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Three-phase boundary motions under constant velocities. I: The vanishing surface tension limit

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In this paper, we deal with the dynamics of material interfaces such as solid–liquid, grain or antiphase boundaries. We concentrate on the situation in which these internal surfaces separate three regions in the material with different physical attributes (e.g. grain boundaries in a polycrystal). The basic two-dimensional model proposes that the motion of an interface Γ_{ij} between regions i and j ($i, j = 1, 2, 3$, $i \neq j$) is governed by the equation

$$V_{ij} = \mu_{ij}(f_{ij}\kappa_{ij} + F_{ij}). \tag{0.1}$$

Here V_{ij}, κ_{ij}, μ_{ij} and f_{ij} denote, respectively, the normal velocity, the curvature, the mobility and the surface tension of the interface and the numbers F_{ij} stand for the (constant) difference in bulk energies. At the point where the three phases coexist, local equilibrium requires that

$$\text{the curves meet at } \textit{prescribed angles}. \tag{0.2}$$

In case the material constants f_{ij} are small, $f_{ij} = \widehat{\varepsilon f_{ij}}$ and $\varepsilon \ll 1$, previous analyses based on the parabolic nature of the equations (0.1) do not provide good qualitative information on the behaviour of solutions. In this case, it is more appropriate to consider the singular case with $f_{ij} = 0$. It turns out that this problem, (0.1) with $f_{ij} = 0$, admits infinitely many solutions. Here, we present results that strongly suggest that, in all cases, a unique solution—'the vanishing surface tension (VST) solution'—is selected by letting $\varepsilon \to 0$. Indeed, a formal analysis of this limiting process motivates us to introduce the concept of *weak viscosity solution* for the problem with $\varepsilon = 0$. As we show, this weak solution is *unique* and is therefore expected to coincide with the VST solution. To support this statement, we present a perturbation analysis and a construction of self-similar solutions; a rigorous convergence result is established in the case of symmetric configurations. Finally, we use the weak formulation to write down a catalogue of solutions showing that, in several cases of physical relevance, the VST solution *differs* from results proposed previously.

1. Introduction

A variety of mathematical models have been devised to investigate the dynamics of interfaces such as solid–liquid, grain or antiphase boundaries. These internal surfaces are, in general, nonequilibrium features of a material: they have a positive excess free energy. Thus, for example, grain boundaries (i.e. boundaries between single crystals in a polycrystal) migrate to reduce the total amount of grain boundary area.

Understanding the evolution of these surfaces is of fundamental importance, not only for its intrinsic interest, but also for its technological significance: they constitute a key factor in determining a wide range of material properties, from mechanical strength to electrical conductivity (see e.g. [**3**]).

The simplest models for interface dynamics can be obtained by neglecting the changes in the bulk (i.e. away from the interfaces) and concentrating solely on the evolution of the internal boundaries. In many instances, these kinds of models provide a good representation of the physics. In this paper, we study a two-dimensional model for the motion of interfaces in multi-phase continua. This model, which applies in particular to grain growth, corresponds to the 'small surface tension' limit of the one derived by Mullins [**10**] and investigated by Bronsard and Reitich [**5**] (cf. equation (1.2) below). Here, we present results that strongly suggest that, even though the 'zero surface tension' problem admits infinitely many solutions, a unique solution—which we shall call 'the vanishing surface tension (VST) solution'—is selected by the limiting process (see Section 2). Indeed, a formal analysis motivates us to introduce the concept of *weak viscosity solution* for the limit problem. As we show, this weak solution is *unique* and is therefore expected to coincide with the VST solution.

Mullins' model [**10**] describes the evolution of three curves in the plane (the internal boundaries) which move according to

$$\text{Normal Velocity} = V = \kappa = \text{Curvature}. \tag{1.1}$$

The three curves are assumed to meet at a point, a 'triple junction' (see Fig. 1.1), with prescribed angles θ_i ($\theta_i = 120°$ in the case of grain boundaries). More generally, (1.1) should be replaced by the equation

$$V = \mu f \kappa - \mu F, \tag{1.2}$$

which describes the evolution of an isotropic interface that is driven by an energy difference F between the bulk phases [**2**]. The material constants f and μ denote, respectively, the energy and mobility of the interface; they may be different for different curves. The existence and uniqueness of solutions for a given initial state was established in [**5**] by means of a parametric formulation of the problem and the analysis of the resulting quasilinear parabolic system of equations with fully

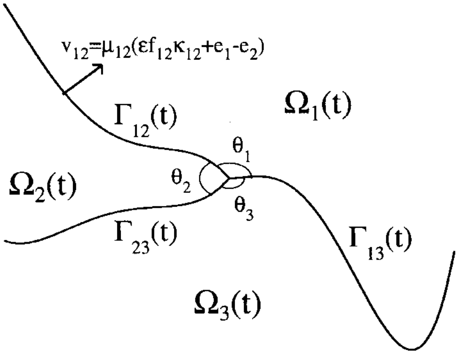

Figure 1.1. The interfaces Γ_{ij} move with normal velocities $V_{ij} = \mu_{ij}(\varepsilon f_{ij}\kappa_{ij} + e_i - e_j)$ and prescribed angles θ_i.

nonlinear boundary conditions. When $\mu f \ll 1$, the system is near degenerate and the analysis in [**5**] does not provide good qualitative information on the behaviour of solutions. For this, it is more appropriate to look into the problem with $\mu f = 0$, that is, a model in which the curves evolve under constant velocities.

The problem of three curves moving under constant velocities and meeting at a triple junction has been previously considered by Taylor [**12**] and Merriman, Bence and Osher [**9**]. It is readily checked that, if no further conditions on the motion are imposed, the solution is not unique (see Fig. 2.1 below). In [**12**] a Huygens' Principle is applied in order to select a particular solution, and a catalogue of solutions depending on the initial configuration is provided. The implementation of the level set approach developed in [**9**] yields solutions which coincide with those in [**12**]. As we shall see, these solutions *do not* always coincide with the VST or weak solutions we present here.

Perhaps the simplest way to motivate the model is to associate to the system an energy functional of the form

$$\begin{aligned}\mathscr{E} &= \varepsilon f_{12}\,\text{length}\,(\Gamma_{12}) + \varepsilon f_{23}\,\text{length}\,(\Gamma_{23}) + \varepsilon f_{31}\,\text{length}\,(\Gamma_{31})\\ &\quad + e_1\,\text{vol}\,(\Omega_1) + e_2\,\text{vol}\,(\Omega_2) + e_3\,\text{vol}\,(\Omega_3),\end{aligned} \tag{1.3}$$

where e_i denotes the (constant) energy density in Ω_i. Then, it is easy to see that the corresponding gradient flow is

$$V_{ij} = \mu_{ij}(\varepsilon f_{ij}\kappa_{ij} + e_i - e_j), \quad \varepsilon \ll 1, \tag{1.4}$$

where V_{ij}, κ_{ij} and μ_{ij} denote, respectively, the normal velocity, the curvature and the mobility of the interface between phase i and phase j. Furthermore, a straightforward calculation shows that, in the absence of dissipation at the triple junction, the requirement that

$$\frac{d\mathscr{E}}{dt} \leqq 0$$

imposes the condition

$$\frac{\sin(\theta_1)}{f_{23}} = \frac{\sin(\theta_2)}{f_{31}} = \frac{\sin(\theta_3)}{f_{12}} \quad \text{(see Fig. 1.1).} \tag{1.5}$$

This condition coincides with the classical condition at triple points that can be found in the materials science literature (see e.g. [**11**]). It was recently shown in [**5**] that (1.5) is also recovered in the small interface-thickness limit of a (diffuse-interface) Allen–Cahn type model for three-phase boundary motion (see also [**1**]). Notice that in the particular case of grain boundaries, we have

$$f_{12} = f_{23} = f_{31}$$

and (1.5) implies that $\theta_i = 120°$.

The remainder of the paper is devoted to understanding the qualitative behaviour of (1.4), (1.5) in the limit $\varepsilon \to 0$. First, in Section 2, we discuss the nonuniqueness of solutions in the case $\varepsilon = 0$. We present a perturbation analysis and a class of self-similar solutions, which suggest that a unique solution (the VST solution) is selected by letting $\varepsilon \to 0$. In Section 3, we introduce the concept of *weak viscosity solution* for the constant-velocities problem ((1.4) with $\varepsilon = 0$). Since the motion is governed by a system of Hamilton–Jacobi equations, the classical notion of viscosity solution can be used away from the triple points. At the triple points, however, and in accordance with (1.5), we need to introduce the idea of *weak angle conditions*. As we shall show, this concept singles out a *unique* solution which coincides with the one obtained in the limit $\varepsilon \to 0$ (Section 4). Finally, in Section 5, we present a preliminary result of convergence of solutions as $\varepsilon \to 0$ in the case where two interfaces are symmetric relative to the third (which is given by a semi-infinite stationary line). A simple example of a nonphysical geometric problem is also discussed at the end of Section 5.

2. Nonuniqueness and the VST solution

As discussed in Section 1, for $\varepsilon > 0$ the equations (1.4) subject to the angle conditions (1.5) admit (in a finite time interval) unique classical solutions for smooth initial data. Of course, when $\varepsilon = 0$, the resulting (first-order) equations

$$V_{ij} = \mu_{ij}(e_i - e_j) \tag{2.1}$$

cannot be constrained by the angle conditions. On the other hand, the equations (2.1), subject only to the requirement that the interfaces meet at a triple point, admit infinitely many solutions. The simplest case in which this is apparent corresponds to that of Figure 2.1, where

$$\mu_{ij} = 1$$

and

$$e_2 - e_1 = \alpha \quad \text{and} \quad e_2 - e_3 = \alpha \quad (\alpha > 0).$$

Figure 2.1 contains two different solutions to the problem. The lighter curves correspond to the initial configuration and the heavier ones to the solution at time $t = 1$. Since the interface velocities are constant, the solution at time $t = T$ can be

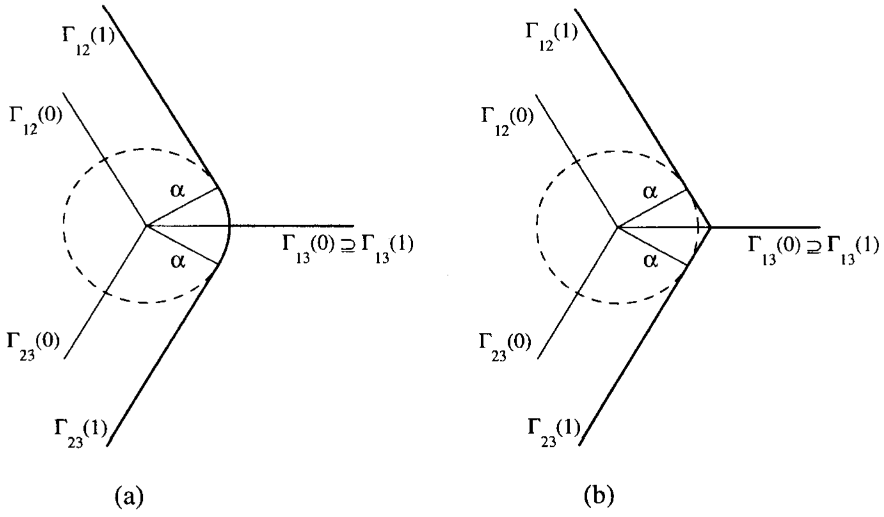

Figure 2.1. (a) The solution proposed in [**12**]. (b) The VST solution.

obtained by simply dilating these graphs. The solution in Figure 2.1(a) corresponds to the construction in [**12**]. However, it is easily checked that the solution that is singled out by taking the limit as $\varepsilon \to 0$ in (1.4) is the one depicted in Figure 2.1(b). Indeed, if the initial configuration satisfies the angle conditions, it is clear that the solution in Figure 2.1(b) is an exact solution of (1.4), (1.5) for $\varepsilon > 0$ (since the interfaces have zero curvature) and therefore it coincides with the limit solution.

More generally, consider the situation of Figure 2.2, where

$$\mu_{21}(e_2 - e_1) = \alpha, \quad \mu_{23}(e_2 - e_3) = \beta \quad \text{and} \quad e_1 = e_3 \quad (\alpha \geqq \beta > 0).$$

Again, Figure 2.2(a) shows the solution proposed in [**12**] (at $t = 1$). The construction is as follows: first draw a parallel line to the initial interface $\Gamma_{12}(0)$ at a distance α, up to the point of tangency with the circle of radius α. Then, follow this circle up to the new position of the triple junction on the x-axis: this defines Γ_{12} at time $t = 1$. To construct Γ_{23} at time $t = 1$, first draw a parallel line to the initial interface $\Gamma_{23}(0)$ at a distance β up to the point of tangency with the circle of radius β and then a line segment from the triple junction which is tangent to this circle (notice that this introduces a corner in Γ_{23}). The construction in Figure 2.2(b) is similar but in building Γ_{12} we do not go all the way down to the x-axis on the circle of radius α. Rather, we propose a solution which, at $t = 1$, contains only part of this arc of circle and then joins Γ_{12} with Γ_{13} with a straight line segment which is tangent to the circle of radius α; the ensuing construction of Γ_{23} is as in Figure 2.2(a). Notice that the length of the arc of circle in the solution is a free parameter, giving rise to infinitely many solutions. The solution we propose, in case

$$\theta_i = 120^\circ \tag{2.2}$$

and

$$0 < \beta < \alpha < 2\beta, \tag{2.3}$$

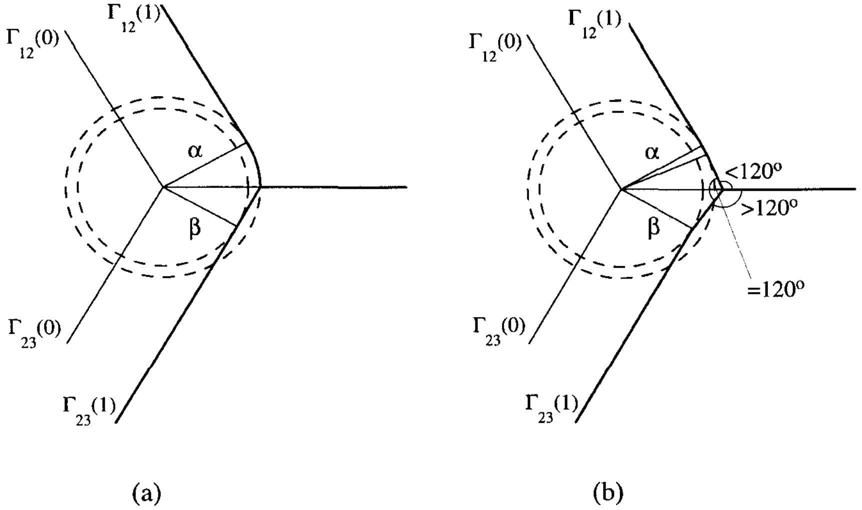

Figure 2.2. (a) The solution proposed in [**12**]. (b) The VST solution; one of the angle conditions is preserved in this solution.

is the one where this length is exactly the one for which the construction results in an angle of 120° between Γ_{12} and Γ_{23}. (Note that if $2\beta < \alpha$ this construction is not possible; in that case, the corresponding solution is the one shown in Fig. 3.4(b)). In the following two subsections, we shall show, through a perturbation analysis and the construction of self-similar solutions, that this is indeed the solution that is selected if we let $\varepsilon \to 0$ in (1.4), (2.2). Further evidence of this fact will be provided in Section 3 after we motivate and define the notion of *viscosity solution* to (2.1) (2.2). As we shall see, the solution depicted in Figure 2.2(b) is the unique viscosity solution of (2.1), (2.2).

Perturbation analysis

As we discussed above, the solution in Figure 2.1(b) is an exact solution of the problem

$$\begin{aligned} V_{21} &= \varepsilon f_{12}\kappa_{21} + \alpha, \\ V_{23} &= \varepsilon f_{23}\kappa_{23} + \alpha, \\ V_{13} &= \varepsilon f_{13}\kappa_{13} \end{aligned} \tag{2.4}$$

subject to the angle conditions (2.2). In order to gain some insight into the structure of solutions, it is natural then to study small perturbations of this problem (see Fig. 2.3):

$$\begin{aligned} V_{21} &= \varepsilon f_{12}\kappa_{21} + \alpha, \\ V_{23} &= \varepsilon f_{23}\kappa_{23} + (\alpha - \delta), \quad \delta \ll 1, \\ V_{13} &= \varepsilon f_{13}\kappa_{13}. \end{aligned} \tag{2.5}$$

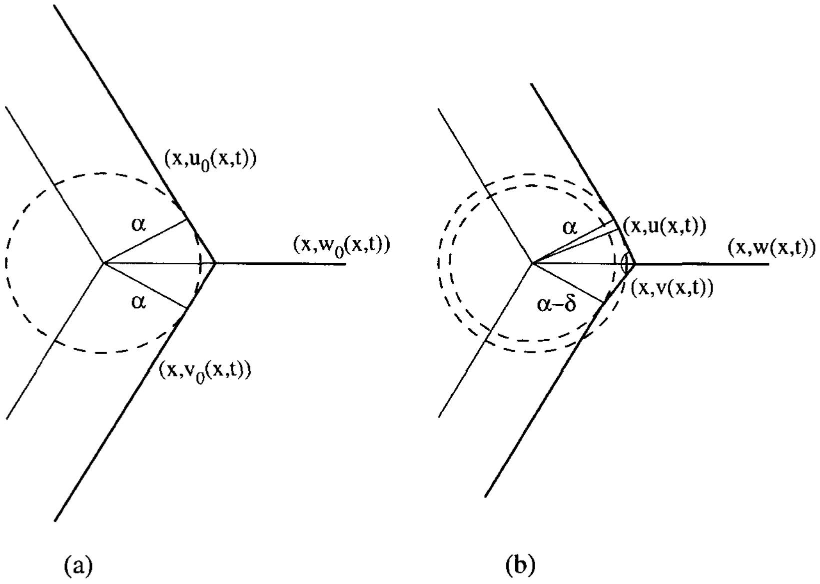

Figure 2.3. Perturbation analysis: (a) exact solution; (b) a perturbation.

In the coordinates of Figure 2.3, the equations (2.5) can be written in the form

$$
\begin{aligned}
u_t &= \varepsilon f_{12}\frac{u_{xx}}{1+u_x^2} + \alpha\sqrt{1+u_x^2}, \quad x < s(t),\\
v_t &= \varepsilon f_{23}\frac{v_{xx}}{1+v_x^2} - (\alpha-\delta)\sqrt{1+v_x^2}, \quad x < s(t) \quad (\delta \ll 1),\\
w_t &= \varepsilon f_{13}\frac{w_{xx}}{1+w_x^2}, \quad x > s(t),
\end{aligned}
\tag{2.6}
$$

where $s(t)$ denotes the (unknown) x-coordinate of the triple junction at time t. The requirement that the solutions meet at the triple junction translates into

$$
\begin{aligned}
u(s(t),t) &= w(s(t),t),\\
v(s(t),t) &= w(s(t),t),
\end{aligned}
\tag{2.7}
$$

and the angle conditions (2.2) become

$$
\begin{aligned}
1 + u_x(s(t),t)v_x(s(t),t) &= -\tfrac{1}{2}\sqrt{1+u_x(s(t),t)^2}\,\sqrt{1+v_x(s(t),t)^2},\\
-1 - u_x(s(t),t)w_x(s(t),t) &= -\tfrac{1}{2}\sqrt{1+u_x(s(t),t)^2}\,\sqrt{1+w_x(s(t),t)^2}.
\end{aligned}
\tag{2.8}
$$

To exact solution for $\delta = 0$ is

$$
\begin{aligned}
u = u_0(x,t) &= -\sqrt{3}x + 2\alpha t,\\
v = v_0(x,t) &= \sqrt{3}x - 2\alpha t,\\
w = w_0(x,t) &\equiv 0,\\
s = s_0(t) &= \frac{2\alpha}{\sqrt{3}}t.
\end{aligned}
\tag{2.9}
$$

The first order in δ

$$
u = u_0 = \delta u_1, \quad v = v_0 + \delta v_1, \quad w = w_0 + \delta w_1, \quad s = s_0 + \delta s_1
$$

and the linearised version of the free-boundary problem (2.6), (2.7), (2.8) is

$$
\begin{aligned}
u_{1t}(x,t) &= \varepsilon\frac{f_{12}}{4}u_{1xx}(x,t) - \alpha\frac{\sqrt{3}}{2}u_{1x}(x,t), \quad x < s_0(t),\\
v_{1t}(x,t) &= \varepsilon\frac{f_{23}}{4}v_{1xx}(x,t) - \alpha\frac{\sqrt{3}}{2}v_{1x}(x,t) + 2, \quad x < s_0(t),\\
w_{1t}(x,t) &= \varepsilon f_{13}w_{1xx}(x,t), \quad x > s_0(t),\\
u_{1x}(s_0(t),t) &= v_{1x}(s_0(t),t)\\
u_{1x}(s_0(t),t) &= 4w_{1x}(s_0(t),t)\\
u_1(s_0(t),t) &= v_1(s_0(t),t) + 2\sqrt{3}s_1(t)\\
u_1(s_0(t),t) &= w_1(s_0(t),t) + \sqrt{3}s_1(t).
\end{aligned}
\tag{2.10}
$$

At this point, the equation (2.10) becomes the key to understanding the behaviour of solutions as $\varepsilon \to 0$. Indeed, the condition

$$u_{1x}(s_0(t), t) = v_{1x}(s_0(t), t)$$

implies that, near the triple junction, the solutions u and v for $\delta \ll 1$ differ from the solution for $\delta = 0$ by a rotation of angle

$$\theta_R \approx -\frac{u_{1x}}{4}\delta \quad (\delta \ll 1). \tag{2.11}$$

In particular, the angle between u and v is preserved in the limit, i.e.

$$Angle\,(u, v) = 120^\circ \tag{2.12}$$

(compare Fig. 2.2(b)).

Self-similar solutions

In what follows, we shall construct self-similar solutions $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ to (cf. (2.6) with $\alpha - \delta = \beta$)

$$\begin{aligned} u_t^\varepsilon &= \varepsilon f_{12}\frac{u_{xx}^\varepsilon}{1+(u_x^\varepsilon)^2} + \alpha\sqrt{1+(u_x^\varepsilon)^2}, \quad x < s^\varepsilon(t), \\ v_t^\varepsilon &= \varepsilon f_{23}\frac{v_{xx}^\varepsilon}{1+(v_x^\varepsilon)^2} - \beta\sqrt{1+(v_x^\varepsilon)^2}, \quad x < s^\varepsilon(t), \\ w_t^\varepsilon &= \varepsilon f_{13}\frac{w_{xx}^\varepsilon}{1+(w_x^\varepsilon)^2}, \quad x > s^\varepsilon(t), \end{aligned} \tag{2.13}$$

subject to the free-boundary conditions (2.7), (2.8). As we shall see, these solutions will converge, as $\varepsilon \to 0$, to a solution of

$$\begin{aligned} u_t &= \alpha\sqrt{1+u_x^2}, \quad x < s(t), \\ v_t &= -\beta\sqrt{1+v_x^2}, \quad x < s(t), \\ w_t &= 0, \quad x > s(t), \end{aligned} \tag{2.14}$$

satisfying (2.12).

The first step in the construction is to assume

$$\begin{aligned} u^\varepsilon(x, t) &= u(x, t) = -a(x - s^\varepsilon(t)), \quad a > 0, \\ v^\varepsilon(x, t) &= v(x, t) = b(x - s^\varepsilon(t)), \quad b > 0, \\ s^\varepsilon(t) &= \sigma t, \quad \sigma > 0. \end{aligned} \tag{2.15}$$

Then, equations (2.13) take on the form

$$\begin{aligned} \sigma a &= \alpha\sqrt{1+a^2}, \\ \sigma b &= \beta\sqrt{1+b^2}, \end{aligned} \tag{2.16}$$

and the angle condition for $(u^\varepsilon, v^\varepsilon)$ becomes (cf. (2.8))

$$1 - ab = -\tfrac{1}{2}\sqrt{1+a^2}\,\sqrt{1+b^2}. \tag{2.17}$$

Solving (2.16), (2.17) for (a, b, σ), we obtain

$$a = \frac{\sqrt{3}\alpha}{2\beta - \alpha}, \quad b = \frac{\sqrt{3}\beta}{2\alpha - \beta},$$
$$\sigma = \frac{2}{\sqrt{3}}\sqrt{\alpha^2 + \beta^2 - \alpha\beta}. \tag{2.18}$$

Next, we want to solve for w^ε by proposing a self-similar form

$$w^\varepsilon(x, t) = \varepsilon\psi\left(\frac{x - \sigma t}{\varepsilon}\right).$$

Then, from (2.13),

$$-\sigma\psi'(z) = f_{13}\frac{\psi''(z)}{1 + (\psi'(z))^2}, \quad z > 0, \tag{2.19}$$

and

$$\psi(0) = 0. \tag{2.20}$$

The requirements that

$$\textit{Angle}\,(u^\varepsilon, w^\varepsilon) = 120^\circ, \quad \textit{Angle}\,(v^\varepsilon, w^\varepsilon) = 120^\circ$$

translates into

$$\psi'(0) = -\eta \quad (\eta > 0), \tag{2.21}$$

where

$$\eta = \tan(\theta_R) = \frac{a - \sqrt{3}}{1 + a\sqrt{3}} = \frac{\sqrt{3} - b}{1 + b\sqrt{3}} = \sqrt{3}\frac{(\alpha - \beta)}{(\alpha + \beta)}. \tag{2.22}$$

It is easily checked that the (unique) solution to (2.19) subject to the initial conditions (2.20) and (2.21) is given by

$$\psi(z) = \frac{f_{13}}{\sigma}\left[\arcsin\left(\frac{\eta}{\sqrt{1 + \eta^2}}e^{-\sigma z/f_{13}}\right) - \arcsin\left(\frac{\eta}{\sqrt{1 + \eta^2}}\right)\right]. \tag{2.23}$$

With this definition, $(u^\varepsilon = u, v^\varepsilon = v, w^\varepsilon)$ is a self-similar solution of (2.13), (2.7) and (2.8) (see Fig. 2.4). Since $\psi(z)$ is uniformly bounded, $(u^\varepsilon, v^\varepsilon, w^\varepsilon) \to (u, v, 0)$ uniformly as $\varepsilon \to 0$. In particular, the angle between u and v is again preserved in the limit, cf. (2.12), see Figure 2.5. Notice that the other angles *are not* preserved.

3. The weak solution

In Section 2 we have shown that, for $0 < \varepsilon \ll 1$, the solutions of (1.4), (1.5) develop boundary layers at the triple junction and that in the limit as $\varepsilon \to 0$ the angle conditions are not necessarily satisfied. In the theory of *viscosity solutions* this phenomenon is well understood and weak formulations of different types of boundary conditions have been developed [6, 7]. In this section, we shall derive a weak formulation of (2.1), (1.5) in the spirit of those obtained in the theory of viscosity

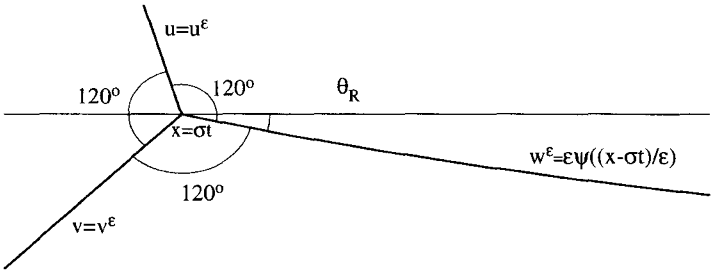

Figure 2.4. A self-similar solution.

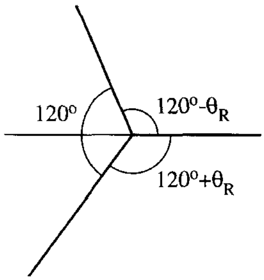

Figure 2.5. The limit ($\varepsilon \to 0$) of self-similar solutions (see Fig. 2.4).

solutions. However, it should be noted here that this system of equations does not have a comparison principle—a property which is essential in the theory of viscosity solutions.

The definition

For each $t \geqq 0$, consider three unbounded, closed and connected regions $\Omega_i(t)$, $i = 1, 2, 3$, satisfying (see Fig. 1.1)

$$\Omega_1(t) \cap \Omega_2(t) \cap \Omega_3(t) = \{T(t)\}, \quad t \geqq 0, \tag{3.1}$$

$$\Omega_i(t) \cap \Omega_j(t) = \Gamma_{ij} = \{p_{ij}(s, t) : s \geqq 0\} \quad (i \neq j), \quad t \geqq 0, \tag{3.2}$$

for some continuous parametrisation $p_{ij} : [0, \infty)^2 \to \mathbb{R}^2$ with

$$\begin{aligned} &p_{ij}(0, t) = T(t), \quad t \geqq 0, \\ &p_{ij}(s, t) \neq p_{ij}(s', t), \quad \text{if } s \neq s'. \end{aligned} \tag{3.3}$$

We assume that near the triple junction $T(t)$ the domains Ω_i satisfy the *local epigraph property*: for each $i \in \{1, 2, 3\}$ and $t_0 \geqq 0$ there exists a neighbourhood $\mathcal{O} \times (t_0 - \delta, t_0 + \delta)$ of $(T(t_0), t_0)$ such that, for $|t - t_0| < \delta$ and in an appropriate coordinate system,

$$T(t) = (s(t), d(t)) \tag{3.4}$$

and there exist continuous functions $u(x, t)$, $w(x, t)$ satisfying

$$\Gamma_{ij} \cap \mathcal{O} = \{(x, y) \in \mathcal{O} : y = u(x, t), x \leqq s(t)\},$$

$$\Gamma_{ik} \cap \mathcal{O} = \{(x, y) \in \mathcal{O} : y = w(x, t), x \geqq s(t)\} \quad (j \neq i \neq k), \tag{3.5}$$
$$\Omega_i \cap \mathcal{O} = \{(x, y) \in \mathcal{O} : y \geqq G(x, t)\},$$

where

$$G(x, t) = \begin{cases} u(x, t), & x \leqq s(t), \\ w(x, t), & x \geqq s(t). \end{cases} \tag{3.6}$$

As for regularity, we only assume that the domains $\Omega_i(\cdot)$ are continuous in the Hausdorff metric, i.e.

$$\lim_{\tau \to t} d_H(\Omega_i(t), \Omega_i(\tau)) = 0, \quad i = 1, 2, 3, \quad t \geqq 0, \tag{3.7}$$

where, for two closed sets $A, B \subset \mathbb{R}^2$,

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y| \right\}. \tag{3.8}$$

If $p_{ij}(\cdot, t)$ is differentiable at $s = 0$ and $|p_{ij_s}(0, t)| \neq 0$, then the angles $\theta_i(t)$ between the curves Γ_{ij} are well defined (see Fig. 1.1). Our goal is to obtain a weak formulation of the angle conditions

$$\theta_i(t) = \theta_i^*, \quad \forall t \geqq 0. \tag{3.9}$$

For this, assume that there exist smooth test functions $\varphi(x, t)$ and $\psi(x, t)$ such that $(s(t_0), t_0)$ is the strict minimiser of $G - F$, where $s(t)$ is defined by (3.4), G is the defining function for Ω_i in the local epigraph condition (3.5) and

$$F(x, t) = \begin{cases} \varphi(x, t), & x \leqq s(t), \\ \psi(x, t), & x \geqq s(t), \end{cases} \tag{3.10}$$

(see Fig. 3.1).

Since φ and ψ are smooth, the angle between their graphs at time t, *Angle* $(\varphi(\cdot, t), \psi(\cdot, t))$, is well defined. If $\theta_i(t_0)$ were defined and satisfied $\theta_i(t_0) \geqq \theta_i^*$, then we would have

$$\textit{Angle}\,(\varphi(\cdot, t), \psi(\cdot, t)) \geqq \theta_i(t_0) \geqq \theta_i^*.$$

Hence we expect to have further restrictions on φ and ψ only when

$$\textit{Angle}\,(\varphi(\cdot, t), \psi(\cdot, t)) < \theta_i^*. \tag{3.11}$$

To derive these additional conditions, suppose that (3.11) holds. Let $\{\Omega_i^\varepsilon : t \geqq 0, i = 1, 2, 3\}$ denote the solution to the curvature-perturbed problem (1.4),

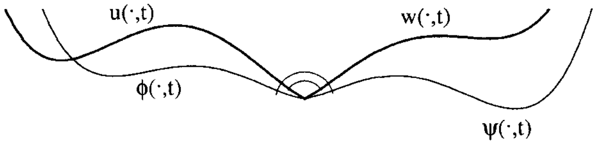

Figure 3.1. The functions ϕ and ψ are test functions for the definition of *Angle* $(u(\cdot, t), w(\cdot, t)) \geqq \theta^*$ **weakly**.

(1.5) and $T^\varepsilon(t)=(s^\varepsilon(t), d^\varepsilon(t))$ the corresponding triple junction. Further, assume that Ω_i^ε satisfies (3.1)–(3.3) and that it has a form similar to that of $\Omega_i(t)$, i.e.

$$\begin{aligned}\Gamma_{ij}^\varepsilon \cap \mathcal{O} &= \{(x, y)\in\mathcal{O} : y = u^\varepsilon(x,t),\ x \leqq s^\varepsilon(t)\},\\ \Gamma_{ik}^\varepsilon \cap \mathcal{O} &= \{(x, y)\in\mathcal{O} : y = w^\varepsilon(x,t),\ x \geqq s^\varepsilon(t)\},\end{aligned} \tag{3.12}$$

for smooth functions u^ε, w^ε satisfying

$$\begin{aligned} u_t^\varepsilon &= \varepsilon\alpha_{ij}\frac{u_{xx}^\varepsilon}{1+(u_x^\varepsilon)^2} + v_{ji}\sqrt{1+(u_x^\varepsilon)^2}, \quad x < s^\varepsilon(t),\\ w_t^\varepsilon &= \varepsilon\alpha_{ik}\frac{w_{xx}^\varepsilon}{1+(w_x^\varepsilon)^2} + v_{ki}\sqrt{1+(w_x^\varepsilon)^2}, \quad x > s^\varepsilon(t),\\ &\mathit{Angle}\,(u^\varepsilon(t), w^\varepsilon(t)) = \theta_i^*, \end{aligned} \tag{3.13}$$

where

$$\alpha_{ij} = \mu_{ij} f_{ij}, \quad v_{ij} = \mu_{ij}(e_i - e_j).$$

Finally, suppose that

$$\lim_{\varepsilon\to 0}\ \sup_{|t-t_0|<\delta} d_H(\Omega_i^\varepsilon(t), \Omega_i(t)) = 0,$$

$$\lim_{\varepsilon\to 0}\ \inf_{|t-t_0|<\delta} |s^\varepsilon(t) - s(t)| + |s^{\varepsilon\prime}(t) - s'(t)| = 0.$$

Set

$$F^\varepsilon(x,t) = \begin{cases}\varphi(x + s(t) - s^\varepsilon(t), t), & x \leqq s^\varepsilon(t),\\ \psi(x + s(t) - s^\varepsilon(t), t), & x \geqq s^\varepsilon(t),\end{cases}$$

$$G^\varepsilon(x,t) = \begin{cases} u^\varepsilon(x,t), & x \leqq s^\varepsilon(t),\\ w^\varepsilon(x,t), & x \geqq s^\varepsilon(t).\end{cases}$$

Then, the local minimisers $(x^\varepsilon, t^\varepsilon)$ of $G^\varepsilon - F^\varepsilon$ converge, as $\varepsilon\to 0$, to $(s(t_0), t_0)$. If $x^\varepsilon = s^\varepsilon(t^\varepsilon)$, we would have

$$\theta_i^* = \mathit{Angle}\,(u^\varepsilon(t^\varepsilon), w^\varepsilon(t^\varepsilon)) \leqq \mathit{Angle}\,(\varphi(t^\varepsilon), \psi(t^\varepsilon)) < \theta_i^*.$$

Hence, $x^\varepsilon \neq s^\varepsilon(t^\varepsilon)$. From (3.13), we then obtain either

$$\varphi_t + (s'(t^\varepsilon) - s^{\varepsilon\prime}(t^\varepsilon))\varphi_x \geqq \varepsilon\alpha_{ij}\frac{\varphi_{xx}}{1+(\varphi_x)^2} + v_{ji}\sqrt{1+(\varphi_x)^2},$$

if $x^\varepsilon < s^\varepsilon(t^\varepsilon)$, or

$$\psi_t + (s'(t^\varepsilon) - s^{\varepsilon\prime}(t^\varepsilon))\psi_x \geqq \varepsilon\alpha_{ik}\frac{\psi_{xx}}{1+(\psi_x)^2} + v_{ki}\sqrt{1+(\psi_x)^2},$$

if $x^\varepsilon > s^\varepsilon(t^\varepsilon)$. Here, φ, ψ and their derivatives are evaluated at $(x^\varepsilon + s(t^\varepsilon) - s^\varepsilon(t^\varepsilon), t^\varepsilon)$. Now, let $\varepsilon \to 0$ and use the assumption that $s^\varepsilon \to s$ in C^1, to conclude

$$\max\{\varphi_t - v_{ji}\sqrt{1+(\varphi_x)^2}, \psi_t - v_{ki}\sqrt{1+(\psi_x)^2}\} \geqq 0 \quad \text{at } (s(t_0), t_0).$$

Since this inequality was derived under the assumption (3.11), we have

$$\max\{Angle\,(\varphi(\cdot, t), \psi(\cdot, t)) - \theta_i^*, \varphi_t - v_{ji}\sqrt{1+(\varphi_x)^2}, \psi_t - v_{ki}\sqrt{1+(\psi_x)^2}\} \geqq 0 \tag{3.14}$$

at $(s(t_0), t_0)$. Notice that the smoothness assumptions on the triple junction were introduced only to derive equation (3.14), which does not itself require the regularity conditions to hold. We are thus led to the following definition.

DEFINITION 3.1. Let $\{\Omega_i(t) : t \geqq 0, i = 1, 2, 3\}$ be unbounded, connected, closed regions satisfying (3.1)–(3.3), (3.7) and the local epigraph condition at the triple junction. We say that

$$\theta_i(t) \geqq \theta_i^* \quad \text{weakly } \forall t > 0,$$

if for any smooth functions φ and ψ such that $G - F$ (cf. (3.6), (3.10)) has a local minimum at $(s(t_0), t_0)$ for some t_0, the inequality (3.14) holds at $(s(t_0), t_0)$.

Similarly, we define the concept of $\theta_i(t) \leqq \theta_i^*$ *weakly*.

DEFINITION 3.2. Let $\{\Omega_i(t) : t \geqq 0, i = 1, 2, 3\}$ be as in Definition 3.1. We say that

$$\theta_i(t) \leqq \theta_i^* \quad \text{weakly } \forall t > 0$$

if for any smooth functions φ and ψ, such that $G - F$ (cf. (3.6), (3.10)) has a local maximum at $(s(t_0), t_0)$ for some t_0, the inequality

$$\min\{Angle(\varphi(\cdot, t), \psi(\cdot, t)) - \theta_i^*, \varphi_t - v_{ji}\sqrt{1+(\varphi_x)^2}, \psi_t - v_{ki}\sqrt{1+(\psi_x)^2}\} \leqq 0 \tag{3.15}$$

holds at $(s(t_0), t_0)$.

DEFINITION 3.3. We say that

$$\theta_i(t) = \theta_i^* \quad \text{weakly } \forall t > 0, \tag{3.16}$$

if $\theta_i(t) \leqq \theta_i^*$ weakly for all $t > 0$ and $\theta_i(t) \geqq \theta_i^*$ weakly for all $t > 0$.

Finally we give the definition of *weak solution* of

$$\text{Normal Velocity of } \Gamma_{ij} = V_{ij} = v_{ij} = \mu_{ij}(e_i - e_j). \tag{3.17}$$

subject to the angle conditions. In this formulation we interpret (3.17) as in the theory of viscosity solutions [**6, 7**], and the angle conditions as in Definition 3.3. Observe that if the parametrisation p_{ij} is differentiable at some (s, t) with $|p_{ij_s}(s, t)| \neq 0$, then (3.17) is equivalent to

$$p_{ij_t} \cdot n_{ij} = v_{ij}, \tag{3.18}$$

where $n_{ij}(s, t)$ denotes the unit vector normal to Γ_{ij} pointing into $\Omega_j(t)$.

DEFINITION 3.4. Let the domains Ω_i be as in Definition 3.1. We shall say that $\{\Omega_i : i = 1, 2, 3\}$ is a weak (viscosity) solution of (3.17) and the angle conditions, if they satisfy (3.16) and they solve (3.17) in the sense of viscosity solutions (see [**4, 8**]).

In Section 4, we shall show that there exists a unique viscosity solution in the sense of the above definition, see Theorems 4.3 and 4.4. In the case of Figure 2.2, it can be easily checked that the weak solution with $\theta_i^* = 120°$ is the one depicted in Figure 2.2(b); in particular, the solution depicted in Figure 2.2(a) *is not* a weak solution in the sense of Definition 3.4. To see this, let us rotate (90° clockwise) the graph in Figure 2.2(a), and define $u(x, t)$ and $w(x, t)$ as in Figure 3.2(a).

If α and β are as in (2.3), then *Angle* $(u(\cdot, t), w(\cdot, t)) > 120°$. We want to show that this angle *is not* less than or equal to 120° weakly, i.e. that the condition (3.15) in Definition 3.2, with $i = 2$, $j = 3$, $k = 1$, $\theta_2^* = 120°$, $v_{12} = -\alpha$ and $v_{32} = -\beta$, *is not* satisfied for some test functions φ and ψ. Let $\varphi_1(x, t)$ and $\psi_1(x, t)$ denote the solution corresponding to Figure 2.2(b), that is, the solution satisfying *Angle* $(\varphi_1(\cdot, t), \psi_1(\cdot, t)) = 120°$, see Figure 3.2(b). Then if $(0, -\alpha t)$ and $(0, -\sigma t)$ denote the positions of the triple junction in Figures 3.2(a) and (b), respectively, the functions

$$\varphi(x, t) = \varphi_1(x, t) + (\sigma - \alpha)t \quad \text{and} \quad \psi(x, t) = \psi_1(x, t) + (\sigma - \alpha)t$$

are admissible test functions for the proposed solution $u(x, t)$, $w(x, t)$. For these test functions we have, at $x = 0$,

$$\varphi_t - v_{32}\sqrt{1 + (\varphi_x)^2} = \varphi_{1_t} + \sigma - \alpha + \beta\sqrt{1 + (\varphi_{1_x})^2} = \sigma - \alpha > 0 \tag{3.19}$$

and

$$\psi_t - v_{12}\sqrt{1 + (\psi_x)^2} = \psi_{1_t} + \sigma - \alpha + \alpha\sqrt{1 + (\psi_{1_x})^2} = \sigma - \alpha > 0. \tag{3.20}$$

Thus, we can slightly change the slopes of φ and ψ and still maintain the above inequalities, while at the same time satisfying

$$\text{Angle}\,(\varphi(\cdot, t), \psi(\cdot, t)) - 120° > 0. \tag{3.21}$$

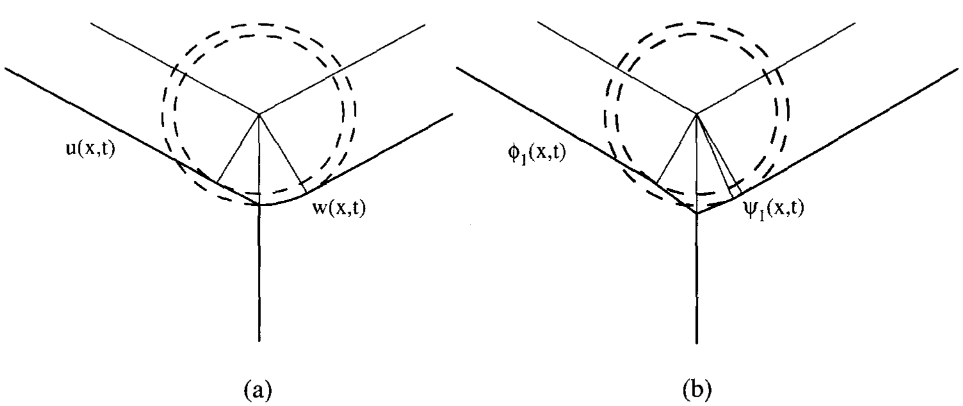

Figure 3.2. (a) The solution proposed in [11] (see also Fig. 2.2) and (b) comparison functions for the weak formulation. The *Angle* $(u(\cdot, t), w(\cdot, t)) > 120°$ but these comparison functions show that *Angle* $(u(\cdot, t), w(\cdot, t))$ *is not* $\leq 120°$ **weakly**.

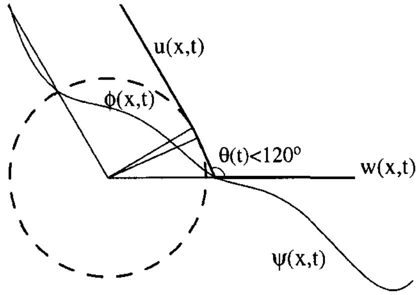

Figure 3.3. The VST solution *does* satisfy the weak formulation: $\theta(t) \geqq 120°$ **weakly**.

Since inequalities (3.19)–(3.21) contradict (3.15) we conclude that this is not a viscosity solution.

Finally, we want to show that the solution in Figure 2.2(b) is a viscosity solution. Since strongly *Angle* $(\Gamma_{12}(t), \Gamma_{23}(t)) = 120°$, *Angle* $(\Gamma_{12}(t), \Gamma_{13}(t)) < 120°$ and *Angle* $(\Gamma_{23}(t), \Gamma_{13}(t)) > 120°$, we need only verify that

$$\textit{Angle}\,(\Gamma_{12}(t), \Gamma_{13}(t)) \geqq 120° \quad \text{and} \quad \textit{Angle}\,(\Gamma_{23}(t), \Gamma_{13}(t)) \leqq 120° \ \textbf{weakly}.$$

We shall only show that the first inequality holds (in the weak sense), since the second inequality can be checked in an analogous manner. For this, consider smooth test functions $\varphi(x, t)$ $(x < s(t))$ and $\psi(x, t)$ $(x > s(t))$ such that $G - F$ (cf. (3.6), (3.10)) has a local minimum at $(s(t_0), t_0)$ for some $t_0 > 0$ (see Fig. 3.3). We want to show that the inequality (3.14) holds at $(s(t_0), t_0)$ with $i = 1$, $j = 2$, $k = 3$, $v_{31} = 0$ and $\theta_1^* = 120°$. If we assume that

$$\textit{Angle}\,(\varphi(\cdot, t_0), \psi(\cdot, t_0)) < 120°,$$

then we need to show that

$$\max\,\{\varphi_t - v_{21}\sqrt{1 + (\varphi_x)^2}, \psi_t\} \geqq 0 \quad \text{at } (s(t_0), t_0). \tag{3.22}$$

Now, since $w(s(t), t) - \psi(s(t), t)$ has a local minimum at $t = t_0$, we have

$$0 = \frac{d}{dt}(w(s(t), t) - \psi(s(t), t))|_{t=t_0} = -\psi_t(s(t_0), t_0) - s'(t_0)\psi_x(s(t_0), t_0)$$

that is,

$$\psi_t(s(t_0), t_0) = -s'(t_0)\psi_x(s(t_0), t_0).$$

But since $0 = \psi(s(t_0), t_0) \geqq \psi(x, t_0)$, it follows that $\psi_x(s(t_0), t_0) \leqq 0$, so that (since $s'(t) > 0$)

$$\psi_t(s(t_0), t_0) \geqq 0$$

and (3.22) holds.

An equivalent formulation

For solutions that are differentiable at the triple junction, like the ones discussed at the end of the previous subsection, the definition of weak solution can be formulated in terms of *algebraic* conditions. In order to derive these conditions, consider a situation like the one in Figure 3.1 where (with $i=1, j=2, k=3$)

$$\Gamma_{12}(t)\cap\mathcal{O}=\{(x,y)\in\mathcal{O}: y=u(x,t), x\leqq s(t)\}$$

and

$$\Gamma_{13}(t)\cap\mathcal{O}=\{(x,y)\in\mathcal{O}: y=w(x,t), x\geqq s(t)\}$$

(cf. (3.5)). Let $\varphi(x,t)$ and $\psi(x,t)$ be smooth test functions and let F and G be defined by (3.10) and (3.6). Assume that $G-F$ has a minimum at $(s(t_0),t_0)$ and that

$$\textit{Angle}\,(\varphi(\cdot,t_0),\psi(\cdot,t_0))\equiv\theta_1<\theta_1^*.$$

Then, using (3.14), we have that $\textit{Angle}\,(u(\cdot,t),w(\cdot,t))\geqq\theta_1^*$ *weakly* at t_0 if and only if

$$\max\{\varphi_t-v_{21}\sqrt{1+(\varphi_x)^2},\psi_t-v_{31}\sqrt{1+(\psi_x)^2}\}\geqq 0\quad\text{at }(s(t_0),t_0). \tag{3.23}$$

Let $T(t)=(s(t),d(t))$ denote the position of the triple junction and n_u, n_w, n_φ and n_ψ the unit normal vectors to the graphs of u, w, φ and ψ at $(s(t_0),d(t_0))$ (pointing towards Ω_1). Since (u,w) is a differentiable solution, we have

$$T_t(t_0)\cdot n_u=v_{21}\quad\text{and}\quad T_t(t_0)\cdot n_w=v_{31}.$$

Thus,

$$T_t(t_0)=An_u+Bn_w, \tag{3.24}$$

where

$$\begin{aligned}A&=\frac{(v_{21}-v_{31}n_u\cdot n_w)}{1-(n_u\cdot n_w)^2}=\frac{(v_{21}+v_{31}\cos(\theta_1))}{\sin^2(\theta_1)},\\ B&=\frac{(-v_{21}n_u\cdot n_w+v_{31})}{1-(n_u\cdot n_w)^2}=\frac{(v_{21}\cos(\theta_1)+v_{31})}{\sin^2(\theta_1)}.\end{aligned} \tag{3.25}$$

On the other hand, since $\varphi(s(t),t)=d(t)$,

$$T_t\cdot n_\varphi=(s',d')\cdot\frac{(-\varphi_x,1)}{\sqrt{1+(\varphi_x)^2}}=\frac{-s'\varphi_x+d'}{\sqrt{1+(\varphi_x)^2}}=\frac{\varphi_t}{\sqrt{1+(\varphi_x)^2}}$$

and similarly for $T_t\cdot n_\psi$, so that (3.23) is equivalent to

$$\max\{T_t\cdot n_\varphi-v_{21},T_t\cdot n_\psi-v_{31}\}\geqq 0\quad\text{at }(s(t_0),t_0).$$

Then, using (3.24), the above condition becomes

$$\max\{An_u\cdot n_\varphi+Bn_w\cdot n_\varphi-v_{21},An_u\cdot n_\psi+Bn_w\cdot n_\psi-v_{31}\}\geqq 0, \tag{3.26}$$

at $(s(t_0),t_0)$. Now, let

$$\textit{Angle}\,(u(\cdot,t_0),\psi(\cdot,t_0))\equiv\eta\geqq\theta_1,$$
$$\textit{Angle}\,(w(\cdot,t_0),\varphi(\cdot,t_0))\equiv\zeta\geqq\theta_1,$$

so that

$$n_u\cdot n_\varphi=\cos(\zeta-\theta_1),$$
$$n_w\cdot n_\varphi=-\cos(\zeta),$$

$$n_u \cdot n_\psi = -\cos(\eta),$$
$$n_w \cdot n_\psi = \cos(\eta - \theta_1).$$

Then, substituting (3.25) and (3.27) into (3.26), we obtain

$$\max\{(\sin(\zeta)\sin(\theta_1) - \sin^2(\theta_1))v_{21} + (-\cos(\zeta)\sin^2(\theta_1) + \sin(\zeta)\cos(\theta_1)\sin(\theta_1))v_{31},$$
$$(-\cos(\eta)\sin^2(\theta_1) + \sin(\eta)\cos(\theta_1)\sin(\theta_1))v_{21} + (\sin(\eta)\sin(\theta_1) - \sin^2(\theta_1))v_{31}\} \geqq 0,$$

or, equivalently,

$$\max\left\{\frac{1}{\sin(\theta_1)}v_{21} + \frac{\cos(\theta_1)}{\sin(\theta_1)}v_{31} - \frac{1}{\sin(\zeta)}v_{21} - \frac{\cos(\zeta)}{\sin(\zeta)}v_{31},\right.$$
$$\left.\frac{1}{\sin(\theta_1)}v_{31} + \frac{\cos(\theta_1)}{\sin(\theta_1)}v_{21} - \frac{1}{\sin(\eta)}v_{31} - \frac{\cos(\eta)}{\sin(\eta)}v_{21}\right\} \geqq 0. \quad (3.27)$$

Thus, since $Angle(\varphi(\cdot, t), \psi(\cdot, t)) = \zeta + \eta - \theta_1$, the condition (3.23) holds if and only if (3.27) is satisfied for all ζ and η with

$$\theta_1 \leqq \zeta, \eta < \pi \quad \text{and} \quad \zeta + \eta - \theta_1 < \theta_1^*.$$

Analogous equivalent conditions can be derived for the definition of $\theta_1 \leqq \theta_1^*$ *weakly* as well as for θ_2 and θ_3. Indeed, if we let

$$H_{ijk}(\lambda) = \frac{1}{\sin(\lambda)}v_{ji} + \frac{\cos(\lambda)}{\sin(\lambda)}v_{ki} \quad (i \neq j \neq k = 1, 2, 3),$$

then we have the following theorem:

THEOREM 3.5. *Assume the domains Ω_i are differentiable at the triple junction. Then $\theta_i(t) \geqq \theta_i^*$ (respectively $\theta_i(t) \leqq \theta_i^*$) weakly if and only if either*

$$\theta_i(t) \geqq \theta_i^* \quad (\textit{respectively } \theta_i(t) \leqq \theta_i^*) \textit{ strongly},$$

or

$$\max(\textit{respectively} \min)\{H_{i,i+1,i+2}(\theta_i(t)) - H_{i,i+1,i+2}(\zeta),$$
$$H_{i,i+2,i+1}(\theta_i(t)) - H_{i,i+2,i+1}(\eta)\} \geqq 0 \ (\textit{respectively} \leqq 0) \quad (3.28)$$

for all ζ and η satisfying

$$\theta_i(t) \leqq \zeta, \eta < \pi \quad \textit{and} \quad \zeta + \eta - \theta_i(t) < \theta_i^*,$$
$$(\textit{respectively } 0 < \zeta, \eta \leqq \theta_i(t) \quad \textit{and} \quad \zeta + \eta - \theta_i(t) > \theta_i^*). \quad (3.29)$$

Here the indices $i+1$ and $i+2$ should be interpreted modulo 3.

A catalogue of weak solutions

In the previous sections we have presented the general definition of weak solutions as well as the equivalent formulation for solutions which are differentiable at the triple junction. We can now attempt to catalogue them according to the interface velocities and initial conditions. Since the behaviour at the triple junction is a local property, we shall restrict our attention to solutions with linear initial data. Furthermore, for simplicity we will assume that the smallest (in magnitude) interface velocity is equal to zero and that $\theta_i^* = 120°$. Under these assumptions, the five possible solutions for *compatible* initial data are presented in Figure 3.4. Again we note here that the solutions in Figure 3.4(a) do not always coincide with the ones derived in [**12**].

Finally, Figure 3.5 shows solutions for *incompatible* initial data. Notice here that if the initial angles are not too far from 120°, then the solution will produce a corner in order to keep an angle at 120° for $t > 0$ (Fig. 3.5(a)). On the other hand, if the initial angles differ from 120° by more than a critical amount (which can be explicitly

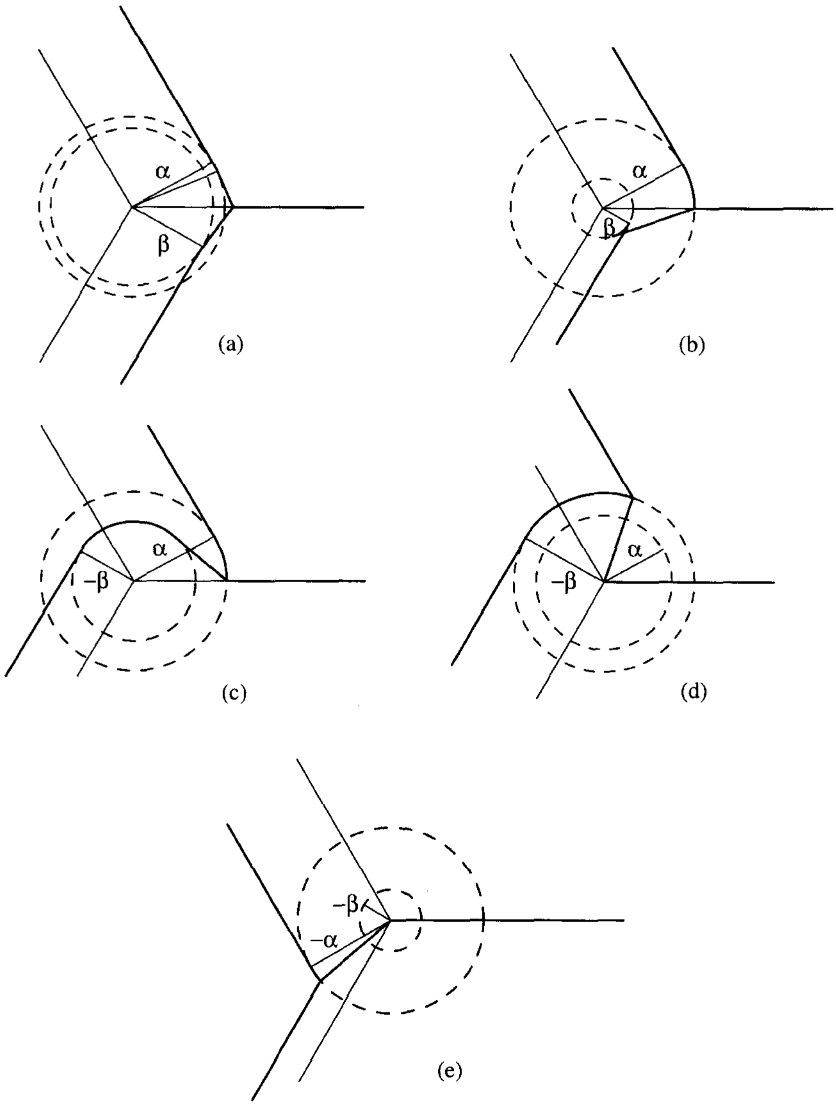

Figure 3.4. A catalogue of weak solutions for compatible initial data: (a) $\alpha > \beta > \alpha/2 > 0$; (b) $\alpha/2 > \beta > 0$; (c) $\alpha > -\beta > 0$; (d) $-\beta > \alpha > 0$; (e) $-\alpha > -\beta > 0$.

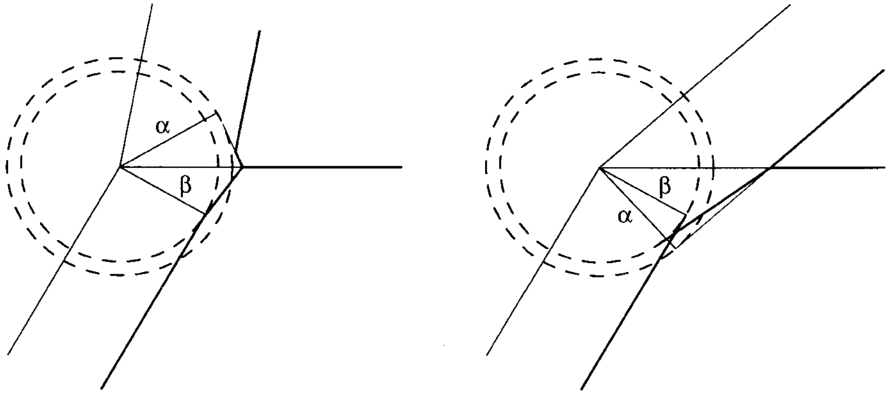

Figure 3.5. The weak solution for incompatible initial data. (a) One of the angle conditions is preserved; (b) no angle condition is preserved in the strong sense.

calculated) then it becomes impossible to keep the angle condition in the strong sense (Fig. 3.5(b)).

4. Existence and uniqueness of weak solutions

In this section, we prove a local existence and uniqueness result for differentiable solutions of the system of Hamilton–Jacobi equations (3.18) subject to the weak angle conditions (3.16).

Existence

For the sake of clarity, we shall restrict ourselves to the case in which

$$v_{21}=\alpha \geqq v_{23}=\beta > v_{13}=0,$$

with

$$\alpha \leqq 2\beta$$

and uniform angle conditions

$$\theta_i(t)=\theta_i^*=\frac{2\pi}{3}, \quad i=1,2,3 \quad t>0. \tag{4.1}$$

In this case the weak angle conditions (3.28) can be simplified, as we show in the following lemma.

LEMMA 4.1. *Suppose that* $\theta_2(t)<\pi$. *Then the angle conditions* (4.1) *hold weakly if, and only if,*

$$\theta_2(t)\in\left\{\frac{2\pi}{3}\right\}\cup[\theta_0,\pi), \tag{4.2}$$

where $\theta_0 \in [2\pi/3, \pi)$ satisfies

$$\cos(\theta_0) = -\frac{(4\alpha\beta - \alpha^2 - \beta^2)}{2(\alpha^2 + \beta^2 - \alpha\beta)}. \tag{4.3}$$

Proof. First we show that $\theta_1(t) \geqq 2\pi/3$ weakly regardless of the value of $\theta_1(t)$. In view of Theorem 3.5, we need to show that, whenever $\theta_1(t) < 2\pi/3$, we have

$$\max\{H_{1,2,3}(\theta_1(t)) - H_{1,2,3}(\xi), H_{1,3,2}(\theta_1(t)) - H_{1,3,2}(\eta)\} \geqq 0 \tag{4.4}$$

for all ξ and η satisfying

$$\theta_1(t) \leqq \xi, \eta < \pi \quad \text{and} \quad \xi + \eta - \theta_1(t) < \theta_1^*.$$

Here,

$$H_{1,2,3}(\xi) = \frac{\alpha}{\sin(\xi)} \quad \text{and} \quad H_{1,3,2}(\eta) = \frac{\alpha\cos(\eta)}{\sin(\eta)}.$$

Then, since $H'_{1,3,2}(\eta) < 0$, we have

$$H_{1,3,2}(\theta_1(t_0)) - H_{1,3,2}(\eta) \geqq 0$$

for $\eta \in [\theta_1(t_0), 2\pi/3]$ and (4.4) holds. The opposite inequality, $\theta_1(t) \leqq 2\pi/3$, as well as the weak equality $\theta_3(t) = 2\pi/3$ can be proved in a similar manner. Thus, in this case, the weak angle conditions impose no restrictions on $\theta_1(t)$ and $\theta_3(t)$.

Finally, we want to show that $\theta_2(t) = 2\pi/3$ weakly if and only if (4.2) holds. Since

$$H_{2,1,3}(\xi) = -\frac{(\alpha + \beta\cos(\xi))}{\sin(\xi)} \quad \text{and} \quad H_{2,3,1}(\eta) = -\frac{(\alpha\cos(\eta) + \beta)}{\sin(\eta)},$$

we have $H'_{2,1,3}(\xi) > 0$ for $\xi < \arccos(-\beta/\alpha)$ and $H'_{2,3,1}(\eta) > 0$ for all η. Since $\arccos(-\beta/\alpha) \geqq 2\pi/3$, it follows from Theorem 3.5 that $\theta_2(t) \geqq 2\pi/3$ weakly if and only if $\theta_2(t) \geqq 2\pi/3$ strongly. On the other hand, $\theta_2(t) \leqq 2\pi/3$ weakly if and only if $\theta_2(t) \leqq 2\pi/3$ strongly or $H_{2,1,3}(\theta_2(t)) \leqq H_{2,1,3}(\xi)$ for all ξ with $2\pi/3 \leqq \xi \leqq \theta_2(t)$. In the latter case, since $H_{2,1,3}(\xi)$ has a maximum at $\xi = \arccos(-\beta/\alpha) > 2\pi/3$, we must have

$$\pi > \theta_2(t) \geqq \theta_0,$$

where $\theta_0 > \arccos(-\beta/\alpha)$ satisfies

$$H_{2,1,3}(\theta_0) = H_{2,1,3}\left(\frac{2\pi}{3}\right)$$

or, equivalently, θ_0 is given by (4.3). □

REMARK 4.2. Notice that the condition (4.2) is in agreement with the construction of weak solutions for incompatible initial data shown in Figure 3.5. Indeed, Figure 3.5(a) corresponds to solutions for which $\theta_2(t) = 2\pi/3$, while the solution in Figure 3.5(b) satisfies $\theta_2(t) \in [\theta_0, \pi)$.

With the help of Lemma 4.1, we will now establish the existence of weak solutions for smooth initial data. First notice that if a solution satisfies

$$\theta_2(t) = \frac{2\pi}{3} \quad \text{strongly} \tag{4.5}$$

then (see Fig. 2.2(b))

$$\theta_1(t)=\frac{\pi}{2}+\bar{\eta}, \quad \theta_3(t)=\frac{\pi}{2}+\hat{\eta}, \tag{4.6}$$

where $\bar{\eta}, \hat{\eta} \in [0, \pi/3]$ are given by

$$\cos(\bar{\eta})=\frac{\alpha}{\sigma}, \quad \cos(\hat{\eta})=\frac{\beta}{\sigma}, \quad \sigma=\frac{2}{\sqrt{3}}\sqrt{\alpha^2+\beta^2-\alpha\beta}$$

and

$$\left|\frac{d}{dt}T(t)\right|=\sigma. \tag{4.7}$$

We consider initial data of the following form

$$\begin{aligned} T(0)&=(0,0),\\ \Gamma_{13}&=\{(x, w_0(x)): x \geqq 0\},\\ \Gamma_{12}&=\{(u_0(y), y): y \geqq 0\},\\ \Gamma_{23}&=\{(v_0(y), y): y \leqq 0\},\end{aligned}$$

where u_0, v_0, w_0 are smooth functions satisfying

$$\begin{aligned} &\text{(i)} && u_0(0)=w_0(0)=v_0(0),\\ &\text{(ii)} && u_0, w_0, v_0 \in C^2(\mathbb{R}) \text{ and their limits as } |x|\to\infty \text{ exist},\\ &\text{(iii)} && w_0'(0)=0\\ &\text{(iv)} && u_0'(0)<\tan(\bar{\eta}),\\ &\text{(v)} && v_0'(0)>-\tan(\hat{\eta}).\end{aligned} \tag{4.8}$$

We look for a solution of the form

$$\begin{aligned} T(t)&=(s(t), w_0(s(t))),\\ \Gamma_{13}&=\{(x, w_0(x)): x \geqq s(t)\},\\ \Gamma_{12}&=\{(u(y,t), y): y \geqq w_0(s(t))\},\\ \Gamma_{23}&=\{(v(y,t), y): y \leqq w_0(s(t))\},\end{aligned}$$

satisfying (4.5) and (4.6). Here $s(t)$ is an increasing function that, in view of (4.7), solves

$$s'(t)=\frac{\sigma}{\sqrt{1+(w_0'(s(t)))^2}}, \tag{4.9}$$

while the functions u and v solve (in the viscosity sense)

$$\begin{aligned} u_t&=\alpha\sqrt{1+u_y^2}, \quad y>w_0(s(t)), \quad t>0,\\ v_t&=\beta\sqrt{1+v_y^2}, \quad y<w_0(s(t)), \quad t>0,\\ u(w_0(s(t)),t)&=v(w_0(s(t)),t)=s(t), \quad t>0.\end{aligned} \tag{4.10}$$

THEOREM 4.3 (Existence). *Let u_0, v_0 and w_0 satisfy* (4.8) *and let $s(t)$ denote the unique solution of* (4.9) *with $s(0)=0$. Then there exist $T>0$ and functions $u(y,t)$ and $v(y,t)$ such that (u,v) is a viscosity solution of* (4.10) *for $t\in(0,T)$ with initial data (u_0,v_0). Moreover, (u,v) is differentiable at $(w_0(s(t)),t)$ for all $t\in(0,T)$ and*

$$\begin{aligned} u_y(w_0(s(t)),t) &= \tan(\bar{\eta}+\theta(t)),\\ v_y(w_0(s(t)),t) &= -\tan(\hat{\eta}-\theta(t)), \end{aligned} \tag{4.11}$$

where $|\theta(t)|<\pi/2$ satisfies $\tan(\theta(t))=w_0'(s(t))$.

Proof. We shall construct the solution $v(y,t)$ in the form of the value function for a deterministic control problem. A construction using the method of characteristics is also possible and we will indicate the connection between these two approaches below. The construction of $u(y,t)$ is similar.

First we write the equation for v in (4.10) in the form

$$v_t = H(v_y), \tag{4.12}$$

where

$$H(p)=\beta\sqrt{1+p^2}=\sup_{|b|\leqq\beta}(-bp+\sqrt{\beta^2-b^2}).$$

The above representation for H suggests that (4.12) is related to the following control problem [7]: fix (y,t) with $y\leqq w_0(s(t))$, $t>0$. Let $\mathscr{A}(y,t)$ denote the collection of all pairs $(\xi(\cdot),\tau)$ satisfying

$$\begin{gathered} \xi:[0,t]\to\mathbb{R},\quad \tau\in[0,t],\\ \xi(r)\leqq w_0(s(r)),\quad \forall r\in[\tau,t],\\ \xi(\tau)=w_0(s(\tau)),\quad \text{if } \tau>0,\\ \xi(t)=y,\\ |\xi'(r)|\leqq\beta,\quad \forall r\in[0,t]. \end{gathered}$$

For $(\xi(\cdot),\tau)\in\mathscr{A}(y,t)$, set

$$J(y,t;\xi(\cdot),\tau)=\int_\tau^t\sqrt{\beta^2-(\xi'(s))^2}\,ds+s(\tau)\chi_{\tau>0}+v_0(\xi(0))\chi_{\tau=0}$$

and

$$v(y,t)=\sup_{(\xi(\cdot),\tau)\in\mathscr{A}(y,t)}J(y,t;\xi(\cdot),\tau).$$

Then $v(y,t)$ is the viscosity solution of (4.12), see e.g. [7]. Furthermore, the maximisers in the definition of v above are straight lines. Hence,

$$v(y,t)=\max_{b\in\mathbb{R}}\{(t-\tau)\sqrt{\beta^2-b^2}+v_0(y-bt)\chi_{\tau=0}+s(\tau)\chi_{\tau>0}\},$$

where for any $b\in\mathbb{R}$, $\tau\in[0,t]$ is defined by

$$\tau=\inf\{\rho\in[0,t]:y+b(r-t)\leqq w_0(s(r)),\quad \forall r\in[\rho,t]\}.$$

Now, observe that whenever $v_0'(0)\leqq 0$, the assumption

$$v_0'(0)>-\tan(\hat{\eta})$$

yields

$$\frac{\beta}{\sigma} < \frac{1}{\sqrt{1 + (v_0'(0))^2}}.$$

Then an elementary analysis using $s'(0) = \sigma$ and the above fact implies that $v(w_0(s(t)), t) = s(t)$ for all sufficiently small $t \geqq 0$. Indeed, there exist $T > 0$, $\delta(t) > 0$ such that for $t \in (0, T]$ and $y \in (w_0(s(t)) - \delta(t), w_0(s(t))]$ we have

$$v(y, t) = V(\tau, t) \equiv s(\tau) + \beta(t - \tau)\cos(\theta(\tau) - \hat{\eta}), \tag{4.13}$$

where $\theta(\cdot)$, $\hat{\eta}$ are as in the statement of the theorem and $\tau = \tau(y, t)$ is the solution of

$$y = w_0(s(\tau)) + \beta(t - \tau)\sin(\theta(\tau) - \hat{\eta}). \tag{4.14}$$

Thus, differentiating (4.13) and (4.14) with respect to y and evaluating at $(w_0(s(t)), t)$, we obtain

$$v_y = (s'(t) - \beta\cos(\theta(t) - \hat{\eta}))\tau_y$$

and

$$1 = (w_0'(s(t))s'(t) - \beta\sin(\theta(t) - \hat{\eta}))\tau_y.$$

Since

$$s'(t) = \sigma\cos(\theta(t)), \quad w_0'(s(t))s'(t) = \sigma\sin(\theta(t)),$$

we have

$$v_y = \frac{\sigma\cos(\theta(t)) - \beta\cos(\theta(t) - \hat{\eta})}{\sigma\sin(\theta(t)) - \beta\sin(\theta(t) - \hat{\eta})}$$

and, since $\cos(\hat{\eta}) = \beta/\sigma$,

$$\begin{aligned} v_y &= \frac{\cos(\theta(t)) - \cos(\hat{\eta})\cos(\theta(t) - \hat{\eta})}{\sin(\theta(t)) - \cos(\hat{\eta})\sin(\theta(t) - \hat{\eta})} \\ &= \frac{\cos(\hat{\eta} + (\theta(t) - \hat{\eta})) - \cos(\hat{\eta})\cos(\theta(t) - \hat{\eta})}{\sin(\hat{\eta} + (\theta(t) - \hat{\eta})) - \cos(\hat{\eta})\sin(\theta(t) - \hat{\eta})} \\ &= -\tan(\hat{\eta} - \theta(t)) \end{aligned} \tag{4.15}$$

and the proof of the theorem is complete.

Finally, let us mention that one could also use the method of characteristics to derive (4.13). Indeed,

$$Y(t; \tau) = w_0(s(\tau)) + \beta(t - \tau)\sin(\theta(\tau) - \hat{\eta})$$

is the characteristic curve for (4.12) emanating from the boundary point $(w_0(s(\tau)), \tau)$. Characteristics emanating from the initial data are

$$\bar{Y}(t; \hat{y}) = \hat{y} + \beta t\frac{v_0'(\hat{y})}{\sqrt{1 + (v_0'(\hat{y}))^2}}$$

and, for $t \leqq t_0(\hat{y})$,

$$v(\bar{Y}(t;\hat{y}),t)=\bar{V}(\hat{y},t)\equiv v_0(\hat{y})+\beta t\frac{1}{\sqrt{1+(v_0'(\hat{y}))^2}}.$$

Now assume that $Y(t;\tau)=\bar{Y}(t;\hat{y})$ for some $0\leqq\tau\leqq t$ and $\hat{y}\leqq 0$. Then, for sufficiently small $t>0$, we have $V(t,\tau)>\bar{V}(\hat{y},t)$ and (4.13) follows. □

Uniqueness

In this section we prove that the solution $(u(y,t), v(y,t), w_0(x))$ constructed in the previous subsection is the *unique* viscosity solution of (3.16) and (3.18). Suppose that there is another solution $\{\hat{\Gamma}_{ij}:t\in[0,T]\}$ of (3.16) and (3.18), satisfying (3.1), (3.2), (3.7). We further assume that there is a neighbourhood $\mathcal{N}$ of $\{(t,0):t\in(0,T]\}$ such that for $i\neq j\in\{1,2,3\}$

$$\hat{p}_{ij}\in C^1(\mathcal{N}),\quad |\hat{p}_{ij,s}|>0\quad\text{on } \mathcal{N}, \tag{4.16}$$

where $\hat{p}_{ij}$ is a parametrisation of the $\hat{\Gamma}_{ij}$'s. Let $\hat{T}(t)$ be the triple point of $\hat{\Gamma}$. Since $\hat{p}_{ij}$ is differentiable on $\mathcal{N}$, for $t\in(0,T]$

$$\hat{T}'(t)\cdot\hat{n}_{13}(0,t)=0,\quad \hat{T}'(t)\cdot\hat{n}_{21}(0,t)=\alpha,\quad \hat{T}'(t)\cdot\hat{n}_{23}(0,t)=\beta. \tag{4.17}$$

Moreover, the angles between the arcs $\hat{\Gamma}_{ij}$ are well defined and by (4.1) we have,

$$\text{if } \theta_2(t)\neq\frac{2\pi}{3}\text{ strongly},\quad\text{then } \theta_2(t)\geqq\theta_0, \tag{4.18}$$

where $\theta_0\in(2\pi/3,\pi)$ is as in (4.3). Since $\theta_2\geqq 2\pi/3$, and $\hat{T}'(t)\cdot\hat{n}_{13}(0,t)=0$, it is elementary to show that

$$\hat{V}(t):=\hat{T}'(t)\cdot\frac{\hat{p}_{13,s}(0,t)}{|\hat{p}_{13,s}(0,t)|}\geqq\sigma>\alpha, \tag{4.19}$$

where σ is as in (4.7).

THEOREM 4.4 (Uniqueness). *$\hat{\Gamma}_{ij}(t)=\Gamma_{ij}(t)$ for all $t\leqq T$ and $i,j=1,2,3$.*

Proof. First we will show that

$$\hat{\Gamma}_{13}(t)\subset\Gamma_{13}(t),\quad\forall t\in[0,T]. \tag{4.20}$$

Fix $t_0>0$. In view of (4.16), there is $\delta=\delta(t_0)>0$ such that the arc

$$S_\delta(t)=\hat{\Gamma}_{13}(t)\cap\{(x,y):|(x,y)-\hat{T}(t_0)|<\delta\}$$

is continuously differentiable for all $t_0\leqq t\leqq t_0+\delta$. Suppose that $X\in S_\delta(t)$ for some $t_0\leqq t\leqq t_0+\delta$ and $X\neq\hat{T}(t)$. Then $X\neq\hat{T}(s)$ for all s near t. Set

$$s(X,t)=\begin{cases} t_0, & \text{if } X\neq\hat{T}(s),\quad\forall s\in[t_0,t],\\ \sup\{s<t:X=\hat{T}(s)\}, & \text{otherwise.}\end{cases}$$

Since the normal velocity at X is zero, we conclude that $X\in\hat{\Gamma}_{13}(s(X,t))$. Hence, for all $t_0\leqq t\leqq t_0+\delta$, we have

$$S_\delta(t)\subset S_\delta(t_0)\cup\{\hat{T}(s):t_0\leqq s\leqq t_0+\delta\}.$$

If, for some $t_0 \leqq t \leqq t_0 + \delta$, $\hat{T}(t) \notin \hat{\Gamma}_{13}(t_0)$, then

$$\frac{\hat{p}_{13,s}(0, t)}{|\hat{p}_{13,s}(0, t)|} = -\frac{\hat{T}'}{|\hat{T}'|},$$

which contradicts (4.19). Hence for all $t \in [t_0, t_0 + \delta]$,

$$\hat{T}(t) \in \hat{\Gamma}_{13}(t_0),$$

and consequently,

$$\hat{T}(t) = \hat{p}_{13}(s(t), t_0),$$

for some $s(t)$.

Now extend $\hat{\Gamma}_{13}(t)$ in the following way:

$$\bar{\Gamma}_{13}(t) = \hat{\Gamma}_{13}(t) \cup \{\hat{p}_{13}(s, t_0) : s \in [0, s(t)]\} \cup \{\hat{T}(t_0) - \tau \hat{p}_{13,s}(s(t_0), t_0) : \tau < 0\}.$$

Then for $t \in [t_0, t_0 + \delta]$, $\bar{\Gamma}_{13}$ solves a two-phase geometric problem with normal velocity zero. By the uniqueness result for the two-phase problem (see e.g. [**4**]), we conclude that $\bar{\Gamma}_{13}(t) = \bar{\Gamma}_{13}(t_0)$, for all $t \in [t_0, t_0 + \delta]$. (4.20) follows from the continuity of $\hat{\Gamma}$.

By (4.20), we conclude that

$$\hat{T}(t) = (x(t), w_0(x(t))).$$

We claim that $x(t)$ is equal to the unique solution of (4.9) and that $\theta_2(t) = 2\pi/3$ for all $t > 0$. Indeed, if $\theta_2(t) \neq 2\pi/3$ for some $t > 0$, then, by (4.18), $\theta_2(t) \neq 2\pi/3$ for all $t > 0$. Moreover, by (4.17) we conclude that $\theta_1 < \pi/2$.

Set

$$d(t) = \sup \{y : (x, y) \in \hat{\Gamma}_{12}(t) \text{ for some } x\}.$$

Since $\theta_1 < \pi/2$, and $\hat{\Gamma}$ is continuous in the Hausdorff topology, we conclude that the maximiser in the above expression is attained. Since $\hat{\Gamma}_{12}$ is a viscosity solution of the equation $V = \alpha$, we conclude that

$$\frac{d}{dt} d(t) \leqq \alpha$$

in the sense of viscosity solutions. Therefore, $d(t) \leqq \alpha t$. But this contradicts (4.19) and we conclude that $\theta_2(t) = 2\pi/3$ for all $t > 0$. Then from (4.17) we obtain that $x(t) = s(t)$. Hence

$$T(t) = \hat{T}(t), \quad \forall t \in [0, T].$$

Finally, we define

$$D = \{(x, y) : x \geqq 0, y \geqq w_0(x)\} \cup \{(x, y) : x \leqq 0, y \geqq 0\}.$$

Then $\hat{\Gamma}_{12}$ and Γ_{12} both solve a two-phase geometric problem with normal velocity α in D satisfying the Dirichlet data

$$\hat{\Gamma}_{12} \cap \partial D = T(t), \quad \forall t \in [0, T].$$

Hence by standard comparison results on viscosity solutions for two-phase problems, [**4**], we conclude that the two solutions coincide.

A similar argument shows that $\hat{\Gamma}_{23} = \Gamma_{23}$. $\square$

5. A simple convergence result

In this section, we prove the convergence of solutions in the symmetric case; in the notation of Section 4, we take $\alpha=\beta$ and $w_0(x)\equiv 0$, $u_0(y)=v_0(-y)$. Without loss of generality, we set $\alpha=1$. Then the solutions to the ε-problem (2.5) have the form,

$$\Gamma^\varepsilon_{13}(t)=\{(x,0):x\geqq s^\varepsilon(t)\},\quad t\geqq 0,$$
$$\Gamma^\varepsilon_{21}(t)=\{(u^\varepsilon(y,t),y):y\geqq 0\},\quad t\geqq 0,$$
$$\Gamma^\varepsilon_{23}(t)=\{(u^\varepsilon(-y,t),y):y\leqq 0\},\quad t\geqq 0,$$

where $s^\varepsilon(t)=u^\varepsilon(0,t)$ and u^ε solves,

$$u^\varepsilon_t=\varepsilon\frac{u^\varepsilon_{yy}}{1+(u^\varepsilon_y)^2}+\sqrt{1+(u^\varepsilon_y)^2}, \tag{5.1}$$

with Neumann data

$$u^\varepsilon_y(0,t)=\frac{1}{\sqrt{3}}. \tag{5.2}$$

We assume that there are constants $0<K\leqq 1/\sqrt{3}\leqq k$, satisfying

$$K\leqq u^\varepsilon_y(y,0)\leqq k,\quad \forall\, y\geqq 0.$$

Then, by the maximum principle, we have

$$K\leqq u^\varepsilon_y(y,t)\leqq k,\quad \forall\, y,t\geqq 0.$$

Hence for every $\varepsilon>0$ the solution exists for all time.

THEOREM 5.1 (Convergence). *As $\varepsilon\to 0$, u^ε converges locally uniformly to the unique viscosity solution of*

$$u_t=\sqrt{1+(u_y)^2}, \tag{5.3}$$

with Neumann boundary data (5.2). *Moreover, for all $t\geqq 0$, $u_y(0,t)$ exists and satisfies* (5.2) *pointwise and*

$$\lim_{\varepsilon\to 0} s^\varepsilon(t)=\frac{2}{\sqrt{3}}t.$$

Proof. Let $\Phi(y)$ be a smooth, convex function satisfying

$$\Phi(0)=0,\quad \Phi_y(0)=\frac{1}{\sqrt{3}},\quad \Phi_y(y)=k,\quad \forall\, y\geqq Y(\Phi), \tag{5.4}$$

with some constant $Y(\Phi)>0$ depending on Φ. For any Φ satisfying (5.4), define

$$\beta(\Phi)=\sup\left\{\frac{\Phi_{yy}}{1+(\Phi_y)^2}+\sqrt{1+(\Phi_y)^2}\right\},$$
$$c^\varepsilon(\Phi)=\sup_{y\geqq 0}\left\{-\varepsilon\Phi\left(\frac{y}{\varepsilon}\right)+ky\right\}.$$

(Observe that the above maximum is achieved at $Y(\Phi)$ and that $c^\varepsilon(\Phi)$ converges to zero as ε tends to zero.) Then, for any fixed $t_0 \geqq 0$, the function

$$\bar{u}^\varepsilon(y,t) = s^\varepsilon(t_0) + \varepsilon\Phi\left(\frac{y}{\varepsilon}\right) + \beta(\Phi)(t-t_0) + c^\varepsilon(\Phi), \quad t \geqq t_0,$$

is a supersolution of (5.1) and (5.2). Since $u^\varepsilon_y(y,t) \geqq k$,

$$u^\varepsilon(y,t_0) \leqq ky + s^\varepsilon(t_0) \leqq \bar{u}^\varepsilon(y,t_0), \quad \forall\, y \geqq 0.$$

Hence, by the maximum principle, $u^\varepsilon \leqq \bar{u}^\varepsilon$ and, in particular, for any $h \geqq 0$,

$$s^\varepsilon(t_0+h) = u^\varepsilon(0,t_0+h) \leqq \bar{u}^\varepsilon(0,t_0+h) = s^\varepsilon(t_0) + \beta(\Phi)h + c^\varepsilon(\Phi).$$

Therefore

$$s^\varepsilon(t_0+h) - s^\varepsilon(t_0) \leqq \beta(\Phi)h + c^\varepsilon(\Phi), \quad \forall\, h \geqq 0, \tag{5.5}$$

for any function Φ satisfying (5.4).

Similarly, let $\hat{\Phi}$ be a smooth, concave function satisfying (5.4) with k replaced by K, i.e. Φ satisfies

$$\hat{\Phi}(0) = 0, \quad \hat{\Phi}_y(0) = \frac{1}{\sqrt{3}}, \quad \hat{\Phi}_y(y) = K, \quad \forall\, y \geqq Y(\hat{\Phi}), \tag{5.6}$$

and set

$$\hat{\beta}(\hat{\Phi}) = \inf\left\{\frac{\hat{\Phi}_{yy}}{1+(\hat{\Phi}_y)^2} + \sqrt{1+(\hat{\Phi}_y)^2}\right\},$$

$$\hat{c}^\varepsilon(\hat{\Phi}) = \sup_{y \geqq 0}\left\{\varepsilon\hat{\Phi}\left(\frac{y}{\varepsilon}\right) - Ky\right\}.$$

Then, arguing as above,

$$s^\varepsilon(t_0+h) - s^\varepsilon(t_0) \geqq \hat{\beta}(\hat{\Phi})h - \hat{c}^\varepsilon(\hat{\Phi}), \quad \forall\, h \geqq 0. \tag{5.7}$$

Now let,

$$s^*(t) = \limsup_{e \to 0, r \to t} s^\varepsilon(r), \quad s_*(t) = \liminf_{\varepsilon \to 0, r \to t} s^\varepsilon(r).$$

Since $c^\varepsilon(\Phi)$ converges to zero as $\varepsilon \to 0$, (5.5) implies that, for any $t, h \geqq 0$, we have

$$s^*(t+h) - s_*(t) \leqq \beta(\Phi)h,$$

for all Φ satisfying (5.4), and similarly by (5.7),

$$s_*(t+h) - s^*(t) \geqq \hat{\beta}(\hat{\Phi})h,$$

for all $\hat{\Phi}$ satisfying (5.6). It is easy to show that the infimum of $\beta(\Phi)$ over all Φ satisfying (5.4) is equal to $2/\sqrt{3}$. Also the supremum of $\hat{\beta}(\hat{\Phi})$ over all $\hat{\Phi}$ is equal to $2/\sqrt{3}$. Hence we have

$$\frac{2}{\sqrt{3}}h \leqq s_*(t+h) - s^*(t) \leqq s^*(t+h) - s_*(t) \leqq \frac{2}{\sqrt{3}}h,$$

for every $t, h \geqq 0$. Therefore

$$s_*(t) = s^*(t) = \frac{2}{\sqrt{3}}\, t,$$

and we conclude that s^ε converges uniformly.

Finally, set

$$u^*(y, t) = \limsup_{\varepsilon \to 0, (z,r) \to (y,t)} u^\varepsilon(z, r), \quad u_*(y, t) = \liminf_{\varepsilon \to 0, (z,r) \to (y,t)} u^\varepsilon(z, r).$$

Then from the theory of viscosity solutions (see e.g. [**6**, **7**]) it follows that u^* is a viscosity subsolution of (5.3), and u_* is a viscosity supersolution of (5.3). Moreover,

$$u^*(0, t) = u_*(0, t) = \frac{2}{\sqrt{3}}\, t.$$

Since there is a unique viscosity solution u of (5.3) satisfying the above (Dirichlet) boundary condition, a standard comparison theorem yields $u^* = u_* = u$. □

In this paper, we have developed a weak theory for geometric equations of the type

$$V_{ij} = \mu_{ij}[e_i - e_j],$$

where V_{ij} is the normal velocity of an interface and $\mu_{ij} > 0$, e_1, e_2, e_3 are given constants. One may also consider more general geometric equations

$$V_{ij} = v_{ij},$$

with given constants v_{ij}. In general, for any given v_{ij}'s there may not be any μ_{ij} and e_i satisfying $v_{ij} = \mu_{ij}[e_i - e_j]$ for every $i \neq j = 1, 2, 3$. In this nonphysical situation, there is no convergence and our theory does not apply. We illustrate this lack of convergence and hence the lack of solutions with a simple example. Consider the problem

$$V_{ij} = 1, \quad i \neq j = 1, 2, 3,$$

with uniform angle conditions and initial data of three half lines meeting at the origin with equal angles. Let $\Gamma^\varepsilon_{ij}(t)$ be the solution of the curvature perturbed problem. By symmetry, each one is obtained from another one by a rotation of 120° and the triple junction remains at the origin. Thus it suffices to consider the evolution of only one of the arcs, Γ_{13}. Suppose that $\Gamma_{13}(0)$ coincides with the x-axis. Then an elementary argument shows that as $\varepsilon \to 0$, $\Gamma_{13}(t)$ spirals around the origin infinitely many times 'converging' to

$$\{(x, y) : x^2 + y^2 \leqq t\} \cup \{(x, t) : x \leqq 0\}.$$

Hence there is no limit of the vanishing surface tension problem. A numerical study of this example can be found in [**9**, Fig. 28].

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