Third order equations from pages 46 and 48, and derived 4th order equations, of www.autodiff.org/Docs/euroad/17thEuroAdWorkshop-MuWang-HigherOrderReverseModeWithMPI.pdf ZTA, July 2016

1 Equations

Given that

$$\frac{\partial f_i}{\partial v_c} = \frac{\partial f_{i+1}}{\partial v_c} + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial f_{i+1}}{\partial v_i}\right) \tag{1}$$

Then

$$\frac{\hat{\partial}}{\hat{\partial}v_b} \left[\frac{\hat{\partial}f_i}{\hat{\partial}v_c} \right] = \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_c} + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_c} \right)
+ \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial f_{i+1}}{\partial v_i} \right)
+ \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_i} \right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i} \right)$$
(2)

It then follows that

$$\frac{\hat{\partial}}{\hat{\partial}v_{a}} \left[\frac{\hat{\partial}}{\hat{\partial}v_{b}} \left(\frac{\hat{\partial}f_{i}}{\hat{\partial}v_{c}} \right) \right] = \frac{\partial^{3}f_{i+1}}{\partial v_{a}\partial v_{b}\partial v_{c}} + \left(\frac{\partial^{3}\phi_{i}}{\partial v_{a}\partial v_{b}\partial v_{c}} * \frac{\partial^{f}f_{i+1}}{\partial v_{i}} \right) + \left(\frac{\partial^{2}\phi_{i}}{\partial v_{a}\partial v_{b}\partial v_{i}} * \frac{\partial^{2}f_{i+1}}{\partial v_{a}\partial v_{b}\partial v_{i}} \right) \\
+ \left(\frac{\partial^{2}\phi_{i}}{\partial v_{a}\partial v_{b}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{c}} \right) + \left(\frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial^{3}f_{i+1}}{\partial v_{a}\partial v_{b}\partial v_{c}} \right) \\
+ \left(\frac{\partial^{2}\phi_{i}}{\partial v_{a}\partial v_{b}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{c}} \right) + \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial^{3}f_{i+1}}{\partial v_{a}\partial v_{i}\partial v_{c}} \right) \\
+ \left(\frac{\partial^{2}\phi_{i}}{\partial v_{a}\partial v_{b}} * \frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{i}} \right) + \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial^{2}\phi_{i}}{\partial v_{a}\partial v_{c}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{i}} \right) \\
+ \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial^{3}f_{i+1}}{\partial v_{a}\partial v_{i}\partial v_{i}} \right) \\
+ \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial^{3}f_{i+1}}{\partial v_{a}\partial v_{i}\partial v_{i}} \right) \\
+ \left(\frac{\partial^{2}\phi_{i}}{\partial v_{a}} * \left[\frac{\partial^{3}f_{i+1}}{\partial v_{i}\partial v_{b}\partial v_{c}} + \left(\frac{\partial^{3}\phi_{i}}{\partial v_{i}\partial v_{b}\partial v_{c}} * \frac{\partial^{4}f_{i+1}}{\partial v_{i}\partial v_{i}} \right) + \left(\frac{\partial^{2}\phi_{i}}{\partial v_{b}\partial v_{c}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{i}} \right) \\
+ \left(\frac{\partial^{2}\phi_{i}}{\partial v_{i}\partial v_{c}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{c}} \right) + \left(\frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial^{3}f_{i+1}}{\partial v_{i}\partial v_{b}\partial v_{c}} \right) \\
+ \left(\frac{\partial^{2}\phi_{i}}{\partial v_{i}\partial v_{b}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{c}} \right) + \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial^{3}f_{i+1}}{\partial v_{i}\partial v_{c}} \right) \\
+ \left(\frac{\partial^{2}\phi_{i}}{\partial v_{i}\partial v_{b}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{c}} \right) + \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial^{2}\phi_{i}}{\partial v_{i}\partial v_{c}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{i}} \right) \\
+ \left(\frac{\partial\phi_{i}}{\partial v_{i}\partial v_{b}} * \frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{c}} \right) + \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial^{2}\phi_{i}}{\partial v_{i}\partial v_{c}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{i}} \right) \\
+ \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{c}} \right) + \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial^{2}\phi_{i}}{\partial v_{i}\partial v_{c}} * \frac{\partial^{2}\phi_{i}}{\partial v_{i}\partial v_{c}} \right) \\
+ \left(\frac{\partial\phi_{i}}{\partial v_{b}} * \frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial\phi_{i}}{\partial v_{c}} * \frac{\partial\phi_{i}}{\partial v_{c}}$$

Which leads to

$$\begin{split} \frac{\partial}{\partial v_{z}} \left\{ \frac{\partial}{\partial v_{0}} \left[\frac{\partial}{\partial v_{0}} \left[\frac{\partial}{\partial v_{0}} \left(\frac{\partial}{\partial v_{0}} \right) \right] \right\} &= \frac{\partial^{4}f_{i+1}}{\partial v_{i}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{z}}{\partial v_{i}\partial v_{0}\partial v_{0}\partial v_{0}} * \frac{\partial^{2}f_{i+1}}{\partial v_{i}\partial v_{0}\partial v_{0}} * \frac{\partial^{2}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}} + \left[\frac{\partial^{2}\phi_{i}}{\partial v_{v}\partial v_{0}\partial v_{0}} * \frac{\partial^{2}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}} * \left[\frac{\partial^{2}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{2}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}} \right) \right] \right\} \\ &+ \left(\frac{\partial^{2}\phi_{i}}{\partial v_{v}\partial v_{v}\partial v_{0}} * \frac{\partial^{2}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}} + \left(\frac{\partial^{2}\phi_{i}}{\partial v_{v}} * \frac{\partial^{2}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}\partial v_{0}\partial v_{0}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}} + \left(\frac{\partial \phi_{i}}{\partial v_{v}\partial v_{0}\partial v_{0}} * \frac{\partial^{4}f_{i+1}}{\partial v_{v}\partial v_{0}\partial v_{0}} * \frac{\partial^{4}f_{i+1}$$

(4)