Electromagnetic Notes

Compiled by Nhat Pham
based on lectures from PHYS20029
and Griffiths' *Introduction to Electrodynamics*

Last update: February 2, 2022

Contents

Contents

1 Introduction

1.1 Recap of first year Electromagnetism

What we covered in first year:

- *Gauss' Law*: Integral over enclosed surface containing an electric field gives the total charge over that surface.

$$\iint_S \boldsymbol{E} \; \mathrm{d}\boldsymbol{S} = \frac{Q}{\epsilon_0} \tag{1.1}$$

- *Ampere's law:* Path of a magnetic field around a line integral is proportional to the current.

$$\oint_P \boldsymbol{B} \; \mathrm{d}\boldsymbol{l} = \mu_0 I \tag{1.2}$$

- *Biot-Savart law:* Magnetic field arising from a small current containing element in the wire. Equivalent in magnetism to *Coulomb's law* in electrostatic.

$$\boldsymbol{B}\left(\boldsymbol{r}\right) = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\boldsymbol{l} \times \left(\boldsymbol{r} - \boldsymbol{r'}\right)}{\left|\boldsymbol{r} - \boldsymbol{r'}\right|^3} \tag{1.3}$$

This course will be concerned with deriving and using the *differential forms* of these integral equations. We will eventually arrive at Maxwell's equations. We will also consider two new fields $\boldsymbol{D}$ and $\boldsymbol{H}$.

Note: Be aware of c.g.s system that changes the formulae as well as the units

2 Electrostatics

2.1 Electrostatics—what you know so far

Definition 2.1.1. Electronic charge is a property that is associated with the fundamental particles, protons (quarks), electrons etc. that occur in nature.

The Coulomb charge is the smallest free charge observed (fractional charges of quarks are smaller but isolated quarks do not appear in nature).

To properly consider the electromagnetic behaviour, we need quantum theory in atomic length scales. E.g., the quantum description of the hydrogen atom is the application of the coulomb potential in Schrödinger's equation. We are only learning classical electrodynamics.

2.2 Coulomb's Law

Definition 2.2.1. The force between two charged particles in S.I. units is

$$F = \frac{q_1 q_2}{4 \pi \epsilon_0 r^2} \hat{r} \tag{2.1}$$

The total force on a test charge Q is the sum of the forces from the other charges in the system. This is the *superposition principle*. I.e., the field from one particle does not change the effect from any other charges in the system.

Definition 2.2.2. Superposition of electric forces applies, such that

$$F = F_1 + F_2 + \ldots \tag{2.2}$$

The electric field E at point r is the force per unit charge exerted on a test charge, such that

$$F = Q_{test} E \tag{2.3}$$

5

From this, we can deduce that the total electric field is the superposition of electric fields of all charges in the system

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \dots \tag{2.4}$$

Note that superposition in this case is not a logical necessity but an experimental fact; if the force is proportional to the square of the charges, then this would not work.

We might ask—what is an electric field? We come to it through an intermediary step in calculating forces, thus we can define it as that. Otherwise, we can treat it as abstract or physical, it does not affect how these particles behave.

2.3 Total charge

Definition 2.3.1. The total charge in a system of discrete (point charges) is

$$Q = \sum_i q_i \tag{2.5}$$

For continuous charge distributions, the sum becomes an integral, and we consider instead the *charge densities*. For each dimension, the charge densities are:

System	Unit charge relation
Line charge	$\lambda \ \mathrm{d}l = \mathrm{d}q$
Surface charge	$\sigma \ \mathrm{d}a' = \mathrm{d}q$
Volume charge	$\rho \ \mathrm{d}\tau' = \mathrm{d}q$

Definition 2.3.2. The total charge in system of continuous charge with a charge density is

$$Q = \int_{body} \mathrm{d}q \tag{2.6}$$

Greek symbols have been used because V is used for potentials.

2.4 Charge densities and fields

Knowing the charge densities and the total charge, we can write Coulomb's law for electric fields. As an example, with 3D charge density, we start with the unit electric field, in differential

form,

$$d\boldsymbol{E}\left(\boldsymbol{r}\right) = \frac{1}{4\pi\epsilon_0 \boldsymbol{r}^2}\rho \; d\tau \tag{2.7}$$

Integrating both sides will give us the resulting electric field.

Definition 2.4.1. Coulomb's law for a continuous charge distribution is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}')}{r^2}\hat{\boldsymbol{r}} \; d\tau \tag{2.8}$$

Example 2.4.1.   1. Derive electric field at a vertical distance above a line of charge.

   2. Derive electric field at a vertical distance above a circular loop and thus electric field from a flat circular disk.

Some takeaways:

If we are asked to find the electric field of a surface or volume:

   1. Split into smaller dimension, *unit length* for a surface, *unit surface* for a volume.

   2. Find the electric field of the smaller dimension shape.

   3. Express the unit charge in terms of charge density.

   4. Integrate over the original shape's limits.

2.5 Gauss' Law

Definition 2.5.1. For any volume or surface that encloses a charge Q then

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\epsilon_0}Q \tag{2.9}$$

If we have high symmetry in the charge distribution, we can integrate over a symmetrical surface to find the electric field. For more complicated situations, Gauss' law *and* the superposition principle if there are still underlying symmetries to be exploited.

Charge distributions that are a superposition of any Gaussian distributions asks us to use the
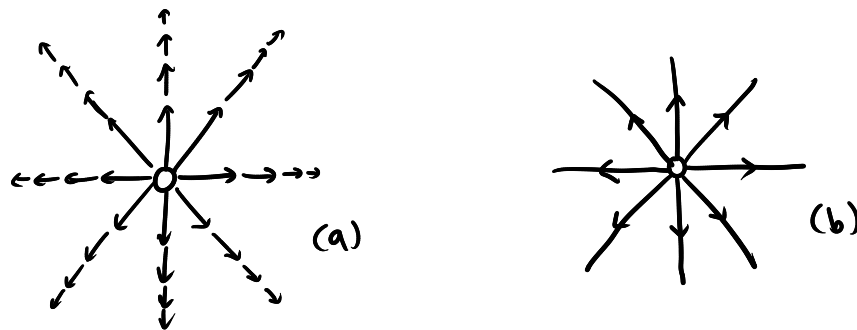
Figure 2.1: (a) shows vector field plot while (b) shows field lines method.

superposition principle to evaluate the integral.

Example 2.5.1.   1. Find flux through faces of a cube of a charge at the corner of a cube. We can resize the cube to centre the charge so that we can use Gauss' law from symmetry

2. Find the field of a uniformly charged solid sphere

3. Find electric field well inside a long cylinder with charge density that varies by perpendicular distance from principle axis.

4. Find the electric field from an infinite sheet with surface charge.

5. Find field of two infinite sheets with surface charge opposite each other (field on either sides and in the middle)

The common strategy seems to be finding symmetry or finding a smaller part of the original geometry and integrate. Draw some field lines to get the feel for the geometry of the problem.

2.6 Drawing fields

The field from a positive charge always point outwards and the magnitude decreases as $1/r^2$. Field lines are represented as arrows that give its direction and whose lengths give its magnitude. Alternative, we can connect the neighbouring arrows and the density of the field lines can represent the strength of the field instead of the length.

Aside: Plotting field lines via Python utilises `matplotlib`.

2.7 Differential form of Gauss' Law

We can re-write Gauss' law by applying the divergence theorem.

Theorem 2.7.1 *The divergence theorem for a vector field $\boldsymbol{X}$ is*

$$\oint_S \boldsymbol{X} \cdot \mathrm{d}\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{X} \, \mathrm{d}V \tag{2.10}$$

For Gauss' Law we can derive from the enclosed charge density in a volume V from equation **??** a new identity. Thus we discover that for an enclosed charge,

$$\frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \int_\tau \rho(\boldsymbol{r}) \, \mathrm{d}\tau \tag{2.11}$$

Combine with the divergence theorem from 2.7.1, we find that

$$\int_\tau \boldsymbol{\nabla} \cdot \boldsymbol{E} \, \mathrm{d}\tau = \int_\tau \frac{\rho(\boldsymbol{r})}{\epsilon_0} \, \mathrm{d}\tau \tag{2.12}$$

Definition 2.7.1. We arrive at the differential form by differentiating both sides with respect to τ

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0} \tag{2.13}$$

This is the first of Maxwell's equation.

2.8 The curl of the electric field

Consider the electric field for a static charge q, in spherical coordinates, the length differential is

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r \, \hat{\boldsymbol{r}} + r \, \mathrm{d}\theta \, \hat{\boldsymbol{\theta}} + r \sin\theta \, \mathrm{d}\phi \, \hat{\boldsymbol{\phi}} \tag{2.14}$$

Since our system is a singular charge with spherical symmetry, the angular differential terms disappear. Thus, we are left with

$$\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \frac{q}{4\pi\epsilon_0 r^2} \, \mathrm{d}r \tag{2.15}$$

Integrating both sides leaves us with

$$\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \frac{q}{4\pi\epsilon_0} \int_a^b \frac{1}{r^2}\,\mathrm{d}r = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{2.16}$$

For a closed loop, the integral evaluates to 0.

Corollary 2.8.1

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0 \tag{2.17}$$

In other words, $\boldsymbol{E}$ is a conservative field. Using Stoke's theorem, we can conclude that

$$\nabla \times \boldsymbol{E} = 0 \tag{2.18}$$

Note: This result only applies for electrostatic fields and not when there are time varying magnetic fields.

2.9 Electrostatic potential

From 2.8 we find that for an electrostatic field, it is conservative. The following applies to any conservative field

Definition 2.9.1. The electrostatic potential as a function of a position vector is the negative of the integral of the electrostatic field along some path from a starting point to another point. Since it is conservative, it is path-independent, hence the difference in potential at two points is the integral evaluated from one point to the other.

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} \tag{2.19}$$

As a result, we have the following identity for the electric potential:

$$\boldsymbol{E} = -\nabla V \tag{2.20}$$

Example 2.9.1. Find the potential inside and outside a uniformly charged spherical shell of radius R. Very similar steps to 3.3.1.

Inside, it is uniform and depends on the radius of the sphere, while outside, it depends on the position of our test charge.

We get to

$$V(z) = \frac{2\pi R \sigma}{2\epsilon_0 z} \left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2} \right] \tag{2.21}$$

and must consider that the second square root term is $z - R$ for test charge outside and $R - z$ for test charge inside.

Thus, we have

$$V(r) = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{r}, & r \geq R \\ \frac{1}{4\pi\epsilon_0} \frac{q}{R}, & r \leq R \end{cases} \tag{2.22}$$

2.10 Notes on the Scalar potential

- Potential is different from potential energy.
- Finding potentials is easier than vector fields.
- Finding vector fields using the relation 2.20.
- There is not an absolute definition of potential—we only observe potential *differences*. Thus, 'zero' potential is arbitrarily defined, usually at infinity. This is for convenience rather than mathematical necessity.
- Potentials also follow the superposition principle.

2.11 Poisson's equation

Substitute 2.20 into the curl of electric field, and we get Poisson's equation.

Definition 2.11.1. Poisson's equation is

$$\boldsymbol{\nabla}^2 V = -\frac{\rho(\boldsymbol{r})}{\epsilon_0} \tag{2.23}$$

2.12 Laplace's equation I

Definition 2.12.1. In the absence of any charges, $\rho(\boldsymbol{r}) = 0$. Thus, Laplace's equation is

$$\boldsymbol{\nabla}^2 V = 0 \tag{2.24}$$

2.13 Point charge

A point charge has spherical symmetry. If we choose our zero potential at infinity and evaluate the integral then we can evaluate the integral to find the electrostatic potential.

Definition 2.13.1. The electrostatic potential of a point charge is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{2.25}$$

Note that point charges do not actually exist, they are there for easier calculations.

2.14 Electrostatic potential for a continuous distribution

For a continuous charge distribution, we can use the superposition principle and sum over all point charges.

Definition 2.14.1. The electrostatic potential for a continuous distribution is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\iiint\limits_{V} \frac{\rho(\boldsymbol{r'})}{r}\,\mathrm{d}\tau' \tag{2.26}$$

For surface and line charge densities, it is the same form, except we are integrating over a surface or a path.

2.15 Summary of equations

Relating V and ρ: Equation 2.23 and equation 2.26.

Relating ρ and $\boldsymbol{E}$: Equation 2.8, equation 2.18 and equation 2.13.

Relating $\boldsymbol{E}$ and V: Equation 2.20 and equation 2.19.

2.16 Boundary conditions

For a thin box passing through a surface charge, as we pass from below to above the surface, there is a discontinuous change in the electric field if the box is infinitesimally thin. This

result is from first year. Since we only care about normal components, because for tangential components to the surface (i.e. sides of the box), this forms a closed loop and the integral is thus 0. We find that the normal component has a discontinuity, so that

$$\boldsymbol{E}_{\text{above}}^{\perp} - \boldsymbol{E}_{\text{below}}^{\perp} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} \tag{2.27}$$

We can integrate this to find the potential above and below. As the integral path length tends to zero, the integral tends to zero, so that the potential above is equal to the potential below.

$$V_{\text{above}} = V_{\text{below}} \tag{2.28}$$

We can also arrange 2.27 to get

Definition 2.16.1. The boundary conditions we have for the electrostatic field due to surface charges

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0} \tag{2.29}$$

Where the partial derivatives represent the normal derivative $\boldsymbol{\nabla}\hat{n}$.

2.17 Work done to move a charge

Definition 2.17.1. The work needed to move a charge in an electric field is

$$W = -Q \int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = Q\left[V(\boldsymbol{b}) - V(\boldsymbol{a})\right] = Q\Delta V \tag{2.30}$$

2.18 The energy of a distribution of point charges

To move one charge to the other, separated by a distance of infinity, to a finite distance r, the work done is

$$W_2 = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{r_{12}} \tag{2.31}$$

The superposition principle is used to find the total work done in moving two or more charges together. It is the sum of the work done to bring each individual pair together.

$$W = \sum_{i=2}^{N} W_i \tag{2.32}$$

To avoid double counting the pairs, we halve the final sum, which yields

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j\neq i}^{N} \frac{q_i q_j}{r_{ij}} \tag{2.33}$$

We can rearrange this equation further using 2.30.

Definition 2.18.1. The work done in moving all these charges from infinity to a finite distance is

$$W = \frac{1}{2} \sum_{i=1}^{N} q_i V(\boldsymbol{r}_i) \tag{2.34}$$

2.19 The energy of a continuous charge distribution

Once again, we can integrate to convert from discrete charges to continuous charges. We can substitute in 2.13 then integrate by parts to get a nicer expression.

Derivation 2.19.1.

$$\begin{aligned}
W &= \frac{\epsilon_0}{2} \int (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \, V \, d\tau \\
&= \frac{\epsilon_0}{2} \left[-\int \boldsymbol{E} \cdot \boldsymbol{\nabla}V \, d\tau + \oint V\boldsymbol{E} \, d\boldsymbol{a} \right] \\
&= \frac{\epsilon_0}{2} \left[-\int \boldsymbol{E}^2 \, d\tau + \oint V\boldsymbol{E} \, d\boldsymbol{a} \right]
\end{aligned} \tag{2.35}$$

This is the *correct* equation, but in theory if you integrate over bigger volumes (as long as it encloses the charge), the contribution from the volume will overtake the contribution from the surface. The first term is the contribution from the volume, while the second is the surface. Since outside our standard sphere (let's say we integrate over all space), $\rho = 0$, our result for W must be the same. But the volume integral grows as E^2 is positive, so the surface integral must decrease. In fact, it decreases by $1/r$. Thus, if we integrate over all space, we are left with the first term only.

Definition 2.19.1. The work done on continuous charge distribution is

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, \mathrm{d}\tau \tag{2.36}$$

This gives us a different idea to think about energy in electrostatics. If we have a series of charges (discrete or continuous). Rather than thinking in terms of the work done by bringing these charges together, we can think of the work as the integral of the electric field over all space. One could say the electric field 'stores' energy.

It is **important** to understand that the energy of the electric field does not obey the superposition principle. If we have two charges and bring them together, the total energy is not the energy in one field plus the energy in the other. This is because it is not a linear relationship.

Example 2.19.1. Find the electrostatic energy of a uniformly charged spherical shell of radius R and total charge q.

Note that we can use either use

$$W = \frac{1}{2} \int \sigma V \, \mathrm{d}a \tag{2.37}$$

or

$$W = \frac{1}{2} \int E^2 \, \mathrm{d}\tau \tag{2.38}$$

to get to the same result.

2.20 The properties of conductors

Definition 2.20.1. Inside a conductor, there are several important properties to remember

- $E = 0$ inside a conductor. If there is a deficit of charge, then it will move until equilibrium is reached.

- $\rho = 0$ inside a conductor. We can have charges inside a conductor, but they must cancel.

- Charges in a conductor are **mobile**. This means they can move freely.

- Any net charge resides on the **surface**.

- A surface of a conductor is an **equipotential**, otherwise charges would move freely until potential equilibrium is reached.

- $\boldsymbol{E}$ is perpendicular to the surface at the surface. Tangential components would mean charges can move along the surface, this is not the case for an equipotential.

2.21 Induced charges

If a positively charged particle is brought closed to a conductor, its field will attract negative charges towards it. These negative charges are closer to the positive particle and thus creates a net attraction. The charges in the conductor move around the surface to cancel the field in the conductor so that at equilibrium there is no $\boldsymbol{E}$ field in the conductor.

The result is a redistribution of charge on the surface of the conductor.

Example 2.21.1. An uncharged spherical conductor has a cavity of arbitrary shape with a charge $+q$ placed somewhere in it. What is the electric field outside of the conductor?

Outside the conductor, using Gauss' Law, for a spherical conductor, it is

$$\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \tag{2.39}$$

Does a strangely shaped cavity affect the distribution of the charge distributed on the surface? The arrangement of negative charges in the inside surface of the cvity that is attracted to the cavity cancels out the asymmetry that is produced by the shape of the cavity. The net effect is that the shape of the cavity does not matter. We can simply argue that we have a uniform surface charge. See 3.4 for relevant theorems.

2.22 Surface charge and the force on a conductor

If we have a *static* field on a conductor, then the boundary conditions for the charge at the surface follow the result from 2.16. We have a discontinuity in the field, and immediately outside the conductor, it is given by 2.27.

We can rearrange for the charge density and express it in terms of the potential

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \tag{2.40}$$

Using the expression for force in 2.3, we can express the force *per unit area* as

$$\boldsymbol{f} = \sigma \boldsymbol{E}_{\text{avg}} \tag{2.41}$$

Because there is a discontinuity, we take the **average** of the electric field above and below (or inside and outside).

For a conductor, this is half of 2.41 since the field is 0 inside and outside, it is given by 2.27.

Definition 2.22.1. The force per unit area for a static field on a conductor is

$$\boldsymbol{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\boldsymbol{n}} \tag{2.42}$$

or as a pressure, it is

$$P = \frac{\epsilon_0}{2}E^2 \tag{2.43}$$

2.23 Capacitors

A capacitor is a device with two conductors separated in space.

Suppose there is a charge $+Q$ and $-Q$ on each conductor respectively. Each conductor is an equipotential, thus we can consider the *potential difference* between them as the integral of the electric field from the negative to the positive conductor.

Since electric field is proportional to the charge 2.3, or more general as 2.8 we can substitute this into the integral, so the potential difference is proportional to the charge

We call this constant of proportionality the *capacitance.*

Definition 2.23.1. The relation between the potential difference and charge is

$$C = \frac{Q}{V} \tag{2.44}$$

Example 2.23.1. Find the capacitance of a parallel plate capacitor made up of two metal sheets of area A separated by a distance d.

Recall that for a linear system like this,

$$E = \frac{V}{d} \tag{2.45}$$

After some algebra, we are left with

$$C = \frac{\epsilon_0 A}{d} \tag{2.46}$$

Example 2.23.2. Find the capacitance of two concentric spherical metallic shells with radii a and b, where $a < b$.

We can use Gauss' Law to determine the electric field for various radii. It is non-zero for $a \leq r \leq b$. As usual, we can determine the potential through integration.

Then we go one step further and establish the capacitance using the non-zero potential between the two shells. This turns out to be

$$C = \frac{4\pi\epsilon_0 ab}{(b-a)} \tag{2.47}$$

2.24 Energy stored in a capacitor

We can ask: how much work is needed to "charge" a capacitor? This is equivalent to moving charges from the positive plate to the negative plate.

We need to work *harder* as we transfer more charge from one plate to the other.

If a plate already as charge q on it, with potential difference between plates $V = q/C$, then from 2.30, the work needed to add a charge of dq is

$$dW = V \, dq = \frac{q}{C} \, dq \tag{2.48}$$

Integrate this equation to get the *total work done* or *total energy stored*. Further, we can use our derived expressions to rearrange.

Definition 2.24.1. The energy stored in a capacitor or work done to charge a capacitor is

$$W = \frac{1}{2}CV^2 \tag{2.49}$$

If we now connect our capacitors to some wires then the capacitor will discharge because of the potential difference between the two plates. Thus, the capacitor is said to be storing energy. In fact, if the space between the plates is air then it would require really big capacitors for a substantial amount of energy. Dielectrics would be a better material.

3 Potentials

3.1 Introduction

To find the electric field or potential from charge densities (as shown in 2.15) is quite difficult as not all of them have simple analytic solutions.

For conductors, we also do not know where the charges are in advance, since the electric field inside is 0.

In most cases, it is better to start from Poisson's equation in 2.11 in regions with charge, and Laplace's equation in 2.12 in regions with no charge.

3.2 Laplace's equation II

3.2.1 3D Laplacian

If we look at the 3D Laplace's equation in full:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{3.1}$$

We see that we have some expressions of requirements. Normally, the second derivative will tell us whether there's a maximum or minimum (positive or negative value). Because this is equal to 0, we cannot have a minimum for all three directions, otherwise, it would not be equal to 0.

Relevantly, we have Earnshaw's theorem, which maintains that a collection of point charges cannot be maintained in a stable stationary equilibrium (impossible to trap a particle in a 3D electric field). If we can trap a particle, then it would mean all points must be a minimum. To follow Laplace's equation, we can only have unstable points (no local maxima or minima)

3.2.2 1D Laplacian

In 1D, it is a relatively straightforward, the first derivate is just a constant, and the function itself is a linear function.

If we take any point on that straight line, and look at two neighbouring points, then the potential is just the average potential of the two surround points.

3.2.3 2D Laplacian

In 2D, like in 3D, it's not possible to write a general solution, since it is not an ODE. However, a useful numerical method yields one result such that the value of V at some point (x, y) is the average of values around (x, y). If it is a circle, then the average value is at the centre.

The average value of a function, in any dimension is given by

$$\bar{f} = \int_A 1^{-1} \int_A f \tag{3.2}$$

Thus, for a circle, the average potential is

$$V_{\text{avg}} = V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl \tag{3.3}$$

By making this circle infinitesimally small, we get a valid result for the potential. We can setup a spreadsheet to perform this calculation.
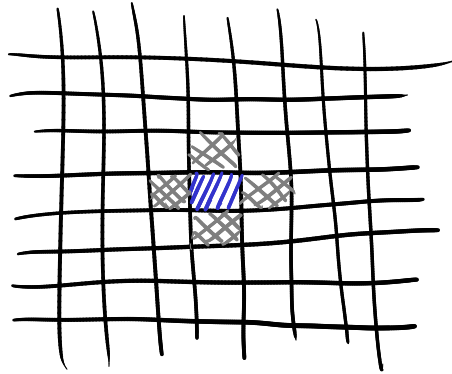

Figure 3.1: Suppose we want to find the potential for the box shaded blue, if we know the potential of its non-diagonal neighbours, then we can average those to find the blue box. We can do this via a relaxation method. If we iteratively find the potential of its neighbours given some boundary conditions, then we can determine the potential in the centre.

3.3 Laplace's equation III

We now focus on the 3D Laplace's equation. We found in the previous Section 3.2 that the potential at some point is just the average of the potential surrounding points.

We can justify this by considering the surrounding points as a sphere with a point charge q outside, and find the average potential over the whole sphere. Let's place our external point charge along the z-axis.

From our previous derivations, we can simply write the down the result. From the origin, our potential due to q is

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{z} \tag{3.4}$$

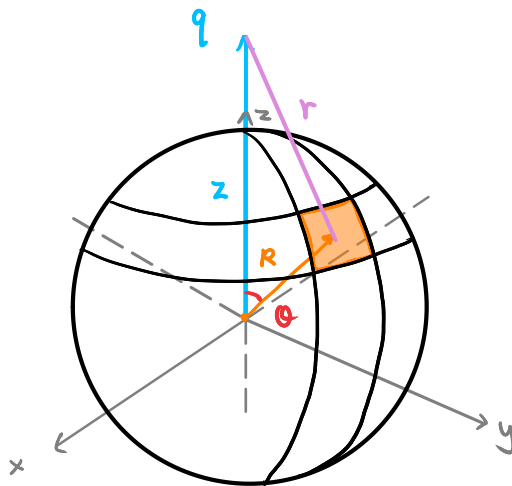Let's go along with the 'average' method and try to get to the same answer.

Figure 3.2: A 3D Laplace's equation set up with a point charge outside a sphere we are considering the potential at the surface shaded orange on the sphere.

Derivation 3.3.1. Immediately, we expect the potential at a point r_0 on the surface to be (imagine q is at the centre of some other sphere).

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r_0} \tag{3.5}$$

We can determine r_0 from the cosine rule.

$$r_0^2 = z^2 + R^2 - 2zR\cos\theta \tag{3.6}$$

From 3.2, we can add the third dimension to this to represent our surface integral as

$$V_{\text{avg}} = \frac{1}{4\pi R^2} \oiint_{\text{sphere}} V \, da \tag{3.7}$$

Remember, this is the average potential of the surface of a sphere of radius R away from the origin.

Do some substitution in spherical coordinates (varying (θ, ϕ) while keeping $r = r_0$), bearing in mind the Jacobian and we will get that the average potential is

$$V_{\text{avg}} = \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \left[z^2 + R^2 - 2zR\cos\theta\right]^{-\frac{1}{2}} R^2 \sin\theta \, d\theta \, d\phi \tag{3.8}$$

We can integrate with respect to ϕ first as there are no ϕ-dependent terms

$$V_{\text{avg}} = \frac{2\pi}{4\pi} \frac{q}{4\pi\epsilon_0} \int_0^\pi \left[z^2 + R^2 - 2zR\cos\theta\right]^{-\frac{1}{2}} \sin\theta \, d\theta \tag{3.9}$$

Then we use inverse chain rule and factorise the terms inside the square roots when evaluating to get to

$$V_{\text{avg}} = \frac{1}{2zR} \frac{q}{4\pi\epsilon_0} \sqrt{z^2 + R^2 - 2zR\cos\theta} \, \Big|_0^\pi = \frac{1}{2zR} \frac{q}{4\pi\epsilon_0} \left[(z+R) - (z-R)\right] \tag{3.10}$$

Finally, we have a nice expression

$$V_{\text{avg}} = \frac{1}{4\pi\epsilon_0} \frac{q}{z} \tag{3.11}$$

What we just calculated is the potential at the centre of the sphere as the average of the sphere of an arbitrary radius R due to a point charge q outside R. We can get this same result if we calculate directly from the external charge itself.

3.4 Uniqueness theorems

Here are the theorems without proofs.

Definition 3.4.1. First Uniqueness Theorem: The solution to Laplace's equation in some volume τ is uniquely determined if V is specified on the boundary surface S.

Definition 3.4.2. Second Uniqueness Theorem: In a volume τ surrounded by conductors and containing a specified charge density, ρ, the electric field is uniquely determined if the *total charge* is given.

Let's consider examples to demonstrate the theorems.

Example 3.4.1. Consider a surface S which encloses some volume with a defined potential V on that surface. The first uniqueness theorem tells us that there is a unique solution to Laplace's equation.

Now, consider 4 charges arranged in a way shown in Figure 3.3. Now we connect them. What happens? From the second uniqueness theorem, we can say that the volume enclosed by the conductors (wires) has a determined total charge (of 0), so we know there is a unique solution for the electric field.
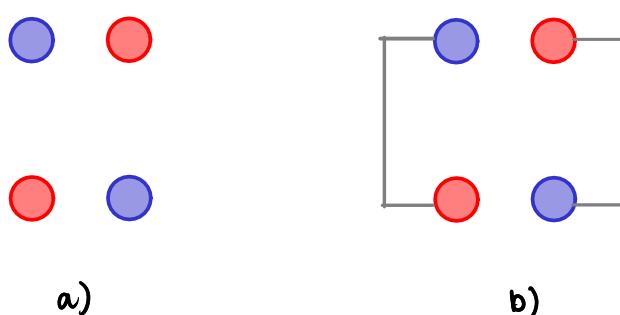

Figure 3.3: An example to demonstrate the second uniqueness theorem.

3.5 Finding the potential—the method of images

Let's consider the classic example: "What is the potential from a positive charge $+q$ sitting at a distance d above a flat and infinite grounded conductor"?

We have the following boundary conditions:

- $V = 0$ when $z = 0$ (level of the grounded conductor)
- $V \rightarrow 0$ a long way from the charge

We can use a trick to solve this problem without solving Laplace's equation. By symmetry, if we remove the conductor and replace it with a negative charge $-q$ at $-d$. At $z = 0$ (like a mirror image), the potential is still 0. From the first uniqueness theorem, we still have the same boundary potential (an invisible conductor), and thus the solution for the potential above the plane will be the same in both cases.

The solution is therefore

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left[\frac{+q}{\sqrt{r^2 + (z-d)^2}} + \frac{-q}{r^2 + (z+d)^2} \right], \; z \geq 0 \tag{3.12}$$

where $r^2 = x^2 + y^2$. This means by using a mirror image of our charge, we can determined the potential strictly above the plane.

What about the surface charge, $\sigma(r)$? Recall in Section 2.16, the surface charge distribution is related by the Equation 2.29, except we only care about what's above the sheet. The direction normal to the surface is $\hat{k}$, thus

$$\sigma = -\left.\frac{\partial V}{\partial z}\right|_{z=0} \tag{3.13}$$

We then evaluate the partial derivative in Equation 3.12, which gives us

$$\sigma(R) = -\frac{qd}{2\pi(r^2 + d^2)^{\frac{3}{2}}} \tag{3.14}$$

From this we can find out the *total* induced charge Q. Using the relation 2.6, we integrate in spherical coordinates (in the x-y plane, this means keeping *theta* constant) to find

$$Q = \int_0^{2\pi} \int_0^\infty -\frac{qd}{2\pi(r^2 + d^2)^{\frac{3}{2}}} r \, dr \, d\phi = -\left.\frac{qd}{2\pi\sqrt{r^2 + d^2}}\right|_0^\infty = -q \tag{3.15}$$

How about the forces and energy in the system? First the force between the two charges is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{2d^2} \hat{z} \tag{3.16}$$

Multiple by $2d$ to get the energy. But this is actually not correct. We have to halve the volume in the actual system because we are only considering $z \geq 0$, thus

$$W = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d} \tag{3.17}$$

We can also get to this result by finding the work needed to bring the charge in from infinity.

$$W = -\int_\infty^d \mathbf{F} \, d\mathbf{l} = -\frac{1}{4\pi\epsilon_0} \int_\infty^d \frac{q^2}{4z^2} \, dz \tag{3.18}$$