Signal and System Chapter 1 Problems

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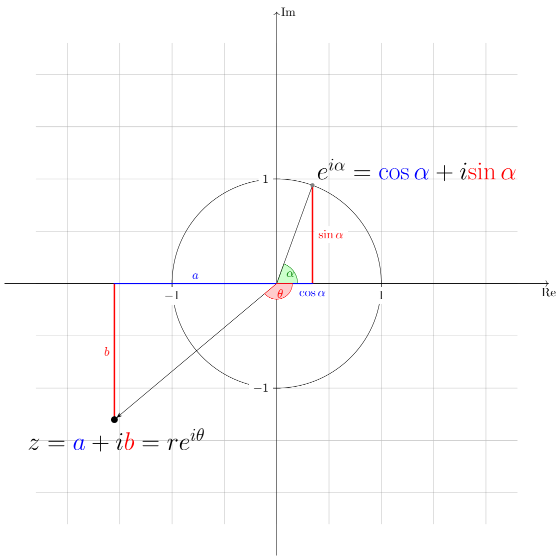
January 18, 2020



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1 Basic Problems With Answers

1.1 Problem 1.1 and Problem 1.2

Problem 1.1

Problem 1.1: Express each of the following complex numbers in *Cartesian* form $x+iy$:

$$\begin{array}{ccc} \frac{1}{2}e^{i\pi} & \frac{1}{2}e^{-i\pi} & e^{i\pi/2} \\ e^{-i\pi/2} & e^{i5\pi/2} & \sqrt{2}e^{i\pi/4} \\ \sqrt{2}e^{i9\pi/4} & \sqrt{2}e^{-i9\pi/4} & \sqrt{2}e^{-i\pi/4} \end{array}$$

Problem 1.2

Problem 1.2: Express each of the following complex numbers in *polar* form $(re^{i\theta}, -\pi < \theta \le \pi)$:

$$\begin{array}{ccc} 5 & -2 & -3i \\ \frac{1}{2}-i\frac{\sqrt{3}}{2} & 1+i & (1-i)^2 \\ i(1-i) & \frac{1+i}{1-i} & \frac{\sqrt{2}+i\sqrt{2}}{1+i\sqrt{3}} \end{array}$$

First, let's figure them out using the Euler's formula. For the nine complex numbers in Problem 1.1, we have:

$$\begin{array}{rcl} \frac{1}{2}e^{i\pi} & = & \frac{1}{2}(\cos(\pi)+i\sin(\pi)) = \frac{1}{2}\cos(\pi) = -\frac{1}{2} \\ \frac{1}{2}e^{-i\pi} & = & \frac{1}{2}(\cos(-\pi)+i\sin(-\pi)) = \frac{1}{2}\cos(-\pi) = -\frac{1}{2} \\ e^{i\pi/2} & = & \cos(\pi/2)+i\sin(\pi/2) = i \\ e^{-i\pi/2} & = & \cos(-\pi/2)+i\sin(-\pi/2) = -i \\ e^{i5\pi/2} & = & \cos(5\pi/2)+i\sin(5\pi/2) = \cos(2\pi+\pi/2)+i\sin(2\pi+\pi/2) = i\sin(\pi/2) = i \\ \sqrt{2}e^{i\pi/4} & = & \sqrt{2}(\cos(\pi/4)+i\sin(\pi/4)) = \sqrt{2}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}) = 1+i \\ \sqrt{2}e^{i9\pi/4} & = & \sqrt{2}(\cos(9\pi/4)+i\sin(9\pi/4)) = 1+i \\ \sqrt{2}e^{-i9\pi/4} & = & \sqrt{2}(\cos(-9\pi/4)+i\sin(-9\pi/4)) = 1-i \\ \sqrt{2}e^{-i\pi/4} & = & \sqrt{2}(\cos(-\pi/4)+i\sin(-\pi/4)) = 1-i \end{array}$$

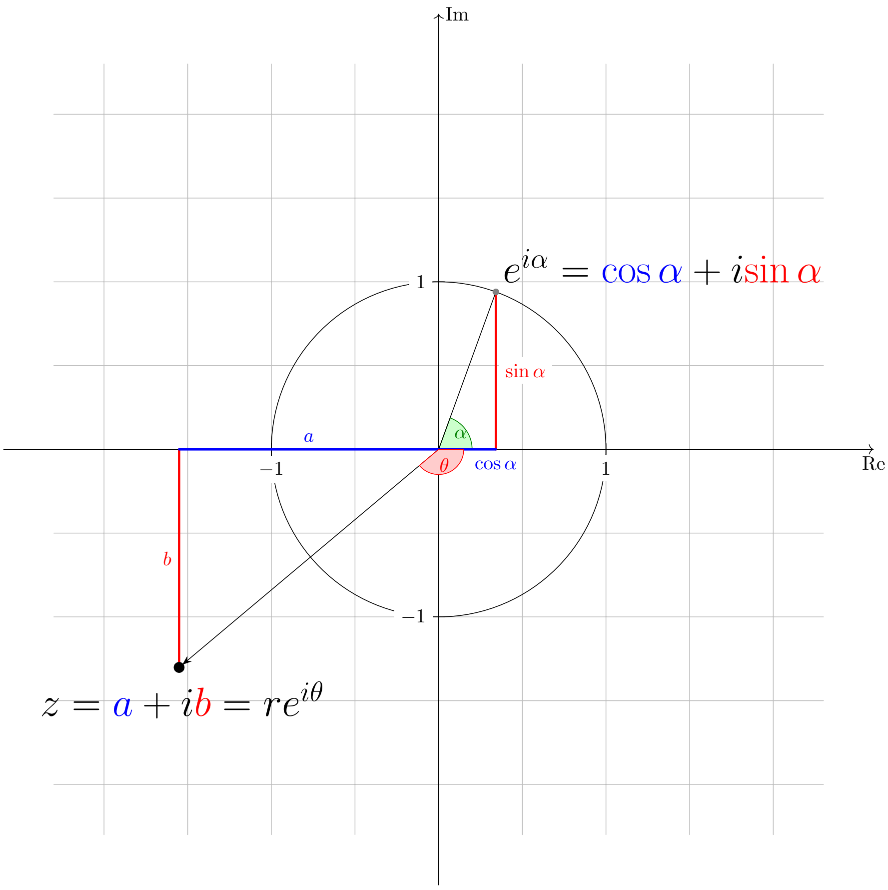
For the nine complex numbers in Problem 1.2 , we have:

$$
\begin{aligned}
5 &= 5e^{i0} = 5e^{i(2\pi n)}, n \in \{\ldots,-2,-1,0,1,2,\ldots\} \\
-2 &= 2e^{i\pi} = -2e^{i0} \\
-3i &= 3e^{-i\pi} \\
\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2} &= e^{-i\pi/3} \\
1+i &= \sqrt{2}e^{i\pi/4} \\
(1-i)^2 &= (\sqrt{2}e^{-i\pi/4})^2 = 2e^{-i\pi/2} \\
i(1-i) &= 1-i = \sqrt{2}e^{-i\pi/4} \\
\tfrac{1+i}{1-i} &= \tfrac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} = e^{i\pi/2} \\
\tfrac{\sqrt{2}+i\sqrt{2}}{1+i\sqrt{3}} &= \tfrac{2e^{i\pi/4}}{2e^{i\pi/3}} = e^{-i\pi/12}
\end{aligned}
$$

Every complex number $a + ib$ can be visualized in the complex plane $\mathbb{C}$. It can be viewed either as the point with the coordinate (a, b) or as a vector starting from $0,0$ to the point (a, b).

Also every complex number $a+ib$ can be represented in the exponential form conveniently by using the Euler's formula $e^{i\alpha} = \cos\alpha + i\sin\alpha$. One complex number have unique Cartesian form but do not have a unique expoential form. Taking $1 + i$ for example, its expoential form can be $\sqrt{2}e^{i(\pi/4+2n\pi)}, n \in \{\ldots,-2,-1,0,1,2,\ldots\}$. When we express a complex number in expoential form, it helps to keep a concept of rotation in mind. In the complex plane $\mathbb{C}$, a complex number will return to itself if it rotates a multiple of 2π around the origin point with radius equal to its modulus. please keep the concept of roatation in mind and it will become increasingly important during our later study.

In the development of complex analysis, $a + ib$ has another form $r\angle\theta$, where r is the modulus and θ the argument (or the angle). Obviously, this notation is not as good as the expoential form, espacialy when we want to do complex analysis such as differentiation and integration. We just mention it here for the sake of completion. The expoential form will be deployed from now on.



Now, let's go back to Problem 1.1 and Problem Problem 1.2. It's easy to figure the answers out using the Euler's formula. Let's do more to show them on the complex plane keeping the concept of rotation in mind.

From Figure 1 , taking $\frac{1}{2}e^{i\pi}$ and $\frac{1}{2}e^{-i\pi}$ for example, in the complex plane, they are the same point $-\frac{1}{2}$ which means $-\frac{1}{2}$ can be reached by rotating $\frac{1}{2}$ with angle π clockwise or with angle $-\pi$ anti-clockwise. Essentially, this is because that $e^{i\theta} = e^{i(\theta+2n\pi)}, n \in \{\ldots,-2,-1,0,1,2,\ldots\}$. It's straightforward that $e^{i\pi} = e^{i(\pi+2(-1)\pi)} = e^{-i\pi}$.

In the end of Problem 1.1 and Problem 1.2, I want to say more about expressing $\frac{1+i}{1-i}$ in its polar form. There are two methods to get the polar form:

1. multiply the fraction's numerator and denominator by $(1+i)$
   So we have:

$$\begin{aligned}\frac{1+i}{1-i} &= \frac{(1+i)(1+i)}{(1-i)(1+i)} \\ &= \frac{2i}{2} = i\end{aligned}$$

2. express the numerator and denominator in expoential form first, then do the following calculation.

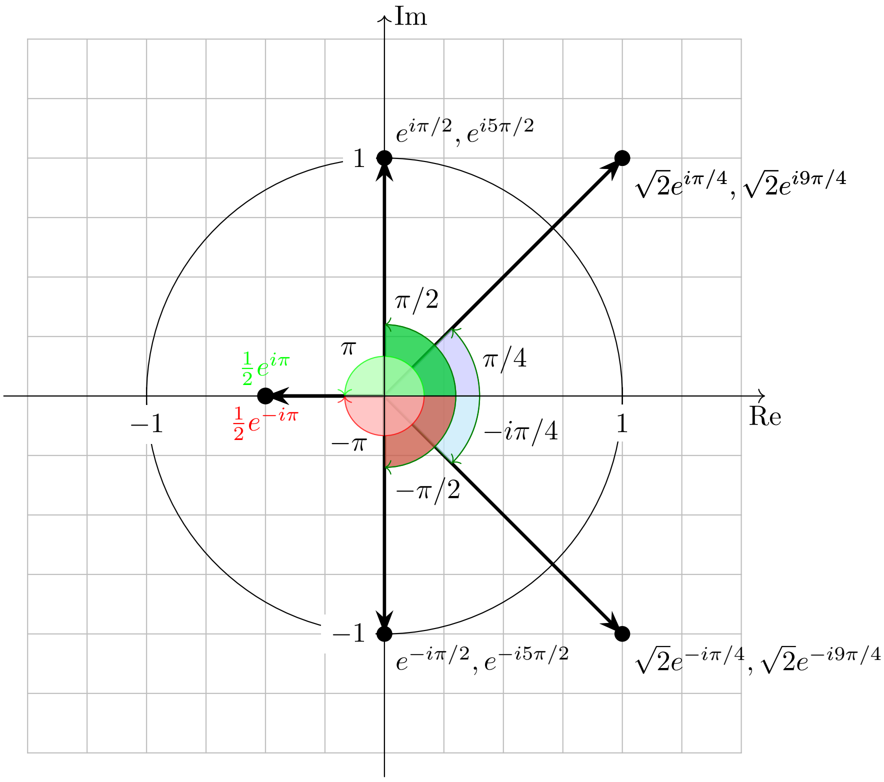


Figure 1: visualized the complex numbers from Problem Problem 1.1 in the complex plan

$$
\begin{aligned}
\frac{1+i}{1-i} &= \frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} \\
&= e^{i2\pi/4} = e^{i\pi/2} = i
\end{aligned}
$$

1.2 Problem 1.3

Problem 1.3

Problem 1.3: Determine the values of P_∞ and E_∞ for each of the following signals:

$$
\begin{array}{llll}
\text{(a)} & x_1(t) = e^{-2t}u(t) & \text{(b)} & x_2(t) = e^{i(2t+\pi/4)} \\
\text{(c)} & x_3(t) = \cos(t) & \text{(d)} & x_1[n] = (\frac{1}{2})^n u[n] \\
\text{(e)} & x_2[n] = e^{i(\pi/2n+\pi/8)} & \text{(f)} & x_3[n] = \cos(\frac{\pi}{4}n)
\end{array}
$$

Before solving this problem, let's review the definition of P_∞ and E_∞. For a continuous time signal $x(t)$, we have

$$
E_\infty = \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2 \mathrm{d}t \tag{1.1}
$$

$$
P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2 \mathrm{d}t = \lim_{T\to\infty}\frac{E_\infty}{2T} \tag{1.2}
$$

For a discrete time signal $x[n]$, we have:

$$
E_\infty = \lim_{N\to\infty}\sum_{n=-N}^{+N}|x[n]|^2 \tag{1.3}
$$

$$
P_\infty = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{+N}|x[n]|^2 = \lim_{N\to\infty}\frac{E_\infty}{2N+1} \tag{1.4}
$$

Equation (1.1)(1.3) are not only mathmatical definitions but also related to physical quantities such as power and energy in a physical system. For an electric circuit, taking the voltage $v(t)$ and current $i(t)$ across a resistor for example, the power at time t can be calculated by:

$$
p(t) = v(t)i(t) = \frac{v^2(t)}{R} \tag{1.5}
$$

Let's go back to equation (1.1) and equation(1.3), if $E_\infty < \infty$ we say that the signal has finite energy otherwise infinite energy. If $P_\infty < \infty$ we say that the signal has finite power otherwise infinite power.

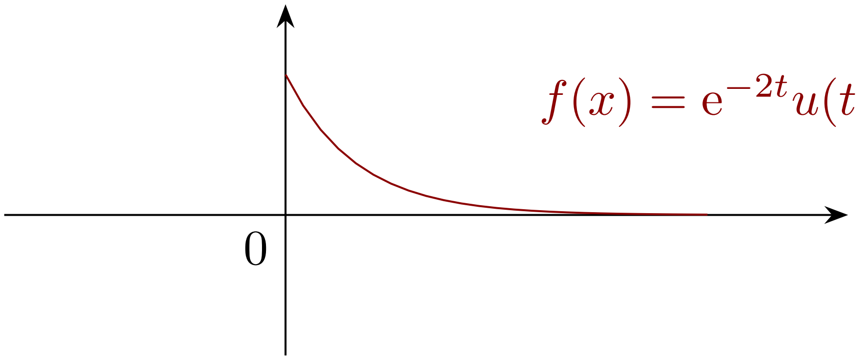
Next let's determine the values of P_∞ and E_∞ for the given signals.

(**a**) : $x_1(t) = e^{-2t}u(t)$:

$$\begin{aligned} E_\infty &= \int_\infty^\infty |e^{-2t}u(t)|^2 \mathrm{d}t \\ &= \int_0^\infty e^{-4t} \mathrm{d}t \\ &= \frac{1}{4} \end{aligned}$$

So, it's straightforward that:

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0 \tag{1.6}$$



$(\mathbf{b})$: $x_2(t) = e^{i(2t+\pi/4)}$
We have $|e^{i(2t+\pi/4)}| = 1$, therefore

$$\begin{aligned} E_\infty &= \int_\infty^\infty |x_2(t) = e^{i(2t+\pi/4)}|^2 \mathrm{d}t \\ &= \infty \end{aligned}$$

For power P_∞, we have:

$$\begin{aligned} P_\infty &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x_2(t)|^2 \mathrm{d}t \\ &= 1 \end{aligned}$$

This signal has constant power. If you keep the concept of rotation mentioned in Problem 1.1, you will notice immediately that all the points generated by $x_2(t)$ lies on the unit circle.
$(\mathbf{c})$: $x_3(t) = \cos(t)$

$$\begin{aligned} E_\infty &= \lim_{T\to\infty} \int_{-\infty}^{\infty} |x_3(t)|^2 \mathrm{d}t \\ &= \int_\infty^\infty \cos(t)^2 \mathrm{d}t = \infty \end{aligned}$$

$$\begin{aligned} P_\infty &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x_3(t)|^2 \mathrm{d}t \\ &= \lim_{T\to\infty} \int_{-T}^{T} \cos(t)^2 \mathrm{d}t \\ &= \lim_{T\to\infty} \int_{-T}^{T} \frac{1+\cos(2t)}{2} \mathrm{d}t = \frac{1}{2} \end{aligned}$$

1.3 Problem 1.4

Problem 1.4

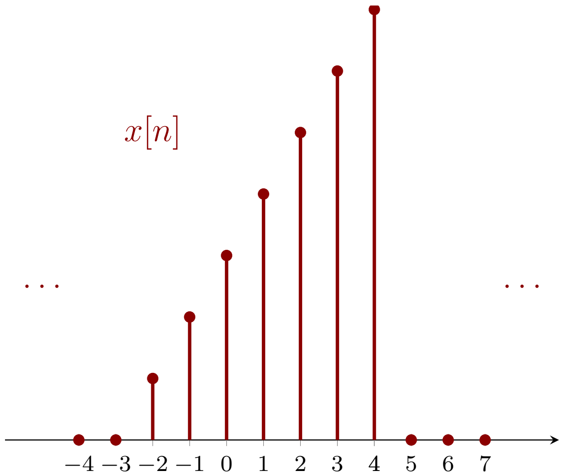
Problem 1.4: Let $x[n]$ be a signal with $x[n] = 0$ for$n < -2$ and $n > 4$, for each signal given below, determine the values of n for which it is guaranteed to be zero.

$$\begin{array}{ll} (\mathbf{a}) & x[n-3] \\ (\mathbf{b}) & x[n+4] \\ (\mathbf{c}) & x[-n] \\ (\mathbf{d}) & x[-n+2] \\ (\mathbf{e}) & x[-n-2] \end{array}$$

For the given signals (**a**) to (**e**), the transformations of the variable n will change the interval in which the signals are zero. For the convenience of calculation, we write the origin signal as:

$$x[m] = 0, m < -2 \quad \text{and} \quad m > 4$$

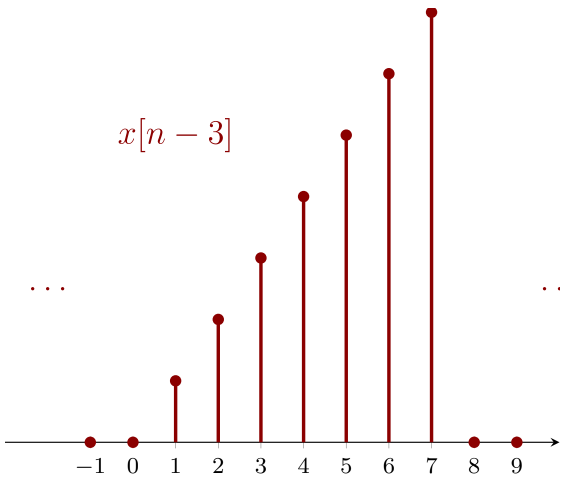
We can visualize $x(m)$ as below (I just give an example, you can name any signal that satisfy $x[m] = 0, m < -2$ and $m > 4$):



(**a**) : $x[n-3]$
For signal (**a**), to get the interval where $x[n-3]=0$, we have:

$$\begin{aligned} t = n-3 &< -2 \\ t = n-3 &> 4 \end{aligned}$$

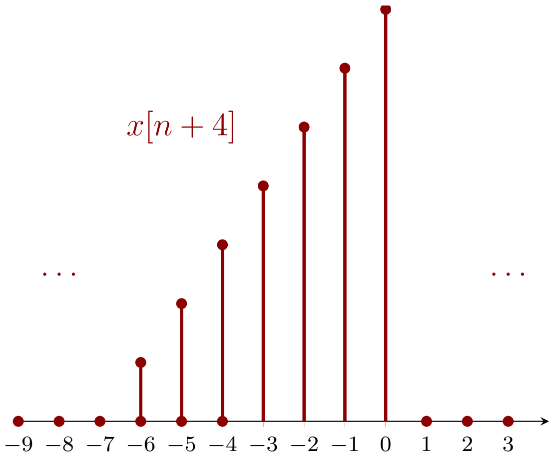
Then, we have $n<1$ and $n>7$ from which we can see that the new signal is a right shift three relative to the origin signal, i.e. new signal is delayed with three.



(**b**) : $x[n+4]$
For signal (**b**), we have:

$$\begin{aligned} t = n+4 &< -2 \\ t = n+4 &> 4 \end{aligned}$$

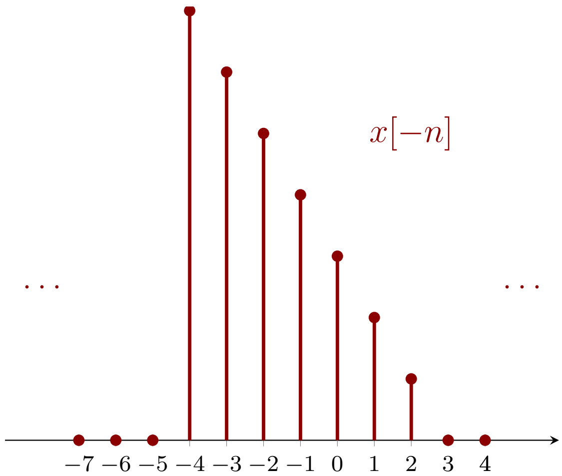
Then, we have $n<-6$ and $n>0$ from which we can see that the new signal is a left shift four relative to the origin singal, i.e. new signal is advanced with four.



$(\mathbf{c}): x[-n]$
For signal (**c**), we have:

$$\begin{aligned} t = -n &< -2 \\ t = -n &> 4 \end{aligned}$$

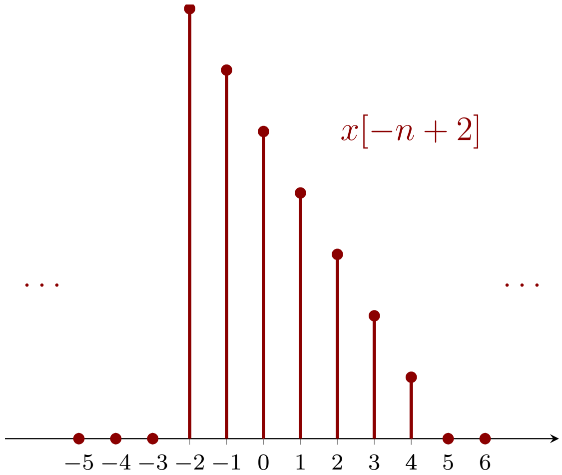
Then, we have $n > 2$ and $n < -4$ from which we can see that the new signal is a reversal of the origin signal.



$(\mathbf{d}): x[-n+2]$
for signal (**d**), we have:

$$\begin{aligned} t = -n+2 &< -2 \\ t = -n+2 &> 4 \end{aligned}$$

Then, we have $n > 4$ and $n < -2$. For $x[-n+2]$, we can first flip the original signal then right shift the flipped signal by 2. Notice the contents in the brackets $-(n+2)$. I would like treat it as $-(n-2)$, by which I know that the minus symbol means reversal and -2 means right shift by 2.



(**e**) : $x[-n-2]$
For signal (**e**), we have:

$$\begin{aligned} t = -n-2 &< -2 \\ t = -n-2 &> 4 \end{aligned}$$

Then, we have $n > 0$ and $n < -6$. To get the new signal, we have to filp the original signal first then left shift the flipped signal by two.