Third order equations from pages 46 and 48, and derived 4th order equations, of
www.autodiff.org/Docs/euroad/17thEuroAdWorkshop-MuWang-HigherOrderReverseModeWithMPI.pdf
ZTA, July 2016

1 Equations

Given that

$$\frac{\hat{\partial} f_i}{\hat{\partial} v_c} = \frac{\partial f_{i+1}}{\partial v_c} + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial f_{i+1}}{\partial v_i} \right) \tag{1}$$

Then

$$\begin{aligned} \frac{\hat{\partial}}{\hat{\partial} v_b} \left[\frac{\hat{\partial} f_i}{\hat{\partial} v_c} \right] = & \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_c} + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_c} \right) \\ & + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial f_{i+1}}{\partial v_i} \right) \\ & + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_i} \right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i} \right) \end{aligned} \tag{2}$$

It then follows that

$$\begin{aligned} \frac{\hat{\partial}}{\hat{\partial} v_a} \left[\frac{\hat{\partial}}{\hat{\partial} v_b} \left(\frac{\hat{\partial} f_i}{\hat{\partial} v_c} \right) \right] = & \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_b \partial v_c} + \left(\frac{\partial^3 \phi_i}{\partial v_a \partial v_b \partial v_c} * \frac{\partial f_{i+1}}{\partial v_i} \right) + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_a \partial v_i} \right) \\ & + \left(\frac{\partial^2 \phi_i}{\partial v_a \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_i} \right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_b \partial v_i} \right) \\ & + \left(\frac{\partial^2 \phi_i}{\partial v_a \partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_c} \right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_i \partial v_c} \right) \\ & + \left(\frac{\partial^2 \phi_i}{\partial v_a \partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i} \right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 \phi_i}{\partial v_a \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i} \right) \\ & + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_a \partial v_i \partial v_i} \right) \\ & + \frac{\partial \phi_i}{\partial v_a} * \left[\frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_c} + \left(\frac{\partial^3 \phi_i}{\partial v_i \partial v_b \partial v_c} * \frac{\partial f_{i+1}}{\partial v_i} \right) + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i} \right) \right. \\ & + \left(\frac{\partial^2 \phi_i}{\partial v_i \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_i} \right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_i} \right) \\ & + \left(\frac{\partial^2 \phi_i}{\partial v_i \partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_c} \right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_c} \right) \\ & + \left(\frac{\partial^2 \phi_i}{\partial v_i \partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i} \right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 \phi_i}{\partial v_i \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i} \right) \\ & \left. + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_i} \right) \right] \end{aligned} \tag{3}$$

Which leads to

$$
\begin{aligned}
\frac{\hat{\partial}}{\hat{\partial} v_z}\left\{\frac{\hat{\partial}}{\hat{\partial} v_a}\left[\frac{\hat{\partial}}{\hat{\partial} v_b}\left(\frac{\hat{\partial} f_i}{\hat{\partial} v_c}\right)\right]\right\} &= \frac{\partial^4 f_{i+1}}{\partial v_z \partial v_a \partial v_b \partial v_c} + \left(\frac{\partial \phi_i}{\partial v_z} * \frac{\partial^4 f_{i+1}}{\partial v_i \partial v_a \partial v_b \partial v_c}\right) \\
&+ Y \\
&+ Z \\
&+ ... \\
&+ \frac{\partial^2 \phi_i}{\partial v_z \partial v_a} * \left[\frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_c} + \left(\frac{\partial^3 \phi_i}{\partial v_i \partial v_b \partial v_c} * \frac{\partial f_{i+1}}{\partial v_i}\right) + \left(\frac{\partial^2 \phi_i}{\partial v_b \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right)\right. \\
&+ \left(\frac{\partial^2 \phi_i}{\partial v_i \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_b \partial v_i}\right) + \left(\frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_b \partial v_i}\right) \\
&+ \left(\frac{\partial^2 \phi_i}{\partial v_i \partial v_b} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_c}\right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_c}\right) \\
&+ \left(\frac{\partial^2 \phi_i}{\partial v_i \partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right) + \left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial^2 \phi_i}{\partial v_i \partial v_c} * \frac{\partial^2 f_{i+1}}{\partial v_i \partial v_i}\right) \\
&+ \left.\left(\frac{\partial \phi_i}{\partial v_b} * \frac{\partial \phi_i}{\partial v_c} * \frac{\partial^3 f_{i+1}}{\partial v_i \partial v_i \partial v_i}\right)\right] \\
&+ W \\
&+ U \\
&+ ...
\end{aligned}
\tag{4}
$$