Worst case RT

$W_d(n)$ $LP_d(H, c)$ — $H = \{h_1, h_2, \ldots, h_n\}$; c: direction for optimization

$\Theta(n)$ 1: If $d=1$, solve directly.

$\Omega(1)$ 2: If $n \le d$, solve directly. (→ just use $O(1)$ for analysis today.)

2.5: $i \leftarrow$ random # btwn 1 & n; swap h_i & h_n

$W_d(n-1)$ 3: $v \leftarrow LP_d(H \setminus \{h_n\}, c)$

$\Theta(d)$ 4: If $v \in h_n$

$\Theta(d)$ 5:   return v

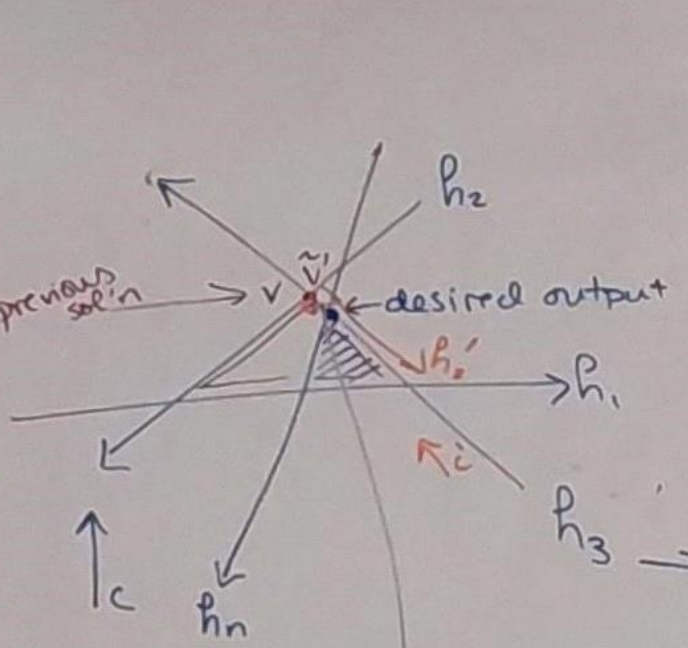
6: else

$\Theta(nd)$ 7: $H' \leftarrow H \setminus \{h_n\}$, projected onto ℓ_n, transformed into $\mathbb{R}^{d-1}$ (gaussian elim!)

$\Theta(d)$ 8: $c' \leftarrow c$, proj. into ℓ_n & then into $\mathbb{R}^{d-1}$

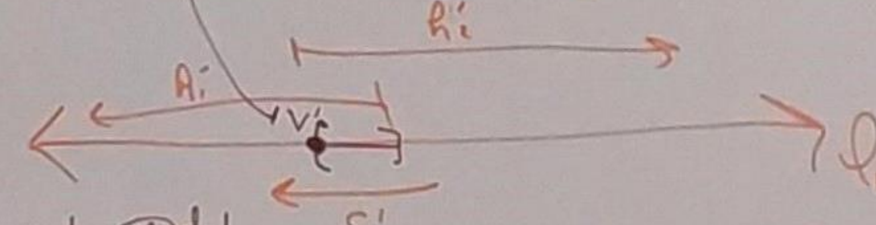
$W_{d-1}(n-1)$ 9: $v' \leftarrow LP_{d-1}(H', c')$

$\Theta(d)$ 10: return v', transformed back into $\mathbb{R}^d$



$h_3 \rightarrow$ defined by $\ell_3 \approx \mathbb{R}^{d-1=1}$

each $h_i \rightarrow h_i'$, an interval $(-\infty, c_i]$ or $[c_i, \infty)$ (a ray)



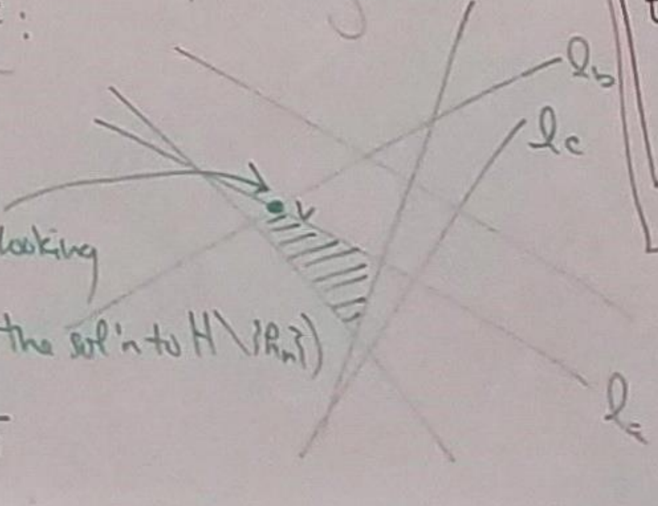
$v' \in \mathbb{R}^{d-1}$ ~~transf. back~~> $v \in \ell_n \subseteq \mathbb{R}^d$

$$W_d(n) = W_d(n-1) + W_{d-1}(n-1) + \Theta(d)$$
$$= \Theta(n^d) \ldots \text{ if } d \text{ const. } \Theta(n^c)$$

Worst-case is realized when… I need to update EVERY time.

But… what are the chances of that happening?

In $\mathbb{R}^2$:



unknown sol'n I'm looking

$\mathbb{P}(v \text{ is not the sol'n to } H \setminus \{h_n\}) = 1 - \frac{2}{n}$

$\boxed{\text{Expected RT: } \Theta(n)}$

$|S| = n$

$T(n)$ QS(S)

$\Theta(1)$ 1: pick random $p \in S$

$\Theta(n)$ 2: $A \leftarrow$ set of elt $< p$

$\Theta(n)$ 3: $B \leftarrow$ set of elts $> p$

$\Theta(n)$ 4: $C \leftarrow$ list of elts $= p$

} total # comparisons prop. to total RT

$T(|A|)$ $T(|B|)$ 5: return QS(A) # C # QS(B)

S (in order) = $[s_1, s_2, \ldots s_{n/2}, \ldots s_n]$

$p = s_n - c$, c const. is also worst case! $\Theta(n^2)$

worst-case pivot $p = s_1$

best case pivot $p = s_{n/2}$

worst case analysis of the algorithm

$$\boxed{\begin{aligned} T(n) &= T(n-1) + T(0) + \Theta(n) \\ &= \Theta(n^2) \end{aligned}}$$

recurrence is:

$$T(n) = 2T\left(\tfrac{n}{2}\right) + \Theta(n) = \Theta(n \log n)$$

Expected RT - take 1

$$T(n) = \Theta(n) + \overbrace{\sum_{i=0}^{n-1}}^{\text{sum over all outcomes}} \underbrace{\frac{1}{n}}_{\text{prob. of outcome}} \overbrace{\left(T(i) + T(n-i-1)\right)}^{\text{the value of outcome}}$$

$$= \Theta(n) + \frac{2}{n} \sum_{i=0}^{n-1} T(i)$$

⋮

gets tricky. But, guess & check. We can prove $T(n) = \Theta(n \log n)$.

note: $|B| = n - |A| - 1$

So, suffices to consider $|A|$.

$|A| \in \{0, 1, 2, \ldots, n-1\}$

and each has $\mathbb{P}(|A| = i) = \frac{1}{n}$

Rand. QS. RT - take 2

Counting the # of comparisons

X = # of total comparisons

X_{ij} = # times s_i and s_j are compared

$$= \begin{cases} 1, & \text{if they are compared} \\ 0, & \text{otherwise} \end{cases}$$

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij} \Rightarrow \mathbb{E}(X) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{E}(X_{ij})$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{\text{outcome } O} \mathbb{P}(X_{ij}=O) \cdot \overbrace{X_{ij}}^{\text{value}}$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(\cancel{\mathbb{P}(X_{ij}=0)} \cdot 0 + \mathbb{P}(X_{ij}=1) \cdot 1 \right)$$

... cont. on next board

$$S = [\overbrace{s_1, s_2, \ldots}^{\text{try again. } X_{ij} \text{ not set}}, s_i, \ldots, \overbrace{s_j, \ldots, s_n}^{\text{try again } X_{ij} \text{ not set}}]$$ ← ordered.

p btwn $\Rightarrow X_{ij} = 0$

$p \in \{s_i, s_j\}$ $\Rightarrow X_{ij} = 1$

p before s_i -OR- p after s_j

not enough info.

Follow the recursion + try again

$$\mathbb{P}\left(\begin{matrix} p \text{ btwn} \\ s_i \text{ and } s_j \end{matrix}\right) = \frac{j-i-1}{n}$$

$$\mathbb{P}\left(\begin{matrix} p = s_i \\ \text{or} \\ p = s_j \end{matrix}\right) = \frac{2}{n}$$

$$\mathbb{P}(X_{ij}=1) = 1 - \mathbb{P}(X_{ij}=0) = \frac{2}{j-i+1}$$

$$\mathbb{P}(X_{ij}=0) = \frac{j-i-1}{j-i+1}$$

.... continuing

$$\mathbb{E}(X) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{P}(X_{ij} = 1)$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-i+1}\right)$$

$$= \sum_{i=1}^{n} 2\left(-1 + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-i+1}\right)$$

$$= \sum_{i=1}^{n} 2\left(H_{n-i+1} - 1\right)$$

$$\leq \sum_{i=1}^{n} 2\left(1 + \ln(n-i+1) - 1\right)$$

$$< \sum_{i=1}^{n} 2(\ln n) = 2n\ln n = \Theta(n\log n) \quad \square$$

Recall: the n-th harmonic #

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$$H_n \in [\ln n, 1 + \ln n]$$

$$\Rightarrow H_n \in \Theta(\log n)$$