KOREA UNIVERSITY

DSBA Transformer survey paper study

A Survey of Transformers

Appendix : Complexity, Parameters, and Scaling

arXiv preprint

DSBA
Data Science & Business Analytics

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00 Complexity and Parameters

Matrix, Complexity, and Parameters

- 행렬 연산과 Complexity, Parameter의 관계를 알아보면…

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 2 + 3 \times 3 \\ 4 \times 1 + 5 \times 2 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \end{bmatrix}$$

$$(2 \times 3) \qquad \underline{(3 \times 1)} \qquad (2 \times 1)$$

$$\boldsymbol{Computational\ Complexity\ (Mult)} = \mathbf{2} \times \mathbf{3} \times \mathbf{1} = \mathbf{6}$$

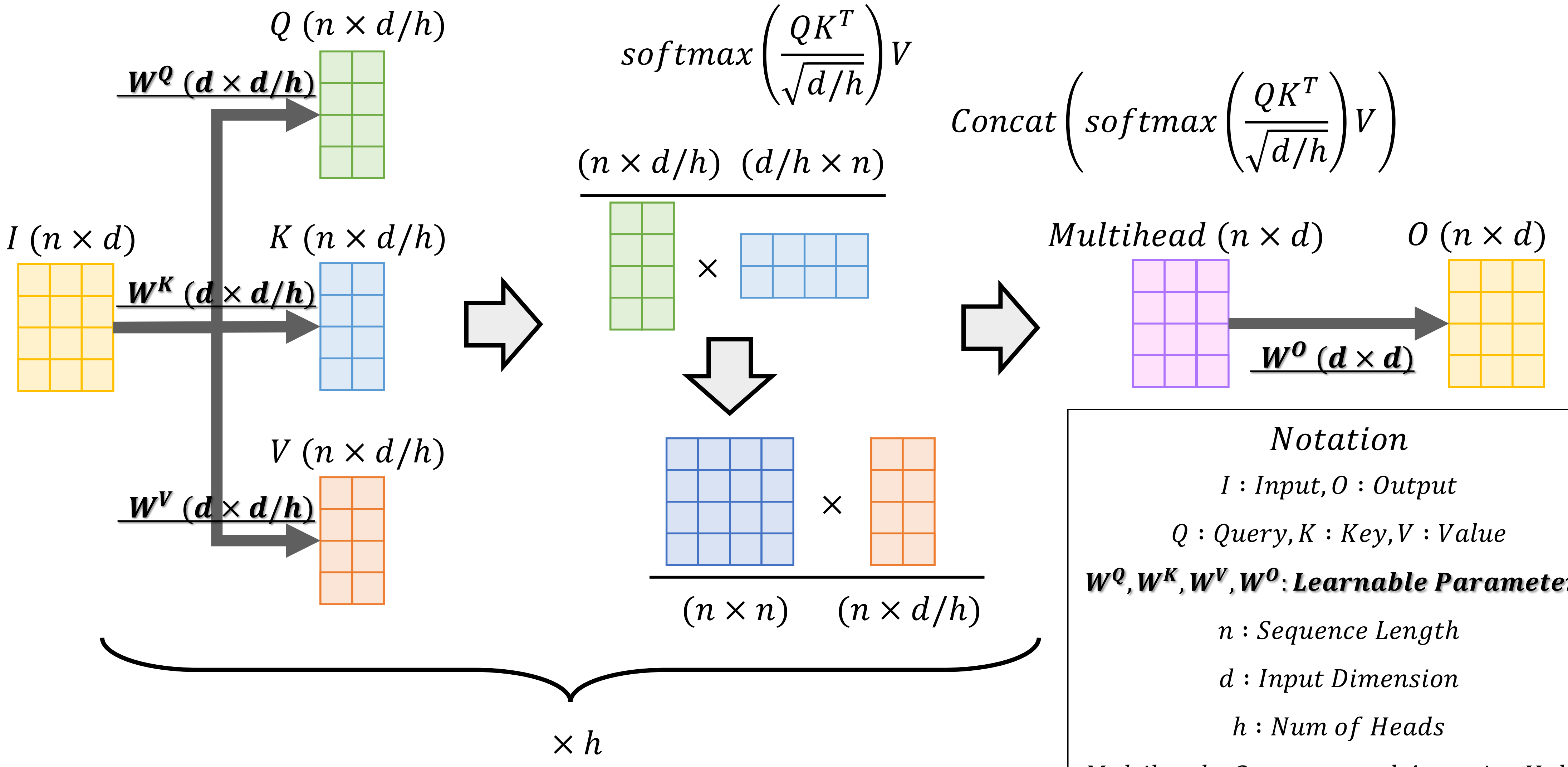
$$\boldsymbol{Parameters} = \mathbf{3} \times \mathbf{1} = \mathbf{3}$$

$$\underline{(3 \times 1)\ size\ matirx\ is\ \boldsymbol{learnable}}$$

00 Complexity and Parameters

Computational Complexity of Self-Attention

Module	Complexity	#Parameters
Self-attention	$O(n^2 \cdot d)$	$4d^2$
Position-wise FFN	$O(n \cdot d^2)$	$8d^2$



Notation

I : Input, O : Output

Q : Query, K : Key, V : Value

$\boldsymbol{W^Q, W^K, W^V, W^O}$: **Learnable Parameters**

n : Sequence Length

d : Input Dimension

h : Num of Heads

Multihead : Concatenated Attention Value

00 Complexity and Parameters

Computational Complexity of Self−Attention

Module	Complexity	#Parameters
Self-attention	$O(n^2 \cdot d)$	$4d^2$
Position-wise FFN	$O(n \cdot d^2)$	$8d^2$

Query Projection : $I\boldsymbol{W^Q} \rightarrow (n \times d) \times \boldsymbol{(d \times d/h)} \rightarrow \frac{nd^2}{h}$

Key Projection : $I\boldsymbol{W^K} \rightarrow (n \times d) \times \boldsymbol{(d \times d/h)} \rightarrow \frac{nd^2}{h}$

Value Projection : $I\boldsymbol{W^V} \rightarrow (n \times d) \times \boldsymbol{(d \times d/h)} \rightarrow \frac{nd^2}{h}$

Self−Attention : $softmax\left(\frac{QK^T}{\sqrt{d/h}}\right)V \rightarrow \frac{n^2d}{h} + \frac{n^2d}{h} = \frac{2n^2d}{h}$

Attention Score : $\frac{QK^T}{\sqrt{d/h}} = A \rightarrow (n \times d/h) \times (d/h \times n) \rightarrow \frac{n^2d}{h}$

Attention Value : $softmax(A)V \rightarrow (n \times n) \times (n \times d/h) \rightarrow \frac{n^2d}{h}$

$\times h$

Output Projection : $Multihead\boldsymbol{W^O} \rightarrow (n \times d) \times \boldsymbol{(d \times d)} \rightarrow nd^2$

Notation

I : $Input$, O : $Output$

Q : $Query$, K : Key, V : $Value$

$\boldsymbol{W^Q, W^K, W^V, W^O}$: ***Learnable Parameters***

n : $Sequence\ Length$

d : $Input\ Dimension$

h : $Num\ of\ Heads$

$Multihead$: $Concatenated\ Attention\ Value$

Multi−Head Self−Attention Computational Complexity = $\boldsymbol{4nd^2 + 2n^2d \rightarrow O(nd^2 + n^2d)}$

Self−Attention Computational Complexity = $\boldsymbol{2n^2d \rightarrow O(n^2d)}$

Num of Parameters = $\underline{\boldsymbol{4d^2}}$

00

Complexity and Parameters

Computational Complexity of Self-Attention

Module	Complexity	#Parameters
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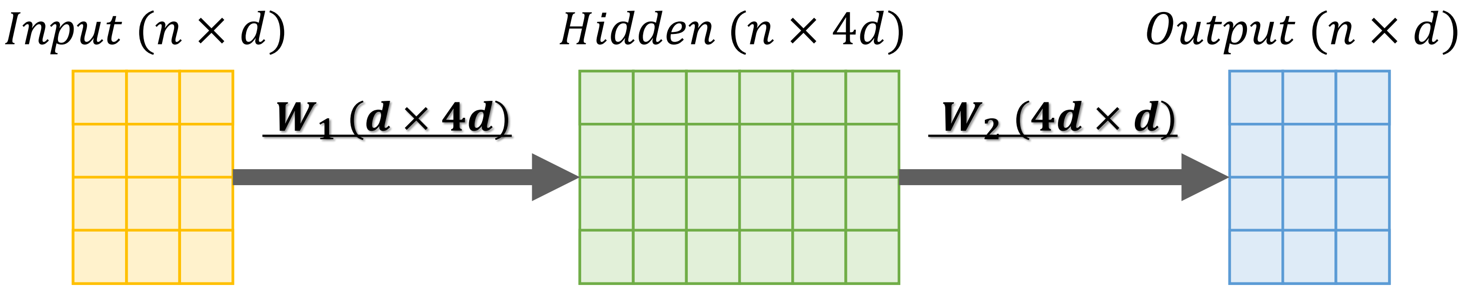
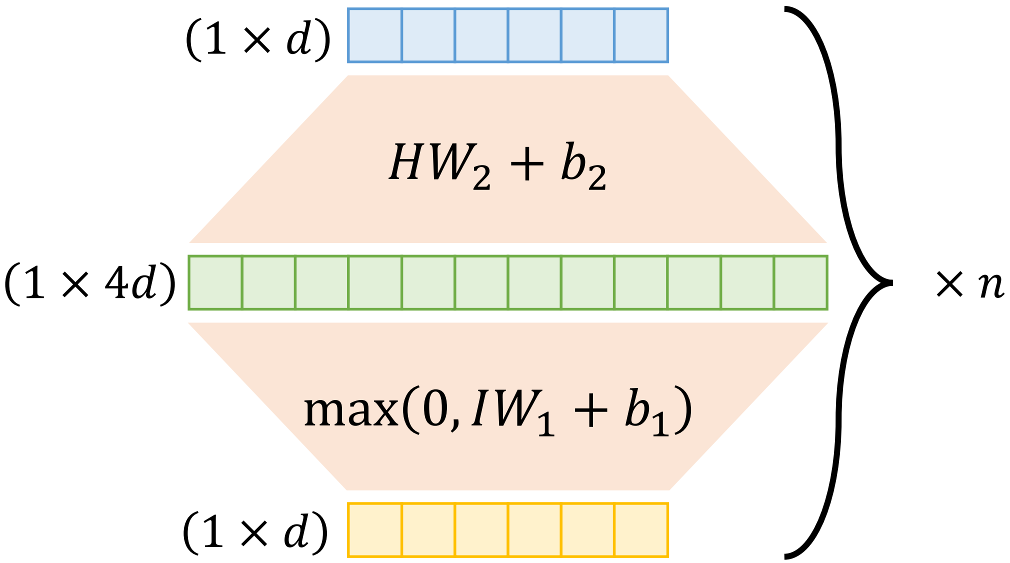
Position-wise FFN	$O(n \cdot d^2)$	$8d^2$



Input to Hidden : $I\boldsymbol{W_1} = H \rightarrow (n \times d) \times (\boldsymbol{d \times 4d}) \rightarrow 4nd^2$

Hidden to Output : $H\boldsymbol{W_2} \rightarrow (n \times 4d) \times (\boldsymbol{4d \times d}) \rightarrow 4nd^2$



⟨Position-wise FFNs⟩

Notation

I : *Input*, H : *Hidden*, O : *Output*

$\boldsymbol{W_1, W_2}$: ***Learnable Parameters***

n : *Sequence Length*

d : *Input Dimension*

Position-wise FFN Computational Complexity = $\boldsymbol{8nd^2} \rightarrow \boldsymbol{O(nd^2)}$

Num of Parameters = $\underline{\boldsymbol{8d^2}}$

00 Complexity and Parameters

Computational Complexity of Self-Attention

https://stackoverflow.com/questions/65703260

Question:

> Therefore, the total complexity of the layer is `O(n^2 d + n d^2)`, which is worse than that of a traditional RNN layer. I obtained the same result for multi-headed attention too, on considering the appropriate intermediate representation dimensionalities (`dk` , `dv`) and finally multiplying by the number of heads `h` .

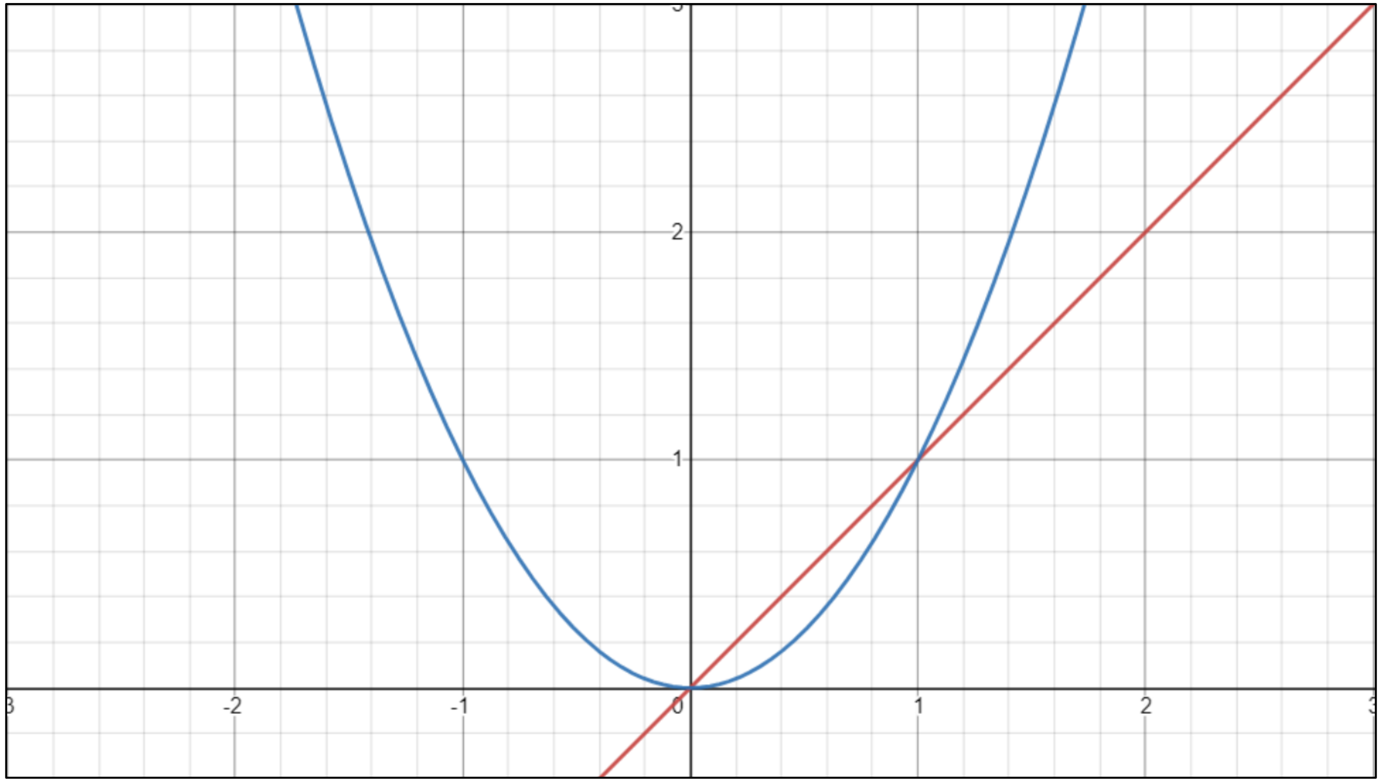
Answer:

> When the original Attention paper was first introduced, it didn't require to calculate `Q` , `V` and `K` matrices, as the values were taken directly from the hidden states of the RNNs, and thus the complexity of Attention layer **is** `O(n^2·d)` .
>
> Now, to understand what `Table 1` contains please keep in mind how most people scan papers: they read title, abstract, then look at figures and tables. Only then if the results were interesting, they read the paper more thoroughly. So, the main idea of the `Attention is all you need` paper was to replace the RNN layers completely with attention mechanism in seq2seq setting because RNNs were really slow to train. If you look at the `Table 1` in this context, you see that it compares RNN, CNN and Attention and highlights the motivation for the paper: using Attention should have been beneficial over RNNs and CNNs. It should have been advantageous in 3 aspects: constant amount of calculation steps, constant amount of operations **and** lower computational complexity for usual Google setting, where `n ~= 100` and

Complexity and Parameters

Computational Complexity of Self-Attention

https://www.desmos.com/calculator?lang=ko

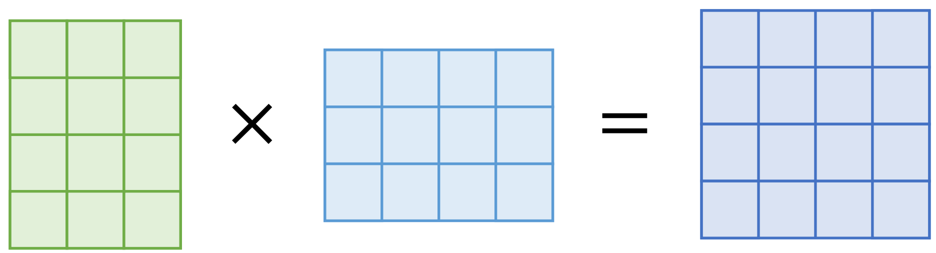


00 Scaling

Scaled Dot-Product Attention

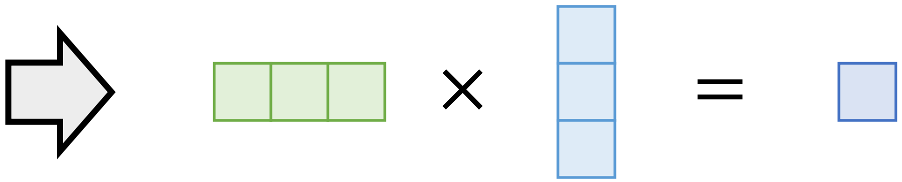
[Algebra of random variables]

$Q\ (n \times d_k)$ $K\ (d_k \times n)$



$Attention\ Score\ Matrix$
$(n \times n)$

$q\ (1 \times d_k)$ $k\ (d_k \times 1)$



$Attention\ Score$
$(scalar)$

$$softmax\left(\frac{QK^T}{\sqrt{d_k}}\right)V$$

$$Attention\ Score = q \cdot k = \sum_{i=1}^{d_k} q_i k_i$$

❖ *if all random variables are independent,*

- ✓ $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$
- ✓ $E[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} E[X_i]$
- ✓ $Var[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} Var[X_i]$
- ✓ $Var[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n}(Var[X_i] + (E[X_i])^2) - \prod_{i=1}^{n}(E[X_i])^2$

$$if\quad q_i \sim E[q_i] = 0, Var[q_i] = 1,$$
$$k_i \sim E[k_i] = 0, Var[k_i] = 1, q_i \perp k_i,$$
$$\rightarrow E[q \cdot k] = 0, Var[q \cdot k] = d_k$$

Scaling

Scaled Dot-Product Attention

He et al., DeBERTa: Decoding-Enhanced BERT with Disentangled Attention, ICLR, 2021

Algorithm 1 Disentangled Attention

Input: Hidden state $\boldsymbol{H}$, relative distance embedding $\boldsymbol{P}$, relative distance matrix $\boldsymbol{\delta}$. Content projection matrix $\boldsymbol{W}_{k,c}$, $\boldsymbol{W}_{q,c}$, $\boldsymbol{W}_{v,c}$, position projection matrix $\boldsymbol{W}_{k,r}$, $\boldsymbol{W}_{q,r}$.

$\boldsymbol{K}_c = \boldsymbol{H}\boldsymbol{W}_{k,c}$, $\boldsymbol{Q}_c = \boldsymbol{H}\boldsymbol{W}_{q,c}$, $\boldsymbol{V}_c = \boldsymbol{H}\boldsymbol{W}_{v,c}$, $\boldsymbol{K}_r = \boldsymbol{P}\boldsymbol{W}_{k,r}$, $\boldsymbol{Q}_r = \boldsymbol{P}\boldsymbol{W}_{q,r}$

$\boldsymbol{A}_{c\to c} = \boldsymbol{Q}_c\boldsymbol{K}_c^\intercal$

for $i = 0, ..., N-1$ **do**

  $\tilde{\boldsymbol{A}}_{c\to p}[i,:] = \boldsymbol{Q}_c[i,:]\boldsymbol{K}_r^\intercal$

end for

for $i = 0, ..., N-1$ **do**

  for $j = 0, ..., N-1$ **do**

    $\boldsymbol{A}_{c\to p}[i,j] = \tilde{\boldsymbol{A}}_{c\to p}[i, \boldsymbol{\delta}[i,j]]$

  end for

end for

for $j = 0, ..., N-1$ **do**

  $\tilde{\boldsymbol{A}}_{p\to c}[:,j] = \boldsymbol{K}_c[j,:]\boldsymbol{Q}_r^\intercal$

end for

for $j = 0, ..., N-1$ **do**

  for $i = 0, ..., N-1$ **do**

    $\boldsymbol{A}_{p\to c}[i,j] = \tilde{\boldsymbol{A}}_{p\to c}[\boldsymbol{\delta}[j,i], j]$

  end for

end for

$\tilde{\boldsymbol{A}} = \boldsymbol{A}_{c\to c} + \boldsymbol{A}_{c\to p} + \boldsymbol{A}_{p\to c}$

$\boldsymbol{H}_o = \text{softmax}(\frac{\tilde{\boldsymbol{A}}}{\sqrt{3d}})\boldsymbol{V}_c$

Output: $\boldsymbol{H}_o$