

Electromagnetic Notes

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based on lectures from PHYS20029
and Griffiths' *Introduction to Electrodynamics*

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1 Introduction

1.1 Recap of first year Electromagnetism

What we covered in first year:

- *Gauss' Law*: Integral over enclosed surface containing an electric field gives the total charge over that surface.

$$\iint_S \mathbf{E} \, dS = \frac{Q}{\epsilon_0} \quad (1.1)$$

- *Ampere's law*: Path of a magnetic field around a line integral is proportional to the current.

$$\oint_P \mathbf{B} \, d\mathbf{l} = \mu_0 I \quad (1.2)$$

- *Biot-Savart law*: Magnetic field arising from a small current containing element in the wire. Equivalent in magnetism to *Coulomb's law* in electrostatic.

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \quad (1.3)$$

This course will be concerned with deriving and using the *differential forms* of these integral equations. We will eventually arrive at Maxwell's equations. We will also consider two new fields \mathbf{D} and \mathbf{H} .

Note: Be aware of c.g.s system that changes the formulae as well as the units

2 Electrostatics

2.1 Electrostatics—what you know so far

Definition 2.1.1. Electronic charge is a property that is associated with the fundamental particles, protons (quarks), electrons etc. that occur in nature.

The Coulomb charge is the smallest free charge observed (fractional charges of quarks are smaller but isolated quarks do not appear in nature).

To properly consider the electromagnetic behaviour, we need quantum theory in atomic length scales. E.g., the quantum description of the hydrogen atom is the application of the coulomb potential in Schrödinger's equation. We are only learning classical electrodynamics.

2.2 Coulomb's Law

Definition 2.2.1. The force between two charged particles in S.I. units is

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \quad (2.1)$$

The total force on a test charge Q is the sum of the forces from the other charges in the system. This is the *superposition principle*. I.e., the field from one particle does not change the effect from any other charges in the system.

Definition 2.2.2. Superposition of electric forces applies, such that

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \dots \quad (2.2)$$

The electric field \mathbf{E} at point r is the force per unit charge exerted on a test charge, such that

$$\mathbf{F} = Q_{test} \mathbf{E} \quad (2.3)$$

From this, we can deduce that the total electric field is the superposition of electric fields of all charges in the system

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \dots \quad (2.4)$$

Note that superposition in this case is not a logical necessity but an experimental fact; if the force is proportional to the square of the charges, then this would not work.

We might ask—what is an electric field? We come to it through an intermediary step in calculating forces, thus we can define it as that. Otherwise, we can treat it as abstract or physical, it does not affect how these particles behave.

2.3 Total charge

Definition 2.3.1. The total charge in a system of discrete (point charges) is

$$Q = \sum_i q_i \quad (2.5)$$

For continuous charge distributions, the sum becomes an integral, and we consider instead the *charge densities*. For each dimension, the charge densities are:

System	Unit charge relation
Line charge	$\lambda \, dl = dq$
Surface charge	$\sigma \, da' = dq$
Volume charge	$\rho \, d\tau' = dq$

Definition 2.3.2. The total charge in system of continuous charge with a charge density is

$$Q = \int_{body} dq \quad (2.6)$$

Greek symbols have been used because V is used for potentials.

2.4 Charge densities and fields

Knowing the charge densities and the total charge, we can write Coulomb's law for electric fields. As an example, with 3D charge density, we start with the unit electric field, in differential

form,

$$d\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2} \rho \, d\tau \quad (2.7)$$

Integrating both sides will give us the resulting electric field.

Definition 2.4.1. Coulomb's law for a continuous charge distribution is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} \, d\tau \quad (2.8)$$

2.4.1 Video examples

1. Derive electric field at a vertical distance above a line of charge.
2. Derive electric field at a vertical distance above a circular loop and thus electric field from a flat circular disk.

Some takeaways:

If we are asked to find the electric field of a surface or volume:

1. Split into smaller dimension, *unit length* for a surface, *unit surface* for a volume.
2. Find the electric field of the smaller dimension shape.
3. Express the unit charge in terms of charge density.
4. Integrate over the original shape's limits.

2.5 Gauss' Law

Definition 2.5.1. For any volume or surface that encloses a charge Q then

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q \quad (2.9)$$

If we have high symmetry in the charge distribution, we can integrate over a symmetrical surface to find the electric field. For more complicated situations, Gauss' law *and* the superposition principle if there are still underlying symmetries to be exploited.

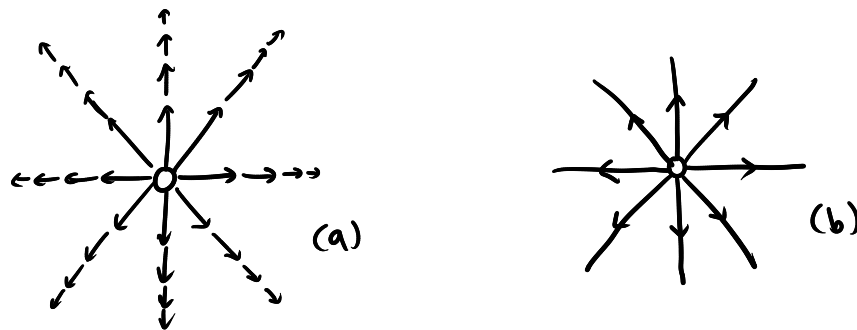


Figure 2.1: (a) shows vector field plot while (b) shows field lines method.

Charge distributions that are a superposition of any Gaussian distributions asks us to use the superposition principle to evaluate the integral.

2.5.1 Video examples

1. Find flux through faces of a cube of a charge at the corner of a cube. We can resize the cube to centre the charge so that we can use Gauss' law from symmetry
2. Find the field of a uniformly charged solid sphere
3. Find electric field well inside a long cylinder with charge density that varies by perpendicular distance from principle axis.
4. Find the electric field from an infinite sheet with surface charge.
5. Find field of two infinite sheets with surface charge opposite each other (field on either sides and in the middle)

The common strategy seems to be finding symmetry or finding a smaller part of the original geometry and integrate. Draw some field lines to get the feel for the geometry of the problem.

2.6 Drawing fields

The field from a positive charge always point outwards and the magnitude decreases as $1/r^2$. Field lines are represented as arrows that give its direction and whose lengths give its magnitude. Alternative, we can connect the neighbouring arrows and the density of the field lines can represent the strength of the field instead of the length.

Aside: Plotting field lines via Python utilises `matplotlib`.

2.7 Differential form of Gauss' Law

We can re-write Gauss' law by applying the divergence theorem.

Theorem 2.7.1 *The divergence theorem for a vector field \mathbf{X} is*

$$\oint_S \mathbf{X} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{X} dV \quad (2.10)$$

For Gauss' Law we can derive from the enclosed charge density in a volume V from ^(-1.5) equation 2.6 a new identity. Thus we discover that for an enclosed charge,

$$\frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \int_{\tau} \rho(\mathbf{r}) d\tau \quad (2.11)$$

Combine with the divergence theorem from 2.7.1, we find that

$$\int_{\tau} \nabla \cdot \mathbf{E} d\tau = \int_{\tau} \frac{\rho(\mathbf{r})}{\epsilon_0} d\tau \quad (2.12)$$

Definition 2.7.1. We arrive at the differential form by differentiating both sides with respect to τ

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0} \quad (2.13)$$

This is the first of Maxwell's equation.

2.8 The curl of \vec{E}

Consider the electric field for a static charge q , in spherical coordinates, the length differential is

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}} \quad (2.14)$$

But since our system is a singular charge with spherical symmetry, the angular differential terms disappear. Thus, we are left with

$$\mathbf{E} \cdot d\mathbf{l} = \frac{q}{4\pi\epsilon_0 r^2} dr \quad (2.15)$$

Integrating both sides leaves us with

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{q}{4\pi\epsilon_0} \int_a^b \frac{1}{r^2} dr = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b} \right) \quad (2.16)$$

For a closed loop, the integral evaluates to 0.

Corollary 2.8.1

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad (2.17)$$

In other words, \mathbf{E} is a conservative field. Using Stoke's theorem, we can conclude that

$$\nabla \times \mathbf{E} = 0 \quad (2.18)$$

Note: This result only applies for electrostatic fields and not when there are time varying magnetic fields.

2.9 Electrostatic potential

From 2.8 we find that for an electrostatic field, it is conservative. The following applies to any conservative field

Definition 2.9.1. The electrostatic potential as a function of a position vector is the negative of the integral of the electrostatic field along some path from a starting point to another point. Since it is conservative, it is path-independent, hence the difference in potential at two points is the integral evaluated from one point to the other.

$$V(\mathbf{b}) - V(\mathbf{a}) = - \int_a^b \mathbf{E} \cdot d\mathbf{l} \quad (2.19)$$

As a result, we have the following identity for the electric potential:

$$\mathbf{E} = -\nabla V \quad (2.20)$$

2.10 Notes on the Scalar potential

- Potential is different from potential energy.
- Finding potentials is easier than vector fields.
- Finding vector fields using the relation 2.20.
- There is not an absolute definition of potential—we only observe potential *differences*. Thus, ‘zero’ potential is arbitrarily defined, usually at infinity. This is for convenience rather than mathematical necessity.
- Potentials also follow the superposition principle.

2.11 Poisson’s equation

Substitute 2.20 into the curl of electric field, and we get Poisson’s equation.

Definition 2.11.1. Poisson’s equation is

$$\nabla^2 V = -\frac{\rho(\mathbf{r})}{\epsilon_0} \quad (2.21)$$

2.12 Laplace’s equation

Definition 2.12.1. In the absence of any charges, $\rho(\mathbf{r}) = 0$. Thus, Laplace’s equation is

$$\nabla^2 V = 0 \quad (2.22)$$

2.13 Point charge

A point charge has spherical symmetry. If we choose our zero potential at infinity and evaluate the integral then we can evaluate the integral to find the electrostatic potential.

Definition 2.13.1. The electrostatic potential of a point charge is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad (2.23)$$

Note that point charges do not actually exist, they are there for easier calculations.

2.14 Electrostatic potential for a continuous distribution

For a continuous charge distribution, we can use the superposition principle and sum over all point charges.

Definition 2.14.1. The electrostatic potential for a continuous distribution is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\mathbf{r}')}{r} d\tau' \quad (2.24)$$

For surface and line charge densities, it is the same form, except we are integrating over a surface or a path.

2.15 Summary of equations linking \vec{E} , V and ρ

Relating V and ρ : Equation 2.21 and equation 2.24.

Relating ρ and \mathbf{E} : Equation 2.8, equation 2.18 and equation 2.13.

Relating \mathbf{E} and V : Equation 2.20 and equation 2.19.

2.16 Boundary conditions

For a thin box passing through a surface charge, as we pass from below to above the surface, there is a discontinuous change in the electric field if the box is infinitesimally thin. This result is from first year. Since we only care about normal components, because for tangential components to the surface (i.e. sides of the box), this forms a closed loop and the integral is thus 0. We find that the normal component has a discontinuity, so that

$$\mathbf{E}_{\text{above}}^\perp - \mathbf{E}_{\text{below}}^\perp = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \quad (2.25)$$

We can integrate this to find the potential above and below. As the integral path length tends to zero, the integral tends to zero, so that the potential above is equal to the potential below.

$$V_{\text{above}} = V_{\text{below}} \quad (2.26)$$

We can also arrange 2.25 to get

Definition 2.16.1. The boundary conditions we have for the electrostatic field due to surface charges

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0} \quad (2.27)$$

Where the partial derivatives represent the normal derivative $\nabla \hat{n}$.

2.17 Work done to move a charge

Definition 2.17.1. The work needed to move a charge in an electric field is

$$W = -Q \int_a^b \mathbf{E} \cdot d\mathbf{l} = Q [V(b) - V(a)] = Q\Delta V \quad (2.28)$$

2.18 The energy of a distribution of point charges

To move one charge to the other, separated by a distance of infinity, to a finite distance r , the work done is

$$W_2 = \frac{1}{4\pi\epsilon_0} \frac{q_2 q_1}{r_{12}} \quad (2.29)$$

The superposition principle is used to find the total work done in moving two or more charges together. It is the sum of the work done to bring each individual pair together.

$$W = \sum_{i=2}^N W_i \quad (2.30)$$

To avoid double counting the pairs, we halve the final sum, which yields

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^N \sum_j^N \frac{q_i q_j}{r_{ij}} \quad (2.31)$$

We can rearrange this equation further using [2.28](#).

Definition 2.18.1. The work done in moving all these charges from infinity to a finite distance is

$$W = \frac{1}{2} \sum_{i=1}^N q_i V(\mathbf{r}_i) \quad (2.32)$$

2.19 The energy of a continuous charge distribution

Once again, we can integrate to convert from discrete charges to continuous charges. We can substitute in 2.13 then integrate by parts to get a nicer expression.

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \int (\nabla \cdot \mathbf{E}) V \, d\tau \\ &= \frac{\epsilon_0}{2} \left[- \int \mathbf{E} \cdot \nabla V \, d\tau + \oint V \mathbf{E} \, d\mathbf{a} \right] \\ &= \frac{\epsilon_0}{2} \left[- \int \mathbf{E}^2 \, d\tau + \oint V \mathbf{E} \, d\mathbf{a} \right] \end{aligned} \quad (2.33)$$

This is the *correct* equation, but in theory if you integrate over bigger volumes (as long as it encloses the charge), the contribution from the volume will overtake the contribution from the surface. The first term is the contribution from the volume, while the second is the surface. Since outside our standard sphere (let's say we integrate over all space), $\rho = 0$, our result for W must be the same. But the volume integral grows as E^2 is positive, so the surface integral must decrease. In fact, it decreases by $1/r$. Thus, if we integrate over all space, we are left with the first term only.

Definition 2.19.1. The work done on continuous charge distribution is

$$W = \frac{\epsilon_0}{2} \int_{\text{allspace}} \mathbf{E}^2 \, d\tau \quad (2.34)$$

This gives us a different idea to think about energy in electrostatics. If we have a series of charges (discrete or continuous). Rather than thinking in terms of the work done by bringing these charges together, we can think of the work as the integral of the electric field over all space. One could say the electric field 'stores' energy.

It is **important** to understand that the energy of the electric field does not obey the superposition principle. If we have two charges and bring them together, the total energy is not the

energy in one field plus the energy in the other. This is because it is not a linear relationship.

2.20 The properties of conductors

Definition 2.20.1. Inside a conductor, there are several important properties to remember

- $\mathbf{E} = 0$ inside a conductor. If there is a deficit of charge, then it will move until equilibrium is reached.
- $\rho = 0$ inside a conductor. We can have charges inside a conductor, but they must cancel out.
- Charges in a conductor are **mobile**. This means they can move freely.
- Any net charge resides on the **surface**.
- A surface of a conductor is an **equipotential**, otherwise charges would move freely until potential equilibrium is reached.
- \mathbf{E} is perpendicular to the surface at the surface. Tangential components would mean charges can move along the surface, this is not the case for an equipotential.

2.21 Induced charges

If a positively charged particle is brought closed to a conductor, its field will attract negative charges towards it. These negative charges are closer to the positive particle and thus creates a net attraction. The charges in the conductor move around the surface to cancel the field in the conductor so that at equilibrium there is no \mathbf{E} field in the conductor.

The result is a redistribution of charge on the surface of the conductor.

2.22 Surface charge and the force on a conductor

If we have a *static* field on a conductor, then the boundary conditions for the charge at the surface follow the result from from 2.16. We have a discontinuity in the field, and immediately outside the conductor, it is given by 2.25.

We can rearrange for the charge density and express it in terms of the potential

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \quad (2.35)$$

Using the expression for force in 2.3, we can express the force *per unit area* as

$$\mathbf{f} = \sigma \mathbf{E}_{\text{avg}} \quad (2.36)$$

Because there is a discontinuity, we take the **average** of the electric field above and below (or inside and outside).

For a conductor, this is half of 2.36 since the field is 0 inside and outside, it is given by 2.25.

Definition 2.22.1. The force per unit area for a static field on a conductor is

$$\mathbf{f} = \frac{1}{2\epsilon_0} \sigma^2 \hat{\mathbf{n}} \quad (2.37)$$

or as a pressure, it is

$$P = \frac{\epsilon_0}{2} E^2 \quad (2.38)$$

2.23 Capacitors

A capacitor is a device with two conductors separated in space.

Suppose there is a charge $+Q$ and $-Q$ on each conductor respectively. Each conductor is an equipotential, thus we can consider the *potential difference* between them as the integral of the electric field from the negative to the positive conductor.

Since electric field is proportional to the charge 2.3, or more general as 2.8 we can substitute this into the integral, so the potential difference is proportional to the charge

We call this constant of proportionality the *capacitance*.

Definition 2.23.1. The relation between the potential difference and charge is

$$C = \frac{Q}{V} \quad (2.39)$$

2.24 Energy stored in a capacitor

We can ask the question: how much work is needed to "charge" a capacitor? This is equivalent to moving charges from the positive plate to the negative plate.

We need to work *harder* as we transfer more charge from one plate to the other.

If a plate already has charge q on it, with potential difference between plates $V = q/C$, then from 2.28, the work needed to add a further charge of dq is

$$dW = V dq = \frac{q}{C} dq \quad (2.40)$$

Integrate this equation to get the *total work done* or *total energy stored*. Further, we can use our derived expressions to rearrange.

Definition 2.24.1. The energy stored in a capacitor or work done to charge a capacitor is

$$W = \frac{1}{2} CV^2 \quad (2.41)$$

If we now connect our capacitors to some wires then the capacitor will discharge because of the potential difference between the two plates. Thus, the capacitor is said to be storing energy. In fact, if the space between the plates is air then it would require really big capacitors for a substantial amount of energy. Dielectrics would be a better material.