Electromagnetic Notes

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based on lectures from PHYS20029
and Griffiths' *Introduction to Electrodynamics*

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Contents

1 Introduction

1.1 Recap of first year Electromagnetism

What we covered in first year:

- *Gauss' Law*: Integral over enclosed surface containing an electric field gives the total charge over that surface.
$$\iint_S \boldsymbol{E} \, \mathrm{d}\boldsymbol{S} = \frac{Q}{\epsilon_0} \tag{1.1}$$

- *Ampere's law:* Path of a magnetic field around a line integral is proportional to the current.
$$\oint_P \boldsymbol{B} \, \mathrm{d}\boldsymbol{l} = \mu_0 I \tag{1.2}$$

- *Biot-Savart law:* Magnetic field arising from a small current containing element in the wire. Equivalent in magnetism to *Coulomb's law* in electrostatic.

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\boldsymbol{l} \times (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \tag{1.3}$$

This course will be concerned with deriving and using the *differential forms* of these integral equations. We will eventually arrive at Maxwell's equations. We will also consider two new fields $\boldsymbol{D}$ and $\boldsymbol{H}$.

Note: Be aware of c.g.s system that changes the formulae as well as the units

2 Electrostatics

2.1 Electrostatics—what you know so far

Definition 2.1.1. Electronic charge is a property that is associated with the fundamental particles, protons (quarks), electrons etc. that occur in nature.

The Coulomb charge is the smallest free charge observed (fractional charges of quarks are smaller but isolated quarks do not appear in nature).

To properly consider the electromagnetic behaviour, we need quantum theory in atomic length scales. E.g., the quantum description of the hydrogen atom is the application of the coulomb potential in Schrödinger's equation. We are only learning classical electrodynamics.

2.2 Coulomb's Law

Definition 2.2.1. The force between two charged particles in S.I. units is

$$\boldsymbol{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} \tag{2.1}$$

The total force on a test charge Q is the sum of the forces from the other charges in the system. This is the *superposition principle*. I.e., the field from one particle does not change the effect from any other charges in the system.

Definition 2.2.2. Superposition of electric forces applies, such that

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \dots \tag{2.2}$$

The electric field $\boldsymbol{E}$ at point r is the force per unit charge exerted on a test charge, such that

$$\boldsymbol{F} = Q_{test}\boldsymbol{E} \tag{2.3}$$

From this, we can deduce that the total electric field is the superposition of electric fields of all charges in the system

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \dots \tag{2.4}$$

Note that superposition in this case is not a logical necessity but an experimental fact; if the force is proportional to the square of the charges, then this would not work.

We might ask—what is an electric field? We come to it through an intermediary step in calculating forces, thus we can define it as that. Otherwise, we can treat it as abstract or physical, it does not affect how these particles behave.

2.3 Total charge

Definition 2.3.1. The total charge in a system of discrete (point charges) is

$$Q = \sum_i q_i \tag{2.5}$$

For continuous charge distributions, the sum becomes an integral, and we consider instead the *charge densities*. For each dimension, the charge densities are:

System	Unit charge relation
Line charge	$\lambda\ \mathrm{d}l = \mathrm{d}q$
Surface charge	$\sigma\ \mathrm{d}a' = \mathrm{d}q$
Volume charge	$\rho\ \mathrm{d}\tau' = \mathrm{d}q$

Definition 2.3.2. The total charge in system of continuous charge with a charge density is

$$Q = \int_{body} \mathrm{d}q \tag{2.6}$$

Greek symbols have been used because V is used for potentials.

2.4 Charge densities and fields

Knowing the charge densities and the total charge, we can write Coulomb's law for electric fields. As an example, with 3D charge density, we start with the unit electric field, in differential form,

$$d\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 \boldsymbol{r}^2}\rho \, d\tau \tag{2.7}$$

Integrating both sides will give us the resulting electric field.

Definition 2.4.1. Coulomb's law for a continuous charge distribution is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{r^2}\hat{\boldsymbol{r}} \, d\tau \tag{2.8}$$

Example 2.4.1. The examples asked us to

1. Derive electric field at a vertical distance above a line of charge.
2. Derive electric field at a vertical distance above a circular loop and thus electric field from a flat circular disk.

Some takeaways:

If we are asked to find the electric field of a surface or volume:

1. Split into smaller dimension, *unit length* for a surface, *unit surface* for a volume.
2. Find the electric field of the smaller dimension shape.
3. Express the unit charge in terms of charge density.
4. Integrate over the original shape's limits.

2.5 Gauss' Law

Definition 2.5.1. For any volume or surface that encloses a charge Q then

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\epsilon_0}Q \tag{2.9}$$

If we have high symmetry in the charge distribution, we can integrate over a symmetrical surface to find the electric field. For more complicated situations, Gauss' law *and* the superposition principle if there are still underlying symmetries to be exploited.

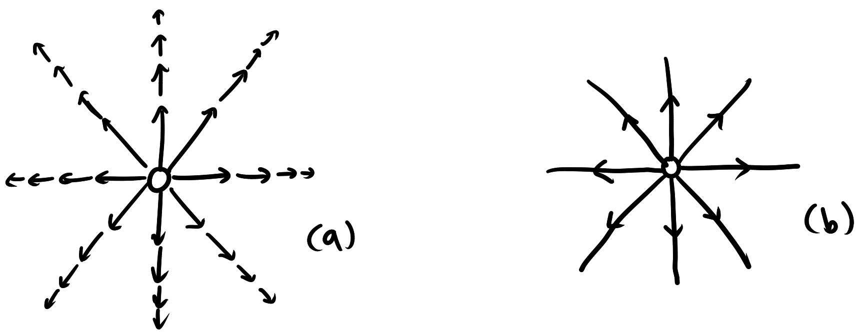


Figure 2.1: (a) shows vector field plot while (b) shows field lines method.

Charge distributions that are a superposition of any Gaussian distributions asks us to use the superposition principle to evaluate the integral.

Example 2.5.1. The examples asked us to

1. Find the flux through faces of a cube of a charge at the corner of a cube. We can resize the cube to centre the charge so that we can use Gauss' law from symmetry
2. Find the field of a uniformly charged solid sphere
3. Find electric field well inside a long cylinder with charge density that varies by perpendicular distance from principle axis.
4. Find the electric field from an infinite sheet with surface charge.
5. Find the field of two infinite sheets with surface charge opposite each other (field on either sides and in the middle)

The common strategy seems to be finding symmetry or finding a smaller part of the original geometry and integrate. Draw some field lines to get the feel for the geometry of the problem.

2.6 Drawing fields

The field from a positive charge always point outwards and the magnitude decreases as $1/r^2$. Field lines are represented as arrows that give its direction and whose lengths give its magnitude. Alternative, we can connect the neighbouring arrows and the density of the field lines can represent the strength of the field instead of the length.

Aside: Plotting field lines via Python utilises `matplotlib`.

2.7 Differential form of Gauss' Law

We can re-write Gauss' law by applying the divergence theorem.

Theorem 2.7.1 *The divergence theorem for a vector field* $\boldsymbol{X}$ *is*

$$\oint_S \boldsymbol{X} \cdot \mathrm{d}\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{X} \, \mathrm{d}V \tag{2.10}$$

For Gauss' Law we can derive from the enclosed charge density in a volume V from equation **??** a new identity. Thus we discover that for an enclosed charge,

$$\frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \int_\tau \rho(\boldsymbol{r}) \, \mathrm{d}\tau \tag{2.11}$$

Combine with the divergence theorem from 2.7.1, we find that

$$\int_\tau \boldsymbol{\nabla} \cdot \boldsymbol{E} \, \mathrm{d}\tau = \int_\tau \frac{\rho(\boldsymbol{r})}{\epsilon_0} \, \mathrm{d}\tau \tag{2.12}$$

Definition 2.7.1. We arrive at the differential form by differentiating both sides with respect to τ

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0} \tag{2.13}$$

This is the first of Maxwell's equation.

2.8 The curl of the electric field

Consider the electric field for a static charge q, in spherical coordinates, the length differential is

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r \, \hat{\boldsymbol{r}} + r \, \mathrm{d}\theta \, \hat{\boldsymbol{\theta}} + r \sin\theta \, \mathrm{d}\phi \, \hat{\boldsymbol{\phi}} \tag{2.14}$$

Since our system is a singular charge with spherical symmetry, the angular differential terms disappear. Thus, we are left with

$$\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \frac{q}{4\pi\epsilon_0 r^2}\,\mathrm{d}r \tag{2.15}$$

Integrating both sides leaves us with

$$\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \frac{q}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}\,\mathrm{d}r = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{2.16}$$

For a closed loop, the integral evaluates to 0.

Corollary 2.8.1

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0 \tag{2.17}$$

In other words, $\boldsymbol{E}$ is a conservative field. Using Stoke's theorem, we can conclude that

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \tag{2.18}$$

Note: This result only applies for electrostatic fields and not when there are time varying magnetic fields.

2.9 Electrostatic potential

From 2.8 we find that for an electrostatic field, it is conservative. The following applies to any conservative field

Definition 2.9.1. The electrostatic potential as a function of a position vector is the negative of the integral of the electrostatic field along some path from a starting point to another point. Since it is conservative, it is path-independent, hence the difference in potential at two points is the integral evaluated from one point to the other.

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} \tag{2.19}$$

As a result, we have the following identity for the electric potential:

$$\boldsymbol{E} = -\boldsymbol{\nabla} V \tag{2.20}$$

Example 2.9.1. Find the potential inside and outside a uniformly charged spherical shell of radius R. Very similar steps to 3.3.1.

Inside, it is uniform and depends on the radius of the sphere, while outside, it depends on the position of our test charge.

We get to

$$V(z) = \frac{2\pi R\sigma}{2\epsilon_0 z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right] \tag{2.21}$$

and must consider that the second square root term is $z - R$ for test charge outside and $R - z$ for test charge inside.

Thus, we have

$$V(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r}, & r \geq R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{R}, & r \leq R \end{cases} \tag{2.22}$$

2.10 Notes on the Scalar potential

- Potential is different from potential energy.
- Finding potentials is easier than vector fields.
- Finding vector fields using the relation 2.20.
- There is not an absolute definition of potential—we only observe potential *differences*. Thus, 'zero' potential is arbitrarily defined, usually at infinity. This is for convenience rather than mathematical necessity.
- Potentials also follow the superposition principle.

2.11 Poisson's equation

Substitute 2.20 into the curl of electric field, and we get Poisson's equation.

Definition 2.11.1. Poisson's equation is

$$\nabla^2 V = -\frac{\rho(\boldsymbol{r})}{\epsilon_0} \tag{2.23}$$

2.12 Laplace's equation I

Definition 2.12.1. In the absence of any charges, $\rho(\boldsymbol{r}) = 0$. Thus, Laplace's equation is

$$\boldsymbol{\nabla}^2 V = 0 \tag{2.24}$$

2.13 Point charge

A point charge has spherical symmetry. If we choose our zero potential at infinity and evaluate the integral then we can evaluate the integral to find the electrostatic potential.

Definition 2.13.1. The electrostatic potential of a point charge is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{2.25}$$

Note that point charges do not actually exist, they are there for easier calculations.

2.14 Electrostatic potential for a continuous distribution

For a continuous charge distribution, we can use the superposition principle and sum over all point charges.

Definition 2.14.1. The electrostatic potential for a continuous distribution is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\boldsymbol{r}')}{r''}\,\mathrm{d}\tau' \tag{2.26}$$

For surface and line charge densities, it is the same form, except we are integrating over a surface or a path.

2.15 Summary of equations

Relating V and ρ: Equation 2.23 and equation 2.26.

Relating ρ and $\boldsymbol{E}$: Equation 2.8, equation 2.18 and equation 2.13.

Relating $\boldsymbol{E}$ and V: Equation 2.20 and equation 2.19.

2.16 Boundary conditions

For a thin box passing through a surface charge, as we pass from below to above the surface, there is a discontinuous change in the electric field if the box is infinitesimally thin. This result is from first year. Since we only care about normal components, because for tangential components to the surface (i.e. sides of the box), this forms a closed loop and the integral is thus 0. We find that the normal component has a discontinuity, so that

$$\boldsymbol{E}^{\perp}_{\text{above}} - \boldsymbol{E}^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} \tag{2.27}$$

We can integrate this to find the potential above and below. As the integral path length tends to zero, the integral tends to zero, so that the potential above is equal to the potential below.

$$V_{\text{above}} = V_{\text{below}} \tag{2.28}$$

We can also arrange 2.27 to get

Definition 2.16.1. The boundary conditions we have for the electrostatic field due to surface charges

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0} \tag{2.29}$$

Where the partial derivatives represent the normal derivative $\boldsymbol{\nabla}\hat{n}$.

2.17 Work done to move a charge

Definition 2.17.1. The work needed to move a charge in an electric field is

$$W = -Q\int_a^b \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} = Q\left[V(\boldsymbol{b}) - V(\boldsymbol{a})\right] = Q\Delta V \tag{2.30}$$

2.18 The energy of a distribution of point charges

To move one charge to the other, separated by a distance of infinity, to a finite distance r, the work done is

$$W_2 = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{r_{12}} \tag{2.31}$$

The superposition principle is used to find the total work done in moving two or more charges together. It is the sum of the work done to bring each individual pair together.

$$W = \sum_{i=2}^{N} W_i \tag{2.32}$$

To avoid double counting the pairs, we halve the final sum, which yields

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j\neq i}^{N} \frac{q_i q_j}{r_{ij}} \tag{2.33}$$

We can rearrange this equation further using 2.30.

Definition 2.18.1. The work done in moving all these charges from infinity to a finite distance is

$$W = \frac{1}{2} \sum_{i=1}^{N} q_i V(\boldsymbol{r}_i) \tag{2.34}$$

2.19 The energy of a continuous charge distribution

Once again, we can integrate to convert from discrete charges to continuous charges. We can substitute in 2.13 then integrate by parts to get a nicer expression.

Derivation 2.19.1.

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \int (\boldsymbol{\nabla} \cdot \boldsymbol{E})\, V\, \mathrm{d}\tau \\ &= \frac{\epsilon_0}{2} \left[-\int \boldsymbol{E} \cdot \boldsymbol{\nabla} V\, \mathrm{d}\tau + \oint V \boldsymbol{E}\, \mathrm{d}\boldsymbol{a} \right] \\ &= \frac{\epsilon_0}{2} \left[-\int \boldsymbol{E}^2\, \mathrm{d}\tau + \oint V \boldsymbol{E}\, \mathrm{d}\boldsymbol{a} \right] \end{aligned} \tag{2.35}$$

This is the *correct* equation, but in theory if you integrate over bigger volumes (as long as it encloses the charge), the contribution from the volume will overtake the contribution from the surface. The first term is the contribution from the volume, while the second is the surface. Since outside our standard sphere (let's say we integrate over all space), $\rho = 0$, our result for W must be the same. But the volume integral grows as E^2 is positive, so the surface integral must decrease. In fact, it decreases by $1/r$. Thus, if we integrate over all space, we are left with the first term only.

Definition 2.19.1. The work done on continuous charge distribution is

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, \mathrm{d}\tau \tag{2.36}$$

This gives us a different idea to think about energy in electrostatics. If we have a series of charges (discrete or continuous). Rather than thinking in terms of the work done by bringing these charges together, we can think of the work as the integral of the electric field over all space. One could say the electric field 'stores' energy.

It is **important** to understand that the energy of the electric field does not obey the superposition principle. If we have two charges and bring them together, the total energy is not the energy in one field plus the energy in the other. This is because it is not a linear relationship.

Example 2.19.1. Find the electrostatic energy of a uniformly charged spherical shell of radius R and total charge q.

Note that we can use either use

$$W = \frac{1}{2} \int \sigma V \, \mathrm{d}a \tag{2.37}$$

or

$$W = \frac{1}{2} \int E^2 \, \mathrm{d}\tau \tag{2.38}$$

to get to the same result.

2.20 The properties of conductors

Definition 2.20.1. Inside a conductor, there are several important properties to remember

- $\boldsymbol{E} = 0$ inside a conductor. If there is a deficit of charge, then it will move until equilibrium is reached.
- $\rho = 0$ inside a conductor. We can have charges inside a conductor, but they must cancel.
- Charges in a conductor are **mobile**. This means they can move freely.
- Any net charge resides on the **surface**.

- A surface of a conductor is an **equipotential**, otherwise charges would move freely until potential equilibrium is reached.
- $\boldsymbol{E}$ is perpendicular to the surface at the surface. Tangential components would mean charges can move along the surface, this is not the case for an equipotential.

2.21 Induced charges

If a positively charged particle is brought closed to a conductor, its field will attract negative charges towards it. These negative charges are closer to the positive particle and thus creates a net attraction. The charges in the conductor move around the surface to cancel the field in the conductor so that at equilibrium there is no $\boldsymbol{E}$ field in the conductor.

The result is a redistribution of charge on the surface of the conductor.

Example 2.21.1. An uncharged spherical conductor has a cavity of arbitrary shape with a charge $+q$ placed somewhere in it. What is the electric field outside of the conductor?

Outside the conductor, using Gauss' Law, for a spherical conductor, it is

$$\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}} \tag{2.39}$$

Does a strangely shaped cavity affect the distribution of the charge distributed on the surface? The arrangement of negative charges in the inside surface of the cvity that is attracted to the cavity cancels out the asymmetry that is produced by the shape of the cavity. The net effect is that the shape of the cavity does not matter. We can simply argue that we have a uniform surface charge. See 3.4 for relevant theorems.

2.22 Surface charge and the force on a conductor

If we have a *static* field on a conductor, then the boundary conditions for the charge at the surface follow the result from 2.16. We have a discontinuity in the field, and immediately outside the conductor, it is given by 2.27.

We can rearrange for the charge density and express it in terms of the potential

$$\sigma = -\epsilon_0\frac{\partial V}{\partial n} \tag{2.40}$$

Using the expression for force in 2.3, we can express the force *per unit area* as

$$\boldsymbol{f} = \sigma\boldsymbol{E}_{\text{avg}} \tag{2.41}$$

Because there is a discontinuity, we take the **average** of the electric field above and below (or inside and outside).

For a conductor, this is half of 2.41 since the field is 0 inside and outside, it is given by 2.27.

Definition 2.22.1. The force per unit area for a static field on a conductor is

$$\boldsymbol{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\boldsymbol{n}} \tag{2.42}$$

or as a pressure, it is

$$P = \frac{\epsilon_0}{2}E^2 \tag{2.43}$$

2.23 Capacitors

A capacitor is a device with two conductors separated in space.

Suppose there is a charge $+Q$ and $-Q$ on each conductor respectively. Each conductor is an equipotential, thus we can consider the *potential difference* between them as the integral of the electric field from the negative to the positive conductor.

Since electric field is proportional to the charge 2.3, or more general as 2.8 we can substitute this into the integral, so the potential difference is proportional to the charge

We call this constant of proportionality the *capacitance*.

Definition 2.23.1. The relation between the potential difference and charge is

$$C = \frac{Q}{V} \tag{2.44}$$

Example 2.23.1. Find the capacitance of a parallel plate capacitor made up of two metal sheets of area A separated by a distance d.

Recall that for a linear system like this,

$$E = \frac{V}{d} \tag{2.45}$$

After some algebra, we are left with

$$C = \frac{\epsilon_0 A}{d} \tag{2.46}$$

Example 2.23.2. Find the capacitance of two concentric spherical metallic shells with radii a and b, where $a < b$.

We can use Gauss' Law to determine the electric field for various radii. It is non-zero for $a \leq r \leq b$. As usual, we can determine the potential through integration.

Then we go one step further and establish the capacitance using the non-zero potential between the two shells. This turns out to be

$$C = \frac{4\pi\epsilon_0 ab}{(b-a)} \tag{2.47}$$

2.24 Energy stored in a capacitor

We can ask: how much work is needed to "charge" a capacitor? This is equivalent to moving charges from the positive plate to the negative plate.

We need to work *harder* as we transfer more charge from one plate to the other.

If a plate already as charge q on it, with potential difference between plates $V = q/C$, then from 2.30, the work needed to add a charge of $\mathrm{d}q$ is

$$\mathrm{d}W = V\,\mathrm{d}q = \frac{q}{C}\,\mathrm{d}q \tag{2.48}$$

Integrate this equation to get the *total work done* or *total energy stored*. Further, we can use our derived expressions to rearrange.

Definition 2.24.1. The energy stored in a capacitor or work done to charge a capacitor is

$$W = \frac{1}{2}CV^2 \tag{2.49}$$

If we now connect our capacitors to some wires then the capacitor will discharge because of the potential difference between the two plates. Thus, the capacitor is said to be storing energy. In fact, if the space between the plates is air then it would require really big capacitors for a substantial amount of energy. Dielectrics would be a better material.

3 Potentials

3.1 Introduction

To find the electric field or potential from charge densities (as shown in 2.15) is quite difficult as not all of them have simple analytic solutions.

For conductors, we also do not know where the charges are in advance, since the electric field inside is 0.

In most cases, it is better to start from Poisson's equation in 2.11 in regions with charge, and Laplace's equation in 2.12 in regions with no charge.

3.2 Laplace's equation II

3.2.1 3D Laplacian

If we look at the 3D Laplace's equation in full:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{3.1}$$

We see that we have some expressions of requirements. Normally, the second derivative will tell us whether there's a maximum or minimum (positive or negative value). Because this is equal to 0, we cannot have a minimum for all three directions, otherwise, it would not be equal to 0.

Relevantly, we have Earnshaw's theorem, which maintains that a collection of point charges cannot be maintained in a stable stationary equilibrium (impossible to trap a particle in a 3D electric field). If we can trap a particle, then it would mean all points must be a minimum. To follow Laplace's equation, we can only have unstable points (no local maxima or minima)

3.2.2 1D Laplacian

In 1D, it is a relatively straightforward, the first derivate is just a constant, and the function itself is a linear function.

If we take any point on that straight line, and look at two neighbouring points, then the potential is just the average potential of the two surround points.

3.2.3 2D Laplacian

In 2D, like in 3D, it's not possible to write a general solution, since it is not an ODE. However, a useful numerical method yields one result such that the value of V at some point (x, y) is the average of values around (x, y). If it is a circle, then the average value is at the centre.

The average value of a function, in any dimension is given by

$$\bar{f} = \int_A 1^{-1} \int_A f \tag{3.2}$$

Thus, for a circle, the average potential is

$$V_{\text{avg}} = V(x, y) = \frac{1}{2\pi R} \oint\limits_{\text{circle}} V \, \mathrm{d}l \tag{3.3}$$

By making this circle infinitesimally small, we get a valid result for the potential. We can setup a spreadsheet to perform this calculation.

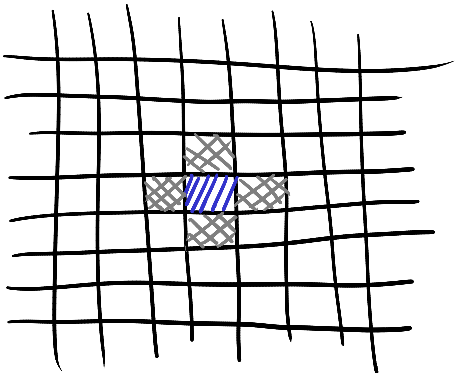


Figure 3.1: Suppose we want to find the potential for the box shaded blue, if we know the potential of its non-diagonal neighbours, then we can average those to find the blue box. We can do this via a relaxation method. If we iteratively find the potential of its neighbours given some boundary conditions, then we can determine the potential in the centre.

3.3 Laplace's equation III

We now focus on the 3D Laplace's equation. We found in the previous Section 3.2 that the potential at some point is just the average of the potential surrounding points.

We can justify this by considering the surrounding points as a sphere with a point charge q outside, and find the average potential over the whole sphere. Let's place our external point charge along the z-axis.

From our previous derivations, we can simply write the down the result. From the origin, our potential due to q is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{z} \tag{3.4}$$

Let's go along with the 'average' method and try to get to the same answer.

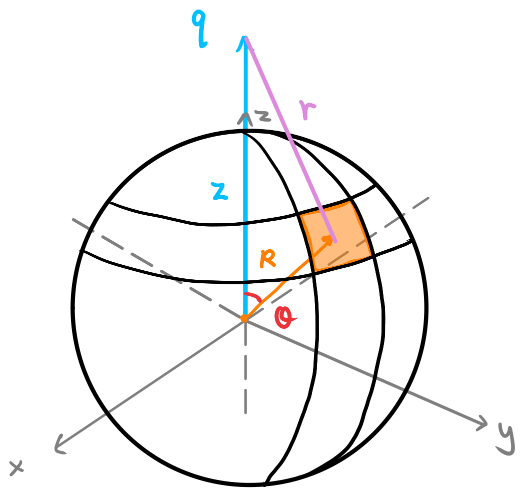


Figure 3.2: A 3D Laplace's equation set up with a point charge outside a sphere we are considering the potential at the surface shaded orange on the sphere.

Derivation 3.3.1. Immediately, we expect the potential at a point r_0 on the surface to be (imagine q is at the centre of some other sphere).

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r_0} \tag{3.5}$$

We can determine r_0 from the cosine rule.

$$r_0^2 = z^2 + R^2 - 2zR\cos\theta \tag{3.6}$$

From 3.2, we can add the third dimension to this to represent our surface integral as

$$V_{\text{avg}} = \frac{1}{4\pi R^2}\oiint_{\text{sphere}} V\,\mathrm{d}a \tag{3.7}$$

Remember, this is the average potential of the surface of a sphere of radius R away from the origin.

Do some substitution in spherical coordinates (varying (θ, ϕ) while keeping $r = r_0$), bearing in mind the Jacobian and we will get that the average potential is

$$V_{\text{avg}} = \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \left[z^2 + R^2 - 2zR\cos\theta\right]^{-\frac{1}{2}} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{3.8}$$

We can integrate with respect to ϕ first as there are no ϕ-dependent terms

$$V_{\text{avg}} = \frac{2\pi}{4\pi} \frac{q}{4\pi\epsilon_0} \int_0^{\pi} \left[z^2 + R^2 - 2zR\cos\theta\right]^{-\frac{1}{2}} \sin\theta \, \mathrm{d}\theta \tag{3.9}$$

Then we use inverse chain rule and factorise the terms inside the square roots when evaluating to get to

$$V_{\text{avg}} = \frac{1}{2zR} \frac{q}{4\pi\epsilon_0} \sqrt{z^2 + R^2 - 2zR\cos\theta} \,\bigg|_0^{\pi} = \frac{1}{2zR} \frac{q}{4\pi\epsilon_0} \left[(z + R) - (z - R)\right] \tag{3.10}$$

Finally, we have a nice expression

$$V_{\text{avg}} = \frac{1}{4\pi\epsilon_0} \frac{q}{z} \tag{3.11}$$

What we just calculated is the potential at the centre of the sphere as the average of the sphere of an arbitrary radius R due to a point charge q outside R. We can get this same result if we calculate directly from the external charge itself.

3.4 Uniqueness theorems

Here are the theorems without proofs.

Definition 3.4.1. First Uniqueness Theorem: The solution to Laplace's equation in some volume τ is uniquely determined if V is specified on the boundary surface S.

Definition 3.4.2. Second Uniqueness Theorem: In a volume τ surrounded by conductors and containing a specified charge density, ρ, the electric field is uniquely determined if the *total charge* is given.

Let's consider examples to demonstrate the theorems.

Example 3.4.1. Consider a surface S which encloses some volume with a defined potential V on that surface. The first uniqueness theorem tells us that there is a unique solution to Laplace's equation.

Example 3.4.2. Now, consider 4 charges arranged in a way shown in Figure 3.3. Now we connect them. What happens? From the second uniqueness theorem, we can say that the volume enclosed by the conductors (wires) has a determined total charge (of 0), so we know there is a unique solution for the electric field.

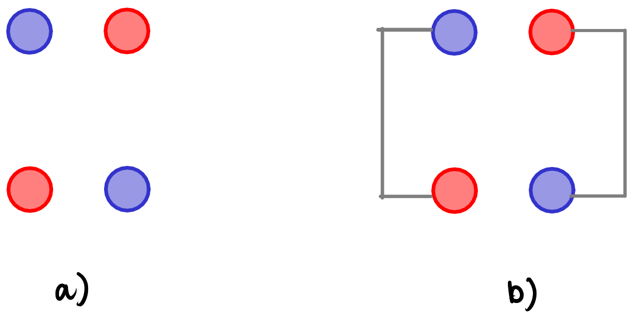


Figure 3.3: An example to demonstrate the second uniqueness theorem.

3.5 Finding the potential—the method of images

Let's consider the classic example: "What is the potential from a positive charge $+q$ sitting at a distance d above a flat and infinite grounded conductor"?

We have the following boundary conditions:

- $V = 0$ when $z = 0$ (level of the grounded conductor)
- $V \to 0$ a long way from the charge

We can use a trick to solve this problem without solving Laplace's equation. By symmetry, if we remove the conductor and replace it with a negative charge $-q$ at $-d$. At $z = 0$ (like a mirror image), the potential is still 0. From the first uniqueness theorem, we still have the same boundary potential (an invisible conductor), and thus the solution for the potential above the plane will be the same in both cases.

The solution is therefore

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{+q}{\sqrt{r^2 + (z-d)^2}} + \frac{-q}{r^2 + (z+d)^2}\right], \; z \geq 0 \tag{3.12}$$

where $r^2 = x^2 + y^2$. This means by using a mirror image of our charge, we can determined the potential strictly above the plane.

What about the surface charge, $\sigma(r)$? Recall in Section 2.16, the surface charge distribution is related by the Equation 2.29, except we only care about what's above the sheet. The direction normal to the surface is $\hat{k}$, thus

$$\sigma = -\frac{\partial V}{\partial z}\bigg|_{z=0} \tag{3.13}$$

We then evaluate the partial derivative in Equation 3.12, which gives us

$$\sigma(R) = -\frac{qd}{2\pi(r^2 + d^2)^{\frac{3}{2}}} \tag{3.14}$$

From this we can find out the *total* induced charge Q. Using the relation 2.6, we integrate in spherical coordinates (in the x-y plane, this means keeping $theta$ constant) to find

$$Q = \int_0^{2\pi}\int_0^{\infty} -\frac{qd}{2\pi(r^2+d^2)^{\frac{3}{2}}} r \,\mathrm{d}r \,\mathrm{d}\phi = -\frac{qd}{2\pi\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q \tag{3.15}$$

How about the forces and energy in the system? First the force between the two charges is

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\boldsymbol{z}} \tag{3.16}$$

Multiply by $2d$ to get the energy. But this is actually not correct. We have to halve the volume in the actual system because we are only considering $z \geq 0$, thus

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d} \tag{3.17}$$

We can also get to this result by finding the work needed to bring the charge in from infinity.

$$W = -\int_{\infty}^{d} \boldsymbol{F} \,\mathrm{d}\boldsymbol{l} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}\,\mathrm{d}z \tag{3.18}$$

3.6 Finding potentials—separation of variables

Note: There will not be any examinable questions in solving PDEs. But we must be comfortable with techniques such as separation of variables. We use this to solve Laplace's equation. Furthermore, we must be able to do this in spherical and cylindrical coordinate systems.

3.7 Multipole expansions—potentials at large distances

If we go far enough away from any charge distribution with a total net charge Q, the potential will always tend to that of a single point charge, i.e.

$$V_{\text{net charge Q}}(r)_{r\to\infty} = \frac{Q}{4\pi\epsilon_0} \to 0 \tag{3.19}$$

But how do we expect the potential to look like at large $\boldsymbol{r}$? For a distribution of charges we can image a hierarchy of structures, starting from a single point charge.

- **Monopole:** Potential tends to $1/r$.
- **Dipole:** Potential tends to $1/r^2$.
- **Quadrupole:** Potential tends to $1/r^3$.
- **Octopole:** Potential tends to $1/r^4$.

Note: We can't reduce a dipole to a monopole, a quadrupole to a dipole, octopole to a quadruple, etc. However, we can imagine representing the potential of any charge distribution as an expansion (sum or superposition) of these multipoles.

Clearly, if we have what appears as a single charge at large distances, the field looks like a point charge, if we have separation of net charge then it will look like the dipole at large distance. If our charge distribution cannot be reduced to a monopole or dipole distribution then the next likely is quadrupolar, etc.

3.7.1 Multipole expansion (Not examinable)

The multipole expansion of the field from a charge distribution is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=r_0}^{\infty}\frac{1}{r^{n+1}}\int_V (r')^n P_n(\cos\alpha)\,\rho(\boldsymbol{r}')\,\mathrm{d}\tau' \tag{3.20}$$

α is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$ and P_n are the Legendre polynomials. The further we go away, the more we can use only the first terms in the series.

3.8 The dipole potential

For two separated charges we can write the potential at some point **P** at $\boldsymbol{r}$ as

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) \tag{3.21}$$

where r_+ and r_- are the distances from the charge to the point **P**, see Figure 3.4. But these two are not centred at the origin of the system. The distance can be calculated from cosine rule.

$$r_\pm^2 = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right) \tag{3.22}$$

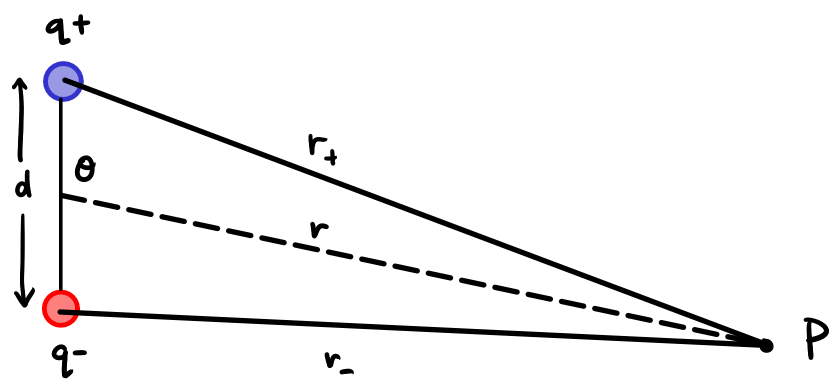


Figure 3.4: The distances in a dipole system

At large r the third term is negligible, reducing this to

$$\frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}} \tag{3.23}$$

Using the binomial expansion, we have

$$\frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \tag{3.24}$$

If we write this expansion for r_+ and r_- separately, and subtract one from the other.

$$\frac{1}{r_+} - \frac{1}{r_-} \simeq \frac{d}{r^2}\cos\theta \tag{3.25}$$

The potential of a dipole measured from its centre for large r is

$$V(\boldsymbol{r}) \simeq \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2} \tag{3.26}$$

where θ is the relative orientation of the dipole to the position vector of $\mathbf{P}$.

Since this seems to imply that we have to work in spherical or polar coordinates, we can avoid this by using the dipole moment, measured as a vector $\boldsymbol{d}$ from one end to the other.

$$\boldsymbol{p} = q\boldsymbol{d} \tag{3.27}$$

Definition 3.8.1. At large r, the potential of a dipole measured from its centre, using the dipole moment is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} \tag{3.28}$$

Note: This is only the approximate potential of the physical dipole we described. If we get too close to the separated charges, this equation no longer applies. We would have to use multipole expansion to get a better approximation.

When we solve problems, it is sensible to align $\boldsymbol{p}$ with an axis to simply the mathematics needed to carry out the calculation.

3.9 The electric field from a dipole

The electric field is the gradient of the potential. We have calculated the potential in Equation 3.28. We need to work out the gradient in spherical coordinates.

Definition 3.9.1. The electric field of a dipole, aligned along the z-axis is

$$\boldsymbol{E}_{\text{dipole}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) \tag{3.29}$$

The coordinate-free version, which does not depend on the orientation of the system, is

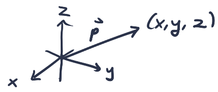
$$\boldsymbol{E}_{\text{dipole}}(r) = \frac{p}{4\pi\epsilon_0 r^3}\left(3\left(\boldsymbol{p}\cdot\hat{\boldsymbol{r}}\right)\hat{\boldsymbol{r}} - \boldsymbol{p}\right) \tag{3.30}$$

See Figure 3.5 for a derivation.

We can see that if the dipole is aligned with the z-axis, the terms from other axis disappears and is what we expect.

Show that the coordinate-free version of $\vec{E}_{dipole}$ is

$$\vec{E}_{dipole} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$



* Start with the general potential:

$$\boxed{V_{dipole} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}}$$

with $\vec{p}(x,yz) = p_x\cdot\hat{x} + p_y\cdot\hat{y} + p_z\cdot\hat{z}$

$$\vec{p} = p\begin{pmatrix} \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{pmatrix}$$

* The field is the gradient

$$\vec{E} = -\vec{\nabla}V$$

* The gradient in spherical coordinates is:

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}$$

* Find $\vec{p}\cdot\hat{r}$ and expand

$$\vec{p}\cdot\hat{r} = (p_x\hat{x} + p_y\hat{y} + p_z\hat{z})\,\hat{r} = \sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta$$

$$\Rightarrow\quad V_{dipole} = \frac{p}{4\pi\varepsilon_0 r^2}\left(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta\right)$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p}{4\pi\varepsilon_0 r^3}\left(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta\right) + \frac{p}{4\pi\varepsilon_0 r^3}\left(\sin\theta\cos\phi - \sin\theta\cos\phi + \sin\theta\sin\phi - \sin\theta\sin\phi + \cos\theta - \cos\theta\right)$$

$$E_\theta = \frac{1}{r}\frac{\partial V}{\partial\theta} = -\frac{p}{4\pi\varepsilon_0 r^3}\left(\cos\theta\cos\phi + \cos\theta\sin\phi - \sin\theta\right) = \frac{p}{4\pi\varepsilon_0 r^3}\left(\sin\theta - \cos\theta\cos\phi - \cos\theta\sin\phi\right)$$

$$E_\phi = \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = -\frac{p}{4\pi\varepsilon_0 r^3}\left(-\sin\phi + \cos\phi\right) = \frac{p}{4\pi\varepsilon_0 r^3}\left(\sin\phi - \cos\phi\right)$$

→ See crossed out or appended terms

$$\vec{E}_{dipole} = \frac{1}{4\pi\varepsilon_0 r^3}\Big(-\vec{p} + 3\underbrace{(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)}_{=\,\vec{p}\cdot\hat{r}}\hat{r}\Big) = \frac{1}{4\pi\varepsilon_0 r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$

as required.

Figure 3.5: Handwritten derivation of the electric field of a dipole

4 Fields in Matter

4.1 Introduction

We might ask: how does an electric field interact with different materials?

- A proper theory requires us to embrace quantum mechanics, we will not do this but stick to the nineteenth-century interpretation.
- Broadly, we have **conductors** (charges can move freely) and **insulators** (charges can move but only in small displacements since they are fixed to atoms).
- Insulators are often called dielectrics but other types exist, such as paraelectric, ferroelectric, piezoelectric.
- There are materials between these extremes: semiconductors. These need to be treated with quantum theory but the action of fields in them is very important in devices.

4.2 Polarised atom

Assume that we understand an atom as neutral with a positive nucleus and surrounding electrons, what happens if an electric field $\boldsymbol{E}$ is placed across it?

The field will try to move the nucleus in one direction and the electrons in the opposite direction. I.e., the centre of the charge distributions separate and they create their own dipole field.

This means if we apply an external field on an atom, we will induce a dipole $\boldsymbol{p}$ in the atom. In other words, the atom becomes polarised.

Definition 4.2.1. A polarised atom has the dipole moment

$$\boldsymbol{p}_{\text{atom}} = \alpha \boldsymbol{E} \tag{4.1}$$

where the constant of proportionality α is the atomic polarisability, given by

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v_{\text{atom}} \tag{4.2}$$

Note: we have assumed here that there is *linear response*. This is fine for 'small' fields but as the field gets large enough it would eventually pull the atom apart and ionise the atom.

Example 4.2.1. Assume a simple model for an atom with a positive charge $q+$ for the nucleus and a sphere of charge $-q$ and radius a to represent the electrons. Calculate the atomic polarisability of this atom.

The net effect of this induces a dipole in the atom. We need to find the field on our displaced nucleus from the electron cloud due to the displacement. We can deduce that if the nucleus is displaced by d, then the sphere with radius d will enclose a charge of Q_{enc} given by

$$Q_{\text{enc}} = \left(\frac{4\pi d^3}{3} \boldsymbol{\nabla} \cdot \frac{4\pi a^3}{3}\right) q = \left(\frac{d}{a}\right)^3 q \tag{4.3}$$

Then, from Gauss' Law, we can calculate the electric field. We can then express $qd\hat{\boldsymbol{r}} = \boldsymbol{p}$ in terms of $\boldsymbol{E}$, whose proportional relationship gives us the atomic polarisability $\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 V_{\text{atom}}$.

4.3 Polarising molecules and crystals

For atoms, polarisation is simple. However, for molecules, the situation is more complicated.

We have chemical bonding (fundamentally electromagnetic interactions on the atomic scale treated using quantum theory).

In this case we may find that the polarisation of the molecule does not necessarily align with the direction of the field. I.e., the relation between $\boldsymbol{E}$ and $\boldsymbol{p}$ is a tensor relationship.

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{4.4}$$

Tensor relationships are common for real materials, especially for molecules and crystals with low symmetry. When we have two vector quantities that are related and it's not the magnitude that changes, but also the direction, we need a tensor relationship.

4.4 Linear polarisation

What happens when we place a material (many atoms) in an electric field?

The simplest idea is to imagine that they all get polarised in the field independently.

If a dipole is induced on each atom, the effect is that we have a lot of dipoles aligned in the field. The material, as a whole, becomes polarised. This combined effect is usually described as a polarisation per *unit volume* $\boldsymbol{P}$.

If the material is made of molecules that already have a dipole moment (water molecules), the net effect is the same, the dipoles still align and we can still describe it in the same way.

Note: There is always thermal energy in our material. The dipoles, induced or not, are always wriggling around. Inevitably, to understand $\boldsymbol{P}$ at a fundamental level, we need to use statistical physics arguments.

4.4.1 How do dipoles align in an electric field?

We can consider the torque on each charge on each dipole due to an external electric field $\boldsymbol{E}$. See Figure 4.1.

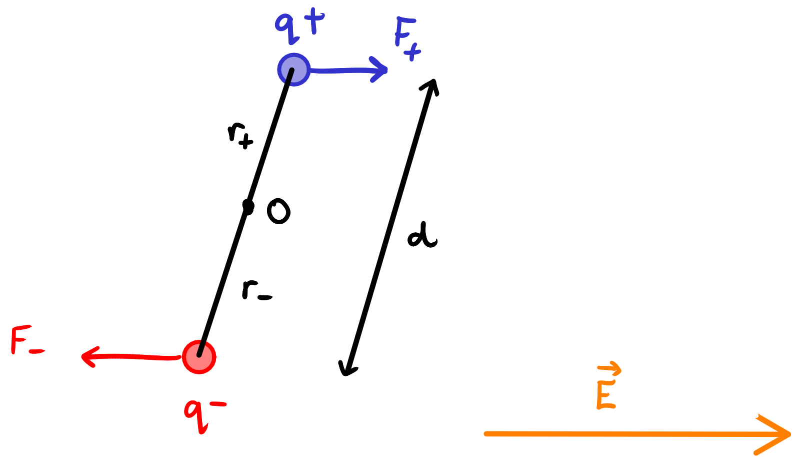


Figure 4.1: The effect of an external electric field on a dipole.

We start with the definition for torque and substitute in the total torque on both charges.

$$\Gamma = (\boldsymbol{r}_+ \times \boldsymbol{F}_+) + (\boldsymbol{r}_- \times \boldsymbol{F}_-) = \left[\frac{\boldsymbol{d}}{2} \times q\boldsymbol{E}\right] + \left[-\frac{\boldsymbol{d}}{2} \times q\boldsymbol{E}\right] \tag{4.5}$$

We can simplify this so that

$$\Gamma = q\boldsymbol{d} \times \boldsymbol{E} \tag{4.6}$$

We recall the definition for the dipole moment from 3.27 and substitute this in to give

Definition 4.4.1. The torque on the dipole in an external electric field is

$$\boldsymbol{\Gamma} = \boldsymbol{p} \times \boldsymbol{E} \tag{4.7}$$

If the field is non-uniform, there will also be a net force on the dipole. This will be

$$\boldsymbol{F} = (\boldsymbol{p} \cdot \boldsymbol{\nabla})\, \boldsymbol{E} \tag{4.8}$$

4.5 The field from a polarised object

To find the field from the whole object we need to 'sum' up the effect of all these dipoles (the principle of superposition)

$$V_{\text{dipole}}\left(\boldsymbol{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p} \cdot \hat{\boldsymbol{r}}}{r^2} \tag{4.9}$$

Let's try to derive an expression for the field from the potential.

Derivation 4.5.1. **Note:** A dipole moment from a volume in our material may be written as $\boldsymbol{p} = \boldsymbol{P}\, \mathrm{d}\tau'$ if we assume a continuous distribution. Then to get a whole material, we can integrate.

$$V\left(\boldsymbol{r}\right) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\boldsymbol{P}\left(\boldsymbol{r}'\right) \cdot \hat{\boldsymbol{r}}}{r^2}\, \mathrm{d}\tau' \tag{4.10}$$

But we notice that

$$\boldsymbol{\nabla}'\left(\frac{1}{r}\right) = \frac{\hat{\boldsymbol{r}}}{r^2} \tag{4.11}$$

So we can write this as

$$V\left(\boldsymbol{r}\right) = \frac{1}{4\pi\epsilon_0}\int_V \boldsymbol{P}\left(\boldsymbol{r}'\right) \cdot \boldsymbol{\nabla}'\left(\frac{1}{r}\right) \mathrm{d}\tau' \tag{4.12}$$

If we integrate by parts, and then apply the divergence theorem, we can rewrite this integral as a surface integral.

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left[\int_V \boldsymbol{\nabla}' \cdot \left(\frac{\boldsymbol{P}}{r}\right) \mathrm{d}\tau' - \int_V \frac{1}{r}\left(\boldsymbol{\nabla}' \cdot \boldsymbol{P}\right) \mathrm{d}\tau'\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{1}{r}\boldsymbol{P} \cdot \mathrm{d}\boldsymbol{a}' - \int_V \frac{1}{r}\left(\boldsymbol{\nabla}' \cdot \boldsymbol{P}\right) \mathrm{d}\tau'\right] \end{aligned} \tag{4.13}$$

Definition 4.5.1. The electric field from a polarised object is

$$V(r) = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{\sigma_b}{r}\,\mathrm{d}a' + \int_V \frac{\rho_b}{r}\,\mathrm{d}\tau'\right] \tag{4.14}$$

where $\sigma_b \equiv \boldsymbol{P}\cdot\hat{\boldsymbol{n}}$ is the 'bound' surface charge and $\rho_b \equiv -\boldsymbol{\nabla}\cdot\boldsymbol{P}$ is the 'bound' charge density.

If we imagine an object that has aligned dipoles, then there will be surface charges, opposite sides will have opposite charges, thus we have an electric field.

Note: Bigger dipoles mean $\rho(r)$ is not constant, so $\boldsymbol{\nabla}\cdot\boldsymbol{P} \neq 0$.

4.6 The electric field displacement D

Consider the charge density within a material as due to the bound charges, ρ_b, that we described in Section 4.5, and that due to anything else (like free electrons), ρ_f.

Now, Gauss' law applied to both charge densities (the total charge density) $\rho = \rho_b + \rho_f$ gives

$$\epsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho = \rho_b + \rho_f = -\boldsymbol{\nabla}\cdot\boldsymbol{P} + \rho_f \tag{4.15}$$

where $\boldsymbol{E}$ is the *total electric field* from both dipoles and free charges.

If we rearrange the left-most and right-most side, we get

$$\boldsymbol{\nabla}\cdot(\epsilon_0\boldsymbol{E} + \boldsymbol{P}) = \rho_f \tag{4.16}$$

Let us define a quantity $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$. Thus, as a result, we have derived the Gauss' law, in differential and integral forms, for the $\boldsymbol{D}$ field.

Definition 4.6.1. Gauss' law for the $\boldsymbol{D}$ field in both forms are

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_f \tag{4.17}$$

and

$$\oint \boldsymbol{D}\cdot\mathrm{d}\boldsymbol{a} = Q_{\text{free}} \tag{4.18}$$

Note: We cannot assume that $\boldsymbol{\nabla}\times\boldsymbol{D} = 0$. So we cannot assume $\boldsymbol{D}$ is determined exclusively by the free charge. Normally, for high symmetry, the curl might be zero. As a consequence, there is also no scalar potential for $\boldsymbol{D}$.

Example 4.6.1. A long straight wire, with line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement.

We write out the divergence relationship between $\boldsymbol{D}$ and ρ_f. Then we use Gauss' Law, and consider a smaller cylinder of radius s and length L inside our insulated wire as our Gaussian surface, ignoring the fields coming from the ends.

Then, the field is simply $\boldsymbol{D}2\pi sL = L$, the rearrange for $\boldsymbol{D}$, keeping in mind it is a vector pointing radially away from the axis of the cylinder.

Outside the rubber, our field is given by $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$, but $\boldsymbol{P} = 0$ outside the rubber. For regions inside the rubber, we cannot determine $\boldsymbol{D}$ as we don't know $\boldsymbol{P}$.

4.7 Boundary conditions for the displacement field

Definition 4.7.1. We establish the boundary conditions for the field $\boldsymbol{D}$ as follows

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \tag{4.19}$$

and

$$\boldsymbol{D}^{\parallel}_{\text{above}} - \boldsymbol{D}^{\parallel}_{\text{below}} = \boldsymbol{P}^{\parallel}_{\text{above}} - \boldsymbol{P}^{\parallel}_{\text{below}} \tag{4.20}$$

Note the discontinuity in the field above and below a surface and that there is also a discontinuity in the parallel field, unlike the electric field.

Derivation 4.7.1. Derive the boundary conditions for $\boldsymbol{D}$.

Consider a sheet with a box that bisects the surface, where the top half is above the surface, and the bottom half below the surface. If we make our Gaussian surface close to the box of charge, then we can use Gauss' Law for our displacement vector. We deduce that $D_{\text{above}} = \frac{\sigma}{2}$ and $D_{\text{below}} = -\frac{\sigma}{2}$. We can rearrange to get Equation 4.19.

To get the expressions for the parallel faces, we consider the side parallel vectors tending to zero as the box gets thinner and thus, our line integral around the box is just for the parallel faces above and below. Write out the equation for $\boldsymbol{D}$, and taking the curl of both sides, then we can use Stoke's theorem to get equation 4.20.

4.8 Linear dielectrics

If we did not know abut atoms, we could still make the observation that the net polarisation is proportional to the field.

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E} \tag{4.21}$$

where χ_e is known as the electric susceptibility and is an example of *linear response*. Hence, we call materials with this behaviour *linear dielectrics*.

This is not always the case (like in ferroelectrics), but it is truer for most materials under *small* fields.

Note: In some low symmetry crystals, $\boldsymbol{P}$ and $\boldsymbol{E}$ may not lie in the same direction and we need to define a susceptibility tensor to establish the relationship between the two fields even if the relationship is linear.

Definition 4.8.1. If we do this, we can generate some relationship

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon_0 \boldsymbol{E} + \epsilon_0 \chi_e \boldsymbol{E} = \epsilon_0 \left(1 + \chi_e\right) \boldsymbol{E} = \epsilon \boldsymbol{E} \tag{4.22}$$

where $\epsilon = (1 + \chi_e)$ is the *permittivity* of the material.

The *relative permittivity* is

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \tag{4.23}$$

Example 4.8.1. Examples from videos

1. A metal sphere of radius a carries a charge Q and is surrounded by a linear dielectric of permittivity ϵ out to a radius b. Find the potential at the centre of the sphere assuming the potential is zero at infinity.
   - To find the potential, we need to know the electric field. We use a Gaussian surface $r > a$ and Gauss' law for the $\boldsymbol{D}$ field. Our field inside the dielectric uses the permittivity ϵ instead of ϵ_0.
   - Write out the potential for each region from the field derived for each region as integrals so that it is in effect the integral of the field from infinity to 0.
2. A parallel plate capacitor is filled with a dielectric with a dielectric constant ϵ_r. What is the change in capacitance?
   - We know that $E = V/d$, we can derive that the capacitance is $C = A\epsilon_0/d$
   - If our gap is filled with a dielectric instead, we can use the $\boldsymbol{D}$ field, giving the relationship $\boldsymbol{D} = \epsilon_0 \epsilon_r \boldsymbol{E} = \sigma$.
   - From $C = Q/V$, we can rewrite using V derived from the relationship for $\boldsymbol{E}$ above.

4.9 Comments on the differences between E and D

Why do we choose to use $\boldsymbol{D}$ rather than $\boldsymbol{E}$?

- If we wish to use $\boldsymbol{E}$ in the presence of a dielectric we would need to determine the bound surface and bulk charge distributions. By defining $\boldsymbol{D}$ we only need to consider the free charges in the system—albeit at the expense of not knowing details about what is happening in the dielectric (in effect we talk about an average macroscopic field in the dielectric).
- As we have the relationship $\boldsymbol{\nabla D} = \rho_{\text{free}}$ and $\oiint \boldsymbol{D}\,\mathrm{d}\boldsymbol{a} = Q_{\text{free}}$ we can use similar techniques to finding $\boldsymbol{D}$ as we did for $\boldsymbol{E}$ (using Gaussian surfaces etc.).
- However, we cannot write

$$\boldsymbol{D}(r) = \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{r}}''}{r''^2}\rho_{\text{free}}\left(\boldsymbol{r}'\right)\mathrm{d}\tau' \tag{4.24}$$

  in other words, there is no Coulomb's law equivalent for the free charge.
- For example, in the case of the electric field we always have $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$, from which it follows $\boldsymbol{E}$ is a conservative field meaning we can express it as the gradient of a scalar field $-\boldsymbol{\nabla} V$. However, for the $\boldsymbol{D}$ field we have $\boldsymbol{\nabla} \times \boldsymbol{D} = \epsilon_0 (\boldsymbol{\nabla} \times \boldsymbol{E}) + \boldsymbol{\nabla} \times \boldsymbol{P} = \boldsymbol{\nabla} \times \boldsymbol{P}$. It is not always the case that $\boldsymbol{\nabla} \times \boldsymbol{P} = 0$. For example when there is some permanent polarisation as for example in an electret.

As a general rule, if our system has high symmetry we may still apply Gauss law as previously. But, if for example the spherical symmetry is broken by a permanent alignment of dipoles such that $\boldsymbol{\nabla} \times \boldsymbol{P} \neq 0$ we cannot just assume $\boldsymbol{D}$ is determined solely by the free charge.

4.10 Energy in dielectrics

The energy needed to charge a capacitor is

$$W = \frac{1}{2}CV^2 \tag{4.25}$$

With a dielectric, we have

$$C = \epsilon_r C_{\text{vac}} \tag{4.26}$$

The work necessary to charge a capacitor increases because the bound charges cancelled off part of the field, so we have to move more free charges to achieve the same potential. In other words, by adding a dielectric in between the capacitor we increase the energy that can be stored for the same amount of potential difference.

If we can make ϵ_r very high and use high voltages (so there is no breakdown in the material) we can make a 'supercapacitor'.

It may also be shown that

$$W = \frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}\,\mathrm{d}\tau \tag{4.27}$$

Remember that in free space, work is given by 2.36.

We could ask why we can't use the free space definition and imagine we bring all the charges together one by one. This would be OK but it doesn't take into account any of the energy needed in stretching the atoms in the dielectric in order to polarise them.

If we imagine we started with an unpolarised dielectric and then bring in the free charges to their final positions then the dielectric will respond by the charges displacing to form the induced dipoles. This extra energy is then taken into account in the work done. This quantity is encapsulated in the $\frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}\,\mathrm{d}\tau$. It is larger than just doing the calculation using the free space equation.

4.11 Forces on dielectric

As with conductors a dielectric is drawn into an electric field. This is best understood by considering a simple parallel plate capacitor into which a dielectric field is partially inserted. See Figure 4.2.

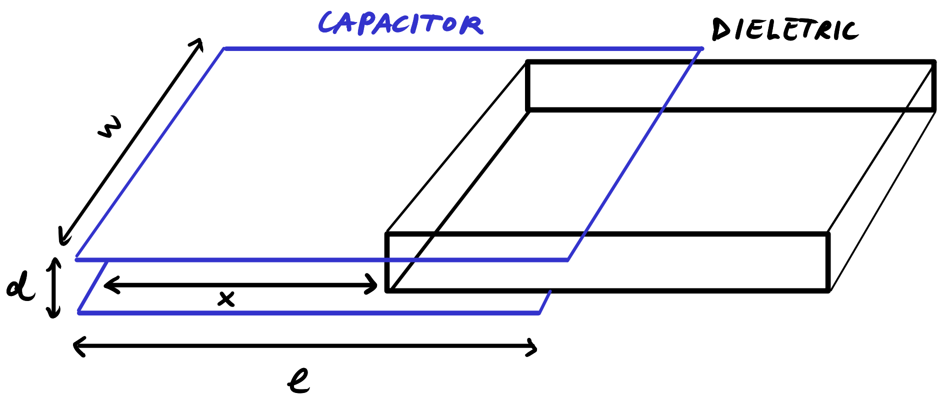


Figure 4.2: A partially inserted dielectric in a capacitor.

In our ideal system there would be no force as the field is perfectly perpendicular to the plates. In reality, it is the small fringing field that allows force to be exerted.

In air, the capacitance is given by

$$C = \frac{\epsilon_0 A}{d} \tag{4.28}$$

while with a dielectric, it is

$$C = \frac{\epsilon_0 \epsilon_r A}{d} \tag{4.29}$$

This fringing field at the edges gives rise to the forces which exerts a force on the dielectric.

In order to pull the dielectric slab out, we need to exert a force F_m on the dielectric, so we do the total amount of work

$$\mathrm{d}W = F_m \, \mathrm{d}x \tag{4.30}$$

Then, the electrical force on the dielectric is $F = -F_m$. Thus, the external force due to the electric field on the slab is

$$F = -\frac{\mathrm{d}W}{\mathrm{d}x} \tag{4.31}$$

Since the energy stored in the capacitor is $E = CV^2/2$, a partially filed capacitor would have a partial area (total capacitance depends on the total dielectric in the capacitors), and thus total energy

$$C = \frac{\epsilon_0 w}{d} \left(\epsilon_r l - \chi_e x\right) \tag{4.32}$$

We can understand this result. The proportion of the capacitor that is still in air is given by the area $A_a = xw$, and the proportion that is dielectric is given by the area $A_d = (l - x)\, w$, thus, we sum up the capacitance for each region. We recall that $\chi_e = 1 - \epsilon_r$.

For a capacitor that is already charged up $W = Q^2/2C$, then

$$F = -\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{1}{2}\frac{Q^2}{C^2}\frac{\mathrm{d}C}{\mathrm{d}x}\frac{1}{2}V^2\frac{\mathrm{d}C}{\mathrm{d}x} \tag{4.33}$$

Note: we are keeping Q as a constant here. If we keep V constant, there would be work from the battery used in changing Q. In this derivation, V changes. The derivative of C w.r.t. x is

$$\frac{\mathrm{d}C}{\mathrm{d}x} = -\frac{\epsilon_0 \chi_e w}{d} \tag{4.34}$$

Definition 4.11.1. The force on a dielectric due to an external electric field is

$$F = -\frac{\epsilon_0 \chi_e w}{d} V^2 \tag{4.35}$$

We can also get the same result if we take V to be constant (maintained by the battery), and Q will change as the dielectric moves.

5 Magnetostatics

5.1 Introduction

So far, we have only considered fixed charges. When charge starts to move, it is subject to an electrostatic force. We have also magically conjured up charge distributions to solve problems without considering how the charges may have arrived in the first place.

The way in which moving charges create time varying fields cannot be considered until we get to Maxwell's equation. The consequences of Maxwell's equations will be considered in later courses.

In this section we will restrict ourselves to the case of electrical currents (mostly conductors) that as you know will give rise to magnetic fields. Moreover, we will for the most part restrict ourselves to steady currents, that will give rise to static magnetic fields—i.e. *magnetostatics*.

We can imagine a situation where a battery sends a current through a very long wire set apart by a distance. Suppose the parallel parts are very long so we can ignore the ends. Current flows up one wire and flows down the other wire in the other direction.

We see empirically that the wire tends to push each other apart. We may conclude there are electrostatic charges that are repulsing each other. But if we bring a test charge, there is no effect on the test charge.

We can also alter the wire so that both currents are now facing the same direction. Now, the wires are attracted to each other.

We conclude then, that moving charges generate magnetic fields. Magnetic fields can be detected by using a compass needle. The direction of the magnetic field lines is determined by the right-hand screw rule.

The magnetic field is always perpendicular to the direction of current flow. We also find that the force exerted is perpendicular to both the magnetic field vector and the direction of the current.

5.2 The Lorentz force law

Definition 5.2.1. The force produced by a charge Q moving with velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$ is expressed as

$$\boldsymbol{F}_{\text{mag}} = Q\,(\boldsymbol{v} \times \boldsymbol{B}) \tag{5.1}$$

In the presence of an electric field $\boldsymbol{E}$, the total charge becomes

$$\boldsymbol{F}_{\text{total}} = Q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{5.2}$$

Note: The use of the cross product removes the need for remembering the right and left-hand rules. If we remember the Lorentz Law, the direction of the force is given by the cross-product convention.

The Lorentz force law is a result of careful observation. It is a fundamental axiom of the theory.

5.3 The work done by magnetic fields

What is the work done on a charge when a magnetic force acts upon it?

$$\mathrm{d}W_{\text{mag}} = \boldsymbol{F}_{\text{mag}} \cdot \mathrm{d}\boldsymbol{l} = Q\,(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}\,\mathrm{d}t = 0 \tag{5.3}$$

Since $\boldsymbol{v} \times \boldsymbol{B}$ is by definition perpendicular to $\boldsymbol{v}$ and $\boldsymbol{B}$ unless they are co-linear, or zero, in which case there is no force. Hence, the dot product is zero.

The consequence is that **magnetic forces do no work**.

This is not uncommon. In a gyroscope, gravity acts to change the direction of the angular momentum vector but does not work. Similarly, central forces do no work.

5.4 Currents

We usually consider a current in a wire and define it as the amount of charge *passing per unit time* at a point in the wire.

We can see how this will be proportional to the velocity $\boldsymbol{v}$ of the charge carriers in the wire and charge q.

Note: We now understand current to be the movement of negatively charged electrons in our wires. The electrons move in the opposite way to the conventional definition of current. In the development of the theory of magnetism the current was assumed to be the movement of positive charges and this has remained.

We all accept and have a conceptual idea of a wire. In practice, unless we say otherwise, we understand a wire as the object that guides the charge carriers but with no physical size of its own.

Along a wire, the current at a particular point P may be thought of as a line charge λ per unit length moving along a wire with velocity $\boldsymbol{v}$.

$$\boldsymbol{I} = \lambda \boldsymbol{v} \tag{5.4}$$

The Lorentz force on a segment of wire is

$$\boldsymbol{F} = \int (\boldsymbol{v} \times \boldsymbol{B})\,\mathrm{d}q = \int (\boldsymbol{v} \times \boldsymbol{B})\,\lambda\,\mathrm{d}l = \int (\boldsymbol{I} \times \boldsymbol{B})\,\mathrm{d}l \tag{5.5}$$

Because the current is constrained to flow along the wire we can write

$$\boldsymbol{I}\,\mathrm{d}l = I\,\mathrm{d}\boldsymbol{l} \tag{5.6}$$

Definition 5.4.1. The force along a wire with a current is

$$\boldsymbol{F} = \int I\,(\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}) \tag{5.7}$$

5.5 Surface and Volume current densities

5.5.1 Surface current density

Currents may not necessarily be thought of as flowing through very thin wires. They may be considered as flowing over a surface or through the bulk of a material (very thick cable). Furthermore, the current may not flow uniformly over the surface on in the bulk.

$$\boldsymbol{K} \equiv \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}l_{\perp}} \tag{5.8}$$

where $\mathrm{d}l$ is perpendicular to the current flow, this is the current per unit length flowing through the surface.

Alternatively

$$\boldsymbol{K} = \sigma \boldsymbol{v} \tag{5.9}$$

where σ is the ‘moving’ surface charge density. The number of carriers move through the surface, but σ doesn’t change. Thus, for a surface,

$$\boldsymbol{F} = \int (\boldsymbol{v} \times \boldsymbol{B})\,\sigma\,\mathrm{d}a = (\boldsymbol{K} \times \boldsymbol{B})\,\mathrm{d}a \tag{5.10}$$

5.5.2 Volume current density

Similarly, we may define a volume current density

$$J \equiv \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}a_\perp} \tag{5.11}$$

where $\mathrm{d}a_\perp$ is a surface area perpendicular to the current flow at the point in the surface. Then we have

$$\boldsymbol{J} = \rho\boldsymbol{v} \tag{5.12}$$

where ρ is the mobile charge density. Similarly,

$$\boldsymbol{F} = \int (\boldsymbol{J} \times \boldsymbol{B})\,\mathrm{d}\tau \tag{5.13}$$

5.6 The continuity equation

From the definition of volume current density in 5.5 we can write

$$I = \int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a} \tag{5.14}$$

Now, from the divergence theorem we have

$$\oint_S \boldsymbol{J} \cdot d\,\mathrm{d}\boldsymbol{a} = \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{J}\,\mathrm{d}\tau \tag{5.15}$$

The left-hand term corresponds to the charge per unit time leaving the enclosed surface and the right-hand term is the corresponding charge leaving the volume per unit time. Now, as charge is conserved this must represent the change in the charge density in the volume per unit time, the minus sign represents the charge density decreasing as the charge flows out.

$$\iiint_V (\boldsymbol{\nabla} \cdot J)\,\mathrm{d}\tau = -\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\,\mathrm{d}\tau = -\iiint_V \left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)\mathrm{d}\tau \tag{5.16}$$

If we equate the kernels of the two integrals, then we get the continuity equation.

Definition 5.6.1. The continuity equation is

$$\boldsymbol{\nabla} \cdot J = -\frac{\partial\rho}{\partial t} \tag{5.17}$$

If ρ is constant, then we have a steady state condition.

For electrostatics, we must have no change in ρ—electrostatic problems are steady state problems.

5.7 The Biot-Savart Law

Magnetostatics refers to the regime where magnetic fields do not vary with time. Consequently, the currents producing these fields must be steady. Hence, we have

$$\frac{\partial \rho}{\partial t} \quad \text{and} \quad \frac{\partial \boldsymbol{J}}{\partial t} = 0 \tag{5.18}$$

We don't ask how we get to this steady state but consider what happens when we are there.

Note: by these definitions a moving point ch arge does not give a steady current—we need a charge distribution.

If we have steady currents then from the continuity equation and $\frac{\partial \rho}{\partial t} = 0$

$$\boldsymbol{\nabla} \cdot J = 0 \tag{5.19}$$

Definition 5.7.1. Putting this all together, we have the equivalent of the integral form of Coulombs law the Biot-Savart Law.

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{I} \times \boldsymbol{r}''}{r''^2} \, \mathrm{d}l' = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}''}{r''^2} \tag{5.20}$$

Where μ_0 is the permeability of free space.

The superposition principle also applies to $\boldsymbol{B}$. It is measured in tesla (T).

5.8 The divergence and curl of B

With the result for $\boldsymbol{B}$ for a long straight wire, we can write

$$\oint \boldsymbol{B} \, \mathrm{d}\boldsymbol{l} = \oint \frac{\mu_0 I}{2\pi s} \, \mathrm{d}l = \frac{\mu_0 I}{2\pi} \oint \mathrm{d}l = \mu_0 I \tag{5.21}$$

where we have taken a circular path of radius s (the result holds true for any closed path enclosing the wire). We can add any number of wires within the loop so that we have

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I_{\text{enc}} \tag{5.22}$$

In terms of the current density $\boldsymbol{J}$ we may express the enclosed current as

$$I_{\text{enc}} = \iint \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a} \tag{5.23}$$

where the integral is over the area enclosed by the loop. We may apply Stokes' theorem to get

$$\mu_0 \iint \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a} = \iint (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{a} \tag{5.24}$$

from which we find

Definition 5.8.1. The curl of the magnetic field is given by Ampere's law in differential form

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{5.25}$$

The curl of the magnetic is not zero, and thus it is not a conservative field, but magnetic forces do no work.

This proof is only for a long straight wire. However, this is indeed the general result. We can quote another result that may be obtained from the Biot-Savart law.

Definition 5.8.2. The divergence of the magnetic field is

$$\boldsymbol{\nabla} \cdot B = 0 \tag{5.26}$$

the physical meaning of this is that there are no magnetic monopoles.

5.9 Maxwell's equation for electrostatics and magnetostatics

Definition 5.9.1. This is a summary of all differential forms

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \frac{\rho}{\epsilon_0} \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} \end{aligned} \tag{5.27}$$

Along with the Lorentz force law, these constitute the most concise formulation of electrostatics and magnetostatics.

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \tag{5.28}$$