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An inverse time-dependent source problem for a time-fractional diffusion equation*

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Abstract

This paper is devoted to identifying a time-dependent source term in a multi-dimensional time-fractional diffusion equation from boundary Cauchy data. The existence and uniqueness of a strong solution for the corresponding direct problem with homogeneous Neumann boundary condition are firstly proved. We provide the uniqueness and a stability estimate for the inverse time-dependent source problem. Then we use the Tikhonov regularization method to solve the inverse source problem and propose a conjugate gradient algorithm to find a good approximation to the minimizer of the Tikhonov regularization functional. Numerical examples in one-dimensional and two-dimensional cases are provided to show the effectiveness of the proposed method.

Keywords: inverse source problem, fractional diffusion equation, conjugate gradient method

(Some figures may appear in colour only in the online journal)

1. Introduction

Fractional diffusion equations are more adequate than integer-order models for describing anomalous diffusion phenomena because fractional order derivatives enable the description of memory and hereditary properties of heterogeneous substances [2, 18]. For the last few decades, fractional diffusion equations have attracted great attention not only from mathematicians and engineers but also from many scientists from fields like biology, physics, chemistry and biochemistry, medicine and finance [9, 17, 18, 26, 45, 46].

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The time fractional diffusion equations arise when replacing the standard time derivative with time fractional derivatives and can be used to describe superdiffusion and subdiffusion phenomena [2, 19, 29, 30]. Direct problems, i.e. initial value problems and initial boundary value problems for time-fractional diffusion equations, have attracted much more attention in recent years, for instance, in regard to the maximum principle [15], some uniqueness and existence results [15, 28], numerical solutions by finite element methods [10] and finite difference methods [13, 21, 50], and exact solutions [16, 18, 40].

However, in some practical situations, a part of boundary data, or initial data, or diffusion coefficient, or source term may not be given and we want to find them by additional measurement data which will yield some fractional diffusion inverse problems. As we know, research on inverse problems for time fractional diffusion equations still lacks wide attention. On the uniqueness of inverse problems, Cheng *et al* in [3] and Li *et al* in [12] gave the uniqueness results for determining the order of the fractional derivative and space-dependent diffusion coefficient in a fractional diffusion equation by using different types of initial data. Sakamoto and Yamamoto in [28] established a few uniqueness results for several inverse problems. Yamamoto *et al* [41] provided a conditional stability for determining a zeroth-order coefficient in a half-order fractional diffusion equation. On numerical computations of inverse problems, Murio in [22] solved a Cauchy problem by a mollification regularization and space marching algorithm. Liu and Yamamoto in [14] proposed a quasi-reversibility method for solving the backward problem of a fractional diffusion equation. Zheng and Wei in [48, 49] solved the Cauchy problems for the time fractional diffusion equations on a strip domain by a Fourier truncation method and a convolution regularization method. Wei and Wang in [36, 37] solved a backward problem and an inverse space-dependent source problem by a modified quasi-boundary value method. In [6, 32, 43, 44], the authors identified the orders of time and space fractional derivatives in time-space fractional partial differential equations.

In this paper we investigate an inverse time-dependent source problem for a time fractional diffusion equation in a general bounded domain.

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficient smooth boundary $\partial\Omega$. Consider the following time-fractional diffusion equation

$$\partial_{0+}^{\alpha} u(x, t) = \Delta u(x, t) + f(x)p(t), \qquad x \in \Omega, \quad t \in (0, T], \quad 0 < \alpha < 1, \tag{1.1}$$

where ∂_{0+}^{α} denotes the Caputo fractional left derivative of order α with respect to t defined by

$$\partial_{0+}^{\alpha} u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^{\alpha}}, \qquad 0 < \alpha < 1, \quad 0 < t \leqslant T,$$

where Γ is the Gamma function and $T > 0$ is a fixed final time. Note that if α tends to 1, the fractional derivative $\partial_{0+}^{\alpha} u$ tends to the first-order derivative u_t, and thus the model (1.1) reproduces the standard diffusion equation.

Suppose unknown function u satisfies the following initial and boundary conditions:

$$u(x, 0) = \phi(x), \qquad x \in \overline{\Omega}, \tag{1.2}$$

$$\frac{\partial u}{\partial n}(x, t) = 0, \qquad x \in \partial\Omega, \quad t \in (0, T], \tag{1.3}$$

where n is the unitary normal vector, exterior to the domain Ω.

If all functions $f(x)$, $p(t)$, $\phi(x)$ are given appropriately, the problem (1.1)–(1.3) is a direct problem. The inverse problem here is to determine the source term $p(t)$ in problem (1.1)–(1.3) from the additional data

$$u(x, t) = g(x, t), \qquad x \in \Gamma, \quad 0 < t \leqslant T, \tag{1.4}$$

where Γ is a nonempty part of $\partial\Omega$.

The inverse source problem mentioned above is an ill-posed problem (refer to section 5). Zhang and Xu [47] considered an inverse space-dependent source problem by using the Cauchy data at one end $x = 0$. Wang *et al* in [35] used a reproducing kernel space method to solve an inverse space-dependent source problem from the final data. Wei and Wang in [37] proposed a modified quasi-boundary value method for identifying the space-dependent source term by using the final data. Tatar *et al* in [31, 33] solved an inverse space-dependent source problem for a space-time fractional diffusion equation. For the integer-order cases, the inverse source problems have been widely studied, see [1, 27] for examples. In [28], Sakamoto and Yamamoto gave a stability estimate for identifying a time-dependent source from the measurement data on an internal point of domain, but no numerical method was provided. Wei and Zhang in [39] proposed a numerical method to solve the inverse time-dependent source problem issued in [28]. In this paper, we focus on a multi-dimensional problem in a general bounded domain by using the boundary Cauchy data. As we know, this investigation for a fractional diffusion equation is the first issue.

Based on the existence and uniqueness of the direct problem, we can formulate the inverse source problem into a regularized variational problem and then deduce the gradient of the regularization functional based on an adjoint problem. Then the conjugate gradient method is employed to solve the variational problem and the discrepancy principle is applied to find a suitable stopping step. We use a finite element method with linear elements to solve the direct problems in each iteration (refer to [37]).

The remainder of this paper is organized as follows. In section 2, we present some preliminaries used in sections 3–5. The existence and uniqueness of a strong solution for the direct problem (1.1)–(1.3) is proved in section 3. In section 4, we provide the uniqueness and a stability estimate for the inverse source problem. In section 5, we transform the inverse problem (1.1)–(1.4) into a Tikhonov regularization problem and deduce the gradient of the regularized functional. Section 6 is devoted to present the conjugate gradients algorithm. The numerical results for six examples are investigated in section 7. Finally, we give a conclusion in section 8.

2. Preliminary

Let $AC[0, T]$ be the space of functions f which are absolutely continuous on $[0, T]$. Throughout this paper, we use the following definitions and propositions given in [11].

Definition 2.1. The Mittag–Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Proposition 2.2. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Then there exists a constant $c = c(\alpha, \beta, \mu) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leqslant \frac{c}{1+|z|}, \qquad \mu \leqslant |\arg(z)| \leqslant \pi.$$

Proposition 2.3. *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad t > 0.$$

Proposition 2.4. *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\partial_{0+}^{\alpha} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda E_{\alpha,1}(-\lambda t^{\alpha}), \quad t > 0.$$

Proposition 2.5. *(See [25].) For $0 < \alpha < 1$, $t > 0$, we have $0 < E_{\alpha,1}(-t) < 1$. Moreover, $E_{\alpha,1}(-t)$ is completely monotonic, that is*

$$(-1)^{n} \frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}} E_{\alpha,1}(-t) \geqslant 0, \quad \forall n \in \mathbb{N}.$$

Proposition 2.6. *For $0 < \alpha < 1$, $\eta > 0$, we have $0 \leqslant E_{\alpha,\alpha}(-\eta) \leqslant \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function with $\eta > 0$.*

Definition 2.7. If $z(t) \in L(0, T)$, then for $\alpha > 0$ the Riemann–Liouville fractional left integral $I_{0+}^{\alpha} f$ and right integral $I_{T-}^{\alpha} f$ are defined by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)\mathrm{d}s}{(t-s)^{1-\alpha}}, \quad 0 < t \leqslant T,$$

and

$$I_{T-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(s)\mathrm{d}s}{(s-t)^{1-\alpha}}, \quad 0 \leqslant t < T.$$

Definition 2.8. Let $z(t) \in AC[0, T]$, then for $0 < \alpha < 1$ the Caputo fractional left derivative $\partial_{0+}^{\alpha} y(t)$ and right derivative $\partial_{T-}^{\alpha} y(t)$ of order α are defined by

$$\partial_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)\mathrm{d}s}{(t-s)^{\alpha}} =: (I_{0+}^{1-\alpha} y')(t), \quad 0 < t \leqslant T,$$

and

$$\partial_{T-}^{\alpha} y(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{y'(s)\mathrm{d}s}{(s-t)^{\alpha}} =: -(I_{T-}^{1-\alpha} y')(t), \quad 0 \leqslant t < T.$$

And for $0 < \alpha < 1$ the Riemann–Liouville fractional left derivative of order α is defined by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{y(s)}{(t-s)^{\alpha}} \mathrm{d}s =: \frac{\mathrm{d}}{\mathrm{d}t} (I_{0+}^{1-\alpha} y)(t), \quad 0 < t \leqslant T. \tag{2.1}$$

Proposition 2.9. *Let $y(t) \in AC[0, T]$. Then the Caputo fractional left derivative $\partial_{0+}^{\alpha} y(t)$ and the Riemann–Liouville fractional left derivative $D_{0+}^{\alpha} y(t)$ exist almost everywhere on $(0, T]$, there is a relationship between the Caputo fractional left derivative and the Riemann–Liouville fractional left derivative*

$$\partial_{0+}^{\alpha} y(t) = D_{0+}^{\alpha} y(t) - \frac{y(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \qquad a.e. \quad t \in (0, T].$$

Lemma 2.10. *(See [38]) For $0 < \alpha < 1$ and $\lambda > 0$, if $q(t) \in AC[0, T]$, we have*

$$\begin{aligned}
&\partial_{0+}^{\alpha} \int_0^t q(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) \mathrm{d}\tau \\
&\quad = q(t) - \lambda \int_0^t q(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) \mathrm{d}\tau, \quad 0 < t \leqslant T.
\end{aligned}$$

In particular, if $\lambda = 0$, we have

$$\partial_{0+}^{\alpha} \int_0^t q(\tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau = \Gamma(\alpha) q(t), \quad 0 < t \leqslant T.$$

Proposition 2.11. *Let $\alpha > 0$, $p \geqslant 1$, $q \geqslant 1$, and $1/p + 1/q \leqslant 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $1/p + 1/q = 1 + \alpha$). If $\varphi(t) \in L^p(0, T)$ and $\psi(t) \in L^q(0, T)$, then*

$$\int_0^T \varphi(t) I_{0+}^{\alpha} \psi(t) \mathrm{d}t = \int_0^T \psi(t) I_{T-}^{\alpha} \varphi(t) \mathrm{d}t.$$

Lemma 2.12. *For $0 < \alpha < 1$, suppose $u(t)$, $v(t) \in AC[0, T]$, then we have*

$$\int_0^T D_{0+}^{\alpha} u(t) v(t) \mathrm{d}t = (I_{0+}^{1-\alpha} u)(T) v(T) + \int_0^T u(t) \partial_{T-}^{\alpha} v(t) \mathrm{d}t.$$

Proof. Note that $|I_{0+}^{1-\alpha} u(t)| \leqslant \|u\|_{C[0,T]} / \Gamma(2-\alpha) t^{1-\alpha}$ for $0 < t \leqslant T$, then define $(I_{0+}^{1-\alpha} u)(0) = 0$, it is easy to prove $I_{0+}^{1-\alpha} u(t) \in C[0, T]$. Since $u(t) \in AC[0, T]$, by proposition 2.9 and the Young theorem, we have

$$D_{0+}^{\alpha} u(t) = \frac{\mathrm{d}}{\mathrm{d}t} I_{0+}^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{u(0)}{t^{\alpha}} + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^{\alpha}} \mathrm{d}s \in L(0, T),$$

therefore $I_{0+}^{1-\alpha} u(t) \in AC[0, T]$. By Lebesgue integration by parts, we have

$$\int_0^T D_{0+}^{\alpha} u(t) v(t) \mathrm{d}t = (I_{0+}^{1-\alpha} u)(T) v(T) - \int_0^T I_{0+}^{1-\alpha} u(t) v'(t) \mathrm{d}t.$$

By proposition 2.11, we have

$$\int_0^T I_{0+}^{1-\alpha} u(t) v'(t) \mathrm{d}t = \int_0^T u(t) I_{T-}^{1-\alpha} v'(t) \mathrm{d}t = -\int_0^T u(t) \partial_{T-}^{\alpha} v(t) \mathrm{d}t.$$

This completes the proof. □

Remark 2.13. Lemma 2.12 is very important for using the adjoint problem approach. There is a similar lemma 2.7 in [11], but there the condition is hard to verify and the result is different from lemma 2.12.

Lemma 2.14. *Suppose $p(t) \in L^\infty(0, T)$, $0 < \alpha < 1$, $\lambda \geqslant 0$, denote*

$$g(t) = \int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha)\mathrm{d}\tau, \qquad t \in (0, T],$$

and define $g(0) = 0$, then $g(t) \in C[0, T]$.

Proof. Consider the case $\lambda > 0$. For $0 < t < T$, by proposition 2.3, we have

$$\begin{aligned}|g(t)| &= \left| \int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha)\mathrm{d}\tau \right| \\ &\leqslant \|p\|_\infty 1/\lambda[1 - E_{\alpha,1}(-\lambda t^\alpha)] \to 0,\ t \to 0^+,\end{aligned}$$

where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T)}$ throughout this paper, hence g is continuous at $t = 0$.

By propositions 2.3 and 2.4, it is not hard to prove that $s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)$ is monotonically decreasing on $(0, \infty)$. For any $t,\ t+h \in (0, T]$, if $h > 0$ we have

$$\begin{aligned}
&|g(t+h) - g(t)| \\
&= \left| \int_0^{t+h} p(\tau)(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t+h-\tau)^\alpha)\mathrm{d}\tau \right. \\
&\quad \left. - \int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha)\mathrm{d}\tau \right| \\
&\leqslant \left| \int_0^t p(\tau)((t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t+h-\tau)^\alpha) \right. \\
&\quad \left. - (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha))\mathrm{d}\tau \right| \\
&\quad + \left| \int_t^{t+h} p(\tau)(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t+h-\tau)^\alpha)\mathrm{d}\tau \right| \\
&\leqslant \|p\|_\infty \int_0^t \ |(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t+h-\tau)^\alpha) \\
&\quad - (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha)|\mathrm{d}\tau \\
&\quad + \|p\|_\infty 1/\lambda[1 - E_{\alpha,1}(-\lambda h^\alpha)] \\
&= \frac{\|p\|_\infty}{\lambda}[1 - E_{\alpha,1}(-\lambda t^\alpha) - E_{\alpha,1}(-\lambda h^\alpha) + E_{\alpha,1}(-\lambda(t+h)^\alpha)] \\
&\quad + \frac{\|p\|_\infty}{\lambda}[1 - E_{\alpha,1}(-\lambda h^\alpha)].
\end{aligned}$$

It is clear we have $\lim_{h\to 0^+} g(t+h) = g(t)$. By a similar deduction, we have $\lim_{h\to 0^-} g(t+h) = g(t)$. Therefore $g \in C[0, T]$.

For the special case $\lambda = 0$, the proof is much more simple. We omit it. □

3. Existence and uniqueness of a strong solution for the direct problem

We denote the eigenvalues of $-\triangle$ with homogeneous Neumann boundary condition as λ_n and the corresponding eigenfunctions as $\varphi_n \in \left\{\psi \in H^2(\Omega);\ \frac{\partial\psi}{\partial n}|_{\partial\Omega} = 0\right\}$, which means we have

$-\triangle\varphi_n = \lambda_n\varphi_n$. Counting according to the multiplicities, we can set: $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ and $\{\varphi_n\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$ (see p 345 in [34], for example). It is clear that φ_0 is a constant. Define the Hilbert scale space as

$$\mathcal{D}((-\triangle + 1)^\gamma) = \{\psi \in L^2(\Omega); \sum_{n=0}^{\infty}(\lambda_n + 1)^{2\gamma}|(\psi, \varphi_n)|^2 < \infty\}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$, and define its norm

$$\|\psi\|_{\mathcal{D}((-\triangle+1)^\gamma)} = \left(\sum_{n=0}^{\infty}(\lambda_n + 1)^{2\gamma}|(\psi, \varphi_n)|^2\right)^{\frac{1}{2}}.$$

The number 1 in the operator $-\triangle + 1$ is not necessary. It may be any nonzero positive number. In fact, the Hilbert scale space $D((-\triangle + \nu)^\gamma)$ is the same for any positive number ν for a fixed γ.

Let us define a strong solution to the direct problem (1.1)–(1.3), then we prove its existence and uniqueness based on the methods in [28].

Definition 3.1. We call $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ such that $\partial_{0+}^\alpha u \in C((0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$ is a strong solution to (1.1)–(1.3) if (1.1) holds in $L^2(\Omega)$ for $0 < t \leqslant T$, (1.2) holds in $L^2(\Omega)$ as $t \to 0^+$ and (1.3) holds in the trace sense.

Theorem 3.2. *If $\phi \in \mathcal{D}((-\triangle + 1)^{\frac{1}{2}})$, $f \in L^2(\Omega)$, $p \in AC[0, T]$, then there exists a unique strong solution to (1.1)–(1.3) and the solution is given by*

$$\begin{aligned} u(x, t) = &\sum_{n=0}^{\infty}(\phi, \varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) + \frac{(f, \varphi_0)}{\Gamma(\alpha)}\int_0^t p(\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau\varphi_0(x) \\ &+ \sum_{n=1}^{\infty}(f, \varphi_n)\int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\varphi_n(x). \end{aligned} \tag{3.1}$$

Moreover, we have the following estimates:

$$\|u\|_{C([0,T];L^2(\Omega))} \leqslant C_1(\|\phi\|_{L^2(\Omega)} + \|p\|_\infty\|f\|_{L^2(\Omega)}), \tag{3.2}$$

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leqslant C_2(\|\phi\|_{\mathcal{D}((-\triangle+1)^{\frac{1}{2}})} + \|p\|_\infty\|f\|_{L^2(\Omega)}), \tag{3.3}$$

where C_1, C_2 are positive constants depending on α, T, Ω.

Proof. Based on proposition 2.4 and lemma 2.10, by the separation of variables, we can obtain a formal solution for the direct problem (1.1)–(1.3) as (3.1). We will show that (3.1) certainly gives a strong solution to (1.1)–(1.3).

In the following proof, we denote C as a generic positive constant. Denote

$$g_n(t) = \int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)d\tau, \quad n \in \mathbb{N} \cup \{0\}, \tag{3.4}$$

then we know

$$|g_n(t)| \leqslant \|p\|_\infty/\Gamma(\alpha+1)t^\alpha \leqslant \|p\|_\infty/\Gamma(\alpha+1)T^\alpha, \quad t \in [0, T], \quad n \in \mathbb{N} \cup \{0\} \tag{3.5}$$

$$|g_n(t)| \leqslant \|p\|_\infty(1 - E_{\alpha,1}(-\lambda_n t^\alpha))/\lambda_n \leqslant \|p\|_\infty/\lambda_n, \quad t \in [0, T], \quad n \in \mathbb{N}. \tag{3.6}$$

(1) We first verify $u \in C([0, T]; L^2(\Omega))$ and $\lim_{t\to 0} \|u(\cdot, t) - \phi(\cdot)\|_{L^2(\Omega)} = 0$. Define

$$u_1 := \sum_{n=0}^{\infty} (\phi, \varphi_n) E_{\alpha,1}(-\lambda_n t^{\alpha}) \varphi_n(x),$$
$$u_2 := (f, \varphi_0) g_0(t) \varphi_0(x),$$
$$u_3 := \sum_{n=1}^{\infty} (f, \varphi_n) g_n(t) \varphi_n(x).$$

Then we have $u(\cdot, t) = u_1(\cdot, t) + u_2(\cdot, t) + u_3(\cdot, t)$. We estimate each term separately. For fixed $t \in [0, T]$, by proposition 2.5 and equation (3.5), we have

$$\|u_1(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} (\varphi, \varphi_n)^2 E^2_{\alpha,1}(-\lambda_n t^{\alpha}) \leqslant \|\varphi\|^2_{L^2(\Omega)}, \tag{3.7}$$

$$\|u_2(\cdot, t)\|^2_{L^2(\Omega)} = (f, \varphi_0)^2 g_0(t)^2 \leqslant \|p\|^2_{\infty} \frac{(f, \varphi_0)^2}{\Gamma^2(\alpha + 1)} t^{2\alpha}, \tag{3.8}$$

and

$$\|u_3(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} (f, \varphi_n)^2 g_n^2(t) \leqslant \|p\|^2_{\infty} \sum_{n=1}^{\infty} \frac{(f, \varphi_n)^2}{\Gamma^2(\alpha + 1)} t^{2\alpha}. \tag{3.9}$$

By equations (3.8) and (3.9), we know

$$\lim_{t\to 0^+} \|u_2(\cdot, t)\|_{L^2(\Omega)} = 0, \quad \lim_{t\to 0^+} \|u_3(\cdot, t)\|_{L^2(\Omega)} = 0. \tag{3.10}$$

Thus, we define $u_2(x, 0) = 0$, $u_3(x, 0) = 0$. From (3.7)–(3.9), we obtain

$$\|u(\cdot, t)\|_{L^2(\Omega)} \leqslant C_1(\|\varphi\|_{L^2(\Omega)} + \|p\|_{\infty} \|f\|_{L^2(\Omega)}), \quad t \in [0, T],$$

where $C_1 = \max\{1, T^{\alpha}/\Gamma(\alpha + 1)\}$.

For t, $t + h \in [0, T]$, we have

$$\begin{aligned} &u(x, t + h) - u(x, t) \\ &= \sum_{n=0}^{\infty} (\phi, \varphi_n)(E_{\alpha,1}(-\lambda_n (t + h)^{\alpha}) - E_{\alpha,1}(-\lambda_n t^{\alpha})) \varphi_n(x) \\ &\quad + (f, \varphi_0)(g_0(t + h) - g_0(t)) \varphi_0(x) + \sum_{n=1}^{\infty} (f, \varphi_n)(g_n(t + h) - g_n(t)) \varphi_n(x) \\ &=: I_1(x, t; h) + I_2(x, t; h) + I_3(x, t; h). \end{aligned}$$

We estimate each term separately. In fact, by proposition 2.5, we have

$$\|I_1(\cdot, t; h)\|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} (\varphi, \varphi_n)^2 |E_{\alpha,1}(-\lambda_n (t + h)^{\alpha}) - E_{\alpha,1}(-\lambda_n t^{\alpha})|^2 \leqslant 4\|\phi\|^2_{L^2(\Omega)},$$

since $\lim_{h\to 0} |E_{\alpha,1}(-\lambda_n (t + h)^{\alpha}) - E_{\alpha,1}(-\lambda_n t^{\alpha})| = 0$ for each $n \in \mathbb{N}$, by using the Lebesgue theorem, we have

$$\lim_{h\to 0} \|I_1(\cdot, t; h)\|^2_{L^2(\Omega)} = 0.$$

By lemma 2.14, we have

$$\lim_{h\to 0} \|I_2(\cdot, t; h)\|^2_{L^2(\Omega)} = 0.$$

Further, by equation (3.5), we have

$$\|I_3(\cdot, t; h)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}(f, \varphi_n)^2(g_n(t+h) - g_n(t))^2 \leqslant C\|p\|^2_\infty\|f\|^2_{L^2(\Omega)}.$$

Similarly, by using the Lebesgue theorem and lemma 2.14, we can prove

$$\lim_{h\to 0}\|I_3(\cdot, t; h)\|^2_{L^2(\Omega)} = 0.$$

Therefore, $u \in C([0, T]; L^2(\Omega))$.

By proposition 2.5, we know

$$\begin{aligned}\|u(\cdot, t) - \phi(\cdot)\|_{L^2(\Omega)} &\leqslant \left(\sum_{n=0}^{\infty}(\phi, \varphi_n)^2(E_{\alpha,1}(-\lambda_n t^\alpha) - 1)^2\right)^{1/2} \\ &\quad + \|u_2(\cdot, t)\|_{L^2(\Omega)} + \|u_3(\cdot, t)\|_{L^2(\Omega)} \\ &\leqslant \|\phi\|_{L^2(\Omega)} + \|u_2(\cdot, t)\|_{L^2(\Omega)} + \|u_3(\cdot, t)\|_{L^2(\Omega)}.\end{aligned}$$

Since $\lim_{t\to 0}(E_{\alpha,1}(-\lambda_n t^\alpha) - 1) = 0$ and (3.10), we have

$$\lim_{t\to 0^+}\|u(\cdot, t) - \phi(\cdot)\|_{L^2(\Omega)} = 0.$$

(2) We verify $\triangle u \in C((0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$ and $u \in L^2(0, T; H^2(\Omega))$ and equation (1.3) holds in the trace sense.

By equation (3.1) and φ_0 is a constant, we know

$$\begin{aligned}-\triangle u(x, t) &= \sum_{n=1}^{\infty}\lambda_n(\phi, \varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) + \sum_{n=1}^{\infty}\lambda_n(f, \varphi_n)g_n(t)\varphi_n(x) \\ &=: v_1(x, t) + v_2(x, t),\end{aligned}$$

where g_n is defined in equation (3.4).

For $0 < t \leqslant T$, by proposition 2.2, we obtain

$$\begin{aligned}\|v_1(\cdot, t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}(\lambda_n(\phi, \varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha))^2 \\ &\leqslant \sum_{n=1}^{\infty}\lambda_n(\phi, \varphi_n)^2\left(\frac{c\sqrt{\lambda_n}}{1 + \lambda_n t^\alpha}\right)^2 \leqslant C\frac{\|\phi\|^2_{\mathcal{D}((-\triangle+1)^{\frac{1}{2}})}}{t^\alpha},\end{aligned} \tag{3.11}$$

where C is a constant depending on α only.

For the second term v_2, by (3.6) we can deduce that

$$\|v_2(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}\lambda_n^2(f, \varphi_n)^2 g_n^2(t) \leqslant \|p\|^2_\infty\sum_{n=1}^{\infty}(f, \varphi_n)^2 \leqslant \|p\|^2_\infty\|f\|^2_{L^2(\Omega)}. \tag{3.12}$$

Since $v_1(x, t)$, $v_2(x, t)$ are convergent in $L^2(\Omega)$ uniformly on $t \in [t_0, T]$ for any given $t_0 > 0$, $\triangle u \in C((0, T]; L^2(\Omega))$ can be obtained similarly to the proof in part (1).

By estimates (3.11)–(3.12), we know $v_1, v_2 \in L^2(0, T; L^2(\Omega))$, hence $\triangle u \in L^2(0, T; L^2(\Omega))$ and $(-\triangle + 1)u \in L^2(0, T; L^2(\Omega))$.

Denote $u_N := \sum_{n=0}^{N}(\phi, \varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) + \sum_{n=0}^{N}(f, \varphi_n)g_n(t)\varphi_n(x)$. It is clear that u_N satisfies $\frac{\partial u_N}{\partial n}|_\Gamma = 0$, by the regularity of the solution for elliptic equations with homogeneous Neumann boundary condition, and for every $t \in (0, T]$ we know

$$\|u_N(\cdot, t) - u_M(\cdot, t)\|_{H^2(\Omega)} \leqslant C\|(-\triangle + 1)(u_N - u_M)\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad N, M \to \infty.$$

Thus $\{u_N\}$ is a Cauchy function sequence in $H^2(\Omega)$. By the completeness of space $H^2(\Omega)$ we have $u = \lim_{N\to\infty} u_N \in H^2(\Omega)$, and $\frac{\partial u}{\partial n}|_{\partial\Omega} = \lim_{N\to\infty} \frac{\partial u_N}{\partial n}|_{\partial\Omega} = 0$ in the trace sense of $H^{\frac{1}{2}}(\partial\Omega)$. Since $\|u_N\|_{H^2(\Omega)} \leqslant C\|(-\triangle + 1)u_N\|_{L^2(\Omega)}$, we have $\|u\|_{H^2(\Omega)} \leqslant C\|(-\triangle + 1)u\|_{L^2(\Omega)}$. Consider $(-\triangle + 1)u \in L^2(0, T; L^2(\Omega))$, we have $u \in L^2(0, T; H^2(\Omega))$. Moreover, we can obtain the following estimate from equations (3.11)–(3.12):

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leqslant C_2(\|\phi\|_{\mathcal{D}((-\triangle+1)^{\frac{1}{2}})} + \|p\|_\infty \|f\|_{L^2(\Omega)}),$$

where $C_2 = C_2(\alpha, T, \Omega)$ is a positive constant depending on α, T, Ω.

(3) We prove that $\partial_{0+}^\alpha u \in C((0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$ and equation (1.1) holds in $L^2(\Omega)$ for $t \in (0, T]$.

By lemma 2.10, we have

$$\begin{aligned}\partial_{0+}^\alpha u(x, t) = &-\sum_{n=1}^{\infty}(\phi, \varphi_n)\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) + (f, \varphi_0)p(t)\varphi_0(x)\\ &+\sum_{n=1}^{\infty}(f, \varphi_n)\left[p(t) - \lambda_n \int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\right]\varphi_n(x)\\ =&\ p(t)f(x) + \triangle u(x, t).\end{aligned}$$

Hence $\partial_{0+}^\alpha u \in C((0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$ and equation (1.1) holds in $L^2(\Omega)$ for $t \in (0, T]$.

(4) We prove the uniqueness of the strong solution to equations (1.1)–(1.3). Under the condition $f(x)p(t) = 0$, $\phi = 0$, we need to prove that the system (1.1)–(1.3) has only a trivial solution. We take the inner product of (1.1) with $\varphi_n(x)$. Using the Green formula and $\frac{\partial\varphi_n}{\partial n}|_{\partial\Omega} = 0$ and setting $u_n(t) \coloneqq (u(\cdot, t), \varphi_n)$, we obtain

$$\begin{cases}\partial_{0+}^\alpha u_n(t) = -\lambda_n u_n(t), & t \in (0, T],\\ u_n(0) = 0.\end{cases}$$

Due to the existence and uniqueness of the ordinary fractional differential equation (see chapter 3 in [11], for example), we obtain that $u_n(t) = 0$, $n = 0, 1, 2, \ldots$. Since $\{\varphi_n\}_{n\in\mathbb{N}\cup 0}$ is an orthonormal basis in $L^2(\Omega)$, we have $u = 0$ in $\Omega \times (0, T]$. Thus the proof is complete. □

Remark 3.3. In theorem 3.2, if $p \in AC[0, T]$ is replaced by $p \in L^2(0, T)$, we can prove u given by equation (3.1) also satisfies $u \in L^2(0, T; H^2(\Omega))$ and the following estimate holds:

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leqslant \bar{C}_2(\|\phi\|_{\mathcal{D}((-\triangle+1)^{\frac{1}{2}})} + \|p\|_{L^2(0,T)}\|f\|_{L^2(\Omega)}). \tag{3.13}$$

Proof. By using Young's inequality for convolution, we have a modified estimate for v_2 in equation (3.12) as follows:

$$
\begin{aligned}
&\int_0^T \|v_2(\cdot, t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty}(f, \varphi_n)^2 \int_0^T \left| \int_0^t p(\tau)\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\mathrm{d}\tau \right|^2 \mathrm{d}t \\
&\leqslant \|p\|^2_{L^2(0,T)} \sum_{n=1}^{\infty}(f, \varphi_n)^2 \left(\int_0^T \lambda_n s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^{\alpha})\mathrm{d}s\right)^2 \\
&\leqslant \|p\|^2_{L^2(0,T)}\,\|f\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.14}
$$

in which we use $\int_0^T \lambda_n s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^{\alpha})\mathrm{d}s = 1 - E_{\alpha,1}(-\lambda_n T^{\alpha}) \leqslant 1$. The rest proof is same with theorem 3.2. □

4. Uniqueness and a stability estimate for the inverse source problem

Theorem 4.1. *Let u_1, u_2 be the solutions of problem (1.1)–(1.3) for $p = p_1, p_2 \in AC[0, T]$ respectively with a fixed $f \in \mathcal{D}((-\Delta + 1)^{\gamma})$ in which $\gamma > \frac{d}{2} + 1$ and $\phi \in \mathcal{D}((-\Delta + 1)^{\frac{1}{2}})$. Assume that there exists a point $x_0 \in \Gamma$ satisfying $\sum_{n=0}^{\infty}(f, \varphi_n)\varphi_n(x_0) \neq 0$. Then there exists a constant C_3 such that*

$$
\|p_1 - p_2\|_{C[0,T]} \leqslant C_3 \|\partial^{\alpha}_{0+}(u_1(x_0, \cdot) - u_2(x_0, \cdot))\|_{C[0,T]}. \tag{4.1}
$$

Proof. In this proof, we denote $u = u_1 - u_2$, $p = p_1 - p_2$, then by theorem 3.2, we know

$$
u(x, t) = \sum_{n=0}^{\infty}(f, \varphi_n)g_n(t)\varphi_n(x)
$$

and

$$
\partial^{\alpha}_{0+}u(x, t) = p(t)\sum_{n=0}^{\infty}(f, \varphi_n)\varphi_n(x) - \sum_{n=1}^{\infty}(f, \varphi_n)\lambda_n g_n(t)\varphi_n(x).
$$

In terms of the regularity of φ_n (see p 348 in [34], for example), we know that

$$
\|\varphi_n\|_{H^{k+2}(\Omega)} \leqslant C(\|\lambda_n\varphi_n\|_{H^k(\Omega)} + \|\varphi_n\|_{H^{k+1}(\Omega)}), \quad k = 0, 1, \cdots.
$$

Note that $-\Delta\varphi_n = \lambda_n\varphi_n$, $\frac{\partial\varphi_n}{\partial n}|_{\partial\Omega} = 0$, we have $\|\nabla\varphi_n\|_{L^2(\Omega)} = \sqrt{\lambda_n}$. By a recursive process, we can prove

$$
\|\varphi_n\|_{H^{2k}} \leqslant C(\lambda_n + 1)^k, \quad \textit{for} \quad n = 0, 1, 2, \ldots.
$$

Thus, for $m > \frac{d}{4}$, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty}\|(f, \varphi_n)\varphi_n\|_{L^{\infty}(\Omega)} &\leqslant C\sum_{n=1}^{\infty}\|(f, \varphi_n)\varphi_n\|_{H^{2m}} \\
&\leqslant C\sum_{n=1}^{\infty}|(f, \varphi_n)|(\lambda_n + 1)^m \\
&\leqslant C\left(\sum_{n=1}^{\infty}\frac{1}{(\lambda_n + 1)^{2\beta}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}(\lambda_n + 1)^{2(m+\beta)}(f, \varphi_n)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $\lambda_n \geqslant Cn^{\frac{2}{d}}$, $n \in N$ (see [4]), then we have $\frac{1}{\lambda_n^{2\beta}} \leqslant C\frac{1}{n^{\frac{4\beta}{d}}}$. If we choose $\beta > \frac{d}{4}$ and $\gamma = m + \beta > \frac{d}{2}$, then by $f \in \mathcal{D}((-L+1)^\gamma)$ and proposition 2.3, for $(x, t) \in \overline{\Omega} \times [0, T]$, we know the series $\sum_{n=0}^{\infty}(f, \varphi_n)\varphi_n(x)$ is convergent on $\overline{\Omega} \times [0, T]$ uniformly. Further, by equation (3.6) we know

$$\sum_{n=1}^{\infty}|(f, \varphi_n)\lambda_n g_n(t)\varphi_n(x)| \leqslant \|p\|_\infty \sum_{n=1}^{\infty}\|(f, \varphi_n)\varphi_n\|_{L^\infty(\Omega)} \leqslant C\|p\|_\infty\|f\|_{\mathcal{D}((-\Delta+1)^\gamma)}.$$

is convergent on $\overline{\Omega} \times [0, T]$ uniformly.

Thus, we have

$$\partial_{0+}^\alpha u(x_0, t) = p(t)\sum_{n=0}^{\infty}(f, \varphi_n)\varphi_n(x_0) - \sum_{n=1}^{\infty}(f, \varphi_n)\lambda_n g_n(t)\varphi_n(x_0) \tag{4.2}$$

for $0 < t \leqslant T$.

Setting

$$Q(t) := \sum_{n=1}^{\infty} -\lambda_n(f, \varphi_n)E_{\alpha,\alpha}(-\lambda_n t^\alpha)\varphi_n(x_0),$$

similarly, if we let $\gamma > \frac{d}{2} + 1$, then $\sum_{n=0}^{\infty}|\lambda_n(f, \varphi_n)E_{\alpha,\alpha}(-\lambda_n t^\alpha)\varphi_n(x)|$ is also convergent uniformly on $\overline{\Omega} \times [0, T]$, thus we can obtain $Q \in C[0, T]$. Inserting it into (4.2), we can deduce

$$|p(t)| \leqslant C\|\partial_{0+}^\alpha u(x_0, \cdot)\|_{C[0,T]} + C\|Q\|_{C[0,T]}\int_0^t (t-\tau)^{\alpha-1}|p(\tau)|\,\mathrm{d}\tau, \quad 0 < t \leqslant T.$$

Applying an inequality of Gronwall type with weakly singular kernel $(t-\tau)^{\alpha-1}$ (e.g. theorem 1 and corollary 2 in [42]), we see

$$|p(t)| \leqslant C_3\|\partial_{0+}^\alpha u(x_0, \cdot)\|_{C[0,T]}, \quad 0 < t \leqslant T,$$

where C_3 is a positive constant depending on f, α, T and Ω.

Thus the proof is completed by $p \in C[0, T]$. □

5. Tikhonov regularization method and the gradient of functional

Although the source term $p(t)$ can be determined uniquely by the measurement data at a point on the boundary in mathematical theory, here we propose a numerical method to find an approximate solution by using the additional condition (1.4).

Define a mapping $\mathcal{A}: p \in AC[0, T] \mapsto u_p|_\Gamma$ where $u_p(x, t)$ is the strong solution of (1.1)–(1.3) with a given function $p(t)$. Since $u_p \in L^2(0, T, H^2(\Omega))$, by the trace theorem, we know $u_p|_\Gamma \in L^2(0, T; H^{3/2}(\Gamma)) \subset L^2(0, T, L^2(\Gamma))$. Then the inverse problem is to solve the following operator equation:

$$\mathcal{A}(p) = g, \tag{5.1}$$

where $\mathcal{A}$ is an operator from $AC[0, T]$ to $L^2(0, T, L^2(\Gamma))$. Note that $C^1[0, T] \subset AC[0, T]$ is dense in $L^2(0, T)$, by remark 3.3, we know the operator $\mathcal{A}$ can be extended continuously onto $L^2(0, T)$. In fact, we have $\mathcal{A}(p) = v|_\Gamma + \mathcal{A}_L p$ where $v(x, t)$ is the solution of (1.1)–(1.3) with $p(t) = 0$, and $\mathcal{A}_L p = u_p|_\Gamma$ where $u_p(x, t)$ is the strong solution of (1.1)–(1.3) with the given function $p(t)$ and $\phi(x) = 0$. Thus $\mathcal{A}_L$ is a linear bounded operator from $L^2(0, T)$ into $L^2(0, T, L^2(\Gamma))$. By theorem 4.1, we know the operator $\mathcal{A}_L$ is injective. The solution of the inverse problem exists if the given data g is in the range $R(\mathcal{A})$ of mapping $\mathcal{A}$, however $R(\mathcal{A})$

is difficult to describe clearly. Based on the theory of linear ill-posed problems, we know if the exact input data g belongs to space $R(\mathcal{A}) + R(\mathcal{A})^{\perp}$, the least squares solution exists. Since the noisy data is usually not in $R(\mathcal{A})$, the existence of a solution for the inverse problem is not important.

According to (3.1), we have

$$\mathcal{A}_L p = \sum_{n=0}^{\infty}(f, \varphi_n)\int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\mathrm{d}\tau\varphi_n(x), \;\; t \in (0, T), \;\; x \in \Gamma$$
$$= \int_0^T p(\tau)(t-\tau)^{\alpha-1}K(t-\tau, x)\mathrm{d}\tau, \quad t \in (0, T), \quad x \in \Gamma,$$

where $K(t, x) := \sum_{n=0}^{\infty}(f, \varphi_n)E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\varphi_n(x)$ for $t > 0$ and 0 for $t \leqslant 0$. Under the condition $f \in \mathcal{D}((-L+1)^{\gamma})$ for $\gamma > \frac{d}{2} + 1$, similar to the proof of $Q \in C[0, T]$ in theorem 4.1, we can obtain $K(t, x) \in C[0, T]$ for any fixed $x \in \Gamma$. Therefore $\mathcal{A}_L$ is a Fredholm integral operator of the first kind with a weakly singular kernel. Thus the operator $\mathcal{A}_L$ is compact from $C[0, T]$ into $C[0, T]$ for $x \in \Gamma$, and is also compact from $L^2(0, T)$ into $L^2(0, T)$ for $x \in \Gamma$ if $\alpha \in (1/2, 1)$ (see p 27 in [23], for example). Thus the inverse source problem we consider here is a linear ill-posed problem.

In order to handle with the possible numerical instability of the inverse problem, we employ the classical Tikhonov regularization method. That is, define a Tikhonov regularization functional

$$\begin{aligned} J(p) &= \frac{1}{2}\|\mathcal{A}(p) - g^{\delta}\|^2_{L^2((0,T)\times\Gamma)} + \frac{\mu}{2}\|p\|^2_{L^2(0,T)} \\ &= \frac{1}{2}\int_0^T\int_{\Gamma}(u_p(x, t) - g^{\delta}(x, t))^2\mathrm{d}S\mathrm{d}t + \frac{\mu}{2}\int_0^T p^2(t)\mathrm{d}t, \end{aligned} \tag{5.2}$$

where $\mu > 0$ is a regularization parameter and g^{δ} is a noisy function of g satisfying $\|g^{\delta} - g\|_{L^2((0,T)\times\Gamma)} \leqslant \delta$. The first term denotes the defect between the exact data and the noisy data and the second term is a penalty term for stabilizing the numerical solution. Then the inverse source problem is transformed into solving the variational problem

$$\min_{p\in L^2(0,T)} J(p). \tag{5.3}$$

It is well-known that the minimizer p_{μ}^{δ} exists uniquely and converges to the exact solution p under a suitable choice of μ; see chapter 5 in [5]. In this paper, the conjugate gradient method (CGM) is used to find the minimizer of functional (5.2). It is well known that the key work is to find the gradient of functional (5.2), which can be obtained by constructing a sensitivity problem and an adjoint problem. The used method is also called the variational adjoint method or the adjoint problem approach [8, 24].

In the following, we suppose that the solution u_p is smooth enough and we give a formal deduction. Let the source term $p(t)$ be perturbed by a small amount $\delta p(t)$, then the forward solution u_p has a small change $w = u_{p+\delta p} - u_p$ which satisfies

Sensitivity problem:

$$\begin{cases} \partial_{0+}^{\alpha} w(x, t) = \Delta w(x, t) + f(x)\cdot\delta p(t), & (x, t) \in \Omega \times (0, T], \\ \dfrac{\partial w(x, t)}{\partial n} = 0, & x \in \partial\Omega, \, t \in (0, T], \\ w(x, 0) = 0, & x \in \overline{\Omega}. \end{cases} \tag{5.4}$$

In order to use the CGM to solve the variational problem, we need to deduce the gradient of the functional $J(p)$.

From (5.2), we have

$$\begin{aligned}\delta J(p) &= J(p+\delta p) - J(p)\\ &= \frac{1}{2}\int_0^T\int_\Gamma (u_p(x,t)+w(x,t)-g^\delta(x,t))^2\mathrm{d}S\mathrm{d}t + \frac{\mu}{2}\int_0^T (p+\delta p)^2\mathrm{d}t\\ &\quad -\frac{1}{2}\int_0^T\int_\Gamma (u_p(x,t)-g^\delta(x,t))^2\mathrm{d}S\mathrm{d}t + \frac{\mu}{2}\int_0^T p^2(t)\mathrm{d}t\\ &= \int_0^T\int_\Gamma (u_p(x,t)-g^\delta(x,t))w(x,t)\mathrm{d}S\mathrm{d}t + \mu\int_0^T p\cdot\delta p\mathrm{d}t\\ &\quad + o(\|w\|_{L^2((0,T)\times\Gamma)} + \|\delta p\|_{L^2(0,T)}).\end{aligned} \tag{5.5}$$

Note that w is the solution of the sensitivity problem. By $w(x,0)=0$, we have $\partial_{0+}^\alpha w(x,t) = D_{0+}^\alpha w(x,t)$. Let $v(x,t)$ be a smooth function. From equation (5.4) and lemma 2.12, we have

$$\begin{aligned}&\int_0^T\int_\Omega f(x)\delta p(t)v(x,t)\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int_\Omega (\partial_{0+}^\alpha w(x,t) - \Delta w(x,t))v(x,t)\mathrm{d}x\mathrm{d}t\\ &= \int_\Omega\int_0^T w(x,t)(\partial_{T-}^\alpha v - \Delta v)(x,t)\mathrm{d}x\mathrm{d}t + \int_\Omega I_{0+}^{1-\alpha}w(x,T)v(x,T)\mathrm{d}x\\ &\quad -\int_0^T\left(\int_\Gamma v(x,t)\frac{\partial w(x,t)}{\partial n}\mathrm{d}S + \int_{\partial\Omega\backslash\Gamma} v(x,t)\frac{\partial w(x,t)}{\partial n}\mathrm{d}S\right.\\ &\quad \left. -\int_\Gamma w(x,t)\frac{\partial v(x,t)}{\partial n}\mathrm{d}S - \int_{\partial\Omega\backslash\Gamma} w(x,t)\frac{\partial v(x,t)}{\partial n}\mathrm{d}S\right)\mathrm{d}t.\end{aligned} \tag{5.6}$$

In order to obtain the gradient of functional $J(p)$, we need to find a function $j(t)$ such that $J(p+\delta p) - J(p) = (j(t), \delta p(t))_{L^2(0,T)} + o(\|\delta p\|_{L^2(0,T)})$. Thus we hope to replace $\int_\Gamma (u_p(x,t) - g^\delta(x,t))w(x,t)\mathrm{d}S$ in (5.5) by δp times a function of t. Therefore, we define an initial boundary value problem for v, called an adjoint problem such that we can obtain $j(t)$ from (5.6).

Adjoint problem:

$$\begin{cases}\partial_{T-}^\alpha v(x,t) = \Delta v(x,t), & (x,t)\in\Omega\times(0,T],\\ \dfrac{\partial v(x,t)}{\partial n} = u_p(x,t) - g^\delta(x,t), & x\in\Gamma,\ t\in(0,T],\\ \dfrac{\partial v(x,t)}{\partial n} = 0, & x\in\partial\Omega\backslash\Gamma,\ t\in(0,T],\\ v(x,T) = 0, & x\in\overline{\Omega}.\end{cases} \tag{5.7}$$

Then we can apply the boundary conditions in (5.4) and (5.7) into (5.6) to obtain

$$\int_0^T\int_\Omega f(x)\delta p(t)v(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Gamma w(x,t)(u_p(x,t) - g^\delta(x,t))\mathrm{d}S\mathrm{d}t. \tag{5.8}$$

This, together with (5.5), implies that

$$\delta J(p) = \int_0^T \int_\Omega f(x)\delta p(t) v(x, t)\mathrm{d}x\mathrm{d}t + \mu \int_0^T p(t)\delta p(t)\mathrm{d}t + o(\|w\|_{L^2((0,T)\times\Gamma)} + \|\delta p\|_{L^2(0,T)}). \tag{5.9}$$

Note that w is a solution of the sensitive problem. By remark 3.3, we have $\|w\|_{L^2((0,T)\times\Gamma)} \leqslant C\|\delta p\|_{L^2(0,T)}$, thus we know $o(\|w\|_{L^2((0,T)\times\Gamma)} + \|\delta p\|_{L^2(0,T)}) = o(\|\delta p\|_{L^2(0,T)})$. By the definition of gradient operator, we obtain

$$J'_p = \int_\Omega f(x)v(x, t)\mathrm{d}x + \mu p(t). \tag{5.10}$$

To be consistent with the direct problem and the sensitive problem, we transform the adjoint problem (5.7) into a forward initial boundary value problem. Let v be the solution of the adjoint problem, define $\tilde{v}(x, t) = v(x, T - t)$, then it is not hard to prove $\partial^\alpha_{T-}v(x, t) = \partial^\alpha_{0+}\tilde{v}(x, T - t)$. Denote $\tau = T - t$, then $\tilde{v}(x, \tau)$ satisfies

$$\begin{cases} \partial^\alpha_{0+}\tilde{v}(x, \tau) = \Delta\tilde{v}(x, \tau), & (x, \tau) \in \Omega \times (0, T], \\ \dfrac{\partial\tilde{v}(x, \tau)}{\partial n} = u_p(x, T - \tau) - g^\delta(x, T - \tau), & x \in \Gamma, \tau \in (0, T], \\ \dfrac{\partial\tilde{v}(x, \tau)}{\partial n} = 0, & x \in \partial\Omega\backslash\Gamma, \tau \in (0, T], \\ \tilde{v}(x, \tau)|_{\tau=0} = 0, & x \in \overline{\Omega}. \end{cases} \tag{5.11}$$

Then in the next section, we solve the forward initial boundary value problem (5.11) instead of (5.7) and then take $v(x, t) = \tilde{v}(x, T - t)$.

6. Conjugate gradient method for the variational problem

We use a CGM to search the minimizer of functional $J(p)$. Let p_k be the kth approximate solution to $p(t)$. Denote

$$p_{k+1} = p_k + \beta_k d_k, \quad k = 0, 1, 2, \cdots, \tag{6.1}$$

where β_k is the step size and d_k is a descent direction in the kth iteration. The CGM use the following iteration formula to updata the descent direction

$$d_k = -J'_{p_k} + \gamma_k d_{k-1}, \tag{6.2}$$

where γ_k is conjugate coefficient calculated by

$$\gamma_k = \frac{\int_0^T (J'_{p_k})^2\mathrm{d}t}{\int_0^T (J'_{p_{k-1}})^2\mathrm{d}t}, \quad \gamma_0 = 0. \tag{6.3}$$

The step size β_k can be obtained in the following deduction. From (5.2), we have

$$J(p_k + \beta_k d_k) = \frac{1}{2}\int_0^T \int_\Gamma (u_{p_k} + \beta_k w_k - g^\delta)^2\mathrm{d}S\mathrm{d}t + \frac{\mu}{2}\int_0^T (p_k + \beta_k d_k)^2\mathrm{d}t, \tag{6.4}$$

where w_k is the solution of the sensitivity problem with $\delta p = d_k$. Let

$$\frac{\mathrm{d}J}{\mathrm{d}\beta_k} = \int_0^T \int_\Gamma (u_{p_k} + \beta_k w_k - g^\delta)w_k\mathrm{d}S\mathrm{d}t + \mu\int_0^T (p_k + \beta_k d_k)d_k\mathrm{d}t = 0, \tag{6.5}$$

we can get a step size

$$\beta_k = -\frac{\int_0^T \int_\Gamma (u_{p_k} - g^\delta) w_k \mathrm{d}S\mathrm{d}t + \mu \int_0^T p_k d_k \mathrm{d}t}{\int_0^T \int_\Gamma w_k^2 \mathrm{d}S\mathrm{d}t + \mu \int_0^T d_k^2 \mathrm{d}t}. \tag{6.6}$$

Therefore, we have the following CGM to solve the minimization problem (5.3).

1. Initialize $p_0 = 0$, and set $k = 0$;
2. Solve the direct problem (1.1)–(1.3) with $p = p_k$, and determine the residual $r_k = (u_{p_k} - g^\delta)|_\Gamma$;
3. Solve the adjoint problem (5.7) and determine the gradient J'_{p_k} by (5.10). If $\mu = 0$, from $v(x, T) = 0$ in (5.7), we know $J'_{p_k}(T) = 0$, thus we cannot update $p_k(T)$. To avoid this case, here we modify $v(x, T) = 2v(x, T - h) - v(x, T - 2h)$ where h is a time step size;
4. Calculate the conjugate coefficient γ_k by (6.3) and the descent direction d_k by (6.2);
5. Solve the sensitivity problem (5.4) for w_k with $\delta p = d_k$;
6. Calculate the step size β_k by (6.6);
7. Update the source term p_{k+1} by (6.1);
8. Increase k by one and go to step (2), repeat the above procedure until a stopping criterion is satisfied.

7. Numerical experiments

In this section, we present the numerical results for six examples in one-dimensional and two-dimensional cases to show the effectiveness of the conjugate gradient algorithm. And the convergence and stability of the algorithm are also investigated.

In our computation, we always set $T = 1$. The noisy data is generated by adding a random perturbation, i.e.

$$g^\delta = g + \varepsilon g \cdot (2 \cdot \mathrm{rand}(\mathrm{size}(g)) - 1).$$

The corresponding noise level is calculated by $\delta = \|g^\delta - g\|_{L^2(\Gamma\times(0,T))}$.

To show the accuracy of the numerical solution, we compute the approximate L^2 error denoted by

$$e_k = \|p_k(t) - p(t)\|_{L^2(0,T)}, \tag{7.1}$$

where $p_k(t)$ is the source term reconstructed at the kth iteration, and $p(t)$ is the exact solution.

The residual E_k at the kth iteration is given by

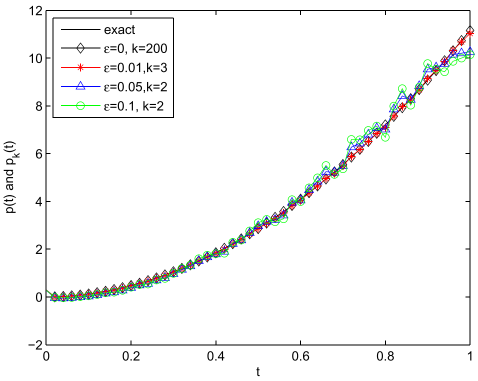
$$E_k = \|u_{p_k} - g^\delta\|_{L^2(\Gamma\times(0,T))}. \tag{7.2}$$

In an iteration algorithm, the most important work is to find a suitable stopping rule. In this study we use the well-known Morozov's discrepancy principle [20], i.e. we choose k satisfying the following inequality:

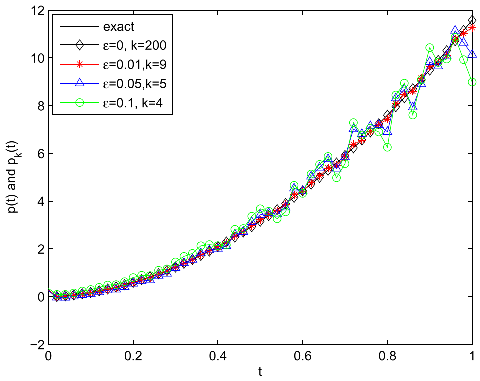
$$E_k \leqslant \tau\delta < E_{k-1}, \tag{7.3}$$

where $\tau > 1$ is a constant and can be taken heuristically to be 1.01, as suggested by Hanke and Hansen [7]. If $\delta = 0$, we take $k = 200$ for one-dimensional examples and $k = 100$ for two-dimensional examples.

In (3.1), we need to know the eigenfunctions of operator $\triangle$ which is very difficult to obtain for high dimensional cases with a general domain. Thus in this paper, we use a finite



(a) $\alpha = 0.3$



(b) $\alpha = 0.7$

Figure 1. The numerical results for example 1 for various noise levels with $\mu = 0$.

element method similar to the one in [37] to solve the direct problem, the sensitive and adjoint problem for the one-dimensional case and two-dimensional case in each iteration instead of using (3.1), and the series solutions of the sensitive and adjoint problems.

7.1. One-dimensional case

Without loss of generality, the space domain Ω is taken as $(0, 1)$ and $\Gamma = 0$ in the one-dimensional case. We consider the following four examples.

Example 1. Let the exact solution for problem (1.1)–(1.3) be $u(x, t) = t^2 \cos(\pi x)$, then $p(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + \pi^2 t^2$ will be reconstructed and the other data can be obtained from the exact solution.

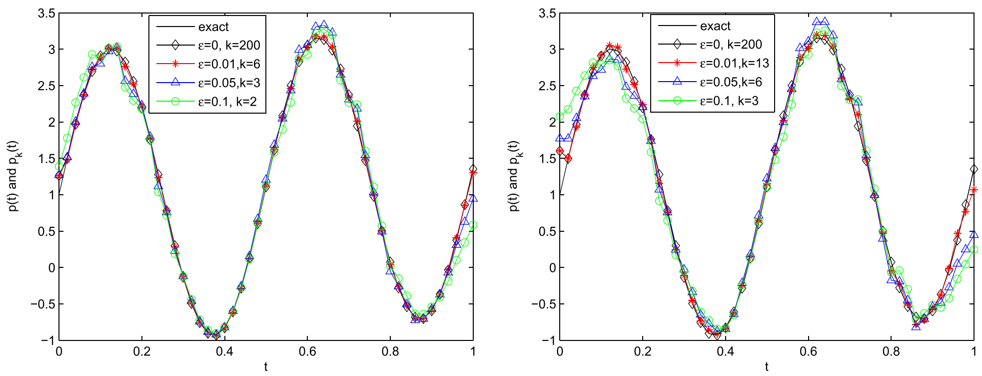
Example 2. In the second example, the analytic solution of problem (1.1)–(1.3) is unknown. We construct the Dirichlet data on boundary Γ by solving the direct problem (1.1)–(1.3) by using a finite element method in which we use the initial data $\varphi(x) = x^{2+\alpha}(1-x)^{2+\alpha}$ and take a source function $f(x) = x^2(1-x)^2 + \cos(\pi x) + 1$ and $p(t) = 2\sin(4\pi t) + \exp(-t) + t$.

Example 3. We test a nonsmooth example with a cusp. Let $p(t) = -|2t - 1| + \alpha$ and the initial data $\varphi(x) = x^{2+\alpha}(1-x)^{2+\alpha}$. Take a source function $f(x) = \cos(\pi x) + 1$. The Dirichlet data $u(0, t)$ is obtained by solving the direct problem (1.1)–(1.3) by a finite element method as the exact input data.

Example 4. We consider a discontinuous example. The source term is

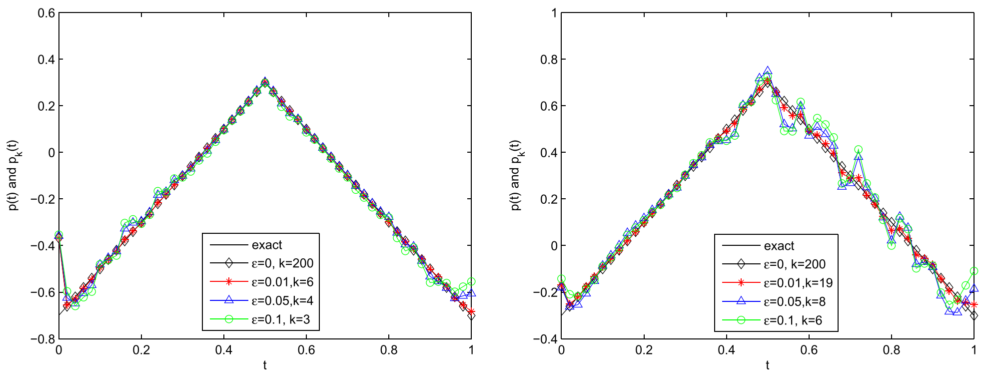
$$p(t) = \begin{cases} 5, & t \in [0.2, 0.8]; \\ 0.5, & t \in [0, 0.2) \bigcup (0.8, 1]. \end{cases} \tag{7.4}$$

Let the initial data be $\varphi(x) = x^{2+\alpha}(1-x)^{2+\alpha}$ and take a source function $f(x) = \cos(\pi x) + 1$. The Dirichlet data $u(0, t)$ is obtained by solving the direct problem (1.1)–(1.3) by a finite element method as the exact input data.



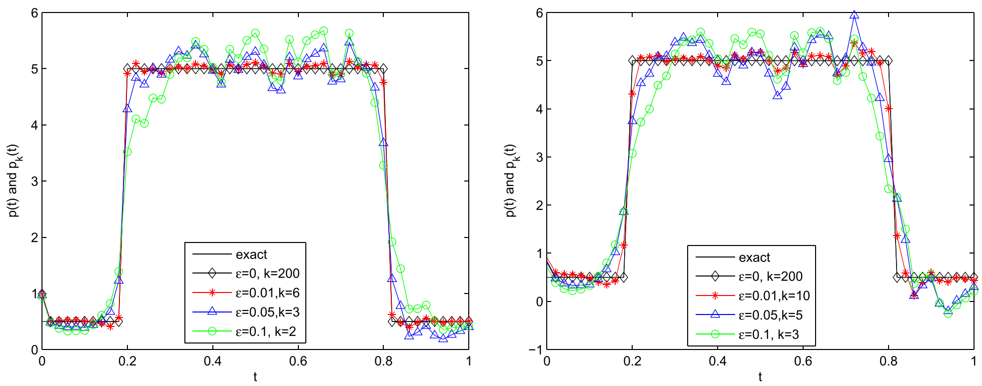
(a) $\alpha = 0.3$ **(b)** $\alpha = 0.7$

Figure 2. The numerical results for example 2 for various noise levels with $\mu = 0$.



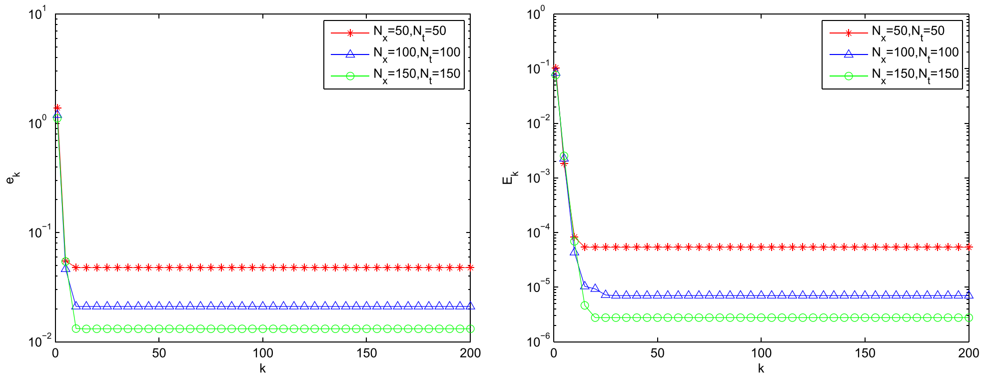
(a) $\alpha = 0.3$ **(b)** $\alpha = 0.7$

Figure 3. The numerical results for example 3 for various noise levels with $\mu = 0$.



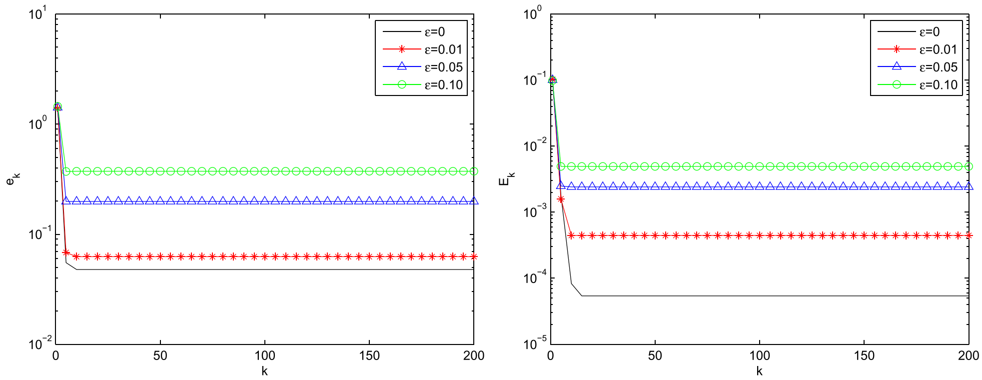
(a) $\alpha = 0.3$ **(b)** $\alpha = 0.7$

Figure 4. The numerical results for example 4 for various noise levels with $\mu = 0$.



(a) e_k **(b)** E_k

Figure 5. The errors e_k and the residuals E_k for example 1 with different grid numbers.



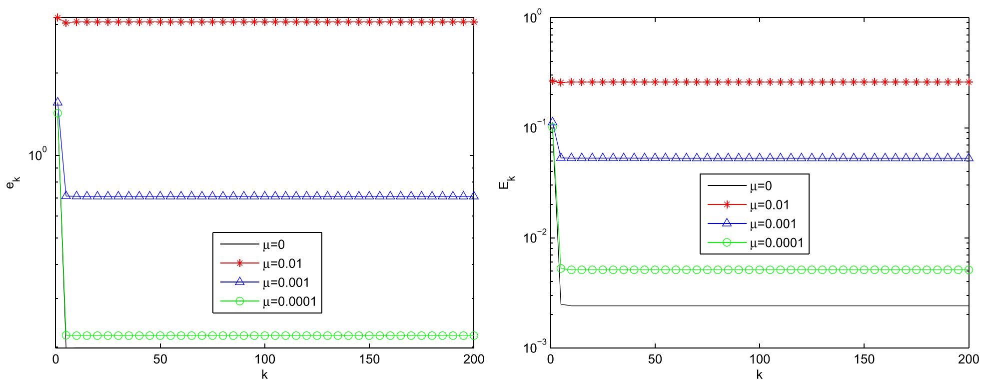
(a) e_k **(b)** E_k

Figure 6. The errors e_k and the residuals E_k for example 1 with various noise levels.

The numerical results for examples 1–4 by using the discrepancy principle for various noise levels in the case of $\alpha = 0.3, 0.7$ are shown in figures 1–4, respectively. We can see that the numerical results for examples 1–4 are quite accurate up to 1% noise added in the exact Dirichlet data $u(0, t)$. For examples 3–4, the numerical results are less accurate and oscillate mildly for $\alpha = 0.7$. The numerical results become less accurate if the fractional orders α and the noise level increase.

In the following, we investigate the convergence and stability of the proposed algorithm. We also show the effectiveness of the regularization parameter in the Tikhonov regularization functional.

In figure 5, we show the approximation error e_k and the residual E_k for iteration steps $k = 1 : 200$ for example 1 with the grid numbers along x and t directions as $(N_x, N_t) \in \{(50, 50), (100, 100), (150, 150)\}$ and $\alpha = 0.5$, $\varepsilon = 0$, $\mu = 0$. We can see that both the error e_k and residual E_k decay with increasing grid numbers. The error e_k decreases very quickly to a stable level with increasing iteration number.



(a) e_k **(b)** E_k

Figure 7. The errors e_k and the residuals E_k for example 1 with different μ.

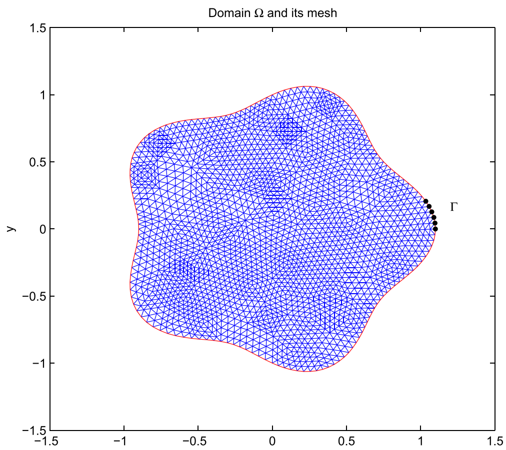


Figure 8. Domain and boundary.

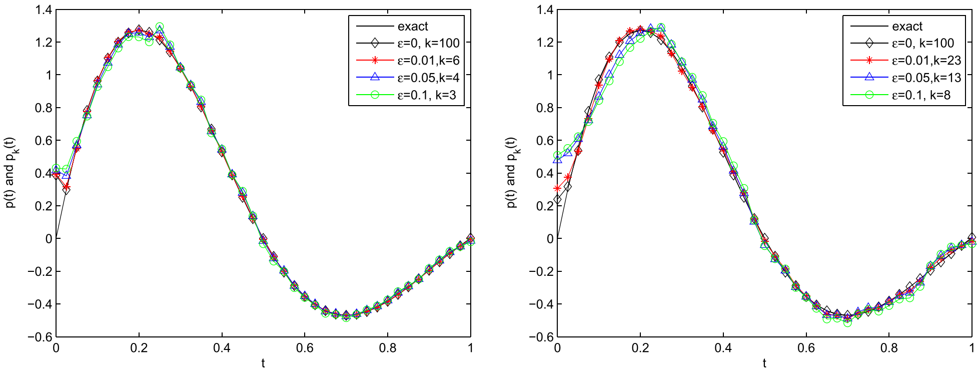
The approximation error e_k and the residual E_k for example 1 with various noise levels are shown in figure 6 with $\alpha = 0.5$, $\mu = 0$, $N_x = N_t = 50$. It can be seen that the numerical error e_k decreases as the noise level in the data decreases.

Next, we show the approximation error e_k and the residual E_k for example 1 with various μ in figure 7 in which we take $\varepsilon = 0.05$, $\alpha = 0.5$. It is observed that the small regularization parameter $\mu = 0, 0.0001$ yields better numerical results than big μ. Thus the regularization parameter μ can be taken as zero in this study. In fact, it is a difficult problem to choose an appropriate regularization parameter.

7.2. Two-dimensional case

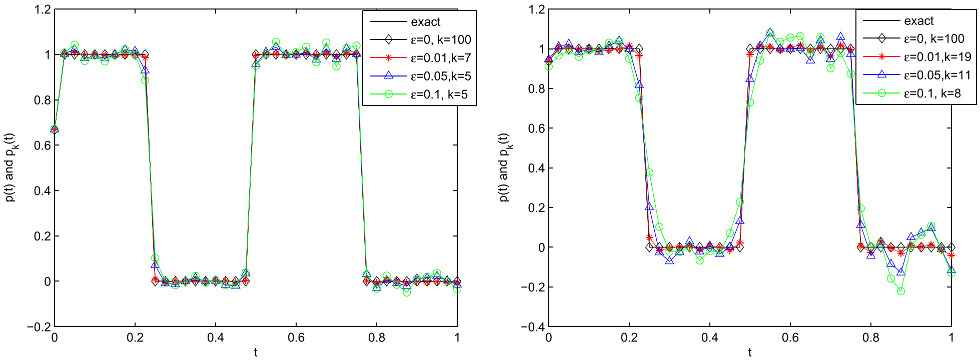
Denote the coordinate as (x, y). Let Ω be a bounded domain with a smooth boundary given by parametrization

$$\partial\Omega = \{(x, y) = r(\theta)(\cos\theta, \sin\theta), \ \ \theta \in [0, 2\pi]\}, \tag{7.5}$$



(a) $\alpha = 0.3$ **(b)** $\alpha = 0.7$

Figure 9. The numerical results for example 5 for various noise levels.



(a) $\alpha = 0.3$ **(b)** $\alpha = 0.7$

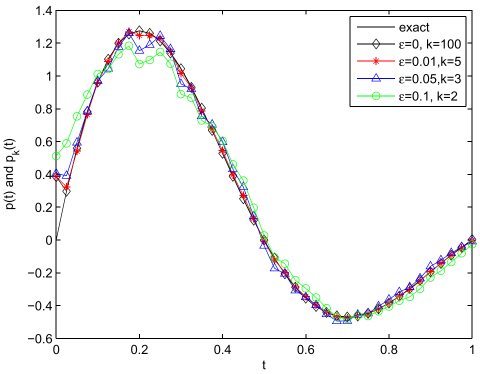
Figure 10. The numerical results for example 6 for various noise levels.

where $r(\theta) = 1 + 0.1\cos 5\theta$. We take the boundary $\Gamma = \partial\Omega \cap \left\{0 \leqslant \theta \leqslant \frac{\pi}{16}\right\}$, see figure 8. We present the numerical results for two examples to show the accuracy and stability of our proposed method.

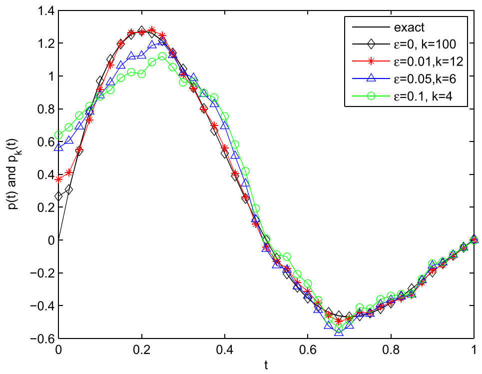
Example 5. We take $\phi(x, y) = \cos(\pi x)\cos(\pi y) + x^2(1 - y)^2$. The exact space-dependent source term is $f(x, y) = \cos(\pi x) + \cos(\pi y) + 1$ and the time-dependent source term is $p(t) = 2\mathrm{e}^{-t}\sin(2\pi t)$. The additional data $u|_\Gamma$ is obtained by solving the direct problem (1.1)–(1.3) in which we use a finite element method; see [37] with the total number of mesh nodes 2205 on Ω and 41 points on $[0, T]$.

Example 6. We consider a discontinuous time-dependent source term

$$p(t) = \begin{cases} 1, & t \in [0, 0.25) \bigcup [0.5, 0.75]; \\ 0, & t \in [0.25, 0.5) \bigcup (0.75, 1]. \end{cases} \tag{7.6}$$



(a) $\alpha = 0.3$



(b) $\alpha = 0.7$

Figure 11. The numerical results for example 5 for various noise levels with $\Gamma = \partial\Omega \cap [0, \pi/128]$.

Table 1. Numerical results for examples 5–6 for different boundary Γ.

	Γ	$0 \leqslant \theta \leqslant \pi/2$,	$0 \leqslant \theta \leqslant \pi/16$	$0 \leqslant \theta \leqslant \pi/64$	$0 \leqslant \theta \leqslant \pi/128$
Example 5	$\min_{1\leqslant k\leqslant 100} e_k$	0.0040	0.0111	0.0224	0.0318
Example 6	$\min_{1\leqslant k\leqslant 100} e_k$	0.0027	0.0089	0.0212	0.0361

Take $\phi(x, y) = 0$ and the space-dependent source term is $f(x, y) = x^2 + y^2 + 1$. The additional data $u|_\Gamma$ is obtained by solving the direct problem (1.1)–(1.3) in which we use a finite element method with the same mesh in example 5.

The numerical results for examples 5–6 by using the discrepancy principle for various noise levels in the case of $\alpha = 0.3, 0.7$ are shown in figures 9 and 10, respectively. We can see that the numerical results for example 5 are in very good agreement with the exact shape and are less accurate for example 6.

In table 1, we show the numerical errors of examples 5–6 for different boundaries Γ with $\alpha = 0.5$ and $\varepsilon = 0.01$. It can be seen that the results still remain good with decreasing size of Γ. In particular, for $\Gamma = \partial\Omega \cap \left\{0 \leqslant \theta \leqslant \frac{\pi}{128}\right\}$, there is only one mesh node on Γ. The numerical results in this case for example 5 are shown in figure 11. We can see the numerical results are good which is in agreement with the theoretical result in theorem 4.1.

8. Conclusions

In this paper, we investigate an inverse time-dependent source problem for a multi-dimensional fractional diffusion equation. The existence and uniqueness for the direct problem and the uniqueness for the inverse problem are both proved. The conjugate gradient method combined with Morozov's discrepancy principle are used to solve the regularized variational problem. Six numerical examples are provided to show that the proposed method is effective and stable.

References

[1] Bao G, Ehlers T A and Li P J 2013 Radiogenic source identification for the helium production-diffusion equation *Commun. Comput. Phys.* **14** 1–20
[2] Berkowitz B, Scher H and Silliman S E 2000 Anomalous transport in laboratory-scale, heterogeneous porous media *Water Resour. Res.* **36** 149–58
[3] Cheng J, Nakagawa J, Yamamoto M and Yamazaki T 2009 Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation *Inverse Problems* **25** 115002
[4] Courant R and Hilbert D 1953 *Methods of Mathematical Physics* vol 1 (CUP Archive) (New York: Interscience Publishers)
[5] Engl H W, Hanke M and Neubauer A 1996 Regularization of inverse problems *Mathematics and its Applications* vol 375 (Dordrecht: Kluwer)
[6] Fan W P, Jiang X Y and Qi H T 2015 Parameter estimation for the generalized fractional element network Zener model based on the Bayesian method *Physica* A **427** 40–9
[7] Hanke M and Hansen P C 1993 Regularization methods for large-scale problems *Surv. Math. Ind.* **3** 253–315
[8] Hasanov A, DuChateau P and Pektaş B 2006 An adjoint problem approach and coarse-fine mesh method for identification of the diffusion coefficient in a linear parabolic equation *J. Inverse Ill-Posed Problems* **14** 435–63
[9] Henry B I, Langlands T A M and Wearne S L 2008 Fractional cable models for spiny neuronal dendrites *Phys. Rev. Lett.* **100** 128103
[10] Jiang Y J and Ma J T 2011 High-order finite element methods for time-fractional partial differential equations *J. Comput. Appl. Math.* **235** 3285–90
[11] Kilbas A A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* vol 204 (Netherlands: Elsevier)
[12] Li G S, Zhang D L, Jia X Z and Yamamoto M 2013 Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation *Inverse Problems* **29** 065014
[13] Lin Y M and Xu C J 2007 Finite difference/spectral approximations for the time-fractional diffusion equation *J. Comput. Phys.* **225** 1533–52
[14] Liu J J and Yamamoto M 2010 A backward problem for the time-fractional diffusion equation *Appl. Anal.* **89** 1769–88
[15] Luchko Y 2011 Maximum principle and its application for the time-fractional diffusion equations *Fract. Calculus Appl. Anal.* **14** 110–24
[16] Luchko Y and Zuo L 2014 θ-function method for a time-fractional reaction-diffusion equation *J. Algerian Math. Soc.* **1** 1–5
[17] Metzler R and Klafter J 2000 Boundary value problems for fractional diffusion equations *Physica* A **278** 107–25
[18] Metzler R and Klafter J 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach *Phys. Rep.* **339** 1–77
[19] Metzler R and Klafter J 2000 Subdiffusive transport close to thermal equilibrium: from the Langevin equation to fractional diffusion *Phys. Rev.* E **61** 6308–11
[20] Morozov V A, Nashed Z and Aries A B 1984 *Methods for Solving Incorrectly Posed Problems* (New York: Springer)
[21] Murio D A 2008 Implicit finite difference approximation for time fractional diffusion equations *Comput. Math. Appl.* **56** 1138–45
[22] Murio D A 2008 Time fractional IHCP with caputo fractional derivatives *Comput. Math. Appl.* **56** 2371–81
[23] Nair M T 2009 *Linear Operator Equations: Approximation and Regularization* (Singapore: World Scientific)
[24] Ou Y H, Hasanov A and Liu Z H 2008 Inverse coefficient problems for nonlinear parabolic differential equations *Acta Math. Sin.* **24** 1617–24
[25] Pollard H 1948 The completely monotonic character of the Mittag–Leffler function $E_\alpha(-X)$ *Bull. Am. Math. Soc.* **54** 1115–6
[26] Raberto M, Scalas E and Mainardi F 2002 Waiting-times and returns in high-frequency financial data: an empirical study *Physica* A **314** 749–55
[27] Sakamoto K and Yamamoto M 2009 Inverse heat source problem from time distributing overdetermination *Appl. Anal.* **88** 735–48

[28] Sakamoto K and Yamamoto M 2011 Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems *J. Math. Anal. Appl.* **382** 426–47
[29] Scalas E, Gorenflo R and Mainardi F 2000 Fractional calculus and continuous-time finance *Physica* A **284** 376–84
[30] Sokolov I M and Klafter J 2005 From diffusion to anomalous diffusion: A century after Einsteins Brownian motion *Chaos* **15** 1–7
[31] Tatar S, Tnaztepe R and Ulusoy S 2015 Determination of an unknown source term in a space-time fractional diffusion equation *J. Fract. Calculus Appl.* **6** 83–90
[32] Tatar S, Tnaztepe R and Ulusoy S 2016 Simultaneous inversion for the exponents of the fractional time and space derivatives in the space-time fractional diffusion equation *Appl. Anal.* **95** 1–23
[33] Tatar S and Ulusoy S 2015 An inverse source problem for a one-dimensional space-time fractional diffusion equation *Appl. Anal.* **94** 2233–44
[34] Taylor M E 2011 Partial differential equations: I. Basic theory *Appl. Math. Sci.* vol 115 2nd edn (New York: Springer)
[35] Wang W Y, Yamamoto M and Han B 2013 Numerical method in reproducing kernel space for an inverse source problem for the fractional diffusion equation *Inverse Problems* **29** 095009
[36] Wei T and Wang J G 2014 A modified quasi-boundary value method for the backward time-fractional diffusion problem *ESAIM* **48** 603–21
[37] Wei T and Wang J G 2014 A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation *Appl. Numer. Math.* **78** 95–111
[38] Wei T and Zhang Z Q 2016 Robin coefficient identification for a time-fractional diffusion equation *Inverse Problems Sci. Eng.* **24** 647–66
[39] Wei T and Zhang Z Q 2013 Reconstruction of a time-dependent source term in a time-fractional diffusion equation *Eng. Anal. Bound. Elem.* **37** 23–31
[40] Wyss W 1986 The fractional diffusion equation *J. Math. Phys.* **27** 2782–5
[41] Yamamoto M and Zhang Y 2012 Conditional stability in determining a zeroth-order coefficient in a half-order fractional diffusion equation by a carleman estimate *Inverse Problems* **28** 105010
[42] Ye H P, Gao J M and Ding Y S 2007 A generalized Gronwall inequality and its application to a fractional differential equation *J. Math. Anal. Appl.* **328** 1075–81
[43] Yu B, Jiang X Y and Qi H T 2015 An inverse problem to estimate an unknown order of a Riemann–Liouville fractional derivative for a fractional stokes first problem for a heated generalized second grade fluid *Acta Mech. Sin.* **31** 153–61
[44] Yu B, Jiang X Y and Wang C 2016 Numerical algorithms to estimate relaxation parameters and Caputo fractional derivative for a fractional thermal wave model in spherical composite medium *Appl. Math. Comput.* **274** 106–18
[45] Yuste S B, Acedo L and Lindenberg K 2004 Reaction front in an $A + B \rightarrow C$ reaction-subdiffusion process *Phys. Rev.* E **69** 036126
[46] Yuste S B and Lindenberg K 2002 Subdiffusion-limited reactions *Chem. Phys.* **284** 169–80
[47] Zhang Y and Xu X 2011 Inverse source problem for a fractional diffusion equation *Inverse Problems* **27** 035010
[48] Zheng G H and Wei T 2010 Spectral regularization method for a Cauchy problem of the time fractional advection-dispersion equation *J. Comput. Appl. Math.* **233** 2631–40
[49] Zheng G H and Wei T 2012 A new regularization method for a Cauchy problem of the time fractional diffusion equation *Adv. Comput. Math.* **36** 377–98
[50] Zhuang P H and Liu F W 2006 Implicit difference approximation for the time fractional diffusion equation *J. Appl. Math. Comput.* **22** 87–99