Randomized Incremental Algorithms

Agenda

- Review LP ☑
- 2D LP R.I.A ☐
- Min enclosing disc R.I.A. ☐

Linear Programming

$c_1x_1 + c_2x_2 + \dots + c_dx_d$ ← objective function maximize.

Subject to

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1d}x_d \le b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2d}x_d \le b_2$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nd}x_d \le b_n$

d-dimension n-# constrains

look for this in run time.

Simplex Algorithm
exponential worst case
fast in practice.

2D linear Programming

$f_{\bar{c}}(p_x, p_y) = c_x p_x + c_y p_y$

$\bar{c} = (c_x, c_y)$

constraints half planes

$c_{11}x + c_{12}y \le b_1 \leftarrow h_1$

$c_{21}x + c_{22}y \le b_2 \leftarrow h_2$

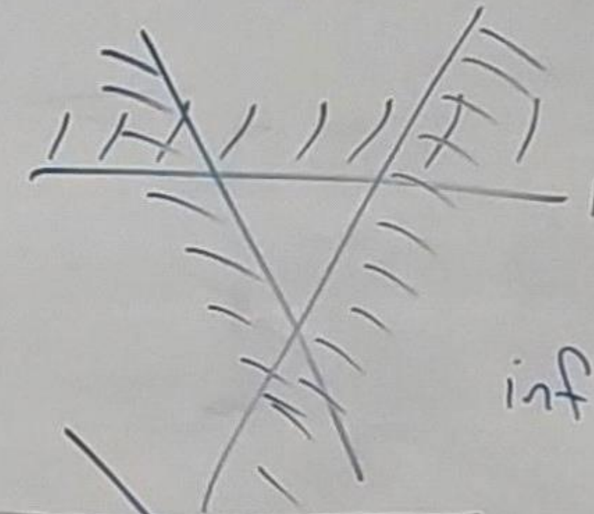
$\vdots$

2D LP $= \{H, \bar{c}\}$

$H = \{h_1, h_2, \dots h_n\}$

4 possibilities

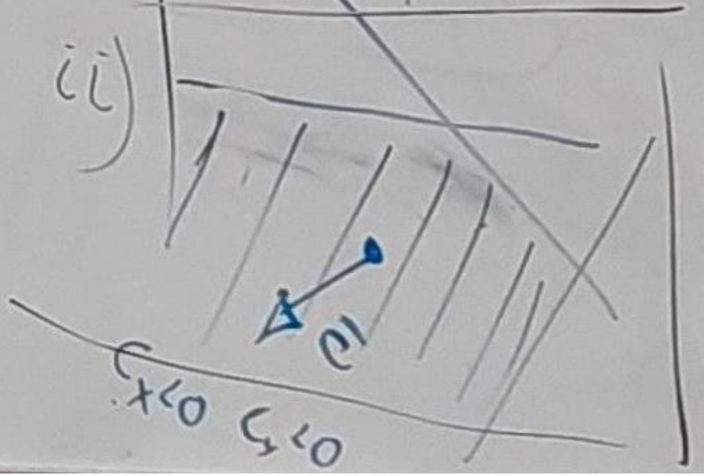
i)



h_1

infeasible

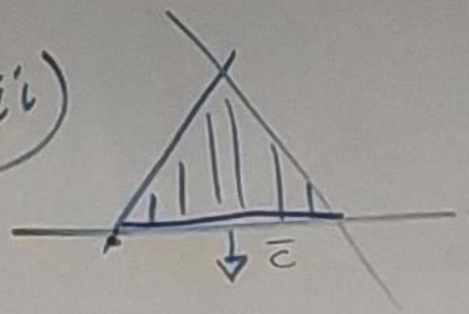
ii)



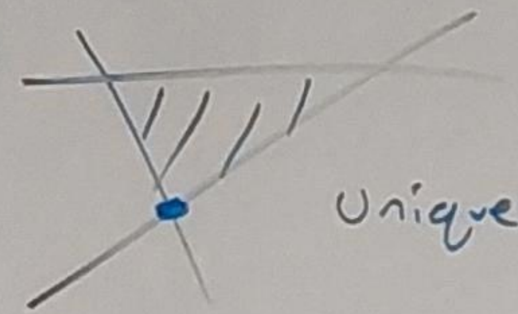
$\bar{c}$

$c_x<0$ $c_y<0$

unbounded no max

iii) 

no unique Solution

iv) 

Unique solution

Fix (ii) Add really "big" m_1 + m_2 bounding box

Fix (iii) rotate slightly avoid no unique Solution.

Algo idea

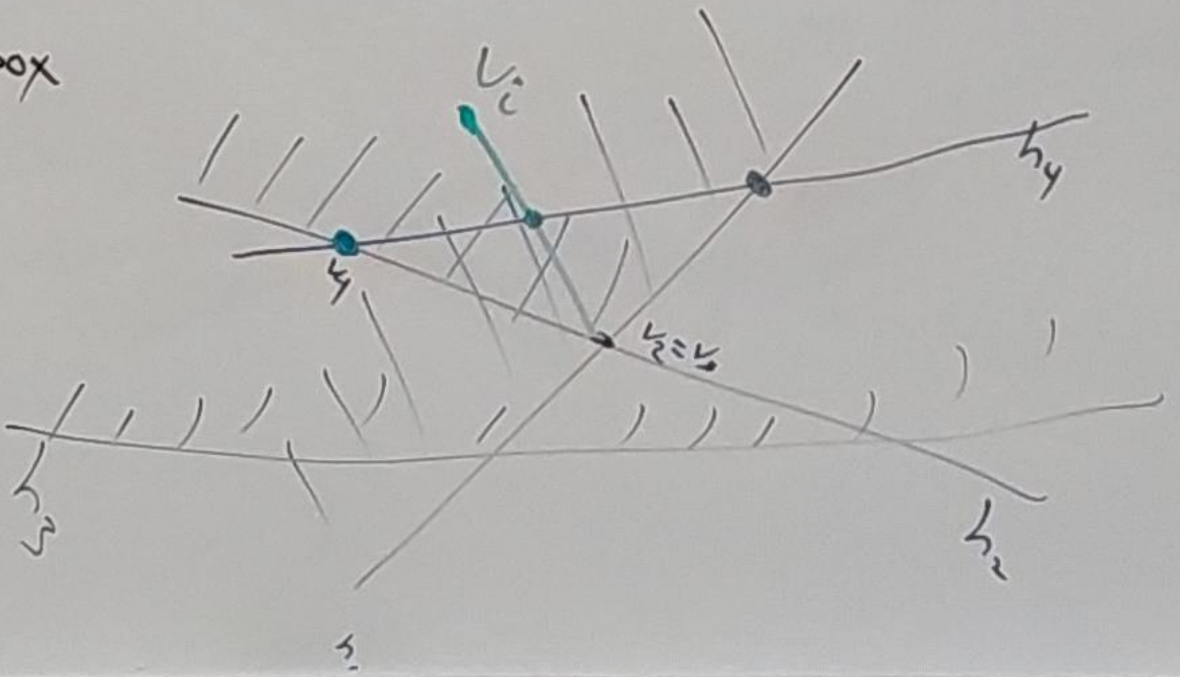
Add planes one at a time

update optimal solution.

C_i - $m_1 \cap m_2 \cap h_1 \dots \cap h_i$

feasible region after i planes are added.

V_i - optimal solution.



Lemma 1 $C_i = m_1 \cap m_2 \cap h_1 \cap \dots \cap h_i$

v_i - current opt.

i) If $v_{i-1} \in h_i$ then $v_i = v_{i-1}$ [No work]

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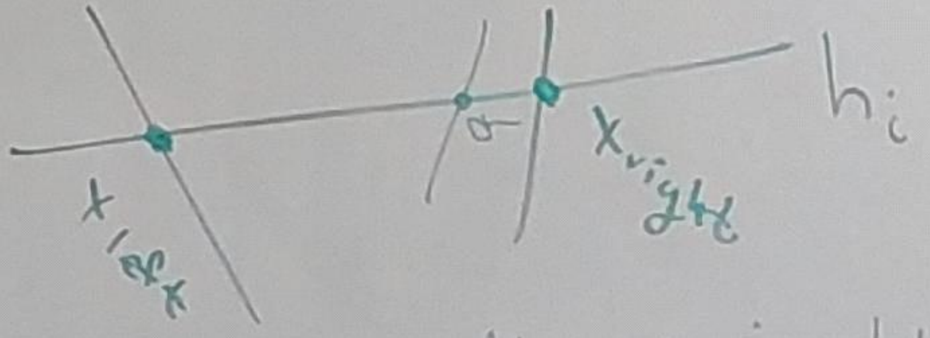
ii) If $v_{i-1} \notin h_i$ $C_i = \emptyset$ or $O(i)$ work

$v_i \in \ell_i$ $\ell_i = \partial(h_i)$

boundary

Do ii) how much work?



$O(i)$

At most i updates.

Randomized Incremental Algorithms

$1 + 2 + 3 + \dots$

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Algo 1 (2)

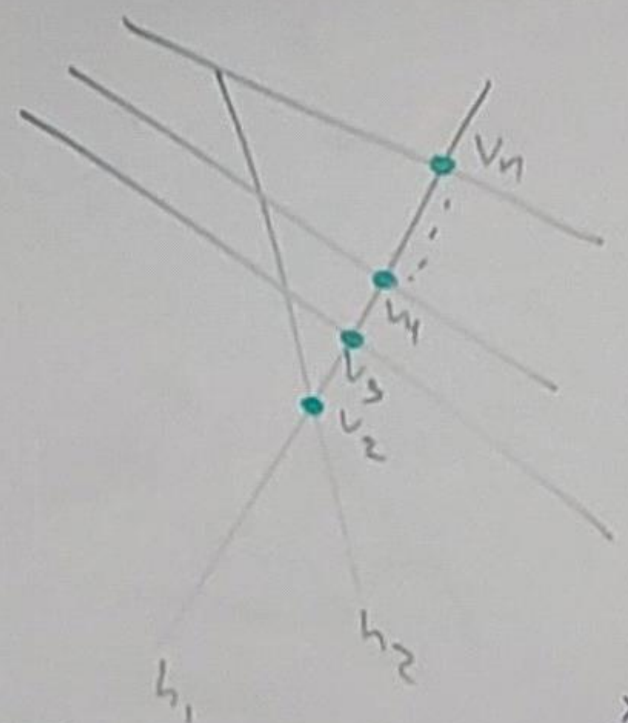
Input $(H, \vec{c})$

Output: If infeasible, say so.
Else max $f_{\vec{c}}(P)$

0) Pick random order of H
1) V_0 opt $m_1 \cap m_2$
2) for $i = 1 \to n$
3) if $v_{i-1} \in h_i \Rightarrow V_i = v_{i-1}$
4) Else V_i on ℓ_i find $O(i)$ time.
5) or report infeasible.

runtime?
worst case.

$$\sum_{i=1}^{n} O(i) = O(n^2)$$



Do better? yes, randomize!

random variable. $X_i := \begin{cases} 1 & v_{i-1} \notin h_i \\ 0 & \text{Else} \end{cases}$

$$E\left(\sum_{i=1}^{n} O(i) \cdot X_i\right) = \sum_{i=1}^{n} O(i) E(X_i)$$

Expected value (average)

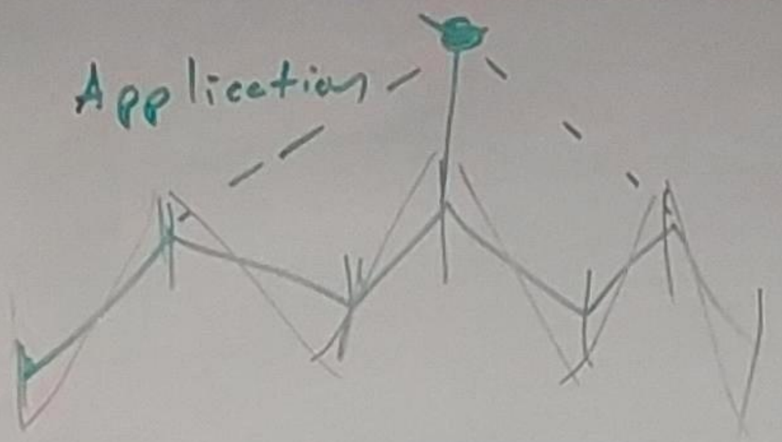
Backwards Analysis:

Probability last plane added makes us move v_i
is $\frac{2}{i}$ at i^{th} stage

$$\sum_{i=1}^{n} O(i)\frac{2}{i} \Rightarrow O(n)$$

Backwards Analysis

See



Lemma 1 $C_i = m_1 \cap m_2 \cap h_1 \cap \dots \cap h_i$

v_i - current opt.

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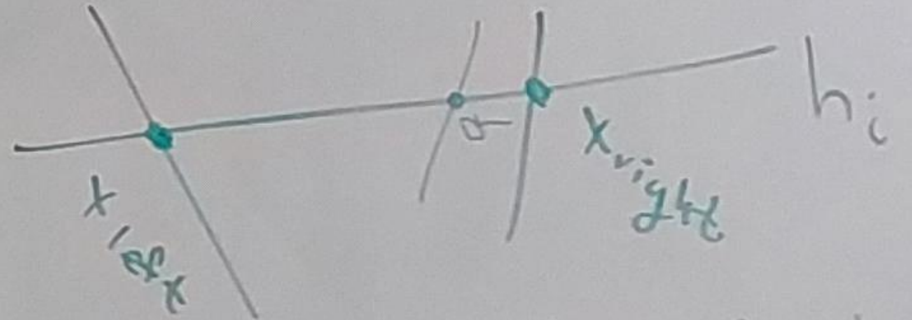
ii) If $v_{i-1} \notin h_i$ $C_i = \emptyset$ or $O(i)$ work

$v_i \in l_i$ $l_i = \partial(h_i)$ ← boundary

Do ii) how much work?



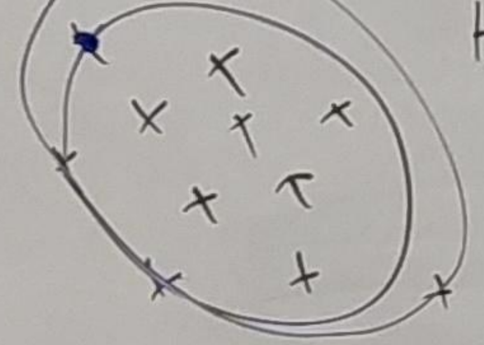
$O(i)$

At most i updates.

Min enclosing disc

n points in Plane - $P = \{p_1 \dots p_n\}$

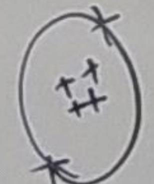
find min radius ball containg P.

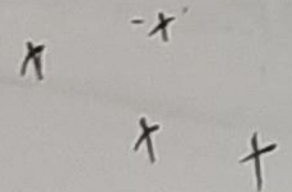


no 4 cocircular

facts: 3 points determine circle

2 points but diameter





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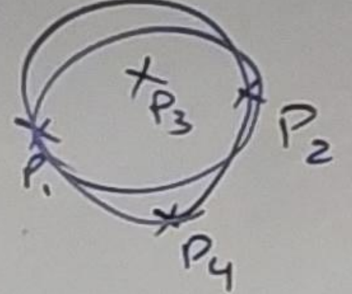
Lemma 1*

If $p_i \in D_{i-1}$ $D_i = D_{i-1}$

Else $p_i \notin D_{i-1}$ p_i on boundary of D_i



MinDisc(P)

In: P

out: D_n

0) P in random order

1) D_2 ball $p_1\ p_2$

2) $i = 3 \to n$

3) if $p_i \in D_{i-1}$ $D_i = D_{i-1}$

4) Else $D_i \leftarrow$ minidisk1(P, p_i)

5) return D_n

Minidisk1(P, q)

know is on boundary

⋮

$j = 2 \to n$

if $p_j \in D_{j-1}$ $D_j = D_{j-1}$

Else $D_j \leftarrow$ minidisc2(P, q, p_j)

return D_n

Minidisk2(P, q_1, q_2)

⋮

$k = 1 \to n$

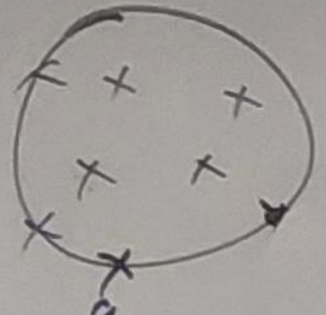
If $p_k \in D_{k-1}$ $D_k = D_{k-1}$

Else compute circle

$D_k = (p_k, q_1, q_2)$

$O(n)$

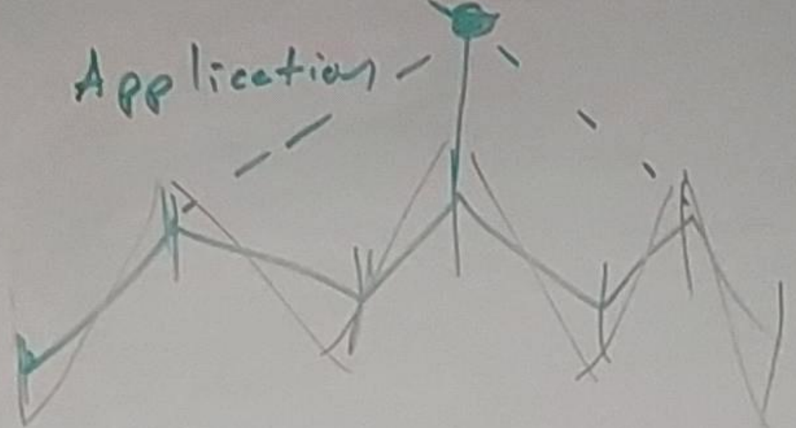
Backwards Analysis



$$O(n) + \sum_{i=2}^{n} O(i)\,\frac{2}{i} \Rightarrow O(n)$$

Minidisk

$$O(n) + \sum_{i=3}^{n} O(i)\,3/i \Rightarrow O(n).$$



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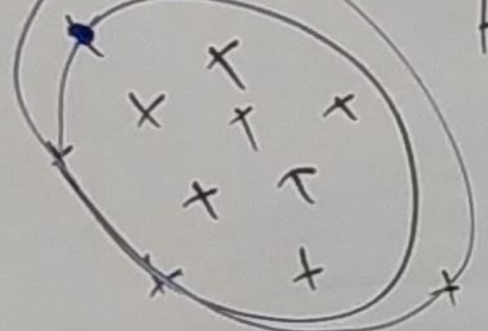

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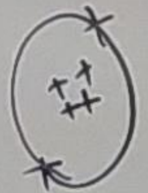

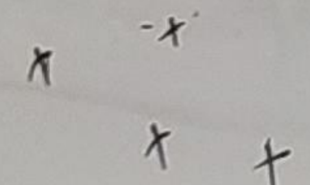

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$v_i \in \ell_i$ $\ell_i = \partial(h_i)$ boundary

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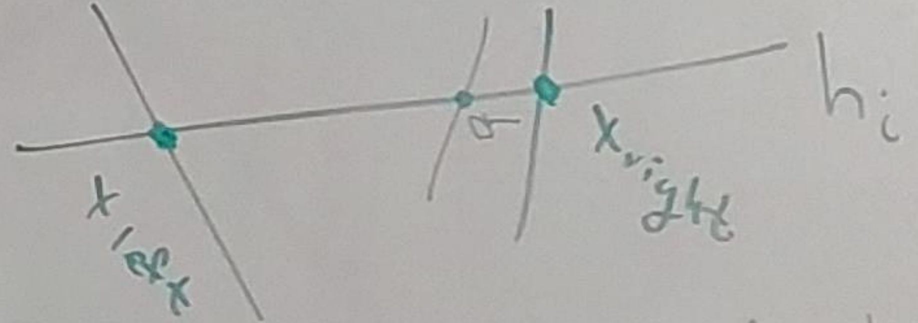

$O(i)$

At most i updates.