Chain Rule of Derivative

The Chain Rule is a fundamental theorem in calculus that is used to find the derivative of a composite function. When a function is composed of other functions, the chain rule allows us to differentiate it with respect to the innermost variable.

Formal defn

If $y = f(g(x))$ where $y = f(u)$ and $u = g(x)$, then the derivative of y with respect to x

$$\boxed{\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}}$$

In simpler terms, this can be expressed as

$$\boxed{\frac{\partial y}{\partial x} = f'(g(x)) \cdot g'(x)}$$

This means that to differentiate a composite function, you first differentiate the outer function with respect to its inner function and then multiply by the derivative of the inner function.

Example

$$y = (\underbrace{3x^2 + 2x + 1}_{u})^5$$

Step 1 : Identify the outer and Inner function

Outer function $f(u) = u^5$

Inner function $u = g(x) = 3x^2 + 2x + 1$

Step 2 : Differentiate the outer function

$$f'(u) = \frac{\partial f}{\partial u} = u^5 = 5u^4 \qquad \frac{\partial(x^5)}{\partial x} = 5x^4$$

Step 3 : Differentiate the Inner function

$$g'(x) = \frac{\partial u}{\partial x} = \frac{\partial(g(x))}{\partial x} = \frac{\partial(3x^2+2x+1)}{\partial x} = 6x+2$$

Step 4 :
$$\frac{\partial y}{\partial x} = f'(g(x)) \cdot g'(x)$$

$$= 5u^4 \cdot (6x+2)$$

$$\boxed{\frac{\partial y}{\partial x} = 5(3x^2+2x+1)^4 \cdot (6x+2)}$$

② Trignometric Function

$y = \sin(4x^3+x)$

$u = g(x) = 4x^3 + x \Rightarrow y = f(g(x))$

$f(x) = \sin(u)$

Step 1 : Identify the outer and Inner function

Outer function : $f(u) = \sin(u)$

Inner function $u = g(x) = 4x^3 + x$

Step 2 :

$$f'(u) = \frac{\partial f}{\partial u} = \cos(u)$$

Step 3 :

$$g'(x) = \frac{\partial u}{\partial x} = \frac{\partial(4x^3+x)}{\partial x} = 12x^2+1$$

Step 4 : Chain Rule

$$\frac{\partial y}{\partial x} = f'(g(x)) \cdot g'(x) = \cos(4x^3 + x) \cdot (12x^2 + 1)$$

$$\boxed{\frac{\partial y}{\partial x} = \cos(4x^3 + x) \cdot (12x^2 + 1)}$$

Example : Composition Of Three Functions

$$y = \underbrace{\sqrt{\sin(\underbrace{3x}_{v})}}_{u}$$

{3 Composite functions}

Step 1 : Identify the functions

Outer function : $f(u) = \sqrt{u} = u^{1/2}$ $\quad u = \sin(3x)$

Middle function : $g(v) = \sin(v)$ $\quad v = 3x$

Inner function $v = h(x) = 3x$.

Step 2 :

$$f'(u) = \frac{1}{2} u^{-1/2} = \frac{1}{2\sqrt{u}}$$

$$g'(v) = \cos v$$

$$h'(x) = 3$$

Step 3 : Apply the Chain Rule

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$u = \sin(3x)$

$$= f'(g(h(x))) \cdot g'((h(x)) \cdot h'(x)$$

$$= \frac{1}{2\sqrt{\sin(3x)}} \cdot \cos(3x) \cdot 3$$

$$\boxed{\frac{\partial y}{\partial x} = \frac{3\cos(3x)}{2\sqrt{\sin(3x)}}}$$

The chain rule is a powerful tool in calculus for differentiating composite functions. By breaking down the differentiation process into manageable steps, it allows us to compute derivatives of complex expressions efficiently. Understanding and applying the chain rule is essential for solving a wide range of problems in mathematics, physics, engineering, economics, and data science.

Applications Of Chain Rule In Data Science

1) Backpropogation In Neural N/W : Training Deep Learning Models

Chain Rule for Calculating the Gradients of Loss function with respect to the weights that are Initialized

2) Gradient Descent Optimization :

Linear Regression → the goal is to minimize a cost function

Update slopes ⟶ Gradient Descent optimizers ⟶ Chain Rule.

③ Chain Rule In F.E

$$z = \log(x^2+y^2)$$

$x \quad z$

$$\boxed{\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x}$$

⇒ How small changes in x & y affect z, Chain Rule.

$y \quad z$

$$\boxed{\frac{\partial z}{\partial y} = \frac{1}{(x^2+y^2)} \cdot 2y}$$

④ Regularization Techniques : Overfitting And Underfitting.