Commutative Algebra and Algebraic Geometry — Lecture 1

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Books

- Main book: Dummit and Foote ch. 15.
- Görtz-Wedhorn ch. 1.
- Atiyah-Macdonald

Musings on the nature of geometry

Manifolds look locally like $\mathbb{R}^n$.

We can consider geometric objects as sets of solutions to polynomial equations in $\mathbb{A}_k^n$. Affine space is especially useful in algebraic geometry.

$\mathbb{A}_k^n$ =affine space of dimension n/k= $\{(x_1, \ldots, x_k) \in k^n\}$.

$$A_{\mathbb{R}}^n = \mathbb{R}^n$$

In $\mathbb{R}$ or $\mathbb{C}$ you get manifolds from this if the solutions are smooth.

Complex manifolds look locally like $\mathbb{C}^n$. Key distinction is holomorphism, which is a far stricter requirement than just (real) differentiability.

Theorem 1. Riemann. Assume X is a compact Riemann surface (locally looks like $\mathbb{C}$). Then a curve in it is algebraic, i.e. it is the complex solution of a system of polynomial equations in some $\mathbb{P}^n(\mathbb{C})$.

e.g. a compact Riemann surface (torus) $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \quad c \to y_2^2 = x^3 - xz^2$.

Projective space is lines through the origin in k^{n+1}:

$$\mathbb{P}_k^n = (k^{n+1} \setminus 0)/\sim$$

where $(a_0, \ldots, a_{n+1}) \sim (\lambda a_0, \ldots, \lambda a_{n+1})$.

$\mathbb{P}^n_k(\mathbb{C})$ is compact.

We study local behavior in this course. One of the nontrivial insights of Grothendieck is having a natural notion of functions on prime ideals.

Noetherian Rings

The most basic type of ring in commutative algebra. Non-notherian rings are wild. For most of this course, let R be a commutative ring with 1. We assume R is a commutative ring.

Recall. A ring is abelian group under addition, group under multiplication (without zero), not generally assumed to be commutative

Recall. R is an integral domain if it has no non-zero zero divisors. This allows cancellation, i.e. $ab = ac \implies b = c$.

Definition 1. R is Noetherian if R satisfies the Ascending Chain Condition on ideals (ACC). That is, if you have an ascending chain of ideals $I_1 \subset I_2 \subset \dots$ it becomes stationary — there exists an n_0 such that $I_n = I_{n_0}$ for all $n \geq n_0$.

Noetherian does not require or imply integral domain.

Definition 2. R is Artinian if R satisfies the Descending Chain Condition on ideals. That is, $J_1 \supset J_2 \supset \dots$ becomes stationary.

Example 1. A finite ring is both Noetherian and Artinian.

Recall. A principal ideal domain is a domain where every ideal can be generated by a single element. R is a PID if R is an integral domain and every ideal in R is principal.

Example 2. $\mathbb{Z}$ is a principal ideal domain.

$$(n) \subset (m) \iff m \mid n.$$

An ascending chain $\dots \mid n_3 \mid n_2 \mid n_1$ in Z gives divisors of n_1. n_1 has finitely many divisors $\implies \mathbb{Z}$ is Noetherian.

The multiples of two contain the multiples of four which contain the multiples of eight, and so on. This is an infinite descending chain and is thus not Artininan.

Example 3. Fields are Noetherian and Artinian since they only have one proper (that is not the whole thing) ideal.

For fields, $0 \neq 1$ and all elements have inverses.

Example 4. $R \times R \times \dots \times R = R^{\mathbb{N}}$. $I_1 = (R, 0, \dots, 0) \subset I_2 = (R, R, 0, \dots) \subset \dots$. Not Noetherian.

Example 5. $k[x_1, \ldots, x_n]$ in infinitely many variables. $(x_n) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, \ldots)$ not Noetherian. But $k[x_1, \ldots, x_n]$ is a unique factorization domain.

Fields $\subset$ Euclidean domains $\subset$ PIDs $\subset$ UFDs $\subset$ integral domains.

Recall. $u \in R$ is a unit if $\exists v \in R$ st $uv = 1$. $R^\times$ is the group of units.

Definition 3. $a \in R$ is irreducible if $a \notin R^\times$ and whenever $a = bc$, either $b \in R^\times$, $c \in R^\times$.

Definition 4. If $a, b \in R^\times$ and $a, b \neq 0$ we say a, b are associate if $\exists u \in R^\times$ st $a = ub$, e.g. 5 and -5 are associate in $\mathbb{Z}$.

Definition 5. R is a UFD if R is a domain and every nonzero nonunit $a \in R$ has a unique factorization.

1. Exists:can write there is a factorization $a = p_1^{e_1} \ldots p_r^{e_r}$ such that p_i irreducible for all i and p_i, p_j are not associate for distinct i, j
2. If there is a second factorization $q_1^{e_1} \ldots q_s^{e_s}$ then $r = s$, and after reordering the factors q_i is associate to p_i etc.

Example 6. $k[x_1, \ldots, x_n]$ is Noetherian. To be covered next time.

Theorem 2. Hilbert Basis Theorem: if R Noetherian then $R[x]$ is also Noetherian.

Lemma 1. Quotients of Noetherian rings are Noetherian: $k[x_1, \ldots, x_k]/I$ is Noetherian for all $\forall k$, $\forall n$ for all ideals I.

$k[x_1, \ldots, x_k]/I$ is important. Sets of solutions to these polynomials are key to algebraic geometry.

Lemma 2. R is Noetherian $\iff$ every ideal is finitely generated.