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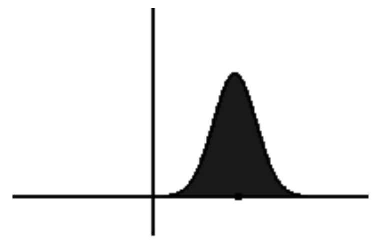
Fokker–Planck equation

In statistical mechanics, the **Fokker–Planck equation** is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, as in Brownian motion. The equation can be generalized to other observables as well.[1] It is named after Adriaan Fokker and Max Planck,[2][3] and is also known as the **Kolmogorov forward equation**, after Andrey Kolmogorov, who independently discovered the concept in 1931.[4] When applied to particle position distributions, it is better known as the **Smoluchowski equation** (after Marian Smoluchowski), and in this context it is equivalent to the convection–diffusion equation. The case with zero diffusion is known in statistical mechanics as the Liouville equation. The Fokker–Planck equation is obtained from the master equation through Kramers–Moyal expansion.



A solution to the one-dimensional Fokker–Planck equation, with both the drift and the diffusion term. In this case the initial condition is a Dirac delta function centered away from zero velocity. Over time the distribution widens due to random impulses.

The first consistent microscopic derivation of the Fokker–Planck equation in the single scheme of classical and quantum mechanics was performed by Nikolay Bogoliubov and Nikolay Krylov.[5][6]

The Smoluchowski equation is the Fokker–Planck equation for the probability density function of the particle positions of Brownian particles.[7]

Contents

One dimension

In one spatial dimension x, for an Itō process driven by the standard Wiener process W_t and described by the stochastic differential equation (SDE)

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$$

with drift $\mu(X_t, t)$ and diffusion coefficient $D(X_t, t) = \sigma^2(X_t, t)/2$, the Fokker–Planck equation for the probability density $p(x, t)$ of the random variable X_t is

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t) p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t) p(x,t)\right].$$

Link between the Itō SDE and the Fokker–Planck equation

In the following, use $\sigma = \sqrt{2D}$.

Define the infinitesimal generator $\mathcal{L}$ (the following can be found in Ref.[8]):

$$\mathcal{L}p(X_t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\mathbb{E}\big[p(X_{t+\Delta t}) \mid X_t = x\big] - p(x)\right).$$

The *transition probability* $\mathbb{P}_{t,t'}(x \mid x')$, the probability of going from (t', x') to (t, x), is introduced here; the expectation can be written as

$$\mathbb{E}(p(X_{t+\Delta t}) \mid X_t = x) = \int p(y)\, \mathbb{P}_{t+\Delta t, t}(y \mid x)\, dy.$$

Now we replace in the definition of $\mathcal{L}$, multiply by $\mathbb{P}_{t,t'}(x \mid x')$ and integrate over dx. The limit is taken on

$$\int p(y) \int \mathbb{P}_{t+\Delta t, t}(y \mid x)\, \mathbb{P}_{t,t'}(x \mid x')\, dx\, dy - \int p(x)\, \mathbb{P}_{t,t'}(x \mid x')\, dx.$$

Note now that

$$\int \mathbb{P}_{t+\Delta t, t}(y \mid x)\, \mathbb{P}_{t,t'}(x \mid x')\, dx = \mathbb{P}_{t+\Delta t, t'}(y \mid x'),$$

which is the Chapman–Kolmogorov theorem. Changing the dummy variable y to x, one gets

$$\int p(x) \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\mathbb{P}_{t+\Delta t, t'}(x \mid x') - \mathbb{P}_{t,t'}(x \mid x')\right)\, dx,$$

which is a time derivative. Finally we arrive to

$$\int [\mathcal{L}p(x)] \mathbb{P}_{t,t'}(x \mid x')\, dx = \int p(x)\, \partial_t \mathbb{P}_{t,t'}(x \mid x')\, dx.$$

From here, the Kolmogorov backward equation can be deduced. If we instead use the adjoint operator of $\mathcal{L}$, $\mathcal{L}^\dagger$, defined such that

$$\int [\mathcal{L}p(x)] \mathbb{P}_{t,t'}(x \mid x')\, dx = \int p(x) [\mathcal{L}^\dagger \mathbb{P}_{t,t'}(x \mid x')]\, dx,$$

then we arrive to the Kolmogorov forward equation, or Fokker–Planck equation, which, simplifying the notation $p(x,t) = \mathbb{P}_{t,t'}(x \mid x')$, in its differential form reads

$$\mathcal{L}^\dagger p(x,t) = \partial_t p(x,t).$$

Remains the issue of defining explicitly $\mathcal{L}$. This can be done taking the expectation from the integral form of the Itō's lemma:

$$\mathbb{E}\big(p(X_t)\big) = p(X_0) + \mathbb{E}\left(\int_0^t \left(\partial_t + \mu\partial_x + \frac{\sigma^2}{2}\partial_x^2\right) p(X_{t'})\,dt'\right).$$

The part that depends on dW_t vanished because of the martingale property.

Then, for a particle subject to an Itō equation, using

$$\mathcal{L} = \mu\partial_x + \frac{\sigma^2}{2}\partial_x^2,$$

it can be easily calculated, using integration by parts, that

$$\mathcal{L}^\dagger = -\partial_x(\mu\cdot) + \frac{1}{2}\partial_x^2(\sigma^2\cdot),$$

which bring us to the Fokker–Planck equation:

$$\partial_t p(x,t) = -\partial_x\big(\mu(x,t)\cdot p(x,t)\big) + \partial_x^2\left(\frac{\sigma(x,t)^2}{2}\,p(x,t)\right).$$

While the Fokker–Planck equation is used with problems where the initial distribution is known, if the problem is to know the distribution at previous times, the Feynman–Kac formula can be used, which is a consequence of the Kolmogorov backward equation.

The stochastic process defined above in the Itō sense can be rewritten within the Stratonovich convention as a Stratonovich SDE:

$$dX_t = \left[\mu(X_t,t) - \frac{1}{2}\frac{\partial}{\partial X_t}D(X_t,t)\right]\,dt + \sqrt{2D(X_t,t)}\circ dW_t.$$

It includes an added noise-induced drift term due to diffusion gradient effects if the noise is state-dependent. This convention is more often used in physical applications. Indeed, it is well known that any solution to the Stratonovich SDE is a solution to the Itō SDE.

The zero-drift equation with constant diffusion can be considered as a model of classical Brownian motion:

$$\frac{\partial}{\partial t}p(x,t) = D_0\frac{\partial^2}{\partial x^2}\left[p(x,t)\right].$$

This model has discrete spectrum of solutions if the condition of fixed boundaries is added for $\{0 \leqslant x \leqslant L\}$:

$$p(0,t) = p(L,t) = 0,$$
$$p(x,0) = p_0(x).$$

It has been shown[9] that in this case an analytical spectrum of solutions allows deriving a local uncertainty relation for the coordinate-velocity phase volume:

$$\Delta x\,\Delta v \geqslant D_0.$$

Here D_0 is a minimal value of a corresponding diffusion spectrum D_j, while Δx and Δv represent the uncertainty of coordinate–velocity definition.

Higher dimensions

More generally, if

$$d\mathbf{X}_t = \boldsymbol{\mu}(\mathbf{X}_t, t)\,dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)\,d\mathbf{W}_t,$$

where $\mathbf{X}_t$ and $\boldsymbol{\mu}(\mathbf{X}_t, t)$ are N-dimensional random vectors, $\boldsymbol{\sigma}(\mathbf{X}_t, t)$ is an $N \times M$ matrix and $\mathbf{W}_t$ is an M-dimensional standard Wiener process, the probability density $p(\mathbf{x}, t)$ for $\mathbf{X}_t$ satisfies the Fokker–Planck equation

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\mu_i(\mathbf{x},t)p(\mathbf{x},t)\right] + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i\,\partial x_j}\left[D_{ij}(\mathbf{x},t)p(\mathbf{x},t)\right],$$

with drift vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)$ and diffusion tensor $\mathbf{D} = \frac{1}{2}\boldsymbol{\sigma\sigma}^{\mathsf{T}}$, i.e.

$$D_{ij}(\mathbf{x},t) = \frac{1}{2}\sum_{k=1}^{M} \sigma_{ik}(\mathbf{x},t)\sigma_{jk}(\mathbf{x},t).$$

If instead of an Itō SDE, a Stratonovich SDE is considered,

$$d\mathbf{X}_t = \boldsymbol{\mu}(\mathbf{X}_t, t)\,dt + \boldsymbol{\sigma}(\mathbf{X}_t, t) \circ d\mathbf{W}_t,$$

the Fokker–Planck equation will read ([8] pag. 129):

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\mu_i(\mathbf{x},t)\,p(\mathbf{x},t)\right] + \frac{1}{2}\sum_{k=1}^{M}\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left\{\sigma_{ik}(\mathbf{x},t)\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left[\sigma_{jk}(\mathbf{x},t)\,p(\mathbf{x},t)\right]\right\}$$

Examples

Wiener process

A standard scalar Wiener process is generated by the stochastic differential equation

$$dX_t = dW_t.$$

Here the drift term is zero and the diffusion coefficient is 1/2. Thus the corresponding Fokker–Planck equation is

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2},$$

which is the simplest form of a diffusion equation. If the initial condition is $p(x, 0) = \delta(x)$, the solution is

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}.$$

Ornstein–Uhlenbeck process

The Ornstein–Uhlenbeck process is a process defined as

$$dX_t = -aX_t dt + \sigma dW_t.$$

with $a > 0$. The corresponding Fokker–Planck equation is

$$\frac{\partial p(x,t)}{\partial t} = a\frac{\partial}{\partial x}\left(x\,p(x,t)\right) + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2},$$

The stationary solution ($\partial_t p = 0$) is

$$p_{ss}(x) = \sqrt{\frac{a}{\pi\sigma^2}}e^{-\frac{ax^2}{\sigma^2}}.$$

Plasma physics

In plasma physics, the distribution function for a particle species s, $p_s(\vec{x},\vec{v},t)$, takes the place of the probability density function. The corresponding Boltzmann equation is given by

$$\frac{\partial p_s}{\partial t} + \vec{v}\cdot\vec{\nabla}p_s + \frac{Z_s e}{m_s}\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\vec{\nabla}_v p_s = -\frac{\partial}{\partial v_i}\left(p_s\langle\Delta v_i\rangle\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i\,\partial v_j}\left(p_s\langle\Delta v_i\,\Delta v_j\rangle\right),$$

where the third term includes the particle acceleration due to the Lorentz force and the Fokker–Planck term at the right-hand side represents the effects of particle collisions. The quantities $\langle\Delta v_i\rangle$ and $\langle\Delta v_i\,\Delta v_j\rangle$ are the average change in velocity a particle of type s experiences due to collisions with all other particle species in unit time. Expressions for these quantities are given elsewhere.[10] If collisions are ignored, the Boltzmann equation reduces to the Vlasov equation.

Computational considerations

Brownian motion follows the Langevin equation, which can be solved for many different stochastic forcings with results being averaged (the Monte Carlo method, canonical ensemble in molecular dynamics). However, instead of this computationally intensive approach, one can use the Fokker–Planck equation and consider the probability $p(\mathbf{v},t)\,d\mathbf{v}$ of the particle having a velocity in the interval $(\mathbf{v},\mathbf{v}+d\mathbf{v})$ when it starts its motion with $\mathbf{v}_0$ at time 0.

Solution

Being a partial differential equation, the Fokker–Planck equation can be solved analytically only in special cases. A formal analogy of the Fokker–Planck equation with the Schrödinger equation allows the use of advanced operator techniques known from quantum mechanics for its solution in a number of cases. Furthermore, in the case of overdamped dynamics when the Fokker–Planck equation contains second partial derivatives with respect to all variables, the equation can be written in the form of a master equation that can easily be solved numerically [11]. In many applications, one is only interested in the steady-state probability distribution $p_0(x)$, which can be found from $\frac{\partial p(x,t)}{\partial t} = 0$. The computation of mean first passage times and splitting probabilities can be reduced to the solution of an ordinary differential equation which is intimately related to the Fokker–Planck equation.

Particular cases with known solution and inversion

In mathematical finance for volatility smile modeling of options via local volatility, one has the problem of deriving a diffusion coefficient $\sigma(\mathbf{X}_t,t)$ consistent with a probability density obtained from market option quotes. The problem is therefore an inversion of the Fokker–Planck equation: Given the density f(x,t) of the option underlying *X* deduced from the option market, one aims at finding the local volatility $\sigma(\mathbf{X}_t, t)$ consistent with *f*. This is an inverse problem that has been solved in general by Dupire (1994, 1997) with a non-parametric solution. Brigo and Mercurio (2002, 2003) propose a solution in parametric form via a particular local volatility $\sigma(\mathbf{X}_t, t)$ consistent with a solution of the Fokker–Planck equation given by a mixture model. More information is available also in Fengler (2008), Gatheral (2008) and Musiela and Rutkowski (2008).

Fokker–Planck equation and path integral

Every Fokker–Planck equation is equivalent to a path integral. The path integral formulation is an excellent starting point for the application of field theory methods.[12] This is used, for instance, in critical dynamics.

A derivation of the path integral is possible in a similar way as in quantum mechanics. The derivation for a Fokker–Planck equation with one variable x is as follows. Start by inserting a delta function and then integrating by parts:

$$\begin{aligned}\frac{\partial}{\partial t}p(x',t) &= -\frac{\partial}{\partial x'}\left[D_1(x',t)p(x',t)\right] + \frac{\partial^2}{\partial {x'}^2}\left[D_2(x',t)p(x',t)\right] \\ &= \int_{-\infty}^{\infty} dx \left(\left[D_1\left(x,t\right)\frac{\partial}{\partial x} + D_2\left(x,t\right)\frac{\partial^2}{\partial x^2}\right]\delta\left(x'-x\right)\right)p(x,t)\,.\end{aligned}$$

The *x*-derivatives here only act on the δ-function, not on $p(x,t)$. Integrate over a time interval ε,

$$p(x',t+\varepsilon) = \int_{-\infty}^{\infty} dx \left(\left(1+\varepsilon\left[D_1(x,t)\frac{\partial}{\partial x} + D_2(x,t)\frac{\partial^2}{\partial x^2}\right]\right)\delta(x'-x)\right)p(x,t) + O(\varepsilon^2).$$

Insert the Fourier integral

$$\delta\left(x'-x\right) = \int_{-i\infty}^{i\infty} \frac{d\tilde{x}}{2\pi i} e^{\tilde{x}(x-x')}$$

for the δ-function,

$$\begin{aligned}p(x',t+\varepsilon) &= \int_{-\infty}^{\infty} dx \int_{-i\infty}^{i\infty} \frac{d\tilde{x}}{2\pi i}\left(1+\varepsilon\left[\tilde{x}D_1(x,t) + \tilde{x}^2 D_2(x,t)\right]\right)e^{\tilde{x}(x-x')}p(x,t) + O(\varepsilon^2) \\ &= \int_{-\infty}^{\infty} dx \int_{-i\infty}^{i\infty} \frac{d\tilde{x}}{2\pi i}\exp\left(\varepsilon\left[-\tilde{x}\frac{(x'-x)}{\varepsilon} + \tilde{x}D_1 f(x,t) + \tilde{x}^2 D_2(x,t)\right]\right)p(x,t) + O(\varepsilon^2).\end{aligned}$$

This equation expresses $p(x',t+\varepsilon)$ as functional of $p(x,t)$. Iterating $(t'-t)/\varepsilon$ times and performing the limit $\varepsilon \to 0$ gives a path integral with action

$$S = \int dt \left[\tilde{x}D_1(x,t) + \tilde{x}^2 D_2(x,t) - \tilde{x}\frac{\partial x}{\partial t}\right].$$

The variables $\tilde{x}$ conjugate to x are called "response variables".[13]

Although formally equivalent, different problems may be solved more easily in the Fokker–Planck equation or the path integral formulation. The equilibrium distribution for instance may be obtained more directly from the Fokker–Planck equation.

See also

- Kolmogorov backward equation
- Boltzmann equation
- Vlasov equation
- Master equation
- Mean field game theory
- Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy of equations
- Ornstein–Uhlenbeck process
- Convection–diffusion equation

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