1 Problem 1.32

Problem 1.1

Let $x(t)$ be a continuous-time signal, and let

$$y_1(t) = x(2t) \text{ and } y_2(t) = x(t/2)$$

The signal $y_1(t)$ represents a speeded up version of $x(t)$ in the sense that the duration of the signal is cut in half. Similarly, $y_2(t)$ represents a slowed down version of $x(t)$ in the sense that the duration of the signal is doubled. Consider the following statements:

1. If $x(t)$ is periodic, then $y_1(t)$ is periodic.
2. If $y_1(t)$ is periodic, then $x(t)$ is periodic.
3. If $x(t)$ is periodic, then $y_2(t)$ is periodic.
4. If $y_2(t)$ is periodic, then $x(t)$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

Problem 1.32a

Let T be the fundamental period of $x(t)$, then we have $x(t+T) = x(t)$. So $y_1(t) = x(2t) = x(2t+T) = x(2(t+\frac{T}{2})) = y_1(t+\frac{T}{2})$, $y_1(t)$ is periodic and the fundamental period is $\frac{T}{2}$ which is consistent with our commonsense that $y_1(t)$ is an speeded up version of $x(t)$.

Problem 1.32b

Let T be the fundamental period of $y_1(t)$, then based on the analysis of Problem 1.32a , we have $x(t)$ is a slowed down version of $y_1(t)$, hense $x(t)$ is periodic and has fundamental period $2T$.

Problem 1.32c

Let T be the fundamental period of $x(t)$, then we have $x(t+T) = x(t)$. So $y_2(t) = x(\frac{t}{2}) = x(\frac{t}{2}+T) = x(\frac{t+2T}{2}) = y_2(t+2T)$, $y_2(t)$ is periodic and the fundamental period is $2T$ which is consistent with our commonsense that $y_2(t)$ is an slowed down version of $x(t)$.

Problem 1.32d

Let T be the fundamental period of $y_2(t)$, then based on the analysis of Problem 1.32c , we have $x(t)$ is a speeded up version of $y_2(t)$, hense $x(t)$ is periodic and has fundamental period $\frac{T}{2}$.

2 Problem 1.33

Problem 2.1

Let $x[n]$ be a discrete-time signal, and let

$$y_1[n] = x[2n] \text{ and } y_2[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

The signals $y_1[n]$ and $y_2[n]$ respectively represent in some sense the speeded up and slowed down versions of $x[n]$. However, it should be noted that the discrete-time notions of speeded up and slowed down have subtle differences with respect to their continuous-time counterparts. Consider the following statements:

1. If $x[n]$ is periodic, then $y_1[n]$ is periodic.
2. If $y_1[n]$ is periodic, then $x[n]$ is periodic.
3. If $x[n]$ is periodic, then $y_2[n]$ is periodic.
4. If $y_2[n]$ is periodic, then $x[n]$ is periodic.

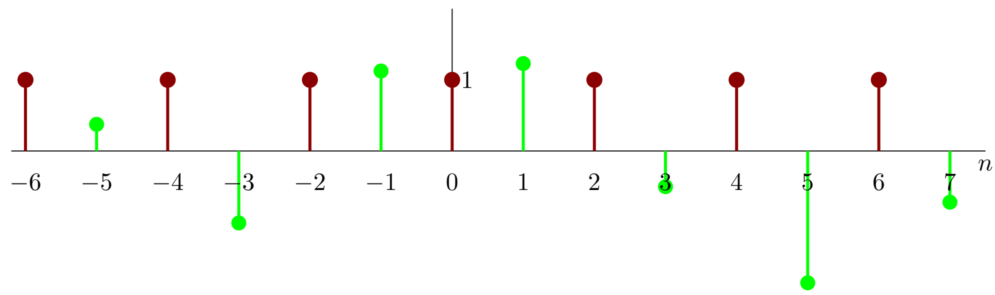
Problem 1.33a

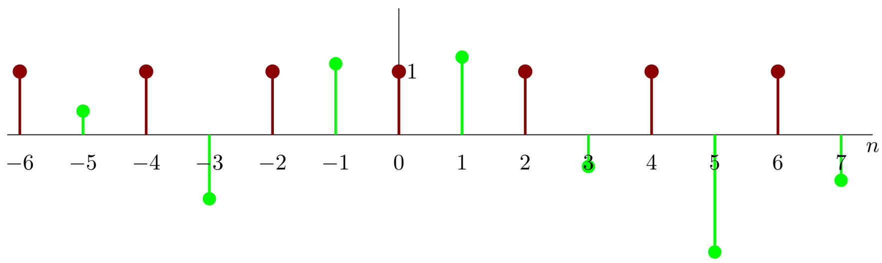
Let N be the fundamental period of $x[n]$, then $x[n+N] = x[n]$, and $y_1[n] = x[2n] = x[2n+N] = y_1[n+\frac{N}{2}]$. If N is even, then $y_1[n]$ has fundamental period $\frac{N}{2}$. It is a little tricky that when N is odd $\frac{N}{2}$ has no meaning. If N is odd, we have to express $y_1[n] = x[2n] = x[2n+2N] = y_1[n+N]$ hence $y_1[n]$ has fundamental period N if N is odd.

Problem 1.33b

We can see $y_1[n]$ is not only a speeded up version but also a down sampling version of $x[n]$. we cannot determine whether or not $y_1[n]$ is periodic based on that $x[n]$ is periodic. We can only tell that the even indexed value of $x[n]$ is period. we know nothing about the odd indexed value of $x[n]$. An example is given below:

The given $x[n]$ is periodic at $2n$ but is random at $2n+1$.





Problem 1.33c

$y_2[n]$ can treated as interpolation of $x[n]$ with zero. So $y_2[n]$ is periodic with fundamental period $2N$ where N is fundamental period of $x[n]$.

Problem 1.33d

Based on analysis of Problem 1.33c , $y_2[n]$ has fundamental period $2N$ where N is the fundamental period of $x[n]$.

Based on the fact that only integer is valid for the independent variable of discrete-time signal, $y_2[n]$ contains all the value of $x[n]$ at the even index whereas at the odd index $y_2[n] = 0$.

3 Problem 1.34

Problem 3.1

In this problem, we explore several of the properties of even and odd signals.

1. Show that if $x[n]$ is an odd signal, then

$$\sum_{n=-\infty}^{+\infty} x[n] = 0$$

2. Show that if $x_1[n]$ is an odd signal and $x_2[n]$ is an even signal, then $x_1[n]x_2[n]$ is an odd signal.
3. Let $x[n]$ be an arbitrary signal with even and odd parts denoted by

$$x_e[n] = \text{Even}\{x[n]\}$$

and

$$x_o[n] = \text{Odd}\{x[n]\}$$

Show that

$$\sum_{n=-\infty}^{+\infty} x^2[n] = \sum_{n=-\infty}^{+\infty} x_e^2[n] + \sum_{n=-\infty}^{+\infty} x_o^2[n]$$

4. Although parts 1-3 have been stated in terms of discrete-time signals, the analogous properties are also valid in continuous time. To demonstrate this, show that

$$\int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} x_e^2(t)dt + \int_{-\infty}^{+\infty} x_o^2(t)dt$$

where $x_e(t)$ and $x_o(t)$ are, respectively, the even and odd parts of $x(t)$.

Problem 1.34a

If $x[n]$ is an odd signal, then we have $x[n]=-x[-n]$ and $x[0]=0$, so

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} x[n] &= \sum_{n=0}^{+\infty} x[n]+\sum_{-\infty}^{0} x[n]\\ &= \sum_{n=0}^{\infty}\big(x[n]+x[-n]\big)\\ &= 0\end{aligned}$$

Problem 1.34b

Because $x_1[n]$ is an odd signal, $x_1[n]=-x_1[-n]$. Because $x_2[n]$ is an even signal, $x_2[n]=x_2[-n]$. we have

$$x_1[n]x_2[n]=-x_1[-n]x_2[-n]$$

i.e. $x_1[n]x_2[n]$ is an odd signal.

Problem 1.34c

$$\begin{aligned}x_e[n] &= \frac{x[n]+x[-n]}{2}\\ x_o[n] &= \frac{x[n]-x[-n]}{2}\end{aligned}$$

Then we have

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} x_e^2[n] &= \sum_{n=-\infty}^{+\infty}\left(\frac{x[n]+x[-n]}{2}\right)^4\\ &= \sum_{n=-\infty}^{+\infty}\left(\frac{x^2[n]+x^2[-n]+2x[n]x[-n]}{2}\right)\end{aligned}$$

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} x_o^2[n] &= \sum_{n=-\infty}^{+\infty}\left(\frac{x[n]-x[-n]}{2}\right)^2\\ &= \sum_{n=-\infty}^{+\infty}\left(\frac{x^2[n]+x^2[-n]-2x[n]x[-n]}{4}\right)\end{aligned}$$

Then we add the above two equations and get

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} x_e^2[n]+\sum_{n=-\infty}^{+\infty} x_o^2[n] &= \sum_{n=-\infty}^{+\infty}\frac{x^2[n]+x^2[-n]}{2}\\ &= \sum_{n=-\infty}^{+\infty} x^2[n]\end{aligned}$$

This conclusion tells us that even we seperate the signal into two parts(even part and odd part), the sum of energy from the even part and odd part is the total energy of the original signal.

Let

$$\begin{aligned}\int_{-\infty}^{+\infty} x^2(t)dt &= \int_{-\infty}^{+\infty} (x_e(t)+x_o(t))^2 dt \\ &= \int_{-\infty}^{+\infty} x_e^2(t)dt + \int_{-\infty}^{+\infty} x_o^2(t)dt + \int_{-\infty}^{+\infty} x_e(t)x_o(t)dt \\ &= \int_{-\infty}^{+\infty} x_e^2(t)dt + \int_{-\infty}^{+\infty} x_o^2(t)dt\end{aligned}$$

Notice that $x_e(t)x_o(t)$ is and odd signal, so $\int_{-\infty}^{+\infty} x_e(t)x_o(t)dt = 0$

4 Problem 1.35

Problem 4.1

Consider the periodic discrete-time exponential time signal

$$x[n] = e^{im(2\pi/N)n}$$

Show that the fundamental period of this signal is

$$N_0 = N/\gcd(m,N)$$

where $\gcd(m,N)$ is the greatest common divisor of m and N—-that is, the largest integer that divides both m and N an integral number of times. For example,

$$\gcd(2,1) = 1, \gcd(2,4) = 2, \gcd(8,12) = 4$$

Note that $N_0 = N$ if m and N have no factors in common.

Proof

Let N_0 the fundamental period of $x[n]$, so that

$$m(2\pi/N)N_0 = l \times 2\pi$$

So, we have $N_0 = \frac{lN}{m}$. If m and N have no factors in common, we have $N_0 = N$ when $l = m$. If m and N have factors in common, the fundamental period should be

$$N_0 = l\frac{N/\gcd(m,N)}{m/\gcd(m,N)}$$

When $l = m/\gcd(m,N)$, we have $N_0 = \frac{N}{\gcd(m,N)}$.