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The Three-Dimensional Navier–Stokes Equations

JAMES C. ROBINSON
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THE THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS

A rigorous but accessible introduction to the mathematical theory of the three-dimensional Navier–Stokes equations, this book provides self-contained proofs of some of the most significant results in the area, many of which can only be found in research papers. Highlights include the existence of global-in-time Leray–Hopf weak solutions and the local existence of strong solutions; the conditional local regularity results of Serrin and others; and the partial regularity results of Caffarelli, Kohn, and Nirenberg.

Appendices provide background material and proofs of some 'standard results' that are hard to find in the literature. A substantial number of exercises are included, with full solutions given at the end of the book. As the only introductory text on the topic to treat all of the mainstream results in detail, this book is an ideal text for a graduate course of one or two semesters. It is also a useful resource for anyone working in mathematical fluid dynamics.

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The Three-Dimensional Navier–Stokes Equations

Classical Theory

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To Moomin – JCR

To Sam & Sofia – JLR

To Dorota – WS

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PART II LOCAL AND PARTIAL REGULARITY

Preface

The purpose of this book is to provide a rigorous but accessible introduction to the mathematical theory of the three-dimensional Navier–Stokes equations, suitable for graduate students, by giving self-contained proofs of what we see as three of the most significant results in the area:

- the existence of global-in-time Leray–Hopf weak solutions, i.e. weak solutions that satisfy the strong energy inequality (Leray, 1934; Hopf, 1951); and the local-in-time existence of strong solutions;
- local regularity results due to Serrin (1962) and others: if

$$u \in L^r((a, b); L^s(U)), \qquad \frac{2}{r} + \frac{3}{s} \leq 1,$$

  then u is spatially smooth within U; and
- the partial regularity result of Caffarelli, Kohn, & Nirenberg (1982) that guarantees that the one-dimensional parabolic Hausdorff measure of the set of space–time singularities of any suitable weak solution is zero.

We end with a result that makes use of many of the properties of solutions that we prove throughout the book, the almost-everywhere uniqueness of the Lagrangian particle trajectories for suitable weak solutions.

We also treat the following topics that often fall into the category of 'standard results' but turn out to be hard to find in the required form:

– the Helmholtz–Weyl decomposition on $\mathbb{T}^3$;
– the L^p boundedness of the Leray projector on $\mathbb{T}^3$;
– estimates on the pressure in the periodic case;
– weak solutions as distributional solutions;
– smoothness and uniqueness for $u \in L^r(0, T; L^s)$, $\frac{2}{r} + \frac{3}{s} = 1$, $3 < s \leq \infty$;
– the local existence of solutions in $H^{1/2}$ and L^3 via energy estimates;

– local estimates for the velocity given the vorticity (via Biot–Savart);
– the validity of the local energy inequality; and
– local and maximal regularity results for solutions of the heat equation.

We assume knowledge of the basic language and results common in the rigorous study of PDEs such as provided by Evans (1998), Renardy & Rogers (2004), or Robinson (2001), among many.

Particularly in the earlier parts of the book, where much of the material is well known and has been presented many times, we have chosen not to clutter the exposition with frequent and exhaustive references. Instead historical discussion and suggestions for further reading are delayed until the Notes at the end of each chapter. After the Notes there is generally a selection of exercises, which either expand on material in the main text or contain steps that would interrupt the flow of an argument. Full solutions are given at the end of the book. This makes the first part of the book, which treats the classical existence, uniqueness, and regularity theory, particularly suitable for a first course on the Navier–Stokes equations.

There are many other books that cover some of the material here: our presentation has been particularly influenced by Chemin, Desjardins, Gallagher, & Grenier (2006); Constantin & Foias (1988); Doering & Gibbon (1995); Galdi (2000, 2011); and Temam (1977, 1983). We touch only briefly on the analysis of the equation in critical spaces, which has been the focus of much research in the last two decades; this topic is covered in detail in the books by Cannone (1995, see also his 2003 review article) and Lemarié-Rieusset (2002).

We would like to thank all our mentors, colleagues, and students, who over the years have fostered our interest in this subject. We would particularly like to acknowledge the academic support and friendship of John Gibbon, Charles Fefferman, and Grzegorz Łukaszewicz. We are very grateful to David McCormick, Wojciech Ożański, Benjamin Pooley, and Mikołaj Sierżęga, who read various drafts of the book and gave us many helpful comments.

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Introduction

The three-dimensional incompressible Navier–Stokes equations form the fundamental mathematical model of fluid dynamics. Derived from basic physical principles under the assumption of a linear relationship between the stress and the rate-of-strain in the fluid, their applicability to real-life problems is undisputed. However, a rigorous mathematical theory for these equations is still far from complete: in particular, there is no guarantee of the global existence of unique solutions.

The aim of this introductory chapter is to give an idea of the derivation of the model and the physical significance of the terms in the equations along with an informal overview of what is currently known and what the problems are in advancing the theory further. Because we are considering the modelling behind the equation the style of presentation in this chapter is somewhat looser than in the rest of the book, which concentrates on what can be proved rigorously about this canonical model.

The Navier–Stokes equations

The Navier–Stokes equations govern the time evolution of the three-component velocity field u and the scalar pressure p in a fluid of uniform density lying within some region $\Omega \subseteq \mathbb{R}^3$: the conservation of linear momentum leads to

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0, \tag{1}$$

while the conservation of mass yields the divergence-free (incompressibility) condition

$$\nabla \cdot u = 0. \tag{2}$$

The parameter $\nu > 0$ is known as the 'kinematic viscosity'; we will discuss this further later, but for now we note that in the main part of the book we will take $\nu = 1$. The case $\nu = 0$ yields the Euler equations for inviscid ('ideal') fluid flow, a model whose mathematical treatment requires very different tools and which will hence play very little part in what follows.

Equation (1) is a system of three equations for the time evolution of the three components of u. However, it should be noticed that in equation (1) there are four unknowns: u_1, u_2, u_3, and p. There is no explicit equation for the time evolution of p, however; rather p has to be chosen to ensure that the incompressibility condition (2) holds. Thus (2) is an indispensable part of the model, since without it the whole system would be underdetermined.

We will consider the problem on three types of domains:

(i) the physical case of a bounded domain $\Omega \subset \mathbb{R}^3$ that has a smooth boundary (in this case we refer to Ω as 'a smooth bounded domain') equipped with Dirichlet (no-slip) boundary conditions $u|_{\partial\Omega} = 0$;

(ii) the whole space $\mathbb{R}^3$ with suitable decay at infinity (e.g. $u \in L^p(\mathbb{R}^3)$ for some $1 \leq p < \infty$); and

(iii) the torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, sometimes described as the case of 'periodic boundary conditions'; for mathematical convenience we will also impose the zero-average condition $\int_{\mathbb{T}^3} u = 0$.

The latter two cases have the advantage that they do not involve any boundaries, which greatly simplifies the analysis in many instances: when we carry out an analysis restricted to these two cases we will refer to the equations 'in the absence of boundaries'. The domains in cases (i) and (iii) have the advantage of being bounded and of allowing the use of the Poincaré inequality ($\|u\|_{L^2} \leq c\|\nabla u\|_{L^2}$); again, there are situations in which a common analysis can be performed, such as the Galerkin-based existence argument we employ in Chapter 4.

Generally our approach in this book is to present the arguments in the situation in which they are simplest (often on the torus) and then to discuss the necessary changes required to deal with the other cases.

When we seek a solution of the Navier–Stokes equations (1)–(2) with an initial condition u_0 we want a divergence-free velocity u (satisfying the boundary conditions) and a pressure p that together[1] satisfy equation (1) for all $t > 0$ and at every point in the domain Ω. Moreover, $u(t)$ must tend to u_0 as t tends to zero.

[1] However, if we take the divergence of (1) then $-\Delta p = \partial_i\partial_j(u_i u_j)$, and so p is determined by u (see Chapter 5). We often consider the equation in a form in which it suffices to solve for u alone.

The main question concerning the model of fluid flow embodied in the 3D Navier–Stokes equations is whether for each sensible initial condition there is exactly one solution defined for arbitrarily large times. Unfortunately, the answer to this question is not known and we are currently unable to rule out the possibility that a perfectly smooth initial condition might evolve according to the 3D Navier–Stokes equations and yet blow up in a finite time.[2] It is not surprising, therefore, that one of the seven Millennium Problems announced by the Clay Institute addresses the question of the global regularity of solutions to the 3D Navier–Stokes equations (Fefferman, 2000): a prize of one million dollars is offered for either a proof of the existence of regular solutions that exist for all $t > 0$ or a counterexample to such an existence theorem (in the absence of boundaries).

Physical derivation of the equations

We now give a very quick idea of the derivation of the equations, and of the meaning of the various terms. More details can be found in Batchelor (1999), Chorin & Marsden (1993), Doering & Gibbon (1995), or Majda & Bertozzi (2002), for example. We assume that the fluid occupies a region $\Omega \subseteq \mathbb{R}^3$.

We begin with the incompressibility condition. Assuming that the density of the fluid ρ is constant, the net flux of mass across the boundary of any spatial region must be zero: in this case, for any sufficiently regular region $U \subset \Omega$ we must have

$$\rho \int_{\partial U} u(x,t) \cdot n \, \mathrm{d}S = 0,$$

where n is the unit outward normal to U. Using the Divergence Theorem we obtain[3]

$$\rho \int_U \nabla \cdot u(x,t) \, \mathrm{d}x = 0.$$

Since this must hold for any sufficiently regular region $U \subset \Omega$, it follows that $\nabla \cdot u = 0$ at every point in Ω.

To derive the equations for the conservation of momentum, we consider a volume of fluid $V(t)$ which we now allow to move with the flow, i.e. we take some initial volume $V(0)$ and consider

$$V(t) = \{X(t; x_0) : \ x_0 \in V(0)\}, \tag{3}$$

[2] In two dimensions such behaviour is not possible, see Section 6.5.

[3] Throughout this book we use $\mathrm{d}x$ to denote the volume element in spatial integrals, rather than $\mathrm{d}V$ or the more cumbersome $\mathrm{d}x^3$.

where $X(t; x_0)$ denotes the solution of the equations for fluid particle trajectories

$$\dot{X}(t) = u(X, t), \qquad X(0) = x_0. \tag{4}$$

(We consider the solutions of these equations in more detail in Chapter 17.)

Note that on a fluid trajectory, for any function $f(x, t)$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f(X(t), t) &= \frac{\partial f}{\partial t}(X(t), t) + \frac{\partial f}{\partial x_j}(X(t), t)\dot{X}_j(t)\\ &= \frac{\partial f}{\partial t}(X(t), t) + \frac{\partial f}{\partial x_j}(X(t), t) u_j(X(t), t),\end{aligned}$$

where we have used the Einstein summation convention (summing over repeated indices). We write the second term in the notationally convenient form $(u \cdot \nabla) f$, and then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(X(t), t) = \partial_t f + (u \cdot \nabla) f,$$

where we write ∂_t for $\partial/\partial t$ and the terms on the right-hand side are evaluated at $(X(t), t)$. Therefore the rate of change of momentum for the fluid in $V(t)$ is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\rho \int_{V(0)} u(X(t; \alpha), t)\,\mathrm{d}\alpha\right\} &= \rho \int_{V(0)} \frac{\mathrm{d}}{\mathrm{d}t} u(X(t; \alpha), t)\,\mathrm{d}\alpha\\ &= \rho \int_{V(t)} \{\partial_t u + (u \cdot \nabla) u\}\,\mathrm{d}x, \end{aligned} \tag{5}$$

noting that due to the incompressibility constraint the factor $|\det(\nabla X)|$ introduced by changing variables in the integral is 1 (this is not immediately obvious and is discussed in more detail in Section 17.1).

We now consider the force on a volume V of the fluid, which is given in terms of the stress tensor σ by the vector F whose ith component is

$$F_i = \int_{\partial V} \sigma_{ij} n_j\,\mathrm{d}S. \tag{6}$$

The particular form of the stress tensor in the Navier–Stokes equations is based on the modelling assumption that σ is a homogeneous, isotropic, and linear function of the 'rate-of-strain tensor' whose components are

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{7}$$

(A fluid for which this is true is called a 'Newtonian fluid'.) In three dimensions this means that σ must be of the form

$$-qI + 2\mu E + \gamma\,\mathrm{tr}(E) I,$$

where μ and γ are fixed scalar constants (the factor of 2 is for later convenience to compensate for the factor of $1/2$ in the definition of E), tr denotes the trace, and q is a scalar that can vary in space and time. Since u is divergence free,

$$\mathrm{tr}(E) = E_{ii} = \nabla \cdot u = 0,$$

and so the stress tensor must take the form

$$\sigma = -qI + 2\mu E. \tag{8}$$

The pressure term $-qI$ provides a uniform force in all directions, while the second term corresponds to forces resulting from differences in the local velocity in the flow, i.e. shear forces, due to the viscous nature of the fluid. To see how this term corresponds to these shear forces we consider the following simple situation, as described by Feynman, Leighton, & Sands (1970). Suppose that we have a fluid contained between two solid plane surfaces and we keep the lower surface stationary while moving the other at a constant speed v (slow enough to prevent any turbulent effects) as in Figure 1.

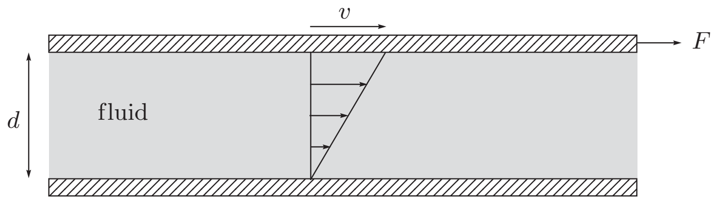

Figure 1. Experimental setup to measure the viscosity of a fluid (after Feynman, Leighton, & Sands, 1970). The fluid is contained between two parallel plates a distance d apart, and the top plate is moved with a constant velocity v.

It is an empirical fact that if the distance between the bottom plane and the upper plane is d then the force F per unit area of the upper plate required to maintain the motion is proportional to the speed v and inversely proportional to the distance d:

$$\frac{F}{A} = \mu \frac{v}{d}.$$

The constant of proportionality μ is known as the dynamic viscosity and is an intrinsic property of the fluid.

Now consider the situation locally, imagining a rectangular cell in the fluid whose faces are parallel with the flow as in Figure 2. In this case the shear force across this cell will be

$$\frac{\delta F}{\delta A} = \mu \, \frac{\delta v}{\delta y},$$

leading to a local shear force $\mu \, \partial v / \partial y$, in line with the form of the stress tensor in (8).

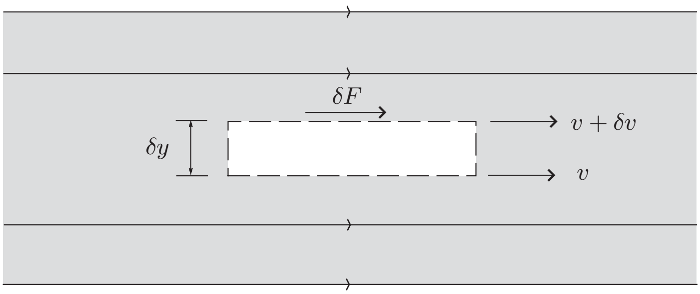

Figure 2. Local stresses caused by local velocity differences. (Figure after Feynman, Leighton, & Sands, 1970.)

By contrast, uniform rotation of a fluid will result in no internal stresses. By performing a Taylor series expansion of the velocity field near a point x_0 (cf. Majda & Bertozzi, 2002) one can write

$$u(x) \simeq u(x_0) + \nabla u(x_0)(x - x_0) = u(x_0) + E(x - x_0) + \frac{1}{2} \, \omega \times (x - x_0),$$

where E is the rate-of-strain tensor from (7) and $\omega = \operatorname{curl} u$ is the vorticity. This offers another indication that the combination of velocity gradients in E is the relevant quantity for determining the local stress.

Given the form of the stress tensor in (8), it follows from (6) that the ith component of the force on a volume V is given (using the Divergence Theorem) by

$$\begin{aligned} \int_{\partial V} \sigma_{ij} n_j \, \mathrm{d}S &= \int_V \partial_j \sigma_{ij} \, \mathrm{d}x \\ &= \int_V \partial_j \left\{ -q\delta_{ij} + \mu(\partial_i u_j + \partial_j u_i) \right\} \mathrm{d}x \\ &= \int_V -\partial_i q + \mu \Delta u_i \, \mathrm{d}x, \end{aligned}$$

writing ∂_j for $\frac{\partial}{\partial x_j}$ and using the fact that $\partial_j u_j = \nabla \cdot u = 0$. So the total force on the volume V is

$$\int_V -\nabla q + \mu \Delta u \,\mathrm{d}x. \tag{9}$$

Therefore if we consider the change in momentum of the fluid in the volume $V(t)$ we have, combining (5) and (9),

$$\rho \int_{V(t)} \{\partial_t u + (u \cdot \nabla)u\} \,\mathrm{d}x = \int_{V(t)} -\nabla q + \mu \Delta u \,\mathrm{d}x.$$

Since $V(t)$ is arbitrary[4] we can deduce that

$$\rho\, \partial_t u - \mu \Delta u + \rho(u \cdot \nabla)u + \nabla q = 0,$$

with

$$\nabla \cdot u = 0. \tag{10}$$

Dividing the first of these equations through by ρ we obtain

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0, \tag{11}$$

where $\nu = \mu/\rho$ is termed the 'kinematic viscosity' and we have set $p = q/\rho$.

Throughout what follows we will consider the equations with $\nu = 1$. We can do this since we are primarily interested in the question of the existence and uniqueness of solutions for all positive times. Note that if the pair $(u(x,t), p(x,t))$ solves (10)–(11) on the time interval $[0, T)$ then (u_ν, p_ν) with

$$u_\nu(x,t) = \nu^{-1} u(x, t/\nu) \qquad \text{and} \qquad p_\nu(x,t) = \nu^{-2} p(x, t/\nu)$$

solves the equations with $\nu = 1$ on the time interval $[0, \nu T)$ but still on the same spatial domain. Therefore if one could prove global existence of smooth solutions ('$T = \infty$') for the choice $\nu = 1$ the same would follow for any other choice of $\nu > 0$.

Rescaling of solutions

There is another 'rescaling transformation' that is particularly significant for the Navier–Stokes equations when they are posed on the whole space, but is also important in the local regularity theory we develop in Part II.

Suppose first that $u(x,t)$ is a solution of the linear heat equation

$$\partial_t u - \Delta u = 0, \qquad x \in \mathbb{R}^3. \tag{12}$$

[4] Given any choice of volume V, we can find an initial volume $V(0)$ such that the recipe in (3) yields $V(t) = V$, by solving (4) backwards from time t to time zero.

It is simple to check that for any $\lambda > 0$ and any $\alpha \in \mathbb{R}$ the function

$$u_{\lambda,\alpha} = \lambda^{\alpha} u(\lambda x, \lambda^2 t) \tag{13}$$

is again a solution of (12).

When we consider the Navier–Stokes equations we need the nonlinear term $(u \cdot \nabla)u$ to transform in the same way as $\partial_t u$ and Δu; it is easy to check that this requires the choice $\alpha = 1$ in (13). Therefore if $u(x, 0) = u_0(x)$ gives rise to a solution $u(x, t)$ of the Navier–Stokes equations on $\mathbb{R}^3$, with corresponding pressure $p(x, t)$, then the rescaled functions

$$u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t) \qquad \text{and} \qquad p_\lambda(x, t) := \lambda^2 p(\lambda x, \lambda^2 t)$$

still solve the equations, but with rescaled initial data $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$. For $\lambda > 1$ this corresponds to shrinking (spatial) distances by a factor of λ^{-1} and increasing the speed by a factor of λ; it is clear that the time (=distance/speed) should therefore shrink by a factor of λ^{-2}.

Note that if an initial condition u_0 gives rise to the solution $u(x, t)$ and the rescaled initial condition $u_{0,\lambda}$ gives rise to the solution $u_\lambda(x, t)$ then

$$u_\lambda \text{ is regular on } [0, T_\lambda] \quad \Leftrightarrow \quad u \text{ is regular on } [0, \lambda^2 T_\lambda]. \tag{14}$$

Spaces of functions in which the norm is unchanged by the rescaling

$$u(x) \mapsto \lambda u(\lambda x) \tag{15}$$

are termed 'critical spaces'. These are the natural spaces in which to try to prove 'small data' results, i.e. the global existence of smooth solutions when the norm of the initial condition is small, since the norm of the data is unaffected by the rescaling transformation (15). We give two examples of such results, in the Sobolev space $\dot{H}^{1/2}$ (Chapter 10) and the Lebesgue space L^3 (Chapter 11).

Critical spaces are important since in some cases local existence in a critical space can be used to deduce global existence. Suppose that X is a critical space and for initial data in X regular solutions exist on the time interval $(0, \tau)$, where $\tau = \tau(\|u_0\|_X)$, i.e. the local existence time depends only on the norm in X. Given $u_0 \in X$ and any $T > 0$, choose λ such that

$$\lambda^2 \tau(\|u_0\|_X) > T;$$

then $\|u_{0,\lambda}\|_X = \|u_0\|_X$ since X is a critical space and the corresponding rescaled solution u_λ is regular on $(0, \tau(\|u_{0,\lambda}\|_X))$. It follows from (14) that the solution u is regular on $(0, \lambda^2 \tau(\|u_0\|_X)) \supset (0, T]$.

None of the local existence results for the 3D Navier–Stokes equations in critical spaces obtain such an existence time depending only on the norm; but this programme has proved successful in other contexts, for example semilinear wave equations (Shatah & Struwe, 1994) and isentropic compressible fluids (Danchin, 2000).

Diffusion, advection, and pressure

The regularity of solutions of the 3D Navier–Stokes equations is determined in a competition between the three terms corresponding to diffusion, advection, and pressure. The first of these terms ($-\Delta u$) contributes a smoothing effect that we expect to help prevent the solutions from blowing up; the second (the nonlinear term $(u \cdot \nabla)u$) corresponds to advection by the fluid and seems to be a source of danger; we will often eliminate the third term (∇p) in our formulation, but its influence remains.

Diffusion: the Laplacian

Let us consider only the first two terms in the equation, i.e. the time derivative and the Laplacian term. It is well known that the solutions of the heat equation

$$\partial_t u - \Delta u = 0$$

are smooth and global in time even if the initial condition is only an element of L^2 (i.e. $\int |u|^2 < \infty$, so that u has finite kinetic energy); see Appendix D. Let us recall the formal reasoning that leads to this result since it will be worthwhile to compare it later with corresponding attempts to prove a similar theorem for the Navier–Stokes equations.

We will consider the equation posed on the torus $\mathbb{T}^3$. Multiplying the heat equation by u and integrating over $\mathbb{T}^3$ it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} |u|^2 + \int_{\mathbb{T}^3} |\nabla u|^2 = 0,$$

and so after an integration in time we obtain the energy equality

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 \,\mathrm{d}s = \frac{1}{2}\|u(0)\|_{L^2}^2.$$

Thus the kinetic energy

$$\frac{1}{2}\|u(t)\|_{L^2}^2 = \frac{1}{2}\int_{\mathbb{T}^3} |u(t,x)|^2 \,\mathrm{d}x$$

is dissipated over time. Moreover, the integral

$$\int_0^T \|\nabla u(s)\|_{L^2}^2 \, \mathrm{d}s$$

(which represents the total energy dissipation) is bounded by the initial energy, so it is easy to deduce that ∇u belongs to L^2 for almost every time.

This allows us to improve the regularity of u by a bootstrapping argument: if we restart the time evolution from some time $\tau > 0$ at which $\nabla u(\tau) \in L^2$ and then multiply the heat equation by $-\Delta u$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} |\nabla u|^2 + \int_{\mathbb{T}^3} |\Delta u|^2 = 0$$

after an integration by parts. From this we can conclude that the L^2 norm of ∇u also decays in time (for $t > \tau$), and that Δu is in L^2 for almost every time $t > \tau$. Continuing in this way we can show that u is smooth for any $t > 0$.

Moreover, it is worth noticing that the energy in the heat equation is dissipated faster in the smaller length scales (i.e. higher frequencies). The easiest way to see this is to solve the heat equation using Fourier series; we can write the solution in the form

$$u(x,t) = \sum_{k \in \mathbb{Z}^3} \hat{u}_k(0) \mathrm{e}^{-|k|^2 t} \mathrm{e}^{\mathrm{i}k \cdot x},$$

when the initial condition u_0 is given by

$$u_0(x) = \sum_{k \in \mathbb{Z}^3} \hat{u}_k(0) \mathrm{e}^{\mathrm{i}k \cdot x}.$$

The larger $|k|$, the smaller the factor $\mathrm{e}^{-|k|^2 t}$, and hence the more rapid the damping effect of the diffusion. Thus the Laplacian tames wild oscillations in the small scales (large $|k|$) and has a smoothing effect on the solution.[5]

Advection: the nonlinear term

The character of the nonlinear term is very much opposite to that of the Laplacian. The simplest model containing a time derivative and a nonlinear term similar to $(u \cdot \nabla)u$ is the scalar Burgers equation

$$u_t + uu_x = 0.$$

[5] It should be noted that there are ODE models $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, that have global solutions, but in which the introduction of diffusion leads to the unexpected blowup of solutions of the PDE $\partial_t u - \Delta u = f(u)$, see Weinberger (1999), for example.

Of course, observations drawn from such an extreme simplification are not directly applicable to the Navier–Stokes equations, but nevertheless it is noteworthy that it is easy, using the method of characteristics, to find examples of smooth initial conditions for which solutions of the Burgers equation develop shocks (jump discontinuities in u) in a finite time (see Section 3.3 in Renardy & Rogers (2004) for example).

There is also another argument for seeing the nonlinear term as the source of potential danger. Counteracting the effect of the Laplacian, which dissipates energy in the smaller scales, the nonlinear term transfers energy from the larger scales to the smaller scales. Indeed, let us again consider the Navier–Stokes equations on the three-dimensional torus and assume (see Doering & Gibbon, 1995) that at a certain time u can be expanded as a finite sum of Fourier modes

$$u(x) = \sum_{|k| \leq N} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}$$

(with the understanding that $\hat{u}_k = 0$ for $|k| > N$). Straightforward computations show that

$$(u \cdot \nabla)u = \sum_{|m| \leq 2N} \hat{v}_m \mathrm{e}^{\mathrm{i}m\cdot x},$$

where $\hat{v}_m$ is given by

$$\hat{v}_m = \sum_{|l| \leq N} (\hat{u}_{m-l} \cdot l)\mathrm{i}\hat{u}_l.$$

Therefore smaller length scales are immediately activated by the nonlinear term. The energy pumped to such small scales could potentially induce fine oscillations that might perhaps lead to the blowup of the gradient of u.

Seen from this perspective the main role of the viscosity coefficient ν in equation (11) is to reflect the strength of the Laplacian in its struggle with the nonlinear term; this idea is in accord with the experimental observation that for large values of viscosity the flow is laminar, easily predictable and very regular, while for small viscosities the flow becomes turbulent, see Batchelor (1999), for example.

The pressure

The effect of the pressure is more subtle, but significant. If we ignore the divergence-free condition and drop the pressure term from the Navier–Stokes equations, we obtain the viscous Burgers equation

$$\partial_t u - \Delta u + (u \cdot \nabla)u = 0.$$

Classical solutions of this equation obey a maximum principle (Kiselev & Ladyzhenskaya, 1957) which can be used as the basis of a global existence proof (see Pooley & Robinson, 2016, for example).

To see this, assume that u is not identically zero, fix $\alpha > 0$, and consider the equation satisfied by $|v|$, where $v(x,t) = \mathrm{e}^{-\alpha t}u(x,t)$, which is

$$\frac{\partial}{\partial t}|v|^2 + 2\alpha|v|^2 + (u\cdot\nabla)|v|^2 - 2v\cdot\Delta v = 0. \tag{16}$$

Noting that $2v\cdot\Delta v = \Delta|v|^2 - 2|\nabla v|^2$ it follows that if $|v|^2$ has a local maximum at $(x,t)\in\mathbb{T}^3\times(0,T]$ then the left-hand side of (16) is strictly positive unless $|v(x,t)| = 0$. Since this is not possible, it must be the case that $\|v(\cdot,t)\|_{L^\infty} \le \|v(\cdot,0)\|_{L^\infty}$, which means that

$$\|u(t)\|_{L^\infty} \le \mathrm{e}^{\alpha t}\|u(0)\|_{L^\infty}.$$

Since $\alpha > 0$ was arbitrary it follows that $\|u(t)\|_{L^\infty} \le \|u(0)\|_{L^\infty}$ for all $t > 0$ (while u remains a classical solution).

The pressure therefore plays an extremely important role in the regularity problem for the Navier–Stokes equations; but in the standard formulation of the Navier–Stokes problem the pressure is eliminated using the observation that the gradient of any function ϕ is orthogonal to any divergence-free function,

$$\nabla\cdot u = 0 \qquad \Rightarrow \qquad \int u\cdot\nabla\phi\,\mathrm{d}x = 0.$$

By projecting the Navier–Stokes equations onto the space of divergence-free functions (where the solution must belong) we therefore obtain an equation in which the pressure plays no obvious role, and we will do this throughout most of Part I of this book. But even though the pressure is an 'invisible' term in the standard formulation of the problem, we cannot forget that it plays a crucial role in the problem of regularity of solutions, and we will see in Part II that it is inescapable when we come to study the equations locally.

Overview of the contents of the book

Part I of this book covers the classical existence results for weak and strong solutions, which date back to Leray (1934) and Hopf (1951), estimates for the pressure, and local existence of solutions in the critical spaces $\dot{H}^{1/2}$ and L^3. Part II treats conditional local regularity results due to Serrin (1962) and others and the partial regularity results for suitable weak solutions due primarily to Caffarelli, Kohn, & Nirenberg (1982).

The appendices provide some useful background material from functional analysis (Appendix A), harmonic analysis (Appendix B), the theory of elliptic and parabolic PDEs (Appendices C and D), and a measurable selection theorem required in the final chapter of the book (Appendix E).

Part I

Weak and Strong Solutions

Overview of Part I

In the first part of the book we treat classical results on the existence of weak and strong solutions and investigate some of the properties of these solutions. With the exception of Chapters 5 (on the pressure) and 11 (local existence in L^3), and parts of Chapter 2 required for these two chapters, none of the results in Part I require any techniques from harmonic analysis.

We recall some basic definitions and results on relevant function spaces in Chapter 1. Chapter 2 introduces the Helmholtz–Weyl decomposition whereby any function $v \in L^2$ can be written as the sum of a divergence-free vector field and a gradient; this decomposition allows us to eliminate the pressure from the governing equations throughout most of Part I. By working in the periodic setting we are able to give a particularly simple explicit proof of the existence of such a decomposition (Theorem 2.6).

In Chapter 3 we give our weak formulation of the Navier–Stokes equations (see Definition 3.3): u is a weak solution if

(i) $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ for every $T > 0$ and
(ii) u satisfies

$$-\int_0^s \langle u, \partial_t \varphi \rangle + \int_0^s \langle \nabla u, \nabla \varphi \rangle + \int_0^s \langle (u \cdot \nabla) u, \varphi \rangle = \langle u_0, \varphi(0) \rangle - \langle u(s), \varphi(s) \rangle$$

for a.e. $s > 0$, for a suitable class of divergence-free test functions φ.

We use the Galerkin method to prove the existence of at least one weak solution in Theorem 4.4; we then show in Theorem 4.6 that the solution we have constructed satisfies the strong energy inequality

$$\frac{1}{2}\|u(t)\|^2 + \int_s^t \|\nabla u(r)\|^2 \, \mathrm{d}r \le \frac{1}{2}\|u(s)\|^2 \qquad \text{for all} \qquad t \ge s$$

for almost every $s \in [0, \infty)$, including $s = 0$. Weak solutions that satisfy the strong energy inequality, termed 'Leray–Hopf weak solutions', turn out to be extremely important in what follows.

The use of divergence-free test functions in (ii), above, eliminates the pressure from the equations. In Chapter 5 we show that in the absence of boundaries the pressure can be obtained, given a weak solution u, by solving the equation $-\Delta p = \partial_i\partial_j(u_i u_j)$, and that the resulting pair (u, p) satisfies the governing equations in the sense of distributions. Using results from harmonic analysis we also show how in the absence of boundaries (on $\mathbb{T}^3$ and $\mathbb{R}^3$) one can estimate p given u, in particular proving that

$$\|p\|_{L^r} \leq C_r \|u\|^2_{L^{2r}}, \qquad 1 < r < \infty.$$

In Chapter 6 we show that if $u_0 \in H^1$ then there is a more regular 'strong' solution that satisfies $u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ for some $T > 0$ (Theorem 6.8). While they retain their regularity these strong solutions are unique in the class of Leray–Hopf weak solutions (Theorem 6.10). In Chapter 7 we use bootstrapping arguments to show that any strong solution is smooth on the time interval (ε, T) (Theorem 7.5). By combining results from these two chapters we show in Theorem 8.14 that Leray–Hopf weak solutions are in fact smooth on a collection of open time intervals of full measure in $(0, \infty)$, and we give an upper bound on the dimension of the set of singular times (if it is not empty). We also show that if u satisfies the 'Serrin condition'

$$u \in L^r(0, T; L^s), \qquad \frac{2}{r} + \frac{3}{s} \leq 1, \qquad 2 \leq r < \infty, \quad 3 < s \leq \infty, \quad (17)$$

then u is smooth on $(0, T]$ (Theorem 8.17); this condition also ensures uniqueness in the class of Leray–Hopf weak solutions (Theorem 8.19).

Chapter 9 shows that for any $T > 0$ the initial data that give rise to a strong solution on $(0, T)$ form an open subset of H^1 (Theorem 9.1), and that the Galerkin approximations used in Chapter 5 to prove existence of a solution must converge strongly in $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ when the limiting solution is strong (Theorem 9.3).

Chapter 10 discusses scaling properties of solutions and defines a critical space as one in which the norm is unaffected by the rescaling $u(x) \mapsto \lambda u(\lambda x)$: one such space is $\dot{H}^{1/2}(\mathbb{R}^3)$, and we show in Corollary 10.2 that initial conditions in $\dot{H}^{1/2}(\mathbb{T}^3)$ lead to regular solutions on $(0, T)$ for some $T > 0$, and global-in-time regular solutions when $\|u_0\|_{\dot{H}^{1/2}}$ is sufficiently small. We prove a similar result in the larger critical space $L^3(\mathbb{T}^3)$ in Theorem 11.4, using the pressure estimates from Chapter 5.

1

Function spaces

In this chapter we introduce our (mostly standard) notation and recall some basic facts about the function spaces we use most often in this book. Less standard material is covered in Section 1.7.1 (Sobolev spaces on $\mathbb{T}^3$) and Section 1.9 (Bochner spaces).

1.1 Domain of the flow

We consider almost exclusively flows in a three-dimensional domain. We focus mainly on two non-physical domains without boundaries:

- the whole space $\mathbb{R}^3$ and
- the three-dimensional torus $\mathbb{T}^3$.

As discussed in the Introduction, we will supplement the problem with appropriate additional conditions: on the whole space we will require some decay at infinity (for example, $u \in L^2(\mathbb{R}^3)$, which corresponds to finite kinetic energy), while on the torus it is convenient to take solutions with zero average, $\int_{\mathbb{T}^3} u = 0$. We concentrate in this chapter on defining function spaces on these domains, taking these constraints into account.

Throughout the book we also state the corresponding results for the physical case

- $\Omega \subset \mathbb{R}^3$ is a simply-connected bounded open set with a smooth boundary, which we call a 'smooth bounded domain' for short; on this kind of domain we always impose[1] the Dirichlet boundary condition $u|_{\partial\Omega} = 0$.

[1] "It turns out – although it is not at all self-evident – that in all circumstances where it has been experimentally checked, *the velocity of a fluid is exactly zero at the surface of a solid*." (Feynman, Leighton, & Sands, 1970, their italics)

We highlight when results in this chapter require bounded domains or the absence of boundaries; an analysis on the torus can take advantage of both of these simplifications, which is why many of the results in the book (particularly in Part I) are proved in this case: the exposition is simplified but the essential difficulties remain.

Functions defined on the torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ can be realised as periodic functions defined on all of $\mathbb{R}^3$, i.e.

$$u(x + 2\pi k) = u(x) \qquad \text{for all} \quad k \in \mathbb{Z}^3; \tag{1.1}$$

it is also convenient at times to identify u with its restriction to the fundamental domain $[0, 2\pi)^3$.

We will usually not distinguish in notation between spaces of scalar and vector-valued functions; when some ambiguity could arise we will use a notation like $[L^p]^n$ (this example denotes n-component vector-valued functions with each component in L^p) and in Chapter 2, in which the distinction between scalar and vector functions is significant, we will define $\mathbb{L}^p := [L^p]^3$.

1.2 Derivatives

We will use the notation ∂_j for the partial derivative corresponding to the jth coordinate. Two combinations of first derivatives will be particularly significant in what follows: if $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field then we can define the divergence of u as

$$\operatorname{div} u = \nabla \cdot u = \partial_i u_i$$

(summing, as ever, over repeated indices) and the curl of u as

$$\operatorname{curl} u = \nabla \wedge u = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix},$$

where $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ are the unit vectors along the coordinate axes. For calculations it is sometimes convenient to express the curl in the form

$$(\operatorname{curl} u)_i = \epsilon_{ijk}\partial_j u_k,$$

where ϵ_{ijk} is the Levi–Civita tensor defined by

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ is an even permutation of } 123 \\ -1 & ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

We also define

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \tag{1.3}$$

The equality

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{1.4}$$

which is useful for proving vector identities such as

$$\operatorname{curl}\operatorname{curl} u = \nabla(\nabla \cdot u) - \Delta u$$

(see Exercise 2.2), can be easily (if painfully) checked by hand.

For higher-order derivatives we will employ multi-index notation. We write

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \qquad \text{and} \qquad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

where the α_i are non-negative integers. For a vector $x = (x_1, \ldots, x_n)$ we define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and similarly we set

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

We write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, n$, and define the factorial $\alpha! = \alpha_1! \cdots \alpha_n!$. With this notation the Leibniz formula for the differentiation of a product can be written as

$$\partial^\alpha(fg) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g),$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

and in a mild abuse of notation we have written 0 for $(0, \ldots, 0)$.

Finally we write

$$|\nabla u| := \left(\sum_{i,j=1}^{3} |\partial_i u_j|^2 \right)^{1/2}.$$

1.3 Spaces of continuous and differentiable functions

Let Ω be a bounded open subset of $\mathbb{R}^n$. We use the following standard notation.

- $C^0(\Omega)$ is the space of continuous functions on Ω. The space $C^0(\overline{\Omega})$ of continuous functions on $\overline{\Omega}$ with norm
$$\|u\|_{C^0(\overline{\Omega})} := \sup_{x\in\overline{\Omega}} |u(x)|$$
is a Banach space.
- $C^{0,\gamma}(\overline{\Omega})$, $0 < \gamma \le 1$, is the space of all uniformly γ-Hölder continuous functions on $\overline{\Omega}$, i.e. functions $f\colon \overline{\Omega} \to \mathbb{R}$ for which there exists $C > 0$ such that
$$|f(x) - f(y)| \le C|x-y|^\gamma \qquad \text{for every} \quad x, y \in \overline{\Omega}.$$
The space $C^{0,\gamma}(\overline{\Omega})$ of γ-Hölder continuous functions on $\overline{\Omega}$ with norm
$$\|u\|_{C^0(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$
is a Banach space.

- $C^k(\Omega)$ is the space of all k-times continuously differentiable functions on Ω. The space $C^k(\overline{\Omega})$, of all k-times continuously differentiable functions with derivatives up to order k continuous on $\overline{\Omega}$ is a Banach space when equipped with the norm
$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{C^0(\overline{\Omega})}.$$
- $C^{k,\gamma}(\overline{\Omega})$, $0 < \gamma \le 1$, consists of all functions $u \in C^k(\Omega)$ for which all the kth derivatives are Hölder continuous with exponent γ, i.e. $\partial^\alpha u \in C^{0,\gamma}(\overline{\Omega})$ for every multi-index α with $|\alpha| = k$.

We will also use the following spaces of smooth (infinitely differentiable) functions.

- $C_c^\infty(\Omega)$ is the space of all smooth functions with compact support in Ω,
$$C_c^\infty(\Omega) := \{\varphi :\ \varphi \in C^\infty(\Omega), \quad \operatorname{supp}\varphi \subset\subset \Omega\},$$
where $A \subset\subset B$ is used to denote the fact that A is a compact subset of B. At times, to maintain a unified notation across all choices of domains, we will also write $C_c^\infty(\mathbb{T}^3)$, but in this case the subscript c is redundant since all functions defined on $\mathbb{T}^3$ have compact support.
- $\mathscr{S}(\mathbb{R}^n)$ is the space of Schwartz functions on $\mathbb{R}^n$, that is the space of all smooth functions such that
$$p_{k,\alpha}(u) := \sup_{x\in\mathbb{R}^n} |x|^k |\partial^\alpha u(x)|$$

is finite for every choice of $k = 0, 1, 2, \ldots$ and multi-index $\alpha \geq 0$. We denote by $\mathscr{S}'(\mathbb{R}^n)$ the space of all tempered distributions on $\mathbb{R}^n$, that is the collection of all bounded linear functionals f on $\mathscr{S}(\mathbb{R}^n)$ that are continuous in the sense that $f(u_n) \to 0$ as $n \to \infty$ if[2] $(u_n) \in \mathscr{S}(\mathbb{R}^n)$ with $p_{k,\alpha}(u_n) \to 0$ as $n \to \infty$ (for all k and α as above). For more details see Friedlander & Joshi (1999), for example.

When X is a Banach space we denote the space of all continuous functions from $[0, T]$ into X by $C([0, T]; X)$; equipped with the norm

$$\|u\|_{C([0,T];X)} := \sup_{t\in[0,T]} \|u(t)\|_X$$

this is again a Banach space. For functions in $C([0, T]; X)$ we have the following version of the Arzelà–Ascoli Theorem, which will be one of the key ingredients in the proof of a simple version of the Aubin–Lions Lemma (Theorem 4.11) that we will use to show the existence of weak solutions of the Navier–Stokes equations in Chapter 4.

Theorem 1.1 (Arzelà–Ascoli Theorem) *Let X be a Banach space and (u_n) a sequence of functions in $C([0, T]; X)$ such that*

(i) *for each $t \in [0, T]$ there exists a compact set $K(t) \subset X$ such that for every $n \in \mathbb{N}$ we have $u_n(t) \in K(t)$;*
(ii) *the functions u_n are equicontinuous: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$*

$$|s - t| \leq \delta \qquad \Rightarrow \qquad \|u_n(s) - u_n(t)\|_X \leq \varepsilon.$$

Then there exists a subsequence (u_{n_k}) and a function $u \in C([0, T]; X)$ such that

$$u_{n_k} \to u \quad \textit{in} \quad C([0, T]; X).$$

1.4 Lebesgue spaces

Given a measurable subset Ω of $\mathbb{R}^n$, a function $f\colon \Omega \to \mathbb{R}$ is (Lebesgue) measurable if $f^{-1}(U)$ is (Lebesgue) measurable for every Borel subset U of $\mathbb{R}$. It is equivalent in this case to require that $f^{-1}(U)$ is measurable for every open set $U \subset \mathbb{R}$, or for every closed set $U \subset \mathbb{R}$.

[2] Here and in what follows we use shorthand notation $(u_n) \in X$ to denote a sequence $(u_n)_{n=1}^{\infty}$ such that $u_n \in X$ for every $n \in \mathbb{N}$.

By $L^p(\Omega)$, where $1 \leq p < \infty$, we denote the standard Lebesgue space of measurable p-integrable (scalar or vector-valued) functions with the norm

$$\|u\|_{L^p} := \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

The space $L^2(\Omega)$ is a Hilbert space when equipped with the inner product

$$\langle u, v \rangle := \int_\Omega u(x) \cdot v(x) \, \mathrm{d}x;$$

since we use this space so frequently we reserve the notation $\| \cdot \|$ for the L^2 norm, and usually omit the L^2 subscript. We use the same notation $\langle f, g \rangle$ to denote $\int_\Omega f(x) \cdot g(x) \, \mathrm{d}x$ whenever $f \cdot g \in L^1$ (see also Section 1.8).

The space $L^\infty(\Omega)$ of essentially bounded functions on Ω is equipped with the standard norm

$$\|u\|_{L^\infty} := \operatorname{ess\,sup} |u| = \inf\{a \in \mathbb{R} : \ \mu(\{x \in \Omega : \ |u(x)| \geq a\}) = 0\},$$

where μ denotes the Lebesgue measure.

For $1 \leq p \leq \infty$ the space $L^p_{\mathrm{loc}}(\Omega)$ consists of those functions that are contained in $L^p(K)$ for every compact subset K of Ω.

We now recall some elementary facts about Lebesgue spaces, and in particular mention some inequalities that will be used frequently in what follows.

Theorem 1.2 (Hölder's inequality) *Let Ω be a measurable set in $\mathbb{R}^n$, either bounded or unbounded. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, where*

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad 1 \leq p, q \leq \infty,$$

then $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)| \, \mathrm{d}x \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

The Hölder inequality can be used in any domain since its proof is just a simple application of Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{for} \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \leq p, q \leq \infty \tag{1.5}$$

(see Exercise 1.1 for a proof of (1.5)). We often use Young's inequality 'with ε': with (p, q) as in (1.5), for any $\varepsilon > 0$ we have

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q;$$

in fact $C(\varepsilon) = q^{-1}(p\varepsilon)^{-q/p}$ but this explicit form is rarely required.

Applying Hölder's inequality with $v \equiv 1$ on Ω shows that Lebesgue spaces are nested on domains of finite measure.

Corollary 1.3 *If $\Omega \subset \mathbb{R}^n$ is a set of finite measure then*

$$L^q(\Omega) \subset L^p(\Omega) \qquad \text{if} \qquad 1 \le p \le q \le \infty.$$

It is important to observe that this is not the case when Ω does not have finite measure (see Exercise 1.2).

In the context of the Navier–Stokes equations we often need to estimate integrals of products of three functions (usually the nonlinear term $(u \cdot \nabla)u$ multiplied by some 'test' function). Therefore in addition to the standard version of Hölder's inequality we will also use the following variant (for the proof see Exercise 1.3).

Theorem 1.4 *Let Ω be a measurable set in $\mathbb{R}^n$. If $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, and $w \in L^r(\Omega)$ with*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \qquad 1 \le p, q, r \le \infty,$$

then $uvw \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)w(x)| \, \mathrm{d}x \le \|u\|_{L^p} \|v\|_{L^q} \|w\|_{L^r}.$$

The two most frequent uses of this in what follows[3] will be with exponents $(6, 2, 3)$,

$$\|uvw\|_{L^1} \le \|u\|_{L^6} \|v\|_{L^2} \|w\|_{L^3}, \tag{1.6}$$

and $(4, 2, 4)$,

$$\|uvw\|_{L^1} \le \|u\|_{L^4} \|v\|_{L^2} \|w\|_{L^4}.$$

We will often estimate the L^6 norm by the norm in H^1 using the 3D Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$ (see Theorem 1.7 or Theorem 1.18), and the L^3 and L^4 norms using the very useful technique of Lebesgue interpolation from the following theorem, whose proof is another application of Hölder's inequality (see Exercise 1.4).

[3] Since we will often be estimating expressions of the form $\int |u||\nabla v||w|$ (see Exercise 1.5) we will usually want to put the L^2 norm on the gradient term; this is the reason for the ordering of the norms in the two examples here.

Theorem 1.5 (Lebesgue Interpolation) *Let Ω be a measurable set in $\mathbb{R}^n$. If $1 \le p \le r \le q \le \infty$ and $u \in L^p(\Omega) \cap L^q(\Omega)$ then $u \in L^r(\Omega)$ with*

$$\|u\|_{L^r} \le \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \quad \textit{where} \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}. \tag{1.7}$$

The case that we will use most frequently will be the interpolation of L^3 between L^2 and L^6,

$$\|u\|_{L^3} \le \|u\|_{L^2}^{1/2} \|u\|_{L^6}^{1/2}. \tag{1.8}$$

It is worth remembering that the Hölder and interpolation inequalities are valid in any measurable subset of $\mathbb{R}^n$, bounded or unbounded, and are dimension independent.

1.5 Fourier expansions

For real-valued functions defined on $\mathbb{T}^3$ it is often useful to consider their Fourier expansion, i.e. to write a function u defined on $\mathbb{T}^3$ in the form

$$u(x) = \sum_{k \in \mathbb{Z}^3} \hat{u}_k \mathrm{e}^{\mathrm{i}k \cdot x}, \tag{1.9}$$

where the coefficients $\hat{u}_k$ are complex numbers satisfying the condition

$$\hat{u}_k = \overline{\hat{u}_{-k}} \qquad \text{for all} \qquad k \in \mathbb{Z}^3, \tag{1.10}$$

imposed to ensure that u is real.

The main issue is the convergence of the series expansion in (1.9). In some cases this is relatively simple: we can write any $u \in L^2(\mathbb{T}^3)$ in the form (1.9), with the understanding that the sum converges in $L^2(\mathbb{T}^3)$, since the collection $\{\mathrm{e}^{\mathrm{i}k \cdot x}\}$ forms an orthogonal basis for (complex-valued functions in) $L^2(\mathbb{T}^3)$; if needed, the Fourier coefficients $\hat{u}_k$ can be computed explicitly via the integral

$$\hat{u}_k = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \mathrm{e}^{-\mathrm{i}k \cdot x} u(x)\, \mathrm{d}x, \tag{1.11}$$

using the orthogonality of the basis elements. When u is an m-component vector-valued function the coefficients $\hat{u}_k$ are elements of $\mathbb{C}^m$ and condition (1.10) holds for all components of $\hat{u}_k$.

The characterisation of functions in $L^2(\mathbb{T}^3)$ in terms of their Fourier coefficients is straightforward: $u \in L^2(\mathbb{T}^3)$ if and only if $\sum |\hat{u}_k|^2 < \infty$. However, in other L^p spaces the situation is somewhat more complicated.

Let (C_N) be an increasing sequence of subsets of $\mathbb{Z}^3$ such that

$$\bigcup_{N=1}^{\infty} C_N = \mathbb{Z}^3.$$

Then

$$\left\| u - \sum_{k \in C_N} \hat{u}_k \mathrm{e}^{\mathrm{i}k \cdot x} \right\|_{L^p} \to 0 \qquad \text{as} \qquad N \to \infty$$

for every $u \in L^p(\mathbb{T}^3)$, $1 \leq p < \infty$, if and only if the maps

$$u \mapsto \sum_{k \in C_N} \hat{u}_k \mathrm{e}^{\mathrm{i}k \cdot x}$$

are uniformly bounded (in N) from $L^p(\mathbb{T}^3)$ into $L^p(\mathbb{T}^3)$. Boundedness of the 'truncated sum operators' implies convergence since trigonometric polynomials are dense in $L^p(\mathbb{T}^3)$, while the converse follows using the Principle of Uniform Boundedness (see Proposition 1.9 in Muscalu & Schlag, 2013, for example).

In the one-dimensional case (functions on $\mathbb{T}$) the Fourier expansions converge in every L^p, $1 < p < \infty$, see Theorem 3.20 and Exercise 3.7 in Muscalu & Schlag (2013), for example. In higher dimensions this can only be guaranteed when $p = 2$ (see Corollary 3.6.10 in Grafakos (2008); the proof relies on a result of Fefferman (1971) for the equivalent phenomenon in the context of the Fourier transform). However, one can obtain convergence in $L^p(\mathbb{T}^3)$, $p \neq 2$, by considering 'square sums' of Fourier components, i.e. using the one-dimensional result repeatedly.

Theorem 1.6 *Let $Q_N = [-N, N]^3 \cap \mathbb{Z}^3$. For every $u \in L^1(\mathbb{T}^3)$ and every $N \in \mathbb{N}$ define*

$$S_N(u) := \sum_{k \in Q_N} \hat{u}_k \mathrm{e}^{\mathrm{i}k \cdot x},$$

where the Fourier coefficients $\hat{u}_k$ are given by (1.11). Then for every $1 < p < \infty$ there is a constant $C_p > 0$, independent of N, such that

$$\|S_N(u)\|_{L^p} \leq C_p \|u\|_{L^p} \quad \textit{for all} \quad u \in L^p(\mathbb{T}^3)$$

and $S_N(u) \to u$ in $L^p(\mathbb{T}^3)$.

1.6 Sobolev spaces $W^{k,p}$

We begin with some well-known results about the Sobolev spaces $W^{k,p}$ for integer values of k. If not stated otherwise we assume below that $\Omega \subseteq \mathbb{R}^n$ is either the whole space or a bounded domain with at least Lipschitz boundary.

We recall that a function $g \in L^1_{\mathrm{loc}}(\Omega)$ is the weak derivative $\partial_i u$ of a function $u \in L^1_{\mathrm{loc}}(\Omega)$ if

$$\int_\Omega g(x)\varphi(x)\,\mathrm{d}x = -\int_\Omega u(x)\partial_i\varphi(x)\,\mathrm{d}x \qquad \text{for all} \quad \varphi \in C_c^\infty(\Omega).$$

Let $1 \le p < \infty$. The space $W^{1,p}(\Omega)$ consists of all those functions $u \in L^p(\Omega)$ all of whose first weak derivatives $\partial_i u$ exist and are in $L^p(\Omega)$ (we often write $\nabla u \in L^p(\Omega)$ for short). The norm in $W^{1,p}(\Omega)$ is given by

$$\|u\|_{W^{1,p}} := \left(\int_\Omega |u(x)|^p + \sum_{i=1}^n \int_\Omega |\partial_i u(x)|^p\,\mathrm{d}x\right)^{1/p}. \tag{1.12}$$

The space $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.

The space $W^{1,p}_0(\Omega)$, with $1 \le p < \infty$ is defined as the closure of $C_c^\infty(\Omega)$ in the norm (1.12). The space $W^{1,p}_0(\Omega)$ is the subset of $W^{1,p}(\Omega)$ that consists of functions vanishing on the boundary (in the sense of trace). Usually we have $W^{1,p}_0(\Omega) \neq W^{1,p}(\Omega)$, but on the whole space these coincide:

$$W^{1,p}_0(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n).$$

Suppose now that $k \ge 2$ and $1 \le p < \infty$. The space $W^{k,p}(\Omega)$ is the space of functions $u \in W^{k-1,p}(\Omega)$ all of whose first weak derivatives also belong to $W^{k-1,p}(\Omega)$. The standard norm in $W^{k,p}(\Omega)$ is given by

$$\|u\|^p_{W^{k,p}} := \sum_{|\alpha|\le k} \|\partial^\alpha u\|^p_{L^p}. \tag{1.13}$$

The space $W^{k,p}_0(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in the $W^{k,p}$ norm.

The following theorem gives the fundamental embedding results for the spaces $W^{1,p}$, valid on any sufficiently regular domain.

Theorem 1.7 *Suppose that $u \in W^{1,p}(\Omega)$, where $\Omega = \mathbb{T}^n$ or $\Omega \subseteq \mathbb{R}^n$ (bounded or unbounded).*

(i) *Sobolev embedding: if $1 \le p < n$ then we have a continuous embedding*

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega), \qquad \text{where} \quad p^* = np/(n-p).$$

Furthermore there exists a constant $c > 0$ such that if $u \in W_0^{1,p}$ then

$$\|u\|_{L^{p^*}} \leq c\|\nabla u\|_{L^p}.$$

(ii) *Morrey's Theorem: if $p > n$ then all functions in $W^{1,p}(\Omega)$ are bounded and continuous on $\overline{\Omega}$ with*

$$W^{1,p}(\Omega) \subset C^{0,1-n/p}(\overline{\Omega}).$$

On a bounded domain we also have the following compact embedding $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$.

Theorem 1.8 (Rellich–Kondrachov Theorem) *Take $p \geq 1$ and let $q \in [1, p^*)$, where $p^* = np/(n-p)$. If Ω is a bounded domain with Lipschitz boundary then from any sequence that is bounded in $W^{1,p}(\Omega)$ we can choose a subsequence that converges strongly in $L^q(\Omega)$.*

On a bounded domain Ω or on $\mathbb{T}^3$ we also have the Poincaré inequality, provided that we include some additional condition (zero boundary data or zero average).

Theorem 1.9 (Poincaré inequality) *Let $1 \leq p < \infty$ and let Ω be a smooth bounded domain. Then the Poincaré inequality*

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p}$$

holds

(i) *for all $u \in W_0^{1,p}(\Omega)$;*
(ii) *for all $u \in W^{1,p}(\Omega)$ such that $\int_\Omega u = 0$; and*
(iii) *for all $u \in W^{1,p}(\mathbb{T}^3)$ such that $\int_{\mathbb{T}^3} u = 0$.*

We do not have such an inequality on $\mathbb{R}^3$: the simple example of the constant function $u \equiv 1$ shows that this inequality does not hold in general, but constant functions are not included in $W^{1,p}(\mathbb{R}^3)$. Taking a non-zero function $u \in C_c^\infty(\mathbb{R}^3)$ and then considering the family $u_k(x) = u(kx)$ shows that the inequality also fails to hold uniformly on $W^{1,p}(\mathbb{R}^3)$.

1.7 Sobolev spaces H^s with $s \geq 0$

Throughout the book we will most often use the L^2-based Sobolev spaces $W^{k,2}$; we adopt the notation

$$H^k(\Omega) := W^{k,2}(\Omega) \qquad \text{and} \qquad H_0^k(\Omega) := W_0^{k,2}(\Omega)$$

for all $k = 1, 2, 3, \ldots$ The advantage of these spaces is that they are Hilbert spaces with the inner product

$$\langle u, v\rangle_{H^k} := \sum_{0\le|\alpha|\le k} \langle \partial^\alpha u, \partial^\alpha v\rangle.$$

The corresponding norm

$$\|u\|_{H^k}^2 = \langle u, u\rangle_{H^k} = \sum_{0\le|\alpha|\le k} \|\partial^\alpha u\|^2 \tag{1.14}$$

clearly coincides with the $W^{k,2}$ norm as defined in (1.13). We refer to this norm as 'the standard H^k norm', but in many situations (as we will now see) it is convenient to use an alternative, but equivalent, norm.

We now recall some basic properties of these spaces when defined on the whole space $\mathbb{R}^3$ and on the torus $\mathbb{T}^3$; we thereby motivate the generalisation of the definition of H^k to non-integer values of k.

1.7.1 Sobolev spaces on $\mathbb{T}^3$

Let us first consider the case of a torus. Since we concentrate on L^2-based Sobolev spaces, it is convenient to work with the Fourier expansion of u,

$$u(x) = \sum_{k\in\mathbb{Z}^3} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}, \qquad \hat{u}_k = \overline{\hat{u}_{-k}}, \tag{1.15}$$

with convergence of the expansion in $L^2(\mathbb{T}^3)$ assured if $\sum_{k\in\mathbb{Z}^3} |\hat{u}_k|^2 < \infty$ (see Section 1.5).

Let u be given by (1.15) and assume for simplicity that u is a scalar function. Then formally

$$\begin{aligned}\|u\|^2 = \int_{\mathbb{T}^3} |u(x)|^2\,\mathrm{d}x &= \int_{\mathbb{T}^3} \sum_{k,l} \hat{u}_k \hat{u}_l \mathrm{e}^{\mathrm{i}(k+l)\cdot x}\,\mathrm{d}x \\ &= \sum_k \int_{\mathbb{T}^3} |\hat{u}_k|^2\,\mathrm{d}x = (2\pi)^3 \sum_k |\hat{u}_k|^2.\end{aligned}$$

Moreover, using the fact that $\partial_j \mathrm{e}^{\mathrm{i}k\cdot x} = \mathrm{i}k_j \mathrm{e}^{\mathrm{i}k\cdot x}$ we can (again formally) compute the gradient of u to obtain

$$\|\nabla u\|^2 = \sum_{j=1}^3 \|\partial_j u\|^2 = (2\pi)^3 \sum_{k\in\mathbb{Z}^3} |k|^2 |\hat{u}_k|^2.$$

It follows that

$$\|u\|_{H^1(\mathbb{T}^3)}^2 = \|u\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla u\|_{L^2(\mathbb{T}^3)}^2 = (2\pi)^3 \sum_{k\in\mathbb{Z}^3} (1 + |k|^2)|\hat{u}_k|^2.$$

Generalising this simple observation, we can define the space $H^s(\mathbb{T}^3)$ for any $s \geq 0$ in the following way. The norm defined in (1.16) is equivalent to the standard definition in (1.14) when $s \in \mathbb{N}$, but the expression makes sense for any $s \geq 0$.

Definition 1.10 For $s \geq 0$ the Sobolev space $H^s(\mathbb{T}^3)$ consists of all functions given by (1.15) for which the norm $\|\cdot\|_{H^s}$ defined by

$$\|u\|_{H^s}^2 := (2\pi)^3 \sum_{k\in\mathbb{Z}^3} (1 + |k|^{2s})|\hat{u}_k|^2 \tag{1.16}$$

is finite.

It is immediate from the definition that $H^{s_2}(\mathbb{T}^3) \subset H^{s_1}(\mathbb{T}^3)$ if $s_2 > s_1$.

It is also useful to consider homogeneous Sobolev spaces, in which we drop the L^2 part of the norm. For any $u \in H^s(\mathbb{T}^3)$ the expression

$$\|u\|_{\dot{H}^s}^2 := (2\pi)^3 \sum_{k\in\dot{\mathbb{Z}}^3} |k|^{2s}|\hat{u}_k|^2, \tag{1.17}$$

where we exclude $k = (0, 0, 0)$ from the summation by setting

$$\dot{\mathbb{Z}}^3 := \{k \in \mathbb{Z}^3 : \ |k| \neq 0\},$$

is clearly well defined. However, $\|\cdot\|_{\dot{H}_s}$ does not define a norm on $H^s(\mathbb{T}^3)$ since it is equal to zero for every constant function. On the other hand, (1.17) does define a norm on the subset of $H^s(\mathbb{T}^3)$ consisting of zero-mean functions. We therefore make the following definition.

Definition 1.11 The space $\dot{L}^2(\mathbb{T}^3)$ consists of all functions $u \in L^2(\mathbb{T}^3)$ that have zero mean,

$$\int_{\mathbb{T}^3} u(x)\,\mathrm{d}x = 0.$$

It is easy to see that $\dot{L}^2(\mathbb{T}^3)$ is given by the collection of all (real) functions with Fourier expansion in which $\hat{u}_0 = 0$, i.e. all real functions of the form

$$\sum_{k\in\dot{\mathbb{Z}}^3} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x} \qquad \text{with} \qquad \sum_{k\in\dot{\mathbb{Z}}^3} |\hat{u}_k|^2 < \infty.$$

Definition 1.12 The homogeneous Sobolev space $\dot{H}^s(\mathbb{T}^3)$ is defined for $s \geq 0$ as

$$\dot{H}^s(\mathbb{T}^3) := H^s(\mathbb{T}^3) \cap \dot{L}^2(\mathbb{T}^3).$$

The nesting of homogeneous spaces, $\dot{H}^{s_2}(\mathbb{T}^3) \subset \dot{H}^{s_1}(\mathbb{T}^3)$ if $s_2 > s_1$, holds in the periodic case, but we will see that the same is not true on the whole space.

It is useful to notice that

$$\|u\|_{\dot H^1}^2 = (2\pi)^3 \sum_{k\in\dot{\mathbb{Z}}^3} |k|^2|\hat u_k|^2 = (2\pi)^3 \sum_{k\in\dot{\mathbb{Z}}^3} |k\hat u_k|^2 = \|\nabla u\|^2$$

and

$$\|u\|_{\dot H^2}^2 = (2\pi)^3 \sum_{k\in\dot{\mathbb{Z}}^3} |k|^4|\hat u_k|^2 = (2\pi)^3 \sum_{k\in\dot{\mathbb{Z}}^3} \left||k|^2\hat u_k\right|^2 = \|\Delta u\|^2.$$

For later use (e.g. in Chapter 10) it is also convenient to express the $\dot H^s$ norm in another way, by writing it as $\|\Lambda^s u\|$, where Λ^s is defined by

$$\Lambda^s u := \sum_{k\in\dot{\mathbb{Z}}^3} |k|^s \hat u_k \mathrm{e}^{\mathrm{i}k\cdot x} \quad \text{when} \quad u = \sum_{k\in\dot{\mathbb{Z}}^3} \hat u_k \mathrm{e}^{\mathrm{i}k\cdot x}. \tag{1.18}$$

Notice that $\Lambda^2 u = -\Delta u$ and so $\Lambda = (-\Delta)^{1/2}$; loosely speaking, one can think of Λ^s as a 'derivative of order s'. In this case the inner product in $\dot H^s$ can be written as

$$\langle u, v\rangle_{\dot H^s} = \langle \Lambda^s u, \Lambda^s v\rangle.$$

1.7.2 Sobolev spaces on $\mathbb{R}^3$

In order to formulate the corresponding definitions of H^s and $\dot H^s$ on the whole space $\mathbb{R}^3$ we replace Fourier series by the Fourier transform. We recall that one can define[4] the Fourier transform of a function $u \in L^1(\mathbb{R}^3)$ by

$$\hat u(\xi) = \int_{\mathbb{R}^3} \mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x} u(x)\,\mathrm{d}x,$$

and then for all $u \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ we have the Plancherel identity

$$\|u\|^2 = \int_{\mathbb{R}^3} |\hat u(\xi)|^2\,\mathrm{d}\xi.$$

Using this identity we can extend the definition of the Fourier transform to all functions in $L^2(\mathbb{R}^3)$ by a density argument.

When $f, \hat f \in L^1(\mathbb{R}^3)$ we can write f as the inverse Fourier transform of $\hat f$,

$$f(x) = (\hat f)^{\check{}}(x) := \int_{\mathbb{R}^3} \mathrm{e}^{2\pi\mathrm{i}\xi\cdot x} \hat f(\xi)\,\mathrm{d}\xi; \tag{1.19}$$

arguing as above, we can extend the operation $\check{}$ to $\hat f \in L^2(\mathbb{R}^3)$ by a density argument; in this case we cannot write the integral term in (1.19), but we still have the identity $f = (\hat f)^{\check{}}$.

[4] There are a variety of definitions, depending on where one chooses to place the factor of 2π. One advantage of this definition is that it makes the Fourier transform an isometry from L^2 into itself, i.e. the Plancherel identity holds as stated here with no multiplicative constant.

Since $\widehat{\partial_i u}(\xi) = 2\pi \mathrm{i}\xi_i \hat{u}(\xi)$ we have

$$\|\nabla u\|^2 = \int_{\mathbb{R}^3} |\widehat{\nabla u}(\xi)|^2 \, \mathrm{d}\xi = (2\pi)^2 \int_{\mathbb{R}^3} |\xi|^2 |\hat{u}(\xi)|^2 \, \mathrm{d}\xi;$$

we can define the space $H^1(\mathbb{R}^3)$ as the subspace of $L^2(\mathbb{R}^3)$ consisting of all functions for which the norm

$$\|u\|^2_{H^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1 + |\xi|^2)|\hat{u}(\xi)|^2 \, \mathrm{d}\xi$$

is finite. This leads to the following definition of the space $H^s(\mathbb{R}^3)$ for non-integer values of $s \geq 0$.

Definition 1.13 Let $s \geq 0$. The Sobolev space $H^s(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \ \|u\|^2_{H^s} < \infty \right\},$$

where

$$\|u\|^2_{H^s} := \int_{\mathbb{R}^3} (1 + |\xi|^{2s})|\hat{u}(\xi)|^2 \, \mathrm{d}\xi. \tag{1.20}$$

As in the periodic case, the norm defined in (1.20) is equivalent to the standard H^k norm when $s = k \in \mathbb{N}$, but s need not be an integer for the definition to make sense. Clearly $H^{s_2}(\mathbb{R}^3) \subset H^{s_1}(\mathbb{R}^3)$ if $s_2 > s_1$.

For any $u \in H^s(\mathbb{R}^3)$ we define the homogeneous H^s norm of u by setting

$$\|u\|^2_{\dot{H}^s} := \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi. \tag{1.21}$$

There are some subtleties in defining the homogeneous space $\dot{H}^s(\mathbb{R}^3)$, since only knowing that $\|u\|_{\dot{H}^s}$ is finite still allows bad behaviour of $\hat{u}$ at the origin in Fourier space.[5] One way to circumvent this is to add an extra condition on $\hat{u}$, as in the following definition.

Definition 1.14 The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ is defined as

$$\left\{ u \in \mathscr{S}' : \ \hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi < \infty \right\},$$

where $\mathscr{S}'$ is the collection of all tempered distributions.

The homogeneous spaces on $\mathbb{R}^3$ are not nested, but we nevertheless have a very useful interpolation property which we now recall. The result follows

[5] We use the definition favoured by Bahouri, Chemin, & Danchin (2011), see also Chemin et al. (2006). This avoids the complexities that arise from problems with attaching meaning to $|\xi|^s \hat{u}$ when one only knows that $\hat{u} \in \mathscr{S}'$, see the discussion in Chapter 6 of Grafakos (2009). The space $\dot{H}^s(\mathbb{R}^n)$ is complete if $s < n/2$ and in this case it can also be defined as the completion of $\mathscr{S}$ in the $\dot{H}^s(\mathbb{R}^n)$ norm; it is not complete when $s > n/2$.