19 Sept 2022

Let $k \leq n$ be integers. (non-negative)

~~Q~~: Given a set of n elements,

Q1: How many subsets does it have?
combinations / the power set

Q2: How many permutations can you make?

Q3: How many subsets of size k are there?

→ small examples (always helpful)

→ the answer & justification.

(note: these questions relate to a field of mathematics called "combinatorics")

<u>Notes</u>: Let S be a set.

1- $\emptyset \subseteq S$

2- $S \subseteq S$

3- $\mathcal{P}(S)$ denote the powerset of S

$$\mathcal{P}(\{a,b,c\}) = \{\emptyset, \{a,b,c\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,b\}, \{b,c\}\}$$

Counting License Plates:

- LPs have 5 characters
  - → 1st 2 are ~~digs~~ digits (10 choices per slot)
  - → ~~2nd~~ last 3 are letters (26 choices per slot)

e.g., 06 ABC
22 DXY

How many LP's can be created?

$$\underbrace{\underline{10}\ \underline{10}}_{\text{100 possibilities, 00 through 99}}\ \underbrace{\underline{26}\ \underline{26}\ \underline{26}}_{} = 10^2 \cdot 26^3 \text{ possible choices}$$

How many LP's if we can't repeat digits or numbers?

$$\underline{10}\ \underline{9}\ \underline{26}\ \underline{25}\ \underline{24} = 10 \cdot 9 \cdot 26 \cdot 25 \cdot 24 = \frac{10!}{8!} \cdot \frac{26!}{23!}$$

"Stars and Bars"

Given a ~~number~~ non-neg INTEGER n, how many ways can we add 3 non-negative integers together to create n (order matters)

e.g., $n=3$

$0+3+0$	
$1+2+0$	$0+2+1$
$2+1+0$	$0+1+2$
$3+0+0$	$0+0+3$
$1+1+1$	

$0+3+0$: | * * * |

$1+2+0$: * | * * |

$\Big\}$ 3 *'s and 2 bars

Given $n+2$ slots, choose 2 of them to be bars

($\equiv$ $n+2$ slots, choose n of them to be stars & rest are bars)

~~Binomial coef~~ I.O.W.: $\binom{n+2}{2} = \binom{n+2}{n}$

More generally: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ $\Big\}$ Binomial coefficients

Let S be a set w/ $n = |S|$.

Let $0 \le k \le n$, ~~[illegible]~~

$\boxed{Q1}$ How many subsets does S have?

$2 \cdot 2 \; 2 \; \ldots \; 2 = 2^n$ choices

$\{0,1\}$ $\{0,1\}$

$\underset{1}{\_\_}$ $\boxed{\underset{2}{\_\_}}$ $\_\_ \ldots \_\_ \; \_\_ \; \underset{n}{\_\_}$ ← indexing our elts of S.

choose 0 if include 2nd elt
1 if don't include 2nd elt.

$$\boxed{2^n = \sum_{k=0}^{n} \binom{n}{k}}$$

$\boxed{Q2}$ How many permutations are there on a set of K elements?

✓

k choices → $k-1$ choices, once the 1st selected

$\underset{1}{\underline{k}}$ $\underset{2}{\underline{k-1}}$ $\underline{k-2}$ $\_\_ \ldots$ $\underset{k}{\underline{1}}$ ← index of the elt.

$\boxed{Q3}$ How many subsets of size k are there?

✓

$\underset{1}{\_\_}$ $\underset{2}{\_\_}$ $\underset{3}{\_\_}$ $\_\_ \ldots$ $\underset{k}{\_\_}$

$k=1$: $\underset{1}{\underline{n}}$

n choices

$k=2$: $\underset{1}{\underline{n}}$ $\underset{2}{\underline{n-1}}$ $= n(n-1) = \frac{n!}{n-2!} = \frac{n!}{(n-k)!}$

counting # of <u>ordered</u> subsets of size k

$k=2$
$S=\{a,b,c\}$

$\frac{a}{1}\ \frac{b}{2}$ same set can come in any permutation!

$\frac{b}{1}\ \frac{a}{2}$ By Q2, we have $k!$

Putting this together, we get

$$\underbrace{\frac{n!}{(n-k)!}}_{\#\text{ of ordered subsets}} \ / \ k!$$

"divided by"

$k!$ → # of repeats of each subset

$$= \frac{n!}{(n-k)!\,k!} =: \binom{n}{k} \text{ unordered subsets.}$$