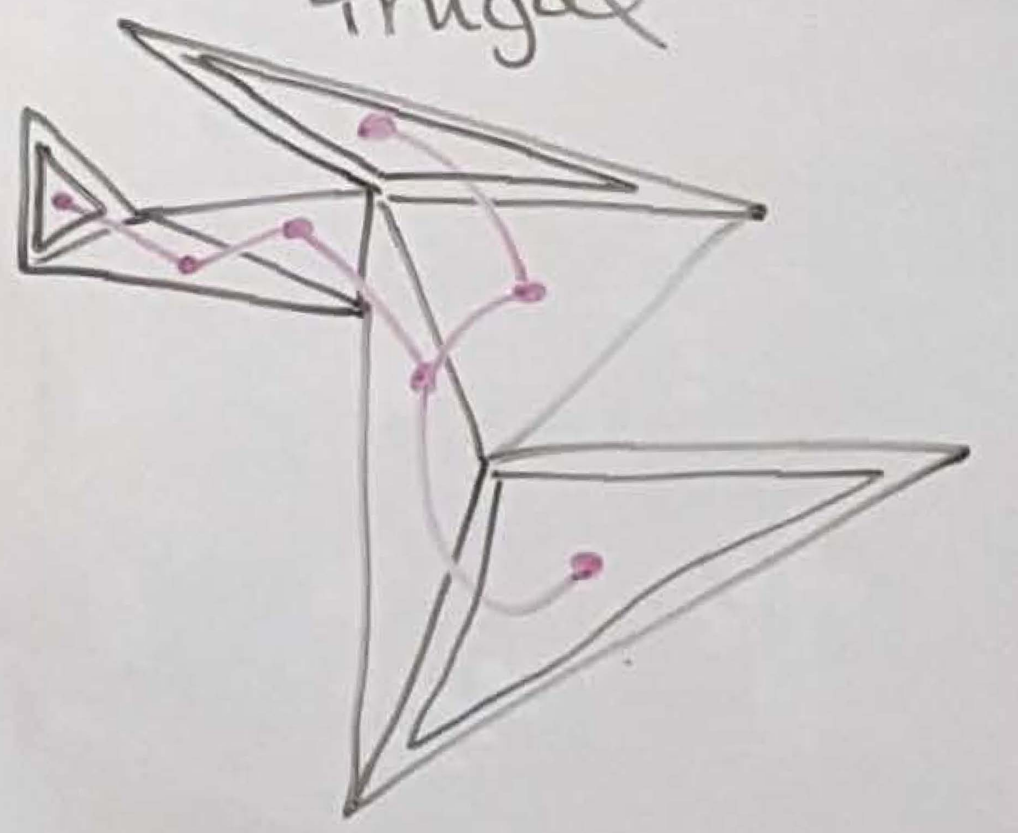
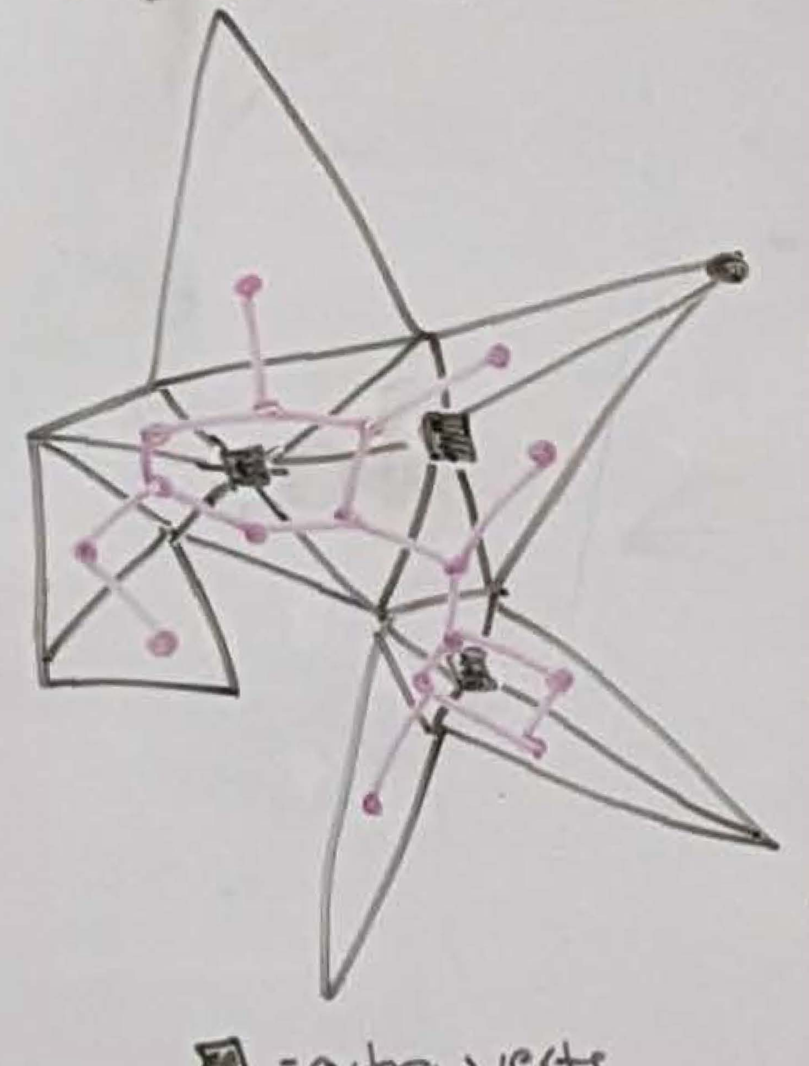
P = polygon

a triangulation of P is frugal if the vertices are exactly the verts of P (i.e., we don't add extra verts)

Frugal



Not Frugal



in pink: the dual

▪ = extra verts

The dual graph is a tree for a frugal Δulation.

Proof: Use induction. Try it!

Primal

"ear"

Dual

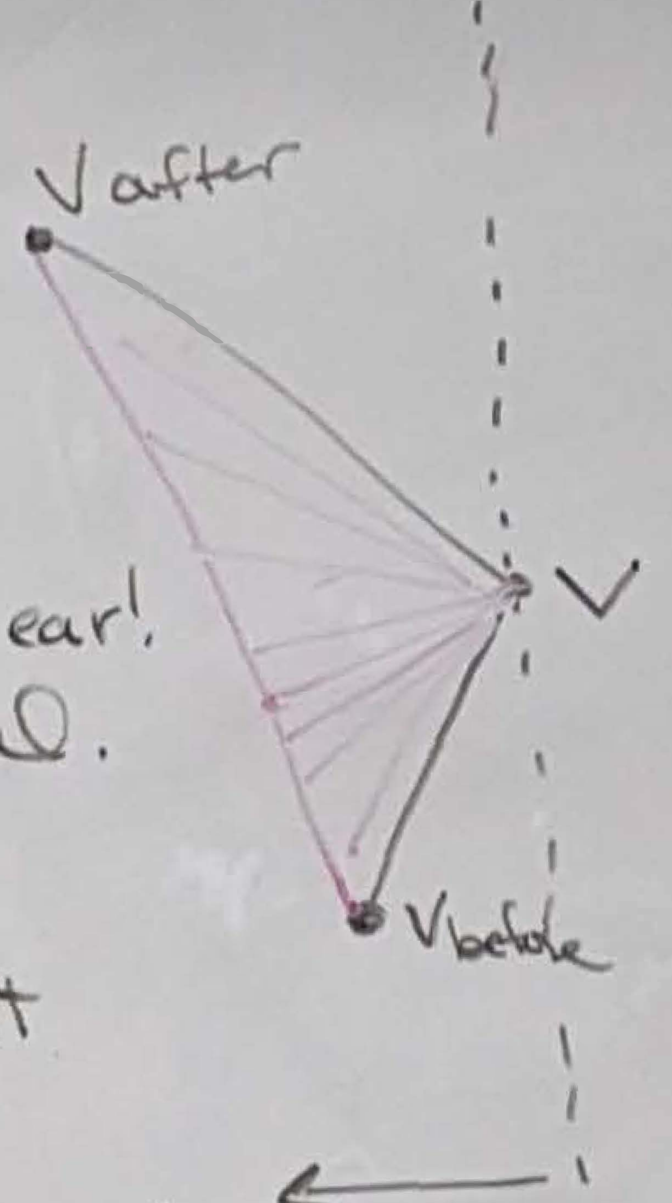
leaf

Lemma: Every polygon has a diagonal.

Proof: Let v be the rightmost vertex of P. Let v_{before} & v_{after} be the verts before + after v, in ccw order.

Case 1: The line segment $\overleftrightarrow{v_{before}\, v_{after}}$ does not intersect P.

$\Rightarrow$ we've found an ear! and a diagonal.



Case 2: The line segment intersects P.

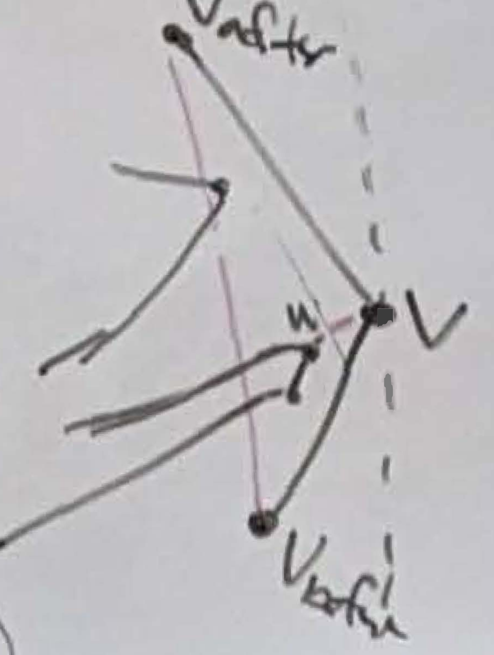
Notice $\forall$ parts of P inside the Δ, there is at least one vertex.

all of P is over on this $\frac{1}{2}$ plane

Let w be the rightmost of these vertices

subclaim: vw is a diagonal.

(insert proof by contradiction. suppose $\exists$ an edge of P intersecting $\overline{wv}$ ... then, w was not rightmost of verts in the triangle)



Cor. 1: Every triangulation of P has an ear.

Cor 2: Every polygon has a frugal triangulation.

The Problem: Triangulating a polygon

Given a polygon P, defined by a simple, closed polygonal chain

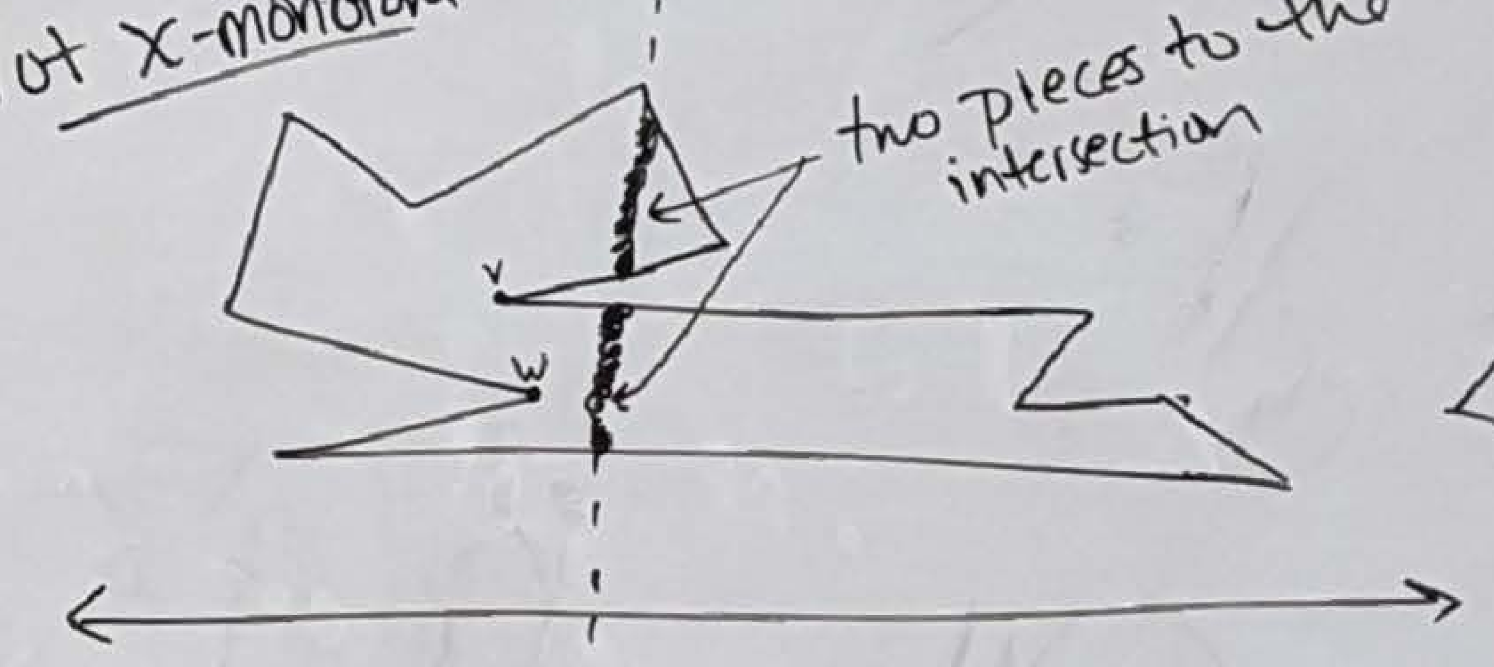
find a frugal triangulation. defined on other board.

Sol'n: Two Steps:
1. Find a decomposition of P into monotone polygons by adding diagonals,
2. Solve the subproblems separately
3. (Put together)

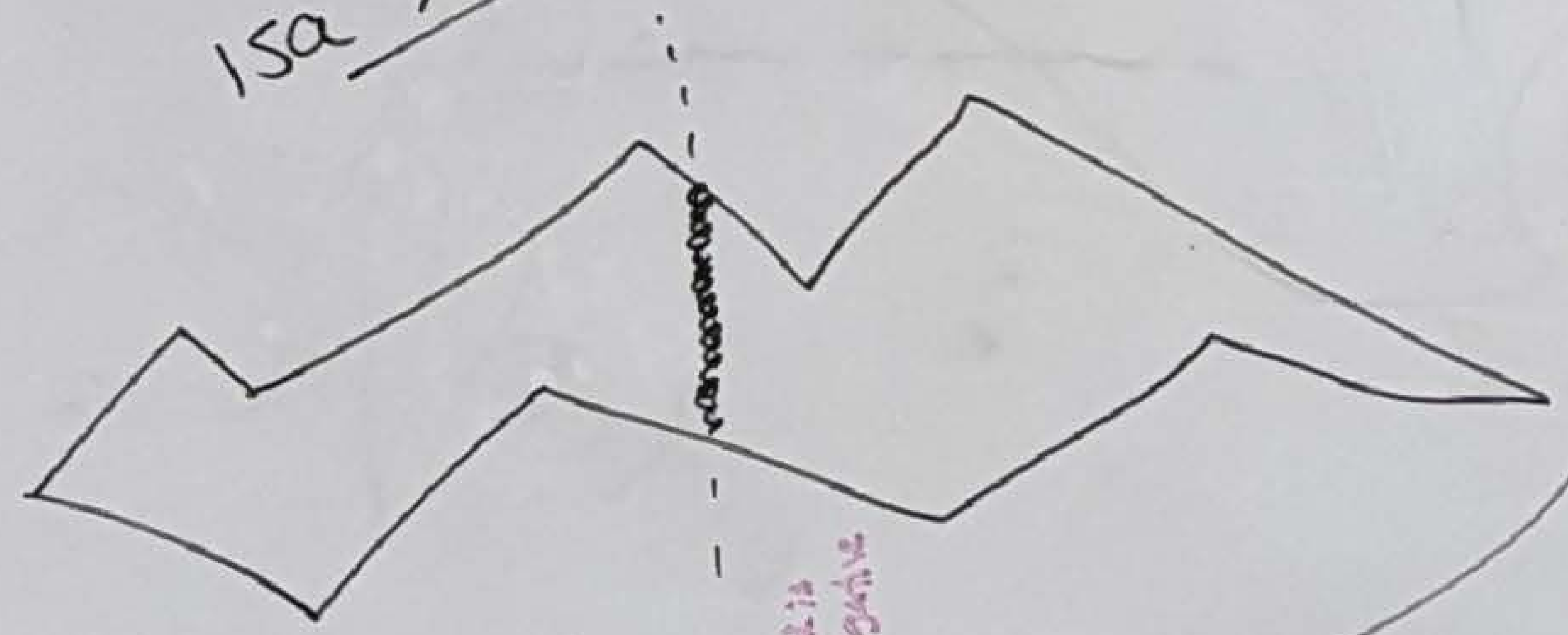
x-Monotone Polygon

$\forall$ vertical line ℓ, $\ell \cap P$ has one connected component

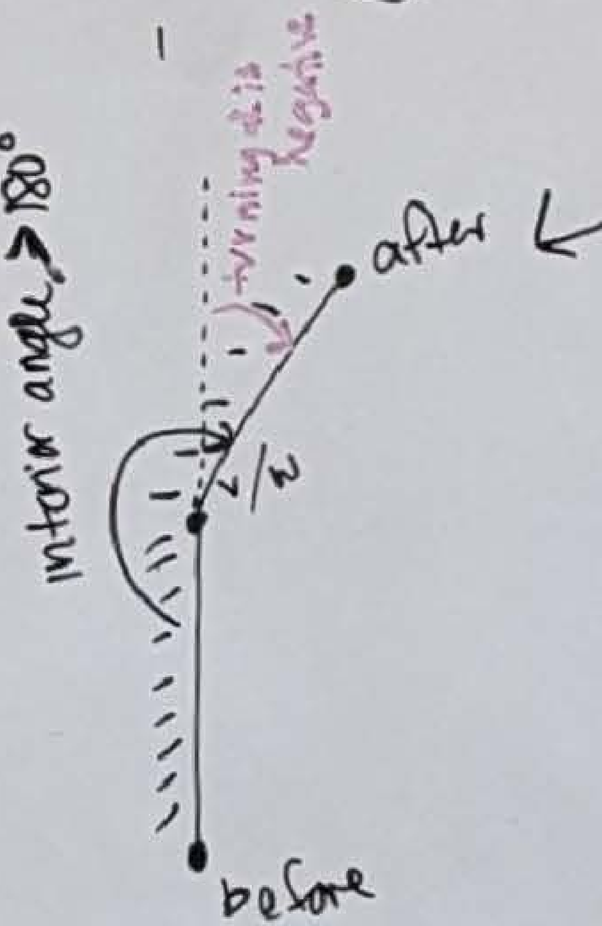
Not x-monotone



is a x-monotone



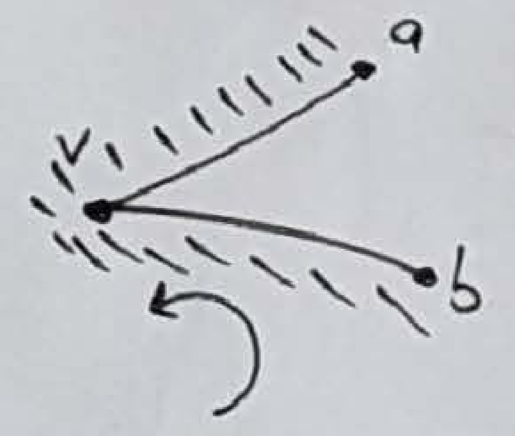
More generally, any vertex with interior $\measuredangle$ greater than 180° is called a reflex vertex.



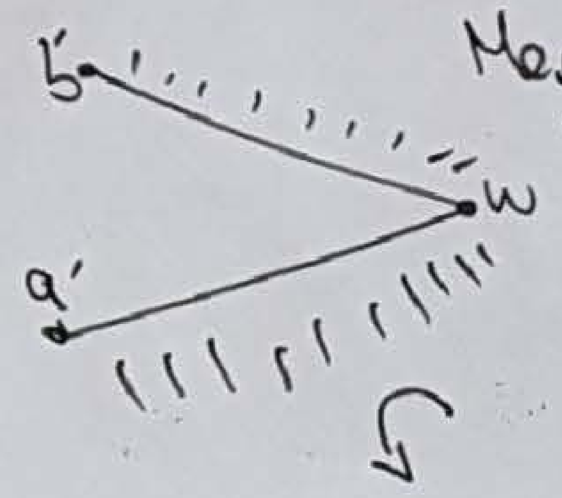
Two types of violations:

Definitions:

1) Split vertex



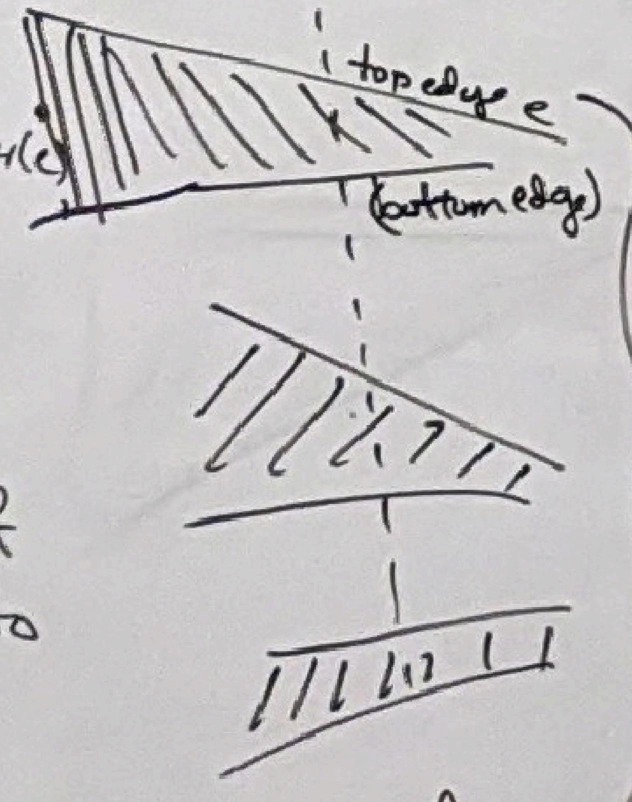
2) Merge vertex



To solve 1:

We use a sweepline!

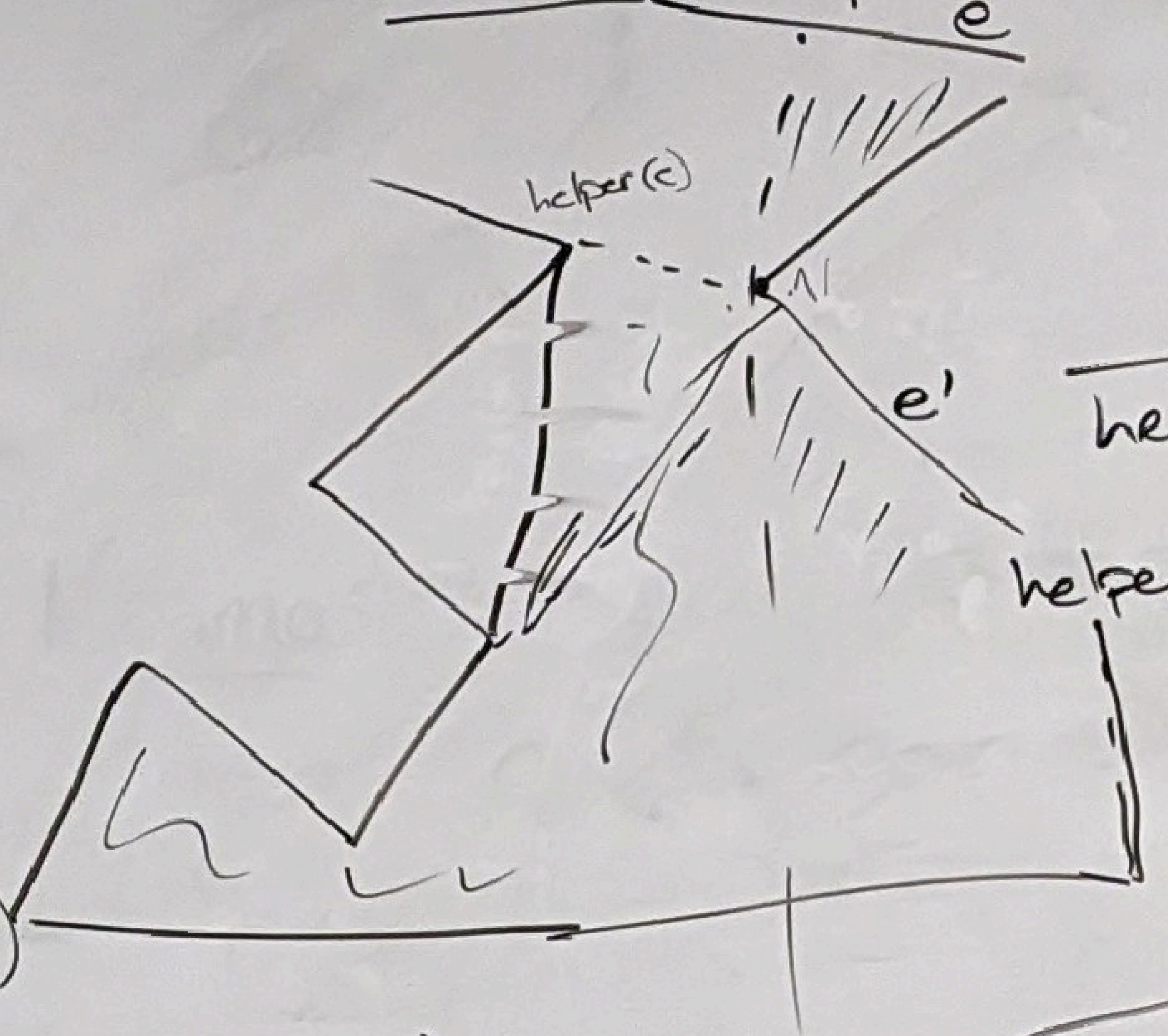
- Events: vertices, sorted by y-coord
  all known, so can store in array / LL
- Sweep line : sorted set of "top edges" (with "helpers"), allowing $\Theta(\log n)$ insertion, deletion, lookup



1st vertex in that "piece" of P that I see to left of ℓ.

generically, $\ell \cap \partial P$ is an even # of points

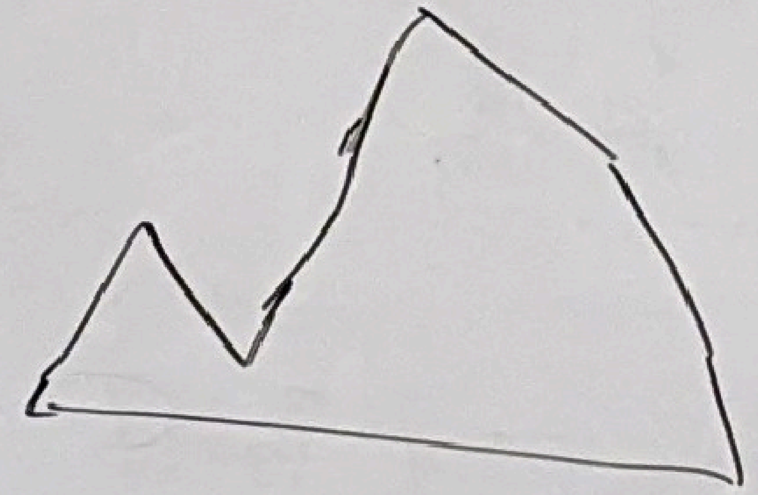
Group 1: Split Vertex

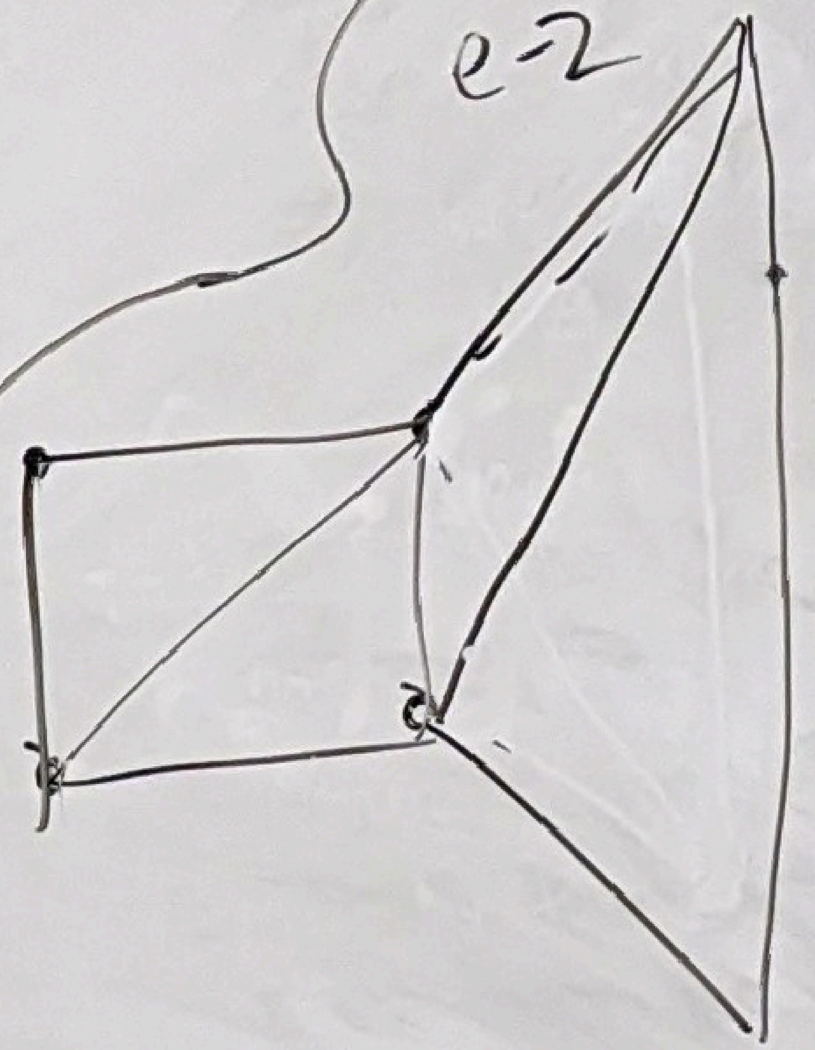


helper(e) v

helper(e') = v

$e \to e, e'$

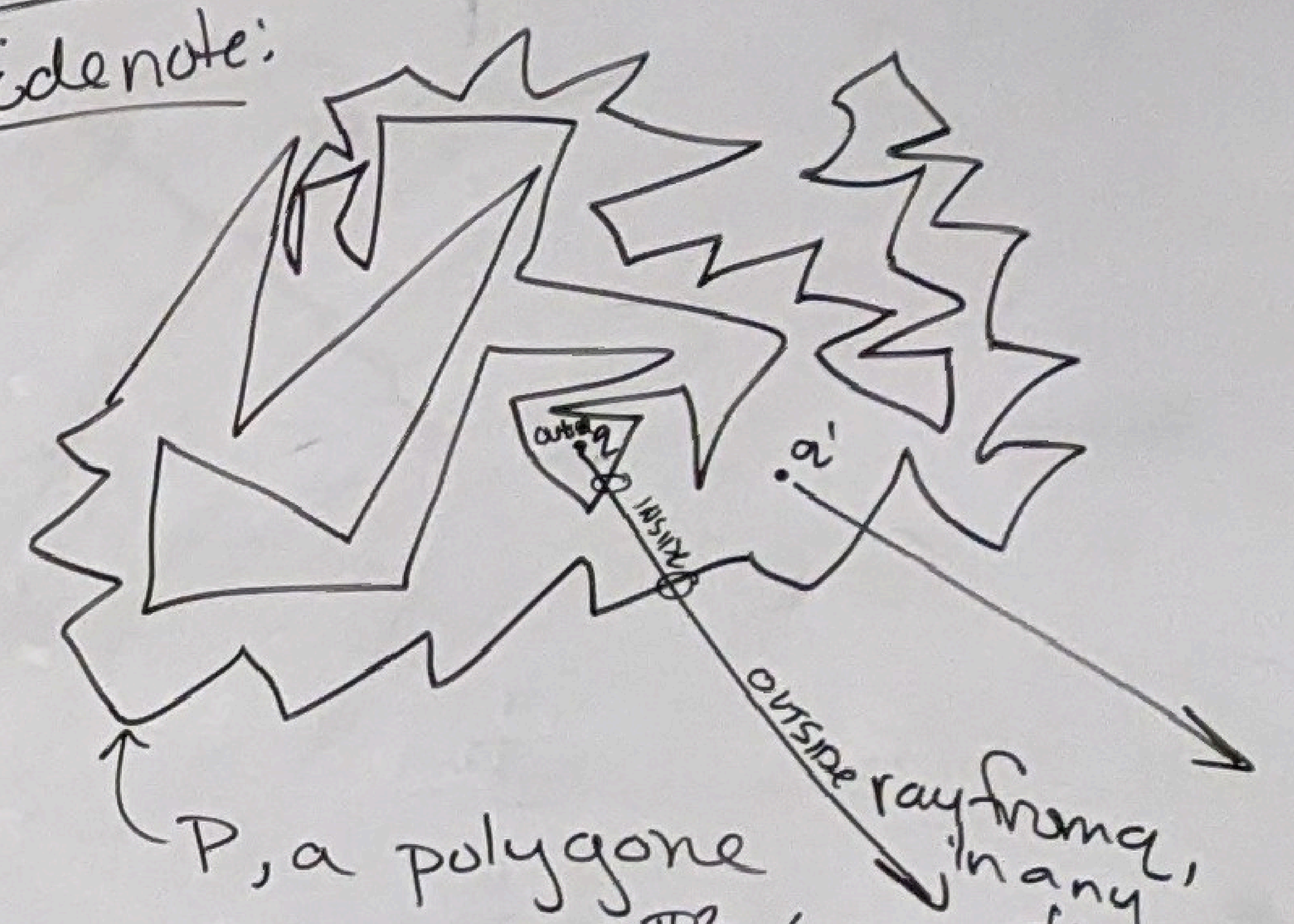




Claim: n-poly, any final triangulation will have
n-2 triangles
n-3 diagonals

Proof: Induction.

Sidenote:



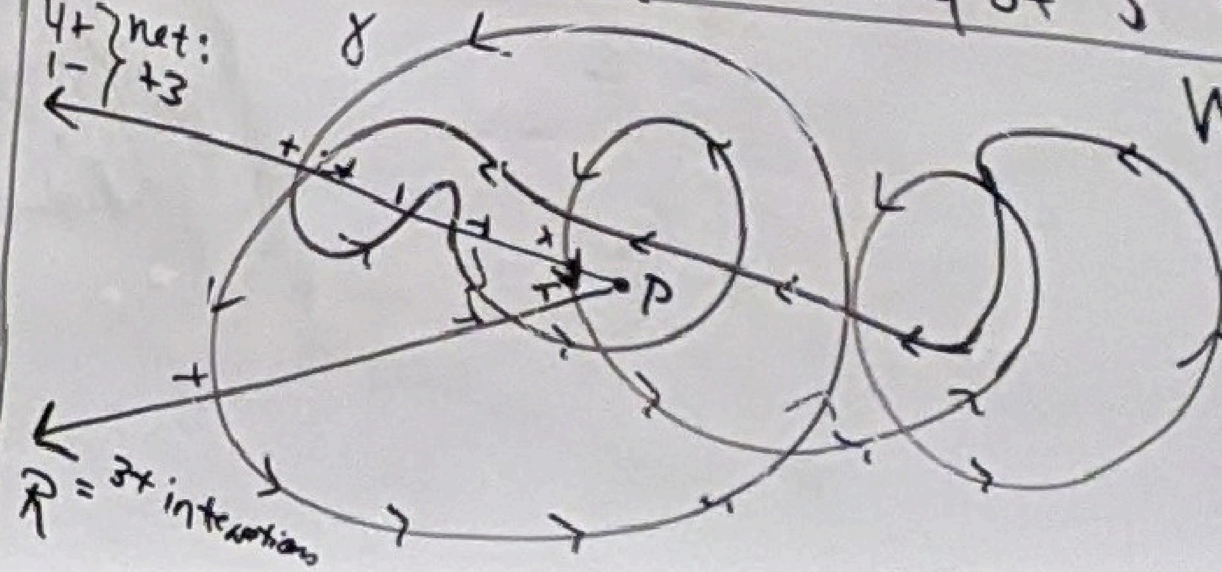
P, a polygon
q, a point $\in \mathbb{R}^2 / \partial P$

Q: Is q inside P or not?

By shooting a $\boxed{\text{ray } R}$ from q, ray from q, in any direction, not intersecting a vertex of P

$q \in P$ iff $R \cap \partial P$ is odd.

works for any embedding of S^1!



Winding # of γ around p