Spitzer identity for Kendall random walks

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Joint work with Barbara Jasiulis-Gołdyn

Kendall Generalized Convolution

Definition 1 (Kendall convolution)

1. $\delta_a \vartriangle_\alpha \delta_b = T_{max\{|a|,|b|\}} \left((1 - z^\alpha)\widetilde{\delta}_1 + z^\alpha \widetilde{\pi}_{2\alpha} \right),$
   where $z = \frac{min\{|a|,|b|\}}{max\{|a|,|b|\}}$, $\alpha \in (0, 1]$, $\widetilde{\pi}_{2\alpha}(dx) = \frac{\alpha}{|x|^{2\alpha+1}} \mathbf{1}_{(1,\infty)}(|x|)dx.$
2. $\nu_1 \vartriangle_\alpha \nu_2(A) = \int_{\mathbb{R}^2} (\delta_x \vartriangle_\alpha \delta_y)(A)\nu_1(dx)\nu_2(dy)$

Kendall random walk

Definition 2 (M. Borowiecka-Olszewska, B.H. Jasiulis-Gołdyn, J.K. Misiewicz, J. Rosiński)

Kendall random walk is a Markov process $\{X_n : n \in \mathbb{N}_0\}$, $X_0 = 0$ *with transition probabilities given by*

$$P_n(x, A) = P(X_{n+k} \in A | X_k = x) = \delta_x \vartriangle_\alpha \nu^{\vartriangle_\alpha n}(A)$$

where measure $\nu \in \mathcal{P}_s$ *is called the unit step distribution.*

Kendall random walk

Definition 3

Stochastic process $\{X_n \colon n \in \mathbb{N}_0\}$ *is a discrete Kendall random walk with parameter* $\alpha > 0$ *and unit step distribution* $\nu \in \mathcal{P}_+$*, if there exist:*

1. (T_k) *i.i.d. r.vs with the distribution* ν*,*
2. (U_k) *i.i.d. r.vs with the uniform distribution on* $[0, 1]$*,*
3. (θ_k) *i.i.d. r.vs from symmetrical Pareto distribution given by the PDF* $\widetilde{\pi}_{2\alpha}(dy) = \alpha|y|^{-2\alpha-1}\mathbf{1}_{[1,\infty)}(|y|)\, dy$*,*

such that (T_k)*,* (U_k) *and* (θ_k) *are independent,* $X_0 = 0, \quad X_1 = T_1$*,*

$$X_{n+1} = M_{n+1} r_{n+1} \left[\mathbf{1}(U_n > \varrho_{n+1}) + \theta_{n+1}\mathbf{1}(U_n < \varrho_{n+1})\right],$$

where θ_{n+1} *and* M_{n+1} *are independent and*
$r_{n+1} = \{sgn(u) : \max\{|X_n|, |T_{n+1}|\} = |u|\}$*,*
$M_{n+1} = \max\{|X_n|, |T_{n+1}|\}, \quad m_{n+1} = \min\{|X_n|, |T_{n+1}|\}, \quad \varrho_{n+1} = \frac{m_{n+1}^{\alpha}}{M_{n+1}^{\alpha}}$.

Kendall random walk

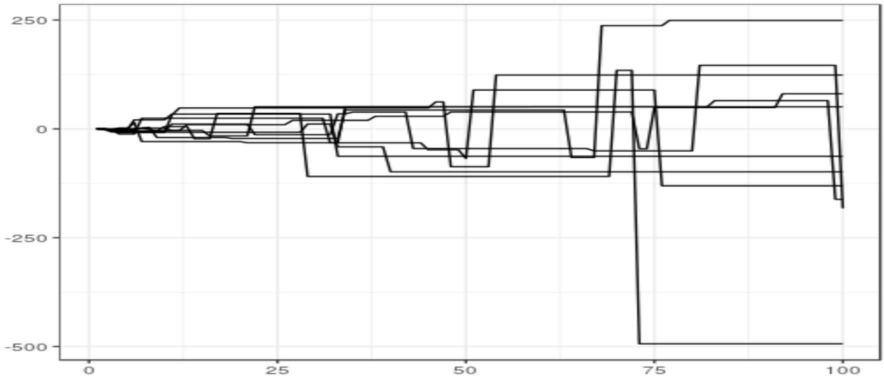

Figure: Example trajectories of Kendall random walk with standard normal unit step distribution.

Williamson transform

Definition 4

Williamson transform is an operation $\nu \to \widehat{\nu}$ given by

$$\widehat{\nu}(t) = \int_{\mathbb{R}} (1 - |xt|^{\alpha})_{+} \, \nu(dx), \quad \nu \in \mathcal{P}_s,$$

where $a_+ = \max(a, 0)$

Theorem 5

Let $\nu_1, \nu_2 \in \mathcal{P}_s$ be probability measures with Williamson transforms $\widehat{\nu_1}, \widehat{\nu_2}$, respectively. Then

$$\int_{\mathbb{R}} (1 - |xt|^{\alpha})_{+} \big(\nu_1 \vartriangle_{\alpha} \nu_2\big)(dx) = \widehat{\nu_1}(t)\widehat{\nu_2}(t).$$

Inverse uniform transform

Theorem 6 (Jasiulis-Gołdyn, Misiewicz, 2017)

Correspondence between measure $\nu \in \mathcal{P}_s$ and its Williamson transform is 1-1. Moreover, if F is the CDF of ν, $\nu(\{0\}) = 0$ and $G(t) = \hat{\nu}(\frac{1}{t})$, we have

$$F(t) = \begin{cases} \frac{1}{2\alpha}\left[\alpha(G(t)+1) + tG'(t)\right] & \text{if} \quad t > 0; \\ 1 - F(-t) & \text{if} \quad t < 0. \end{cases}$$

at continuity points of F.

Notation

- $\tau_a^+ = \inf\{i \geq 0 : X_i > a\}$
- $\tau_a^- = \inf\{i \geq 0 : X_i < a\}$
- $\min \emptyset = \infty$
- G(x): Williamson transform of the unit step distribution taken at $1/t$
- $H(x) = 2F(x) - 1 - G(x)$
- $\Psi(t) = (1 - |t|^\alpha)_+$

Useful lemmas

Lemma 7 (Jasiulis-Gołdyn, Misiewicz)

$$\begin{aligned} \delta_x \vartriangle_\alpha \delta_y(0,t) &:= \frac{1}{2}\left(1 - \left|\frac{xy}{t^2}\right|^\alpha\right) \mathbf{1}_{\{|x|<t,|y|<t\}} \\ &= \frac{1}{2}\left[\Psi\left(\frac{x}{t}\right) + \Psi\left(\frac{y}{t}\right) - \Psi\left(\frac{x}{t}\right)\Psi\left(\frac{y}{t}\right)\right] \mathbf{1}_{\{|x|<t,|y|<t\}} \end{aligned}$$

Lemma 8 (Jasiulis-Gołdyn, Misiewicz)

$$\begin{aligned} &\delta_x \vartriangle_\alpha \nu\,(0,t) = P_1(x,[0,t)) \\ = &\left[\Psi\left(\frac{x}{t}\right)\left(F(t) - \frac{1}{2}\right) + \frac{1}{2}G(t) - \frac{1}{2}\Psi\left(\frac{x}{t}\right)G(t)\right] \mathbf{1}_{\{|x|<t\}} \\ = &\frac{1}{2}\left[\Psi\left(\frac{x}{t}\right)H(t) + G(t)\right] \mathbf{1}_{\{|x|<t\}} \end{aligned}$$

Special case: a = 0

Theorem 9 (Jasiulis-Gołdyn, Misiewicz)

R.v. τ_0^+ *(and by the symmetry of K.R.W.* τ_0^- *) has a geometric distribution* $P(\tau_0^+ = k) = \frac{1}{2^k}$, $k = 1, 2, \cdots$.
Moment generating function of τ_0^+ *is given by*

$$\mathbf{E}s^{\tau_0^+} = \frac{\frac{s}{2}}{1 - \frac{s}{2}}, \quad 0 \leqslant s < 2.$$

Special case: a = 0

- $\Phi_n(t) := \mathbf{P}\{X_1 \leqslant 0, \ldots X_{n-1} \leqslant 0, 0 < X_n < t\} = \frac{1}{2^n}\, G(t)^{n-1}\Big[2n\Big(F(t) - \frac{1}{2}\Big) - (n-1)G(t)\Big]$

Theorem 10 (Jasiulis-Gołdyn, Misiewicz)

$$\begin{aligned}\mathbf{P}\left\{X_{\tau_0^+} < t\right\} &= \sum_{k=1}^{\infty} \mathbf{P}\left\{X_{\tau_0^+} < t, \tau_0^+ = k\right\} \\ &= \sum_{k=1}^{\infty} \Phi_k(t) = \frac{1}{(2-G(t))^2}\left[4F(t) - 2 - G(t)^2\right]\end{aligned}$$

Special case: a = 0

- Let $\{X_n \colon n \in \mathbb{N}_0\}$ be the K.R.W.
- Let $N_{s/2}$ be a geometric r.v. independent of (X_n).
- We define compound geometric Kendall r.v.:
  $Z_{s/2} = \sum_{k=1}^{\infty} X_k \mathbf{1}_{\{N_{s/2}=k\}}$
- Notice that $\mathbf{E}\Psi\left(X_k/t\right) = G(t)^k$ i $\mathbf{E}\Psi(Z_{s/2}/t) = \frac{G(t)\left(1-\frac{s}{2}\right)}{1-\frac{s}{2}G(t)}$.

Theorem 11 (Jasiulis-Gołdyn, Misiewicz)

Let $\{X_n : n \in \mathbb{N}_0\}$ be the K.R.W with unit step distribution $X_1 \sim F$ such that $F(0) = \frac{1}{2}$. Then

$$\mathbf{E}s^{\tau_0^+}\Psi\left(uX_{\tau_0^+}\right) = \frac{\frac{s}{2}}{1-\frac{s}{2}} \cdot \frac{(1-\frac{s}{2})G(1/u)}{1-\frac{s}{2}G(1/u)} = \mathbf{E}s^{\tau_0^+}\,\mathbf{E}\Psi\left(uZ_{s/2}\right).$$

Distribution of the first ladder moment

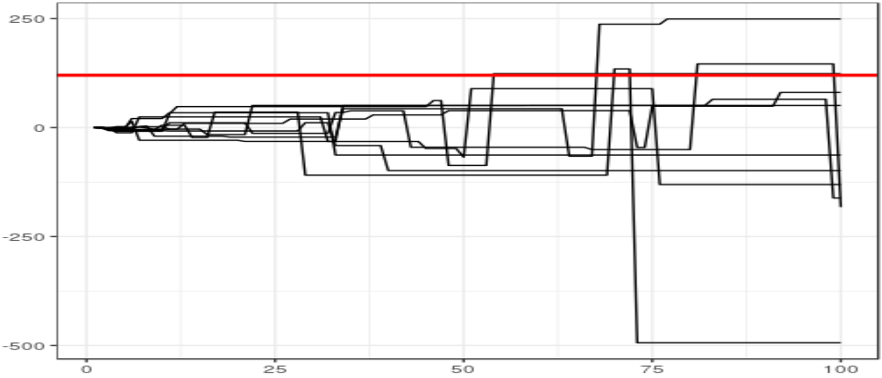

Figure: Illustration of the problem

Distribution of the first ladder moment

Theorem 12 (Jasiulis-Gołdyn, Staniak)

Distribution of τ_a^+ is given by

$$\mathbb{P}(\tau_a^+ = n) = A(a)\mathbb{P}(\tau_0^+ = n) + B(a)n(1 - G(a))^2 G(a)^{n-1} + C(a)G(a)^{n-1}(1 - G(a))$$

where

$$\begin{cases} A(a) & = 1 + \frac{H(a)}{(2G(a)-1)^2} - \frac{G(a)}{(2G(a)-1)} \\ B(a) & = \frac{H(a)}{(2G(a)-1)(1-G(a))} \\ C(a) & = \frac{G(a)}{2G(a)-1} - \frac{H(a)}{(2G(a)-1)^2}\frac{G(a)}{(1-G(a))}. \end{cases}$$

It is a convex combination of two geometric distribution (par: $\frac{1}{2}$ and $G(a)$) and a shifted negative binomial distribution (par: 2 and $G(a)$).

Proof

- $\mathbb{P}(\tau_a^+ = n) = \mathbb{P}(X_0 \leq a, X_1 \leq a, \dots, X_{n-1} \leq a, X_n > a)$
  $= \int_{-\infty}^{a} \dots \int_{-\infty}^{a} \int_a^{\infty} P_1(x_{n-1}, dx_n) P_1(x_{n-2}, dx_{n-1}) \dots P_1(0, dx_1)$
- Define
  $I_1 = \int_a^{\infty} P_1(x_{n-1}, dx_n) = \frac{1}{2} - \frac{1}{2}\left[\Psi(\frac{x_{n-1}}{a}) H(a) + G(a)\right] \mathbf{1}(|x_{n-1}| < a)$

Proof: recurrence equations

- Assume that I_j is of the form
  $I_j = A_j + \mathbf{1}(|x_{n-j}| < a)\left[\Psi\left(\frac{x_{n-j}}{a}\right) H(a)B_j + C_j G(a)\right]$
- $I_{j+1} = \int_{-\infty}^{a} I_j \, (\delta_{x_{n-j-1}} \triangle_\alpha \nu)(d_{x_{n-j}})$
- $\begin{cases} A_{j+1} = & \frac{1}{2}A_j, \\ B_{j+1} = & \frac{1}{2}A_j + G(a)(B_j + C_j), \\ C_{j+1} = & \frac{1}{2}A_j + C_j G(a) \end{cases}$
- $A_1 = \frac{1}{2}$, $B_1 = C_1 = -\frac{1}{2}$

Solutions

- $$A_j = \left(\frac{1}{2}\right)^j$$

- $$B_j = G(a)^{j-1} B_1 + \sum_{m=2}^{j} G(a)^{j-m} \left(\frac{1}{2}\right)^m + \sum_{m=1}^{j-1} G(a)^{j-m} C_m$$
$$= \sum_{m=1}^{j} G(a)^{j-m} C_m$$

- $$C_j = G(a)^{j-1} C_1 + \sum_{m=2}^{j} G(a)^{j-m} \left(\frac{1}{2}\right)^m = \sum_{m=1}^{j} G(a)^{j-m} \left(\frac{1}{2}\right)^m - G(a)^{j-1}$$

Closed-form solutions

- $A_j = (\frac{1}{2})^j$
- $B_j = G(a)^{j-1}\frac{(1-G(a)(2G(a)-1)-G(a)}{(2G(a)-1)^2}j + \left(\frac{1}{2}\right)^j \frac{1}{(2G(a)-1)^2}$
- $C_j = \frac{G(a)^{j-1}(1-G(a))-2^{-j}}{2G(a)-1} = G(a)^{j-1}\frac{1-G(a)}{2G(a)-1} + \left(\frac{1}{2}\right)^j \frac{-1}{2G(a)-1}$

$$\mathbb{P}(\tau_a^+ = n) = A(a)\mathbb{P}(\tau_0^+ = n) + B(a)n(1 - G(a))^2 G(a)^{n-1} + C(a)G(a)^{n-1}(1 - G(a))$$

Useful lemma

Lemma 13 (Jasiulis-Gołdyn, Staniak)

With the notation $M(a,t) = H(a)\Psi\left(\frac{a}{t}\right) + \left(1 - \Psi\left(\frac{a}{t}\right)\right) G(a)$ *the following equality holds*

$$
\begin{aligned}
& \int_{-\infty}^{a} \Psi\left(\frac{x}{t}\right) (\delta_y \vartriangle_\alpha \nu)(dx) \\
= \; & \frac{1}{2}\Psi\left(\frac{y}{a}\right) \left[H(a)\Psi\left(\frac{a}{t}\right) + \left(1 - \Psi\left(\frac{a}{t}\right)\right) G(a)\right] \\
+ \; & \frac{1}{2} G(a)\Psi\left(\frac{a}{t}\right) \mathbb{1}(|y| < a) + \frac{1}{2}\Psi\left(\frac{y}{t}\right) G(t) \\
= \; & \frac{1}{2}\Psi\left(\frac{y}{a}\right) M(a,t) + \frac{1}{2} G(a)\Psi\left(\frac{a}{t}\right) \mathbb{1}(|y| < a) + \frac{1}{2}\Psi\left(\frac{y}{t}\right) G(t).
\end{aligned}
$$

First ladder height

$$\mathbb{P}(X_{\tau_a^+} \leq t) = \sum_{k=1}^{\infty} P(X_k \leq t, \tau_a^+ = k) = \sum_{k=1}^{\infty} \Phi_n^a(t)$$

$$\Phi_n^a(t) = \frac{H_1(t)}{2} I(n,a,t) + \frac{G(t)}{2} II(n,a,t) - \frac{H_1(a)}{2} I(n,a,a) - \frac{G(a)}{2} II(n,a,a)$$

dla

$$\begin{aligned} I(n,a,t) &:= \int_{-\infty}^{a} \dots \int_{-\infty}^{a} \Psi\left(\frac{x_{n-1}}{t}\right)(\delta_{x_{n-2}} \triangle_\alpha \nu)(dx_{n-1}) \dots \nu(dx_1), \\ II(n,a,t) &:= \int_{-\infty}^{a} \dots \int_{-\infty}^{a} \mathbb{1}(|x_{n-1}| < t)(\delta_{x_{n-2}} \triangle_\alpha \nu)(dx_{n-1}) \dots \nu(dx_1) \end{aligned}$$

Recurrence equations

For $I(k, a, t)$ we define

$$\begin{cases} I_1 & := I_1(x_2) = \int_{-\infty}^{a} \Psi\left(\frac{x_{n-1}}{t}\right)(\delta_{x_{n-2}} \vartriangle_\alpha \nu)(dx_{n-1}), \\ I_{j+1} & := I_{j+1}(x_{n-j-2}) = \int_{-\infty}^{a} I_j \; (\delta_{x_{n-j-2}} \vartriangle_\alpha \nu)(dx_{n-j-1}), \; 1 \le j \le n-2. \end{cases}$$

$$\begin{aligned} I_j &= \frac{1}{2} M(a,t) \Psi\left(\frac{x_{k-j-1}}{a}\right) A_j^I + \frac{1}{2} G(a) H(a) \Psi\left(\frac{a}{t}\right) \Psi\left(\frac{x_{n-j-1}}{a}\right) B_j^I \\ &+ \frac{1}{2} G(a) \Psi\left(\frac{a}{t}\right) \mathbb{1}(|x_{n-j-1}| < a) C_j^I + \Psi\left(\frac{x_{n-j-1}}{t}\right) D_j^I \end{aligned}$$

Recurrence equations

$$\begin{cases} A^l_{j+1} & = G(a)A^l_j + D^l_j, \\ B^l_{j+1} & = G(a)B^l_j + C^l_j, \\ C^l_{j+1} & = G(a)C^l_j + D^l_j, \\ D^l_{j+1} & = \left(\frac{G(t)}{2}\right) D^l_j. \end{cases}$$

$$\begin{aligned} I(n,a,t) &= G(a)^{n-1}\left[(n-1)\frac{H(a)\Psi\left(\frac{a}{t}\right)}{2G(a)-G(t)} + \frac{G(a)+H(a)\Psi\left(\frac{a}{t}\right)}{2G(a)-G(t)}\right. \\ &- \left.\frac{2G(a)H(a)\Psi\left(\frac{a}{t}\right)}{(2G(a)-G(t))^2}\right] \\ &+ \left(\frac{G(t)}{2}\right)^{n-1}\left[1 - \frac{G(a)+H(a)\Psi\left(\frac{a}{t}\right)}{2G(a)-G(t)} + \frac{2G(a)H(a)\Psi\left(\frac{a}{t}\right)}{(2G(a)-G(t))^2}\right] \end{aligned}$$

Recurrence equations

For $II(n,a,t)$ we define

$$\begin{cases} J_1 & := J_1(x_{n-2}) = \int_{-\infty}^{a} \mathbb{1}(|x_{n-1}|<t)(\delta_{x_{n-2}} \vartriangle_\alpha \nu)(dx_{n-1}), \\ J_{j+1} & := J_{j+1}(x_{n-j-2}) = \int_{-\infty}^{a} J_j \, (\delta_{x_{n-j-2}} \vartriangle_\alpha \nu)(dx_{n-j-1}), \ 1 \le j \le n-2. \end{cases}$$

$$\begin{aligned} J_j &= \frac{1}{2}H(t)\Psi\left(\frac{x_{k-j-1}}{t}\right)A_j^J + \frac{1}{2}G(t)\mathbb{1}(|x_{k-j-1}|<t)B_j^J \\ &+ \Psi\left(\frac{x_{k-j-1}}{a}\right)C_j^J + \frac{1}{2}G(a)\mathbb{1}(|x_{k-j-1}|<a)D_j^J. \end{aligned}$$

Recurrence equations

$$
\begin{cases}
A_{j+1}^J &= \frac{G(t)}{2}\left(A_j^J + B_j^J\right), \\
B_{j+1}^J &= \frac{G(t)}{2} B_j^J, \\
C_{j+1}^J &= \frac{1}{2}H(t)\frac{1}{2}M(a,t)A_j^J + \frac{1}{2}H(a)\frac{1}{2}G(t)B_j^J + C_j^J G(a) + \frac{1}{2}G(a)H(a)D_j^J, \\
D_{j+1}^J &= \frac{1}{2}H(t)\Psi\left(\frac{a}{t}\right)A_j^J + \frac{G(t)}{2}B_j^J + G(a)D_j^J.
\end{cases}
$$

Recurrence equations

$$
\begin{aligned}
II(n,a,t) &= \frac{H(t)}{2}(n-1)\left(\frac{G(t)}{2}\right)^{n-2} + \left(\frac{G(t)}{2}\right)^{n-1} \\
+ \quad & G(a)^{n-2}\left[\frac{H(a)}{2} + (n-2)A(a,t) + \frac{2B(a,t)}{2G(a)-G(t)} + \frac{4C(a,t)G(a)}{(2G(a)-G(t))^2}\right] \\
- \quad & \left(\frac{G(t)}{2}\right)^{n-2}\left[\frac{2B(a,t)}{2G(a)-G(t)} + 2C(a,t)\left(\frac{G(t)}{(2G(a)-G(t))^2} + \frac{n-1}{2G(a)-G(t)}\right)\right. \\
+ \quad & \frac{G(a)}{2}\left[G(a)^{n-1}\left[\frac{2}{2G(a)-G(t)} + \frac{2H(t)\Psi\left(\frac{a}{t}\right)}{(2G(a)-G(t))^2}\right]\right. \\
+ \quad & \left(\frac{G(t)}{2}\right)^{n-2}\left[\frac{G(t)(G(t)-H(t)\Psi\left(\frac{a}{t}\right)-2G(a))}{(2G(a)-G(t))^2} - (n-1)\frac{H(t)\Psi\left(\frac{a}{t}\right)}{2G(a)-G(t)}\right]\Bigg]
\end{aligned}
$$

Distribution of the maximum of K.R.W.

Lemma 14

Let $\{X_n : n \in \mathbb{N}_0\}$ *be the K.R.W. Then the distribution of* $M_n = \max_{0 \le i \le n} X_i$ *is given by*

$$\begin{aligned} &\mathbb{P}(M_n \le t) \\ &= A(t)\mathbb{P}(\tau_0^+ = n) + B(t)\frac{G(t)}{1-G(t)}(1-G(t))^2 nG(t)^{n-1} \\ &+ (B(t)+C(t))\frac{G(t)}{1-G(t)}G(t)^{n-1}(1-G(t)) \end{aligned}$$

for A*,* B *i* C *defined earlier and* $t > 0$.

Simulations: R tool

https://github.com/mstaniak/kendallRandomPackage

install.packages("KendallRandomWalks")

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