# On numerical schemes for phase-field models for electrowetting with electrolyte solutions

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We present an energy-stable, decoupled discrete scheme for a recent model (see [1]) supposed to describe electrokinetic phenomena in two-phase flow with general mass densities. This model couples momentum and Cahn–Hilliard type phase-field equations with Nernst–Planck equations for ion density evolution and an elliptic transmission problem for the electrostatic potential.
The transport velocities in our scheme are based on the old velocity field updated via a discrete time integration of the force densities. This allows to split the equations into three blocks which can be treated sequentially: The phase-field equation, the equations for ion transport and electrostatic potential, and the Navier–Stokes type equations. By establishing a discrete counterpart of the continuous energy estimate, we are able to prove the stability of the scheme.



## 1 Introduction

Electric fields may change the wetting behavior of conductive liquids or electrolyte solutions. In 2012 E. Campillo-Funollet, G. Grün and F. Klingbeil used energy-methods – namely Onsager's variational principle – to derive a thermodynamically consistent model (see [1]) describing these phenomena. Considering two immiscible fluids and $M$ species in a fixed domain $\Omega$ and an electrostatic potential $V$ in a larger domain $\Omega^* \supset \Omega$, they defined the energy of the system as

$$\int_\Omega \tfrac{1}{2}\rho(\phi)|\mathbf{u}|^2 + \int_\Omega \left[\tfrac{\delta}{2}|\nabla\phi|^2 + \tfrac{1}{\delta}W(\phi)\right] + \sum_{i=1}^{M}\int_\Omega \left[g(\omega_i) + \beta_i(\phi)\omega_i\right] + \int_{\Omega^*} \tfrac{1}{2}\varepsilon(\mathbf{x},\phi)|\nabla V|^2 + \int_{\partial\Omega}\gamma(\phi)\,, \tag{1}$$

with the volume-averaged velocity-field $\mathbf{u}$ and the mass density $\rho$ which depends linearly on the phase-field parameter $\phi$. The evolution of the fluid-fluid-interface is governed by the Ginzburg-Landau free energy $\left[\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}W(\phi)\right]$ with a polynomial double-well potential $W(\phi) = \frac{1}{4}\left(1-\phi^2\right)^2$ and a small parameter $\delta$ controlling the width of the interface region. For the ease of notation we set $\delta \equiv 1$. The evolution of the number densities of the species $\{\omega_i\}_{i=1,\dots,M}$ is governed by the entropic function $g(s) := s\log s - s$ and bounded functions $\beta_i$ referring to the solubility of the species. The dielectric permittivity $\varepsilon$ may depend on the fluids and may vary between $\Omega$ and $\Omega^*\backslash\Omega$. The last contribution to the energy is the fluid-solid interfacial energy which prescribes the contact angle. Based on this energy, the following set of equations was derived:

$$\rho\partial_t\mathbf{u} + \left((\rho\mathbf{u} + \rho'\mathbf{j})\cdot\nabla\right)\mathbf{u} - \operatorname{div}\left\{2\eta(\phi)\mathbf{Du}\right\} + \nabla p = \mu_\phi\nabla\phi + \sum_{j=1}^{M}\mu_{\omega_j}\nabla\omega_j - \sum_{j=1}^{M} z_j\omega_j\nabla V \quad \text{in } \Omega\,, \tag{2a}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{2b}$$

$$\partial_t\phi + \mathbf{u}\cdot\nabla\phi - \operatorname{div}\left\{m(\phi)\nabla\mu_\phi\right\} = 0 \quad \text{in } \Omega\,, \tag{2c}$$

$$\mu_\phi = W'(\phi) - \Delta\phi + \sum_{j=1}^{M}\beta_j'(\phi)\omega_j - \tfrac{1}{2}\partial_\phi\varepsilon|\nabla V|^2 \quad \text{in } \Omega\,, \tag{2d}$$

$$\partial_t\omega_i + \mathbf{u}\cdot\nabla\omega_i - \operatorname{div}\left\{k_i(\phi)\tfrac{1}{g''(\omega_i)}\nabla\left(\mu_{\omega_i} + z_iV\right)\right\} = R_i(\phi,\omega_1,\dots,\omega_M,V) \quad \text{in } \Omega\,, \tag{2e}$$

$$\mu_{\omega_i} = g'(\omega_i) + \beta_i(\phi) \quad \text{in } \Omega\,, \tag{2f}$$

$$-\operatorname{div}\left\{\varepsilon\nabla V\right\} = \sum_{j=1}^{M} z_j\omega_j\chi_\Omega \quad \text{in } \Omega^*\,, \tag{2g}$$

for all $i \in \{1,\dots,M\}$, together with the boundary conditions $\mathbf{u} = 0$, $\frac{\partial\phi}{\partial\mathbf{n}} = -\gamma'(\phi)$, $\frac{\partial\mu_{\omega_1}}{\partial\mathbf{n}} = \dots = \frac{\partial\mu_{\omega_M}}{\partial\mathbf{n}} = \frac{\partial\mu_\phi}{\partial\mathbf{n}} = 0$ on $\partial\Omega$ and $V = \bar{V}$ on $\partial\Omega^*$. We used the notation $\mathbf{j} := -m(\phi)\nabla\mu_\phi$ and denoted the chemical potentials by $\mu_\phi$ and $\{\mu_{\omega_i}\}_{i=1,\dots,M}$, the mobility by $m$, the viscosity by $\eta$ and the charge of the different species by $z_i$ for $i = 1,\dots,M$. To describe the reactions

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between the species a generalization of the mass action law was applied. Denoting the stoichiometric coefficients by $\zeta_i$, the reaction rates can be written as

$$R_i\left(\phi,\omega_1,...,\omega_M,V\right) := \zeta_i \left[ e^{\sum_{\zeta_j<0}|\zeta_j|\left(\mu_{\omega_j}+z_jV\right)} - e^{\sum_{\zeta_j\geq 0}\zeta_j\left(\mu_{\omega_j}+z_jV\right)} \right]. \tag{3}$$

As this model consists of multiple coupled, non-linear equations, a discrete scheme has to be judged not only by its stability but also by its computational costs. To accommodate those demands, we present a discrete scheme for the case of equal dielectric permittivities, which allows to split the equations into three blocks which can be treated sequentially (see Section 2). In Section 3, we investigate the scheme with regard to stability and non-negativity of the species concentrations. Simulations showing the practicability of this approach are presented in Section 4.

## 2 Discrete scheme

Concerning the electrostatic potential we will allow for inhomogeneous Dirichlet boundary data on $\Gamma_D \subset \partial\Omega^*$ and homogeneous Neumann boundary data on $\partial\Omega^*\backslash\Gamma_D$. We will assume that there exists a sufficiently smooth continuation of the Dirichlet data on $\Omega^*$ (again denoted by $\bar{V}$) and that $\operatorname{dist}\left(\Omega,\Gamma_D\right) \geq c > h > 0$.
We divide the time interval $I := [0,T)$ in subintervals $I_n := [t_n, t_{n+1})$ with $t_{n+1} := t_n + \tau_n$ for time increments $\tau_n > 0$ and $n = 0,...,N-1$. For simplicity, we take $\tau_n \equiv \tau$ for $n = 0,...,N-1$. Furthermore, we assume $\mathcal{T}_h$ ($\mathcal{T}_h^*$) to be a quasiuniform triangulation on $\Omega$ ($\Omega^*$) with simplicial elements in the sense of [2] with $\mathcal{T}_h \subset \mathcal{T}_h^*$ and define the following finite dimensional spaces: For the approximation of the phase-field $\phi$, the species concentrations $\{\omega_i\}_{i=1,...,M}$ and the chemical potentials $\{\mu_{\omega_i}\}_{i=1,...,M}$ and $\mu_\phi$ we introduce the space $U_h$ of continuous, piecewise linear finite element functions on $\mathcal{T}_h$. For the approximation of the electrostatic potential $V$ we use the space $U_{0,h}^*$ defined as the space of continuous, piecewise linear finite element functions on $\mathcal{T}_h^*$ satisfying $\psi^* = 0$ on $\Gamma_D$ for all $\psi^* \in U_{0,h}^*$. We want to emphasize that $U_{0,h}^*$ restricted to $\mathcal{T}_h$ is $U_h$. The velocity field $\mathbf{u}$ and the pressure $p$ are approximated by standard Taylor–Hood elements with $\mathbf{u} = 0$ on $\partial\Omega$. We denote the corresponding spaces by $\mathbf{W}_h$ and $S_h$. The nodal interpolation operator projecting $C^0(\Omega)$ onto $U_h$ is denoted by $\mathcal{I}_h\{.\}$.
As the derivatives of the entropic function $g(s)$ become singular for $s \searrow 0$, we introduce some regularization. Following [3] we define $g_\nu$ such that $g_\nu''(s) = \max\{\nu, s\}^{-1}$ for all $s \in \mathbb{R}$. To allow for a chain rule of the form $\left(g''(s)\right)^{-1}\nabla g'(s) = \nabla s$, we introduce the operator $\Xi_\nu(s)$ which approximates the inverse of $g_\nu''$ locally by a diagonal matrix. The exact definition of this operator can be found e.g. in [3] and [4]. As there is no mechanism ensuring $\omega_i \geq 0$ ($i = 1,...,M$), we apply a third regularization. Defining a smooth cutoff function $r$ satisfying $r(s) = s$ for all $s \in \mathbb{R}_0^+$ and $r > -C$ for some $C > 0$, we replace $\beta_i(\phi)\omega_i$ by $\beta_i(\phi)\, r(\omega_i)$ for all $i$. We also apply a cut-off function on the mass density to guarantee $\rho > 0$ (cf. [5]).
Denoting the difference quotient $\frac{f(a^{n+1})-f(a^n)}{a^{n+1}-a^n}$ by $\frac{\delta f}{\delta a}\left(a^{n+1},a^n\right)$ and the backward time difference quotient by $\partial_\tau^- a^{n+1} := \frac{1}{\tau}\left(a^{n+1}-a^n\right)$, we end up with the following scheme:
For given $\phi^n$, $\{\omega_i^n\}_{i=1,...,M}$ in $U_h$ and $\mathbf{u}^n \in \mathbf{W}_h$, we first compute $\phi^{n+1}$, $\mu_\phi^{n+1} \in U_h$ solving

$$\int_\Omega \mathcal{I}_h\left\{\partial_\tau^-\phi^{n+1}\theta\right\} - \int_\Omega \left\langle \mathbf{u}_\phi^\star, \nabla\theta\right\rangle \phi^n + \int_\Omega \mathcal{I}_h\left\{m\left(\phi^n\right)\right\}\left\langle \nabla\mu_\phi^{n+1}, \nabla\theta\right\rangle = 0 \quad \forall\theta \in U_h\,, \tag{4a}$$

$$\begin{aligned}\int_\Omega \mathcal{I}_h\left\{\mu_\phi^{n+1}\psi\right\} &= \int_\Omega \left\langle\nabla\phi^{n+1},\nabla\psi\right\rangle + \int_\Omega \mathcal{I}_h\left\{W'\left(\phi^{n+1},\phi^n\right)\psi\right\} \\ &\quad + \sum_{j=1}^M \int_\Omega \mathcal{I}_h\left\{\tfrac{\delta\beta_j}{\delta\phi}\left(\phi^{n+1},\phi^n\right) r\left(\omega_j^n\right)\psi\right\} + \int_{\partial\Omega}\mathcal{I}_h\left\{\tfrac{\delta\gamma}{\delta\phi}\left(\phi^{n+1},\phi^n\right)\psi\right\} \quad \forall\psi\in U_h\,,\end{aligned} \tag{4b}$$

with $\mathbf{u}_\phi^\star := \mathbf{u}^n - \tau\frac{\phi^n}{\rho^n}\nabla\mu_\phi^{n+1}$ and $W'(.,.)$ denoting a convex-concave decomposition of the double-well potential. Other suitable discretizations of the $W'$ were discussed in [5].
In a second step we take care of the species and the electrostatic potential by computing $\left\{\omega_i^{n+1}\right\}_{i=1,...,M}$, $\left\{\mu_{\omega_i}^{n+1}\right\}_{i=1,...,M} \in \left(U_h\right)^M$ and $\left(V^{n+1}-\bar{V}^{n+1}\right) \in U_{0,h}^*$ solving

$$\begin{aligned}&\int_\Omega \mathcal{I}_h\left\{\partial_\tau^-\omega_i^{n+1}\theta_i\right\} - \int_\Omega \left\langle \mathbf{u}_{\omega_i,M}^\star, \nabla\theta_i\right\rangle\omega_i^n + \int_\Omega k_i\left(\phi^{n+1}\right)\left\langle \Xi_\nu\left(\omega_i^{n+1}\right)\nabla\mathcal{I}_h\left\{\mu_{\omega_i}^{n+1}+z_iV^{n+1}\right\},\nabla\theta_i\right\rangle \\ &\qquad = \int_\Omega \mathcal{I}_h\left\{R_i\left(\phi^{n+1},\omega_1^{n+1},...,\omega_M^{n+1},V^{n+1}\right)\theta_i\right\} \qquad \text{for all } \theta_i\in U_h,\ i=1,...,M\,,\end{aligned} \tag{5a}$$

$$\int_\Omega \mathcal{I}_h\left\{\mu_{\omega_i}^{n+1}\psi_i\right\} = \int_\Omega \mathcal{I}_h\left\{\left(g_\nu'\left(\omega_i^{n+1}\right)+\beta_i\left(\phi^{n+1}\right)r'\left(\omega_i^{n+1}\right)\right)\psi_i\right\} \qquad \text{for all } \psi_i\in U_h,\ i=1,...,M\,, \tag{5b}$$

$$\int_{\Omega^*}\varepsilon(\mathbf{x})\left\langle\nabla V^{n+1},\nabla\psi^*\right\rangle = \int_\Omega \mathcal{I}_h\left\{\sum_{j=1}^M z_j\omega_j^{n+1}\psi^*\right\} \qquad \text{for all } \psi^*\in U_{0,h}^*\,, \tag{5c}$$

 

with $\mathbf{u}^{\star}_{\omega_i,M} := \mathbf{u}^n - \tau M \frac{\omega_i^n}{\rho^n} \nabla \mathcal{I}_h \left\{ \mu_{\omega_i}^{n+1} + z_i V^{n+1} \right\}$.
In the last step we finally update the velocity field by computing $\mathbf{u}^{n+1} \in \mathbf{W}_h$ and $p^{n+1} \in S_h$ satisfying

$$\begin{aligned}
&\int_\Omega \left\langle \partial_\tau^- \left( \rho^{n+1} \mathbf{u}^{n+1} \right), \mathbf{w} \right\rangle - \tfrac{1}{2} \int_\Omega \partial_\tau^- \rho^{n+1} \left\langle \mathbf{u}^{n+1}, \mathbf{w} \right\rangle + \int_\Omega 2\mathcal{I}_h \left\{ \eta \left( \phi^{n+1} \right) \right\} \mathbf{D}\mathbf{u}^{n+1} : \mathbf{D}\mathbf{w} \\
&+ \tfrac{1}{2} \int_\Omega \rho^n \left\langle \mathbf{u}^n, \left( \nabla \mathbf{u}^{n+1} \right)^T \mathbf{w} \right\rangle - \tfrac{1}{2} \int_\Omega \mathcal{I}_h \left\{ \rho' \left( \phi^{n+1} \right) m \left( \phi^n \right) \right\} \left\langle \nabla \mu_\phi^{n+1}, \left( \nabla \mathbf{u}^{n+1} \right)^T \mathbf{w} \right\rangle \\
&- \tfrac{1}{2} \int_\Omega \rho^n \left\langle \mathbf{u}^n, \left( \nabla \mathbf{w} \right)^T \mathbf{u}^{n+1} \right\rangle + \tfrac{1}{2} \int_\Omega \mathcal{I}_h \left\{ \rho' \left( \phi^{n+1} \right) m \left( \phi^n \right) \right\} \left\langle \nabla \mu_\phi^{n+1}, \left( \nabla \mathbf{w} \right)^T \mathbf{u}^{n+1} \right\rangle - \int_\Omega p^{n+1} \operatorname{div} \mathbf{w} \\
&\qquad = - \int_\Omega \phi^n \left\langle \nabla \mu_\phi^{n+1}, \mathbf{w} \right\rangle - \sum_{i=1}^M \int_\Omega \omega_i^n \left\langle \nabla \left( \mu_{\omega_i}^{n+1} + z_i V^{n+1} \right), \mathbf{w} \right\rangle \quad \text{for all } \mathbf{w} \in \mathbf{W}_h \,, \qquad (6a) \\
&\qquad \int_\Omega q \operatorname{div} \mathbf{u}^{n+1} = 0 \qquad \text{for all } q \in S_h \,. \qquad (6b)
\end{aligned}$$

We want to emphasize that blocks (4), (5) and (6) can be solved sequentially. The main ingredient for the presented partially decoupled scheme is the choice of the transport velocities $\mathbf{u}_\phi^\star$ and $\mathbf{u}_{\omega_i,M}^\star$ $(i = 1, ..., M)$. Using the idea presented in [6], [7] and [8], we update $\mathbf{u}^n$ by a discrete time integration of the force densities. Hence we are able to postpone the computation of $\mathbf{u}^{n+1}$ until the very end without losing stability. For a pure Cahn–Hilliard–Navier–Stokes system convergence of a discrete scheme based on this decoupling technique was proven in [5].

## 3 Properties

As one may treat (4)-(6) sequentially, the existence of solutions can be shown individually for each block by standard arguments (e.g. Brouwer's fixed point theorem). Overall, the discrete solutions satisfy the following stability result.

**Theorem 3.1** *Let* $\left\{ \left( \phi^n, \mu_\phi^n, \omega_1^n, \mu_{\omega_1}^n, ..., \omega_M^n, \mu_{\omega_M}^n, V^n, \mathbf{u}^n \right) \right\}_{n=0,...,N}$ *be a solution of* (4)-(6)*. Under the assumptions of Section 2 the estimate*

$$\begin{aligned}
&\int_\Omega \tfrac{1}{2} \rho^N |\mathbf{u}^N|^2 + \int_\Omega \tfrac{1}{2} |\nabla \phi^N|^2 + \int_\Omega \mathcal{I}_h \left\{ W \left( \phi^N \right) \right\} + \sum_{i=1}^M \int_\Omega \mathcal{I}_h \left\{ g \left( \omega_i^N \right) + \beta_i \left( \phi^N \right) r \left( \omega_i^N \right) \right\} + \int_{\partial\Omega} \mathcal{I}_h \left\{ \gamma \left( \phi^N \right) \right\} \\
&+ \int_{\Omega^*} \tfrac{1}{2} \mathcal{I}_h \left\{ \varepsilon \left( \mathbf{x} \right) \right\} |\nabla V^N|^2 + \tau \sum_{n=0}^{N-1} \int_\Omega \mathcal{I}_h \left\{ m \left( \phi^n \right) \right\} |\nabla \mu_\phi^{n+1}|^2 + \tau \sum_{n=0}^{N-1} \int_\Omega 2\mathcal{I}_h \left\{ \eta \left( \phi^{n+1} \right) \right\} |\mathbf{D}\mathbf{u}^{n+1}|^2 \\
&\qquad + \sum_{n=0}^{N-1} \sum_{i=1}^M \int_\Omega \mathcal{I}_h \left\{ k_i \left( \phi^{n+1} \right) \right\} \left\langle \Xi_\nu \left( \omega_i^{n+1} \right) \nabla \left( \mu_{\omega_i}^{n+1} + z_i V^{n+1} \right), \nabla \left( \mu_{\omega_i}^{n+1} + z_i V^{n+1} \right) \right\rangle \leq C \qquad (7)
\end{aligned}$$

*holds true with* $C > 0$ *depending on the given initial and boundary data.*

Proof. To prove this regularity result we start by testing (4a) by $\mu_\phi^{n+1}$, (4b) by $\partial_\tau^- \phi^{n+1}$, (6a) by $\mathbf{u}^{n+1}$ and (6b) by $p^{n+1}$. Due to the splitting approach the convective term from (4a) does not cancel out with the corresponding term on the right hand side of (6a). By applying Young's inequality, we are able to absorb this difference in the arising dissipative terms. We continue by testing (5a) by $\left( \mu_{\omega_i}^{n+1} + z_i V^{n+1} \right)$ and (5b) by $\partial_\tau^- \omega_i$ for $i = 1, ..., M$ and by applying a similar absorption argument to the convective parts of the species equations. The reaction terms can be controlled by using the monotonicity of the exponential function. By subtracting (5c) at $t = t_n$ from (5c) at $t = t_{n+1}$, testing the resulting equation by $\frac{1}{\tau} \left( V^{n+1} - \bar{V}^{n+1} \right)$ and summation, we obtain the following bound for the left hand side of (7):

$$C \left( \phi^0, \omega_1^0, ..., \omega_M^0, V^0, \mathbf{u}^0 \right) + \tau \sum_{n=0}^{N-1} \int_{\Omega^*} \varepsilon \left\langle \nabla \partial_\tau^- V^{n+1}, \nabla \bar{V}^{n+1} \right\rangle - \tau \sum_{n=0}^{N-1} \sum_{j=1}^M \int_\Omega \mathcal{I}_h \left\{ z_j \partial_\tau^- \omega_j^{n+1} \bar{V}^{n+1} \right\} . \qquad (8)$$

The last two terms represent the external forces induced by the prescribed electrostatic potential on $\Gamma_D$. We introduce some function $\bar{W}^{n+1} \in U_{0,h}^*$ for all $n \leq N-1$ satisfying $\bar{W}^{n+1} = \bar{V}^{n+1}$ in $\Omega$. Due to the regularity of the triangulation and $\operatorname{dist}(\Omega, \Gamma_D) \geq c$, the estimate $\|\nabla \bar{W}^{n+1}\|_{L^2(\Omega^*)} \leq C \|\bar{V}\|_{H^1(\Omega^*)}$ holds true (see e.g. [9]). Therefore, discrete integration by parts with respect to time and a discrete version of Gronwall's lemma (cf. [10, Lemma 4.2.3]) yield the desired result. □

Although we can not expect the species concentrations to stay non-negative, we are still able to prove the existence of a lower, $\nu$-dependent bound for the case of non-negative initial data. Following the lines of [3, Theorem 6.1] one may use Theorem 3.1 to derive $\min \omega_j^n g_\nu' (\nu) \leq \frac{1}{h} C$ for $j \in \{1, ..., M\}$. Noting $g_\nu' (\nu) = \log \nu \leq 0$, we may conclude

$$\min_j \min \omega_j^n \geq \tfrac{C}{h g_\nu'(\nu)} \,. \qquad (9)$$

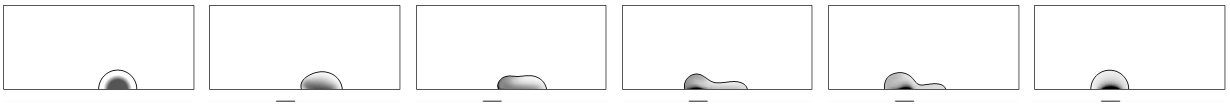

**Fig. 1:** Scenario 1: Moving droplet at $t = 0$, $t = 0.003$, $t = 0.005$, $t = 0.008$, $t = 0.011$, $t = 0.025$ (from left to right).

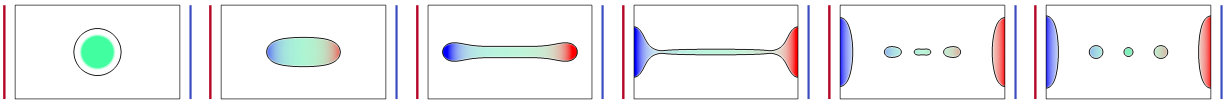

**Fig. 2:** Scenario 2: Electrolyte droplet in an electric field at $t = 0$, $t = 0.12$, $t = 0.25$, $t = 0.41$, $t = 0.53$, $t = 0.56$ (from left to right).

**Remark 3.2** As we are not aware of a straight forward extension of the decoupling strategy to the case of a fluid dependent dielectric permittivity, we have to drop the decoupling of (4) and (5) in that case. Nevertheless, the decoupling from (6) stays valid along with the stability result and the lower bound for the species concentrations.

## 4 Numerical experiments

The presented scheme was implemented in the inhouse code EconDrop developed by G. Grün, F. Klingbeil and the author (cf. [11], [1], [12], [5]). To verify the interplay of the different equations, we decided on the following two-dimensional scenarios (see Figure 1 and 2) with a constant liquid-solid interfacial energy density $\gamma$ prescribing a contact angle of $90°$:
In the first scenario, we place a semicircular shaped droplet ($\phi = 1$) with radius 1 and center $(1, 0)$ in a domain $\Omega := (-5, 5) \times (0, 4.375)$. In the Figure 1, the contour of droplet is marked by the black line. Both fluids share the same mass density ($\rho \equiv 0.0001$) and the same viscosity $\eta \equiv 0.0005$. The dielectric permittivity is constant ($\varepsilon \equiv 1$). Regarding the species concentration (indicated by the background colour), we start with a concentration of 10 in smaller semicircle (radius=0.5), which decays linearly to avoid concentration jumps in the initial data. The considered species carry a positive charge $z_1 = 1$ and are assumed to be soluble only in the droplet ($\beta_1(1) = 1$) and nearly insoluble in the ambient liquid ($\beta_1(-1) = 10$). The coefficient $k_1$ is set to 5, allowing for fast species movement. After a short period of time, we activate an electrode with a prescribed potential of $-20$ at $[-1.5, -0.5] \times \{-0.625\}$. In consequence, the ions move towards the electrode and entrain the droplet to its new position.
In the second scenario, we investigate the interplay of different species. We consider three kinds of species with the properties $\beta_1(-1) = 5$, $\beta_1(1) = 1$, $k_1 \equiv 2$, $z_1 = 0$, $\zeta_1 = -2$, $\beta_2(-1) = 5$, $\beta_2(1) = 1$, $k_2 \equiv 2$, $z_2 = 1$, $\zeta_2 = 2$, $\beta_3(-1) = 5$, $\beta_3(1) = 1$, $k_3 \equiv 2$, $z_3 = -1$ and $\zeta_3 = 2$, i.e. the species are soluble only in the droplet and the first, uncharged one may decompose into positive and negative charged ions. The species concentrations are indicated by the background colours in Figure 2 ($\omega_1 \hat{=}$green, $\omega_2 \hat{=}$red and $\omega_3 \hat{=}$blue). We place a circular shaped droplet with radius 0.5 and center $(0, 0)$ in a domain $\Omega = (-1.75, 1.75) \times (-1, 1)$. Again mass density and viscosity are chosen constant ($\rho \equiv 0.0001$, $\eta \equiv 0.05$). The dielectric permittivity is 1 in $\Omega$ and 2 in $\Omega^* \backslash \Omega$. The initial setup for concentrations is similar to the one in the first scenario. We start with $\omega_1 = 10$ in a small circle with radius 0.1 and center $(0, 0)$ and $\omega_2 \equiv 0 \equiv \omega_3$. To avoid concentration jumps, $\omega_1$ decays linearly. Prescribing an electrostatic potential of $V = \pm 5$ at $\mathbf{x} = \mp 2$ (indicated by the red and blue lines in Figure 2), we excite a reaction producing charged species by decomposing the uncharged one. The emerging ions move towards the electrodes. As they are both only soluble in the droplet, they cause the droplet to stretch and finally to split. To ease such deformations of the droplet, we reduced the surface tension by weighting the Ginzburg–Landau free energy by 0.1.

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