If the optical constants of doubly refracting media are to be observed, the plates are to be mounted for rotation around an axis parallel to the slit and passing symmetrically between the faces. Otherwise the wedge effect is liable to produce excessive distortion in the results. If columns are to be used, plane parallel faces are to be cut at a definite angle to the optic axes, as the columns cannot be rotated.

The position of minimum thickness of the plate or crystal may be recognized by the aid of two rotations, at right angles to each other, and the reversal of motion of the interference rings. It is thus easy to place the axis of rotation at right angles to the beam.

The interferometer used in the above experiments was an improvised apparatus, made of 1/4 inch gas-pipe through which water continually circulated. The angle between the component or interfering beams being but 30°, or less, the micrometer was sufficiently close to the observing telescope to admit of easy manipulation. The arms have since been elongated to over 1 metre in length each, and the available free depth below the beam is 15 centimetres. There seems to be no difficulty of increasing these dimensions in any degree. In spite of the lightness of the apparatus, tremor of ellipses does not seriously hamper the observations. Suppose now a glass column 1 metre long with plane parallel end faces is placed in one of the beams. The micrometer displacement which restores the centre of ellipses to the fiducial sodium line will be of the order of 50 centimetres, measurable, so far as the interferences are concerned, to 5×10^{-5} centimetre. Hence $\mu-1$ must be measurable with an accuracy of 1 part in a million.

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LXXXIX. *On the Shape of the Capillary Surface formed by the External Contact of a Liquid with a Cylinder of Large Radius.* By ALLAN FERGUSON, *B.Sc.* (*Lond.*), *Assistant Lecturer in Physics in the University College of North Wales, Bangor**.

THE following analysis is an attempt to find an approximate value for the first integral of the differential equation of the capillary surface formed when a cylinder is dipped vertically into a liquid. The writer's attention has

* Communicated by Prof. E. Taylor Jones.

been recently devoted to the photography of the capillary surface formed by pendent drops*, and it seemed possible that photographs of the meridional curve of the capillary surface in external contact with a cylinder might give useful information as to the angle of contact of a liquid and solid. A few preliminary experiments have been made which indicate the feasibility of the formulæ developed below. The present paper will, however, be devoted to the discussion of these formulæ, leaving the experimental results for future presentation.

In the paper referred to above, several approximations to the outline of the capillary curve have been discussed. To these may be added the equations

$$a^2=q^2+\frac{2qa^2}{\mu}-\frac{a^2}{3e}(2\sqrt{2}-1),$$

and

$$a=\frac{k}{\sqrt{2}\cos\frac{\omega}{2}}+\frac{a^2}{\mu\sqrt{2}\cos\frac{\omega}{2}}-\frac{a^2}{3l\sqrt{2}\cos^2\frac{\omega}{2}}\left(1-\sin^3\frac{\omega}{2}\right),$$

originally due to Poisson†, and modified by Magie‡. All of them have reference to the capillary surfaces formed by drops, pendent or sessile; the writer has not found any discussion of the particular capillary curve which forms the subject of the present paper.

In the discussion which follows it is important to bear in mind that the angle of contact of the liquid and solid is assumed to be acute; that the radius (r) of the cylinder is assumed to be large in comparison with the other quantities to be measured; and in the discussion of the signs of the various quantities involved—a point which has an important bearing on the shape of the resulting formulæ—the signs will always be chosen in accordance with the diagram given in fig. 1 below.

Let Y′ O′ O″ Y″ be the trace of the cylindrical surface, of which OY is the axis, immersed in the fluid. Let OY be vertical and OX any horizontal line drawn in the "level" surface of the fluid.

The capillary surface will be a surface of revolution about

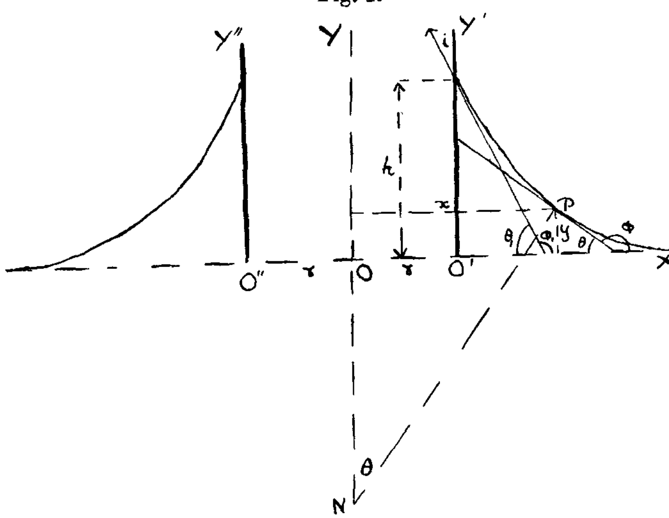
* Phil. Mag. March 1912.
† *Nouvelle Théorie*, p. 217.
‡ Phil. Mag. vol. xxvi. (1888).

OY, and at any point P (x, y) the equation to the surface is

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{y}{a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

where $a^2 = \frac{T}{g\rho}$, and a is therefore of the dimensions of a

Fig. 1.



length. R_1 and R_2, the principal radii of curvature at P, are given by

$$R_1 = \frac{(1+y_1^2)^{3/2}}{y_2},$$

and

$$R_2 = PN = \frac{x}{\sin\theta} = \frac{x\sqrt{1+y_1^2}}{-y_1},$$

remembering that

$$\tan\theta = -\tan\phi = -y_1.$$

Hence, substituting in (i.) and clearing of fractions we have for the differential equation of the meridional curve

$$a^2 x y_2 - a^2 y_1(1+y_1^2) = xy(1+y_1^2)^{3/2},$$

or

$$a^2 x \frac{dp}{dx} - a^2 p(1+p^2) = xy(1+p^2)^{3/2}. \quad . \quad . \quad . \quad \text{(ii.)}$$

Referred to parallel axes through O′ as origin, (ii.) becomes

$$a^2(x-r)\frac{dp}{dx}-a^2p(1+p^2)=(x-r)y(1+p^2)^{3/2}, \quad . \quad \text{(iii.)}$$

or

$$\frac{a^2x}{r}\frac{dp}{dx}-a^2\frac{dp}{dx}-\frac{a^2p}{r}(1+p^2)=\frac{xy}{r}(1+p^2)^{3/2}-y(1+p^2)^{3/2}, \text{(iv.)}$$

giving when $r=\infty$

$$a^2\frac{dp}{dx}=y(1+p^2)^{3/2}, \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}$$

or

$$\frac{y}{a^2}=\frac{1}{\text{R}_1},$$

which is obviously the correct equation to the capillary surface in contact with a plane wall.

To obtain an approximate solution of (iv.) on the assumption that r is large, take the value of $\frac{dp}{dx}$ in (v.)—which is exactly true when r is infinite—and substitute in the first term on the left-hand side of (iv.), which is a small term. We thus obtain

$$a^2\frac{dp}{dx}+\frac{a^2p}{r}(1+p^2)=y(1+p^2)^{3/2},$$

or putting $\frac{dp}{dx}=p\cdot\frac{dp}{dy}$,

$$\frac{dp}{dy}+\frac{1+p^2}{r}=\frac{(1+p^2)^{3/2}}{a^2p}y. \quad . \quad . \quad . \quad . \quad \text{(vi.)}$$

Again, if r be infinite, (vi.) becomes

$$\frac{dp}{dy}=\frac{(1+p^2)^{3/2}}{a^2p}y,$$

giving on integration,

$$y^2=2a^2\left(1-\frac{1}{\sqrt{1+p^2}}\right), \quad . \quad . \quad . \quad . \quad \text{(vii.)}$$

the integration constant being obtained from the fact that y and p vanish together.

Also, when $y=h$, $p=\tan\phi_1=-\cot i$ (*vide* fig. 1) giving

$$h^2=2a^2(1-\sin i), \quad . \quad . \quad . \quad . \quad \text{(viii.)}$$

the usual expression for the height to which a liquid rises against a plane wall.

Now write (vi.) in the form

$$\frac{a^2p}{(1+p^2)^{3/2}}\frac{dp}{dy}+\frac{a^2}{r}\cdot\frac{p}{\sqrt{1+p^2}}=y, \quad . \quad . \quad . \quad \text{(ix.)}$$

and substitute in the second term on the left-hand side of (ix.) the value of $\frac{p}{\sqrt{1+p^2}}$ given by (vii.).

We have at once from (vii.)

$$p=\pm y\frac{\sqrt{4a^2-y^2}}{2a^2-y^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(x.)}$$

and if the angle of contact be acute y^2 is always less than $2a^2$ (*vide* equation (viii.)); also from fig. 1, p is continuously negative. Hence in (x.) above, the negative sign must be chosen, and

$$\frac{p}{\sqrt{1+p^2}}=-\frac{y\sqrt{4a^2-y^2}}{2a^2}.$$

Substituting in (ix.) we have

$$\frac{a^2pdp}{(1+p^2)^{3/2}}=\left(y+\frac{y\sqrt{4a^2-y^2}}{2r}\right)dy,$$

which, integrating, and remembering that when $y=0, p=0$, gives

$$a^2\left(1-\frac{1}{\sqrt{1+p^2}}\right)-\frac{4a^3}{3r}=\frac{y^2}{2}-\frac{1}{6r}(4a^2-y^2)^{3/2}. \quad \text{(x. } a\text{)}$$

As before, when $y=h$, $p=-\cot i$, giving

$$a^2(1-\sin i)-\frac{4a^3}{3r}=\frac{h^2}{2}-\frac{4a^3}{3r}\left(1-\frac{h^2}{4a^2}\right)^{3/2}. \quad . \quad . \quad \text{(xi.)}$$

In this equation, h, r, and a^2 are capable of experimental determination, and i can therefore be calculated.

Equation (xi.) can be put into a very simple form by expanding the quantity under the radical sign, and neglecting

$\frac{h^4}{16a^4}$ and higher powers; the expansion is legitimate, as h^2 is always less than $2a^2$ for acute angles of contact. This gives at once

$$2a^2(1-\sin i)=h^2\left(1+\frac{a}{r}\right),$$

or, if h_0 be the height to which the liquid rises against a *plane* wall, then, very approximately,

$$h^2=h_0{}^2\left(1-\frac{a}{r}\right).$$

Equation (xi.) should, however, always be used in computing $\sin i$ from numerical data.

A solution of (vi.) in a different form can be obtained in the following manner.

Putting $p=\tan\phi$, substituting, and taking, for reasons similar to those already given, the negative sign in extracting the square root of $\sec^6\phi$, (vi.) becomes

$$\frac{d\phi}{dy}+\frac{1}{r}=-\frac{4}{a^2\sin\phi}. \quad . \quad . \quad . \quad . \quad \text{(xii.)}$$

Putting the right-hand side of (xii.) equal to zero, and integrating,

$$r\phi=-y+c. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiii.)}$$

Assuming c to be a function of y, as in the ordinary method of variation of parameters,

$$\frac{d\phi}{dy}=\frac{1}{r}\frac{dc}{dy}-\frac{1}{r}; \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv.)}$$

substituting in (xii.) the values of c and $\frac{d\phi}{dy}$ given by (xiii.) and (xiv.) we have

$$\frac{1}{r}\frac{dc}{dy}=-\frac{y}{a^2\sin\frac{c-y}{r}},$$

or

$$ydy=-\frac{a^2}{r}\sin\frac{c-y}{r}\,dc,$$

$$=-\left(\frac{a^2}{r}\sin\frac{c}{r}-\frac{y}{r^2}\cos\frac{c}{r}\right)dc.$$

If we neglect the term involving $\frac{y}{r^2}$ on the right-hand side of the above equation, we then have

$$ydy = -\frac{a^2}{r}\sin\frac{c}{r}\,.\,dc, \quad . \quad . \quad . \quad . \quad \text{(xv.)}$$

giving on integration

$$c = r\cos^{-1}\frac{y^2 - c_1}{2a^2}.$$

Substituting in (xiii.), we obtain

$$\frac{y^2 - c_1}{2a^2} = \cos\left(\phi + \frac{y}{r}\right),$$

and when $y=0$, $\phi=\pi$, giving

$$\frac{y^2 - 2a^2}{2a^2} = \cos\left(\phi + \frac{y}{r}\right) = \cos\left(\tan^{-1}p + \frac{y}{r}\right). \quad \text{(xvi.)}$$

As before, when $y=h$, $p=-\cot i$, and therefore

$$\frac{2a^2 - h^2}{2a^2} = \sin\left(i + \frac{h}{r}\right), \quad . \quad . \quad . \quad \text{(xvii.)}$$

a convenient equation to determine i.

Equations (xvii.) and (xi.) represent approximations of different orders of accuracy, of which (xi.) is the closer. The neglect of the term involving $\frac{y}{r^2}$ in forming equation (xv.) is evidently of a lower order of accuracy than the processes which result in equation (xi.). The difference between the two results can readily be shown by computing the value of $\frac{2a^2 - h^2}{2a^2}$ as given for the two equations.

If we take, for simplicity, a liquid of zero contact-angle, a simple calculation shows that

$$\left(\frac{2a^2 - h^2}{2a^2}\right)_{\text{xvii}} - \left(\frac{2a^2 - h^2}{2a^2}\right)_{\text{xi}}$$
$$= \frac{h}{r}\left(1 - \frac{h}{2a}\right),$$

a small difference which becomes less important as r increases. The preliminary experiments which have been carried out are in agreement with this result. The approximation of equation (xvii.) is specially convenient, as a knowledge of the slope of the tangent at any point enables a^2—and therefore the surface-tension—to be calculated by means of the simple equation (xvi.), with much greater ease than would be the case were equation (x. a) to be used.

A knowledge of a^2 is, of course, necessary before i can be computed. This can be obtained from a photograph of the capillary curve by means of equation (xvi.) above, or directly by measuring, at the same time that the photograph is taken, the weight (mg) necessary to balance the pull due to surface-tension on a vertical plate of known perimeter p, just touching the surface of the liquid. We then have

$$p\mathrm{T}\cos i = mg,$$

or

$$a^2 \cos i = \frac{m}{p\rho} = k, \quad . \quad . \quad . \quad . \quad \text{(xviii.)}$$

where k is a known quantity. We can now eliminate a^2 between (xviii.) and (xi.), or (xviii.) and (xvii.). The first elimination gives a cubic for i; the second gives a much simpler form, thus:—

Substituting for a^2 in (xvii.), we have

$$1 - \frac{h^2 \cos i}{2k} = \sin\left(i + \frac{h}{r}\right),$$

$$= \sin i + \frac{h}{r} \cos i,$$

whence

$$\sin i + \mathrm{A} \cos i - 1 = 0,$$

where

$$\mathrm{A} \equiv \frac{h}{r} + \frac{h^2}{2k}.$$

This gives finally

$$\cos i = \frac{2\mathrm{A}}{\mathrm{A}^2 + 1},$$

giving i in terms of known quantities.

Having determined $\cos i$, equation (xviii.) gives a^2, or T.

In a future communication the writer hopes to give the results of experiments based on the formulæ detailed above.

University College of North Wales, Bangor,
June 1912.