CORRECTNESS OF THE CODE

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This note is the correctness of spTest.jl, namely that it will pick up all cycles for base b when low$=0$ and high$=b-1$.

Let f_b be a map where $f_b([x,y,z]_b) = [\max(x,y), \min(x,y), z]_b$, and $\leq_b$ an ordering where $[x_0,x_1,x_2]_b \leq_b [y_0,y_1,y_2]_b$ if and only if

$$f_b([x_0,x_1,x_2]_b) \leq f_b([y_0,y_1,y_2]_b).$$

Now suppose $n_1,\ldots,n_k$ be a cycle of length k such that n_1 is minimal respect to $\leq_b$. (Note that n_i's have at most 3 digits so the above ordering make sense on n_i's). Moreover, we know that at least one n_i has at most two digits and so n_1 has only two digits. So let $n_1 = [u,v]_b$, and $x_0 = \max(u,v)$ and $y_0 = \min(u,v)$.

Firstly we have $[u,v]_b = n_1 \leq_b n_2 = u^2+v^2 = x_0^2+y_0^2$ by the minimality of n_1, and since $f_b(n) \leq n$, we have

$$x_0 + by_0 = [x_0,y_0,0] = f_b([u,v,0]) \leq f_b(n_2) \leq n_2 = x_0^2 + y_0^2.$$

And so

$$\begin{aligned} x_0 + by_0 &\leq x_0^2 + y_0^2 \\ y_0(b-y_0) &\leq x_0^2 - x_0 \\ y_0(b-y_0) + \tfrac{1}{4} &\leq \left(x_0 - \tfrac{1}{2}\right)^2 \\ \tfrac{1}{2} + \sqrt{y_0(b-y_0)+\tfrac{1}{4}} &\leq x_0 \end{aligned}$$

And so in line 71 in spTest.jl, the for loop

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for x=max(y,ceil(Int,sqrt(y*b-y^2+1/4)+1/2)):b-1
```

will eventually hit $y = y_0$ and $x = x_0$.

And now suppose $[x_0,y_0,0]_b$ has already been seen. Then by the ordering of the for loop, we must have reached $[x_0,y_0,0]_b$ from some $[x_0',y_0',0]_b$ with $x_0' \geq y_0'$ and $[x_0,y_0,0]_b \geq [x_0',y_0',0]_b$. So now we know we have reached $[x_0,y_0,0]_b$ in the while loop at line 79 where we started at $[x_0',y_0',0]_b$ (which may or may not be $[x_0,y_0,0]_b$). We also note that $f_b([x_0,y_0,0]_b) = [x_0,y_0,0]_b$ and $f_b([x_0',y_0',0]_b) = [x_0',y_0',0]_b$, and so $[x_0',y_0',0]_b \leq_b [x_0,y_0,0]_b$ as well. Finally, we will not break out of the while loop prematurely at line 85 since the inequality is exactly the $\leq_b$ ordering, and we picked n_1 to be the minimal n_i in the cycle and $[x_0',y_0',0]_b \leq_b [x_0,y_0,0]_b$. If $f(n_1) = n_1$, then eventually we will hit $n = n_1$ again, and otherwise we will hit $n = n_2$ again and we will detect this loop.