Wiener-Hopf factorization for extremal Markovian sequences connected with the Kendall convolution

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Generalized convolution of the Kendall type

Definition 1 (Kendall convolution)

1. $\delta_a \vartriangle_\alpha \delta_b = T_{max\{|a|,|b|\}} \left((1 - z^\alpha) \widetilde{\delta}_1 + z^\alpha \widetilde{\pi}_{2\alpha} \right),$ where $z = \frac{min\{|a|,|b|\}}{max\{|a|,|b|\}}$, $\alpha \in (0, 1]$, $\widetilde{\pi}_{2\alpha}(dx) = \frac{\alpha}{|x|^{2\alpha+1}} \mathbf{1}_{(1,\infty)}(|x|) dx$.
2. $\nu_1 \vartriangle_\alpha \nu_2(A) = \int_{\mathbb{R}^2} (\delta_x \vartriangle_\alpha \delta_y)(A) \nu_1(dx) \nu_2(dy)$

Kendall random walk

Definition 2 (Rosiński et al.)

The Kendall random walk is a Markov process $\{X_n : n \in \mathbb{N}_0\}$ *with* $X_0 = 0$ *and the transition probabilities*

$$P_n(x, A) = P(X_{n+k} \in A | X_k = x) = \delta_x \vartriangle_\alpha \nu^{\vartriangle_\alpha n}$$

where measure $\nu \in \mathcal{P}_s$ *is called the step distribution.*

Kendall random walk representation

Theorem 1 (Jasiulis-Gołdyn)

The Markov process $\{X_n : n \in \mathbb{N}_0\}$ *with step distribution* $Y_i \sim \nu$ *has the following properties:* $X_0 = 0$, $X_1 = Y_1$,

$$X_{n+1} = v(|X_n|, |Y_{n+1}|)u(X_n, Y_{n+1})\theta_n^{Q_n} a.e.,$$

where θ_n *is independent of* $v(|X_n|, |Y_{n+1}|)u(X_n, Y_{n+1})$, $v(x, y) = x \vee y$, $u(x, y) = \frac{x \wedge y}{x \vee y}$,

$$u(x, y) = \begin{cases} sgn(x), & |x| \geq |y|, \\ sgn(y), & |x| < |y|, \end{cases}$$

and

$$\mathbb{P}(Q_n = k | X_n, Y_{n+1}) = \begin{cases} \left(z(|X_n|, |Y_{n+1}|)\right)^{\alpha}, & k = 1, \\ 1 - \left(z(|X_n|, |Y_{n+1}|)\right)^{\alpha}, & k = 0. \end{cases}$$

Williamson transform

Definition 3

Williamson transform Operation $\nu \to \widehat{\nu}$ given by

$$\widehat{\nu}(t) = \int_{\mathbb{R}} \left(1 - |xt|^{\alpha}\right)_{+} \nu(dx), \quad \nu \in \mathcal{P}_s,$$

where $a_+ = a$ if $a \geqslant 0$ and $a_+ = 0$ otherwise.

Theorem 2

Let $\nu_1, \nu_2 \in \mathcal{P}_s$ be probability measures with Williamson transforms $\widehat{\nu_1}, \widehat{\nu_2}$. Then

$$\int_{\mathbb{R}} (1 - |xt|^{\alpha})_{+} \big(\nu_1 \vartriangle_{\alpha} \nu_2\big)(dx) = \widehat{\nu_1}(t)\widehat{\nu_2}(t).$$

Inverse Williamson transform

Theorem 3 (Jasiulis-Gołdyn, Misiewicz, 2017)

The correspondence between measure $\nu \in \mathcal{P}_s$ and its Williamson transform is $1-1$. Moreover, denoting by F the cumulative distribution function of ν, $\nu(\{0\}) = 0$ and $G(t) = \hat{\nu}(\frac{1}{t})$, we have

$$F(t) = \begin{cases} \frac{1}{2\alpha}\left[\alpha(G(t)+1) + tG'(t)\right] & \text{if} \quad t > 0; \\ 1 - F(-t) & \text{if} \quad t < 0. \end{cases}$$

except for the countable many $t \in \mathbb{R}$.

Definitions

- $\tau_a^+ = \inf\{i \geq 0 : X_i > a\}$
- $\tau_a^- = \inf\{i \geq 0 : X_i < a\}$
- Convention: $\min \emptyset = \infty$
- Notation: $H(x) = 2F(x) - 1 - G(x)$
- Notation: $\Psi(t) = (1 - |t|^\alpha)_+$

Useful lemmas

Lemma 4 (Jasiulis-Gołdyn, Misiewicz)

$$\begin{aligned}
\delta_x \vartriangle_\alpha \delta_y(0,t) &:= \frac{1}{2}\left(1 - \left|\frac{xy}{t^2}\right|^\alpha\right) \mathbf{1}_{\{|x|<t,|y|<t\}} \\
&= \frac{1}{2}\left[\Psi\left(\frac{x}{t}\right) + \Psi\left(\frac{y}{t}\right) - \Psi\left(\frac{x}{t}\right)\Psi\left(\frac{y}{t}\right)\right] \mathbf{1}_{\{|x|<t,|y|<t\}}
\end{aligned}$$

Lemma 5 (Jasiulis-Gołdyn, Misiewicz)

$$\begin{aligned}
&\delta_x \vartriangle_\alpha \nu(0,t) = P_1(x,[0,t)) \\
&= \left[\Psi\left(\frac{x}{t}\right)\left(F(t) - \frac{1}{2}\right) + \frac{1}{2}G(t) - \frac{1}{2}\Psi\left(\frac{x}{t}\right)G(t)\right] \mathbf{1}_{\{|x|<t\}} \\
&= \frac{1}{2}\left[\Psi\left(\frac{x}{t}\right)H(t) + G(t)\right] \mathbf{1}_{\{|x|<t\}}
\end{aligned}$$

Useful lemmas

Lemma 6 (Jasiulis-Gołdyn, Staniak)

$$F_n(t) = \begin{cases} \frac{1}{2\alpha}\left[\alpha(G(t)^n + 1) + tnG(t)^{n-1}G'(t)\right] & , t > 0, \\ 1 - F_n(-t) & , t < 0. \end{cases}$$

Lemma 7 (Jasiulis-Gołdyn, Staniak)

$$\begin{aligned} & \delta_x \vartriangle_\alpha \nu^n(0,t) = P_n(x,[0,t)) \\ = & \left[\Psi\left(\frac{x}{t}\right)\left(F_n(t) - \frac{1}{2}\right) + \frac{1}{2}G(t)^n - \frac{1}{2}\Psi\left(\frac{x}{t}\right)G(t)^n\right]\mathbf{1}_{\{|x|<t\}} \end{aligned}$$

Results for a = 0

Theorem 8 ((Jasiulis-Gołdyn, Misiewicz))

R. v. τ_0^+ *(and, by symmetry of the Kendall random walk, also variable* τ_0^-*) has geometric distribution* $P(\tau_0^+ = k) = \frac{1}{2^k}$, $k = 1, 2, \cdots$.
Generating function for τ_0^+ *is given by*

$$\mathbf{E}s^{\tau_0^+} = \frac{\frac{s}{2}}{1 - \frac{s}{2}}, \quad 0 \leqslant s < 2.$$

More results

- $\Phi_n(t) := \mathbf{P}\{X_1 \leqslant 0, \ldots X_{n-1} \leqslant 0, 0 < X_n < t\} = \frac{1}{2^n}\, G(t)^{n-1}\Big[2n\Big(F(t) - \frac{1}{2}\Big) - (n-1)G(t)\Big]$

Theorem 9 (Jasiulis-Gołdyn, Misiewicz)

$$\mathbf{P}\left\{X_{\tau_0^+} < t\right\} = \sum_{k=1}^{\infty} \mathbf{P}\left\{X_{\tau_0^+} < t, \tau_0^+ = k\right\} = \sum_{k=1}^{\infty} \Phi_k(t) = \frac{1}{(2-G(t))^2}\left[4F(t) - 2 - G(t)^2\right]$$

Even more results

- Let $\{X_n \colon n \in \mathbb{N}_0\}$ be the Kendall random walk
- Let $N_{s/2}$ be geometric r.v. independent of (X_n).
- Define geometric-Kendall r.v.: $Z_{s/2} = \sum_{k=1}^{\infty} X_k \mathbf{1}_{\{N_{s/2}=k\}}$
- Notice that $\mathbf{E}\Psi\left(X_k/t\right) = G(t)^k$ and $\mathbf{E}\Psi(Z_{s/2}/t) = \frac{G(t)\left(1-\frac{s}{2}\right)}{1-\frac{s}{2}G(t)}$.

Theorem 10 (Jasiulis-Gołdyn, Misiewicz)

Let $\{X_n : n \in \mathbb{N}_0\}$ be a random walk under the Kendall convolution with the unit step $X_1 \sim F$ such that $F(0) = \frac{1}{2}$. Then

$$\mathbf{E}s^{\tau_0^+}\Psi\left(uX_{\tau_0^+}\right) = \frac{\frac{s}{2}}{1-\frac{s}{2}} \cdot \frac{(1-\frac{s}{2})G(1/u)}{1-\frac{s}{2}G(1/u)} = \mathbf{E}s^{\tau_0^+}\,\mathbf{E}\Psi\left(uZ_{s/2}\right).$$

Distribution of the first ladder moment

Theorem 11 (Jasiulis-Gołdyn, Staniak)

The distribution of τ_a^+ is given by

$$
\begin{aligned}
&\mathbb{P}(\tau_a^+ = k) \\
&= \left(\frac{1}{2}\right)^{k-1} \left[\frac{F(a)(2G(a)-1)^2 + G(a)[2F(a)-2-G(a)^2]}{(2G(a)-1)^2}\right] \\
&+ G(a)^{k-1} \frac{(k-1)(2G(a)-1-2G(a)^2}{(2G(a)-1)^2} \\
&+ G(a)^{k-1} \frac{(1-G(a))(2F(a)-1)(2G(a)-1)^2}{(2G(a)-1)^2}
\end{aligned}
$$

Proof

- $\mathbb{P}(\tau_a^+ = k) = \mathbb{P}(X_0 \leq a, X_1 \leq a, \ldots, X_{k-1} \leq a, X_k > a)$
  $= \int_{-\infty}^{a} \ldots \int_{-\infty}^{a} \int_{a}^{-\infty} P_1(x_{k-1}, dx_k) P_1(x_{k-2}, dx_{k-1}) \ldots P_1(0, dx_1)$
- Define
  $I_1 = \int_a^{\infty} P_1(x_{k-1}, dx_k) = \frac{1}{2} - \frac{1}{2} \left[\Psi(\frac{x_{k-1}}{a}) H(a) + G(a) \right] \mathbf{1}(|x_{k-1}| < a)$

Proof: recurrence equations

- Assume that I_j is of the form
  $I_j = A_j + \mathbf{1}(|x_{k-j}| < a) \left[\Psi\left(\frac{x_{k-j}}{a}\right) H(a) B_j + C_j G(a)\right]$
- $I_{j+1} = \int_{-\infty}^{a} I_j \, (\delta_{x_{k-j-1}} \vartriangle_\alpha \nu)(d_{x_{k-j}})$
- $\begin{cases} A_{j+1} = & \frac{1}{2} A_j, \\ B_{j+1} = & \frac{1}{2} A_j + G(a)(B_j + C_j), \\ C_{j+1} = & \frac{1}{2} A_j + C_j G(a) \end{cases}$
- $A_1 = \frac{1}{2}$, $B_1 = C_1 = -\frac{1}{2}$

Solutions

- $$A_j = \left(\frac{1}{2}\right)^j$$

- $$B_j = G(a)^{j-1}B_1 + \sum_{m=2}^{j} G(a)^{j-m}\left(\frac{1}{2}\right)^m + \sum_{m=1}^{j-1} G(a)^{j-m}C_m$$
$$= \sum_{m=1}^{j} G(a)^{j-m}C_m$$

- $$C_j = G(a)^{j-1}C_1 + \sum_{m=2}^{j} G(a)^{j-m}\left(\frac{1}{2}\right)^m = \sum_{m=1}^{j} G(a)^{j-m}\left(\frac{1}{2}\right)^m - G(a)^{j-1}$$

Closed form solutions

- $A_j = (\frac{1}{2})^j$
- $B_j = G(a)^{j-1} \frac{(1-G(a)(2G(a)-1)-G(a)}{(2G(a)-1)^2} j + \left(\frac{1}{2}\right)^j \frac{1}{(2G(a)-1)^2}$
- $C_j = \frac{G(a)^{j-1}(1-G(a))-2^{-j}}{2G(a)-1} = G(a)^{j-1} \frac{1-G(a)}{2G(a)-1} + \left(\frac{1}{2}\right)^j \frac{-1}{2G(a)-1}$

Corollary

[Jasiulis-Gołdyn, Staniak]

Lemma 1

Generating function for τ_a^+ is given by

$$
\begin{aligned}
H(s) &= \frac{\frac{s}{2}}{1-\frac{s}{2}}\left[2F(a) + \frac{4G(a)^2(2F(a)-1)}{(2G(a)-1)^2}\right] \\
&+ \frac{s}{(1-sG(a))^2}\left[\frac{(2G(a)-1-2G(a)^2)H}{(2G(a)-1)^2}\right] \\
&+ \frac{s}{1-sG(a)}\left[\frac{(2F(a)-1)(G(a)+1)-G(a)(2G(a)-1-2G(a)^2)}{(2G(a)-1)^2}\right]
\end{aligned}
$$

Mean of the first ladder moment

Lemma 2

Mean of the random variable τ_a^+ is given by

$$
\begin{aligned}
\mathbb{E}\tau_a^+ &= \frac{1}{4}\left[2F(a) + \frac{4G(a)^2(2F(a)-1)}{(2G(a)-1)^2}\right] \\
&+ \frac{1-G(a)^2}{(1-G(a))^4}\left[\frac{(2G(a)-1-2G(a)^2)H}{(2G(a)-1)^2}\right] \\
&+ \frac{1}{(1-G(a))^2}\left[\frac{(2F(a)-1)(G(a)+1)-G(a)(2G(a)-1-2G(a)^2)}{(2G(a)-1)^2}\right.
\end{aligned}
$$

In progress

- $\mathbb{P}(X_{\tau_a^+} < t, \tau_a^+ = k)$: complicated formula
- $\mathbb{E}s^{\tau_a^+}\Psi(uX_{t_a^+})$: the final goal

More

- Renewal theory for Kendall convolution
- Limit properties of Kendall random walks
- More analogies with classical theory
- Applications

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