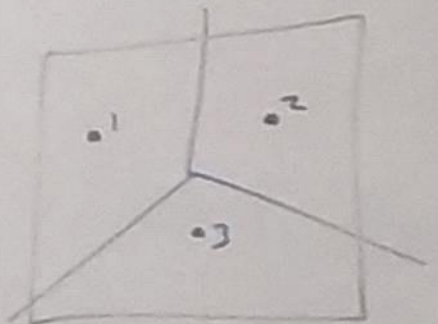
Voronoi Diagrams

P in $\mathbb{R}^2$



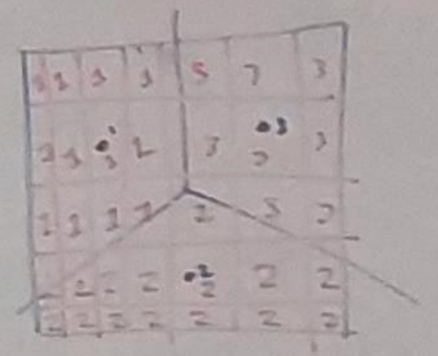
continuous domain

Choose a pt $q \in \mathbb{R}^2$, decide which $p \in P$ is closest to q

Voronoi Diagram

* Cells are convex

Pixels in Computer

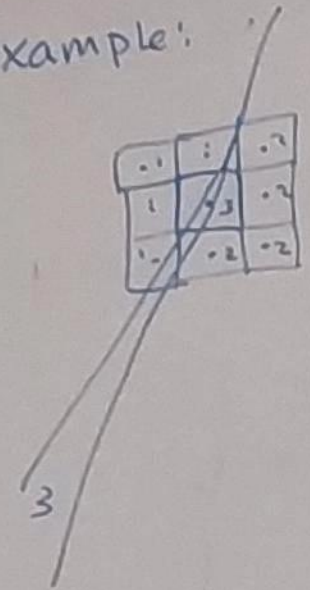


Choose a pixel (i,j) and label it w/ the closest site to its center.

Discrete Voronoi Diagram

*cells can be disconnected!

→ example:



RIC for trapezoid maps (input: S, a set of segments)

INITIALIZE: $\square$ } bounding box for $\subseteq \mathbb{R}^2$ that we care about

RANDOMLY ORDER $S = \{s_1, s_2, \dots, s_n\}$ labeled by the random order

At any step, we have trap of $(S_{i-1} := \{s_1, s_2, \dots, s_{i-1}\})$

Add back s_i → update the trapezoids $\Theta(k_i)$, where k_i = # of traps in S_i touching s_i.

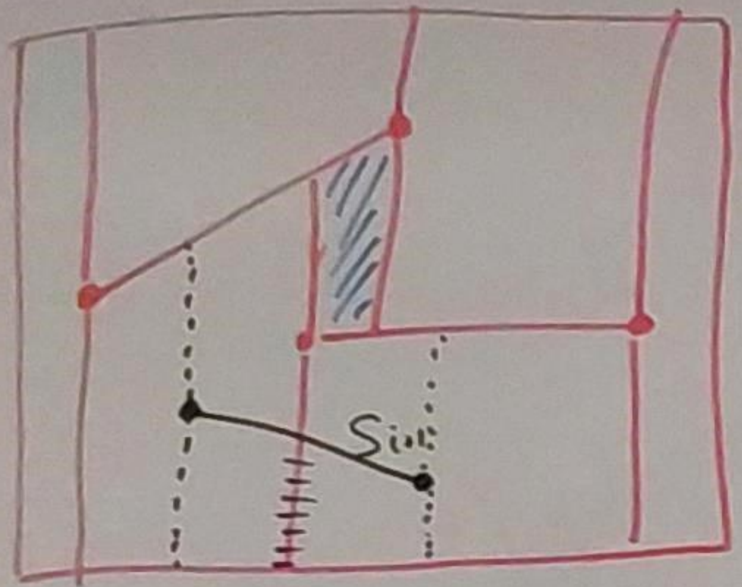
→ update the search tree data structure $\Theta(k_i)$

At worst: k_i is $O(i) \Rightarrow$ algorithm is $O(n^2)$

At best: k_i is $\Theta(1) \Rightarrow$ algorithm is $\Theta(n)$ if always "best-case" scenario

We'll show $E(k_i) = \Theta(1) \Rightarrow$ expected analysis is $\Theta(n)$ runtime

T_i = the trap. of S_i



Claim: $\mathbb{E}(k_i) = \Theta(1)$

Proof:

$$\mathbb{E}(k_i) = \frac{1}{i} \sum_{s_j \in S} k_j$$

$$= \frac{1}{i} \sum_{s_j \in S_i} \sum_{t \in T_i} \mathbb{1}(s_j \text{ touches } t)$$

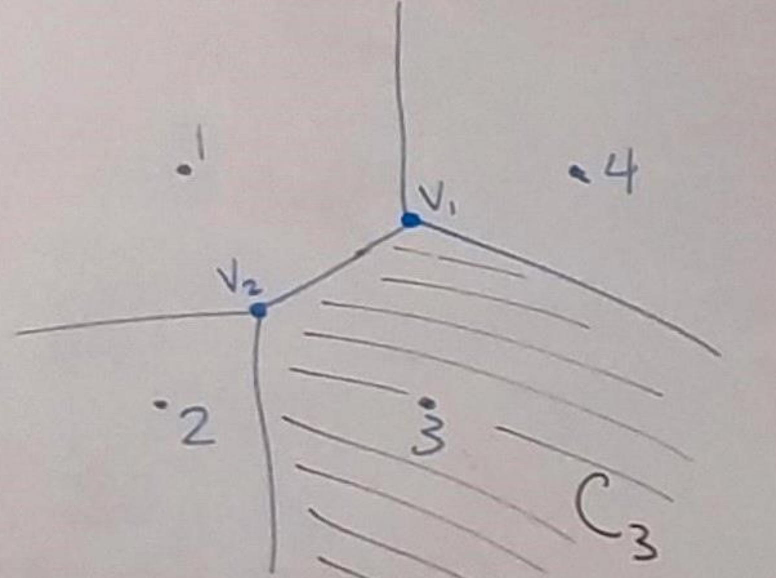
$$= \frac{1}{i} \sum_{t \in T_i} \left(\sum_{s_j \in S_i} \mathbb{1}(s_j \text{ touches } t) \right)$$

$$\leq \frac{1}{i} \sum_{t \in T_i} 4 \longleftarrow \text{since each trap. is defined by 4 sides.}$$

$$= \frac{4}{i}(3i+1) = \Theta(1) \quad \blacksquare$$

S - set of sites in $\mathbb{R}^d$

V-Dgm: A {cellulation / partition} of $\mathbb{R}^d$ defined by nearest neighbors



v_1 = closest to {1, 3, 4}, equidistant to those 3

C_3 = closest to {3}. unique closest elt of our sites

codim 0 cells (2-cells in $\mathbb{R}^2$, 3-cells in $\mathbb{R}^3$) have a unique closest

codim 1 cells have 2 equidist. closest

comdim 2 cells have 3 equidist closest..

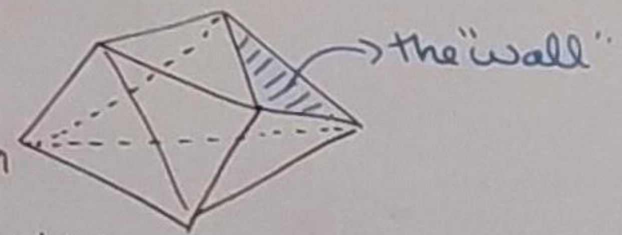
$\mathbb{R}^3$

each site has a 3-cell

"walls" of 3-cells are 2-cells + have 2 unique neighbors (codim 1)

"edges of the walls" are 1-cells (codim 2 here) have 3 equidist NN

Vertices have 4 equidist NN



Could be a voronoi cell

The dual structure to voronoi diagram is the Delaunay Δulation.

Voronoi in $\mathbb{R}^d$	Delaunay
cell V_i for site s_i	a pt s_i ← we use the site itself
$\exists$ wall between V_i & V_j	an edge (s_i, s_j) ← we use a straight
$\bar{V}_i \cap \bar{V}_j \cap \bar{V}_k \neq \emptyset$	a triangle $[s_i, s_j, s_k]$

$\mathbb{R}^2$ stops here

$\bigcap_{i=1}^{k} \bar{V}_i \neq \emptyset$	Simplex $[s_1, s_2, \ldots, s_k]$

we stay in the "flat" / affine space

$\bar{V}_i$:= the closure of V_i, V_i along w/ its boundary

Properties/Observations of DT

(1) Over all Δulations of S,
DT maximizes the min angle
(over all angles in all Δs.

Σ = the set of all angles in all Δs.

$(\sigma_1, \sigma_2, \dots, \sigma_N)$ orders ∡s from smallest to largest

The DT maximizes $(\sigma_1, \sigma_2, \dots, \sigma_N)$ wrt lexicographic order over all Δulations.

(Proof in DMLN-11)

(2) Empty Circle Property

(a) $[abc] \in DT(S) \iff$ circumcircle of $\{a,b,c\}$ is empty (no pts of S are inside it)

(b) $[ab]$ an edge in $DT(S) \iff$ circle of diameter $\overline{ab}$ is empty.

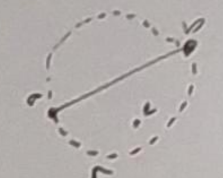
(3) Closest Pair Property

$\{a,b\}$ are closest pair in S

$\Downarrow$

(a,b) is an edge in $DT(S)$

by (2b)



would be a contradiction

(4) MST $\subseteq DT_1(S)$

one-skeleton

Proof: Let T be a MST for $CG(S)$.

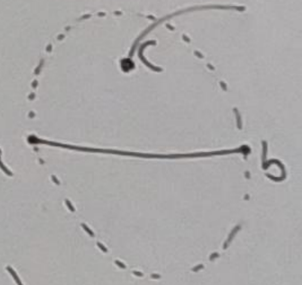
complete Euclidean-dist graph whose verts are S

Suppose, by contradiction, that $T \not\subseteq DT_1(S)$.

✓ they have same vertex set

Thus, there must be some edge $(a,b) \in T$ that is not in $DT_1(S)$.

$\therefore \exists c \in S$ inside:



Consider $T' = T \setminus \{(a,b)\}$

WLOG, we have 2 conn. comp, one with $\{a,c\}$ and one with $\{b\}$

$T'' = T' \cup \{(b,c)\}$

$w(T'') = w(T) - w(ab) + w(bc)$

$< w(T)$ ※

DTs & CHs

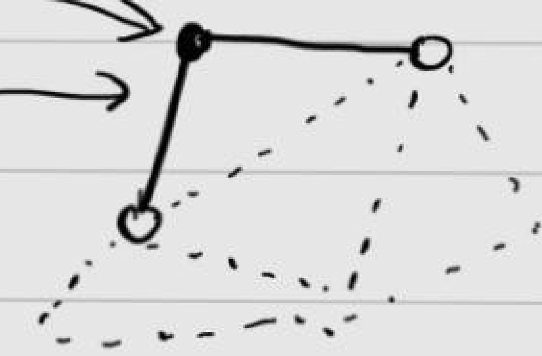
① Unbounded cells of $VD(S)$

$\Updownarrow$

vertex of $CH(S)$

$\Updownarrow$

endpoint of "boundary" edge of $DT(S)$



a) A boundary edge sees one Δ

b) An interior edge sees two Δs.

② $DT(S)$ is the $\downarrow$ projection of $LCH(S^{\uparrow})$

- $S^{\uparrow}$: S lifted to parabola $z = x^2 + y^2$