Noncentral Skewed HEAVY Model for High-Dimensional Realized Returns and Covariances Matrices

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Abstract

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JEL subject classifications:

Keywords:

1 Introduction

2 Noncentral HEAVY GAS skew tF Model

Assumptions:

(A1) The vector of asset returns $\mathbf{y}_t \in \mathbb{R}^p$ conditionally on the observation set $\mathcal{F}_{t-1}$ is assumed to follow a normalized multivariate skew Student's t distribution with density given by

$$f_y(\mathbf{y}_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1}; \nu_0) = 2 f_{T_{\nu_0}}(\mathbf{y}_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1}) F_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top \mathbf{y}_t}{(\nu_0 - 2 + \mathbf{y}_t^\top \mathbf{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0 + p}\right), \tag{2.1}$$

where the function $F_{T_{\nu_0}}(\cdot)$ denotes the c.d.f. of the standard univariate t-distribution with ν_0 degrees of freedom and $f_{T_{\nu_0}}(\cdot)$ is density of the (standardized) multivariate central t distribution with ν_0 degrees of freedom given by

$$f_{T_{\nu_0}}(\mathbf{y}_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1}) = \frac{\Gamma[(\nu_0+p)/2]}{\Gamma[\nu_0/2][\pi(\nu_0-2)]^{p/2}} |\mathbf{\Sigma}_t|^{-1/2} \left(1 + \frac{\mathbf{y}_t^\top \mathbf{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0 - 2}\right)^{-(\nu_0+p)/2}. \tag{2.2}$$

The vector $\boldsymbol{\alpha}$ stays for the skewness parameters.

Basic properties of multivariate skew t:

(i) $\boldsymbol{\alpha} = \mathbf{0}$ implies the central multivariate t-distribution with ν_0 degrees of freedom.

(ii) $\lim_{\nu_0 \to \infty} f_{T_{\nu_0}}(\mathbf{y}_t) = 2\phi_{\mathbf{0},\mathbf{\Sigma}_t}(\mathbf{y}_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1})\Phi(\boldsymbol{\alpha}^\top \mathbf{y}_t)$, i.e., the multivariate skew t tends to the multivariate skew normal as $\nu_0 \to \infty$.

(iii) Let $\sigma_{t,jj}$ is the j-th diagonal element of $\mathbf{\Sigma}_t$. Then $\sigma_{t,jj}^{-1} y_{t,j}^2 \sim F(1, \nu_0)$ and it is independent of $\boldsymbol{\alpha}$ and of $\mathcal{F}_{t-1}$.

(iv) the quadratic form $p^{-1}\mathbf{y}_t^\top \mathbf{\Sigma}_t^{-1}\mathbf{y}_t \sim F(p, \nu_0)$ and it is independent of $\boldsymbol{\alpha}$ and of $\mathcal{F}_{t-1}$.

(v) At last, the multivariate skew t-distribution with ν_0 degrees of freedom obeys the following stochastic representation

$$\mathbf{y}_t \overset{d}{\sim} \frac{\mathbf{x}_t}{\sqrt{W_t/\nu_0}}$$

with $\mathbf{x}_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1} \sim \mathcal{SN}(\mathbf{0}, \mathbf{\Sigma}_t, \boldsymbol{\alpha})$ (multivariate skew normal) and $W_t \sim \chi^2_{\nu_0}$ independent of $\mathbf{x}_t$. Or, using the stochastic representation of $\mathbf{x}_t$, we get

$$\mathbf{y}_t \overset{d}{\sim} \frac{1}{\sqrt{W_t/\nu_0}}\left((\mathbf{\Sigma}_t^{-1} + \boldsymbol{\alpha}\boldsymbol{\alpha}^\top)^{-1/2}\mathbf{z}_t + \frac{\mathbf{\Sigma}_t\boldsymbol{\alpha}}{(1+\boldsymbol{\alpha}^\top\mathbf{\Sigma}_t\boldsymbol{\alpha})^{1/2}}|u_t|\right), \tag{2.3}$$

where $\mathbf{z}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, $u_t \sim \mathcal{N}(0,1)$ and W_t are independent.

(vi) The first two moments are given by

$$\mathbb{E}(\mathbf{y}_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1}) = \sqrt{\frac{\nu_0}{\pi}}\frac{\Gamma\left(\frac{\nu_0-1}{2}\right)}{\Gamma\left(\frac{\nu_0}{2}\right)}\frac{\mathbf{\Sigma}_t\boldsymbol{\alpha}}{(1+\boldsymbol{\alpha}^\top\mathbf{\Sigma}_t\boldsymbol{\alpha})^{1/2}}, \quad \nu_0 > 1 \tag{2.4}$$

$$\mathbb{C}\mathrm{ov}(\mathbf{y}_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1}) = \frac{\nu_0}{\nu_0-2}\mathbf{\Sigma}_t - \frac{\nu_0}{\pi}\left(\frac{\Gamma\left(\frac{\nu_0-1}{2}\right)}{\Gamma\left(\frac{\nu_0}{2}\right)}\right)^2\frac{\mathbf{\Sigma}_t\boldsymbol{\alpha}\boldsymbol{\alpha}^\top\mathbf{\Sigma}_t}{1+\boldsymbol{\alpha}^\top\mathbf{\Sigma}_t\boldsymbol{\alpha}}, \quad \nu_0 > 2\,. \tag{2.5}$$

(vii) Thus, the multivariate skew t can be seen as a generalization of a non-central multivariate t-distribution.

(A2) The realized covariance matrix RC_t conditionally on $\mathcal{F}_{t-1}$ is assumed to follow a rescaled matrix variate non-central F distribution with degrees of freedom ν_1, ν_2 and non-centrality matrix $\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t$ with density given by

$$
\begin{aligned}
&f_{RC}(RC_t|\mathbf{\Sigma}_t,\mathbf{\Omega}_t,\mathcal{F}_{t-1};\nu_1,\nu_2)\\
&= K(\nu_1,\nu_2)\left|\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t^{-1}\right|^{\nu_1/2}\frac{|RC_t|^{(\nu_1-p-1)/2}}{\left|\mathbf{I}+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t^{-1}RC_t\right|^{(\nu_1+\nu_2)/2}}\\
&\quad\times e^{\mathrm{tr}\left(-\frac{1}{2}\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t\right)}{}_1F_1\left(\frac{\nu_1+\nu_2}{2};\frac{\nu_1}{2};\frac{1}{2}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t\mathbf{\Sigma}_t^{-1}RC_t[\mathbf{I}+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t^{-1}RC_t]^{-1}\right)
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&= K(\nu_1,\nu_2)\left(\frac{\nu_1}{\nu_2-p-1}\right)^{p\nu_1/2}|RC_t|^{-(\nu_2+p+1)/2}|\mathbf{\Sigma}_t|^{(\nu_1+2\nu_2)/2}\left|\mathbf{\Sigma}_tRC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t\right|^{-(\nu_1+\nu_2)/2}\\
&\quad\times e^{\mathrm{tr}\left(-\frac{1}{2}\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t\right)}{}_1F_1\left(\frac{\nu_1+\nu_2}{2};\frac{\nu_1}{2};\frac{1}{2}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t[\mathbf{\Sigma}_tRC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\right)
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&= \frac{K(\nu_1,\nu_2)}{\Gamma_p[(\nu_1+\nu_2)/2]}\left(\frac{\nu_1}{\nu_2-p-1}\right)^{p\nu_1/2}|RC_t|^{-(\nu_2+p+1)/2}|\mathbf{\Sigma}_t|^{(\nu_1+2\nu_2)/2}e^{\mathrm{tr}\left(-\frac{1}{2}\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t\right)}\\
&\quad\times\int\limits_{\mathbf{V}>\mathbf{O}}\exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_tRC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V}
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&= \frac{K(\nu_1,\nu_2)}{\Gamma_p[(\nu_1+\nu_2)/2]}\left(\frac{\nu_1}{\nu_2-p-1}\right)^{p\nu_1/2}|RC_t|^{-(\nu_2+p+1)/2}|\mathbf{\Sigma}_t|^{(\nu_1+2\nu_2)/2}\left|\mathbf{\Sigma}_tRC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t\right|^{-(\nu_1+\nu_2)/2}\\
&\quad\times 2^{-(\nu_1+\nu_2)/2}|\mathbf{\Omega}_t|^{-(\nu_1+\nu_2)/2}e^{\mathrm{tr}\left(-\frac{1}{2}\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t\right)}\int\limits_{\mathbf{V}>\mathbf{O}}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}\\
&\quad\times{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_tRC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V},
\end{aligned}
\tag{2.9}
$$

where $K(\nu_1,\nu_2)=\frac{\Gamma_p[(\nu_1+\nu_2)/2]}{\Gamma_p[\nu_1/2]\Gamma_p[\nu_2/2]}$ with $\Gamma_p(x)=\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(x-(i-1)/2)$ for $x>\frac{p-1}{2}$ is the multivariate Gamma function. The function ${}_1F_1(\cdot)$ is the matrix variate hypergeometric function

$$
{}_qF_k(a_1,\ldots,a_k;b_1,\ldots,b_k;\mathbf{A})=\sum_{i=0}^{\infty}\sum_{l}\frac{(a_1)_l\ldots(a_q)_l}{(b_1)_l\ldots(b_k)_l}\frac{C_l(\mathbf{A})}{i!}\,, \tag{2.10}
$$

where $\mathbf{A}$ is a $p\times p$ symmetric matrix, $\sum_l$ denotes the summation over all partitions l, $(c)_l=\frac{\Gamma_p(c,l)}{\Gamma_p(c)}$ with $\Gamma_p(c,l)=\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma(c+i_j-(j-1)/2)$ for $c>(p-1)/2-i_m$ and $l=(i-1,\ldots,i_m)$ with $i_1\geq i_2\geq\ldots i_m\geq 0$ and $\sum_j^p i_j=i$; $C_l(\mathbf{A})$ stands for the zonal polynomial.

We used Gupta & Nagar (2000), Theorem 1.6.2 to get from (2.7) to (2.8),

$$
\begin{aligned}
&{}_1F_1\left(\frac{\nu_1+\nu_2}{2};\frac{\nu_1}{2};\frac{1}{2}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t[\mathbf{\Sigma}_tRC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\right)\\
&=\frac{1}{\Gamma_p[(\nu_1+\nu_2)/2]}\left|\mathbf{\Sigma}_tRC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t\right|^{\frac{\nu_1+\nu_2}{2}}\\
&\times\int\limits_{\mathbf{V}>\mathbf{O}}\exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_tRC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V},
\end{aligned}
$$

and similarly to get from (2.7) to (2.9). (2.8) is useful to derive the score w.r.t. $\mathbf{\Sigma}_t$, (2.9) is useful to derive the score w.r.t. $\mathbf{\Omega}_t$.

Basic properties of non-central matrix variate F distribution:

(i) Obviously, if $\mathbf{\Omega}_t \equiv \mathbf{O}$ the distribution becomes the central one and the density function simplifies to

$$f_{RC}(RC_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1}; \nu_1, \nu_2) = K(\nu_1, \nu_2) \frac{\left|\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t^{-1}\right|^{\nu_1/2} |RC_t|^{(\nu_1-p-1)/2}}{\left|\mathbf{I} + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t^{-1} RC_t\right|^{(\nu_1+\nu_2)/2}}. \tag{2.11}$$

(ii) For $\nu_2 \to \infty$ it collapses to a non-central Wishart distribution with ν_1 degrees of freedom.

(iii) Let $\mathbf{\Omega}_t = \mathbf{\Lambda}_t \mathbf{\Lambda}_t^\top$ where $\mathbf{\Lambda}_t$ is a $p \times \nu_1$ matrix. The matrix F distribution obeys the following stochastic representation

$$RC_t \overset{d}{\sim} \frac{\nu_2 - p - 1}{\nu_1} \mathbf{\Sigma}_t^{1/2} \mathbf{V}_t^{1/2} \mathbf{T}_t^{-1} \mathbf{V}_t^{1/2} \mathbf{\Sigma}_t^{1/2}, \tag{2.12}$$

where $\mathbf{V}_t = (\mathbf{X}_t + \mathbf{\Sigma}_t^{-1/2}\mathbf{\Lambda}_t)(\mathbf{X}_t + \mathbf{\Sigma}_t^{-1/2}\mathbf{\Lambda}_t)^\top$ has a standard non-central Wishart distribution with ν_1 degrees of freedom and non-centrality matrix $\mathbf{\Sigma}_t^{-1/2}\mathbf{\Lambda}_t\mathbf{\Lambda}_t^\top\mathbf{\Sigma}_t^{-1/2}$, $\mathbf{X}_t \sim \mathcal{N}_{p,\nu_1}(\mathbf{O}, \mathbf{I}_p \otimes \mathbf{I}_{\nu_1})$ is matrix variate standard normal, $\mathbf{V}_t^{1/2}$ denotes a symmetric square root of $\mathbf{V}_t$ and $\mathbf{T}_t$ is a standard Wishart matrix with ν_2 degrees of freedom.

(iv) The mean of $RC_t|\mathbf{\Sigma}_t, \mathcal{F}_{t-1}$ is equal to $\mathbf{\Sigma}_t + \frac{1}{\nu_1}\mathbf{\Omega}_t$.

(A3) Dynamics for $\mathbf{\Sigma}_t$ and $\mathbf{\Omega}_t$:

$$\mathbf{\Sigma}_t = \mathbf{\Xi} + \alpha \mathbf{S}_{t-1} + \beta \mathbf{\Sigma}_{t-1} \quad \text{GAS recursion} \tag{2.13}$$

$$\mathbf{\Omega}_t = \mathbf{M} RC_{t-1} \mathbf{M}^\top \quad \text{WAR specification} \tag{2.14}$$

The matrix $\mathbf{M}$ could be assumed of some simple structure (diagonal, sparse etc.); $\mathbf{\Xi}$ is the unconditional covariance matrix and $\mathbf{S}_t$ is the scaled score matrix.

(A4) Independence of error processes:
We assume that all five error processes $\{\mathbf{z}_t\}$, $\{u_t\}$, $\{W_t\}$, $\{\mathbf{V}_t\}$ (or $\{\mathbf{X}_t\}$), and $\{\mathbf{T}_t\}$ in (2.3) and (2.12) are mutually independent. Assumption (A4) in particularly implies that the return vector $\mathbf{y}_t$ and the realized covariance matrix RC_t conditionally on $\mathcal{F}_{t-1}$ are independent.

2.1 t-noncentral-spiked-F scalar (p, q, r) model

The model of the t-noncentral-spiked-F (p, q, r) model dynamics (Eq. 2.9 and 2.10) are given by

$$\mathbf{\Sigma}_t = \mathbf{\Xi} + \sum_{i=1}^{p} \alpha_i \mathbf{S}_{\mathbf{\Sigma}_{t-i}} + \sum_{j=1}^{q} \beta_i \mathbf{\Sigma}_{t-i}$$

$$\mathbf{\Omega}_t = \sum_{i=1}^{r} \mathbf{M}_i \mathbf{S}_{\mathbf{\Omega}_{t-i}} \mathbf{M}_i',$$

where $\mathbf{\Xi}$ is a positive definite p by p matrix, α_i and β_i are scalars and $\mathbf{M}_i = \mathbf{m}_i\mathbf{m}_i'$ and $\mathbf{m}_i$ is a p by 1 vector, such that $\mathbf{\Omega}_t$ is rank 1 and the noncentral-F becomes the spiked noncentral-F.

Since (assuming stationarity),

$$\begin{aligned}
\mathbf{E}[RC_t] &= \mathbf{E}[\Sigma_t] + \frac{1}{\nu_1}\mathbf{E}[\Omega_t] \\
&= \frac{\mathbf{\Xi}}{(1-\sum_{j=1}^{q}\beta_i)} + \frac{1}{\nu_1}\mathbf{E}\left[\sum_{i=1}^{r}\mathbf{M}_i RC_{t-i}\mathbf{M}_i'\right] \\
&= \frac{\mathbf{\Xi}}{(1-\sum_{j=1}^{q}\beta_i)} + \frac{1}{\nu_1}\sum_{i=1}^{r}\mathbf{M}_i\mathbf{E}\left[RC_t\right]\mathbf{M}_i' \\
&\Leftrightarrow \\
\mathbf{\Xi} &= \left(1-\sum_{i=1}^{q}\beta_i\right)\left(\mathbf{E}[RC_t] - \frac{1}{\nu_1}\sum_{i=1}^{r}\mathbf{M}_i\mathbf{E}[RC_t]\mathbf{M}_i'\right),
\end{aligned}$$

we can use

$$\hat{\mathbf{\Xi}} = \left(1-\sum_{i=1}^{q}\beta_i\right)\left(\frac{1}{T}\sum_{t=1}^{T}RC_t - \frac{\sum_{j=1}^{r}\mathbf{M}_j\frac{1}{T}\sum_{t=1}^{T}RC_t\mathbf{M}_j'}{\nu_1}\right).$$

The the only parameters to estimate via ML are α_i, β_i and $\mathbf{m}_i$, such that the estimation is fast and the model is applicable in higher dimensions.

2.2 t-noncentral-F full (p, q, r) model

The model of the t-noncentral-spiked-F (p, q, r) full model dynamics (Eq. 2.9 and 2.10) are given by

$$\begin{aligned}
\mathbf{\Sigma}_t &= \mathbf{\Xi} + \sum_{i=1}^{p}\mathbf{A}_i RC_{t-i}\mathbf{A}_i' + \sum_{j=1}^{q}\mathbf{B}_j\mathbf{\Sigma}_{t-i}\mathbf{B}_j' \\
\mathbf{\Omega}_t &= \sum_{i=1}^{r}\mathbf{M}_i RC_{t-i}\mathbf{M}_i,
\end{aligned}$$

where $\mathbf{A}_i$, $\mathbf{B}_i$ and $\mathbf{M}_i$ are p by p matrices and $\mathbf{\Xi}$ is a positive definite p by p matrix. All parameters are estimated via one-step ML.

3 Likelihood function and score matrix

Let $\text{vech}(\mathbf{A}) = (a_{11}, ..., a_{k1}, ..., a_{ii}, ..., a_{ki}, ...a_{kk})^\top$ for an arbitrary symmetric matrix $\mathbf{A} = (a_{ij})$. In the following we make use of the operator **vec** defined by $\text{vec}(\mathbf{A}) = (a_{11}, ..., a_{k1}, ..., a_{1i}, ..., a_{ki}, ...a_{kk})^\top$. For the properties of the operators **vech** and **vec** we refer to Harville (1997). Moreover, let $\mathbf{D}_k$ be a $k^2 \times k(k+1)/2$ duplication matrix such that $\mathbf{D}_k\text{vech}(A) = \text{vec}(A)$ and $\mathbf{D}_k^+ = (\mathbf{D}_k^\top\mathbf{D}_k)^{-1}\mathbf{D}_k^\top$ with the property $\mathbf{D}_k^+\text{vec}(A) = \text{vech}(A)$. Finally, the symbol $\otimes$ stand for the Kronecker product.

In the derivation of the likelihood we use following two results (see, Harville (1997, p. 368))

$$\frac{\partial(\text{vec}(\mathbf{A}^{-1}))^\top}{\partial\text{vech}(\mathbf{A})} = -\mathbf{D}_k^\top(\mathbf{A}^{-1}\otimes\mathbf{A}^{-1})\mathbf{D}_k^{+\top}\mathbf{D}_k^\top\,. \tag{2.15}$$

and

$$\frac{\partial\log|\mathbf{A}|}{\partial\text{vech}(\mathbf{A})} = \mathbf{D}_k^\top\text{vec}(\mathbf{A}^{-1})\,, \tag{2.16}$$

where (2.16) follows from Magnus and Neudecker (1999, p. 171) since

$$\mathrm{d}\log|\mathbf{A}| = \mathrm{tr}(\mathbf{A}^{-1}\,\mathrm{d}\mathbf{A}) = \mathrm{vec}(\mathbf{A}^{-1})^\top\ \mathrm{dvec}(\mathbf{A}) = \mathrm{vec}(\mathbf{A}^{-1})^\top\mathbf{D}_k\ \mathrm{dvech}(\mathbf{A})$$

The conditional log-likelihood of the considered model is given by

$$\mathcal{L}_t = \mathcal{L}_{t;1} + \mathcal{L}_{t;2} \tag{2.17}$$

with

$$\mathcal{L}_{t;1} = \log f_y(\mathbf{y}_t|\boldsymbol{\Sigma}_t, \mathcal{F}_{t-1};\nu_0)$$

and

$$\mathcal{L}_{t;2} = \log f_{RC}(RC_t|\boldsymbol{\Sigma}_t, \boldsymbol{\Omega}_t, \mathcal{F}_{t-1};\nu_1,\nu_2)$$

3.1 Skew-t score

3.1.1 Score w.r.t. $\boldsymbol{\Sigma}_t$

Then, we compute

$$\begin{aligned}
\frac{\partial\mathcal{L}_{t;1}}{\partial\mathrm{vech}(\boldsymbol{\Sigma}_t)} &= -\frac{1}{2}\frac{\partial\log|\boldsymbol{\Sigma}_t|}{\partial\mathrm{vech}(\boldsymbol{\Sigma}_t)} - \frac{\nu_0+p}{2}\frac{\partial\log\left(1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}\right)}{\partial\mathrm{vech}(\boldsymbol{\Sigma}_t)} \\
&+ \frac{\partial\log\left(F_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)\right)}{\partial\mathrm{vech}(\boldsymbol{\Sigma}_t)} \\
&= -\frac{1}{2}\mathbf{D}_p^\top\mathrm{vec}(\boldsymbol{\Sigma}_t^{-1}) - \frac{\nu_0+p}{2}\left(1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}\right)^{-1}\frac{1}{\nu_0-2}\frac{\partial(\mathrm{vec}(\boldsymbol{\Sigma}_t^{-1}))^\top}{\partial\mathrm{vech}(\boldsymbol{\Sigma}_t)}(\mathbf{y}_t\otimes\mathbf{y}_t) \\
&- \frac{1}{2}\frac{f_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}{F_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}\frac{\sqrt{\nu_0+p}\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{3/2}}\frac{\partial(\mathrm{vec}(\boldsymbol{\Sigma}_t^{-1}))^\top}{\partial\mathrm{vech}(\boldsymbol{\Sigma}_t)}(\mathbf{y}_t\otimes\mathbf{y}_t) \\
&= -\frac{1}{2}\mathbf{D}_p^\top\mathrm{vec}(\boldsymbol{\Sigma}_t^{-1}) + \frac{1}{2}\frac{\sqrt{\nu_0+p}}{\sqrt{\nu_0-2}}\left(1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}\right)^{-1} \\
&\times \left(\frac{\sqrt{\nu_0+p}}{\sqrt{\nu_0-2}} + \frac{f_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}{F_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t/(\nu_0-2)}{\sqrt{1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}}}\right) \\
&\times \mathbf{D}_p^\top(\boldsymbol{\Sigma}_t^{-1}\otimes\boldsymbol{\Sigma}_t^{-1})\mathbf{D}_p^{+\top}\mathbf{D}_p^\top(\mathbf{y}_t\otimes\mathbf{y}_t) \\
&= -\frac{1}{2}\mathbf{D}_p^\top\mathrm{vec}(\boldsymbol{\Sigma}_t^{-1}) + \frac{1}{2}\frac{\sqrt{\nu_0+p}}{\sqrt{\nu_0-2}}\left(1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}\right)^{-1} \\
&\times \left(\frac{\sqrt{\nu_0+p}}{\sqrt{\nu_0-2}} + \frac{f_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}{F_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t/(\nu_0-2)}{\sqrt{1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}}}\right)\mathbf{D}_p^\top(\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t\otimes\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t),
\end{aligned}$$

where in the last equality we used the following properties:

(i) $\mathbf{D}_k\mathbf{D}_k^+ = \frac{1}{2}(\mathbf{I}_{k^2}+\mathbf{K}_k)$,

(ii) $\mathbf{K}_k = \mathbf{K}_k^\top = \mathbf{K}_k^{-1}$,

(iii) $\mathbf{K}_k(\mathbf{A} \otimes \mathbf{a}) = (\mathbf{a} \otimes \mathbf{A})$ for an arbitrary $k \times n$ matrix $\mathbf{A}$ and an arbitrary $k \times 1$ vector $\mathbf{a}$.

Now removing vech operators and denoting

$$\mathbf{Y}_t = \frac{1}{2}\boldsymbol{\Sigma}_t^{-1}\left(\mathbf{y}_t\mathbf{y}_t^\top \omega_t - \boldsymbol{\Sigma}_t\right)\boldsymbol{\Sigma}_t^{-1} \tag{2.18}$$

with

$$\omega_t = \frac{\sqrt{\nu_0+p}}{\sqrt{\nu_0-2}}\left(1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}\right)^{-1}\left(\frac{\sqrt{\nu_0+p}}{\sqrt{\nu_0-2}} + \frac{f_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}{F_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}\frac{\boldsymbol{\alpha}^\top\mathbf{y}_t/(\nu_0-2)}{\sqrt{1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}}}\right)$$

we obtain

$$\frac{\partial\mathcal{L}_{t;1}}{\partial\boldsymbol{\Sigma}_t} = 2\mathbf{Y}_t - \mathbf{Y}_t \circ \mathbf{I}\,.$$

3.1.2 Score w.r.t. $\boldsymbol{\alpha}_t$ (if needed and if we will decide to replace $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha}_t$)

After replacing $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha}_t$ we get

$$\begin{aligned}\frac{\partial\mathcal{L}_{t;1}}{\partial\boldsymbol{\alpha}_t} &= -\frac{1}{2}\frac{\partial\log|\boldsymbol{\Sigma}_t|}{\partial\boldsymbol{\alpha}_t} - \frac{\nu_0+p}{2}\frac{\partial\log\left(1+\frac{\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}{\nu_0-2}\right)}{\partial\boldsymbol{\alpha}_t}\\ &+ \frac{\partial\log\left(F_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}_t^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)\right)}{\partial\boldsymbol{\alpha}_t}\\ &= \frac{f_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}_t^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}{F_{T_{\nu_0+p}}\left(\frac{\boldsymbol{\alpha}_t^\top\mathbf{y}_t}{(\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t)^{1/2}}\sqrt{\nu_0+p}\right)}\frac{\sqrt{\nu_0+p}}{\sqrt{\nu_0-2+\mathbf{y}_t^\top\boldsymbol{\Sigma}_t^{-1}\mathbf{y}_t}}\mathbf{y}_t\end{aligned}$$

3.2 Noncentral-F Score

3.2.1 Score w.r.t. $\boldsymbol{\Sigma}_t$

For derivation of the noncentral matrix F score w.r.t. $\boldsymbol{\Sigma}_t$, p.d.f. representation (2.8) is used. The Log-Likelihood function w.r.t $\boldsymbol{\Sigma}$ is

$$\begin{aligned}\mathcal{L}_{t;2} = &\frac{\nu_1+2\nu_2}{2}\log\left(|\boldsymbol{\Sigma}_t|\right) - \frac{1}{2}\mathrm{tr}(\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\Omega}_t)\\ &+\log\left(\int_{\mathbf{V}>\mathbf{O}}\exp\left(-\frac{1}{2}\mathrm{tr}((\boldsymbol{\Sigma}_t R C_t^{-1}\boldsymbol{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\boldsymbol{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}\,{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\boldsymbol{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V}\right).\end{aligned} \tag{2.19}$$

Then the score obtains as

$$
\begin{aligned}
&\frac{\partial \mathcal{L}_{t;2}}{\partial \mathrm{vech}(\mathbf{\Sigma}_t)} \\
&= \left(\frac{\nu_1}{2}+\nu_2\right)\frac{\partial \log|\mathbf{\Sigma}_t|}{\partial \mathrm{vech}(\mathbf{\Sigma}_t)} - \frac{1}{2}\frac{\partial \mathrm{tr}\left(\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t\right)}{\partial \mathrm{vech}(\mathbf{\Sigma}_t)} \\
&+ \frac{\partial \log\left(\int_{\mathbf{V}>\mathbf{O}} \exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V}\right)}{\partial \mathrm{vech}(\mathbf{\Sigma}_t)}
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
&= \left(\frac{\nu_1}{2}+\nu_2\right)\mathbf{D}_p^\top \mathrm{vec}(\mathbf{\Sigma}_t^{-1}) + \frac{1}{2}\mathbf{D}_p^\top(\mathbf{\Sigma}_t^{-1}\otimes\mathbf{\Sigma}_t^{-1})\mathrm{vec}(\mathbf{\Omega}_t) \\
&+ \frac{1}{\int_{\mathbf{V}>\mathbf{O}} \exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V})} \\
&\times \frac{\partial\left(\int_{\mathbf{V}>\mathbf{O}} \exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V}\right)}{\partial \mathrm{vech}(\mathbf{\Sigma}_t)}
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
&= \left(\frac{\nu_1}{2}+\nu_2\right)\mathbf{D}_p^\top \mathrm{vec}(\mathbf{\Sigma}_t^{-1}) + \frac{1}{2}\mathbf{D}_p^\top(\mathbf{\Sigma}_t^{-1}\otimes\mathbf{\Sigma}_t^{-1})\mathrm{vec}(\mathbf{\Omega}_t) \\
&+ \frac{1}{\int_{\mathbf{V}>\mathbf{O}} \exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V})} \\
&\times \int_{\mathbf{V}>\mathbf{O}} \frac{\partial \exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)}{\partial \mathrm{vech}(\mathbf{\Sigma}_t)}|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V}
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
&= \left(\frac{\nu_1}{2}+\nu_2\right)\mathbf{D}_p^\top \mathrm{vec}(\mathbf{\Sigma}_t^{-1}) + \frac{1}{2}\mathbf{D}_p^\top(\mathbf{\Sigma}_t^{-1}\otimes\mathbf{\Sigma}_t^{-1})\mathrm{vec}(\mathbf{\Omega}_t) \\
&- \frac{1}{2}\mathbf{D}_p^\top \frac{\int_{\mathbf{V}>\mathbf{O}} \mathbf{Z}(\mathbf{V})|\mathbf{V}|^{\frac{\nu_2}{2}} \exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V}}{\int_{\mathbf{V}>\mathbf{O}} |\mathbf{V}|^{\frac{\nu_2}{2}} \exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)\mathrm{d}\mathbf{V}}
\end{aligned} \tag{2.23}
$$

$$
= \left(\frac{\nu_1}{2}+\nu_2\right)\mathbf{D}_p^\top \mathrm{vec}(\mathbf{\Sigma}_t^{-1}) + \frac{1}{2}\mathbf{D}_p^\top(\mathbf{\Sigma}_t^{-1}\otimes\mathbf{\Sigma}_t^{-1})\mathrm{vec}(\mathbf{\Omega}_t) - \frac{1}{2}\mathbf{D}_p^\top \mathbf{Z}\left(\frac{\mathrm{E}\left[\mathbf{V}|\mathbf{V}|^{\frac{\nu_2}{2}}\right]}{\mathrm{E}[|\mathbf{V}|^{\frac{\nu_2}{2}}]}\right) \tag{2.24}
$$

$$
\approx \left(\frac{\nu_1}{2}+\nu_2\right)\mathbf{D}_p^\top \mathrm{vec}(\mathbf{\Sigma}_t^{-1}) + \frac{1}{2}\mathbf{D}_p^\top(\mathbf{\Sigma}_t^{-1}\otimes\mathbf{\Sigma}_t^{-1})\mathrm{vec}(\mathbf{\Omega}_t) - \frac{1}{2}\mathbf{D}_p^\top \mathbf{Z}\left(\frac{(\bar{\mathbf{\Sigma}}+\frac{\bar{\mathbf{\Omega}}}{\nu_1})|\mathbf{I}+\frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{\Omega}}}{\nu_1}|^{\frac{\nu_2}{2}}(\nu_1+\nu_2)}{e^{\mathrm{tr}\left(-\frac{1}{2}\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{\Omega}}\right)}{}_1F_1\left((\nu_1+\nu_2)/2,\,\frac{1}{2}\nu_1,\,\frac{1}{2}\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{\Omega}}\right)}\right),
$$

see Appendix for last step

where in (2.23)

$$
\begin{aligned}
&\frac{\partial\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)}{\partial \mathrm{vech}(\mathbf{\Sigma}_t)} \\
&= -\frac{1}{2}\mathbf{D}_p^\top\left((\mathbf{V}\otimes RC_t^{-1})\mathrm{vec}(\mathbf{\Sigma}_t) + (RC_t^{-1}\otimes\mathbf{V})\mathrm{vec}(\mathbf{\Sigma}_t) + \frac{\nu_1}{\nu_2-p-1}\mathrm{vec}(\mathbf{V})\right) && (2.25) \\
&\equiv -\frac{1}{2}\mathbf{D}_p^\top \mathbf{Z}(\mathbf{V}) && (2.26)
\end{aligned}
$$

and the expectation in (2.24) are w.r.t. $\mathbf{V} \sim$ noncentral Wishart, since

$$
\exp\left(-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t)\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\mathbf{V}\right)
$$

is the kernel of the p.d.f. of the noncentral Wishart distribution with covariance matrix

$$
\bar{\mathbf{\Sigma}} = \left(\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t\right)^{-1},
$$

non-centrality matrix

$$
\bar{\mathbf{\Omega}} = \left(\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t\right)^{-1}\frac{\nu_1}{\nu_2-p-1}\mathbf{\Omega}_t\left(\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t\right)^{-1}
$$

and

$$\mathbf{E}[\mathbf{V}] = \nu_1 \bar{\mathbf{\Sigma}} + \bar{\mathbf{\Omega}} \equiv \mathbf{Q}.$$

3.2.2 Score w.r.t. $\mathbf{\Omega}_t$

For derivation of the noncentral matrix F score w.r.t. $\mathbf{\Omega}_t$, p.d.f. representation (2.9) is used. The Log-Likelihood function w.r.t $\mathbf{\Omega}_t$ is

$$\begin{aligned}\mathcal{L}_{t;2} = & -\frac{\nu_1+\nu_2}{2}\log|\mathbf{\Omega}_t| - \frac{1}{2}\mathrm{tr}\left(\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t\right) \\ & + \log\left(\int_{\mathbf{V}>\mathbf{O}} \exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}} {}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V}\right).\end{aligned} \tag{2.27}$$

Then the score obtains as

$$\begin{aligned}\frac{\partial\mathcal{L}_{t;2}}{\partial\mathrm{vech}(\mathbf{\Omega}_t)} \\ = & -\frac{\nu_1+\nu_2}{2}\frac{\partial\log|\mathbf{\Omega}_t|}{\partial\mathrm{vech}(\mathbf{\Omega}_t)} - \frac{1}{2}\frac{\partial\mathrm{tr}\left(\mathbf{\Sigma}_t^{-1}\mathbf{\Omega}_t\right)}{\partial\mathrm{vech}(\mathbf{\Omega}_t)} \\ & + \frac{\partial\log\left(\int_{\mathbf{V}>\mathbf{O}}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V}\right)}{\partial\mathrm{vech}(\mathbf{\Omega}_t)}\end{aligned} \tag{2.28}$$

$$\begin{aligned} = & -\frac{\nu_1+\nu_2}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Omega}_t^{-1}) - \frac{1}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Sigma}_t^{-1}) \\ & + \frac{1}{\int_{\mathbf{V}>\mathbf{O}}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V}} \\ & \times\frac{\partial\int_{\mathbf{V}>\mathbf{O}}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V}}{\partial\mathrm{vech}(\mathbf{\Omega}_t)}\end{aligned} \tag{2.29}$$

$$\begin{aligned} = & -\frac{\nu_1+\nu_2}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Omega}_t^{-1}) - \frac{1}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Sigma}_t^{-1}) \\ & + \frac{\int_{\mathbf{V}>\mathbf{O}}\frac{\partial\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)}{\partial\mathrm{vech}(\mathbf{\Omega}_t)}|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V}}{\int_{\mathbf{V}>\mathbf{O}}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1+\nu_2-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V}}\end{aligned} \tag{2.30}$$

$$\begin{aligned} = & -\frac{\nu_1+\nu_2}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Omega}_t^{-1}) - \frac{1}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Sigma}_t^{-1}) \\ & + \frac{1}{2}\mathbf{D}_p^\top(\mathbf{\Omega}_t^{-1}\otimes\mathbf{\Omega}_t^{-1}) \\ & \times\frac{\int_{\mathbf{V}>\mathbf{O}}\mathrm{vec}(\mathbf{V})|\mathbf{V}|^{\frac{\nu_2}{2}}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V}}{\int_{\mathbf{V}>\mathbf{O}}|\mathbf{V}|^{\frac{\nu_2}{2}}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)\mathrm{d}\mathbf{V}}\end{aligned} \tag{2.31}$$

$$= -\frac{\nu_1+\nu_2}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Omega}_t^{-1}) - \frac{1}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Sigma}_t^{-1}) + \frac{1}{2}\mathbf{D}_p^\top(\mathbf{\Omega}_t^{-1}\otimes\mathbf{\Omega}_t^{-1})\mathrm{vec}\left(\frac{\mathrm{E}\left[\mathbf{V}|\mathbf{V}|^{\frac{\nu_2}{2}}\right]}{\mathrm{E}[|\mathbf{V}|^{\frac{\nu_2}{2}}]}\right) \tag{2.32}$$

$$\approx -\frac{\nu_1+\nu_2}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Omega}_t^{-1}) - \frac{1}{2}\mathbf{D}_p^\top\mathrm{vec}(\mathbf{\Sigma}_t^{-1}) + \frac{1}{2}\mathbf{D}_p^\top(\mathbf{\Omega}_t^{-1}\otimes\mathbf{\Omega}_t^{-1})\mathrm{vec}\left(\frac{(\bar{\mathbf{\Sigma}}+\frac{\bar{\mathbf{\Omega}}}{\nu_1})\left|\mathbf{I}+\frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{\Omega}}}{\nu_1}\right|^{\frac{\nu_2}{2}}(\nu_1+\nu_2)}{e^{\mathrm{tr}\left(-\frac{1}{2}\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{\Omega}}\right)}{}_1F_1\left((\nu_1+\nu_2)/2,\frac{1}{2}\nu_1,\frac{1}{2}\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{\Omega}}\right)}\right),$$

see Appendix for last step

where the expectation in (2.32) are w.r.t. $\mathbf{V}\sim$ noncentral Wishart, since

$$\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{\Omega}_t^{-1}\mathbf{V})\right)|\mathbf{V}|^{\frac{\nu_1-p-1}{2}}{}_0F_1\left(\frac{\nu_1}{2};\frac{1}{4}\frac{\nu_1}{\nu_2-p-1}[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t+\frac{\nu_1}{\nu_2-p-1}\mathbf{\Sigma}_t]^{-1}\mathbf{V}\right)$$

is the kernel of the p.d.f. of the noncentral Wishart distribution with covariance matrix

$$\bar{\mathbf{\Sigma}} = \mathbf{\Omega}_t,$$

non-centrality matrix

$$\bar{\mathbf{\Omega}} = \frac{\nu_1}{\nu_2 - p - 1}\mathbf{\Omega}_t[\mathbf{\Sigma}_t RC_t^{-1}\mathbf{\Sigma}_t + \frac{\nu_1}{\nu_2 - p - 1}\mathbf{\Sigma}_t]^{-1}\mathbf{\Omega}_t$$

and

$$\mathbf{E}[\mathbf{V}] = \nu_1\bar{\mathbf{\Sigma}} + \bar{\mathbf{\Omega}} \equiv \mathbf{Q}.$$

3.3 Total Score

Finally, the score matrix is calculated by

$$\nabla_{\mathbf{\Sigma}_t} = \frac{\partial \mathcal{L}_t}{\partial \mathbf{\Sigma}_t} = \frac{\partial \mathcal{L}_{t;1}}{\partial \mathbf{\Sigma}_t} + \frac{\partial \mathcal{L}_{t;2}}{\partial \mathbf{\Sigma}_t}\,, \tag{2.33}$$

and

$$\nabla_{\mathbf{\Omega}_t} = \frac{\partial \mathcal{L}_t}{\partial \mathbf{\Omega}_t} = \frac{\partial \mathcal{L}_{t;1}}{\partial \mathbf{\Omega}_t} + \frac{\partial \mathcal{L}_{t;2}}{\partial \mathbf{\Omega}_t}\,. \tag{2.34}$$

We then scale $\nabla_{\mathbf{\Sigma}_t}$ as in Opschoor et al.,

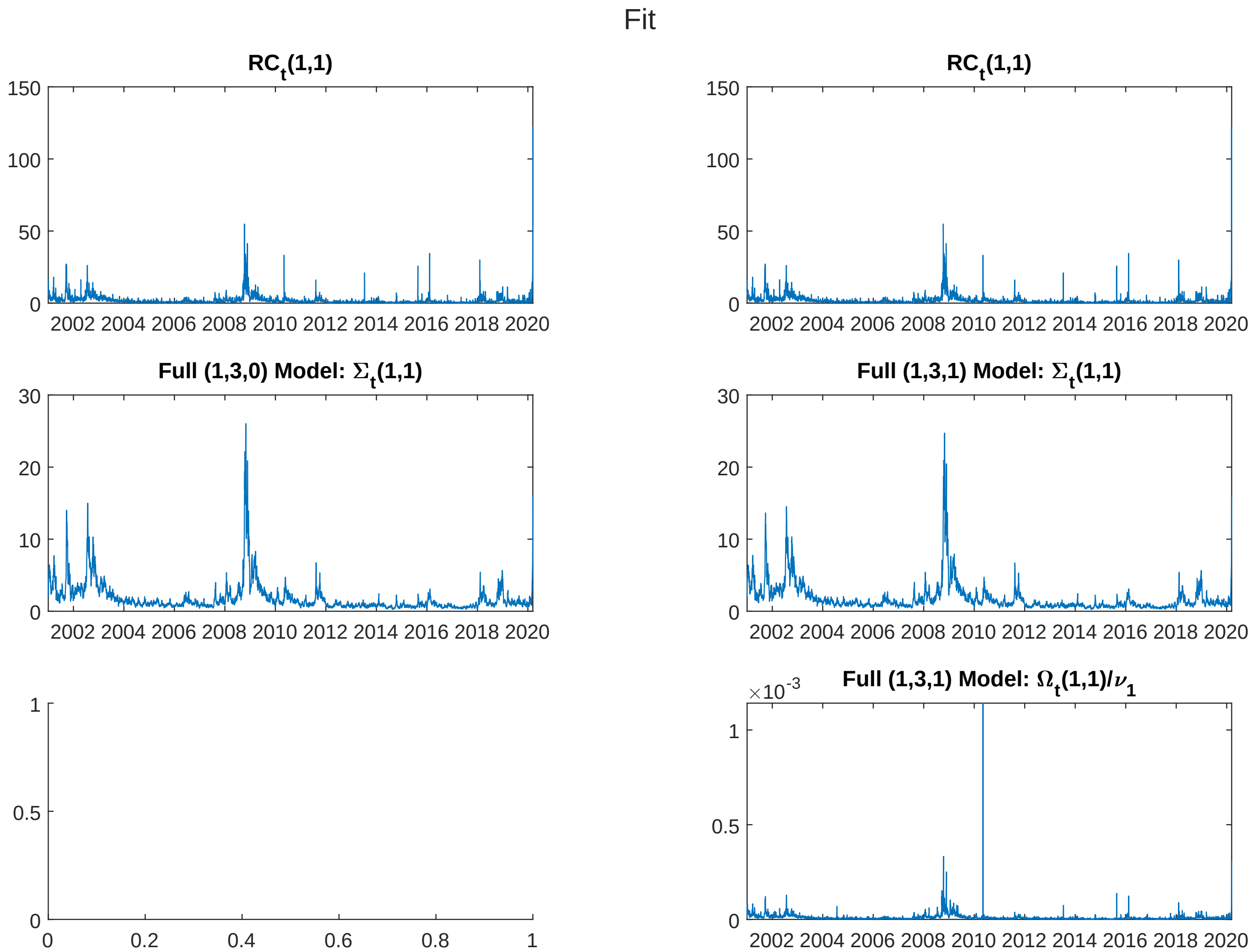
$$\mathbf{S}_{\mathbf{\Sigma}_t} = \nabla_{\mathbf{\Sigma}_t} \tag{2.35}$$

4 Empirical Application

$r = 0$ is the baseline Opschoor t-F model. Data is $p = 5$ (Assets: BA/HD/JPM/PFE/PG as in Opschoor Table 4 Panel A.1) from Jan 2001 to Mar 2020 for $T \approx 4500$.

(p, q, r)	$dim(\theta)$	$\mathcal{L}(\hat{\theta})$	BIC	$\sum_{i=1}^{q} \beta_i$	NaN	e^*_{11}	e^*_{21}	e^*_{31}	e^*_{41}	e^*_{51}	e^*_{22}	e^*_{32}	e^*_{42}	e^*_{52}	e^*_{33}	e^*_{43}	e^*_{53}	e^*_{44}	e^*_{54}	e^*_{55}
t-noncentral-spiked-F scalar (p,q,r) model. Estimation with intercept targeting.																				
$(1,1,0)$	20	-20788	41746	0.988																
$(1,1,1)$	25	-20712	41636	0.987																
$(1,1,2)$	30	-20669	41593	0.987																
$(1,2,0)$	21	-20519	41215	0.988																
$(1,2,1)$	26	-20451	41123	0.988																
$(1,2,2)$	31	-20403	41069	0.987																
$(1,3,0)$	22	-20400	40987	0.988																
$(1,3,1)$	27	-20334	40898	0.989																
$(1,3,2)$	32	-20289	40849	0.988																
$(2,1,0)$	21	-20430	41039	0.991																
$(2,1,1)$	26	-20364	40949	0.991																
$(2,1,2)$	31	-20317	40898	0.991																
$(2,2,0)$	22	-20113	40412	0.999																
$(2,2,1)$	27	-20031	40292	0.999																
$(2,2,2)$	32	-20007	40286	0.999																
$(2,3,0)$	23	-20062	40318	1																
$(2,3,1)$	28	-19981	40200	1																
$(2,3,2)$	33	-19952	40184	1																
$(3,1,0)$	22	-20320	40827	0.993																
$(3,1,1)$	27	-20252	40732	0.993																
$(3,1,2)$	32	-20210	40692	0.993																
$(3,2,0)$	23	-20055	40306	1																
$(3,2,1)$	28	-19975	40187	1																
$(3,2,2)$	33	-19945	40170	1																
$(3,3,0)$	24	-20055	40314	1																
$(3,3,1)$	29	-19975	40195	1																
$(3,3,2)$	34	-19942	40172	1																

t-noncentral-F full (p,q,r) model. Full Likelihood Optimization.


Fit
RC_t(1,1)
RC_t(1,1)
Full (1,3,0) Model: Σ_t(1,1)
Full (1,3,1) Model: Σ_t(1,1)
Full (1,3,1) Model: Ω_t(1,1)/ν_1
×10^-3
2002 2004 2006 2008 2010 2012 2014 2016 2018 2020

5 Appendix

Let's derive $\mathrm{E}\left[\mathbf{V}|\mathbf{V}|^h\right]$ if $\mathbf{V} \sim W(\mathbf{\Sigma}, \nu)$.

First, note that if $\mathbf{S} \sim W(\nu + 2h, \mathbf{\Sigma})$, then we know that

$$\mathrm{E}[\mathbf{S}] = \int\limits_{\mathbf{S}>\mathbf{0}} \mathbf{S} \frac{|\mathbf{S}|^{\frac{1}{2}((\nu+2h)-p-1)} e^{-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma})^{-1}\mathbf{S})}}{2^{((\nu+2h)p)/2}|\mathbf{\Sigma}|^{(\nu+2h)/2}\Gamma_p((\nu+2h)/2)} d\mathbf{S} = (\nu+2h)\mathbf{\Sigma}. \tag{2.36}$$

Thus

$$\int\limits_{\mathbf{S}>\mathbf{0}} \mathbf{S}|\mathbf{S}|^{\frac{1}{2}(\nu+2h-p-1)} e^{-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma})^{-1}\mathbf{S})} d\mathbf{S} = (\nu+2h)\mathbf{\Sigma} 2^{((\nu+2h)p)/2}|\mathbf{\Sigma}|^{(\nu+2h)/2}\Gamma_p((\nu+2h)/2). \tag{2.37}$$

Using (2.37) we can now derive $\mathrm{E}\left[\mathbf{V}|\mathbf{V}|^h\right]$ if $\mathbf{V} \sim W(\mathbf{\Sigma}, \nu)$,

$$\begin{aligned}
\mathrm{E}[\mathbf{V}|\mathbf{V}|^h] &= \int\limits_{\mathbf{V}>\mathbf{0}} \mathbf{V}|\mathbf{V}|^h \frac{|\mathbf{V}|^{\frac{1}{2}(\nu-p-1)} e^{-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma})^{-1}\mathbf{V})}}{2^{(\nu p)/2}|\mathbf{\Sigma}|^{\nu/2}\Gamma_p(\nu/2)} d\mathbf{V} && (2.38)\\
&= \frac{1}{2^{(\nu p)/2}|\mathbf{\Sigma}|^{\nu/2}\Gamma_p(\nu/2)} \int\limits_{\mathbf{V}>\mathbf{0}} \mathbf{V}|\mathbf{V}|^{\frac{1}{2}(\nu+2h-p-1)} e^{-\frac{1}{2}\mathrm{tr}((\mathbf{\Sigma})^{-1}\mathbf{V})} d\mathbf{V} && (2.39)\\
&\overset{2.37}{=} \frac{1}{2^{(\nu p)/2}|\mathbf{\Sigma}|^{\nu/2}\Gamma_p(\nu/2)} (\nu+2h)\mathbf{\Sigma} 2^{((\nu+2h)p)/2}|\mathbf{\Sigma}|^{(\nu+2h)/2}\Gamma_p((\nu+2h)/2) && (2.40)\\
&= \mathbf{\Sigma}|\mathbf{\Sigma}|^h 2^{hp}(\nu+2h)\frac{\Gamma_p((\nu+2h)/2)}{\Gamma_p(\nu/2)}. && (2.41)
\end{aligned}$$

Now, if $\mathbf{V} \sim$ noncentral-$W(\mathbf{\Sigma}, \mathbf{\Omega}, \nu)$

$$\frac{\mathrm{E}\left[\mathbf{V}|\mathbf{V}|^{\frac{\nu_2}{2}}\right]}{\mathrm{E}[|\mathbf{V}|^{\frac{\nu_2}{2}}]} = \frac{\mathrm{E}\left[\mathbf{V}|\mathbf{V}|^{\frac{\nu_2}{2}}\right]}{\frac{2^{(\nu_2/2)p}\Gamma_p((\nu_1+\nu_2)/2)|\mathbf{\Sigma}|^{\frac{\nu_2}{2}}}{\Gamma_p(\nu_1/2)} e^{\mathrm{tr}\left(-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{\Omega}_t\right)} {}_1F_1\left(\frac{1}{2}\nu_1 + \frac{\nu_2}{2}, \frac{1}{2}\nu_1, \frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{\Omega}_t\right)}$$

Gupta and Nagar (Theorem 3.5.6, p. 119))

$$\approx \frac{\left(\mathbf{\Sigma} + \frac{\mathbf{\Omega}}{\nu_1}\right)\left|\mathbf{\Sigma} + \frac{\mathbf{\Omega}}{\nu_1}\right|^{\frac{\nu_2}{2}} 2^{(\nu_2/2)p}(\nu_1+\nu_2)\frac{\Gamma_p((\nu_1+\nu_2)/2)}{\Gamma_p(\nu_1/2)}}{\frac{2^{(\nu_2/2)p}\Gamma_p((\nu_1+\nu_2)/2)|\mathbf{\Sigma}|^{\frac{\nu_2}{2}}}{\Gamma_p(\nu_1/2)} e^{\mathrm{tr}\left(-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{\Omega}_t\right)} {}_1F_1\left(\frac{1}{2}\nu_1 + \frac{\nu_2}{2}, \frac{1}{2}\nu_1, \frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{\Omega}_t\right)}$$

$\mathrm{E}\left[\mathbf{V}|\mathbf{V}|^{\frac{\nu_2}{2}}\right]$ is derived in (2.41) for $\mathbf{V} \sim W(\mathbf{\Sigma} + \mathbf{\Omega}/\nu, \nu) \approx$ noncentral-$W(\mathbf{\Sigma}, \mathbf{\Omega}, \nu)$

$$= \frac{(\mathbf{\Sigma} + \frac{\mathbf{\Omega}}{\nu_1})\left|\mathbf{I} + \frac{\mathbf{\Sigma}^{-1}\mathbf{\Omega}}{\nu_1}\right|^{\frac{\nu_2}{2}}(\nu_1+\nu_2)}{e^{\mathrm{tr}\left(-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{\Omega}\right)} {}_1F_1\left((\nu_1+\nu_2)/2, \frac{1}{2}\nu_1, \frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{\Omega}\right)}. \tag{2.42}$$