

STUDENT PROJECT: A STEP TO SINGULAR FOLIATIONS THROUGH SMOOTH PARTITIONIFOLDS

Foliations describe how a manifold can be decomposed into a disjoint union of submanifolds of the same dimension, known as *leaves*. Naturally, when attempting to define a singular foliation, one is led to consider a decomposition into submanifolds—now possibly of varying dimensions—still referred to as leaves. We call such a decomposition a *partitionifold*. The aim of this project is to pave a path toward the theory of singular foliations through the natural framework of partitionifolds.

Objective

Establish a natural transition from foliations to the notion of singular foliations by studying the properties of "partitionifolds" and assessing to what extent this constitutes a suitable notion of singular foliation.

Prerequisites

- Basic differential geometry (manifolds, vector fields, singular distributions).
- Foliation theory.
- Nagano-Sussmann theorem; Stefan-Sussmann theorem.

Project Outline

1. Theoretical Review

- Partitionifolds and leaves.
- Tangent vector fields to a partitionifold.
- Smooth partitionifolds and Nagano-Sussmann theorem .
- Stefan-Sussman singular foliations.

2. Core Tasks

- Define the notion of partitionifold of a manifold M and the notion of leaves.
- Study the main properties of partitionifolds.

- Study the set of vector fields that are tangent to the leaves and show that their flows are symmetries.
- Define smooth partitionifold (compatibility conditions with its tangent vector fields).
- Describe smooth partitionifolds in terms of their tangent vector fields (Nagano-Sussmann theorem).

3. Compute concrete examples and counter-examples

4. Make a conclusion

Deliverables

- A short report ($\sim 10 - 15$ pages) summarizing:
 - Theoretical background.
 - Detailed computations and illustrations.
 - Geometric interpretations.
- A seminar talk (10-15 minutes) summarizing the findings.

Learning Outcomes

By completing this project, students will:

- Make a smooth transition from (regular) foliations to foliations with singularities;
- Understand the various possible definitions of singular foliations, which may a priori lead to parallel theories of singular foliations;
- Develop intuition about how singularities arise in foliation theory and how to model them in an abstract framework.