16 Sept 2022

A recursive formula is a formula that uses smaller instances of itself.

$T: \mathbb{N} \to \mathbb{R}$

recursive form

e.g.,
$$T(n) = \begin{cases} 1, & n = 1 \\ T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + n, & n > 1 \end{cases}$$

$$\cancel{T(n) = \begin{cases} 0, & n = 0 \\ T(n/2), & n > 0 \end{cases}}$$

if $T: \mathbb{R} \to \mathbb{R}$ goes on forever!
(e.g.) $T(5) = T(2.5)$
$= T(1.25)$
...

or if $T: \mathbb{N} \to \mathbb{R}$ not well-defined.

SORTING

asymptotic form

$$\boxed{T(n) \text{ is } \Theta(n \log n)}$$

This is what we want!

asymptotic recursive form

$T: \mathbb{N} \to \mathbb{R}$

$$T(n) = \begin{cases} \Theta(1), & n = 1 \\ 2T(\lceil \frac{n}{2} \rceil) + \Theta(n), & n > 1 \end{cases}$$

note: often we drop floor / ceil. here

Closed form:

$T(n) = n \log n - 1$ (exact # of steps)

We've seen this before!

$$\left.\begin{aligned} \sum_{k=1}^{n} k &= \sum_{k=1}^{n-1} k + \sum_{k=n}^{n} k \\ &= \sum_{k=1}^{n-1} k + n \\ f(n) &= f(n-1) + n \end{aligned}\right\} \text{recursive form}$$

$$\boxed{\sum_{k=1}^{n} n-k+1} = \sum_{k=1}^{n} k = \frac{n(n-1)}{2} \quad \Big\} \text{ closed form}$$

$$\sum_{k=1}^{n} k = \Theta(n^2) \quad \Big\} \text{ asymptotic form}$$

e.g., BUBBLE SORT

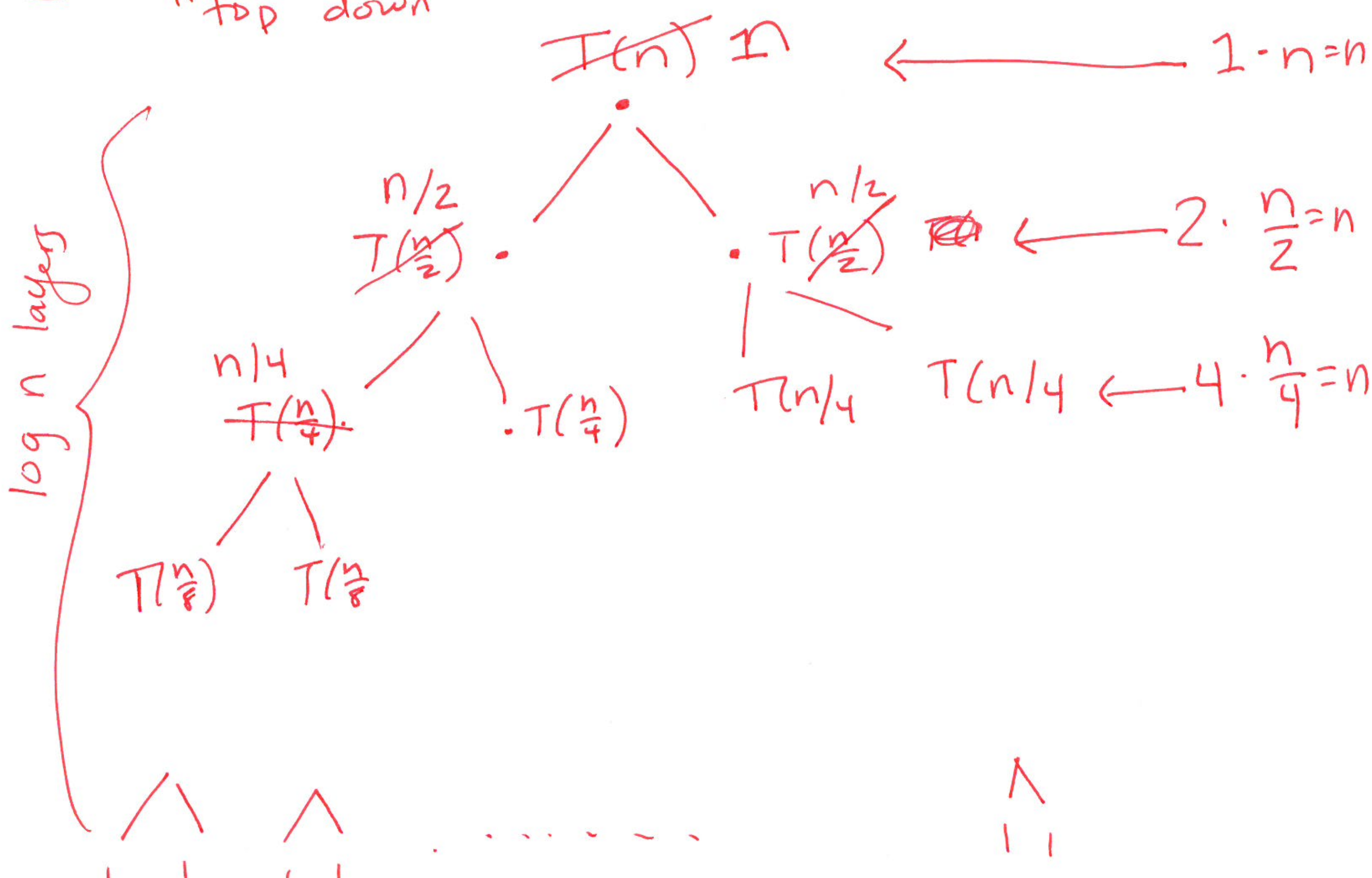
I have a recurrence relation.

Now what? How do I solve it?

1 Recursion Tree

"top down"

$T(n) = 2T(\frac{n}{2}) + n$



TOTAL: logn layers, each of which costs n

$\therefore \Theta(n \log n)$

(2) "bottom up"

$T(1) = 1$

$T(2) = 2 \cdot T(1) + 2 = \underline{2 \cdot 1 + 2} = 4$

$T(3) = 2T(2) + 3 = 2 \cdot 4 + 3 = 11$

$= 2 \cdot (2 \cdot 1 + 2) + 3$

do we see a pattern here?

(3) Guess / Check (via induction)

(4) Master's Method

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

the cost of putting those subproblems together

how many subproblems

each subproblem has cost / size $\frac{n}{b}$ (not n-1, etc)

Merge Sort: $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$

Binary Search: $T(n) = 1 \cdot T\left(\frac{n}{2}\right) + \Theta(1)$

In-Class Exercise 03

CSCI 432

September 16, 2022

$\log_b a = c$

$\iff b^{\log_b a} = b^c$

$\iff a = b^c$

Name:
Who did you work with today?

Master's Theorem

Master's theorem allows us to quickly solve recurrence relations of the form:

$$T(n) = aT(n/b) + f(n),$$

where $a, b \in \mathbb{N}$ such that $a \geq 1$ and $b > 0$ and $f(n)$ is asymptotically positive. Then, we can determine the closed-form of $T(n)$ as follows:

case 1 case 2 case 3

? ?

$n^{\log_b a-\varepsilon}$ $n^{\log_b a}$ $n^{\log_b a+\varepsilon}$

1. IF there exists $\varepsilon \in \mathbb{R}_+$ such that $f(n) \in O(n^{\log_b a-\varepsilon})$, THEN $T(n) \in \Theta(n^{\log_b a})$.
2. IF there exists $\varepsilon \in \mathbb{R}_+$ such that $f(n) \in \Theta(n^{\log_b a})$, THEN $T(n) \in \Theta(n^{\log_b a} \log n)$.
3. IF (1) there exists $\varepsilon \in \mathbb{R}_+$ such that $f(n) \in \Omega(n^{\log_b a+\varepsilon})$ and (2) there exists $c \in (0,1)$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $af(n/b) \leq cf(n)$, THEN $T(n) \in \Theta(f(n))$.

	a	b	$\log_b a$	$n^{\log_b a}$	$f(n)$	Potential Case?	ε, if Case 1 or 3	Closed Form
$T(n) = T(n/2) + 1$								
$T(n) = 2T(n/4) + \sqrt{n}$								
$T(n) = 2T(n/4) + n$								
$T(n) = 2T(n/4) + n^2$								
$T(n) = 3T(n/3) + \Theta(1)$								

Remember, Case 3 has an additional condition to check! Do that in the space provided below, or on the back of this page.

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