Commutative Algebra and Algebraic Geometry — Lecture 3

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Modules cont'd

R is commutative with 1 unless specified otherwise.

Recall. Let M on R-module $R \times M \to M$.

There is no concept analogous to normal subgroups. You are allowed to take quotient at any time, due to commutatitvity.

Definition 1. A function $f : M \to N$ of R-modules is a homomorphism of R-modules (or R-linear map) if we have

1. $f(\underbrace{rm}_{\text{M action}}) = \underbrace{rf(m)}_{\text{N action}}$

2. It's a homomorphism of the underlying Abelian group. That is, $f(m+n) = f(m) + f(n)$.

Let $X \subset M$. Then X *generates* M if $\forall m \in M \exists$ a finite number $r_1, \ldots, r_n \in R$ and $x_1, \ldots, x_n \in X$ st element $m = \sum r_i x_i$. Has to be finite because infinite sums not generally defined in rings, i.e. no general notion of convergence.

If $X = \{x_1, \ldots, x_n\}$ write $M = (x_1, \ldots, x_n) = Rx_1 + \ldots + Rx_n$.

M is *finitely generated* if M is generated by X for some finite $X \subset M$.

M is *cyclic* if $M = (x)$ for some $x \in M$. This is the analog of principal ideals.

$X \subset M$ is a basis (or free set of generators) if for all $m \in M$ there exists a unique decomposition $m = \sum r_i x_i$ where $r_i \in R \quad x_i \in X, \quad r_i, x_i \neq 0$.

M is *free* if there exists a basis for M.

Example 1. $R = k =$ field $\implies M$ is vector space $\implies M$ free.

Example 2. $R = \mathbb{Z}$ and M finitely generated, then Mfree $\iff M \cong \mathbb{Z}^k$

Example 3. $R = \mathbb{Z}$. $\mathbb{Z}/n\mathbb{Z}$ not free since $n \cdot 1 = 0$ but cyclic generated by $1 \in \mathbb{Z}/n\mathbb{Z}$.

Definition 2. Torsion. $M \in R$-module. $\mathrm{Tor}(M) = \{m \in M \mid \exists r \in R \text{ st } rm = 0, r \neq 0\}$

Exercise 1. R domain $\implies$ $\mathrm{Tor}(M)$ is a submodule.

M is *torsion-free* if $\mathrm{Tor}(M) = (0)$.

M is torsion if $M = \mathrm{Tor}(M)$.

Example 4. $k = \mathbb{R} =$ field. There is no torsion in a vector space.

If $R = \mathbb{Z}$, M Abelian group $\implies$ $\mathrm{Tor}(M) = \{m \in M \mid \text{order of } M \text{ finite }\}$.

km additively $\leftrightarrow$ m^k multiplicatively

$km = 0 \leftrightarrow m^k = 1$

$\mathrm{Tor}(\mathbb{Z}^k) = (0)$. Every finite Abelian group is a torsion $\mathbb{Z}$-module.

Torsion and freeness are opposite extremes.

Let X any set. $R^X = \{\text{collection of } \textbf{finite} \text{ sums } \sum r_i x_i\} = \bigoplus_{x \in X} Rx$ (or $F(X)$ in Dummit-Foote). This is the free module generated by X.

Universal property: inclusion $\iota : X \to R^X, x \mapsto x$. For all $M \in R$-module, for all set maps $f : X \to M$ there exists a unique extension $F : R^X \to M$ st $F(x) > f(x)$.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & R^X \arrow[d, "\exists!"] \\ & M \end{tikzcd}$$

X is just a set. The unique map is a module homomorphism.

Remark. For groups, Free(x)= free group on X. F_n: see chap 6 of DF. Non-abelain for $n \geq 2$.

Observe. Every module is quotient of a free module. E.g. Take $X = M$ as a set. $f = \mathrm{Id}$.

$F : R^X \to M$ is surjective $\iff$ $f(X)$ generates M.

Assume $R^X \xrightarrow{F} M$. Then $\mathrm{Ker}F$ = module of relations. M is finitely presented if the re exists a generated set X st 1) X is finite 2) $\mathrm{Ker}F$ is finitely generated.

$(*)$ If R Noetherian and M is finitely generated $\implies$ all submodules of M are finitely generated. So R Noetherian implies every finitely generated module is finitely presented.

.e.g free groups $\rightsquigarrow$ presentation of groups. $D_8 = \langle r, s \mid r^4 = s^2 = e, srs^{-1} = r^3 = r^{-1} \rangle$

$F_2 \subseteq$ Free $(r, s) \twoheadrightarrow D_8$ (arrow means surjective?)

Ker = (normal subgroup generated by r^4, s^2, srs^{-1})

ACC for $M \in R$-module means $\forall M_1 \subset M_2 \subset \ldots \subset M_k \subset M$ (ascending chain of R modules $\subset M$) there exists t st $M_t = M_{t+1} = \ldots$

Definition 3. M is Noetherian if M satisfies ACC. R is Noetherian as a ring iff R is Noetherian R-module, all submodules of R are ideals of R. Lemma about Noetherian rings works for R-modules. That is,

Lemma 1. TFAE:

1. M satisfies ACC
2. Every nonempty collection of submodules has a maximal element
3. Every submodule is finitely generated

Proof. Same as earlier.

Lemma 1 implies $(*)$.

Theorem 1. Classification of finitely generated modules over a PID. Let R be a PID and M an R-module, finitely generated. Then $\exists k \in \mathbb{Z}$ st

$$M \cong \mathbb{R}^k \oplus \mathrm{Tor}(M)$$

A PID is by definition an integral domain, so this is a submodule.

$$\mathrm{Tor}(M) \cong \underbrace{R/(a_1) \oplus \ldots \oplus R/(a_k)}_{\text{each cyclic torsion generated by 1}}$$

with nonunits $a_i \in R$, $a_1 \mid a_2 \mid \ldots \mid a_k$.

Remark: $R^k \cong R^X$ with $|X| = k$. R^k is like $\underbrace{R \oplus \ldots \oplus R}_{\text{k times}}$ each R is cyclic.

In particular: every finitely generated R-module is a direct sum of cyclic modules. Importantly, M is torsion free $\iff$ M is free.

Example 5. If $R = \mathbb{Z}$: fundamental theorem of finitely generated Abelian groups.

M an R-module $\Leftrightarrow (V, T)$ V is vector space over k and $T : V \to V$.

$\dim_k V < \infty \iff$ free part $k[x]^n = 0$.

Jordan normal form of $T \leftrightarrow M \cong k[x]/a_1(x) \oplus \ldots \oplus k[x]/a_n(x)$

The proof of Thm 1 is more boring than hard.

Exact sequences of modules

Given a homomorphism of modules, $\to M_{i-1} \xrightarrow{f} M_i \xrightarrow{g} M_{i+1}$ is exact at M_i if $\mathrm{Ker}\, g = \mathrm{Im}\, f$.

e.g. $0 \to M' \xrightarrow{f} M$ exact $\iff$ f injective

$M \xrightarrow{g} M'' \to 0$ exact $\iff$ g surjective

$\underbrace{0 \xrightarrow{f} M' \xrightarrow{g} M \to M'' \to 0}_{\text{(short exact sequence)}}$ is exact $\iff$ $M/\mathrm{Im} f \cong M''$ $\quad (\mathrm{Im} f \cong M')$

Given $M \xrightarrow{f} M''$ a *section* of f is a map $s : M'' \to M$ st $f \circ s = Id_{M''}$.
Example 6. A vector field is a section of a tangent bundle.

$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is *split* if there exists section $s : M'' \to M$ of g. Then s is called *splitting* and we get $M \cong M' \oplus M''$.

In general, say M is an extension of M'' by M'.

Analogy with arbitrary groups: G group, N normal in G. $H = G/N$. Having this data is the same as having $1 \to N \to G \to \underbrace{G/N}_{=H} \to 1$ "multiplicatively written short exact sequence"

Again, split if $\exists$ group homomorphism $s : H \to G$.

Observe: Split for groups does not in general imply $G \cong N \times G/N$. Only for Abelian groups. If split, then we have $G = N \cdot s(H), \quad N \neg s(H) = \{1\} \iff G = N \rtimes H$ (semidirect product)

$f : M \to N$ have of R-modules $\mathrm{coker}(f) = N/\mathrm{Im} f$.

Observe: For groups cokernel may not exist.

This part was transcribed in haste. Read the book instead.

$\mathrm{Hom}(M, N) = \{R\text{-linear maps } f : M \to N\}$, an Abelian group.

$$\begin{tikzcd}
& L \arrow[d] \arrow[dr] & & & \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[dr] & M'' \arrow[r] \arrow[d] & 0 \\
& & & L' &
\end{tikzcd}$$

$$M' \xrightarrow{f} M \qquad \rightsquigarrow \qquad \mathrm{Hom}(L,M) \xrightarrow{f_*} \mathrm{Hom}(L,M')$$

$(\alpha : L \to M'$, and $f_*(\alpha) = f \circ \alpha)$

$$M \xrightarrow{g} M'' \longrightarrow \mathrm{Hom}(M'',L') \xrightarrow{g^*} \mathrm{Hom}(M,L)$$

$(\beta : M'' \to L', g^*(\beta) = \beta \circ g)$

$(*)$ $\quad 0 \to M' \to M \to M'' \to 0$ exact $\implies 0 \to \mathrm{Hom}(L,M') \to \mathrm{Hom}(L,M) \xrightarrow{(\dagger)} \mathrm{Hom}(L,M'')$ exact but maybe not " $\to 0$". $(\dagger)$ maybe not surjective

Definition 4. L is projective $\iff \forall 0 \to M' \to M \to M'' \to 0, \mathrm{Hom}(L,M) \to \mathrm{Hom}(L,M'')$ surjective.

Dually, apply $\mathrm{Hom}(\_,L)$ to $(*)$ get

$$\mathrm{Hom}(M',L) \leftarrow \mathrm{Hom}(M,L) \leftarrow \mathrm{Hom}(M'',L) \leftarrow 0$$

exact but maybe not "$0 \leftarrow$ "

Definition 5. L is injective $\iff \forall(*)$ $\mathrm{Hom}(\_,L)(*)$ again short exact sequence

Thoughts:

- Free $\implies$ projective
- Projective $\iff$ direct sum of free
- R PID $\implies$ M finitely generated (M projective $\iff$ M free)

$R = k[x_1,\dots,x_n]$ $n > 2$ not PID. Serre's question: every finitely generated projective module over a polynomial ring is free. Quillen–Suslin theorem proves Serre's problem.