\
= \;& \frac{1}{2}\Psi\left(\frac{y}{a}\right)\left[H(a)\Psi\left(\frac{a}{t}\right) + \left(1 - \Psi\left(\frac{a}{t}\right)\right)G(a)\right] \\
+ \;& \frac{1}{2}G(a)\Psi\left(\frac{a}{t}\right)\mathbb{1}(|y| < a) + \frac{1}{2}\Psi\left(\frac{y}{t}\right)G(t) \\
= \;& \frac{1}{2}\Psi\left(\frac{y}{a}\right)M(a,t) + \frac{1}{2}G(a)\Psi\left(\frac{a}{t}\right)\mathbb{1}(|y| < a) + \frac{1}{2}\Psi\left(\frac{y}{t}\right)G(t).
\end{aligned}$$

Results for a = 0

Theorem 7 (Jasiulis-Gołdyn, Misiewicz)

R. v. τ_0^+ (and, by symmetry of the Kendall random walk, also variable τ_0^-) has geometric distribution $P(\tau_0^+ = k) = \frac{1}{2^k}$, $k = 1, 2, \cdots$.
Generating function for τ_0^+ is given by

$$\mathbf{E}s^{\tau_0^+} = \frac{\frac{s}{2}}{1 - \frac{s}{2}}, \quad 0 \leqslant s < 2.$$

More results

- $\Phi_n(t) := \mathbf{P}\{X_1 \leqslant 0, \ldots X_{n-1} \leqslant 0, 0 < X_n < t\} = \frac{1}{2^n} G(t)^{n-1}\left[2n\left(F(t) - \frac{1}{2}\right) - (n-1)G(t)\right]$

Theorem 8 (Jasiulis-Gołdyn, Misiewicz)

$$\begin{aligned} \mathbf{P}\left\{X_{\tau_0^+} < t\right\} &= \sum_{k=1}^{\infty} \mathbf{P}\left\{X_{\tau_0^+} < t, \tau_0^+ = k\right\} \\ &= \sum_{k=1}^{\infty} \Phi_k(t) = \frac{1}{(2-G(t))^2}\left[4F(t) - 2 - G(t)^2\right] \end{aligned}$$

Even more results

- Let $\{X_n \colon n \in \mathbb{N}_0\}$ be the Kendall random walk
- Let $N_{s/2}$ be geometric r.v. independent of (X_n).
- Define geometric-Kendall r.v.: $Z_{s/2} = \sum_{k=1}^{\infty} X_k \mathbf{1}_{\{N_{s/2}=k\}}$
- Notice that $\mathbf{E}\Psi\left(X_k/t\right) = G(t)^k$ and $\mathbf{E}\Psi(Z_{s/2}/t) = \frac{G(t)\left(1-\frac{s}{2}\right)}{1-\frac{s}{2}G(t)}$.

Theorem 9 (Jasiulis-Gołdyn, Misiewicz)

Let $\{X_n : n \in \mathbb{N}_0\}$ be a random walk under the Kendall convolution with the unit step $X_1 \sim F$ such that $F(0) = \frac{1}{2}$. Then

$$\mathbf{E}s^{\tau_0^+}\Psi\left(uX_{\tau_0^+}\right) = \frac{\frac{s}{2}}{1-\frac{s}{2}} \cdot \frac{(1-\frac{s}{2})G(1/u)}{1-\frac{s}{2}G(1/u)} = \mathbf{E}s^{\tau_0^+}\,\mathbf{E}\Psi\left(uZ_{s/2}\right).$$

Distribution of the first ladder moment

Theorem 10 (Jasiulis-Gołdyn, Staniak)

The distribution of τ_a^+ *is given by*

$$\mathbb{P}(\tau_a^+ = k) = \left(\frac{1}{2}\right)^{k-1} \left[\frac{F(a)(2G(a)-1)^2 + G(a)[2F(a) - 2 - G(a)^2]}{(2G(a)-1)^2}\right] + G(a)^{k-1} \frac{(k-1)(2G(a)-1) - 2G(a)^2}{(2G(a)-1)^2} + G(a)^{k-1} \frac{(1-G(a))(2F(a)-1)(2G(a)-1)^2}{(2G(a)-1)^2}$$

Proof

- $\mathbb{P}(\tau_a^+ = k) = \mathbb{P}(X_0 \leq a, X_1 \leq a, \ldots, X_{k-1} \leq a, X_k > a)$
  $= \int_{-\infty}^{a} \ldots \int_{-\infty}^{a} \int_a^{\infty} P_1(x_{k-1}, dx_k) P_1(x_{k-2}, dx_{k-1}) \ldots P_1(0, dx_1)$
- Define
  $I_1 = \int_a^{\infty} P_1(x_{k-1}, dx_k) = \frac{1}{2} - \frac{1}{2} \left[\Psi(\frac{x_{k-1}}{a}) H(a) + G(a) \right] \mathbf{1}(|x_{k-1}| < a)$

Proof: recurrence equations

- Assume that I_j is of the form
  $I_j = A_j + \mathbf{1}(|x_{k-j}| < a)\left[\Psi\left(\frac{x_{k-j}}{a}\right) H(a)B_j + C_j G(a)\right]$
- $I_{j+1} = \int_{-\infty}^{a} I_j\ (\delta_{x_{k-j-1}} \vartriangle_\alpha \nu)(d_{x_{k-j}})$
- $\begin{cases} A_{j+1} = & \frac{1}{2}A_j, \\ B_{j+1} = & \frac{1}{2}A_j + G(a)(B_j + C_j), \\ C_{j+1} = & \frac{1}{2}A_j + C_j G(a) \end{cases}$
- $A_1 = \frac{1}{2}$, $B_1 = C_1 = -\frac{1}{2}$

Solutions

- $$A_j = \left(\frac{1}{2}\right)^j$$

- $$B_j = G(a)^{j-1}B_1 + \sum_{m=2}^{j} G(a)^{j-m}\left(\frac{1}{2}\right)^m + \sum_{m=1}^{j-1} G(a)^{j-m}C_m$$
$$= \sum_{m=1}^{j} G(a)^{j-m}C_m$$

- $$C_j = G(a)^{j-1}C_1 + \sum_{m=2}^{j} G(a)^{j-m}\left(\frac{1}{2}\right)^m = \sum_{m=1}^{j} G(a)^{j-m}\left(\frac{1}{2}\right)^m - G(a)^{j-1}$$

Closed form solutions

- $A_j = (\frac{1}{2})^j$
- $B_j = G(a)^{j-1}\frac{(1-G(a)(2G(a)-1)-G(a)}{(2G(a)-1)^2}j + \left(\frac{1}{2}\right)^j \frac{1}{(2G(a)-1)^2}$
- $C_j = \frac{G(a)^{j-1}(1-G(a))-2^{-j}}{2G(a)-1} = G(a)^{j-1}\frac{1-G(a)}{2G(a)-1} + \left(\frac{1}{2}\right)^j \frac{-1}{2G(a)-1}$

Corollary

[Jasiulis-Gołdyn, Staniak]

Lemma 1

Generating function for τ_a^+ is given by

$$\begin{aligned} H(s) &= \frac{\frac{s}{2}}{1-\frac{s}{2}}\left[2F(a) + \frac{4G(a)^2(2F(a)-1)}{(2G(a)-1)^2}\right] \\ &+ \frac{s}{(1-sG(a))^2}\left[\frac{(2G(a)-1-2G(a)^2)H}{(2G(a)-1)^2}\right] \\ &+ \frac{s}{1-sG(a)}\left[\frac{(2F(a)-1)(G(a)+1)-G(a)(2G(a)-1-2G(a)^2)}{(2G(a)-1)^2}\right] \end{aligned}$$

Mean of the first ladder moment

Lemma 2

Mean of the random variable τ_a^+ is given by

$$
\begin{aligned}
\mathbb{E}\tau_a^+ &= \frac{1}{4}\left[2F(a) + \frac{4G(a)^2(2F(a)-1)}{(2G(a)-1)^2}\right] \\
+ & \frac{1-G(a)^2}{(1-G(a))^4}\left[\frac{(2G(a)-1-2G(a)^2)H}{(2G(a)-1)^2}\right] \\
+ & \frac{1}{(1-G(a))^2}\left[\frac{(2F(a)-1)(G(a)+1)-G(a)(2G(a)-1-2G(a)^2)}{(2G(a)-1)^2}\right]
\end{aligned}
$$

Distribution of the first ladder height

$$\mathbb{P}(X_{\tau_a^+} \leq t) = \sum_{k=1}^{\infty} P(X_k \leq t, \tau_a^+ = k)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_1 \leq a, \ldots, X_{k-1} \leq a, a < X_k \leq t)$$

$$= \sum_{k=1}^{\infty} \int_{-\infty}^{a} \cdots \int_{-\infty}^{a} \int_{a}^{t} (\delta_{x_{k-1}} \triangle_\alpha \nu)(dx_k)(\delta_{x_{k-2} \triangle_\alpha \nu})(dx_{k-1}) \ldots \nu(dx_1).$$

Distribution of the first ladder height

$$
\begin{aligned}
& \tilde{\Phi}_k(t) \\
= \; & \int_{-\infty}^{a} \cdots \int_{-\infty}^{a} \left(\int_a^t (\delta_{x_{k-1}} \vartriangle_\alpha \nu)(dx_k) \right) (\delta_{x_{k-2} \vartriangle_\alpha \nu})(dx_{k-1}) \ldots \nu(dx_1) \\
= \; & \frac{1}{2} \int_{-\infty}^{a} \cdots \int_{-\infty}^{a} \Bigg(\left[H(t) \Psi\left(\frac{x_{k-1}}{t} \right) + G(t) \right] \mathbb{1}(|x_{k-1}| < t) \\
- \; & \left[H(a) \Psi\left(\frac{x_{k-1}}{a} \right) + G(a) \right] \mathbb{1}(|x_{k-1}| < a) \Bigg) (\delta_{x_{k-2} \vartriangle_\alpha \nu})(dx_{k-1}) \ldots \nu(dx_1)
\end{aligned}
$$

Distribution of the first ladder height

$$\begin{aligned}
I(k) &:= \int_{-\infty}^{a} \dots \int_{-\infty}^{a} \Psi\left(\frac{x_{k-1}}{t}\right)(\delta_{x_{k-2}} \triangle_\alpha \nu)(dx_{k-1}) \dots \nu(dx_1),\\
II(k) &:= \int_{-\infty}^{a} \dots \int_{-\infty}^{a} \mathbb{1}(|x_{k-1}| < t)(\delta_{x_{k-2}} \triangle_\alpha \nu)(dx_{k-1}) \dots \nu(dx_1),\\
III(k) &:= \int_{-\infty}^{a} \dots \int_{-\infty}^{a} \Psi\left(\frac{x_{k-1}}{a}\right)(\delta_{x_{k-2}} \triangle_\alpha \nu)(dx_{k-1}) \dots \nu(dx_1),\\
IV(k) &:= \int_{-\infty}^{a} \dots \int_{-\infty}^{a} \mathbb{1}(|x_{k-1}| < a)(\delta_{x_{k-2}} \triangle_\alpha \nu)(dx_{k-1}) \dots \nu(dx_1).
\end{aligned}$$

Recurrence equations

$$\begin{cases} I_1 & = \int_{-\infty}^{a} \Psi\left(\frac{x_{k-1}}{t}\right) (\delta_{x_{k-2}} \triangle_\alpha \nu)(dx_{k-1}), \\ I_{j+1} & = \int_{-\infty}^{a} I_j \; (\delta_{x_{k-j-2}} \triangle_\alpha \nu)(dx_{k-j-1}), \; 1 \le j \le k-2. \end{cases}$$

$$\begin{aligned} I_j \quad = & \quad \frac{1}{2} M(a) \Psi\left(\frac{x_{k-j-1}}{a}\right) A_j^l + \frac{1}{2} G(a) H(a) \Psi\left(\frac{a}{t}\right) \Psi\left(\frac{x_{k-j-1}}{a}\right) B_j^l \\ + & \quad \frac{1}{2} G(a) \Psi\left(\frac{a}{t}\right) \mathbb{1}(|x_{k-j-1}| < a) C_j^l + \Psi\left(\frac{x_{k-j-1}}{t}\right) D_j^l \end{aligned}$$

Recurrence equations

$$\begin{cases} A_{j+1}^{l} & = A_j^l G(a) + D_j^l, \\ B_{j+1}^{l} & = G(a) B_j^l + C_j^l, \\ C_{j+1}^{l} & = G(a) C_j^l + D_j^l, \\ D_{j+1}^{l} & = \left(\frac{G(t)}{2}\right) D_j^l. \end{cases}$$

Recurrence equations

$$\begin{cases} J_1 & = \int_{-\infty}^{a} \mathbb{1}(|x_{k-1}| < t)(\delta_{x_{k-2}} \triangle_\alpha \nu)(dx_{k-1}), \\ J_{j+1} & = \int_{-\infty}^{a} J_j \, (\delta_{x_{k-j-2}} \triangle_\alpha \nu)(dx_{k-j-1}), \; 1 \le j \le k-2. \end{cases}$$

$$\begin{aligned} J_j &= \frac{1}{2} H(t) \Psi\left(\frac{x_{k-j-1}}{t}\right) A_j^J + \frac{1}{2} G(t) \mathbb{1}(|x_{k-j-1}| < t) B_j^J \\ &+ \Psi\left(\frac{x_{k-j-1}}{a}\right) C_j^J + \frac{1}{2} G(a) \mathbb{1}(|x_{k-j-1}| < a) D_j^J. \end{aligned}$$

Recurrence equations

$$\begin{cases} A^J_{j+1} & = \frac{G(t)}{2}\left(A^J_j + B^J_j\right) \\ B^J_{j+1} & = \frac{G(t)}{2} B^J_j \\ C^J_{j+1} & = \frac{1}{2}H(t)\frac{1}{2}M(a,t)A^J_j + \frac{1}{2}H(a)\frac{1}{2}G(t)B^J_j + C^J_j G(a) + \frac{1}{2}G(a)H(a)D^J_j \\ D^J_{j+1} & = \frac{1}{2}H(t)\Psi\left(\frac{a}{t}\right)A^J_j + \frac{G(t)}{2}B^J_j + G(a)D^J_j \end{cases}$$

Recurrence equations

$$\begin{cases} K_1 & = \int_{-\infty}^{a} \Psi\left(\frac{x_{k-1}}{a}\right)(\delta_{x_{k-2}} \triangle_\alpha \nu)(dx_{k-1}), \\ K_{j+1} & = \int_{-\infty}^{a} K_j \; (\delta_{x_{k-j-2}} \triangle_\alpha \nu)(dx_{k-j-1}), \; 1 \leq j \leq k-2. \end{cases}$$

$$K_j = A_j^K \Psi\left(\frac{x_{k-j-1}}{a}\right)$$

$$K_j = G(a)^j \Psi\left(\frac{x_{k-j-1}}{a}\right).$$

Recurrence equations

$$\begin{cases} L_1 & = \int_{-\infty}^{a} \mathbb{1}(|x_{k-1}| < a)(\delta_{x_{k-2}} \triangle_\alpha \nu)(dx_{k-1}), \\ L_{j+1} & = \int_{-\infty}^{a} L_j \, (\delta_{x_{k-j-2}} \triangle_\alpha \nu)(dx_{k-j-1}), \; 1 \le j \le k-2. \end{cases}$$

$$L_j = A_j^L \Psi \left(\frac{x_{k-j-1}}{a} \right) H(a) + \mathbb{1}(|x_{k-j-1}| < a) G(a) B_j^L.$$

$$\begin{cases} A_{j+1}^L & = G(a)(A_j^L + B_j^L) \\ B_{j+1}^L & = G(a) B_j^L \end{cases}$$

Bibliography

1. **A. Lachal**, *A note on Spitzer identity for random walk*, *Statistics & Probability Letters*, **78**(2), 97-–108, 2008.
2. **T. Nakajima**, *Joint distribution of first hitting time and first hitting place for random walk*, *Kodai Math. J.* **21**, 192–200, 1998.
3. **B.H. Jasiulis-Gołdyn, J.K. Misiewicz**, *Kendall random walk, Williamson transform and the corresponding Wiener-Hopf factorization*, in press: *Lith. Math. J.*, 2016, arXiv: http://arxiv.org/pdf/1501.05873.pdf.
4. **B.H. Jasiulis-Gołdyn, M. Staniak**, *Spitzer identity for Kendall random walk*, 2017, in preparation.
5. **M. Borowiecka-Olszewska, B.H. Jasiulis-Gołdyn, J.K. Misiewicz, J. Rosiński**, *Lévy processes and stochastic integrals in the sense of generalized convolutions*, *Bernoulli*, **21**(4), 2513–2551, 2015.