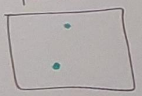
Some Basics:

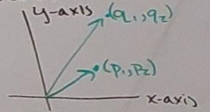
$\mathbb{R}^d \equiv \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{d\text{-times}}$

$p \in \mathbb{R}^d$ can be "a point" ← the location
"a vector" ← the direction (+ magnitude)

points



vectors



$\alpha \in \mathbb{R}^1$ is often called "a scalar"

things I can do:

$u, v \in \mathbb{R}^d$

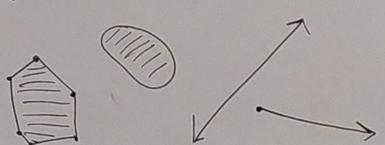
$u + v = (u_1 + v_1, u_2 + v_2, \dots, u_d + v_d) \in \mathbb{R}^d$

$u \cdot v = \sum_{i=1}^{d} u_i \cdot v_i \in \mathbb{R}$

$\alpha u = (\alpha u_1, \alpha u_2, \dots, \alpha u_d) \in \mathbb{R}^d$

Def$^{\underline{n}}$: A set $S \subseteq \mathbb{R}^d$ is <u>convex</u> iff $\forall s_1, s_2 \in S$, the line segment $\overline{s_1 s_2}$ is also contained in S.

e.g.,

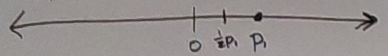


$P \subset \mathbb{R}^d$, $|P|$ is finite.

$\therefore P = \{p_1, p_2, \dots, p_n\}$

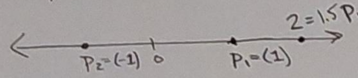
- $q \in \mathbb{R}^d$ is a <u>linear combination</u> of points in P iff $\exists$ a coef. vector $\alpha \in \mathbb{R}^d$ s.t. ("such that")
  $q = \sum_{i=1}^{|P|} \alpha_i p_i$
- an <u>affine</u> combination of vectors in P is a <u>lin. combo</u> $\sum \alpha_i p_i$ such that
  $\sum \alpha_i = 1$.
  "creating a whole new point by combining the other pts. α_i tells us how much of p_i to take"
- a <u>convex combo</u> is an <u>affine combo</u> such that $\alpha_i \geq 0 \ \forall i$.

$P=\{p_1\}\subset\mathbb{R}^1$



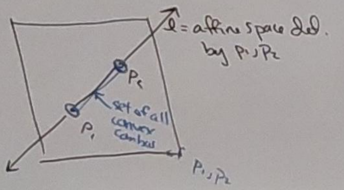
- $\mathrm{linspan}(P)=\mathbb{R}^1$
- $\mathrm{aff\ span}(P)=\bullet$
- $\mathrm{convex\ span}(P)=\bullet$

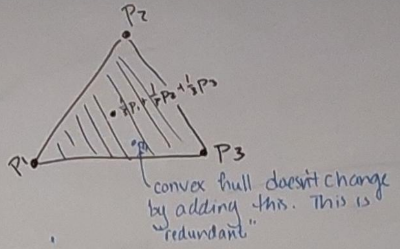
$P=\{p_1,p_2\}\subset\mathbb{R}^1$



- $\mathrm{lin\ span}(P)=\mathbb{R}^1$
- $\mathrm{aff\ span}(P)=\mathbb{R}^1$ = line def. by p_1,p_2
- $\mathrm{convex\ span}(P)$ = line seg. def by p_1,p_2

$P=\{p_1,p_2\}\subset\mathbb{R}^2$



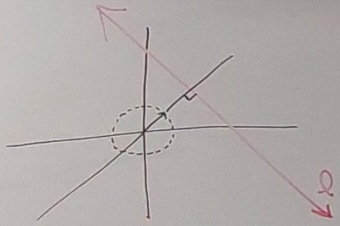


$CH(\{p_1,p_2,p_3\}) = CH(\{p_1,p_2,p_3,p_4\})$

3 equivalent definitions of (Planar) Convex Hull: Given a finite set $P\subset\mathbb{R}^2$

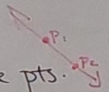
1. Smallest convex set that contains P.
   ↳ area
2. The set of all convex combinations of points in P.
3.

EXAMPLES:

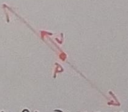


How to define ℓ?

① Choose 2 pts & def. ℓ to be the set of aff combos of those pts.



② a pt $\in \ell$ and a vector to indicate direction



③ If not vertical: $y = mx + b$

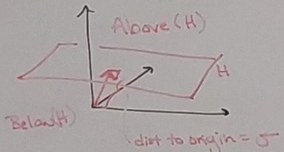
④ direction u & scalar σ
↳ perp. to ℓ ↳ "height"

$$h(u,\sigma) = \{x \in \mathbb{R}^d \mid u \cdot x = \sigma\}$$

Direction = a unit vector

• $S^{d-1} \subset \mathbb{R}^d$ is the set of directions in $\mathbb{R}^d$.

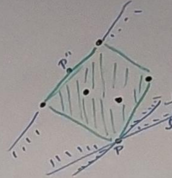
I use h for "hyperplane"



Half Spaces (defined by hyperplanes)

$\mathring{h}^{\uparrow}(u,\sigma) = \{x \in \mathbb{R}^d \mid u \cdot x > \sigma\}$ "open" doesn't include the boundary

$\bar{h}^{\uparrow}(u,\sigma) = \{x \in \mathbb{R}^d \mid u \cdot x \geq \sigma\}$ "closed" includes the boundary



• = point set P
/ = convex hull H

Given a convex shape (e.g., H above) and a point $p \in \partial(H)$, (the boundary of)

we call ℓ a support line if H is entirely on one side of ℓ and $p \in \ell$

INPUT

- Let p, q, r be 3 random pts in $\mathbb{R}^2$

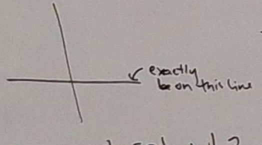
Q What is the probability that the 3 points are on the same line?

A: The prob. is 0!

→ move origin to 1st point

→ spin so 2nd pt is on x-axis

Same Q: What is Prob that 3rd pt is also on the x-axis?



$$\frac{\mu(x\text{-axis})}{\mu(\mathbb{R}^2)} = 0$$

NOTE:

If I'm given a pt set w/ 3 colin points, I can jiggle it + get arbitrarily close, but no longer co-linear.

General Position:

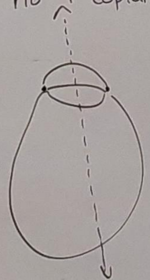
The liscence to assume that our input is "nice" aka avoiding degenerate input.

→ want: Prob of nice input = 1

e.g., no 3^+ colinear points

no 4^+ co-circular points

no 4^+ coplanar points in $\mathbb{R}^{3+}$



3 equivalent definitions of (Planar) Convex Hull: Given a finite set $P \subset \mathbb{R}^2$

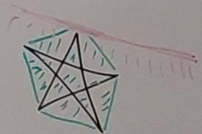
① Smallest convex set that contains P.

↳ area

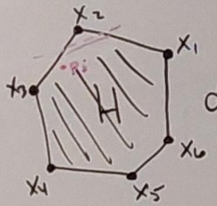
② The set of all convex combinations of points in P.

③ The intersection of all closed halfplanes containing P.

EXAMPLES:



Output: CCW list of verts def. $CH(P) =: H$



define H by $\partial(H)$.

e.g.,

$\langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$

OR

$\langle x_4, x_5, x_6, x_1, x_2, x_3 \rangle$

} define the same shape!

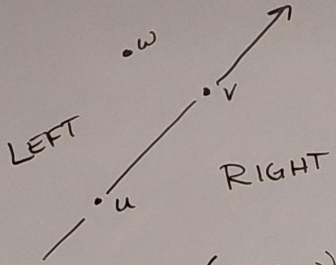
<u>Lemma 1</u>

① $x_i \in P$ proof by contradiction

② $\overleftrightarrow{x_i x_{i+1}}$ is a support line

③ 3 in a row makes a left turn.

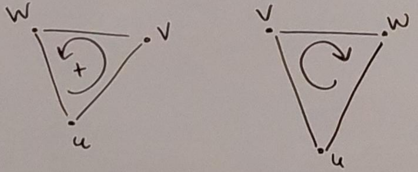
x_i, x_{i+1}, x_{i+2} is a "Left"



orientation orient(u, v, w)

- start at u, point toward v, which way is w?

Left = positive direction



Jack's favorite determinant

$$\text{orient}(u,v,w) = \text{sign}\left(\det\begin{pmatrix} 1 & u_x & u_y \\ 1 & v_x & v_y \\ 1 & w_x & w_y \end{pmatrix}\right)$$