signal and system chapter 1 problems

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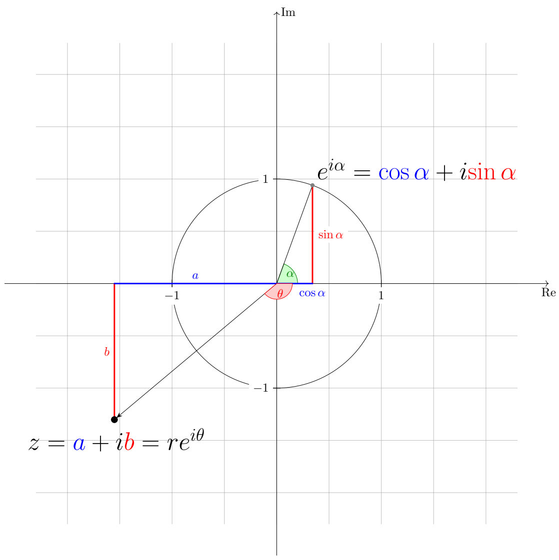

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1 Basic Problems With Answers

Problem 1.1

roblem 1.1: Express each of the following complex numbers in *Cartesian* form $x+iy$:

$$\begin{array}{ccc} \frac{1}{2}e^{i\pi} & \frac{1}{2}e^{-i\pi} & e^{i\pi/2} \\ e^{-i\pi/2} & e^{i5\pi/2} & \sqrt{2}e^{i\pi/4} \\ \sqrt{2}e^{i9\pi/4} & \sqrt{2}e^{-i9\pi/4} & \sqrt{2}e^{-i\pi/4} \end{array}$$

Problem 1.2

roblem 1.2: Express each of the following complex numbers in *polar* form $(re^{i\theta}, -\pi < \theta \le \pi)$:

$$\begin{array}{ccc} 5 & -2 & -3i \\ \frac{1}{2}-i\frac{\sqrt{3}}{2} & 1+i & (1-i)^2 \\ i(1-i) & \frac{1+i}{1-i} & \frac{\sqrt{2}+i\sqrt{2}}{1+i\sqrt{3}} \end{array}$$

First, let's figure them out using the Euler's formula. For the nine complex numbers in Problem 1.1, we have:

$$\begin{aligned}
\frac{1}{2}e^{i\pi} &= \frac{1}{2}(\cos(\pi)+i\sin(\pi)) = \frac{1}{2}\cos(\pi) = -\frac{1}{2} \\
\frac{1}{2}e^{-i\pi} &= \frac{1}{2}(\cos(-\pi)+i\sin(-\pi)) = \frac{1}{2}\cos(-\pi) = -\frac{1}{2} \\
e^{i\pi/2} &= \cos(\pi/2)+i\sin(\pi/2) = i \\
e^{-i\pi/2} &= \cos(-\pi/2)+i\sin(-\pi/2) = -i \\
e^{i5\pi/2} &= \cos(5\pi/2)+i\sin(5\pi/2) = \cos(2\pi+\pi/2)+i\sin(2\pi+\pi/2) = i\sin(\pi/2) = i \\
\sqrt{2}e^{i\pi/4} &= \sqrt{2}(\cos(\pi/4)+i\sin(\pi/4)) = \sqrt{2}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}) = 1+i \\
\sqrt{2}e^{i9\pi/4} &= \sqrt{2}(\cos(9\pi/4)+i\sin(9\pi/4)) = 1+i \\
\sqrt{2}e^{-i9\pi/4} &= \sqrt{2}(\cos(-9\pi/4)+i\sin(-9\pi/4)) = 1-i \\
\sqrt{2}e^{-i\pi/4} &= \sqrt{2}(\cos(-\pi/4)+i\sin(-\pi/4)) = 1-i
\end{aligned}$$

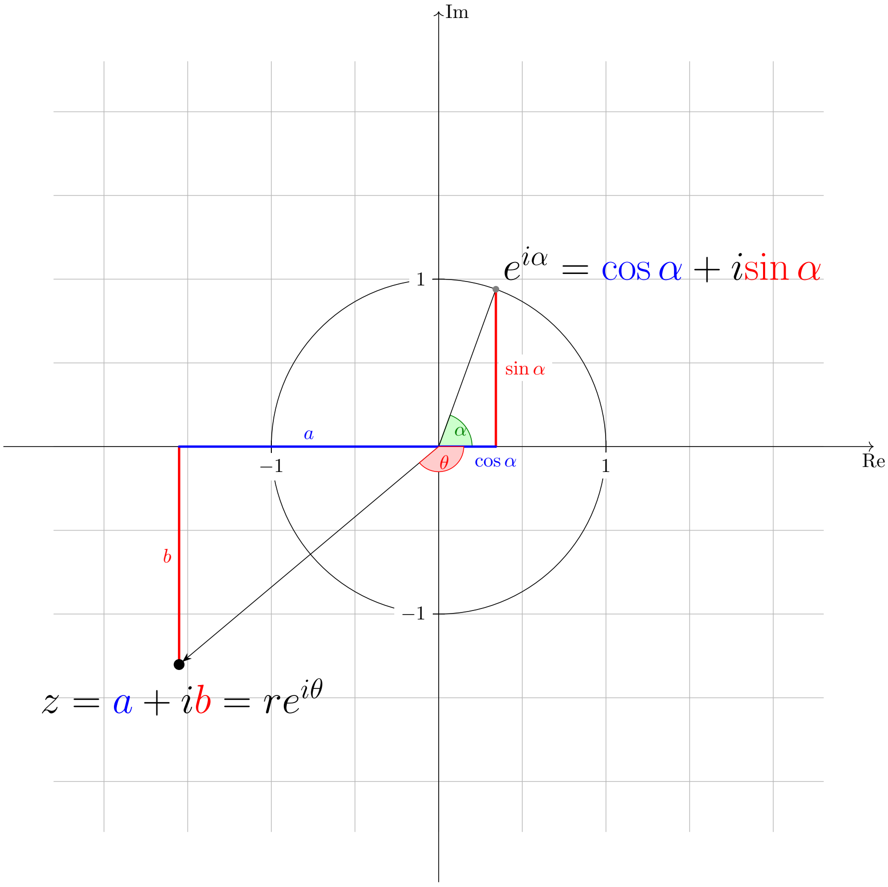
For the nine complex numbers in Problem 1.2 , we have:

$$
\begin{aligned}
5 &= 5e^{i0} = 5e^{i(2\pi n)}, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
-2 &= 2e^{i\pi} = -2e^{i0} \\
-3i &= 3e^{-i\pi} \\
\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2} &= e^{-i\pi/3} \\
1+i &= \sqrt{2}e^{i\pi/4} \\
(1-i)^2 &= (\sqrt{2}e^{-i\pi/4})^2 = 2e^{-i\pi/2} \\
i(1-i) &= 1-i = \sqrt{2}e^{-i\pi/4} \\
\tfrac{1+i}{1-i} &= \tfrac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} = e^{i\pi/2} \\
\tfrac{\sqrt{2}+i\sqrt{2}}{1+i\sqrt{3}} &= \tfrac{2e^{i\pi/4}}{2e^{i\pi/3}} = e^{-i\pi/12}
\end{aligned}
$$

Every complex number $a + ib$ can be visualized in the complex plane $\mathbb{C}$. It can be viewed either as the point with the coordinate (a, b) or as a vector starting from $0, 0$ to the point (a, b).

Also every complex number $a+ib$ can be represented in the exponential form conveniently by using the Euler's formula $e^{i\alpha} = \cos\alpha + i\sin\alpha$. One complex number have unique Cartesian form but do not have a unique expoential form. Taking $1 + i$ for example, its expoential form can be $\sqrt{2}e^{i(\pi/4+2n\pi)}, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. When we express a complex number in expoential form, it helps to keep a concept of rotation in mind. In the complex plane $\mathbb{C}$, a complex number will return to itself if it rotates a multiple of 2π around the origin point with radius equal to its modulus. please keep the concept of roatation in mind and it will become increasingly important during our later study.

In the development of complex analysis, $a + ib$ has another form $r\angle\theta$, where r is the modulus and θ the argument (or the angle). Obviously, this notation is not as good as the expoential form, espacialy when we want to do complex analysis such as differentiation and integration. We just mention it here for the sake of completion. The expoential form will be deployed from now on.



Now, let's go back to Problem 1.1 and Problem Problem 1.2. It's easy to figure the answers out using the Euler's formula. Let's do more to show them on the complex plane keeping the concept of rotation in mind.

From Figure 1 , taking $\frac{1}{2}e^{i\pi}$ and $\frac{1}{2}e^{-i\pi}$ for example, in the complex plane, they are the same point $-\frac{1}{2}$ which means $-\frac{1}{2}$ can be reached by rotating $\frac{1}{2}$ with angle π clockwise or with angle $-\pi$ anti-clockwise. Essentially, this is because that $e^{i\theta} = e^{i(\theta+2n\pi)}, n \in \{\ldots,-2,-1,0,1,2,\ldots\}$. It's straightforward that $e^{i\pi} = e^{i(\pi+2(-1)\pi)} = e^{-i\pi}$.

In the end of Problem 1.1 and Problem 1.1, I want to say more about expressing $\frac{1+i}{1-i}$ in its polar form. There are two methods to get the polar form:

1. multiply the fraction's numerator and denominator by $(1+i)$
   So we have:

$$\begin{aligned} \frac{1+i}{1-i} &= \frac{(1+i)(1+i)}{(1-i)(1+i)} \\ &= \frac{2i}{2} = i \end{aligned}$$

2. express the numerator and denominator in expoential form first, then do the following calculation.

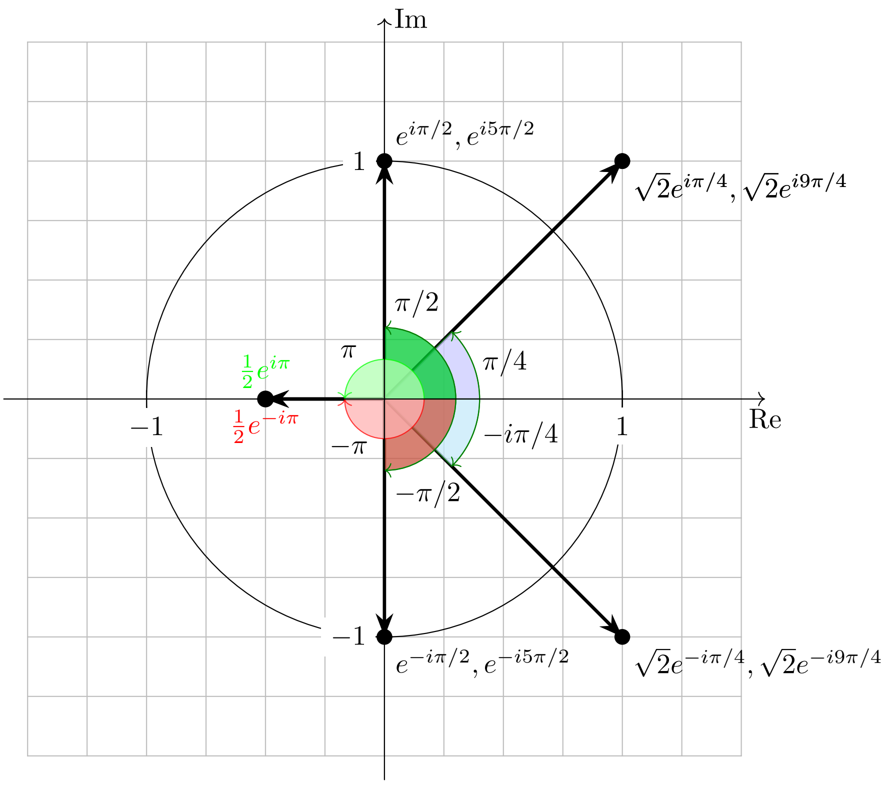


Figure 1: visualized the complex numbers from Problem Problem 1.1 in the complex plan

$$\begin{aligned}\frac{1+i}{1-i} &= \frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} \\ &= e^{i2\pi/4} = e^{i\pi/2} = i\end{aligned}$$