# Electromagnetic Notes

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based on lectures from PHYS20029
and Griffiths' *Introduction to Electrodynamics*

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# Contents

# 1 Introduction

## 1.1 Recap of first year Electromagnetism

What we covered in first year:

- *Gauss' Law*: Integral over enclosed surface containing an electric field gives the total charge over that surface.

$$\int_S \boldsymbol{E} \; \mathrm{d}\boldsymbol{S} = \frac{Q}{\epsilon_0} \tag{1.1}$$

- *Ampere's law:* Path of a magnetic field around a line integral is proportional to the current.

$$\oint_P \boldsymbol{B} \; \mathrm{d}\boldsymbol{l} = \mu_0 I \tag{1.2}$$

- *Biot-Savart law:* Magnetic field arising from a small current containing element in the wire. Equivalent in magnetism to *Coulomb's law* in electrostatic.

$$\boldsymbol{B}\left(\boldsymbol{r}\right) = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\boldsymbol{l} \times \left(\boldsymbol{r} - \boldsymbol{r'}\right)}{\left|\boldsymbol{r} - \boldsymbol{r'}\right|^3} \tag{1.3}$$

This course will be concerned with deriving and using the *differential forms* of these integral equations. We will eventually arrive at Maxwell's equations. We will also consider two new fields $\boldsymbol{D}$ and $\boldsymbol{H}$.

**Note:** Be aware of c.g.s system that changes the formulae as well as the units

# 2 Electrostatics

## 2.1 Electrostatics—what you know so far

**Definition 2.1.1.** Electronic charge is a property that is associated with the fundamental particles, protons (quarks), electrons etc. that occur in nature.

The Coulomb charge is the smallest free charge observed (fractional charges of quarks are smaller but isolated quarks do not appear in nature).

To properly consider the electromagnetic behaviour, we need quantum theory in atomic length scales. E.g., the quantum description of the hydrogen atom is the application of the coulomb potential in Schrödinger's equation. We are only learning classical electrodynamics.

## 2.2 Coulomb's Law

**Definition 2.2.1.** The force between two charged particles in S.I. units is

$$\boldsymbol{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} \tag{2.1}$$

The total force on a test charge $Q$ is the sum of the forces from the other charges in the system. This is the *superposition principle*. I.e., the field from one particle does not change the effect from any other charges in the system.

**Definition 2.2.2.** Superposition of electric forces applies, such that

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \dots \tag{2.2}$$

The electric field $\boldsymbol{E}$ at point $r$ is the force per unit charge exerted on a test charge, such that

$$\boldsymbol{F} = Q_{test}\boldsymbol{E} \tag{2.3}$$

From this, we can deduce that the total electric field is the superposition of electric fields of all charges in the system

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \dots \tag{2.4}$$

Note that superposition in this case is not a logical necessity but an experimental fact; if the force is proportional to the square of the charges, then this would not work.

We might ask—what is an electric field? We come to it through an intermediary step in calculating forces, thus we can define it as that. Otherwise, we can treat it as abstract or physical, it does not affect how these particles behave.

## 2.3 Total charge

**Definition 2.3.1.** The total charge in a system of discrete (point charges) is

$$Q = \sum_i q_i \tag{2.5}$$

For continuous charge distributions, the sum becomes an integral, and we consider instead the *charge densities*. For each dimension, the charge densities are:

| System | Unit charge relation |
|---|---|
| Line charge | $\lambda \, \mathrm{d}l = \mathrm{d}q$ |
| Surface charge | $\sigma \, \mathrm{d}a' = \mathrm{d}q$ |
| Volume charge | $\rho \, \mathrm{d}\tau' = \mathrm{d}q$ |

**Definition 2.3.2.** The total charge in system of continuous charge with a charge density is

$$Q = \int_{body} \mathrm{d}q \tag{2.6}$$

Greek symbols have been used because $V$ is used for potentials.

## 2.4 Charge densities and fields

Knowing the charge densities and the total charge, we can write Coulomb's law for electric fields. As an example, with 3D charge density, we start with the unit electric field, in differential form,

$$\mathrm{d}\boldsymbol{E}\left(\boldsymbol{r}\right) = \frac{1}{4\pi\epsilon_0 \boldsymbol{r}^2}\rho \; \mathrm{d}\tau \tag{2.7}$$

Integrating both sides will give us the resulting electric field.

**Definition 2.4.1.** Coulomb's law for a continuous charge distribution is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{r^2}\hat{\boldsymbol{r}} \; \mathrm{d}\tau \tag{2.8}$$

Video example 1 derives electric field at a vertical distance above a line of charge. Video example 2 derives electric field at a vertical distance above a circular loop and thus electric field from a flat circular disk. Some takeaways:

If we are asked to find the electric field of a surface or volume:

1. Split into smaller dimension, *unit length* for a surface, *unit surface* for a volume.
2. Find the electric field of the smaller dimension shape.
3. Express the unit charge in terms of charge density.
4. Integrate over the original shape's limits.

## 2.5 Gauss' Law

**Definition 2.5.1.** For any volume or surface that encloses a charge $Q$ then

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a} = \frac{1}{\epsilon_0}Q \tag{2.9}$$

If we have high symmetry in the charge distribution, we can integrate over a symmetrical surface to find the electric field. For more complicated situations, Gauss' law *and* the superposition principle if there are still underlying symmetries to be exploited.

Charge distributions that are a superposition of any Gaussian distributions asks us to use the superposition principle to evaluate the integral.

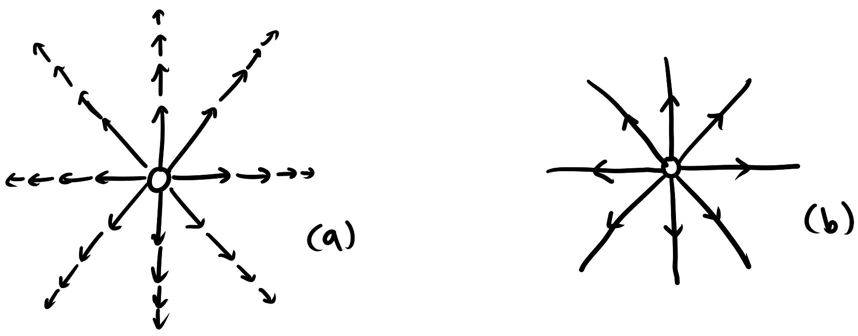


Figure 2.1: (a) shows vector field plot while (b) shows field lines method.

### 2.5.1 Video examples

1. Find flux through faces of a cube of a charge at the corner of a cube. We can resize the cube to centre the charge so that we can use Gauss' law from symmetry
2. Find the field of a uniformly charged solid sphere
3. Find electric field well inside a long cylinder with charge density that varies by perpendicular distance from principle axis.
4. Find the electric field from an infinite sheet with surface charge.
5. Find field of two infinite sheets with surface charge opposite each other (field on either sides and in the middle)

The common strategy seems to be finding symmetry or finding a smaller part of the original geometry and integrate. Draw some field lines to get the feel for the geometry of the problem.

## 2.6 Drawing fields

The field from a positive charge always point outwards and the magnitude decreases as $1/r^2$. Field lines are represented as arrows that give its direction and whose lengths give its magnitude. Alternative, we can connect the neighbouring arrows and the density of the field lines can represent the strength of the field instead of the length.

**Aside:** Plotting field lines via Python utilises `matplotlib`.

## 2.7 Differential form of Gauss' Law

We can re-write Gauss' law by applying the divergence theorem.

**Theorem 2.7.1** *The divergence theorem for a vector field* $\boldsymbol{X}$ *is*

$$\oint_S \boldsymbol{X} \cdot \mathrm{d}\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{X} \, \mathrm{d}V \tag{2.10}$$

For Gauss' Law we can derive from the enclosed charge density in a volume $V$ from**(-1.5) equation 2.6 a new identity. Thus we discover that for an enclosed charge,

$$\frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \int_\tau \rho(\boldsymbol{r}) \, \mathrm{d}\tau \tag{2.11}$$

Combine with the divergence theorem from 2.7.1, we find that

$$\int_\tau \boldsymbol{\nabla} \cdot \boldsymbol{E} \, \mathrm{d}\tau = \int_\tau \frac{\rho(\boldsymbol{r})}{\epsilon_0} \, \mathrm{d}\tau \tag{2.12}$$

**Definition 2.7.1.** We arrive at the differential form by differentiating both sides with respect to $\tau$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0} \tag{2.13}$$

This is the first of Maxwell's equation.

## 2.8 The curl of $\vec{E}$

Consider the electric field for a static charge $q$, in spherical coordinates, the length differential is

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r \, \hat{\boldsymbol{r}} + r \, \mathrm{d}\theta \, \hat{\boldsymbol{\theta}} + r \sin\theta \, \mathrm{d}\phi \, \hat{\boldsymbol{\phi}} \tag{2.14}$$

But since our system is a singular charge with spherical symmetry, the angular differential terms disappear. Thus, we are left with

$$\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \frac{q}{4\pi\epsilon_0 r^2} \, \mathrm{d}r \tag{2.15}$$

Integrating both sides leaves us with

$$\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \frac{q}{4\pi\epsilon_0} \int_a^b \frac{1}{r^2}\, \mathrm{d}r = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \tag{2.16}$$

For a closed loop, the integral evaluates to 0.

**Corollary 2.8.1**

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0 \tag{2.17}$$

*In other words,* $\boldsymbol{E}$ *is a conservative field. Using Stoke's theorem, we can conclude that*

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \tag{2.18}$$

**Note:** This result only applies for electrostatic fields and not when there are time varying magnetic fields.