Commutative Algebra and Algebraic Geometry — Lecture 4

Martin Svanberg

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Loose ends

Is $\mathrm{Hom}(M, N)$ an R-module? Only if R commutative.

Example 1. $R = F$ and $M = V$, a finite dimensional vector space, $N = W$. Look at $\mathrm{Hom}(V, W)$, a vector space of dimension $\dim(V)\dim(W)$. If $b_1, \ldots, b_m$ is a basis for V, $c_1, \ldots, c_n$ basis for W. Then $V \cong F^n$, $W \cong F^m$ and $\mathrm{Hom}(F^n, F^m)$ is the set of $m \times n$ matrices.

Example 2. $R = F[G]$ and F a field, G a finite group. "Group ring" or "group algebra", an example of noncommutative rings. If G acts on a set, then G also acts on Functions(X,F).

If you define the action as $(g \cdot f)(x) = f(gx)$ then it fails, but $(g \cdot f)(x^{-1})$ does work. So Functions(X, F) is a vector space and a G-module, but this we cannot do for general rings.

Observe: in finite dimensional vector spaces, a maximal linearly independent set is a basis. This is not true for modules. Linear independence is defined as before, but a maximally linearly independent set might not be a basis. For example, (1) is a maximal linearly independent set of Z-module, but so is (2). It is free of rank 1 (cyclic), but it is not a generating set.

Consider $\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n$, where $A \in M_n(\mathbb{Z})$. $\mathrm{Im}(A)$ is a submodule of $\mathbb{Z}^n$. Assume $\det A \neq 0$. Equivalent to $\mathrm{Im} A$ being a finite index in $\mathbb{Z}^n$. Then $|\mathbb{Z}^n/\mathrm{Im}(A)| = |\det A|$

There is also $\mathbb{Q}^n \xrightarrow{A} \mathbb{Q}^n$, isomorphism when $\det A$ nonzero. Then $|\det A| = 1$.

Definition 1. The rank of an R-module M is the size of a maximally linearly independent set in M.

Injective and projective modules

The basic test: short exact sequences. $0 \to M' \xrightarrow{f} M \xrightarrow{g-} M'' \to 0$. So $M/M' \cong M''$ by the first isomorphism theorem.

Then we looked at $\mathrm{Hom}(L, \_)$ and $\mathrm{Hom}(\_, L)$.

Relating to surjectivity: $o \to \mathrm{Hom}(L, M') \to \mathrm{Hom}(L, M) \to \mathrm{Hom}(L, M'')$ does it go to zero? $(\ldots \to 0?\ )$

Relating to injectivity: $0 \leftarrow \mathrm{Hom}(M', L) \leftarrow \mathrm{Hom}(M, L) \leftarrow \mathrm{Hom}(M'', L)$ $(0 \leftarrow \ldots?)$

Projective if $\mathrm{Hom}(L, *)$ is exact for all short exact sequences $*$. Injective if $\mathrm{Hom}(*, L)$ is exact for all short exact sequences $*$.

Lemma 1. Link between projective and free. L is projective iff L is a direct factor of a free module. i.e., there exists M free and N such that $M = N \oplus L$. Corollary, free implies projective. Other corollary, every module is a quotient of projective module. This last corollary is what people mean when they say that the category of R-modules has enough projectives.

There is not an equally nice relationship for injectivity, even though injectivity is dual to projectivity.

Categories

A category $\mathcal{C}$ is a pair $(\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C})$. $\mathrm{Ob}\,\mathcal{C}$ class of objects, $\mathrm{Mor}\,\mathcal{C}$ class of sets of morphisms: $\mathrm{Hom}_{\mathcal{C}}(A, B)$ where $A, B \in \mathrm{Ob}\,\mathcal{C}$.

$\mathrm{Hom}(*, *)$ are sets, and composition is $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$.

We have:

(i) Composition is associative.

(ii) If $A \neq C$ or $B \neq D$ then $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(C, D) \neq \varnothing$.

(iii) For all A, there exists an identity morphism $1_A \in \mathrm{Hom}(A, A)$ such that $f \cdot 1_A = f$ and $1_A \cdot g = g$, that is $\underbrace{\mathrm{Hom}(A, A)}_{1_A} \times \underbrace{\mathrm{Hom}(A, B)}_{g} \to \mathrm{Hom}(A, B)$.

Also, $\mathrm{Hom}(C, A) \times \mathrm{Hom}(A, A) \to \mathrm{Hom}(C, A)$.

For groups, objects are groups and morphisms are group homomorphisms. For modules, objects are all R-modules and morphisms are all R-linear maps $\mathrm{Hom}(M, N)$.

Functors

A (covariant) functor $\mathcal{F}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$ is an object that associates to each $A \in \mathrm{Ob}\,\mathcal{C}$ a $\mathcal{F}(A) \in \mathrm{Ob}\,\mathcal{D}$. To morphisms f it associates $\mathcal{F}(f) \in \mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$, a morphism in the category $\mathcal{D}$.

We require

(i) $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$ for $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$

(ii) $(1_A) = 1_{\mathcal{F}(A)}$ for all $A \in \mathrm{Ob}\,\mathcal{C}$

$\mathcal{F}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ if $f \mapsto \mathcal{F}(f)$ where $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathcal{F}(f) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(B), \mathcal{F}(A))$ and $\mathcal{F}(fg) = \mathcal{F}(g)\mathcal{F}(f)$.

Afine algebraic sets form a category with morphisms of algebraic sets.

Finitely generated k-algebras form a category with morphisms of k-algebras.

If V, W be affine algebraic sets. Then the functor $V \mapsto k[V]$ and $(\varphi : V \to W) \mapsto k[W] \xrightarrow{\varphi^*} k[V]$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr] & W \arrow[d, "f"] \\ & k \end{tikzcd}$$

This is a contravariant functor.

The category of R-modules are abelian categories. What can we do in abelian categories Ab?

We can take kernels. If $f : A \to B$ where $A, B \in \text{Ab}$. (We write this instead of writing Ob). Then $\text{Ker } f \in \text{Ab}$. And $\text{Im } f \in \text{Ab}$. And $\text{Coker } f = B/\text{Im } f \in \text{Ab}$. But this doesn't work in Grp, the category of groups, since we require the image to be normal in that case.

We can do direct sums. If A, B are abelian groups, then their direct sum is also an abelian group. Same goes for abelian categories.

$\text{Hom}(A, B)$ is itself an abelian group for $A, B \in \text{Ab}$. Again, doesn't work in the category Grp.

So, if we can take Ker, Im, Coker, Coim, can take direct sums and Hom is an abelian group, then we have an abelian category.

Observe: Categories need not deal with sets. This complicates the actual definition, but we will not bother with that.

Example 3. R-modules are abelian categories. Grp is not abelian category.

Morita Equivalence

Definition 2. Two rings are Morita equivalent if the categories R-mod and S-mod are equivalent as categories.

Definition 3. $\mathcal{C}, \mathcal{D}$ are isomorphic if there exists $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{H} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{F} \circ \mathcal{H} = 1_{\mathcal{D}}$ and $\mathcal{M} \circ \mathcal{F} = 1_{\mathcal{C}}$.

(Infinity categories: lol. Morphisms between morphisms between morphisms ... between functors)

Example 4. Let R be any commutative ring with 1 and let $S = M_n(R)$. Highly noncommutative. We still have the categories of left/right R-modules. R, S are Morita equivalent.

Definition 4. A natural transformation $\eta : \mathcal{F} \to \mathcal{H}$ between two covariant functors $\mathcal{F}, \mathcal{H} : \mathcal{C} \to \mathcal{D}$ a rule that associates to every $A \in \mathcal{C}$ a $\eta_A \in \text{Hom}(\mathcal{F}A, \mathcal{H}A)$, for all $(A, B) \in \text{Ob } \mathcal{C}$ and for all $f \in \text{Hom}_{\mathcal{C}}(A, B)$ such that

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\eta_A} & \mathcal{H}(A) \\ \downarrow{\scriptstyle \mathcal{F}(f)} & & \downarrow{\scriptstyle \mathcal{H}(f)} \\ \mathcal{F}(B) & \xrightarrow{\eta_B} & \mathcal{H}(B) \end{array}$$

commutes.

Definition 5. Categories $\mathcal{C}, \mathcal{D}$ are equivalent if there exists $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are each naturally isomoprhic to the identity functor. i.e. there exists a natural transformation $\eta : \mathcal{F} \circ \mathcal{G} \to 1_{\mathcal{D}}$ with an inverse.