Robust optimization algorithms

15.C57/15.C571/6.C57/6.C571/IDS.C57: Optimization

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Solving a robust optimization problem

- Nominal optimization problem:

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} \leq b_i, \ \forall i = 1, \cdots, m \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Robust formulation with an infinite number of constraints:

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall \boldsymbol{a}_i \in \mathcal{U}_i, \ \forall i = 1, \cdots, m \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Reformulation with a finite number of (ill-structured) constraints

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall i = 1, \cdots, m \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

$\rightarrow$ How to develop a tractable solution to the robust formulation?

Reformulation via a direct method

Principles

- The robust formulation is not directly implementable due to ill-structured constraints or an infinite number of constraints

$$\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i \iff \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall \boldsymbol{a}_i \in \mathcal{U}_i$$

- Suppose that we can solve the inner maximization problem directly

$$\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} = g(\boldsymbol{x})$$

→ The following two problems are equivalent:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall i \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

$$\begin{aligned} \max_{\boldsymbol{x},\boldsymbol{p}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & g(\boldsymbol{x}) \leq b_i, \ \forall i \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

→ Analytical characterization of $\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x}$ yields a tractable reformulation for well-structured, norm-based uncertainty sets

Norm uncertainty sets: motivation

- Uncertainty sets to capture small deviations around nominal values
  - Example: local element-wise deviations

$$\mathcal{U}_i = \{\boldsymbol{a}_i \in \mathbb{R}^n : a_{ij} = \overline{a}_{ij} + \sigma_{ij} u_{ij};\ -\Gamma \leq u_{ij} \leq \Gamma,\ \forall j = 1, \cdots, n\}$$

  - Example: global perturbations from central limit theorem

$$\mathcal{U}_i = \left\{\boldsymbol{a}_i \in \mathbb{R}^n : a_{ij} = \overline{a}_{ij} + \sigma_{ij} u_{ij};\ -\Gamma\sqrt{n} \leq \sum_{j=1}^{n} u_{ij} \leq \Gamma\sqrt{n}\right\}$$

- The norm uncertainty sets captures these deviations through
  - perturbation $\boldsymbol{u}_i$, with small norm controlling extent of uncertainty
  - scaling factor Δ_i, reflecting relative uncertainty across components

Definition (Norm uncertainty sets)

Let $\boldsymbol{\Delta}_i \in \mathbb{R}^{k_i \times n}, \rho \in \mathbb{R}_+$. The norm uncertainty set is:

$$\mathcal{U}_i = \left\{\boldsymbol{a}_i \in \mathbb{R}^n : \boldsymbol{a}_i = \overline{\boldsymbol{a}}_i + \boldsymbol{\Delta}_i^\top \boldsymbol{u}_i,\ \|\boldsymbol{u}_i\| \leq \rho,\ \boldsymbol{u}_i \in \mathbb{R}^{k_i}\right\}$$

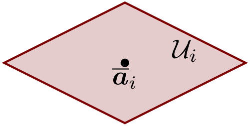
Norm uncertainty sets: visualization

$$\mathcal{U}_i = \left\{ \boldsymbol{a}_i \in \mathbb{R}^n : \ \boldsymbol{a}_i = \overline{\boldsymbol{a}}_i + \boldsymbol{\Delta}_i^\top \boldsymbol{u}_i, \ \|\boldsymbol{u}_i\| \leq \rho, \ \boldsymbol{u}_i \in \mathbb{R}^{k_i} \right\}$$

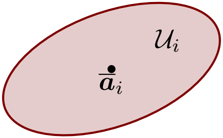
- Focus on the ℓ_p norm and ℓ_∞ norm:

$$\begin{cases} \|\boldsymbol{x}\|_p = \left(\displaystyle\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \\ \|\boldsymbol{x}\|_\infty = \displaystyle\max_{j=1,\cdots,n} |x_j| \end{cases}$$

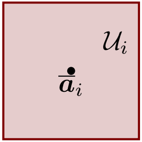
ℓ_1 **norm**
[Budget uncertainty]



ℓ_2 **norm**
[Ellipsoidal uncertainty]



ℓ_∞ **norm**
[Box uncertainty]



Robust counterpart: ellipsoidal uncertainty sets (1/3)

$$\mathcal{U}_i = \left\{ \boldsymbol{a}_i \in \mathbb{R}^n : \ \boldsymbol{a}_i = \overline{\boldsymbol{a}}_i + \boldsymbol{\Delta}_i^\top \boldsymbol{u}_i, \ \|\boldsymbol{u}_i\|_2 \leq \rho, \ \boldsymbol{u}_i \in \mathbb{R}^{k_i} \right\}$$

- Robust optimization formulation

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall i = 1, \cdots, m \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Inner problem:

$$Z_i^* = \max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} = \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \max_{\|\boldsymbol{u}_i\|_2 \leq \rho} \boldsymbol{u}_i^\top (\boldsymbol{\Delta}_i \boldsymbol{x})$$

- Claim: The maximization problem satisfies:

$$\max_{\|\boldsymbol{u}_i\|_2 \leq \rho} \boldsymbol{u}_i^\top (\boldsymbol{\Delta}_i \boldsymbol{x}) = \rho \|(\boldsymbol{\Delta}_i \boldsymbol{x})\|_2$$

Robust counterpart: ellipsoidal uncertainty sets (2/3)

$$\max_{\|\boldsymbol{u}_i\|_2 \leq \rho} \boldsymbol{u}_i^\top (\boldsymbol{\Delta}_i \boldsymbol{x})$$

- The maximization problem admits the following solution $\boldsymbol{u}_i^* = \rho \frac{\boldsymbol{\Delta}_i \boldsymbol{x}}{\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2}$
  - $\boldsymbol{u}_i^*$ is feasible because

$$\|\boldsymbol{u}_i^*\|_2 = \rho \frac{\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2}{\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2} = \rho$$

  - $\boldsymbol{u}_i^*$ achieves an objective value of

$$(\boldsymbol{u}_i^*)^\top (\boldsymbol{\Delta}_i \boldsymbol{x}) = \rho \frac{\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2^2}{\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2} = \rho \|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2$$

  - $\boldsymbol{u}_i^*$ is optimal: per Cauchy–Schwarz inequality, for any feasible $\boldsymbol{u}_i$,

$$\boldsymbol{u}_i^\top (\boldsymbol{\Delta}_i \boldsymbol{x}) \leq \|\boldsymbol{u}_i\|_2 \|(\boldsymbol{\Delta}_i \boldsymbol{x})\|_2 \leq \rho \|(\boldsymbol{\Delta}_i \boldsymbol{x})\|_2$$

- This proves that $\max_{\|\boldsymbol{u}_i\|_2 \leq \rho} \boldsymbol{u}_i^\top (\boldsymbol{\Delta}_i \boldsymbol{x}) = \rho \|(\boldsymbol{\Delta}_i \boldsymbol{x})\|_2$, hence

$$Z_i^* = \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2$$

Robust counterpart: ellipsoidal uncertainty sets (3/3)

Proposition (Robust counterpart: ellipsoidal uncertainty sets)

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \textit{s.t.} \quad & \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2 \leq b_i, \ \forall i = 1, \cdots, m \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Interpretation: safety buffer $\rho\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2$ to account for uncertainty
  - The larger the uncertainty, the larger the buffer
- The size of the robust counterpart is the same as the size of the nominal problem: n variables, m constraints
- The robust counterpart with an ellipsoidal uncertainty set is a second-order cone optimization problem
  - Special class of convex optimization, which can be solved efficiently

→ The robust counterpart is nearly as tractable as the nominal linear optimization problem

Robust counterpart: norm uncertainty sets (1/2)

Definition (Dual norm)

Consider a norm $\|\cdot\|$. The corresponding dual norm is defined as:

$$\|\boldsymbol{s}\|_* = \max_{\|\boldsymbol{x}\|\leq 1} |\boldsymbol{s}^\top \boldsymbol{x}| = \max_{\|\boldsymbol{x}\|\neq \boldsymbol{0}} \frac{|\boldsymbol{s}^\top \boldsymbol{x}|}{\|\boldsymbol{x}\|}$$

- The dual norm is an important notion in linear algebra, measuring how much one can "stretch" a vector by multiplying it with a unit vector
- Inner problem:

$$\begin{aligned}
\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} &= \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \max_{\|\boldsymbol{u}_i\|\leq \rho} \boldsymbol{u}_i^\top (\boldsymbol{\Delta}_i \boldsymbol{x}) \\
&= \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \max_{\|\boldsymbol{v}_i\|\leq 1} \rho \cdot \boldsymbol{v}_i^\top (\boldsymbol{\Delta}_i \boldsymbol{x}) \\
&= \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \cdot \|\boldsymbol{\Delta}_i \boldsymbol{x}\|_*
\end{aligned}$$

- Therefore, we can reformulate the robust constraint:

$$\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i \iff \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \cdot \|\boldsymbol{\Delta}_i \boldsymbol{x}\|_* \leq b_i$$

Robust counterpart: norm uncertainty sets (2/2)

Proposition (Robust counterpart: norm uncertainty sets)

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ s.t. \quad & \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \|\boldsymbol{\Delta}_i \boldsymbol{x}\|_* \leq b_i, \ \forall i = 1, \cdots, m \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Same interpretation: safety buffer $\rho\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_*$ to account for uncertainty
- The dual norm of ℓ_p is ℓ_q, with $q = \frac{p}{p-1}$: $\|\boldsymbol{s}\|_* = \|\boldsymbol{s}\|_q$
  - The dual norm of the ℓ_∞ norm is the ℓ_1 norm.

    $\|\boldsymbol{u}_i\|_\infty \leq \rho \rightarrow$ Robust counterpart: $\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_1 \leq b_i$
  - The dual norm of the ℓ_2 norm is the ℓ_2 norm.

    $\|\boldsymbol{u}_i\|_2 \leq \rho \rightarrow$ Robust counterpart: $\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_2 \leq b_i$
  - The dual norm of the ℓ_1 norm is the ℓ_∞ norm.

    $\|\boldsymbol{u}_i\|_1 \leq \rho \rightarrow$ Robust counterpart: $\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho\|\boldsymbol{\Delta}_i \boldsymbol{x}\|_\infty \leq b_i$

Reformulation using duality

Principles

- The robust formulation is not directly implementable due to ill-structured constraints or an infinite number of constraints

$$\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i \iff \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall \boldsymbol{a}_i \in \mathcal{U}_i$$

- What if we could transform the inner maximization problem into a minimization problem: $\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} = \min_{\boldsymbol{p}_i \in \mathcal{V}_i} g(\boldsymbol{p}_i)$?

$$\min_{\boldsymbol{p}_i \in \mathcal{V}_i} g(\boldsymbol{p}_i) \leq b_i \iff \exists \boldsymbol{p}_i \in \mathcal{V}_i : g(\boldsymbol{p}_i) \leq b_i$$

$\rightarrow$ The following two problems are equivalent:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \min_{\boldsymbol{p}_i \in \mathcal{V}_i} g(\boldsymbol{p}_i) \leq b_i, \ \forall i \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} \max_{\boldsymbol{x}, \boldsymbol{p}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & g(\boldsymbol{p}_i) \leq b_i, \ \forall i \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & \boldsymbol{p}_i \in \mathcal{V}_i \end{aligned}$$

$\rightarrow$ Use of strong duality to transform $\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x}$ into $\min_{\boldsymbol{p}_i \in \mathcal{V}_i} g(\boldsymbol{p}_i)$

Robust counterpart: polyhedral uncertainty sets (1/3)

$$
\begin{aligned}
\max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall i = 1, \cdots, m \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

- Example: uncertainty set from central limit theorem

$$
\mathcal{U}_i = \left\{ \boldsymbol{a}_i \in \mathbb{R}^n : -\Gamma\sqrt{n} \leq \sum_{j=1}^{n} \frac{a_{ij} - \overline{a}_{ij}}{\sigma_{ij}} \leq \Gamma\sqrt{n} \right\}
$$

Definition (Polyhedral uncertainty sets)

Let $\boldsymbol{D}_i \in \mathbb{R}^{\ell_i \times n}$, $\boldsymbol{d}_i \in \mathbb{R}^{\ell_i}$. The polyhedral uncertainty set is:

$$
\begin{aligned}
\mathcal{U}_i = \big\{ \boldsymbol{a}_i \in \mathbb{R}^n : \ & \boldsymbol{a}_i = \overline{\boldsymbol{a}}_i + \boldsymbol{u}_i, \\
& \boldsymbol{D}_i \boldsymbol{u}_i \leq \boldsymbol{d}_i, \\
& \boldsymbol{u}_i \in \mathbb{R}^n \big\}
\end{aligned}
$$

Robust counterpart: polyhedral uncertainty sets (2/3)

- Primal inner problem:

$$\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} = \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \begin{array}[t]{rl} \max & \boldsymbol{u}_i^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{D}_i \boldsymbol{u}_i \leq \boldsymbol{d}_i \\ & \boldsymbol{u}_i \in \mathbb{R}^n \end{array}$$

- Dual inner problem:

$$\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} = \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \begin{array}[t]{rl} \min & \boldsymbol{p}_i^\top \boldsymbol{d}_i \\ \text{s.t.} & \boldsymbol{p}_i^\top \boldsymbol{D}_i = \boldsymbol{x}^\top \\ & \boldsymbol{p}_i \geq \boldsymbol{0} \end{array}$$

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- Therefore, we can reformulate the robust constraint:

$$\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i \iff \exists \boldsymbol{p}_i \geq \boldsymbol{0} \ : \ \boldsymbol{p}_i^\top \boldsymbol{D}_i = \boldsymbol{x}^\top, \ \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \boldsymbol{p}_i^\top \boldsymbol{d}_i \leq b_i$$

Robust counterpart: polyhedral uncertainty sets (3/3)

Proposition (Robust counterpart: polyhedral uncertainty sets)

$$
\begin{aligned}
\max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\textit{s.t.} \quad & \boldsymbol{p}_i^\top \boldsymbol{D}_i = \boldsymbol{x}^\top, \ \forall i = 1, \cdots, m \\
& \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \boldsymbol{p}_i^\top \boldsymbol{d}_i \leq b_i, \ \forall i = 1, \cdots, m \\
& \boldsymbol{p}_i \geq \boldsymbol{0}, \ \forall i = 1, \cdots, m \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

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- The robust counterpart is a linear optimization problem
- The size of the robust counterpart depends on the size of the nominal problem and the size of the polyhedral uncertainty sets
  - Nominal problem: n variables, m constraints
  - Robust counterpart: $n + \sum_{i=1}^m \ell_i$ variables, $m + nm$ regular constraints and $n + \sum_{i=1}^m \ell_i$ non-negativity constraints.
- Generally, the robust counterpart is tractable in practice

Robust counterpart: summary

$$
\begin{aligned}
\max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x} \le b_i, \ \forall i \\
& \boldsymbol{x} \ge \boldsymbol{0}
\end{aligned}
$$

Uncertainty set	$\mathcal{U}_i^0$	Robust counterpart	Tractability
Box	$\lVert\boldsymbol{u}_i\rVert_\infty \le \rho$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \lVert \boldsymbol{\Delta}_i \boldsymbol{x}\rVert_1 \le b_i$	Linear
Ellipsoidal	$\lVert\boldsymbol{u}_i\rVert_2 \le \rho$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \lVert \boldsymbol{\Delta}_i \boldsymbol{x}\rVert_2 \le b_i$	Conic
Budget	$\lVert\boldsymbol{u}_i\rVert_1 \le \Gamma$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \Gamma \lVert \boldsymbol{\Delta}_i \boldsymbol{x}\rVert_\infty \le b_i$	Linear
Budget-box	$\begin{cases} \lVert\boldsymbol{u}_i\rVert_\infty \le \rho \\ \lVert\boldsymbol{u}_i\rVert_1 \le \Gamma \end{cases}$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \lVert \boldsymbol{y}\rVert_1 +$ $\Gamma \lVert \boldsymbol{\Delta}_i \boldsymbol{x} - \boldsymbol{y}\rVert_\infty \le b_i$	Linear
Polyhedral	$\boldsymbol{D}_i \boldsymbol{u}_i \le \boldsymbol{d}_i$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \boldsymbol{p}_i^\top \boldsymbol{d}_i$ $\boldsymbol{p}_i^\top \boldsymbol{D}_i = \boldsymbol{x}^\top$ $\boldsymbol{p}_i \ge \boldsymbol{0}$	Linear

Cutting planes algorithm

Intuition

$$
\begin{aligned}
\max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall \boldsymbol{a}_i \in \mathcal{U}_i, \ \forall i = 1, \cdots, m \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

- Suppose that, for any $\boldsymbol{x}$, we can solve $\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x}$ efficiently

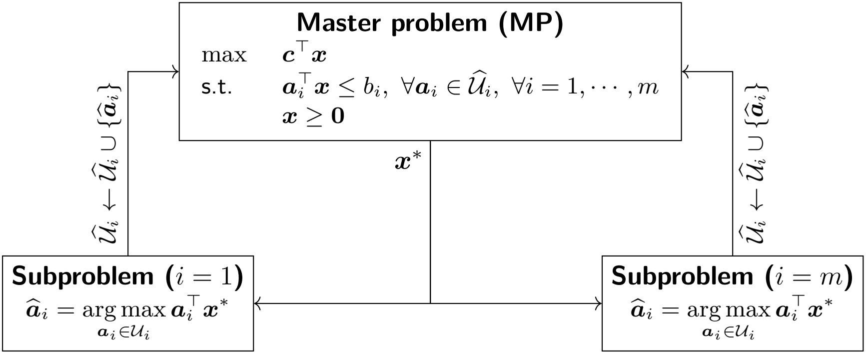
→ Add violated constraints iteratively, or prove that none exists

1. Solve the nominal problem; find solution $\boldsymbol{x}_0$
2. Solve $\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x}_0$; solution $\widehat{\boldsymbol{a}}_i$
3. If $\widehat{\boldsymbol{a}}_i \boldsymbol{x}_0 \leq b_i$ for all $i = 1, \cdots, m$, `STOP`: the solution is optimal
4. Otherwise, add the corresponding constraints: $\widehat{\boldsymbol{a}}_i^\top \boldsymbol{x} \leq b_i$
5. Solve the following problem (dual simplex); find solution $\boldsymbol{x}_1$; iterate.

$$
\begin{aligned}
\max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \overline{\boldsymbol{a}}_i^\top \boldsymbol{x} \leq b_i, \ \forall i = 1, \cdots, m \\
& \widehat{\boldsymbol{a}}_i^\top \boldsymbol{x} \leq b_i, \ \forall i = 1, \cdots, m \\
& \boldsymbol{x} \geq \boldsymbol{0}.
\end{aligned}
$$

Cutting planes method

- Maintain restricted uncertainty sets $\widehat{\mathcal{U}}_i \subseteq \mathcal{U}_i$ for all $i = 1, \cdots, m$
- Iterate between two components
  - Master problem: derive solution $\boldsymbol{x}^*$ based on restricted uncertainty sets
  - Subproblem: prove that $\boldsymbol{x}^*$ is feasible (hence, optimal) or expand the restricted uncertainty sets with new robust constraints $\widehat{\boldsymbol{a}}_i^\top \boldsymbol{x} \leq b_i$



Cutting planes algorithm

Algorithm

Inputs: *Vectors* $\mathbf{b}$ *and* $\mathbf{c}$, *uncertainty sets* $\mathcal{U}_1, \cdots, \mathcal{U}_m$, *tolerance* $\varepsilon > 0$

Output: *Solution* $\boldsymbol{x}^*$

1. *Initialize the master problem as the nominal problem:*
$$\widehat{\mathcal{U}}_1 = \{\overline{\boldsymbol{a}}_1\}, \cdots, \widehat{\mathcal{U}}_m = \{\overline{\boldsymbol{a}}_m\}$$
2. *Solve master problem*
   - *If infeasible,* `STOP`*: the robust problem is infeasible*
   - *If unbounded, find* $\boldsymbol{x}^*$, *such that* $\mathbf{c}^\top \boldsymbol{x}^* < -M$ (M *large)*
   - *Otherwise, find optimal solution* $\boldsymbol{x}^*$
3. *For each uncertain row* $i = 1, \cdots, m$, *solve subproblem*
   - *Compute* $\widehat{\boldsymbol{a}}_i = \arg\max\limits_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x}^*$
   - *If* $\widehat{\boldsymbol{a}}_i^\top \boldsymbol{x} > b_i + \varepsilon$, *add constraint* $\widehat{\boldsymbol{a}}_i^\top \boldsymbol{x} \leq b_i$ *to the master problem*
4. *If no constraints were added,* `STOP`*:* $\boldsymbol{x}^*$ *is the optimal robust solution. Otherwise, go to Step 2.*

Back to the nurse scheduling optimization problem

$$z_{qjid} = \# \text{ nurses from tier } q \text{ in position } j \text{ during shift } i \text{ on day } d$$

- Multi-objective formulation, with hyperparameters μ_1, μ_2
  - Minimizing number of nurse-shifts scheduled
  - Minimizing insufficiency from target nurse-to-patient ratio (npr)
  - Minimizing deviation in staffing levels from current schedule

$$\begin{aligned}
\min \quad & \sum_{q\in\mathcal{Q}}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\sum_{d\in\mathcal{D}} z_{qjid} + \mu_1 \cdot npr + \mu_2 \cdot \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\sum_{d\in\mathcal{D}} \Delta z_{jie} \\
\text{s.t.} \quad & npr \geq \sum_{j\in\mathcal{J}}\sum_{s\in\mathcal{S}}\sum_{w\in\mathcal{W}} \lambda_w \left(h_{jsw} - \sum_{q\in\mathcal{Q}}\sum_{i\in\mathcal{I}}\sum_{d\in\mathcal{D}} \sigma_{id}^{sw} z_{qjid} \right), \ \forall \boldsymbol{h} \in \mathcal{U} \\
& \Delta z_{jid} \geq \left| z_{jid}^{cur} - \sum_{q\in\mathcal{Q}} z_{qjid} \right|, \ \forall j \in \mathcal{J},\ i \in \mathcal{I},\ d \in \mathcal{D} \\
& \boldsymbol{z} \in \mathcal{Z}
\end{aligned}$$

Cutting planes algorithm for nurse scheduling optimization

Algorithm

Initialization:

- *Uncertainty set from historical demand* $\mathcal{U}^0 \leftarrow \{\boldsymbol{h}^{hist}\}$
- *Initiate the master problem based on* $\mathcal{U}^0$
- *Define iteration counter* $\kappa \leftarrow 1$ *and tolerance* η

1. *Solve master problem, and obtain a solution* $\boldsymbol{z}^\kappa$
2. *Define*

$$npr_\kappa(\boldsymbol{h}, \boldsymbol{z}^\kappa) = \sum_{j\in\mathcal{J}} \sum_{s\in\mathcal{S}} \sum_{w\in\mathcal{W}} \lambda_w \left(h_{jsw} - \sum_{q\in\mathcal{Q}} \sum_{i\in\mathcal{I}} \sum_{d\in\mathcal{D}} \sigma_{id}^{sw} z_{qjid}^\kappa \right)$$

3. *Maximize insufficiency:* $\boldsymbol{h}^\kappa \leftarrow \arg\max_{\boldsymbol{h}\in\mathcal{U}} npr_\kappa(\boldsymbol{h}, \boldsymbol{z}^\kappa)$
4. *Update the uncertainty set:* $\mathcal{U}^\kappa \leftarrow \left\{\boldsymbol{h}^{hist}, \boldsymbol{h}^1, \cdots, \boldsymbol{h}^\kappa\right\}$
4. *Update iteration counter:* $\kappa \leftarrow \kappa + 1$
5. *If* $(npr^\kappa - npr^{\kappa-1})/npr^{\kappa-1} < \eta$, *STOP. Otherwise, go to Step 1.*

Conclusion

Two methods for solving robust optimization problems

Cutting plane methods

- Iterations between a master problem and a subproblem
- Applicable to any uncertainty set as long as $\max\limits_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x}$ is solvable

Master problem (MP)

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}_i^\top \boldsymbol{x} \le b_i, \ \forall \boldsymbol{a}_i \in \widehat{\mathcal{U}}_i, \ \forall i \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

$\updownarrow$

Subproblem

$$\widehat{\boldsymbol{a}}_i = \arg\max_{\boldsymbol{a}_i \in \mathcal{U}_i} \boldsymbol{a}_i^\top \boldsymbol{x}^*$$

Reformulation methods

- Reformulation via a robust counterpart, using strong duality
- Tractable robust counterparts for a wide range of uncertainty sets

Uncertainty set	$\mathcal{U}_i^0$	Robust counterpart	Tractability
Box	$\\|\boldsymbol{u}_i\\|_\infty \le \rho$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \\|\boldsymbol{\Delta}_i \boldsymbol{x}\\|_1 \le b_i$	Linear
Ellipsoidal	$\\|\boldsymbol{u}_i\\|_2 \le \rho$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \\|\boldsymbol{\Delta}_i \boldsymbol{x}\\|_2 \le b_i$	Conic
Budget	$\\|\boldsymbol{u}_i\\|_1 \le \Gamma$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \Gamma \\|\Delta_i \boldsymbol{x}\\|_\infty \le b_i$	Linear
Budget-box	$\begin{cases} \\|\boldsymbol{u}_i\\|_\infty \le \rho \\ \\|\boldsymbol{u}_i\\|_1 \le \Gamma \end{cases}$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \rho \\|\boldsymbol{y}\\|_1 +$ $\Gamma \\|\Delta_i \boldsymbol{x} - \boldsymbol{y}\\|_\infty \le b_i$	Linear
Polyhedral	$\boldsymbol{D}_i \boldsymbol{u}_i \le \boldsymbol{d}_i$	$\overline{\boldsymbol{a}}_i^\top \boldsymbol{x} + \boldsymbol{p}_i^\top \boldsymbol{d}_i$ $\boldsymbol{p}_i^\top \boldsymbol{D}_i = \boldsymbol{x}^\top$ $\boldsymbol{p}_i \ge \boldsymbol{0}$	Linear

- Computational benchmarking: both methods are competitive[1]
  - Cutting planes methods tend to dominate with polyhedral uncertainty
  - Reformulation methods tend to dominate with ellipsoidal uncertainty

[1] Bertsimas, Dunning, Lubin. Reformulation versus cutting-planes for robust optimization: A computational study. 2016.

Summary

Takeaway

Robust optimization can be solved with two methods:

1. *Reformulation methods via a robust counterpart, using strong duality.*
2. *Cutting planes: iteration between a master problem and a subproblem (cutting planes can also be applied in the absence of strong duality)*

Takeaway

Robust optimization problems exhibit strong tractability in practice, for a wide range of optimization problems and uncertainty sets.

Takeaway

Robust solutions can provide significant benefits by protecting against real-world data perturbations and changes in the operating environment.