Page Rank (overview)

Def

Review

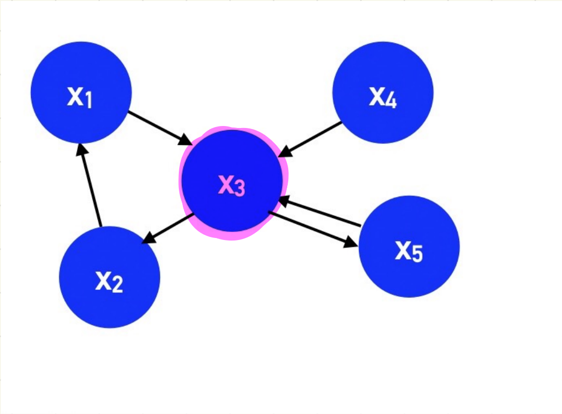
Prestige Centrality – a node is prestigious if prestigious point to it

Let $N_i = \{x_j \in V : d(x_i, x_j) = 1\}$ ← all verts 1 hop from x_i

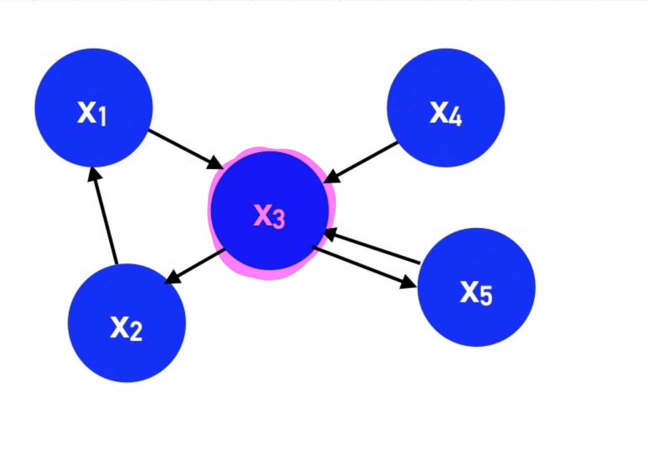
The prestige of x_i is

$$p(x_i) = \sum_{x_j \in N_i} p(x_j)$$

Example:



$$p(x_3) = p(x_1) + p(x_5) + p(x_4)$$



Review

$p(x_1) = p(x_2)$

$p(x_2) = p(x_3)$

$p(x_3) = p(x_1) + p(x_5) + p(x_4)$

$p(x_4) = 0$

$p(x_5) = p(x_3)$

5×5

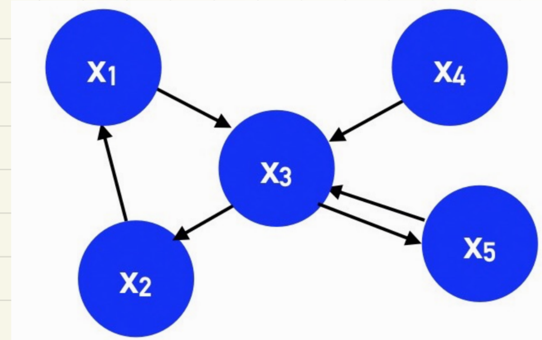
Graph as Matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$

5×1

Lin Alg & Graphs

Review



$p(x_1) = p(x_2)$

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$p(x_4) = 0$

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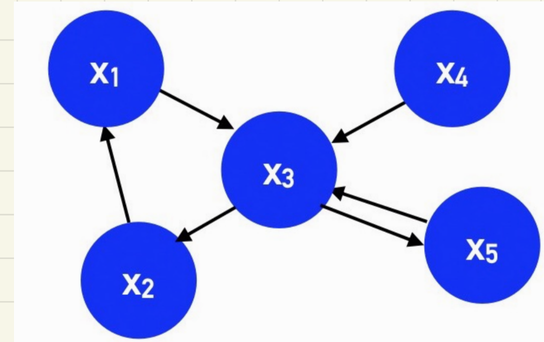
$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$

How to write $p(x_3)$ w/ mat-vec?

$$p(x_3) = p_1(x_1) + p(x_4) + p(x_5) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$

Review

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$



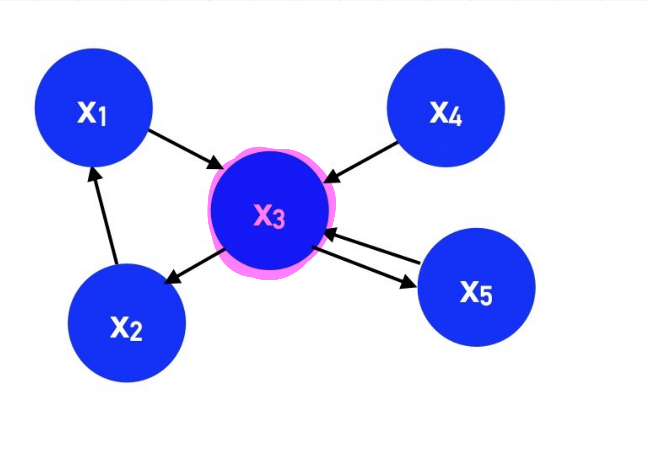
Repeat for each $p(x_i)$

$$p(x_1) = A^T_1 p$$
$$p(x_2) = A^T_2 p$$
$$p(x_3) = A^T_3 p$$
$$p(x_4) = A^T_4 p$$
$$p(x_5) = A^T_5 p$$

$$\longrightarrow p = A^T p$$

how do we solve for p?

Review



$$p(x_1) = p(x_2)$$
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5×5

Graph as Matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$

5×1

Review

Formally

pick p_0 (how, TBD)

$$p_1 = A^T p_0$$

$$p_2 = A^T p_1 = A^T (A^T p_0) = (A^T)^2 p_0$$

$$p_3 = A^T p_2 = A^T (A^T)^2 p_0 = (A^T)^3 p_0$$

$$\vdots$$

$$p_k = A^T p_{k-1} = (A^T)^k p_0$$

This process is called ==power iteration== and p converges to the ==dominant eigenvector== of A^T (for any $p_0 \neq 0$)

Review

<u>$PowerIteration(A,\epsilon)$</u>

$k = 0$

$p_0 = 1 \in R^n$

Repeat:

$k = k + 1$

$p_k = A^T p_{k-1}$

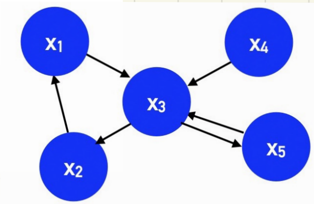
$i = argmax_j \{p_k[j]\}$

$p_k = \frac{1}{p_k[i]} p_k$

Until $||p_k - p_{k-1}|| \leq \epsilon$

$p = \frac{1}{||p_k||} p_k$

Return p



$$p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} \longrightarrow i = 3 \longrightarrow p_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 1 \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} \longrightarrow i = 2 \longrightarrow p_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 1 \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{5}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \longrightarrow i = 3 \longrightarrow p_3 = \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \\ 1 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$\vdots$$

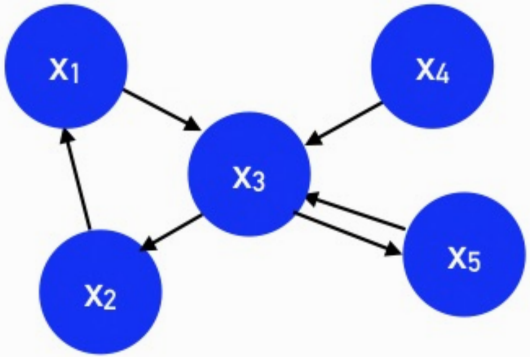
Prestige + Normalization

how likely am I to end up somewhere in a random walk

$$\boxed{p = N^T p}$$

$$A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$N^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$



$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} \rightarrow p(x_5) = \frac{1}{2} p(x_3)$$

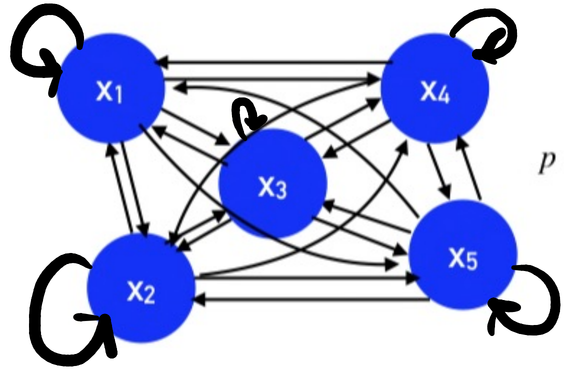
Prestige + Normalization + Random jumps

avoid getting stuck
in a sink

$$\boxed{p = N_r p}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$N_r = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$



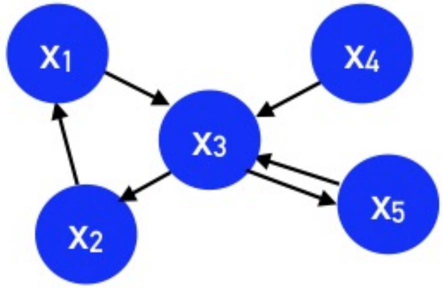
$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} \rightarrow p(x_3) = \frac{1}{5}p(x_1) + \frac{1}{5}p(x_2) + \frac{1}{5}p(x_3) + \frac{1}{5}p(x_4) + \frac{1}{5}p(x_5)$$

Page Rank = Prestige + Normalization + Random jumps

$$\boxed{p = (1-\alpha)N^T p + \alpha N_r p}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = (1-\alpha)\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} + \alpha\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

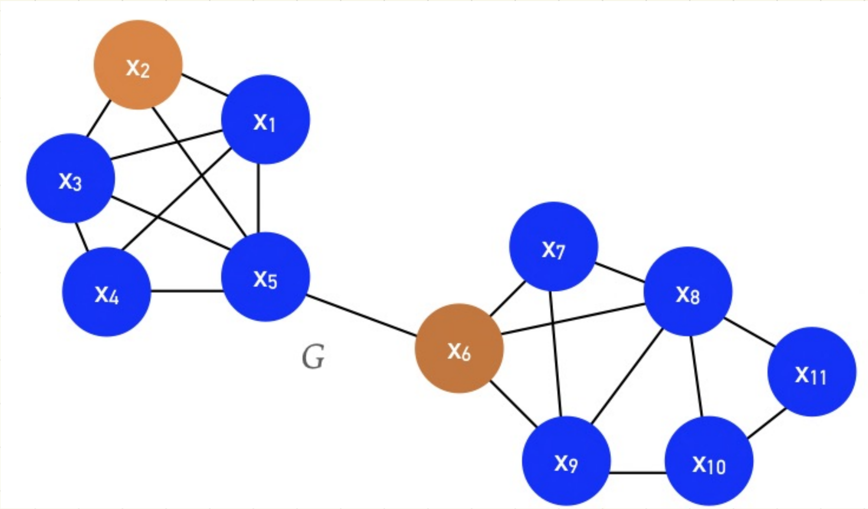
Clustering

Coefficent

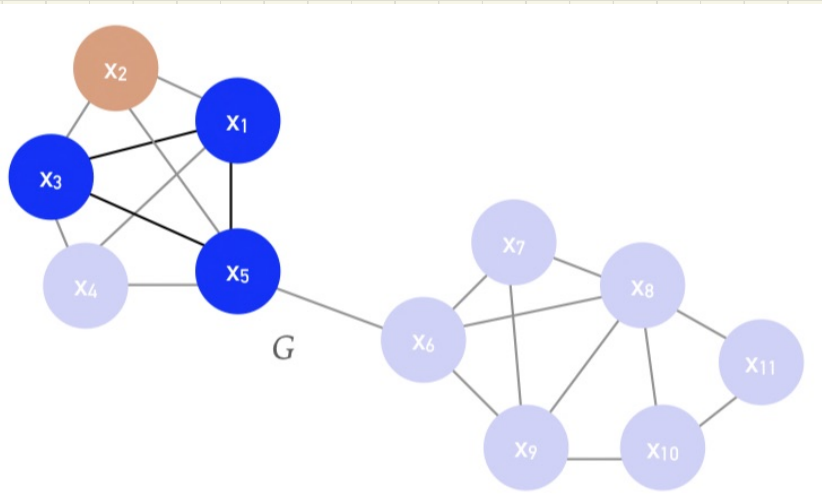
Informal Def

Clustering coeff measures cohesiveness of node's neighborhood

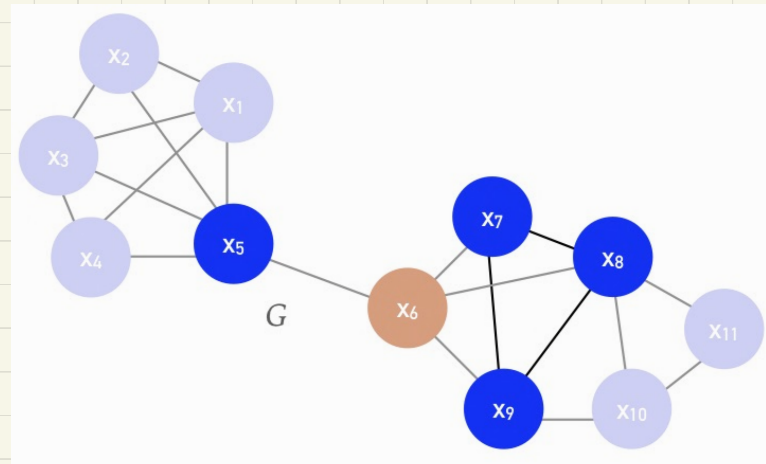
Example



X_2 has 3 neighbors
all neighbors connected



X_6 has 4 neighbors
in 2 components



Formal Def

Let $G_i = (U_i, E_i)$ be the subgraph induced by the neighbors of node x_i.

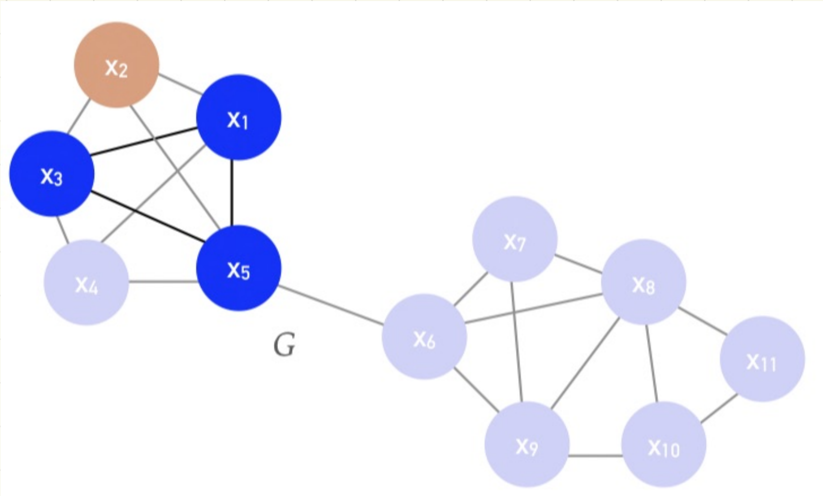
Let $n_i = |U_i|$ $\quad m_i = |E_i|$

The clustering coeff of x_i is

$$\frac{m_i}{\binom{n_i}{2}} = \frac{\text{\# of edges among neighbors of } x_i}{\text{\# of possible edges among neighbors of } x_i}$$

example: Clust coeff of x_2

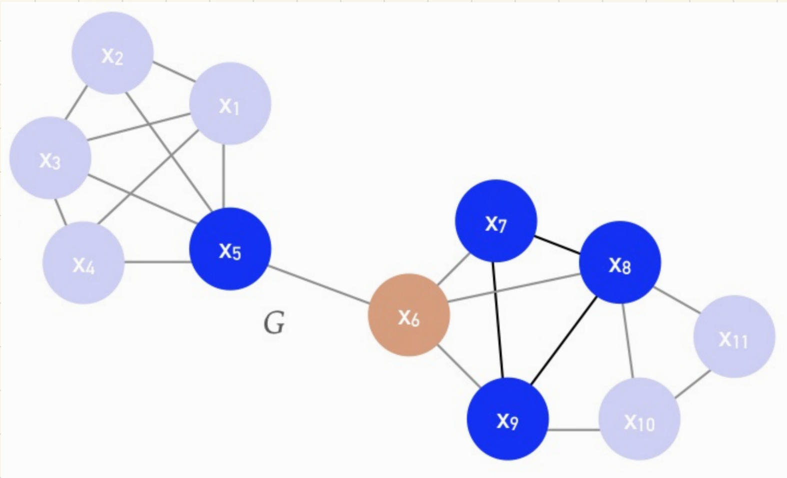
$$\frac{\text{\# of edges among neighbors of } x_i}{\text{\# of possible edges among neighbors of } x_i}$$



$$\frac{\text{\# of edges} \ldots x_2}{\text{\# of possible edges} \ldots x_2}$$

$=$

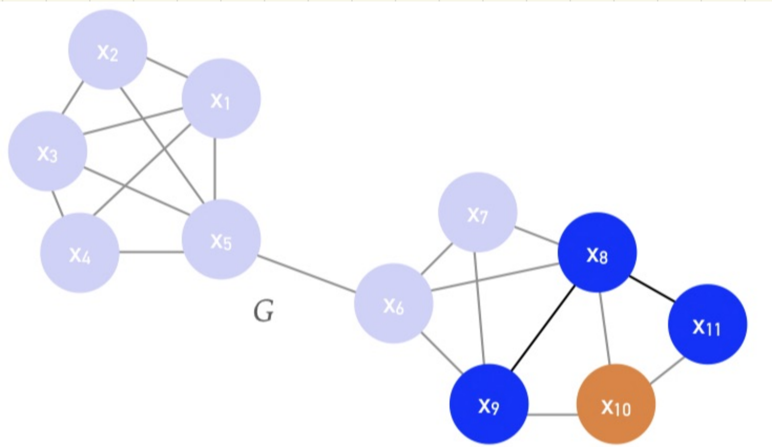
example: Clustering coeff of x_6



$$\frac{\text{\# of edges} \ldots x_6}{\text{\# of possible edges} \ldots x_6}$$

$=$

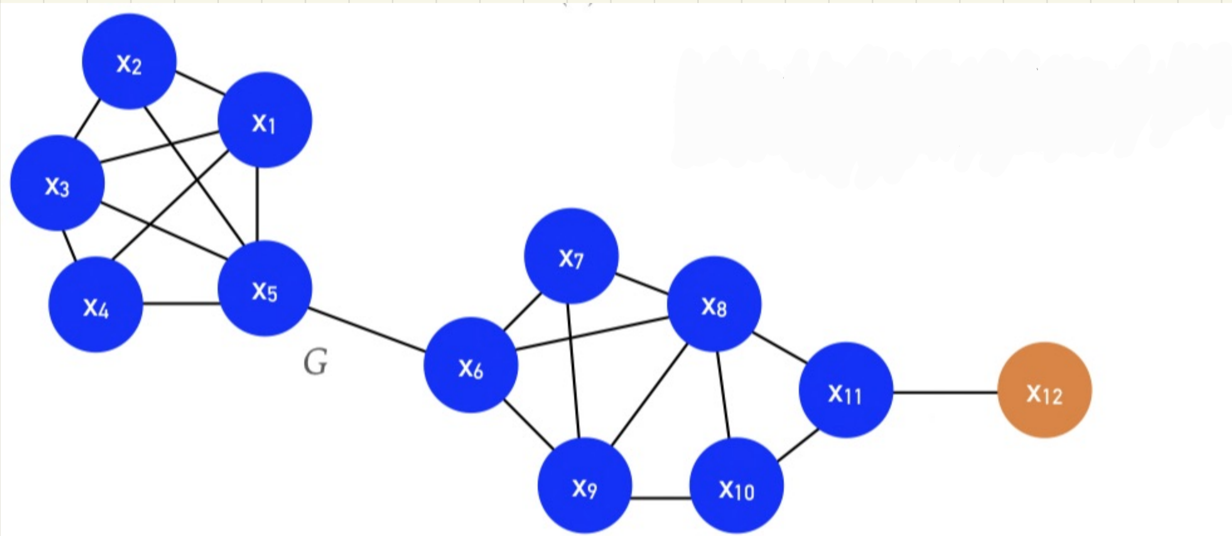
example: Clustering coeff of x_{10}



$$\frac{\text{\# of edges} \ldots x_{10}}{\text{\# of possible edges} \ldots x_{10}}$$

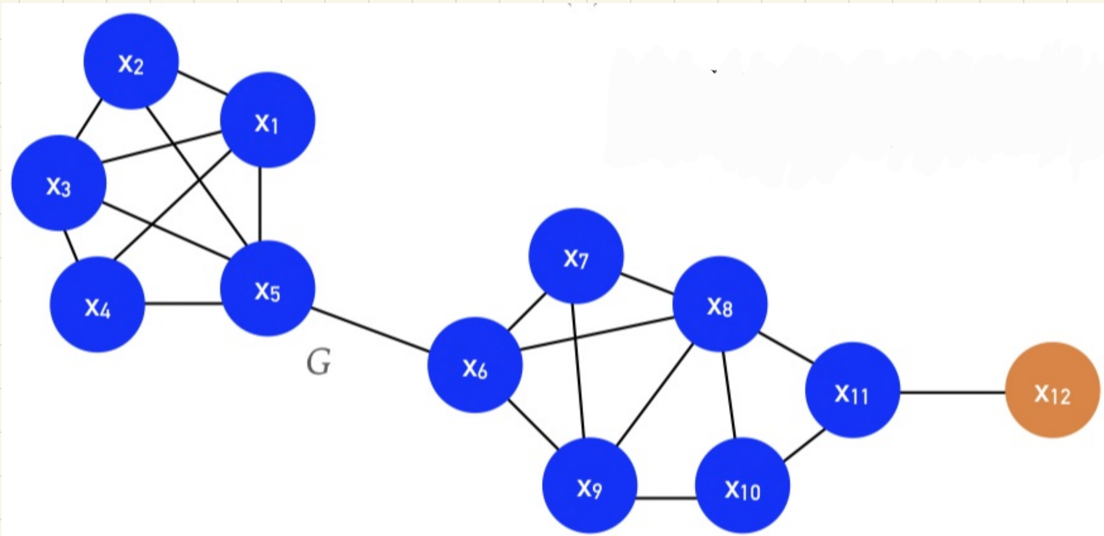
$=$

What if node has deg less than 2?



$$\frac{\#\text{ of edges} \dots x_{12}}{\#\text{of possible edges} \dots x_{12}} =$$

What if node has deg less than 2?

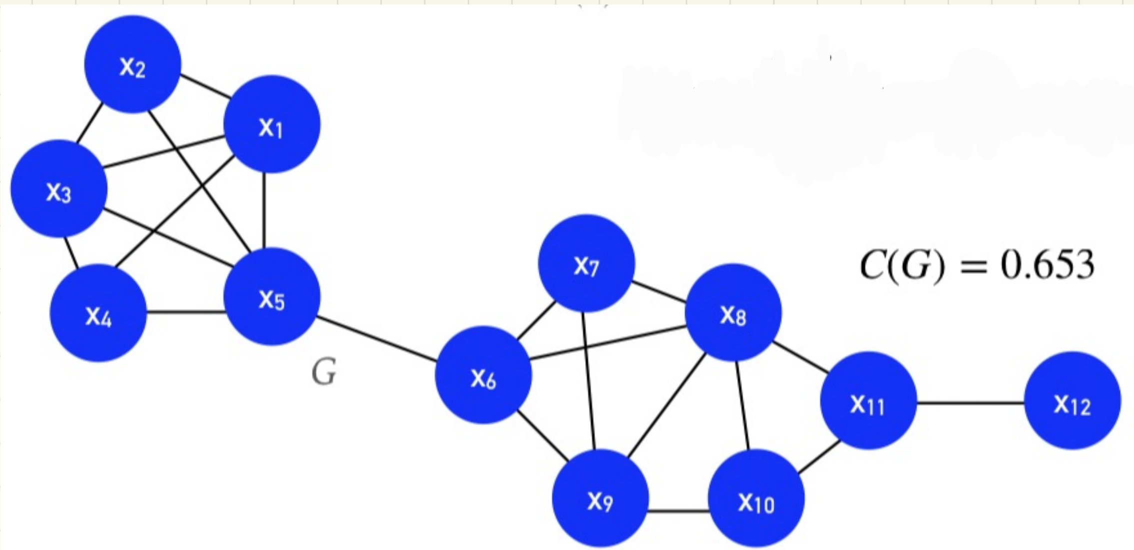


Define clust Coeff as 0 for nodes w/ degree less than two

$$\frac{\#\text{ of edges} \dots x_{12}}{\#\text{of possible edges} \dots x_{12}} = \cancel{\frac{0}{0}}$$

Given graph G clust coeff of G:

$$C(G) = \frac{1}{n}\sum_{i=1}^{n} \text{clust coeff } x_i$$



Coding using

networkX !!