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Computational solutions of three-dimensional advection-diffusion equation using fourth order time efficient alternating direction implicit scheme

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To develop an efficient numerical scheme for three-dimensional advection diffusion equation, higher order ADI method was proposed. 2nd and fourth order ADI schemes were used to handle such problem. Von Neumann stability analysis shows that Alternating Direction Implicit scheme is unconditionally stable. The accuracy and efficiency of such schemes was depicted by two test problems. Numerical results for two test problems were carried out to establish the performance of the given method and to compare it with the others Typical methods. Fourth order ADI method were found to be very efficient and stable for solving three dimensional Advection Diffusion Equation. The proposed methods can be implemented for solving non-linear problems arising in engineering and physics. [http://dx.doi.org/10.1063/1.4996341]

1. INTRODUCTION

In this paper we are dealing with 3-D advection-diffusion equation. Many mathematical models non-linear differential equations plays a very important role in various physical biological and chemical phenomena.[1] Advection-diffusion equation have been applied in many areas of science and engineering, including economics, fluid dynamics, forecasting, astrophysics, oceanography, meteorology, etc.[2,3] To make a mathematical model of physical phenomena, it is essential to understand the behavior of equation. Furthermore advection-diffusion problems have also a key role in computational fluid dynamics to simulate flow problems.[1,3–5] Development of such new proposed schemes with accurate stable and efficient results of advection-diffusion equation is of dynamic importance. 3-D advection-diffusion equation is given by the following equation.[6]

$$u_t + \xi_x u_x + \xi_y u_y + \xi_z u_z = \eta_x u_{xx} + \eta_y u_{yy} + \eta_z u_{zz} \tag{1}$$

where $(x, y, z, t) \in \Omega \times (0\ T]$
with initial conditions
$u(x; y; z; 0) = u_0(x; y; z), \quad (x, y, z) \in \Omega$
The Dirichlet boundary conditions are given by
$u(a; y; z; t) = f_1(x; y; z; t), \quad u(b; y; z; t) = f_2(x; y; t)$
$u(x; c; z; t) = g_1(x; y; z; t), \quad u(x; d; z; t) = g_2(x; y; z; t)$
$u(x; y; e; t) = h_1(x; y; z; t), \quad u(x; y; f; t) = h_2(x; y; z; t)$
where $(x, y, z, t) \in \Omega \times (0\ T]$, $\Omega = \{(x, y, z) : a \le x \le b, c \le y \le d, e \le x \le f\}$ is a cubic domain in R^3, (0, T] is the time interval. $u_0, f_1, f_2, g_1, g_2, h_1, h_2$ are given sufficiently smooth functions and u(x, y, z, t) may represent heat, diffusion, etc. where as ξ_x, ξ_y, ξ_z are constants denotes the convective velocities and the constant η_x, η_y, η_z denotes diffusion coefficients in the direction of (x,y,z) direction respectively.[6]

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As advection-diffusion equation is probably one of the simplest non-linear PDE for which it is possible to obtain an exact solution.[7] Also depending on the magnitude of the various terms in advection-diffusion equation, it behaves as an elliptic, parabolic or hyperbolic PDE, consequently.[8] Recent years researchers did a lot of work on one and two-dimensional convection-diffusion equations, using various finite difference scheme, finite element schemes,[9] higher order finite difference schemes,[8] Adomain decomposition method, finite volume methods[10,11] (2007), Cubic B Spline etc. Dehghan and Mohebbi[12] presented the fourth-order compact finite difference scheme which is unconditionally stable and gave an accuracy of fourth order in space and time. Furthermore Spotz and Carey[13] presented an extension of higher order compact difference techniques which is conditionally stable for steady-state[14] to the time-dependent problems with accuracy of order $t^r + h^4, r \leq 2$. The shooting method Roberts and Shipman[15] moreover Welty[16] developed the graphical methods, Wazwaz[17] worked on modified domain decomposition method, the Cole-Hopf transformation Fletcher,[7] Islam[18] presents radial basis function collocation method, differential transform method Liu and Hou.[19] Mittal and jiwari[20] presents the differential quadrature method for solving two- dimensional convection-diffusion equations. Zhu[21] used this work and find the solution of advection-diffusion equation by Adomian decomposition Thomas,[22] Morton and Mayers,[23] Forsythe and Wasow[24] used this work to solve the advection-diffusion equation.

Many different researchers used advection-diffusion equation to develop new algorithms and to test various existing algorithms. For exact solution of such non-linear problem, researchers used Hopf-Cole transformation to linearize the advection-diffusion equations into parabolic partial differential equation.[25] Three-dimensional advection-diffusion equations, scientists worked a lot by using various finite difference techniques and a lot of method developed. In recent research Dehghan[26] developed both finite difference schemes fully explicit and and fully implicit with constants coefficients. Thongmoom[27] found the solution of three-dimensional advection-diffusion equation using finite difference schemes. Prieto[28] enhance the finite difference method by solving equation (1) explicitly. Moreover (Ge et al., 2013) derived unconditionally stable, highly accurate $O(k^2 + h^4)$ exponential fourth order compact ADI method for solving three dimensional computational problems to demonstrate the high accuracy and efficiency and to show its superiority over the classical Douglas-Gunn ADI scheme and the Karaas high order ADI scheme.[29]

1.1. Problem 1

From literature review, we found that earlier work done on[30] advection -diffusion equation in one and two dimensions using various finite difference schemes. We extended our work to enhance our knowledge towards three-dimensional advection-diffusion equation.[29,30] Two test problems were taken to understand the numerical solution with finite difference schemes. By setting some parameters with arbitrary constants in bounded domain $\Omega = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$. Exact solution of the above three dimensional equation is[6]

$$u(x, y, z, t) = e^{t+x+y+z} \tag{2}$$

where (x,y,z) $\in \Omega$, $t > 0$ and the coefficients $\xi_x = 2, \xi_y = 1, \xi_{z=1}$ and $\eta_x = \eta_y = \eta_z = 1$.[31] Boundary conditions and initial conditions can be taken from exact solution of u(x,y,z,t).

Solving three and two dimensional unsteady Convection-diffusion equations Kalita et al., 2002[32] and Kara[29] derived a higher order compact schemes with weight time discretization respectively. To develop highly accurate results with less time, there have been attempts to develop higher order compact ADI methods. Michell and Faireweather obtained a high order split formula for two dimensional diffusion problems in Mitchell et al., 1964 and later in Dai et al., 2002. Furthermore for higher order compact ADI method Karaa and Zhang,[29] developed those methods for solving two dimensional convection-diffusion equations which gave an accuracy of 2nd order in time and fourth order in space. Additionally in 2006 Karaa developed three dimensional scheme for convection-diffusion equation.[29] S.A.Mahdi derived a fourth order compact exponential ADI finite difference scheme for 3-D unsteady state convection-diffusion equations.[33] In this paper, we derived a higher order finite difference scheme which gave an accuracy of second order in time and fourth order in space, for 3-D unsteady state advection-diffusion equations. In this scheme we have used all points using fourth order compact finite formulas derived from the Taylor's series. The scheme is unconditionally stable with initial values. Numerical experiments for newAlternating Direction Implicit fourth order method showed that this scheme is very efficient and makes a good agreement with the exact solution. The derived results was compared with Karaa ADI fourth order compact scheme.[29] To develop the numerical solutions of 3D advection-diffusion equation with the help of ADI, two test problems have been studied to illustrate the accuracy and stability of the method by obtaining the error norm. Test problems are given to establish the applicability and accuracy of the proposed method.

2. NUMERICAL METHODS

Numerical solution of the three dimensional non-linear advection diffusion equation is in a cubic domain Ω. To define step sizes choose integers L and M and P $hx = (b - a)/L$, $hy = (d - c)/M$ and $h_z = (f - e)/P$ in x, y and z directions respectively. Define Partition up to the interval [a, b] and [c, d] into L and M equal parts of width h_x and h_y and the interval [e,f] in to P equal parts of width h_z. Meshing through a grid by drawing height (z direction) vertical and horizontal lines through the points with coordinates (x_l, y_m, zp), where $x_l = a + lh_x$ for each $l = 0, 1, 2, \ldots L$, $y_m = c + mh_y$ for each $m = 0, 1, 2, \ldots M$ and $z_p = e + ph_z$ for each $p = 0, 1, 2, \ldots P$ also the lines $x = x_l$, $y = y_m$ and $z = z_p$ are grid lines, and their intersections are the mesh points of the grid. For each mesh point in the interior of the grid, (x_l, y_m), z_p, for $l = 1, 2, \ldots, L - 1$, $m = 1, 2, \ldots, M - 1$ and $p = 1, 2, \ldots .P - 1$, applied different algorithms to approximate the non linear solution to the problem in equation (1) (Ref. 6) also we assume $t_n = nk, n = 0, 1, \ldots .\ NT$ where t is the time.

2.1. Second order alternating direction implicit scheme

We apply Alternating Direction Implicit finite difference scheme to equation by integrating equation (1) in the compact way. Before apply ADI scheme in above equation first define some basic formulas:

$$\left.\begin{aligned}
u_t &= \frac{u^{n+1}_{l,m,p} + u^n_{l,m,p}}{dt} \qquad u = \frac{u^{n+1}_{l,m,p} + u^n_{l,m,p}}{2} \qquad u_z = \frac{u^{n+1}_{l,m,p+1} - u^{n+1}_{l,m,p-1} + u^n_{l,m,p+1} - u^n_{l,m,p-1}}{4h_z}\\
u_x &= \frac{u^{n+1}_{l+1,m,p} - u^{n+1}_{l-1,m,p} + u^n_{l+1,m,p} - u^n_{l-1,m,p}}{4h_x} \qquad u_y = \frac{u^{n+1}_{l,m+1,p} - u^{n+1}_{l,m-1,p} + u^n_{l,m+1,p} - u^n_{l,m-1,p}}{4h_y}\\
\delta^2_x \hat{u} &= \frac{\hat{u}_{l+1,m,p} - 2\hat{u}_{l,m,p} + \hat{u}_{l-1,m,p}}{h^2_x} \qquad \delta^2_y \hat{u} = \frac{\hat{u}_{l,m+1,p} - 2\hat{u}_{l,m,p} + \hat{u}_{l,m-1,p}}{h^2_y}\\
\delta^2_z \hat{u} &= \frac{\hat{u}_{l,m,p+1} - 2\hat{u}_{l,m,p} + \hat{u}_{l,m,p-1}}{h^2_z}
\end{aligned}\right\}$$

The derivation of alternating direction implicit scheme are as follows:

Sweep in x-direction

$$\left.\frac{u^*_{l,m,p}}{dt} - \frac{u^n_{l,m,p}}{dt} = \frac{\eta_x}{2}\{\delta^2_x(u^*_{l,m,p} + u^n_{l,m,p})\} + \eta_y\delta^2_y u^n_{l,m,p} + \eta_z\delta^2_z u^n_{l,m,p} - F(u^n_{l,m,p})\right\} \tag{3}$$

where

$F(u^n_{l,m,p}) = \{\xi_x\delta_x u^n_{l,m,p} + \xi_y\delta_y u^n_{l,m,p} + \xi_z\delta_z u^n_{l,m,p}\}$

Sweep in y-direction

$$\left.\frac{u^{**}_{l,m,p}}{dt} - \frac{\eta_y}{2}\delta^2_y u^{**}_{l,m,p} = \frac{u^*_{l,m,p}}{k} - \frac{\eta_y}{2}\delta^2_y u^n_{l,m,p}\right\} \tag{4}$$

Sweep in z-direction

$$\left.\frac{u^{n+1}_{l,m,p}}{dt} - \frac{\eta_z}{2}\delta^2_z u^{n+1}_{l,m,p} = \frac{u^{**}_{l,m,p}}{k} - \frac{\eta_z}{2}\delta^2_z u^n_{l,m,p}\right\} \tag{5}$$

Where $u^*_{l,m,p}$ $u^{**}_{l,m,p}$ defines similarly to $u^{n+1}_{l,m}$. This method is unconditionally stable. The method has accuracy $O(k^2 + h^2)$, Thomas algorithm was used to solve tridiagonal system. finally the scheme makes tridiagonal family of linear system. Iterative methods was carried out to solved this system. The trick used in constructing the Alternating Direction Implicit scheme is to split time step in to three, and apply three different stencils in each half time step, therefore to increment time by one time step in each grid point, we first compute both of these stencils are chosen such that the resulting linear system is tridiagonal. To obtain the numerical solution, we need to solve a non-linear tridiagonal system at each time step. The performance was done by using iterative method and Thomas algorithm. To get high time efficiency, fourth order Alternating Direction Implicit method was used discussed later. Same parameters were used in this method as described above. The derivation of fourth order ADI scheme, we have following steps;

2.2. Fourth order alternating direction implicit scheme

To develop fourth order Alternating Direction Implicit finite difference scheme, to get more accurate and efficient results for the advection diffusion equation, integrating equation (1) in a compact way. Before apply fourth order ADI scheme in above equation, higher order finite difference formulas were defined[34]

$$\left.\begin{aligned}
u_t &= \frac{u^{n+1}_{l,m,p} + u^n_{l,m,p}}{dt}\\
u &= \frac{u^{n+1}_{l,m,p} + u^n_{l,m,p}}{2}\\
\delta_x u^n_{l,m,p} = u_x &= \frac{1}{12h_x}\{-u^n_{l+2,m,p} + 8u^n_{l+1,m,p} - 8u^n_{l-1,m,p} + u^n_{l-2,m,p}\}\\
\delta_y u^n_{l,m,p} = u_y &= \frac{1}{12h_y}\{-u^n_{l,m+2,p} + 8u^n_{l,m+1,p} - 8u^n_{l,m-1,p} + u^n_{l,m-2,p}\}\\
\delta_z u^n_{l,m,p} = u_z &= \frac{1}{12h_z}\{-u^n_{l,m,p+2} + 8u^n_{l,m,p+1} - 8u^n_{l,m,p-1} + u^n_{l,m,p-2}\}\\
\delta^2_x u^n_{l,m,p} = u_{xx} &= \frac{1}{12h_x^2}\{-u^n_{l+2,m,p} + 16u^n_{l+1,m,p} - 30u^n_{l,m,p} + 16u^n_{l-1,m,p} - u^n_{l-2,m,p}\}\\
\delta^2_y u^n_{l,m,p} = u_{yy} &= \frac{1}{12h_y^2}\{-u^n_{l,m+2,p} + 16u^n_{l,m+1,p} - 30u^n_{l,m,p} + 16u^n_{l,m-1,p} - u^n_{l,m-2,p}\}\\
\delta^2_z u^n_{l,m,p} = u_{zz} &= \frac{1}{12h_z^2}\{-u^n_{l,m,p+2} + 16u^n_{l,m,p+1} - 30u^n_{l,m,p} + 16u^n_{l,m,p-1} - u^n_{l,m,p-2}\}
\end{aligned}\right\} \tag{6}$$

When apply Crank-Nicholson by replacing each u as an average of the above formulas fourth order central difference scheme becomes:

$$\left.\begin{aligned}
u_x &= \frac{1}{12*2h_x}\{-(u^{n+1}+u^n)_{l+2,m,p} + 8(u^{n+1}+u^n)_{l+1,m,p} - 8(u^{n+1}+u^n)_{l-1,m,p} + (u^{n+1}+u^n)_{l-2,m,p}\}\\
u_y &= \frac{1}{12*2h_y}\{-(u^{n+1}+u^n)_{l,m+2,p} + 8(u^{n+1}+u^n)_{l,m+1,p} - 8(u^{n+1}+u^n)_{l,m-1,p} + (u^{n+1}+u^n)_{l,m-2,p}\}\\
u_z &= \frac{1}{12*2h_z}\{-(u^{n+1}+u^n)_{l,m,p+2} + 8(u^{n+1}+u^n)_{l,m,p+1} - 8(u^{n+1}+u^n)_{l,m,p-1} + (u^{n+1}+u^n)_{l,m,p-2}\}
\end{aligned}\right\} \tag{7}$$

Truncation error of fourth order first derivative central difference formula in x direction is $\frac{-h^4}{30}\frac{\partial u^5}{\partial x^5}$, same error was in y and z- direction. The truncation error of fourth order second derivative central difference formula in x-direction is $\frac{-h^4}{90}\frac{\partial u^6}{\partial x^6}$ same results in y and z direction. The derivation of fourth order Alternating Direction Implicit scheme applied on advection-diffusion equation are as follows:

Sweep in x-direction

$$\left.\frac{u^*_{l,m,p}}{dt} - \frac{u^n_{l,m,p}}{dt} = \frac{\eta_x}{2}\{\delta^2_x(u^*_{l,m,p} + u^n_{l,m,p})\} + \eta_y\delta^2_y u^n_{l,m,p} + \eta_z\delta^2_z u^n_{l,m,p} - F(u^n_{l,m,p})\right\} \tag{8}$$

By using equation (7) $F(u^n_{l,m,p})$ is as follows:

$F(u^n_{l,m,p}) = \{\xi_x\delta_x u^n_{l,m,p} + \xi_y\delta_y u^n_{l,m,p} + \xi_z\delta_z u^n_{l,m,p}\}$

Sweep in y-direction

$$\left.\frac{u^{**}_{l,m,p}}{dt} - \frac{\eta_y}{2}\delta^2_y u^{**}_{l,m,p} = \frac{u^*_{l,m,p}}{k} - \frac{\eta_y}{2}\delta^2_y u^n_{l,m,p}\right\} \tag{9}$$

Sweep in z-direction

$$\left.\frac{u_{l,m,p}^{n+1}}{dt} - \frac{\eta_z}{2}\delta_z^2 u_{l,m,p}^{n+1} = \frac{u_{l,m,p}^{**}}{k} - \frac{\eta_z}{2}\delta_z^2 u_{l,m,p}^{n}\right\} \tag{10}$$

Where $u_{l,m,p}^{*}$ $u_{l,m,p}^{**}$ defines similarly to $u_{l,m}^{n+1}$. This method is also unconditionally stable. Accuracy of this method is 2nd order in time and fourth order in space $O(k^2 + h^4)$. Thomas algorithm was used to solve this system, using this algorithm family of linear system were solved. Time efficient Iterative methods and deep meshing would carried out to solved this system. Advection diffusion equation firstly, we applied implicit method in the x-direction and explicit in y-direction and z direction to get better accuracy from level n to a level n^* and secondly applied implicit method in y-direction and explicit method in x-direction and z-direction from level n^* to level n^{**}. In third step implicit method was carried out in z-direction and explicit method was used in x-direction and y-direction from level n^{**} to level n^{n+1}. Parameters, operators, coefficients and notations that are defined above was used for this fourth order Alternating Direction Implicit method. The compact ADI scheme of three dimensional advection-diffusion equation produced very accurate, stable and time efficient results. To get high time efficiency, fourth order alternating direction implicit method is playing a very important part and giving us very good results.

Stability Analysis

To construct stability analysis for the three dimensional advection-diffusion equation. By using von neuman stability analysis, the standard von-neuman stability analysis leads to the amplification factor of Douglas ADI can be found as.

$$Lambda = \frac{1 + (a_x + a_y + a_z) + (a_x a_z + a_y a_z + a_x a_z) + a_z a_x a_y}{(1 + a_x)(1 + a_y)(1 + a_z)} \tag{11}$$

we see that $lambda \leq 1$ without any constraints always hold true, i.e. This shows that the scheme is unconditionally stable.

where $a_x = 2sin^2 \frac{1}{2} k_x h_x$ $\quad a_y = 2sin^2 \frac{1}{2} k_y h_y$ $\quad a_z = 2sin^2 \frac{1}{2} k_z h_z$

1.2. Problem:2

In this problem the cubic domain of three dimensional nonlinear advection-diffusion equation (1) is given as $\Omega = [(x, y, z) : 0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2]$. Exact solution of the above two dimensional equation is

$$u(x, y, z, t) = \frac{1}{(4t + 1)^{\frac{3}{2}}} e^{\{\frac{-1}{\eta(1+4t)}\}\{(x-\xi t-0.5)^2+(y-\xi t-0.5)^2+(z-\xi t-0.5)^2\}} \tag{12}$$

where (x,y,z) $\in \Omega$, $t > 0$ and $\eta_x = \eta_y = \eta_z = \eta = 0.01$ *and* $\xi_x = \xi_y = \xi_z = \xi = 0.8$ are coefficients of this problem. Boundary conditions and initial conditions can be taken from exact solution of u(x,y,z,t). where Ω is a cubic domain in R^3. The main objective of the paper is to find efficient solution of unknown $u(x, y, z, t)$. Two test problems were described to understand the numerical solution by taking efficient finite difference schemes up to order 4. Advection diffusion equation has been extensively studied to describe various kinds of phenomena which can be seen from equation (Ref. 8).

Algorithm

Clearly, the system is tridiagonal and can be solved with Thomas algorithm. In general a tridiagonal system can be written as,

$$a_l x_{l-1} + b_l x_l + c_l x_{l+1} = S_l, \quad with \;\; a_1 = c_l = 0$$

above system can be written as in a matrix-vector form,

$$Au = S$$

Where A is a coefficient matrix. Right hand side is column vector which is known. Our main goal is to find the resultant vector u. Now we have

$$A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & \cdots 0 \\ a_2 & b_2 & c_2 & 0 & 0 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_{n-1} & b_{n-1} & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots & a_n & b_n \end{bmatrix}$$

$$\underline{u} = [u_1, u_2, u_3, \cdots, u_n]^t$$
$$\underline{S} = [s_1, s_2, s_3, \cdots, s_n]^t$$

where $b_l = \frac{1}{k} + \frac{\eta_x}{h_x}$ .... $l = 1, 2, \ldots.\ n$, $a_l = \frac{-\eta}{2h_x}$ .... $l = 2, \ldots.\ n$, $c_l = \frac{-\eta}{2h_x}$ .... $l = 1, 2, \ldots.\ n-1$ same for y and z-direction. By equating both sides of the $Au = S$, = Implementation of Thomas algorithm are shown in results.

Numerical solution of penta-diagonal-system in general can be written as :

$$a_l x_{l-2} + b_l x_{l-1} + c_l x_l + d_l x_{l+1} + e_l x_{l+2} = S_l, \quad \text{with } a_1 = a_2 = b_1 = d_n = e_{n-1} = e_n = 0 \tag{13}$$

The above system in (13) can be written as in a matrix-vector form,

$$Au = S$$

Where A is a coefficient matrix. Right hand side is column vector which is known. Our main goal is to find the resultant vector u. Now we have

$$A = \begin{bmatrix} c_1 & d_1 & e_1 & 0 & 0 & \cdots 0 \\ b_2 & c_2 & d_2 & e_2 & 0 & \cdots 0 \\ a_3 & b_3 & c_3 & d_3 & e_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & e_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & d_{n-1} \\ \cdots & \cdots & \cdots & a_n & b_n & c_n \end{bmatrix} \tag{14}$$

$$\underline{u} = [u_1, u_2, u_3, \cdots, u_n]^t \tag{15}$$
$$\underline{S} = [s_1, s_2, s_3, \cdots, s_n]^t \tag{16}$$

where $c_l = \frac{1}{k} + 30\frac{\eta_x}{24h_x^2}$ .... $l = 1, 2, \ldots.n$, $d_l = -16\frac{\eta_x}{24h_x^2}$ .... $l = 2, \ldots.n-1$, $e_l = \frac{\eta_x}{24h_x^2}$ .... $l = 1, 2, \ldots.n - 2$, $b_l = -16\frac{\eta_x}{24h_x^2}$ .... $l = 2, \ldots.n$, $c_l = \frac{\eta_x}{24h_x^2}$ .... $l = 3, \ldots.n$ same for y and z-direction.

By equating both sides of the $Au = S$, = The solution to the linear system Au = S was found in three steps by using forward and then backward substitution at the end we got the value of the unknown u(x,y,x,t).The computational results of penta-diagonal system were carried out by using two test problems discussed later.

3. ERROR NORMS

The accuracy and consistency of the schemes is measured in terms of error norms specially L_2 and L_∞ which are defined as:

$$L_\infty = |U_{i,j,k} - u_{i,j,k}| \tag{17}$$

$$L_2 = \sqrt{\frac{\sum_{i,=1}^{L} (|U_i - u_i|)^2}{L}} \tag{18}$$

where u(x,y,t) and U(x,y,t) denote the numerical and exact solutions at the grid point (x_l, y_m, z_p, t_n).

4. RESULTS AND DISCUSSION

Numerical results were presented on a uniform grid. Two methodologies were performed using the same Alternating Direction Implicit scheme. In Table I numerical results was performed, compared with the analytical results by changing typical mesh points at (0.125, 0.125, 0.0563), (0.125, 0.0875, 0.0563) etc; with grids size 25×25, time step dt = 1.953125e-04, step size = 0.00625, time level t = 0.01. Difference of analytical and approximate results were calculated in Table I. Error norm were calculated in Table II by changing step size and time step with fixed grid size 25 × 25, for the unknown u(x,y,z,t). Computational time was also calculated using matlab 15.2b produced very refine results. In literature existing results was compared with the obtained results. For convergence criteria L_2, L_∞ norm was taking an account for the unknown u(x,y,t). M.Tamseer et al. (2016), M.D.Compas (2014) and S.Karra were considered this problem in (2004) and (2006).[6,8,29] Error analysis of L_2 norm at time step dt = 0.01, step size = 0.1, 0.05, grid size = 30 × 30 at different time level (0.1,0.2,0.3...) in Table III problem 1. Fourth order compact ADI was used in two dimensional coupled convection-diffusion equations recently researchers worked on three dimensional using this scheme. In figure 1 comparison of analytical and numerical results of ADI at fixed grid size 25 × 25, time level = 0.01, time step dt = 7.8124e-04 along z-direction z = 0.0875 at height level 15. Increasing grid size got more accurate results matched, best agreement with the exact solution. To develop approximate result at dt = 0.1, step size = 0.1 at z level 10 with fixed grid size = 25 × 25, in figure 2. To attain more refine and better results matched with the exact solution by changing grid size, deep meshing produced excellent result in figure 3 at time step = 0.125, h = 0.025, grid size 21 × 21 at z = 0.2750, height level = 12 in z-direction. In Table IV error analysis showed L∞ norm by changing step size and time step with fixed grid size 25 × 25, CPU timing showed that how much Alternating Direction Implicit scheme

TABLE I. Comparison of Analytical and Exact solution at different typical mesh points with time step dt=1.953125e-04, h=0.00625 at t=0.01 and grid size=25×25 for unknown u(x,y,t).

	Numerical solution of ADI $O(k^2 + h^2)$ Problem 1		
	Solution Comparison		
(Typical mesh points)	Approx-u	Exact-U	abs(U-u)
(0.0125,0.0125,0.0563)	1.0846	1.0955	0.0109
(0.0125,0.0875,0.0563)	1.1691	1.1809	0.0117
(0.0188,0.0188,0.0563)	1.0983	1.1093	0.0110
(0.0875,0.0125,0.0563)	1.1691	1.1809	0.0117
(0.0125,0.0563,0.0563)	1.1331	1.1445	0.0114
(0.1438,0.0875,0.0563)	1.3331	1.3465	0.0134
(0.1313,0.1375,0.0563)	1.3840	1.3979	0.0139
(0.0313,0.0375,0.0563)	1.1331	1.1445	0.0114
(0.0938,0.0625,0.0563)	1.2368	1.2492	0.0124
(0.1250,0.0063,0.0563)	1.2062	1.2184	0.0121

TABLE II. Error analysis by changing step size at different time level with fixed grid size=25×25 for unknown u(x,y,t). **Self time** is the time spent in a function excluding the time spent in its child functions. Self time also includes overhead resulting from the process of profiling. **Total time** is a total time spent in a function.

		Numerical solution of ADI $O(k^2 + h^2)$ Problem 1		
		Solution Comparison		
dt	h	L_∞	selftime	total time
0.1	0.1	3.127e-02	47.662 s	49.662 s
0.05	0.05	3.164e-02	86.448s	89.445s
0.025	0.025	8.145e-03	107.44s	111.34s
0.0125	0.02	9.213e-03	189.45s	198.34s
0.005	0.01	2.234e-04	225.87s	256.79s
0.0025	0.015	2.123e-04	449.23s	512.31s
0.001	0.001	2.097e-04	667.12s	713.12s

TABLE III. Error analysis L_2 norm by changing time level with step size=0.1, 0.05, time step=0.01 and grid size=30×30 for unknown u(x,y,t).

	Numerical solution of ADI $O(k^2 + h^2)$ Problem 2	
	Solution Comparison	
time level	step size=0.1	step size=0.05
(0.1)	1.7656e-04	1.0023e-05
(0.2)	1.3145e-04	3.9766e-06
(0.3)	1.0234e-04	2.9987e-06
(0.4)	9.6734e-05	2.8765e-06
(0.5)	8.4508e-05	2.7654e-06
(0.6)	6.9834e-05	4.9834e-06
(0.7)	3.9976e-05	3.9812e-06
(0.8)	2.0976e-05	1.2344e-06
(0.9)	8.2234e-06	3.0817e-07
(1.0)	2.0093e-06	3.1134e-07

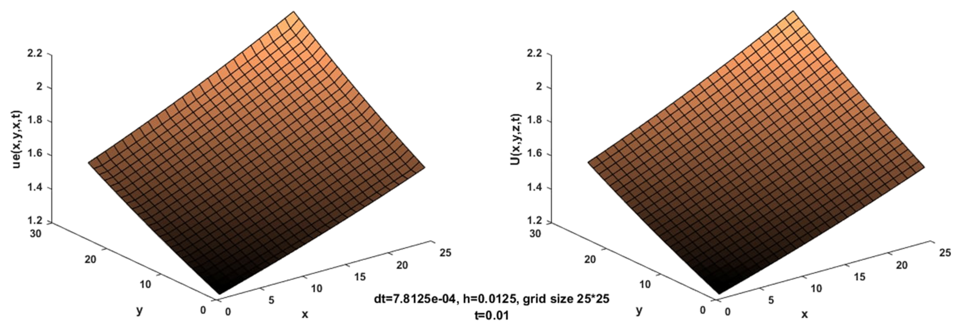

FIG. 1. analytical and numerical results of ADI at z=0.0875(:,:,15), t=0.01,dt=7.8125e-04, step size=0.0125 at grid size 25×25 problem 1 where U(x,y,z,t) denotes the exact solution and ue(x,y,z,t) represents the approximate results of the given problem.

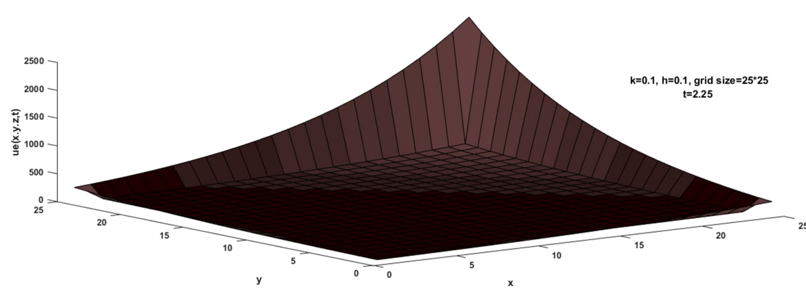

FIG. 2. numerical results of ADI at z=0.0563(:,:,10), t=2.25, grid size=25×25, dt=0.1, step size=0.1 problem 1.

was efficient w.r.t time. Self time is the time shows excluding the time spent in its child function and total time is total time spent in a function. Furthermore comparison analytical and numerical solution using Alternating Direction Implicit scheme at dt = 1.9531e-04, h = 0.00625, t = 0.01 at height 10 in z -direction at fixed grid size 25 × 25 explained in figure 4. Moreover in figure 5 and 6 analytical and numerical results of problem 1 and problem 2 was compared. In figure 5 grid size increased 25 × 25 at time step = 1.9531e-04, step size = 0.00625, time = 0.01, z = 0.1188 at height level = 20. Grid size 21 × 21, with times step = 3.125e-03, step size = 0.025, t = 0.01 and z = 0.01 at height level 5 presented in figure 6. Figures 5 and 6 presents excellent results matched approximately with the exact one for the unknown u(x,y,z,t). In Table V L_∞ norm was calculate using ADI fourth order scheme by changing time step dt at grid size 25 × 25, t = 1.25 and h = 0.025. In this table results was compared with the results of Douglas-Gun ADI Scheme and Karaa ADI Scheme with same parameters. Results showed the same behavior corresponds to the results already explained in the literature. The obtained

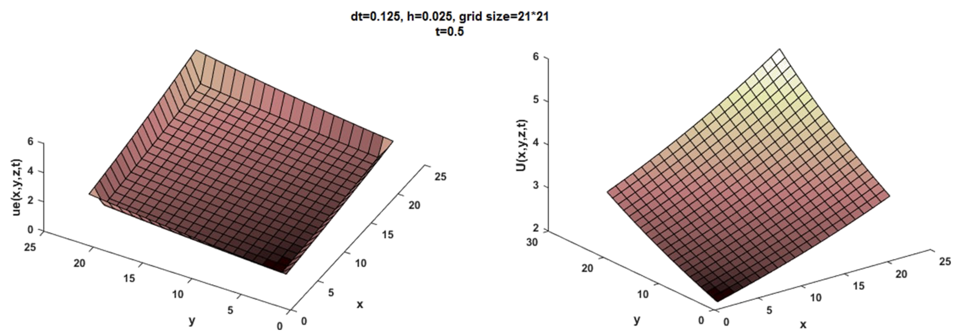

FIG. 3. analytical and numerical results of ADI at z=0.2750(:,:,12), t=0.5, dt=0.125, step size=0.025 at grid size 21 × 21 problem 1.

TABLE IV. Error analysis by changing step size at different time level with fixed grid size=25×25 for unknown u(x,y,t). **Self time** is the time spent in a function excluding the time spent in its child functions. Self time also includes overhead resulting from the process of profiling. **Total time** is a total time spent in a function.

Numerical solution of ADI $O(k^2 + h^4)$ Problem 1				
Solution Comparison				
time step	step size	L_∞	selftime	total time
0.1	0.1	5.9134e-04	87.662s	49.662 s
0.05	0.05	2.4906e-04	132.448s	89.445s
0.01	0.025	5.9734e-06	201.44s	123.34s
0.005	0.02	5.3123e-07	302.45s	192.34s
0.001	0.01	2.2234e-06	340.87s	243.79s

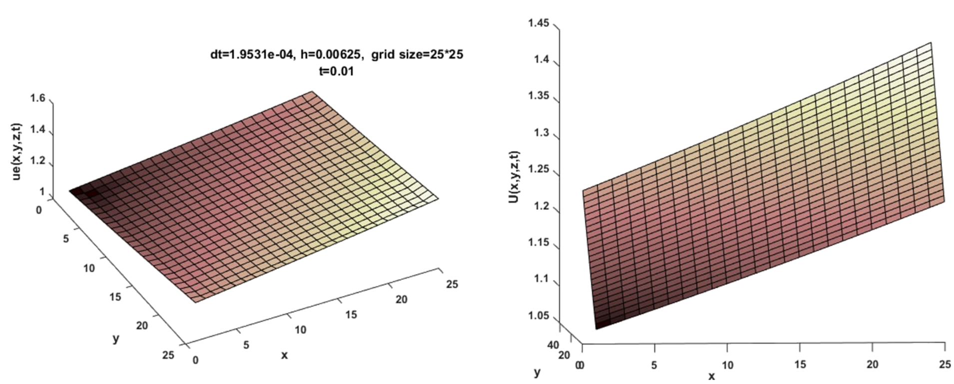

FIG. 4. analytical and numerical results of ADI at z=0.0563(:,:,10), t=0.01, dt=1.9531e-04, step size=0.00625 at grid size 25 × 25 problem 1.

solutions were better than those obtained in earlier studies (Karaa et al. (2006), M.M.Gupta et al. (1984), Kara et al. (2004), Harfash et al. (2010)).

In problem 2, considering equation (1) over the domain $[0, 2] \times [0, 2] \times [0, 2]$, boundary and initial conditions were taken from the exact solution showed stable results time step and increasing grid size (refine mesh size). In Table VI L_2 error norm was calculated and compared their results with M.D.Compas,[8] results using fourth order ADI Implicit scheme at time 1.25, h = 0.05, grid size = 20 × 20, attained stable results. L_2 norm results was compared with the results of Samir Karaa by changing time step at grid size 30 × 30, step size = 0.5, time t = 1 in Table VII. Analysis explained that results makes a good combination with the analytical solution. Furthermore in figure (7, 8) approximate solution was obtained using fourth order Alternating Direction Implicit scheme by changing time step = (6.25e-02, 0.125) with same step size = 0.025, time = 0.25 and grid size 21 × 21

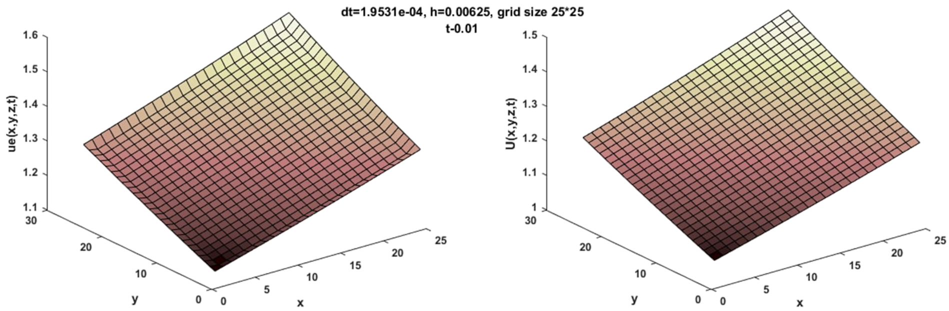

FIG. 5. analytical and numerical results of ADI at z=0.1188(:,:,20), t=0.01,dt=1.9531e-04, step size=0.00625 at grid size 101×101 problem 1.

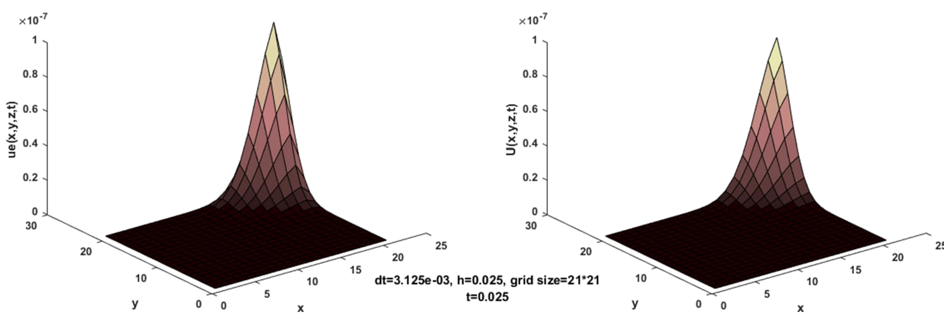

FIG. 6. analytical and numerical results of ADI at z=0.1(:,:,5), t=0.025, dt=3.125e-03, step size=0.025 at grid size 21×21 problem 2.

TABLE V. Calculating Error L_2 norm of ADI $O(k^2 + h^4)$ using different parameters for unknown values $u(x, t)$ at grid size 25×25.

Errors for different time step at R=$\frac{dt}{h^2}$ at t=1.25,h=0.025,dt=$h_1 \times R$ where h_1 = 0.000625 Problem 2			
Solution Comparison			
dt	*Douglas – Gunn ADIscheme*[35]	*KaraasADIscheme*[29]	present work
0.003125	5.599e-04	5.184e-05	6.361e-06
0.00625	5.764e-04	5.268e-05	2.553e-05
0.0125	5.991e-04	6.978e-05	5.645e-05
0.025	7.329e-04	2.035e-04	2.917e-04

TABLE VI. Calculating Error L_2 norm of ADI $O(k^2 + h^4)$ using different parameters for unknown values $u(x, t)$ at grid size 20×20.

Errors for different time step at R=$\frac{dt}{h^2}$ at t=1.25,h=0.05,dt=$h_1 \times R$ where h_1 = 0.0025 Problem 2		
Solution Comparison		
dt	*M.D.Campos*[8]	present work
0.0125	4.900e-05	5.4431e-06
0.025	5.431e-05	6.4432e-06
0.05	8.152e-05	7.1324e-05
0.1	2.413e-04	3.9976e-04

at height level = 10 in z-direction. In figure (9) analytical and numerical results was compared at dt = 0.0125, step size = 0.05, time = 0.25 with fixed grid size = 25×25. Analytical and numerical results was compared at time step = 0.0125, t = 1.25, step size = 0.025 with grid size = 25×25 in figure 10.

TABLE VII. Calculating Error L_2 norm of ADI $O(k^2 + h^4)$ using different parameters for unknown values $u(x, t)$ at grid size 30×30.

	Errors for different time step at h=0.5,time=1 Problem 2	
	Solution Comparison	
dt	*SamirKaraa*[29]	present work
0.1	1.40E-03	1.123e-03
0.001	1.41E-03	1.114e-03
0.002	1.41E-03	1.21e-03
0.001	1.41E-03	1.21e-03

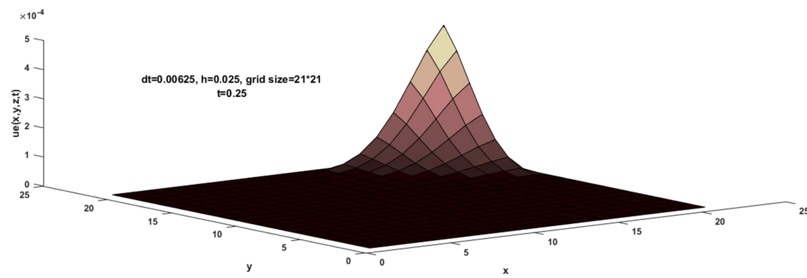

FIG. 7. numerical results of ADI at z=0.2250(:,:,10), t=0.25, dt=6.25e-02, step size=0.025 at grid size 21×21 problem 2.

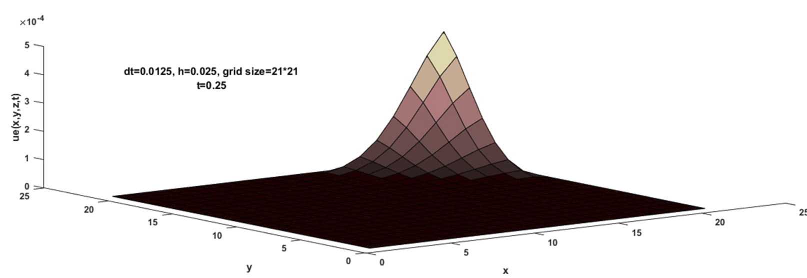

FIG. 8. numerical results of ADI at z=0.2250(:,:,10), t=0.25, dt=0.125, step size=0.025 at grid size 21×21 problem 2.

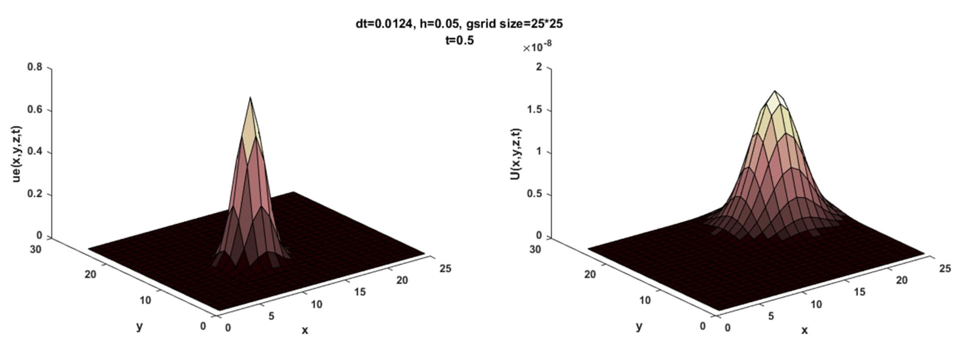

FIG. 9. analytical and numerical results of ADI at z=0.4500(:,:,12), t=0.25, dt=0.0125, step size=0.05 at grid size 25×25 problem 2.

In Table VI increasing time step error increased and by decreasing step size in Table VII stable results were obtained. Moreover figures presents very refine results in time efficient manner. Table V showed the same behavior by increasing time step error increased. Table II and III showed that by increasing time step and time level error reduced and the solution corresponds towards the exact solution.

Results gained from both methodologies using ADI scheme at very small time step and step size produced very accurate results due to error reduction. By increasing time level sharp edges were

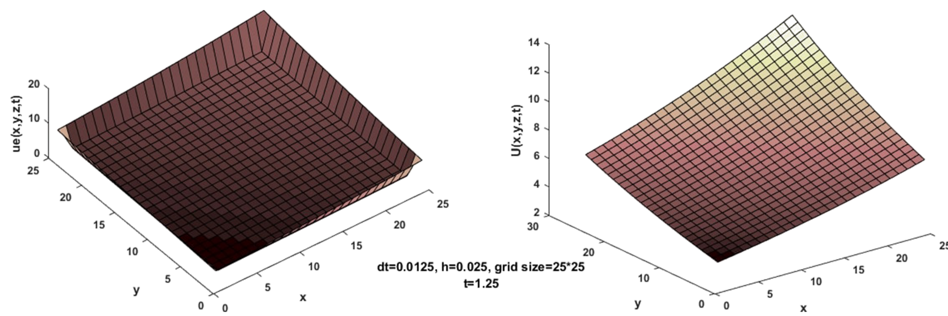

FIG. 10. analytical and numerical results of ADI at z=0.1(:,:,5), t=1.25, dt=0.0125, step size=0.025 at grid size 25 × 25 problem 1.

removed. The obtained results are very interesting to understand the efficiency of the ADI scheme using both methodologies. The corresponding graphical representation for the solution of unknown u(x, y, z, t) was presented in Figures (9) and (10). Moreover From graphical illustrations, obtained numerical results give steady state solution and the scheme is unconditionally stable.

Fourth order ADI scheme gives more accuracy as compared with the 2nd ADI schemes reduces to the 2nd order ADI scheme and solve the system very efficiently. The truncation error: O $(k^2 + h^4)$. Fourth order ADI implementation is computationally in a time efficient manner. Fourth order Alternating Direction Implicit method was very efficient and easy method for implementation in recent research and is applicable for simple and ideal problems presents good results.

4.1. Conclusion

In this research paper a numerical treatment for three dimensional advection-diffusion equation was discussed by means of Alternating Direction Implicit scheme. Calculated results showed excellent agreement towards the analytical solution. Two test problems were taken to illustrate the accuracy efficiency and stability of the schemes. Numerical results showed that the proposed fourth order compact finite difference schemes is more efficient and produced more accurate results than the second order finite difference scheme. Both finite difference schemes are unconditionally stable and highly accurate. For convergence L_2 *and* L_∞ norms were treated towards zero when time step was increased. The proposed methods can be used to stimulate the numerical solution of the advection diffusion equation, showed in two test problems. The approach used in this paper we observed that fourth order Alternating Direction Implicit scheme is very good ,excellent, accurate and stable for the solutions of heat transfer equations.

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[1] J. Li and Y. Tung Chen, *Computational Partial Differential Equations Using Matlab* (Chapman and Hall book, CRC Press, Newyork, 2008).
[2] J. N. Noye, *Nonlinear Partial Differential Equations in Engineering* (North-Holland Publishing Comp. Conference in Queen's College, University of Melbourne, Australia, 1981).
[3] W. F. Ames, *Non-linear Partial Differential Equations in Engineering* (Academic Press, New York, 1965).
[4] W. F. Ames, *Finite Difference Methods For Partial Differential Equations* (Academic Press, New York, 1969).
[5] W. E. Schiesser and G. W. Griffiths, *A Compendium of Partial Differential Equation Models* (Cambridge University Press, 2009).
[6] M. T. Tamseer, V. K. Srivastava, and P. D. Mishra, "Numerical simulation of three dimensional advection-diffusion equations by using modified cubic B-spline differential quadrature method," Asia Pacific Journal of Engineering Science and Technology (2016).

[7] C. A. Fletcher, "Generating exact solutions of the two-dimensional Burgers equations," International Journal for Numerical Methods in Fluids **3**, 213–216 (2016).
[8] M. D. Campos and E.C. Romao, "Higher-order finite difference method applied to the solution of the three -dimensional heat transfer equation and to the study of heat ex changers," Engenharia Trmica (Thermal Engineering) **13**(2) (2014).
[9] S. D. Dhawam, S. K. Kapoor, and S. K. Kumar, "Numerical method for advection diffusion equation using FEM and B-splines," Journal Of Computer Science **3**, 429–2378 (2012).
[10] W. Malalasekara and Versteeg, *An Introduction to Computational Fluid Dynamics Method*. Pearson, Edition 2: 2007.
[11] S. M. Mazumder, *Numerical methods for partial differential equations: finite difference and finite volume methods* (Academic Press, New York, 2015).
[12] M. Dehghan and A. Mohebbi, "High-order compact boundary value method for the solution of unsteady convection-diffusion problems," Math. Comput. Simulat. **79**, 683–699 (2008).
[13] W. F. Sportz and G. F. Carey, "Extension of high-order compact schemes to time-dependent problems," Numer. Methods Partial Diff.Eq. **17**, 657–672 (2001).
[14] W. F. Sportz, "High-order compact finite difference schemes for computational mechanics," PhD Thesis. University of Texas at Austin, Austin, TX, 1995.
[15] S. M. Robert and J. S. Shipman, "Two point boundary value problems. A shooting methods" (American Elsevier, 1972).
[16] J. R. Welty, C. E. Wilson, and G. L. Rorrer, *Fundamental of Heat and Mass Transfer*, Edition 4 (Wiley, 2001).
[17] A. M. Wazwaz, "The numerical solution of sixth order boundary value problems by the modified decomposition method," Applied Mathematics and Computation **18**, 311–325 (2001).
[18] S. Islam, B. Sarler, R. Vertnik, and G. Kosec, "Radial basis function collocation method for the numerical solution of the two-dimensional transient nonlinear coupled Burgers equations," Applied Mathematical Modelling **36**, 1148–1160 (2012).
[19] J. C. Liu and G. L. Hou, "Numerical solutions of the space and time fractional coupled Burgers equations by generalized differential transform method," Applied Mathematics and Computation **217**, 7001–7008 (2011).
[20] R. C. Mittal and R. Jiwari, "Differential quadrature method for two-dimensional Burgers' equations," International Journal for Computational Methods in Engineering Science and Mechanics **10**, 450–459 (2009).
[21] H. Zhu, H. Shu, and M. Ding, "Numerical solutions of two-dimensional Burgers' equations by discrete Adomian decomposition method," Computers and Mathematics with Applications **60**, 840–848 (2010).
[22] J. W. Thomas, *Numerical Partial Differential Equations. Finite Difference Methods*, Springer-Verlag, 1995.
[23] K. W. Morton and D. F. Mayers, *Numerical Solution of Partial Differential Equations*. Cambridge University Press, 1994.
[24] G. E. Forsythe and W. R. Wasow, *Finite Difference Methods for Partial Differential Equations*. Literary Licensing., 2013.
[25] V. K. Srivastava, Asutosh, and M. Tamsir, "Generating exact solution of three dimensional coupled unsteady nonlinear generalized viscous Burgers equations," International Journal of Modern Mathematical Sciences **5**, 1–13 (2013).
[26] M. Dehghan, "Numerical solution of the three-dimensional advection-diffusion equation," Applied Mathematics and Computation **150**, 1–19 (2004).
[27] M. Thongmoon, R. Mckibbin, and S. Tangmanee, "Numerical solution of a 3-D advection-dispersion model for pollutant transport," Thai Journal of Mathematics **5**, 91–108 (2007).
[28] U. Prieto, J. J. Benito Muoz, and L. G. Corvinos, "Application of the generalized finite difference method to solve the advection-diffusion equation," Journal of Computational and Applied Mathematics **235**, 1849–1855 (2011).
[29] S. Kara and J. Zang, "High order ADI method for solving unsteady convection-diffusion problems," J. Comput. Phys **29**, 1–9 (2004).
[30] S. Kara, "A high order compact ADI method for solving three-dimensional unsteady convection-diffusion problems," Numer. Methods Partial Differential Eqs. **22**, 1–11 (2006).
[31] R. C. Mittal and A. Tripathi, "Numerical solutions of two-dimensional Burgers equations using modified Bi-cubic B-spline finite elements," Dept of Math, Emerald Insight, 2, April 2014.
[32] J. C. Kalita, D. C. Dalal, and A. K. Dass, "A class of higher order compact schemes for the unsteady two dimensional convection-diffusion equation with variable convection coefficients," Int. J. Numer. Methods Fluids. **29**, 1111–1131 (2002).
[33] S. A. Mahdi, "Fourth order exponential compact alternating direction implicit (ADI) scheme for solving three-dimensional convection-diffusion equations" (submitted for publication).
[34] A. A. Akbar Fatah Bajellan, *Computation of the Convection-Diffusion Equation by the Fourth Order Compact Finite Difference Method*. zmir Institute of Technology, Jan, 2015.
[35] Y. Ge, Z. F. Tian, and J. Zhang, "An exponential high-order compact ADI method for 3D unsteady convection diffusion problems," Numerical Methods for Partial Differential Equations **29**, 186–205 (2013).