Decision Models - Assignment 1

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Exercise 1

Answer the following questions:

1. **Can an LP model have more than one optimal solution? Is it possible for an LP model to have exactly two optimal solutions? Why or why not?**
   Yes, an LP problem can have more than one optimal solution. This could happen when the objective function is parallel to one of the problem constraints that defines on the edge (or face, in higher dimensions) of the feasible region. In this case, the entire segment of feasible points along that edge yields the same optimal value and might be an optimal solution. However, an LP problem can NOT have two exact optimal solutions. This is because the set of optimal solutions (feasible region) in linear programming is always convex. If two points x_1 and x_2 are both optimal, then every convex combination of these two points (that is, any point of the form $\lambda x_1 + (1-\lambda)x_2$, where $0 \le \lambda \le 1$) is also optimal. Therefore, between any two optimal solutions, there are infinitely many other optimal solutions.

2. **Are the following objective functions for an LP model equivalent?**
   $$\begin{array}{ll} \max & (2x_1 + 3x_2 - x_3) \\ \min & (-2x_1 - 3x_2 + x_3) \end{array}$$
   Yes, the two objective functions are equivalent, because maximizing a linear function is equivalent to minimizing its negative. If both objective functions are applied to the same set of constraints, they will lead to the same optimal values for the decision variables x_1, x_2, x_3. The only difference will be the objective function value, which will have opposite signs. So yes, both formulations will produce identical optimal solutions, even though the objective values will be numerically opposite.

3. **Which of the following constraints are not linear or cannot be included as a constraint in a linear programming problem?**
   The constraints that are not linear and therefore cannot be included in an LP problem are:

- $2x_1 + \sqrt{x_2} \geq 60$ because it contains a square root, which is a non-linear function.
- $\frac{3x_1+2x_2x_1-3x_3}{x_1+x_2+x_3} \leq 0.9$ because it contains a product of variables and at least a variable in the denominator, both of which make it non-linear.
- $3x_1^2 + 7x_2 \leq 45$ because it contains a quadratic term x_1^2, which is not linear.

Only the constraints $2x_1 + x_2 - 3x_3 \geq 50$ and $4x_1 - \frac{1}{2}x_2 = 75$ are linear.

Exercise 2 A construction materials company is looking for a way to maximize profit for transportation of their goods. The company has a train available with 4 wagons. The weight and surface capacities of each wagon are shown in the following table:

Wagon j	Weight capacity C_j^w (ton)	Surface capacity C_j^s (m^2)
1	10	5000
2	8	4000
3	12	8000
4	6	2500

When stocking the wagons, the company can choose among 3 types of cargo, each with its own specifications, as shown in the following table:

Cargo type i	Availability a_i (ton)	Unit surface occupied s_i (m^2/ton)	Unit profit p_i (€/ton)
1	20	500	3500
2	10	300	2500
3	18	400	2000

The company has to decide how many tons of each cargo type should be loaded on train wagons in order to maximize profit.

1. **Define the decision variables for the problem described above.**
   Let:

   $$x_{ij} = \text{number of tons of cargo type i loaded into type j wagon}$$

   For $i = \{1, 2, 3\}$ and $j = \{1, 2, 3, 4\}$.

   The decision variables are the combination of cargo types and wagons:

- x_{11}
- x_{12}
- x_{13}
- x_{14}
- x_{21}
- x_{22}
- x_{23}
- x_{24}
- x_{31}
- x_{32}
- x_{33}
- x_{34}

Define the objective function for the problem described above.

In order to maximize the total profit, which depends on how many tons of cargo are loaded into each wagon. Let:

- $p_1 = 3500$/ton (cargo type 1)
- $p_2 = 2500$/ton (cargo type 2)
- $p_3 = 2000$/ton (cargo type 3)

If the tons of cargo must be split between the four wagons j, then the objective function is:

$$\text{Maximize Z} = \sum_{i=1}^{3}\sum_{j=1}^{4} p_i x_{ij} = 3500(x_{11}+x_{12}+x_{13}+x_{14})+2500(x_{21}+x_{22}+x_{23}+x_{24})+2000(x_{31}+x_{32}+x_{33}+$$

2. **Define the constraints for the problem described above.**

   - **Weight capacity constraint**:
     For each wagon $j = 1, 2, 3, 4$, the total weight loaded cannot exceed its capacity C_j^w:

     $$\sum_{i=1}^{3} x_{ij} \leq C_l^w \quad \text{for} \quad j = 1, 2, 3, 4$$

     The weight capacity constraints are:

     – $x_{11} + x_{21} + x_{31} \leq 10$ (wagon 1)
     – $x_{12} + x_{22} + x_{32} \leq 8$ (wagon 2)
     – $x_{13} + x_{23} + x_{33} \leq 12$ (wagon 3)
     – $x_{14} + x_{24} + x_{34} \leq 6$ (wagon 4)

- **Surface capacity constraints**:
  Each cargo type occupies a specific surface per ton. The total surface occupied on each wagon cannot exceed its surface capacity C_j^s:

$$\sum_{i=1}^{3} s_i x_{ij} \leq C_j^s \quad \text{for} \quad j = 1, 2, 3, 4$$

  Given the unit surfaces $s_1 = 500$, $s_2 = 300$ and $s_3 = 400$, the surface capacity constraints are:
  - $500x_{11} + 300x_{21} + 400x_{31} \leq 5000$
  - $500x_{12} + 300x_{22} + 400x_{32} \leq 4000$
  - $500x_{13} + 300x_{23} + 400x_{33} \leq 8000$
  - $500x_{14} + 300x_{24} + 400x_{34} \leq 2500$
- **Availability constraints**
  Each cargo type i has a limited availability a_i, so the total amount loaded across all wagons cannot exceed it:

$$\sum_{j=1}^{4} x_{ij} \leq a_i \quad \text{for } i = 1, 2, 3$$

  Given the availabilities:
  - $x_{11} + x_{12} + x_{13} + x_{14} \leq 20$
  - $x_{21} + x_{22} + x_{23} + x_{24} \leq 10$
  - $x_{31} + x_{32} + x_{33} + x_{34} \leq 18$
- **Non-negativity constraints**:

$$x_{ij} \geq 0 \quad \text{for all } i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4$$

3. **Write and solve the model with AMPL.**

```
set CARGO := 1 2 3;
set WAGON := 1 2 3 4;

param profit :=
1 3500
2 2500
3 2000;

param surface_req :=
1 500
2 300
3 400;

param avail :=
```

```
1 20
2 10
3 18;

1 10
2 8
3 12
4 6;

param surfaceCap :=
1 5000
2 4000
3 8000
4 2500;
```

Now let's define the model:

```
set CARGO;
set WAGON;

param profit {CARGO};
param surface_req {CARGO};
param avail {CARGO};
param weightCap {WAGON};
param surfaceCap {WAGON};

var x {CARGO, WAGON} >= 0;

maximize TotalProfit:
    sum {i in CARGO, j in WAGON} profit[i] * x[i,j];

subject to WeightCapacity {j in WAGON}:
    sum {i in CARGO} x[i,j] <= weightCap[j];

subject to SurfaceCapacity {j in WAGON}:
    sum {i in CARGO} surface_req[i] * x[i,j] <= surfaceCap[j];

subject to Availability {i in CARGO}:
    sum {j in WAGON} x[i,j] <= avail[i];

solve;
display x, TotalProfit;
```

After running the model in AMPL with CPLEX as the solver, we obtain:

- **An optimal objective value of** 107000: this indicates the maximum total profit (in monetary units) that the company can achieve under the given constraints (weight limits, surface limits, and cargo availability).

```
context:  set  >>> CARGO; <<<
ampl: option solver cplex;
ampl: solve;
CPLEX 22.1.1:             CPLEX 22.1.1: optimal solution; objective 107000
13 simplex iterations
ampl: display x, TotalProfit;
x :=
1 1    4
1 2    0.5
1 3   12
1 4    3.5
2 1    0
2 2    7.5
2 3    0
2 4    2.5
3 1    6
3 2    0
3 3    0
3 4    0
;

TotalProfit = 107000
```

- **A set of $x[i, j]$ values** that specify how many tons of cargo type i should be loaded onto wagon j.

Because all these constraints are satisfied and the objective value cannot be improved further, the solution is optimal.

One may notice that for certain cargo types, the solution does not use up all available tons. This is normal in a profit-maximization problem: if other cargo types yield a higher profit per unit of surface or weight, the solver tends to fill the wagons with the more profitable types (subject of course to the constraints).

Exercise 3

Solve the following Integer Linear Programming problem with the Branch and Bound method:

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 10 \\ & 5x_1 + 3x_2 \leq 39 \\ & x_1 \geq 0, \quad x_2 \geq 0 \\ & x_1, x_2 \in Z \end{aligned}$$

We first ignore the integer restrictions (i.e., $x_1, x2 \in Z$) and solve the continuous linear programming problem:

$$\max \quad 2x_1 + 5x_2 \qquad \text{s.t.} \quad \begin{cases} x_1 + x_2 \leq 10 \\ 5x_1 + 3x_2 \leq 39 \\ x_1, x_2 \geq 0 \end{cases}$$

The feasible region is defined by the intersection of those constraints that form the system.

We examine the key vertices:

1. **Intercept on the x_2-axis**:

   Set $x_1 = 0 \Longrightarrow x_2 \leq 10$ and $3x_2 \leq 39 \Longrightarrow x_2 \leq 13$.

   The binding constraint here is $x_2 \leq 10$.

   Thus, vertex: $(0, 10)$.

   Objective value: $2(0) + 5(10) = 50$.

2. **Intercept on the x_1-axis**:

   Set $x_2 = 0 \Longrightarrow x_1 \leq 10$ and $5x_1 \leq 39 \Longrightarrow x_1 \leq 7.8$.

   Thus, vertex: $(7.8, 0)$.

   Objective value: $2(7.8) + 5(0) = 15.6$.

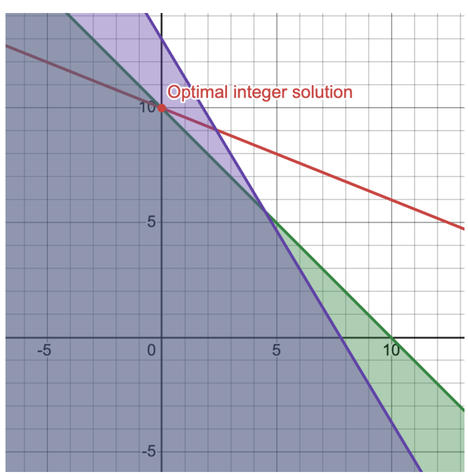
3. **Intersection of $x_1 + x_2 = 10$ and $5x_1 + 3x_2 = 39$**:

   Solve:
   $$\begin{cases} x_1 + x_2 = 10 \quad \Rightarrow \quad x_2 = 10 - x_1, \\ 5x_1 + 3(10 - x_1) = 39. \end{cases}$$

   Then,
   $$5x_1 + 30 - 3x_1 = 39 \quad \Longrightarrow 2x_1 = 9 \quad \Longrightarrow x_1 = 4.5$$

   and
   $$x_2 = 10 - 4.5 = 5.5$$

   Objective value: $2(4.5) + 5(5.5) = 9 + 27.5 = 36.5$



Comparing the objective values at the vertices:

- $(0, 10)$ yields 50,

- $(4.5, 5.5)$ yields 36.5,
- $(7.8, 0)$ yields 15.6.

The maximum value is 50 at $(0, 10)$.

Since in the continuous relaxation the optimal solution is $(0, 10)$, we now check the integrality:

- $x_1 = 0$ and $x_2 = 10$ are both integers.

Because the solution satisfies the integer conditions, there is no need for further branching. In a typical branch and bound process, if the solution obtained from the continuous relaxation had non-integer values, one would create additional subproblems (branches). However, in this case, the relaxed solution is already feasible for the integer program.

So, the **integer optimal solution** is:

$$x_1^* = 0, \quad x_2^* = 10, \quad \text{Optimal Objective Value} = 50$$

Thus, the best strategy is to set $x_1 = 0$ and $x_2 = 10$, achieving the maximum profit of 50.