Third order equations from pages 46 and 48, and derived 4th order equations, of
`www.autodiff.org/Docs/euroad/17thEuroAdWorkshop-MuWang-HigherOrderReverseModeWithMPI.pdf`
ZTA, July 2016

1 Equations

Given that

$$\frac{\hat{\partial} f_i}{\hat{\partial} v_c} = \frac{\partial f_{i+1}}{\partial v_c} + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial f_{i+1}}{\partial v_i}\right) \tag{1}$$

Then

$$\frac{\hat{\partial}}{\hat{\partial} v_b}\left[\frac{\hat{\partial} f_i}{\hat{\partial} v_c}\right] = \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_c} + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial f_{i+1}}{\partial v_i}\right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_i}\right)$$
$$+ \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_c}\right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right) \tag{2}$$

It then follows that

$$\frac{\hat{\partial}}{\hat{\partial} v_a}\left[\frac{\hat{\partial}}{\hat{\partial} v_b}\left(\frac{\hat{\partial} f_i}{\hat{\partial} v_c}\right)\right] = \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_b \partial v_c} + \left(\frac{\partial^3 \phi_i}{\partial v_a \partial v_b \partial v_c} * \frac{\partial f_{i+1}}{\partial v_i}\right) + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_a \partial v_i}\right)$$
$$+ \left(\frac{\partial^2 \phi_i}{\partial v_a \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_i}\right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_b \partial v_i}\right)$$
$$+ \left(\frac{\partial^2 \phi_i}{\partial v_a \partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_c}\right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_i \partial v_c}\right)$$
$$+ \left\{\left[\left(\frac{\partial^2 \phi_i}{\partial v_a \partial v_c} * \frac{\partial \phi_i}{\partial v_b}\right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^2 \phi_i}{\partial v_a \partial v_b}\right)\right] * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right\}$$
$$+ \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_i \partial v_i}\right)$$
$$+ \frac{\partial \phi_i}{\partial v_a} * \left[\frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_c} + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_i}\right)\right.$$
$$\left. + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_c}\right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_i}\right)\right] \tag{3}$$

Which leads to

$$
\frac{\hat{\partial}}{\hat{\partial} v_z}\left\{\frac{\hat{\partial}}{\hat{\partial} v_a}\left[\frac{\hat{\partial}}{\hat{\partial} v_b}\left(\frac{\hat{\partial} f_i}{\hat{\partial} v_c}\right)\right]\right\} = \frac{\partial^4 f_{i+1}}{\partial v_z \partial v_a \partial v_b \partial v_c} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^4 f_{i+1}}{\partial v_i \partial v_a \partial v_b \partial v_c}\right)
$$

$$
+ \left(\frac{\partial^4 \phi_i}{\partial v_z \partial v_a \partial v_b \partial v_c} * \frac{\partial f_{i+1}}{\partial v_i}\right) + \left\{\frac{\partial^3 \phi_i}{\partial v_a \partial v_b \partial v_c} * \left[\frac{\partial^2 f_{i+1}}{\partial v_z \partial v_i} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right)\right]\right\}
$$

$$
+ \left(\frac{\partial^3 \phi_i}{\partial v_z \partial v_a \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_i}\right) + \left\{\frac{\partial^2 \phi_i}{\partial v_a \partial v_c} * \left[\frac{\partial^3 f_{i+1}}{\partial v_z \partial v_b \partial v_i} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_i}\right)\right]\right\}
$$

$$
+ \left(\frac{\partial^2 \phi_i}{\partial v_z \partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_b \partial v_i}\right) + \left\{\frac{\partial \phi_i}{\partial v_c} * \left[\frac{\partial^4 f_{i+1}}{\partial v_z \partial v_a \partial v_b \partial v_i} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^4 f_{i+1}}{\partial v_i \partial v_a \partial v_b \partial v_i}\right)\right]\right\}
$$

$$
+ \left(\frac{\partial^3 \phi_i}{\partial v_z \partial v_a \partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_c}\right) + \left\{\frac{\partial^2 \phi_i}{\partial v_a \partial v_b} * \left[\frac{\partial^3 f_{i+1}}{\partial v_z \partial v_i \partial v_c} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_c}\right)\right]\right\}
$$

$$
+ \left(\frac{\partial^2 \phi_i}{\partial v_z \partial v_b} * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_i \partial v_c}\right) + \left\{\frac{\partial \phi_i}{\partial v_b} * \left[\frac{\partial^4 f_{i+1}}{\partial v_z \partial v_a \partial v_i \partial v_c} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^4 f_{i+1}}{\partial v_i \partial v_a \partial v_i \partial v_c}\right)\right]\right\}
$$

$$
+ \left\{\left[\left(\frac{\partial^3 \phi_i}{\partial v_z \partial v_a \partial v_b} * \frac{\partial \phi_i}{\partial v_c}\right) + \left(\frac{\partial^2 \phi_i}{\partial v_a \partial v_b} * \frac{\partial^2 \phi_i}{\partial v_z \partial v_c}\right)\right] * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right\}
$$

$$
+ \left\{\frac{\partial^2 \phi_i}{\partial v_a \partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \left[\frac{\partial^3 f_{i+1}}{\partial v_z \partial v_i \partial v_i} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_i}\right)\right]\right\}
$$

$$
+ \left\{\left[\left(\frac{\partial^2 \phi_i}{\partial v_z \partial v_b} * \frac{\partial^2 \phi_i}{\partial v_a \partial v_c}\right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^3 \phi_i}{\partial v_z \partial v_a \partial v_c}\right)\right] * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right\}
$$

$$
+ \left\{\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 \phi_i}{\partial v_a \partial v_c} * \left[\frac{\partial^3 f_{i+1}}{\partial v_z \partial v_i \partial v_i} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_i}\right)\right]\right\}
$$

$$
+ \left\{\left[\left(\frac{\partial^2 \phi_i}{\partial v_z \partial v_b} * \frac{\partial \phi_i}{\partial v_c}\right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 \phi_i}{\partial v_z \partial v_c}\right)\right] * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_i \partial v_i}\right\}
$$

$$
+ \left\{\frac{\partial \phi_i}{\partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \left[\frac{\partial^4 f_{i+1}}{\partial v_z \partial v_a \partial v_i \partial v_i} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^4 f_{i+1}}{\partial v_i \partial v_a \partial v_i \partial v_i}\right)\right]\right\}
$$

$$
+ \frac{\partial^2 \phi_i}{\partial v_z \partial v_a} * \left[\frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_c} + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_i}\right)\right.
$$

$$
+ \left.\left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_c}\right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_i}\right)\right]
$$

$$
+ W
$$

$$
+ \frac{\partial \phi_i}{\partial v_a} * \left[\frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_c} + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_i}\right)\right.
$$

$$
+ \left.\left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_c}\right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_i}\right)\right]
$$

$$
+ Y
$$

$$
+ \left(\frac{num}{den} * \frac{num}{den}\right) + \left\{\frac{num}{den} * \left[\frac{num}{den} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{num}{den}\right)\right]\right\}
$$

$$
+ Z
$$

$$
+ \left\{\left[\left(\frac{num}{den} * \frac{num}{den}\right) + \left(\frac{num}{den} * \frac{num}{den}\right)\right] * \frac{num}{den}\right\}
$$

$$
+ \left\{\frac{num}{den} * \frac{num}{den} * \left[\frac{num}{den} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{num}{den}\right)\right]\right\}
$$

$$
+ ...
$$

(4)