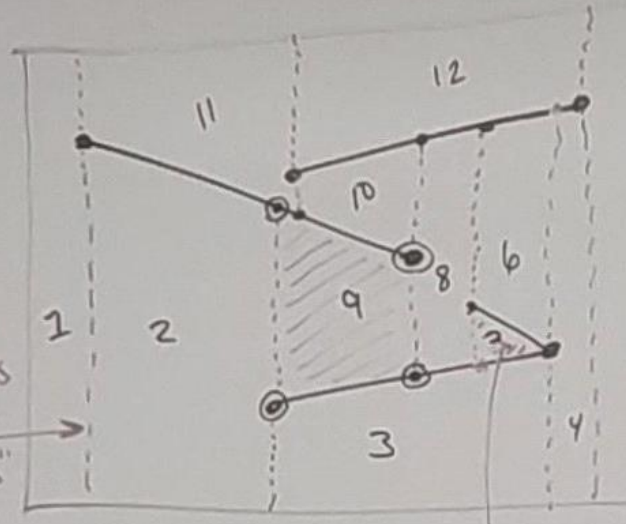
Lessons 8+9: Trapezoidations & Point Location

in $\mathbb{R}^2$, let $G=(V,E)$ be an embedded graph
$E \to S$ in lecture notes

$\neg P$ assumption: no 2 verts share x-coord
→ no vertical edges

Point Location: Given a query point q and a embedded graph G, find the "face" (2-cell) that contains q.
Ideally: G has size n, want $\Theta(\log n)$

vertical segments we add are called "walls"



a degenerate trapezoid

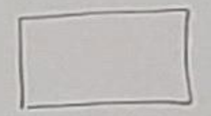
INPUT: 4 segments, 7 endpoints
$T(S)$ has 11 traps + 1 Δ = 12 cells
15 vertices = 11 on BB + 4 on edge

① create a bounding box
↳ this step is for convenience of not dealing w/ infinite/unbounded cells.

② from each endpoint, shoot "bullets" up + down

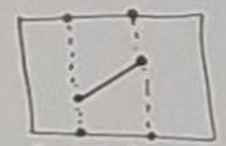
Small Examples:

n=0



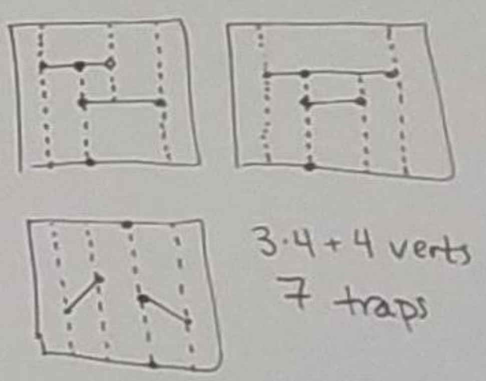
4 verts, 1 trap

n=1



3·2+4 verts
4 trap

n=2



3·4+4 verts
7 traps

The 2 cells are either ① trapezoid: top/bottom are pieces of input line segments (or BB)
left/right are vertical lines defined by an endpoint of a segment.
② degenerate: triangle when 2 edges have a common endpoint

Q: How many verts + traps are in the trapezoidation of S, where $|S| = n$?

endpoints + new ones we introduce

try this first!

$\boxed{\# \text{ segments} = n}$

$\# \text{endpoints} \le 2n$, if shared allowed

$\boxed{\# \text{endpoints} = 2n}$, if no sharing / completely disjoint edges/segments

can you upper/lower bound the # of verts or traps?

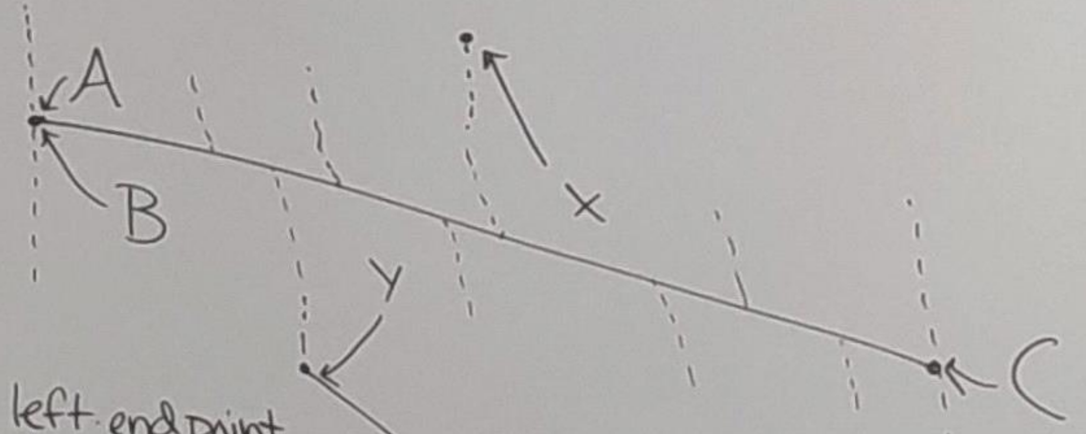
the trapezoidation map of S.

<u>Lemma</u>: $\mathcal{T}(S)$ has $6n+4$ vertices
(exact if no Δ's, upper bound OTW)

has $3n+1$ traps
(exact if no Δ's, upper bound OTW)

<u>Proof</u>: We start w/ n segments, which has $2n$ endpoints.
Each endpoint contributes (up to) 3 vertices in $\mathcal{T}(S)$, one from the endpoint itself, one from where each bullet lands.
Plus, we have 4 verts from bounding box
$\therefore 3(2n) + 4 = 6n+4$ vertices.

Charge each trap to the endpoint that defines it's left vertical wall.



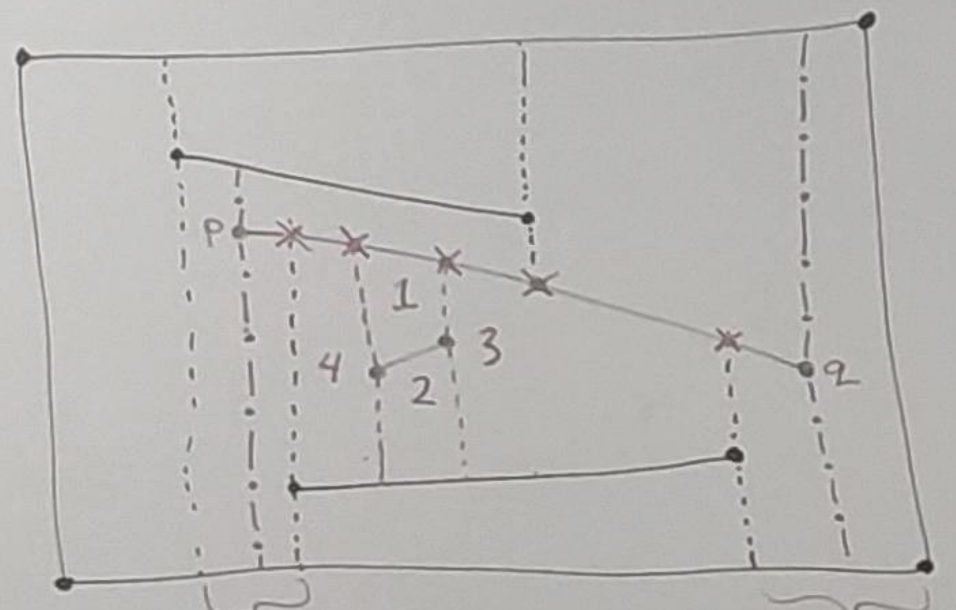
So, each left endpoint has 2 traps + each right endpoint has 1
$\therefore$ 3/segment, plus one "unlabeled" trap
$\Rightarrow 3n+1$ traps.

$\square$

$S = \{s_1, s_2, \dots, s_n\}$, randomly ordered

RIC: We've solved for $S_i = \{s_1, s_2, \dots, s_i\}$

want to add s_{i+1}

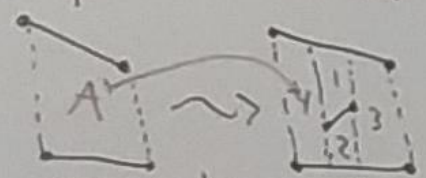


1 trap ~> 3 traps
added 2 new ones

1 trap ~> 3 traps
added 2 trap
1 charged to the new edge

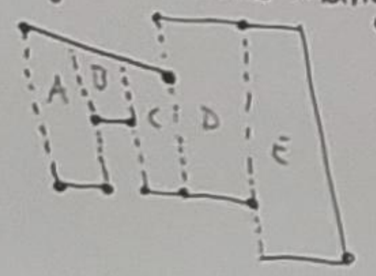
Cases:

1) next segment is in one cell



one cell becomes 4

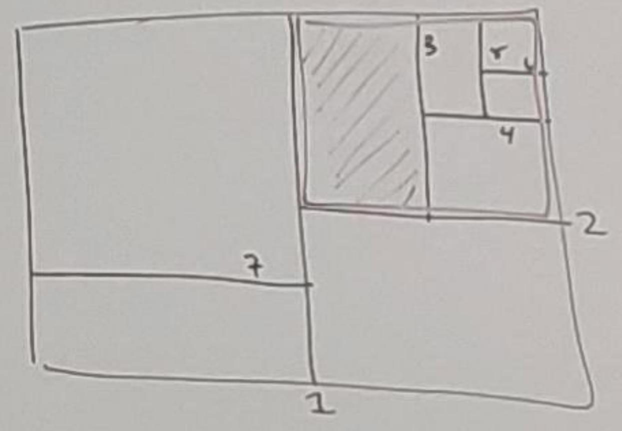
2) new segment crosses many cells

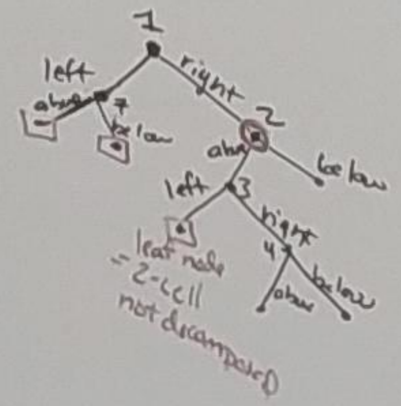


Left: add vert lines
Right: add vert lines
in between: block bullets

Sidenote: Quad tree

→ each internal node rep
above/below or left/right
of a particular horiz/vert line.





For point location w/ trap maps:

Buid a binary DAG w/ 3 types of nodes:
↳ a binary tree w/ shared subtrees

(i) leaf nodes represent trapezoids

(ii) internal node rep. to left/right of a given x-value (from an endpoint)

(iii) internal node rep above/below a (partial) line segment

note: only asked if x-coord is between x-coords of the line segment

note 2: need both x + y-coords to answer this q.