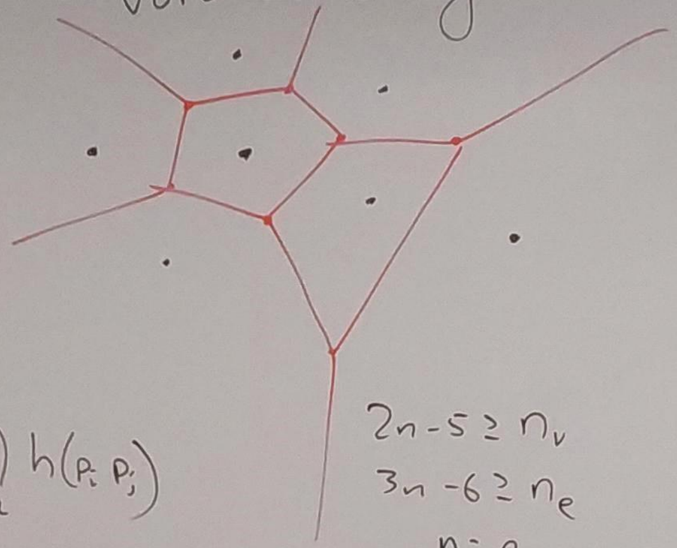
Voronoi Diagrams - Fortune's Algorithm

Recap -

$P - \{p_1 \dots p_n\}$ - sites

$V(p_i) = \{q \in \mathbb{R}^2 \mid p_i \text{ closest site to } q\}$

voronoi cells



vertices at center of empty circumcircles

- faces are convex
- $\forall$ v $\in$ Voronoi $\deg(v) = 3$
- unbounded cell $\Rightarrow$ on convex hull

$V(p_i) = \bigcap_{j \neq i} h(p_i p_j)$

$\Rightarrow$ $O(n^2 \log n)$ deterministic or $O(n^2)$ expected

$2n - 5 \geq n_v$

$3n - 6 \geq n_e$

$n = n_f$

All subsets of 3 points

check each other point to see if in circumcircle

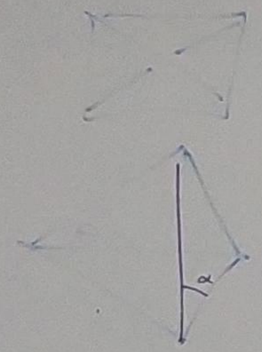
if empty

$\Rightarrow$ vertex in voronoi

$O(n^4)$

Delaunay triangulation

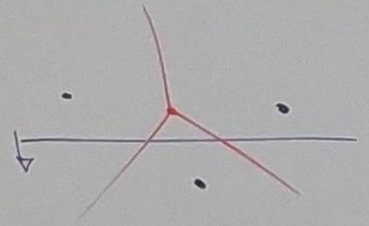
- Best - max min angle



Fortune's Algo.

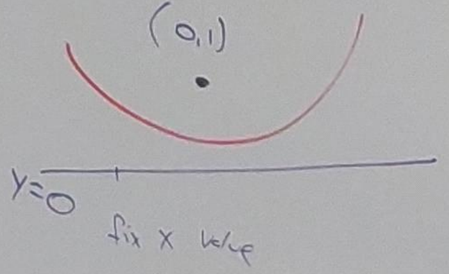
Sweep line

Problem: Sites below the sweep line may generate a vertex above the sweep line.



Solution: Add another line beach line.

Beach line - records points that are equal distance to site above sweep line & the sweep line.



what is the shape

$$(x-0)^2+(y-1)^2=(x-x)^2+(y-0)^2$$

$$x^2+y^2-2y+1=y^2$$

$$\frac{x^2+1}{2}=y$$

Get parabola!

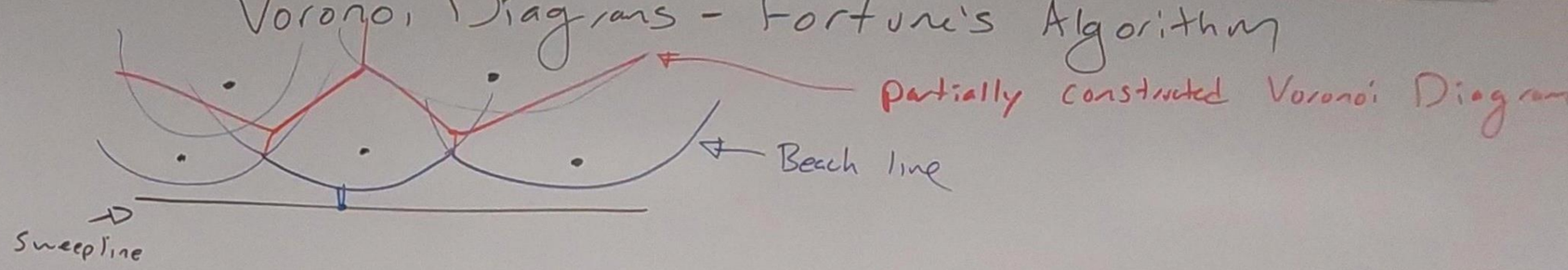
Voronoi Diagrams - Fortune's Algorithm

3 parts of a Sweep line

- Sweep line status - y-coordinate.
  Here: Beach line status

- Events
  Here: site event - site
  Vertex event - add vertex to Voronoi diagram.

- Update Rules



Above beach line - "safe"
Don't need to update

Intersection of parabolas on beach line are on Voronoi edges.

beach line x-monotone
(passes vertical line test)

lower envelope of parabolas

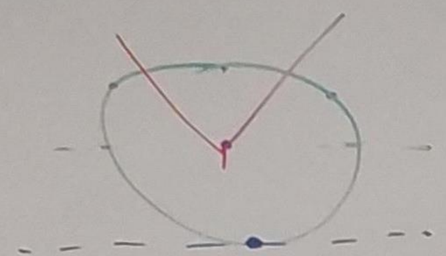
Data structure for Beach line
- only need to store sites in order by x-value

Dictionaries or binary search tree.
(Balanced)

Events

Site event

Pass new site
⇒ add new parabola to beach line.



known ahead of time
Sort by y-value put in priority Queue.

Fortune's Algo.

Vertex event

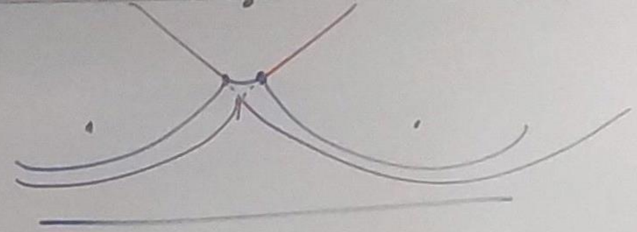
Parabola shrinks to 0

⇒ Add new vertex to Voronoi diagram

$P_i P_j P_k$ on beach line

Consider circumcircle bottom see if empty

if yes y-value of circle bottom is vertex event.



Operations -

- determine parabola above x-value
  Add arc to beach line.
  (find break points of beach line.)
  remove arc from beach line
- predecessor & successor on beach line.

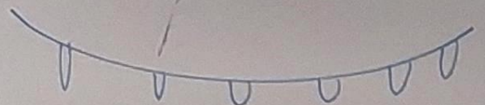
All $O(\log n)$

Voronoi Diagrams - Fortune's Algorithm

Site events

Add parabola

⇒ split an arc



$\langle P_1 P_2 P_j P_3 P_4 \rangle \rightarrow \langle P_1 P_2 P_j' P_i P_j'' P_3 P_4 \rangle$

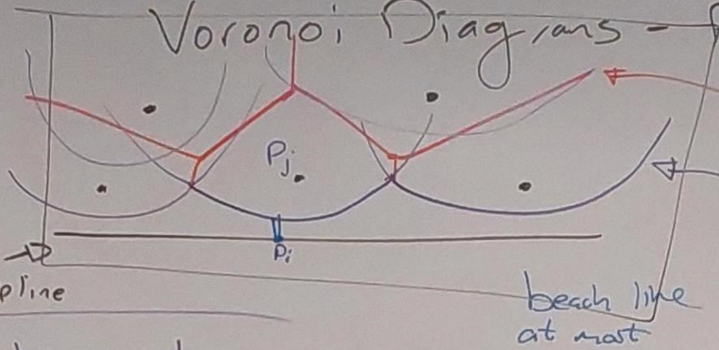
Delete event $P_2 P_j P_3$

Add 2 events

$P_2 P_j' P_i$ + $P_i P_j'' P_3$

Don't need

$P_j' P_i P_j''$



Partially constructed Voronoi Diagram

Beach line

Vertex event

Delete P_j from beach line.

$\langle P_1 P_i P_j P_4 P_2 \rangle \rightarrow \langle P_1 P_i P_4 P_2 \rangle$

Delete events

$P_1 P_i P_j$ $P_i P_j P_4$ $P_j P_4 P_2$

Add events

$P_1 P_i P_4$ + $P_i P_4 P_2$

$O(n \log n)$ Sort by y-value

$O(n)$ events

$O(n)$ - parabolas.

$O(\log n)$ work at each event

⇒ $\Theta(n \log n)$ runtime.

Can we do better?

No. unbounded cells on $conv(P)$.

Voronoi $\rightarrow$ convex hull $\rightarrow$ Sorting

$\Omega(n \log n)$