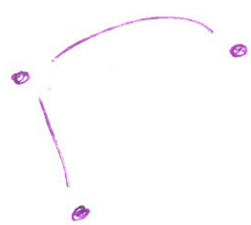
26 August 2022

<u>Claim:</u> A tree with n vertices has n-1 edges.

<u>Def'ns:</u>

- A <u>graph</u> $G=(V,E)$ is ~~a set of nodes or~~ a set V that we call <u>nodes</u> or <u>vertices</u> and a set E of unordered pairs of vertices that we call <u>edges</u>.



- A <u>multigraph</u> allows self-loops + multiple edges between the same pair of verts (ie., E is a multiset)

- A <u>set</u> does not allow the same object twice, ie.,

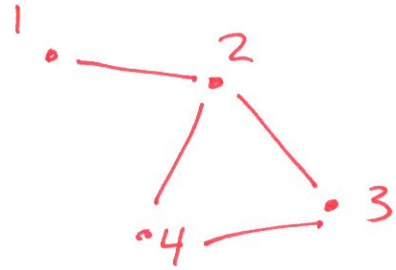
$$\{\{a\},\{a\},\{b\}\} = \{\{a\},\{b\}\}$$

- A digraph has directed edges
$[a,b] \neq [b,a]$

- 

} same or isomorphic graphs, but different <u>labeled</u> graphs.

• a tree is an acyclic graph that is connected (that is, a graph for which there does not exist a nontrivial path that starts & stops at the same vertex & does not repeat edges

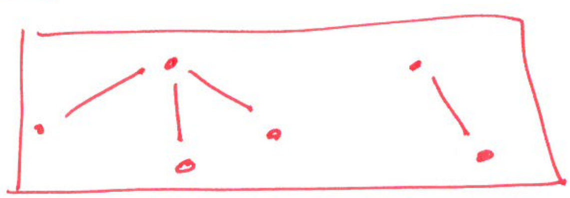


path
2 → 3 → 4 → 2

path
1 → 2 → 1

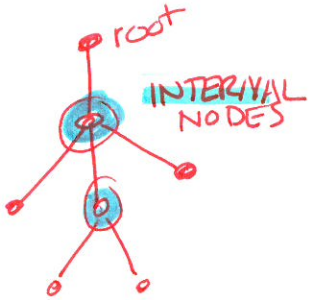
→ sometimes trees have a designated root node. Then, every other vertex/node is an "descendant" of that root. We can think all edges as pointing down from "parent" to "child"

• A forest is a set of trees



• A leaf is a node in a tree with only one "neighbor" (aka - incident to exactly one edge)

Lemma 1 Every tree with ≥ 2 vertices has at least one leaf node.



Proof: by contradiction, using the fact it's acyclic

LEMMA 2 Removing an edge from a tree with ≥ 2 verts results in two smaller trees

each has less vertices than the original

Proof: Case analysis / proof by contradiction.

Lemma 3 If T is a tree with ≥ 2 verts, then T has at least one edge.

Claim: A tree with n vertices has n - 1 edges.

Restatement: $\forall n \in \mathbb{Z}$ s.t. $n \geq 0$, INDUCTION

~~1 ... ORS a ...~~

$\forall$ trees T with n vertices, T has n - 1 edges.

Claim: $\forall x \in S$
Proof
Let $x \in S$.

generalizing from the general particular

PROOF: We proceed by induction.

For the base case $n = 1$. There is only one tree w/ 1 vertex + that tree has $0 = 1 - 1$ edges.

Let $\boxed{k \geq 1}$,

assume that "$\forall$ trees T with $n \leq k$ verts T has $k-1$ edges"

Now, let $n = k + 1$.

WTS: "$\forall$ trees T w/ $k+1$ verts, T has k edges"

Let $T = (V, E)$ be a tree with $k+1$ vertices.

Since $k \geq 1$, we know $k+1 \geq 1+1 = 2$.

$\therefore$ by $\boxed{\text{Lemma 3}}$, T has at least one edge.

Let $e \in E$. $T \setminus \{e\} = T_1 \sqcup T_2$ by $\boxed{\text{Lemma 2}}$.

Notationally, let $T_1 = (V_1, E_1)$
$T_2 = (V_2, E_2)$.

Since no verts removed/duplicated,
$|V_1| + |V_2| = |V|$ and $|V_1| \lneq |V|$ and $|V_2| \lneq |V|$

By I.A, we know:

$$|E_1| = |V_1| - 1 \text{ and } |E_2| = |V_2| - 1.$$

~~Because~~ Since no edges are in both E_1 and E_2 and since e was the only edge removed from T,

$$|E| = \underset{\substack{\uparrow \\ \text{for } e}}{1} + |E_1| + |E_2|$$

$$= \cancel{1} + |V_1| - \cancel{1} + |V_2| - 1$$

$$= |V| - 1$$, as was to be shown.

$\square$

Challenge: try this using $\boxed{\text{Lemma 1}}$ instead.