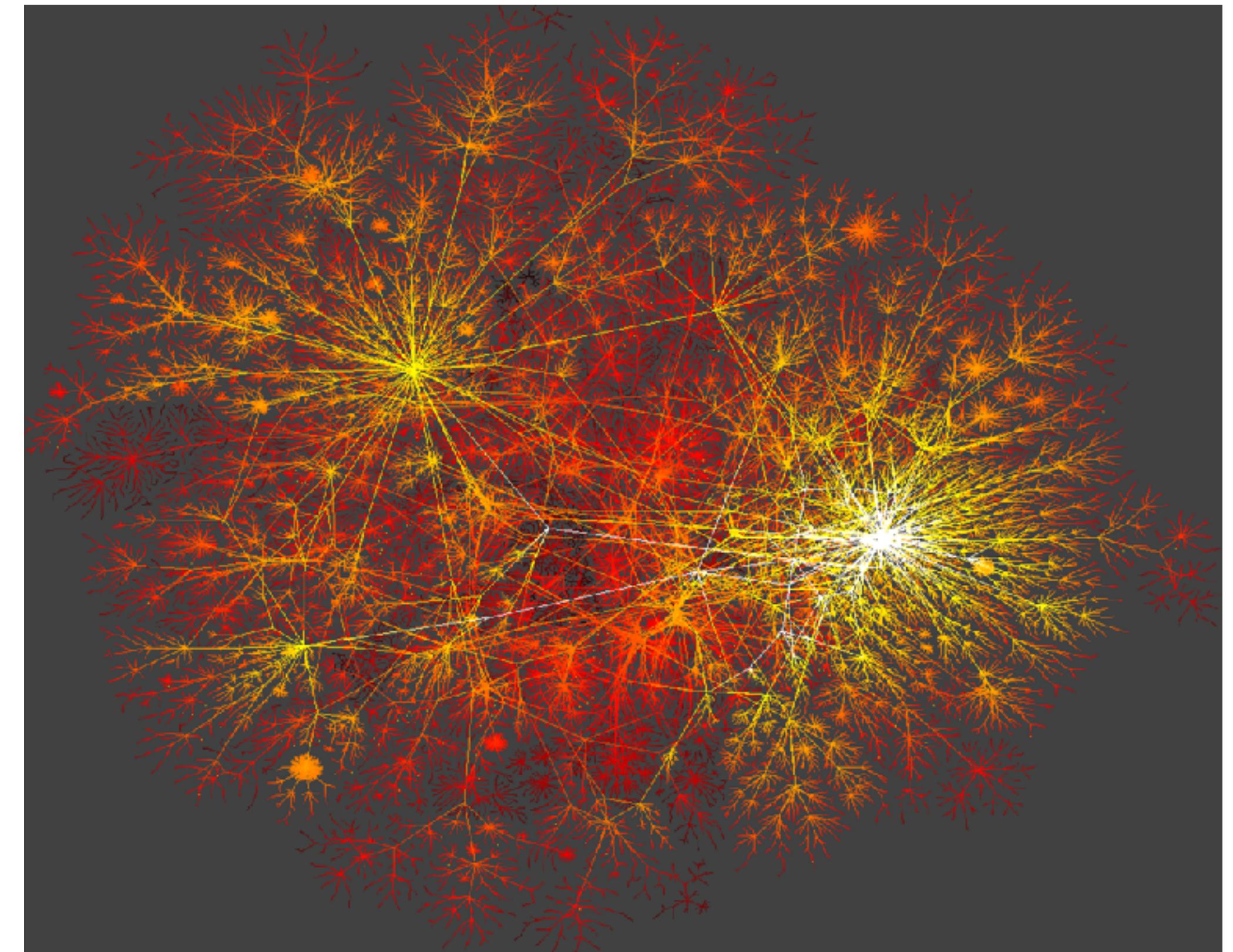


Lecture 19: Introduction to network science

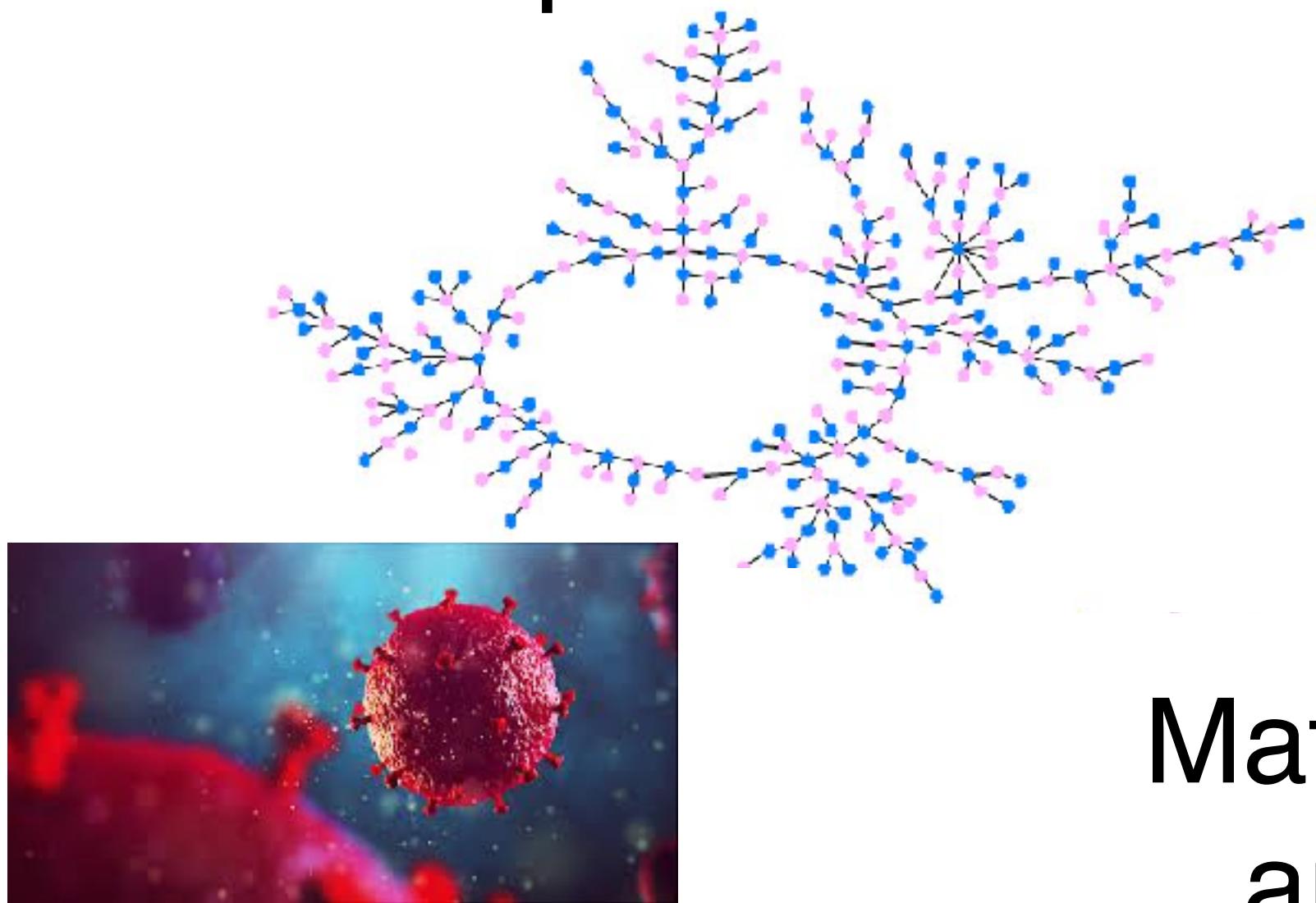
Instructor: Michael Szell

Nov 8, 2023



Today you will learn about networks

Why networks
are important



Mathematical concepts
and data structures

$$I = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 3 \\ 4 \end{pmatrix} \quad J = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$



After a stroke, how fast do you arrive in the hospital?

Mentimeter

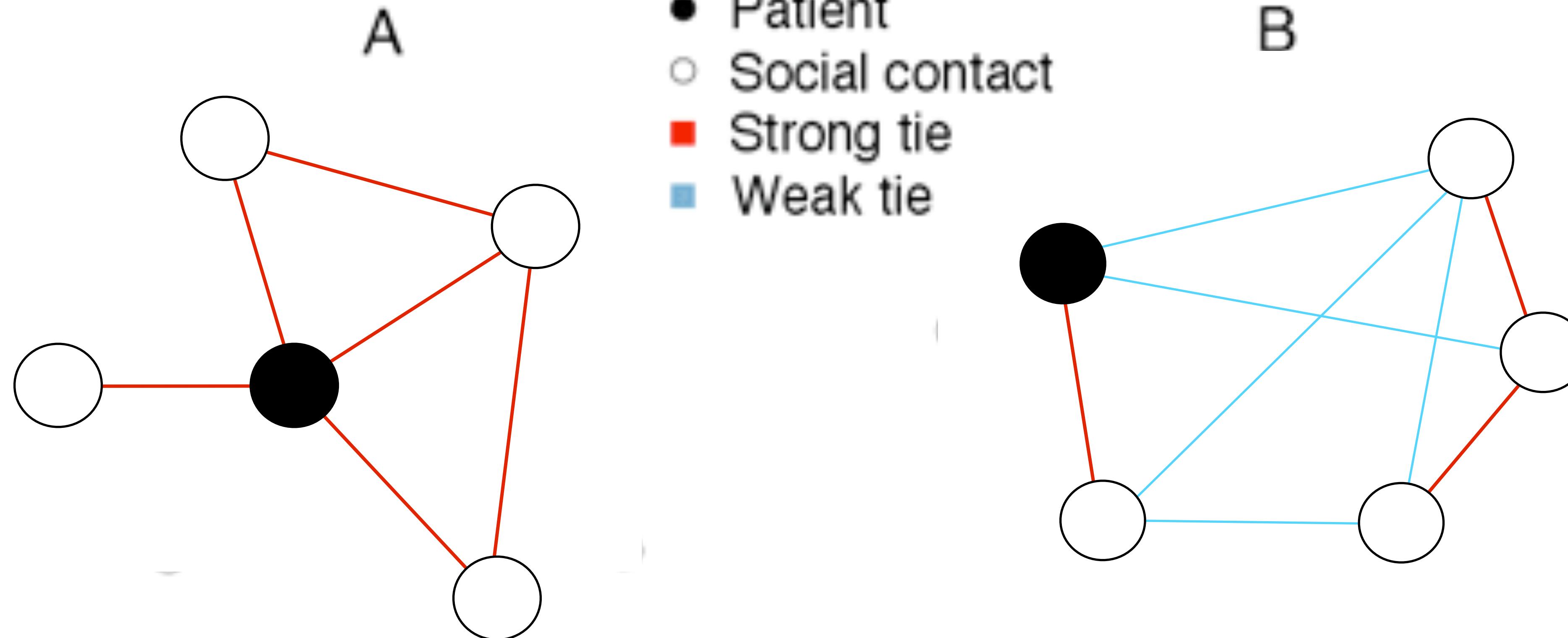


© 2013 American Stroke Association

Stræk. Snak. Smil.

What matters most?

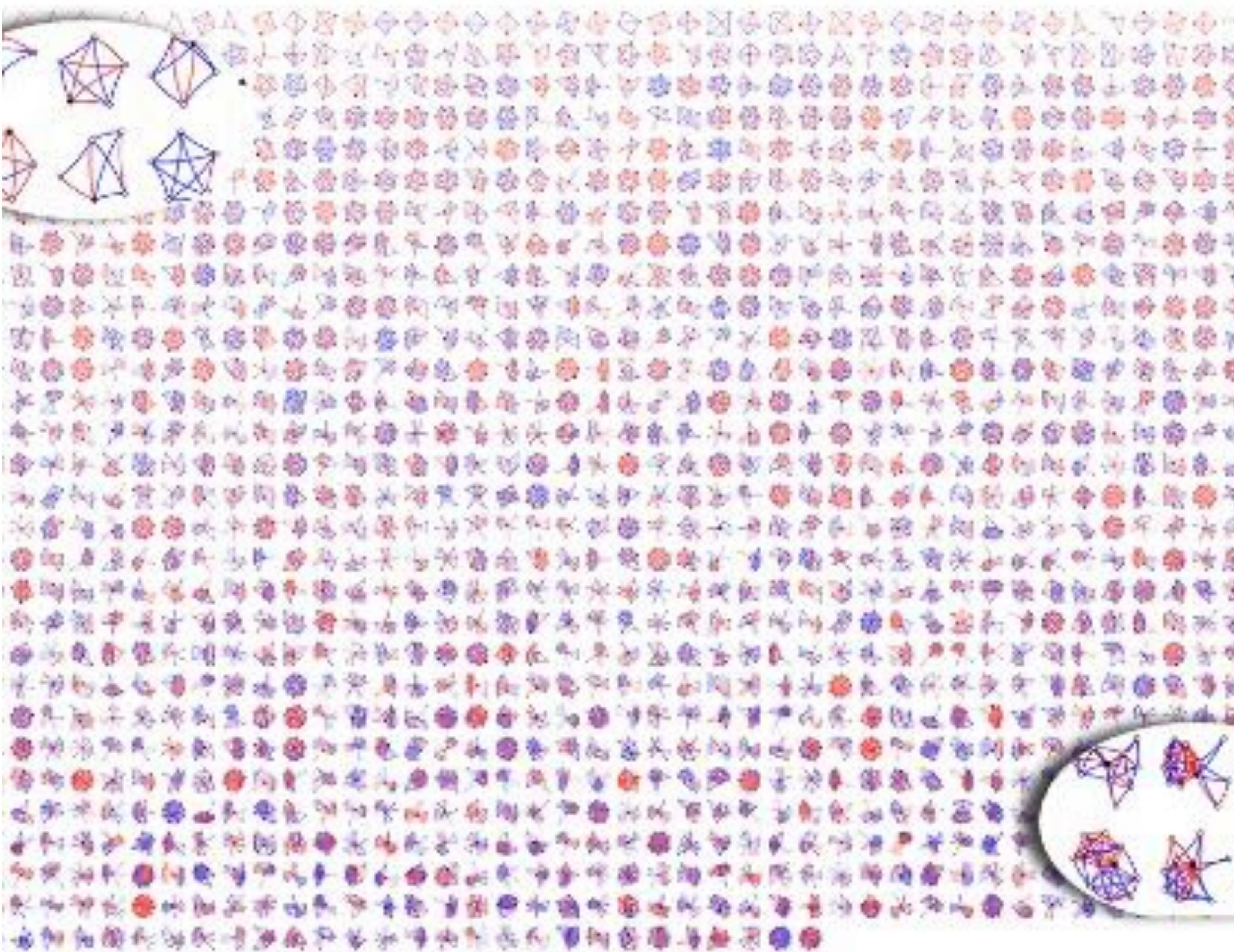
Your social network matters. But how?



Who arrives first in the hospital?

Video
(removed for privacy reasons)

Your social network matters

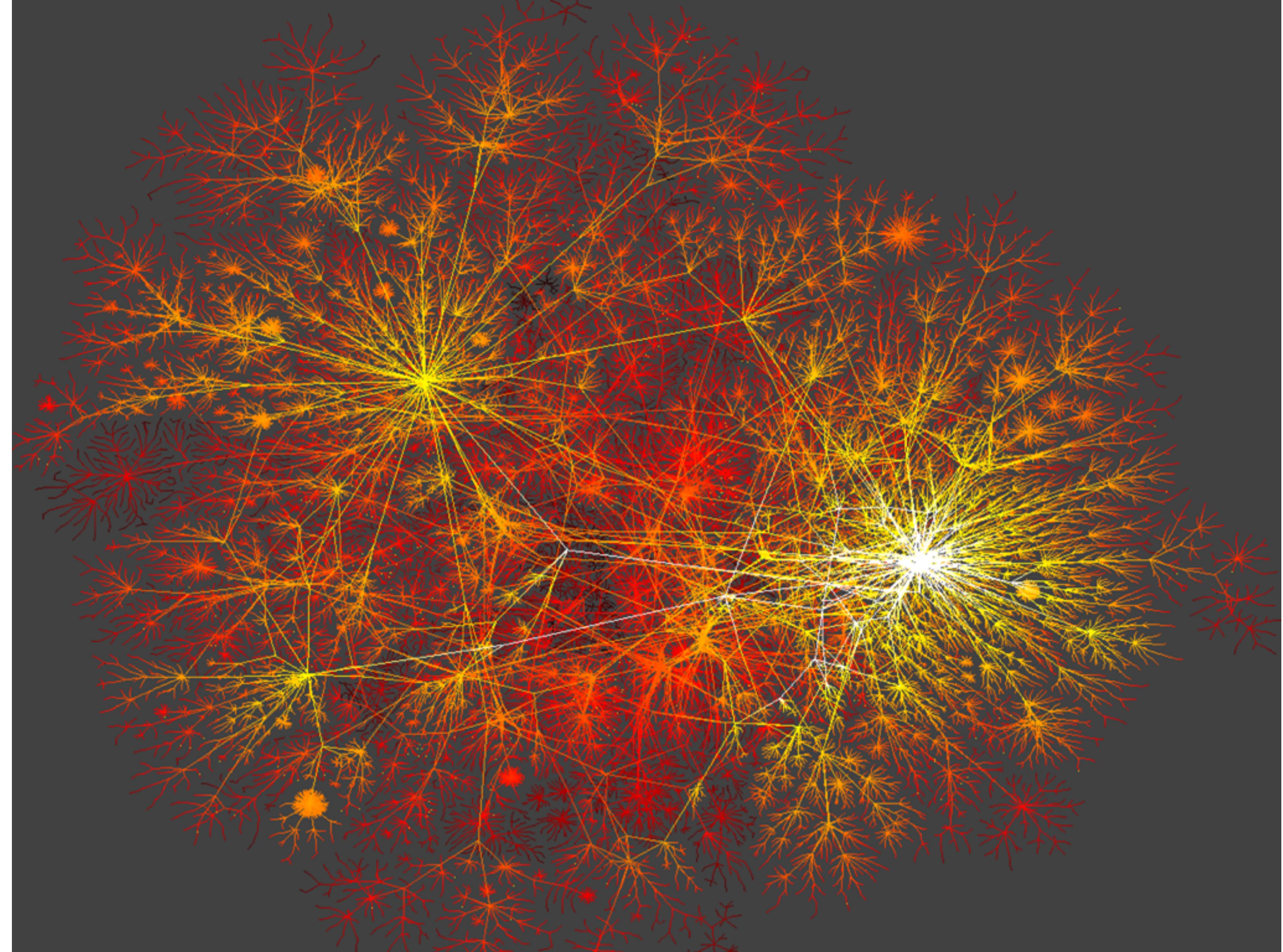


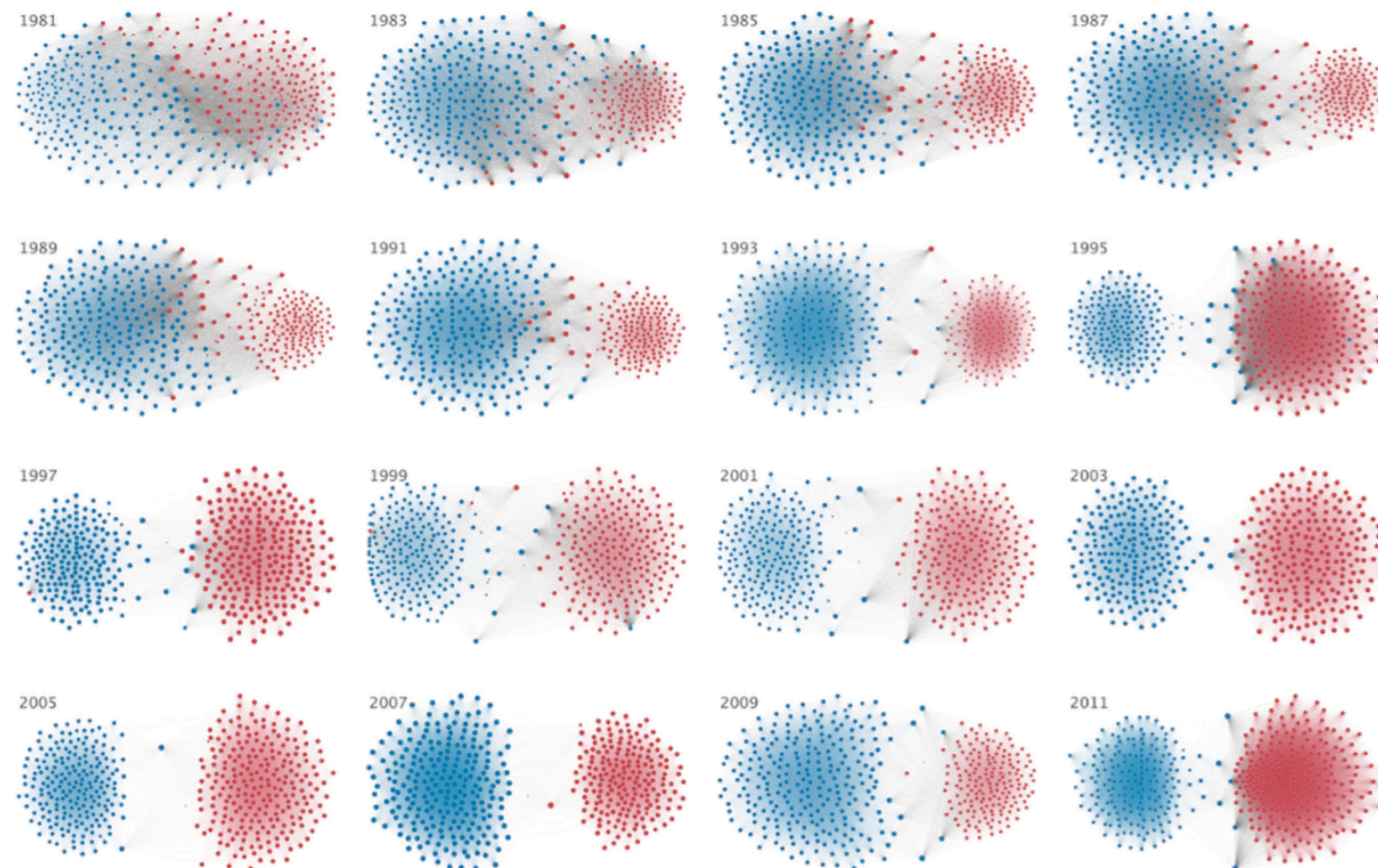
**Networks describe
interactions or relations**



facebook

December 2010

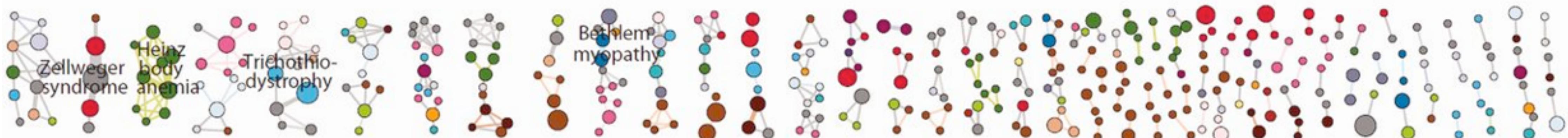
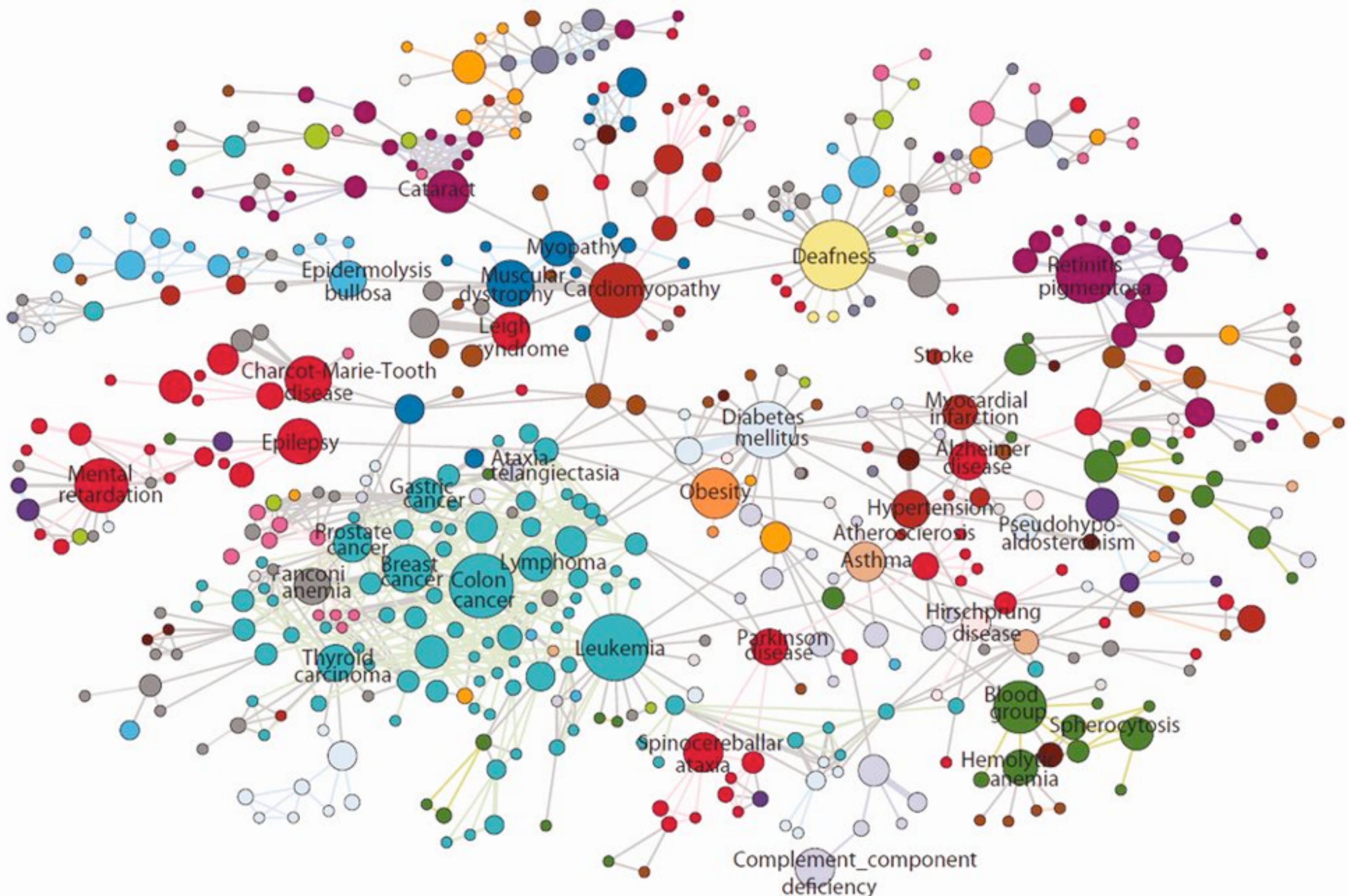
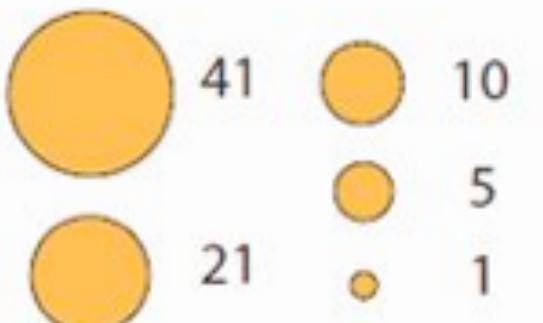


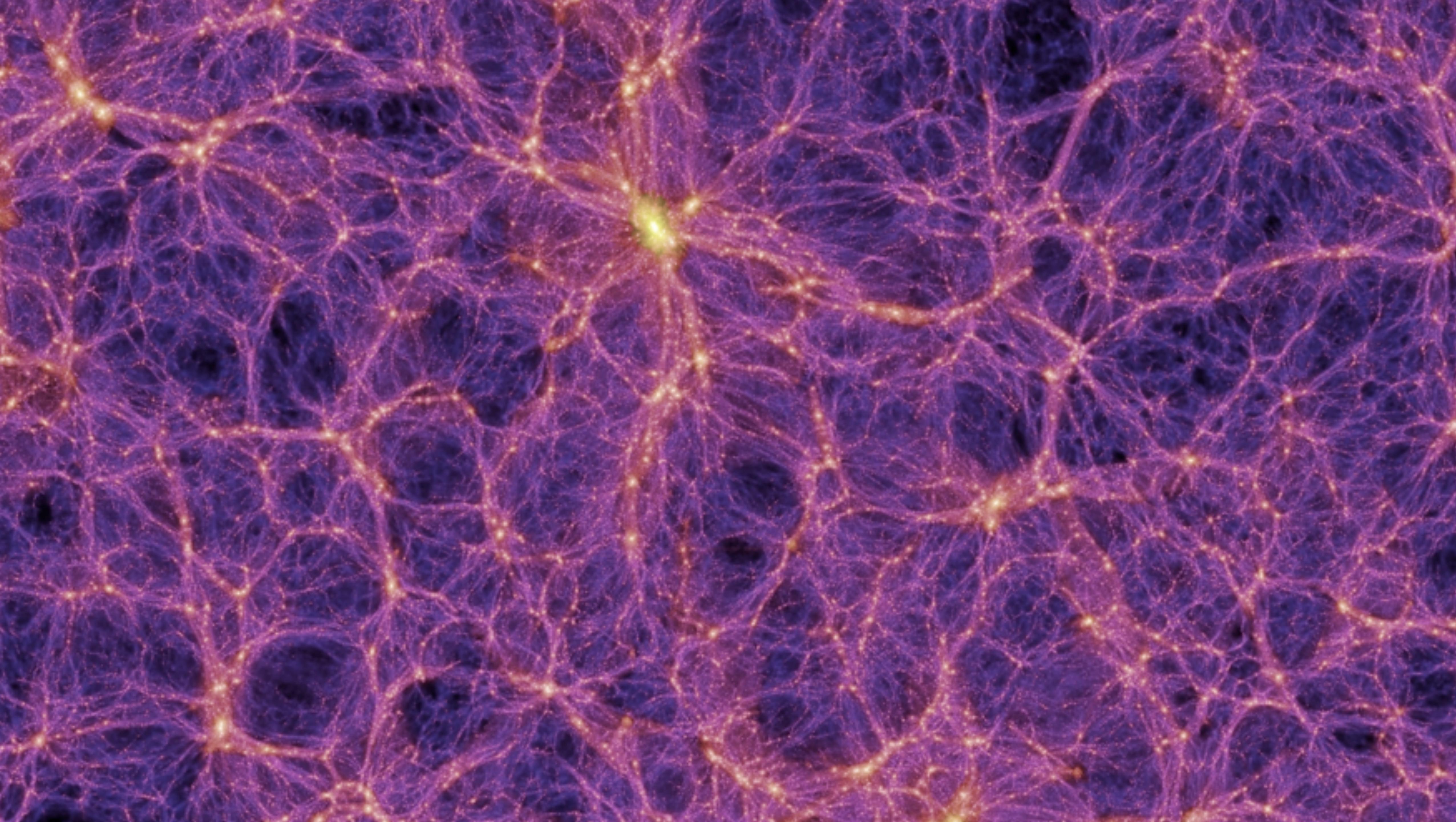


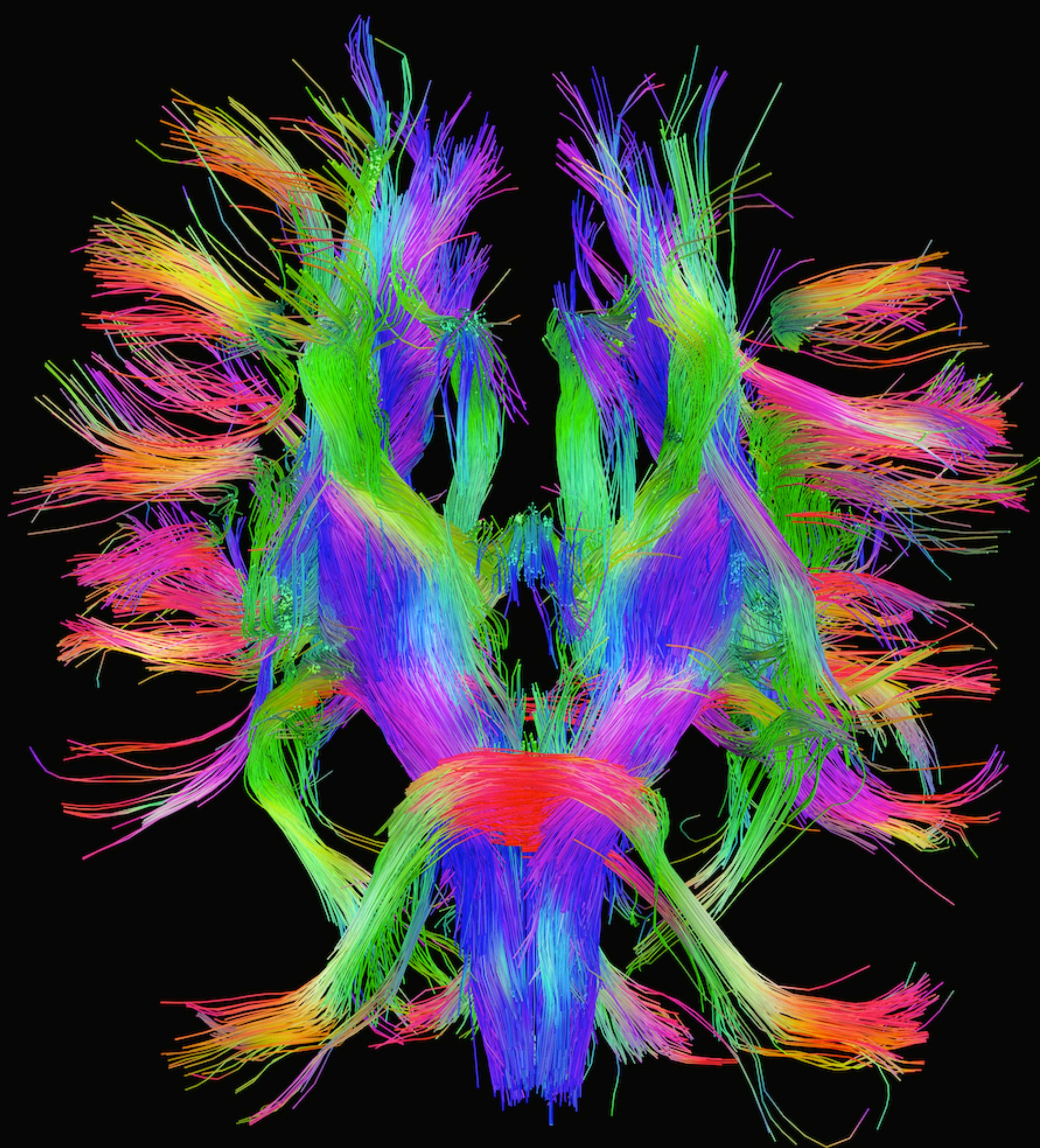
Disorder Class

- Bone
- Cancer
- Cardiovascular
- Connective tissue
- Dermatological
- Developmental
- Ear, Nose, Throat
- Endocrine
- Gastrointestinal
- Hematological
- Immunological
- Metabolic
- Muscular
- Neurological
- Nutritional
- Ophthalmological
- Psychiatric
- Renal
- Respiratory
- Skeletal
- multiple
- Unclassified

Node size

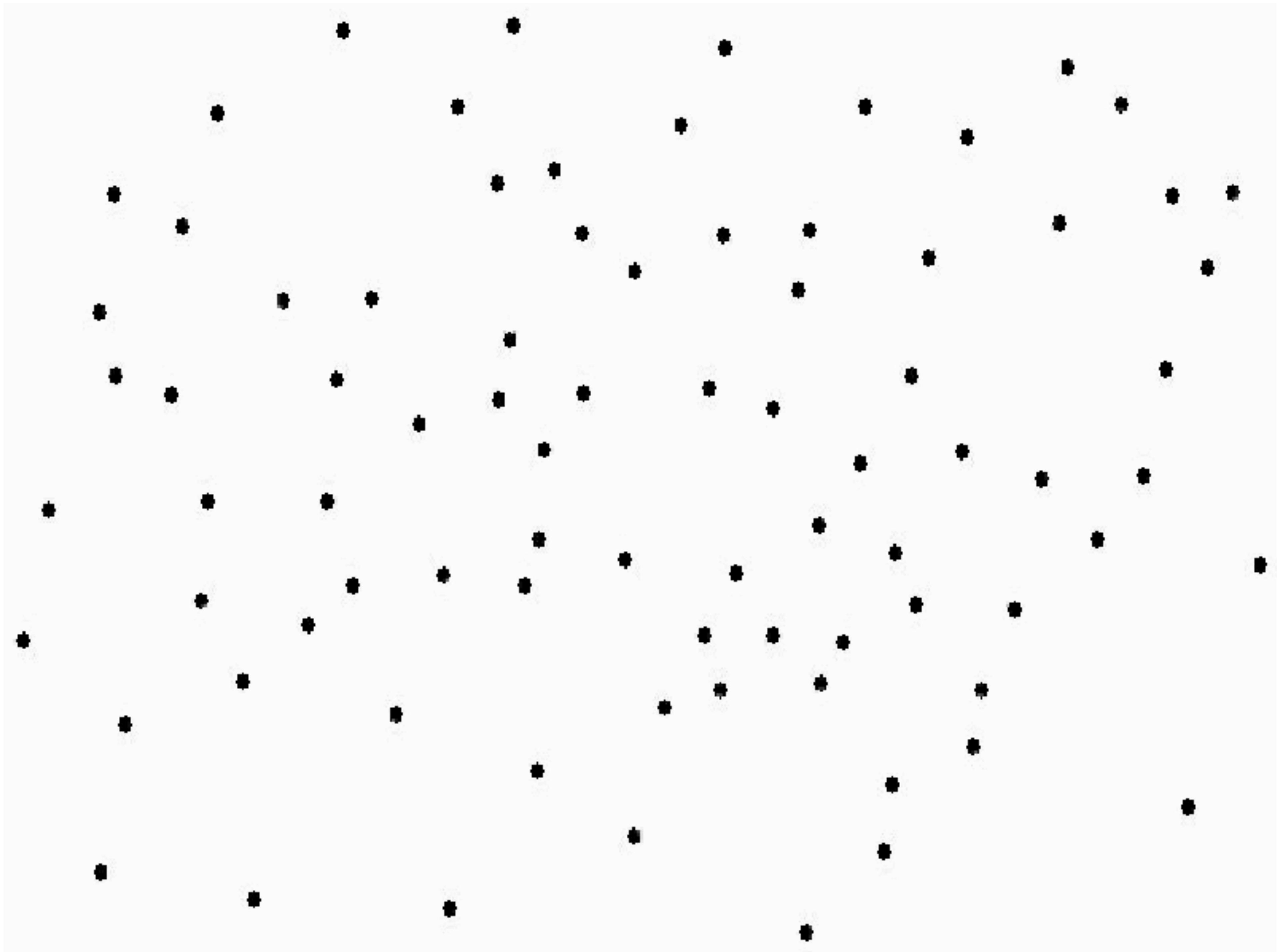


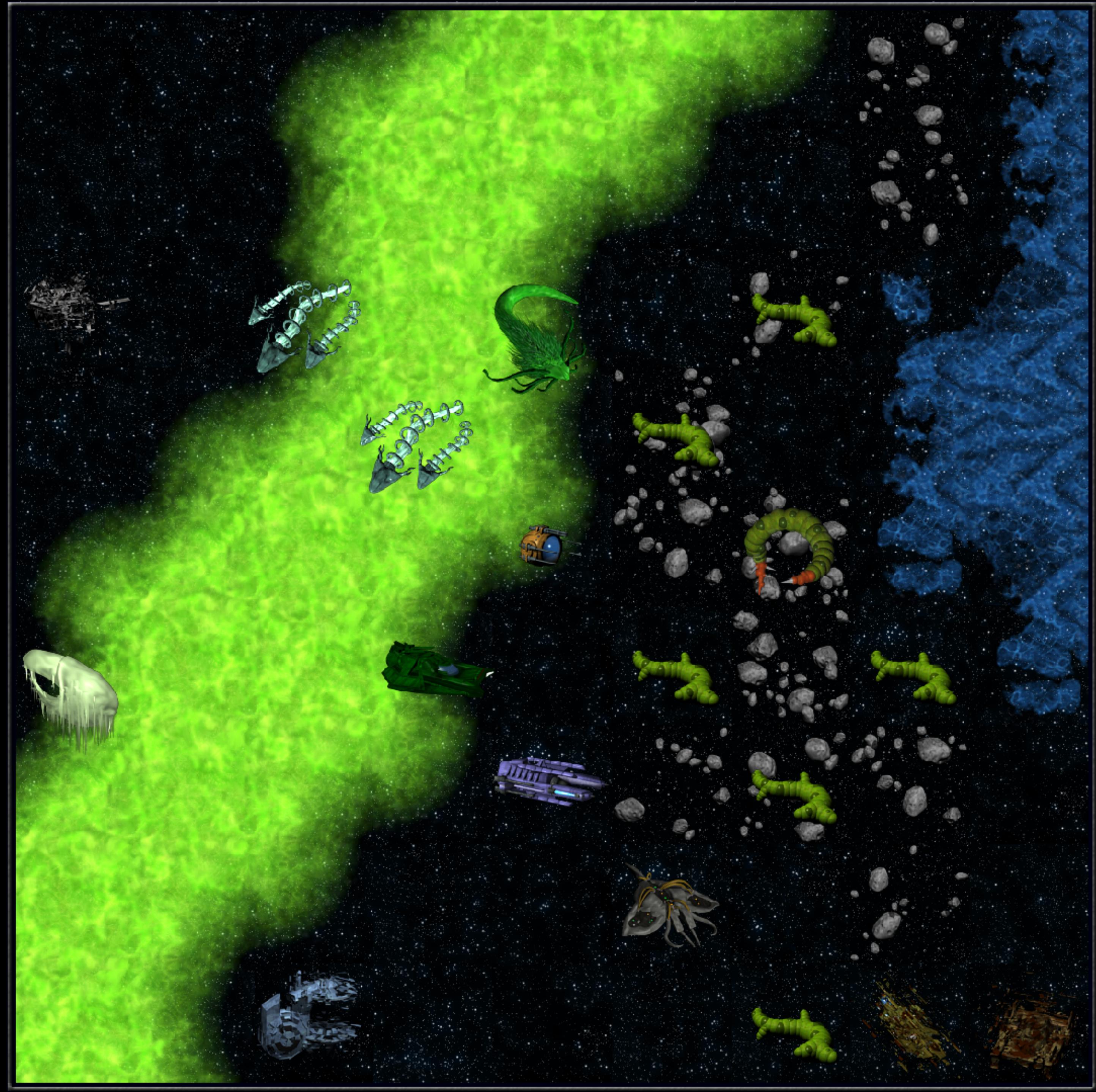


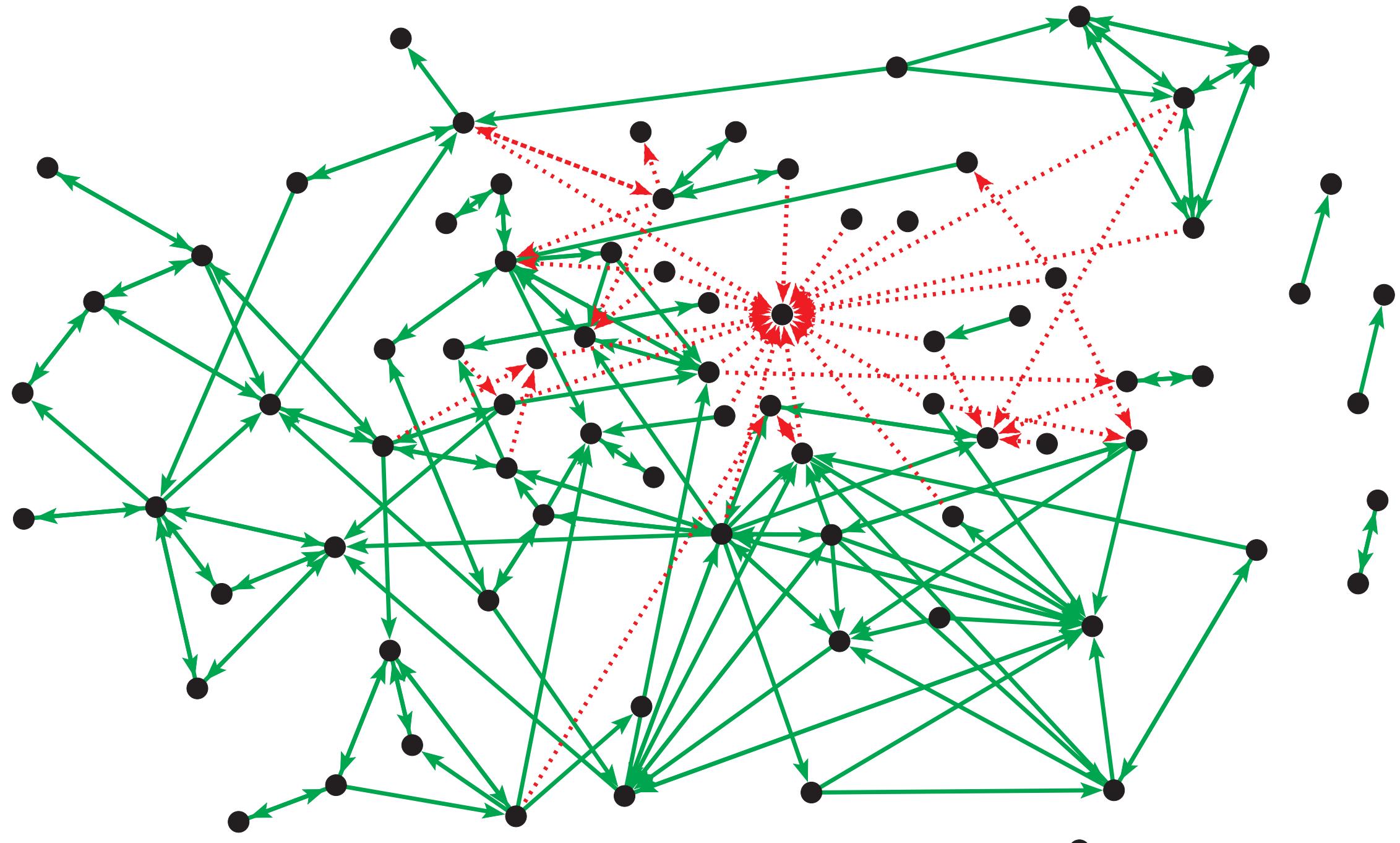




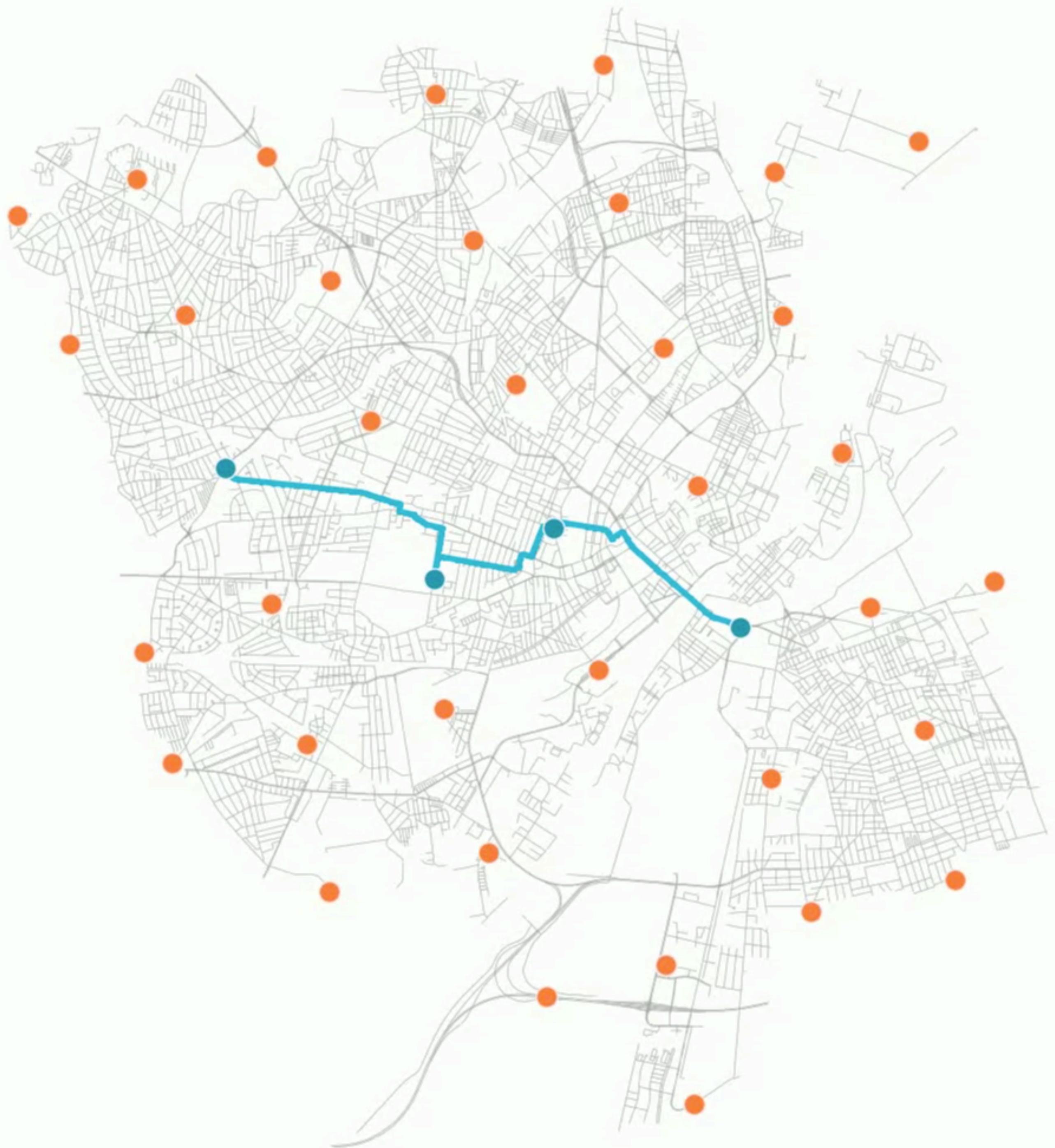
"Those who were trained to fly didn't know the others.
One group of people did not know the other group."











Networks are the backbones of complex systems

Many single units



Strong, nonlinear interactions

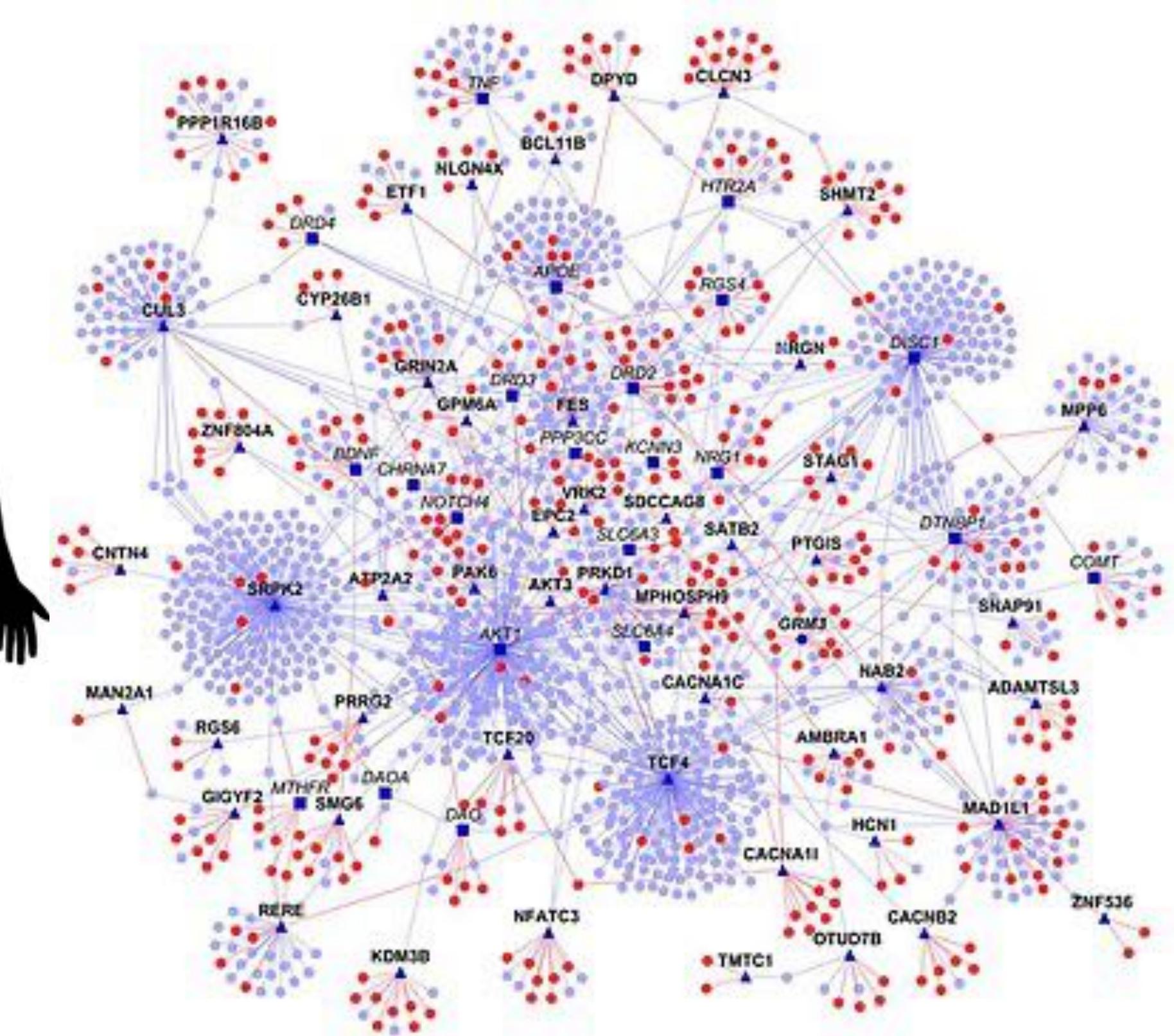
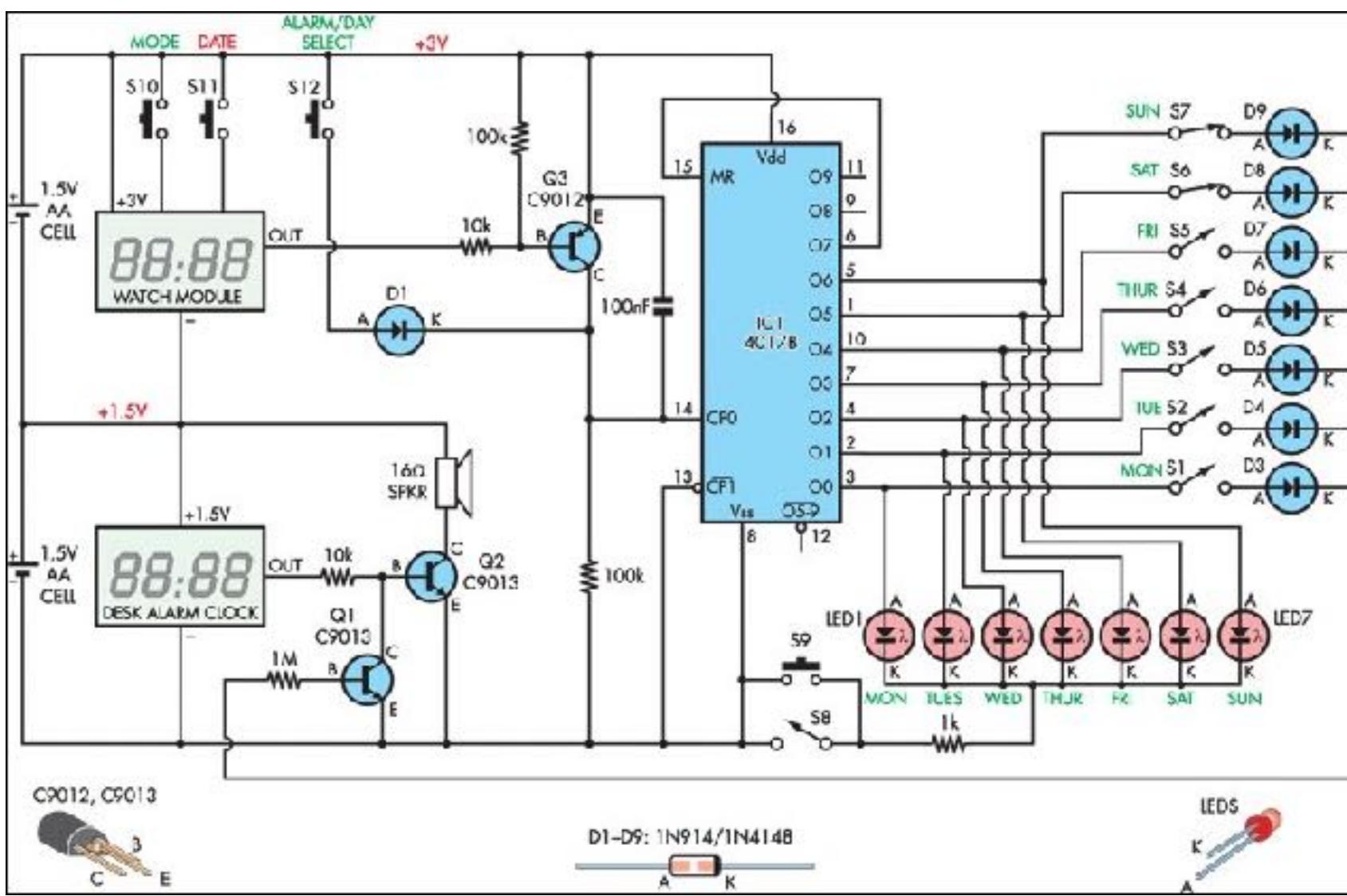


Emergence of collective behavior

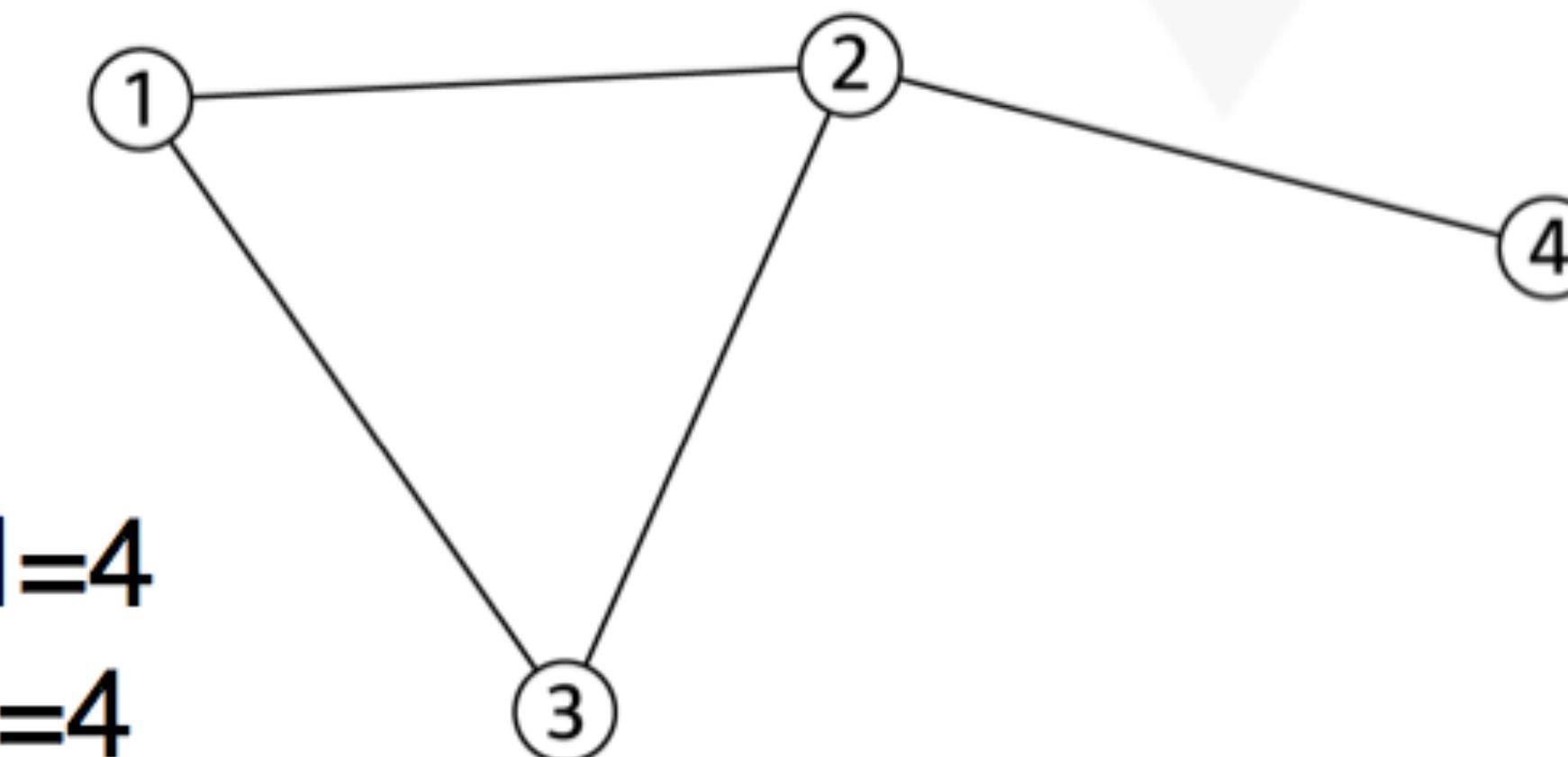
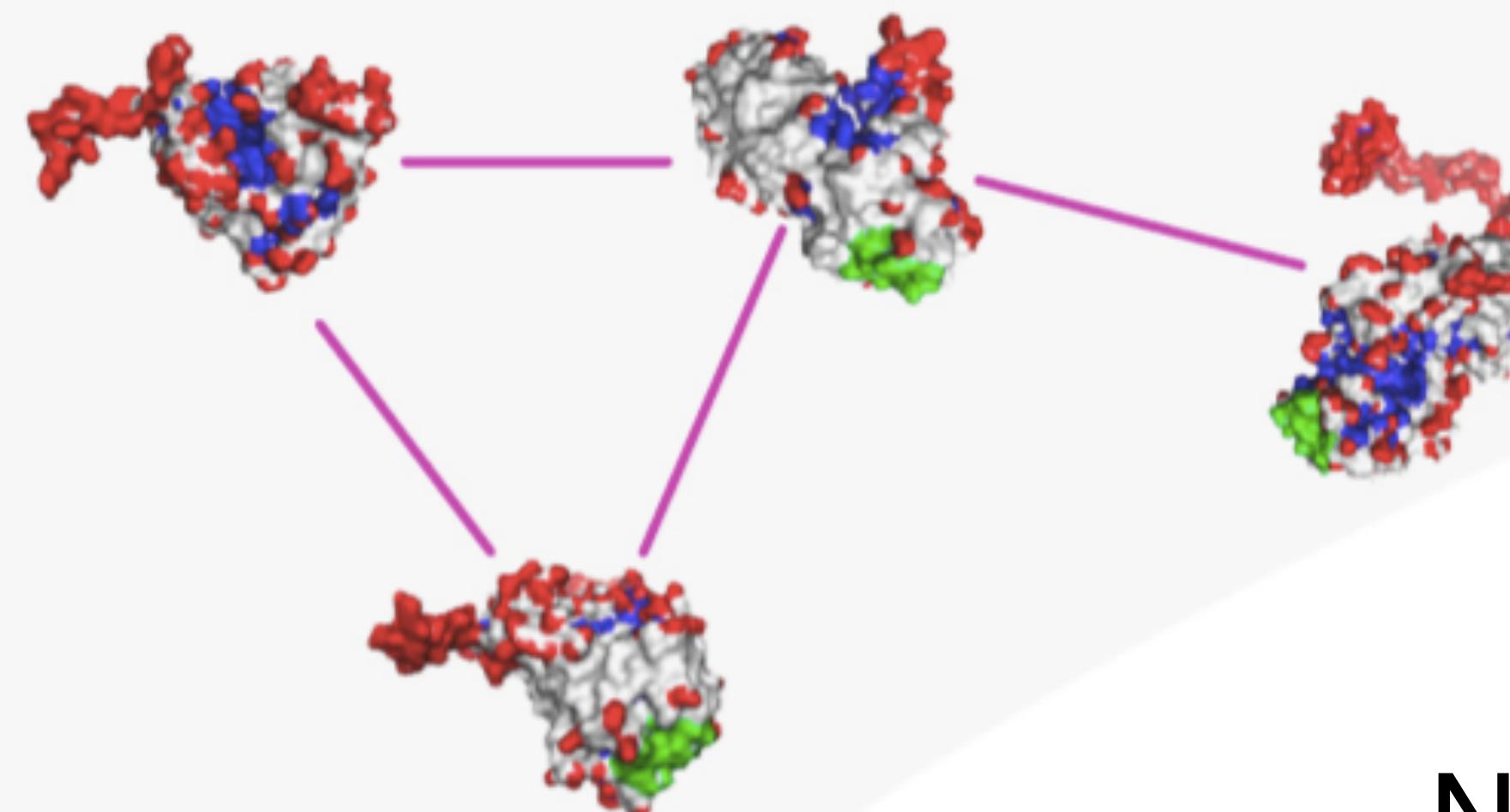
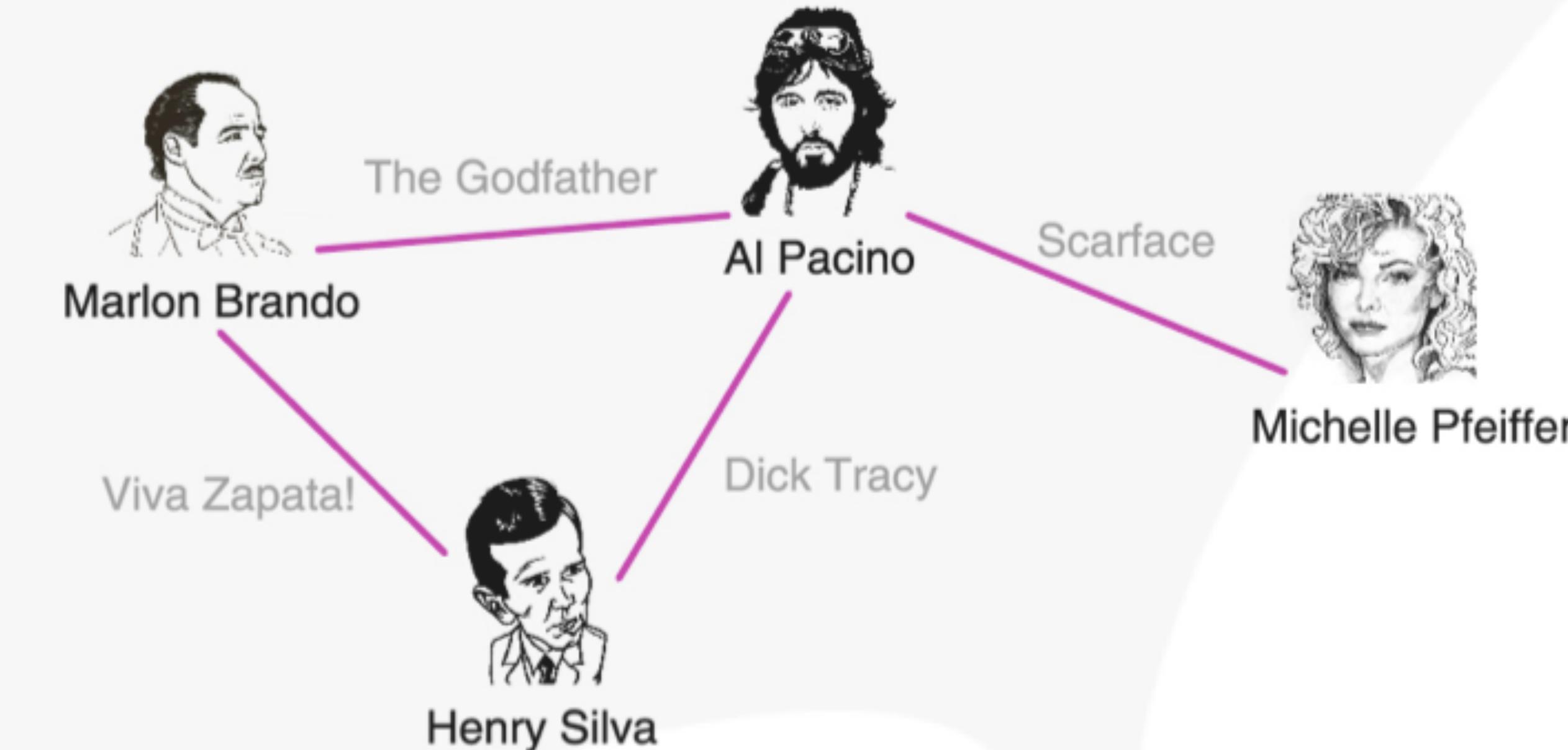
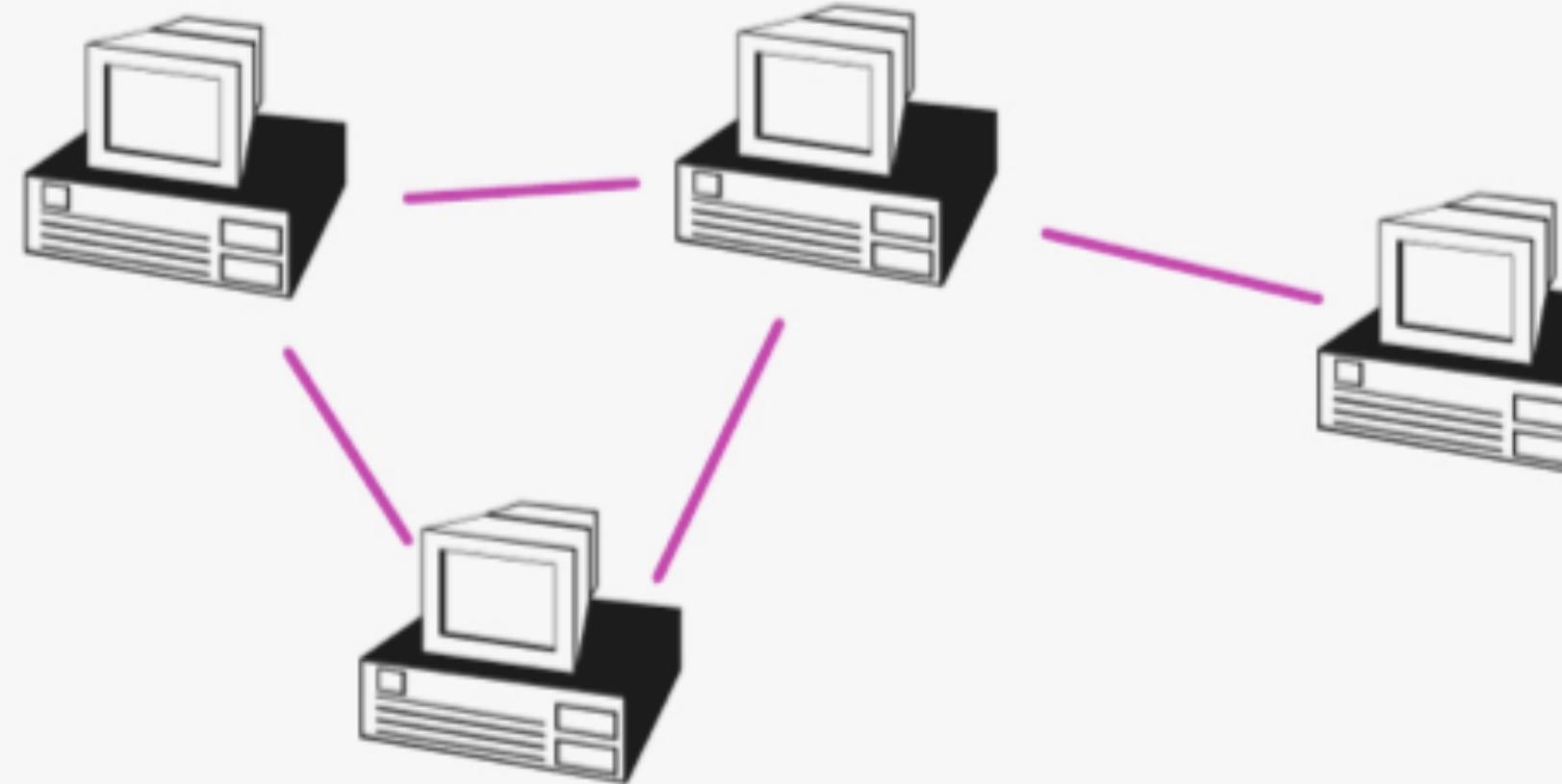


Networks are the maps of complex systems

We now have the data to see and study them

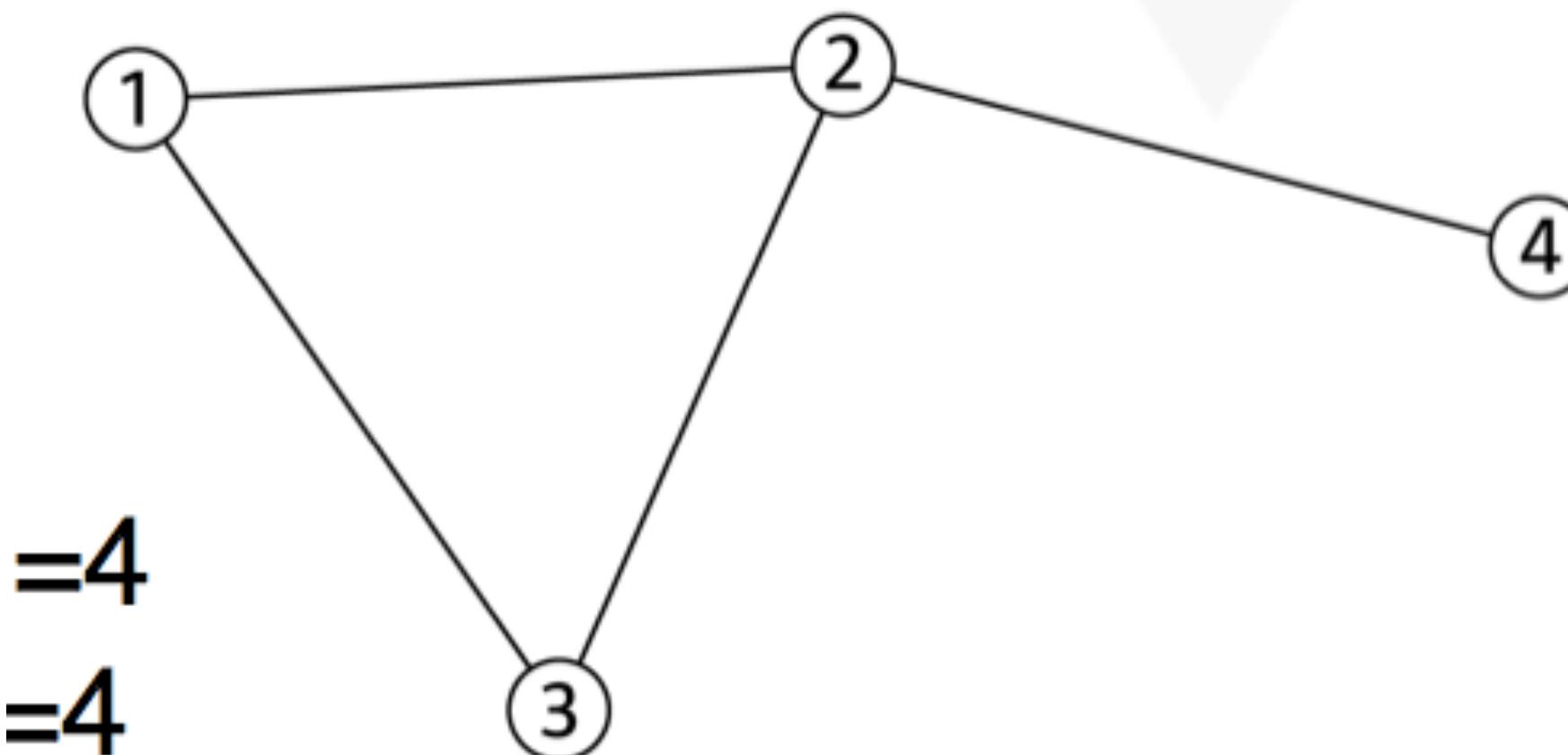
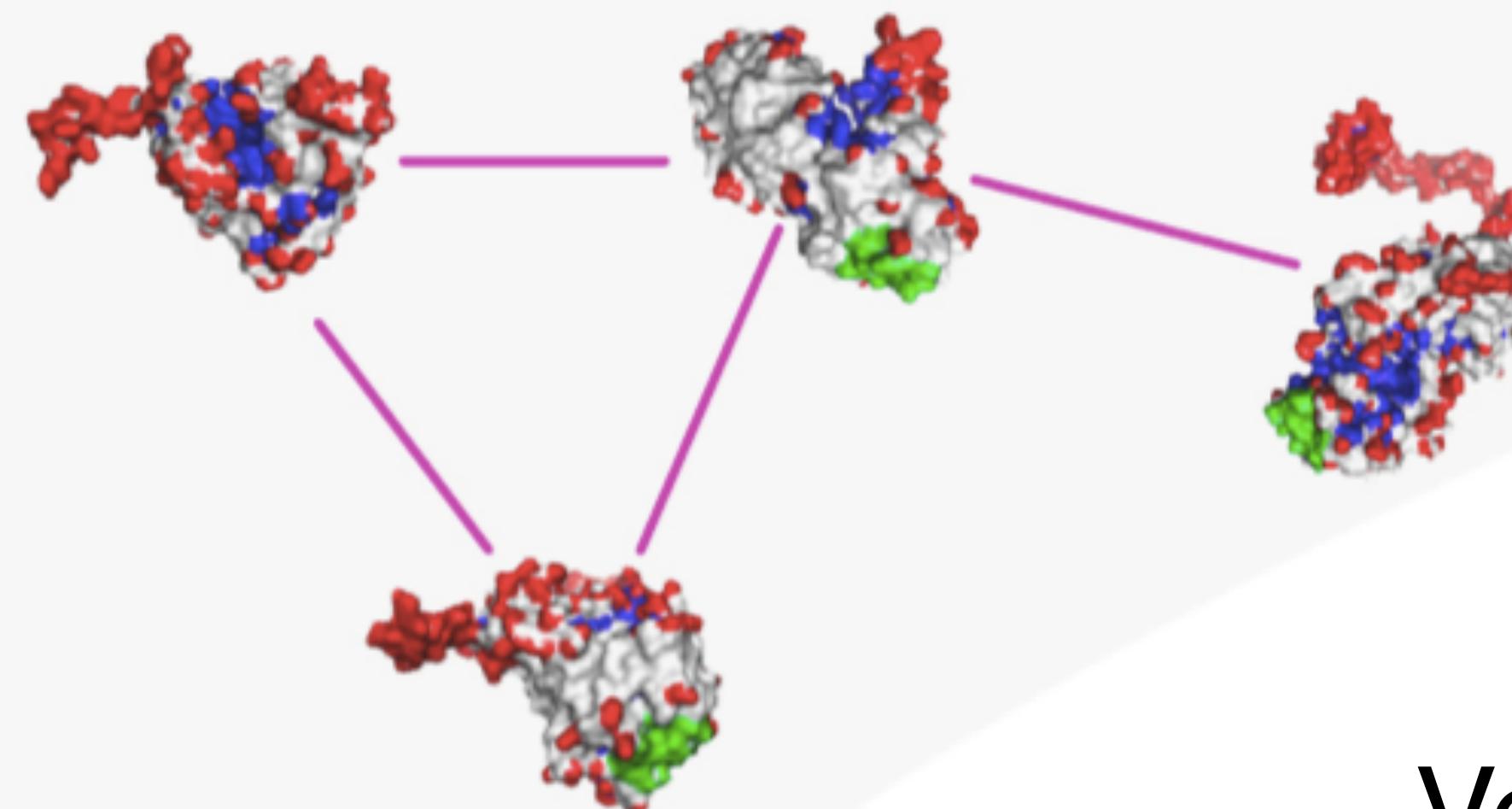
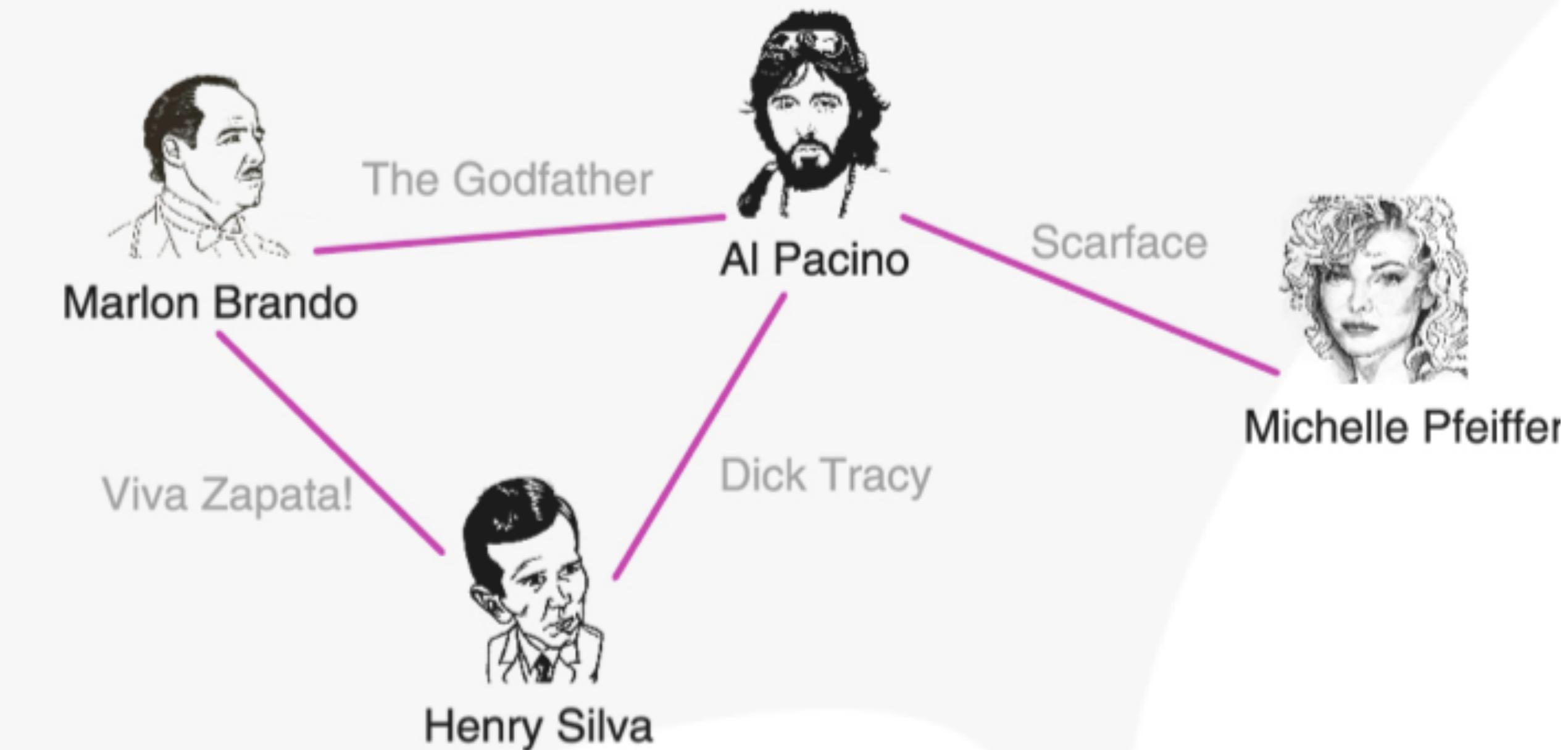
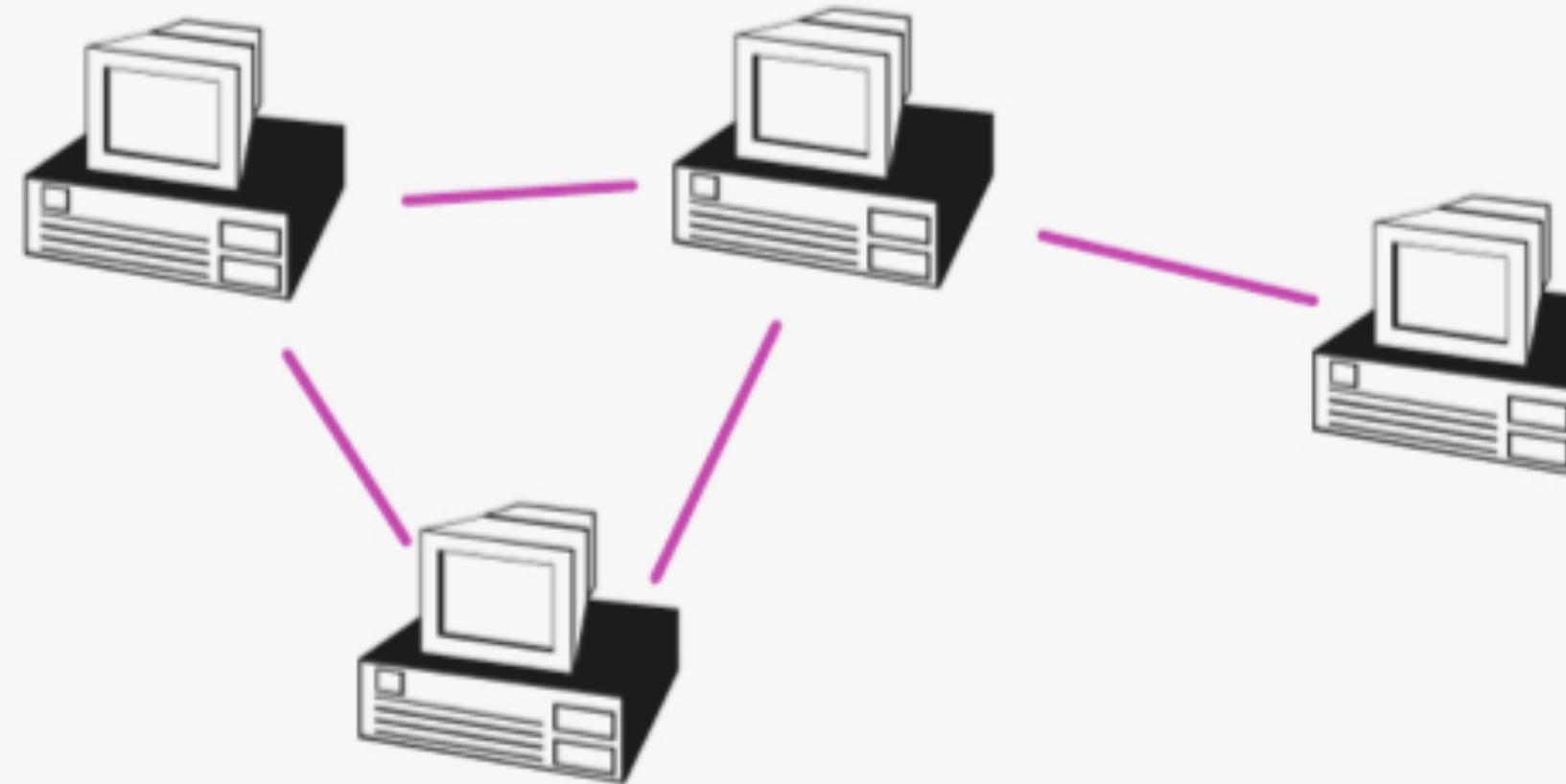


Networks are a common language for different applications



Nodes $N=4$
Links $L=4$
Graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$

Network = Graph + real-world meaning

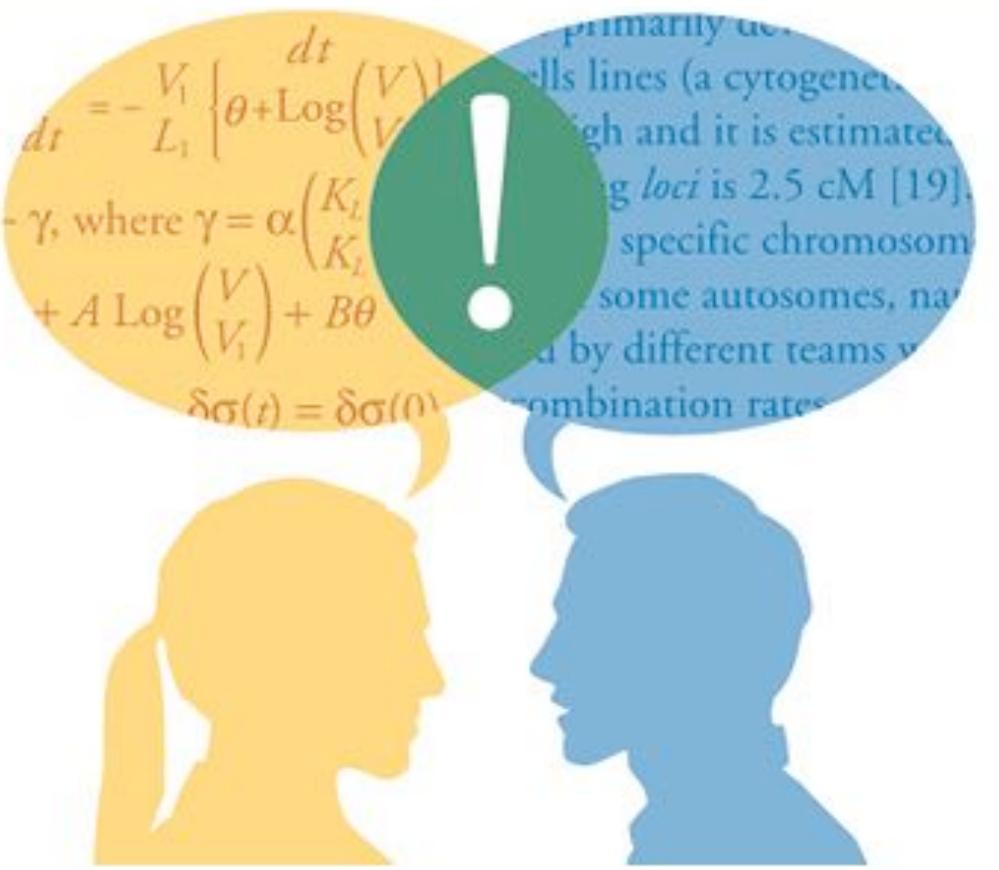


Vertices $V=4$

Edges $E=4$

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

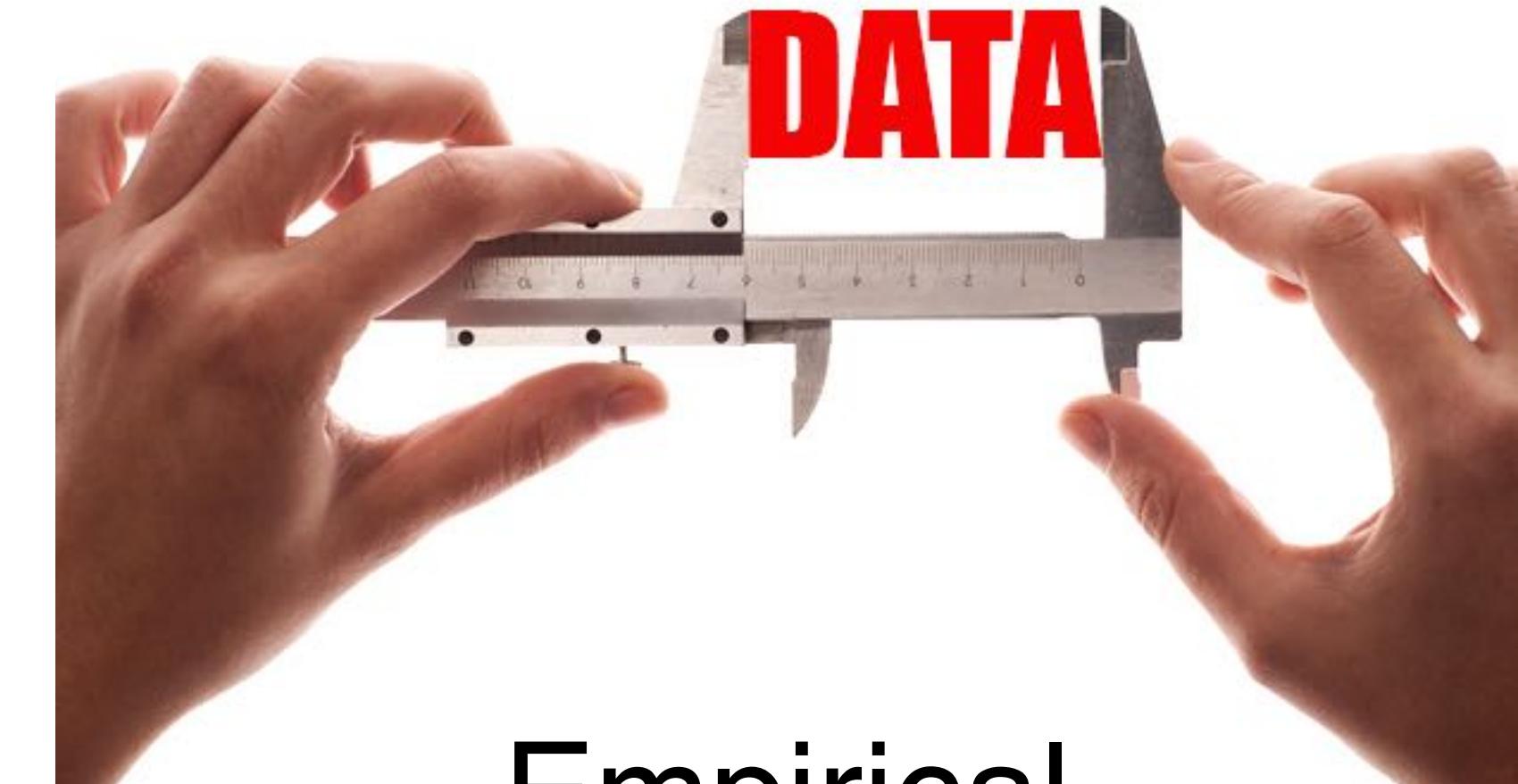
Network science is a part of Data science. It is:



Interdisciplinary



Quantitative,
mathematical



Empirical



Computational

Networks have a huge economic impact

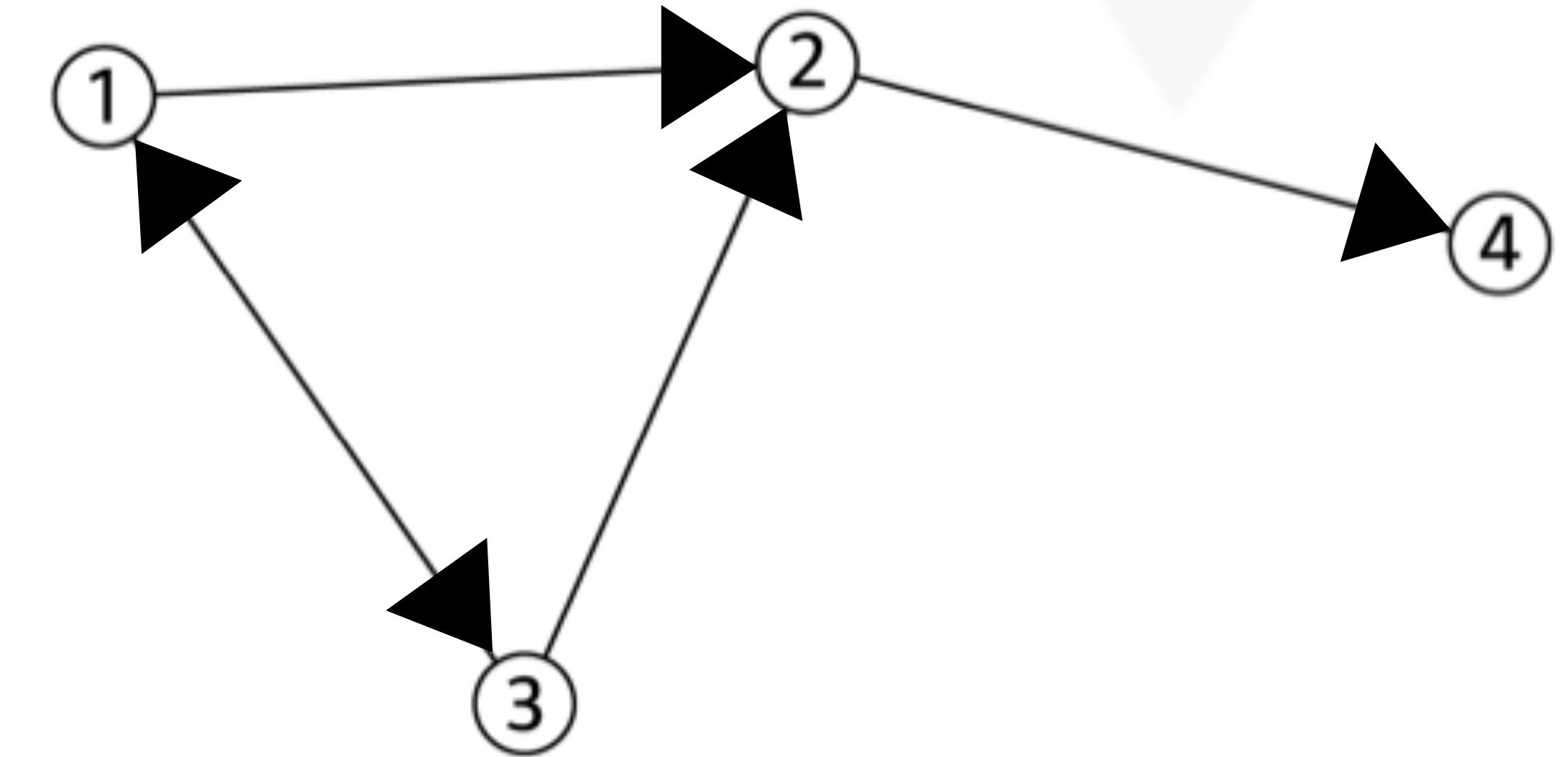


Data is the new oil



A directed graph (**digraph**) has links with a direction

Nodes $N=4$
Links (Arcs) $L=5$



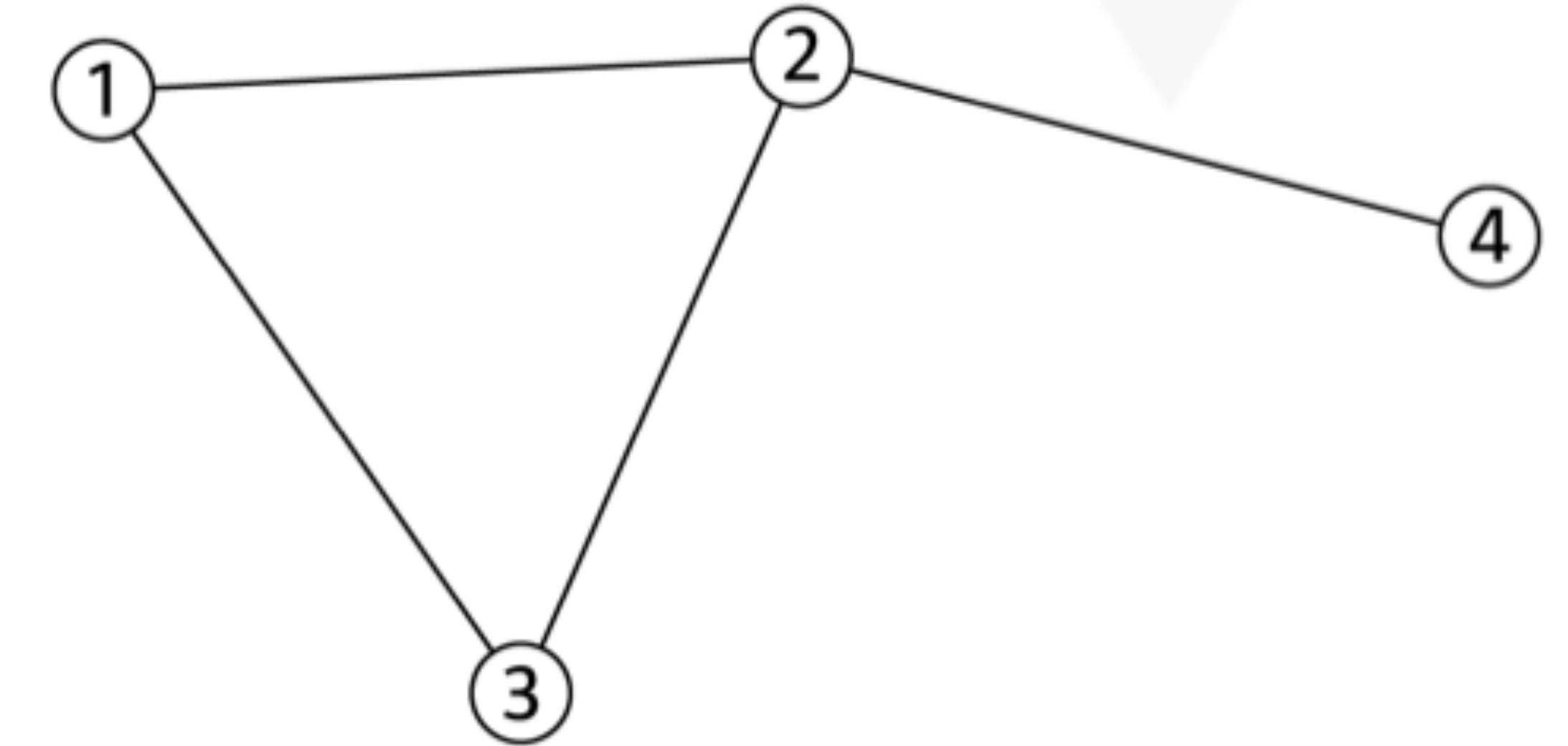
The degree k_i of a node i is the number of incident links

$$k_1 = 2$$

$$k_2 = 3$$

$$k_3 = 2$$

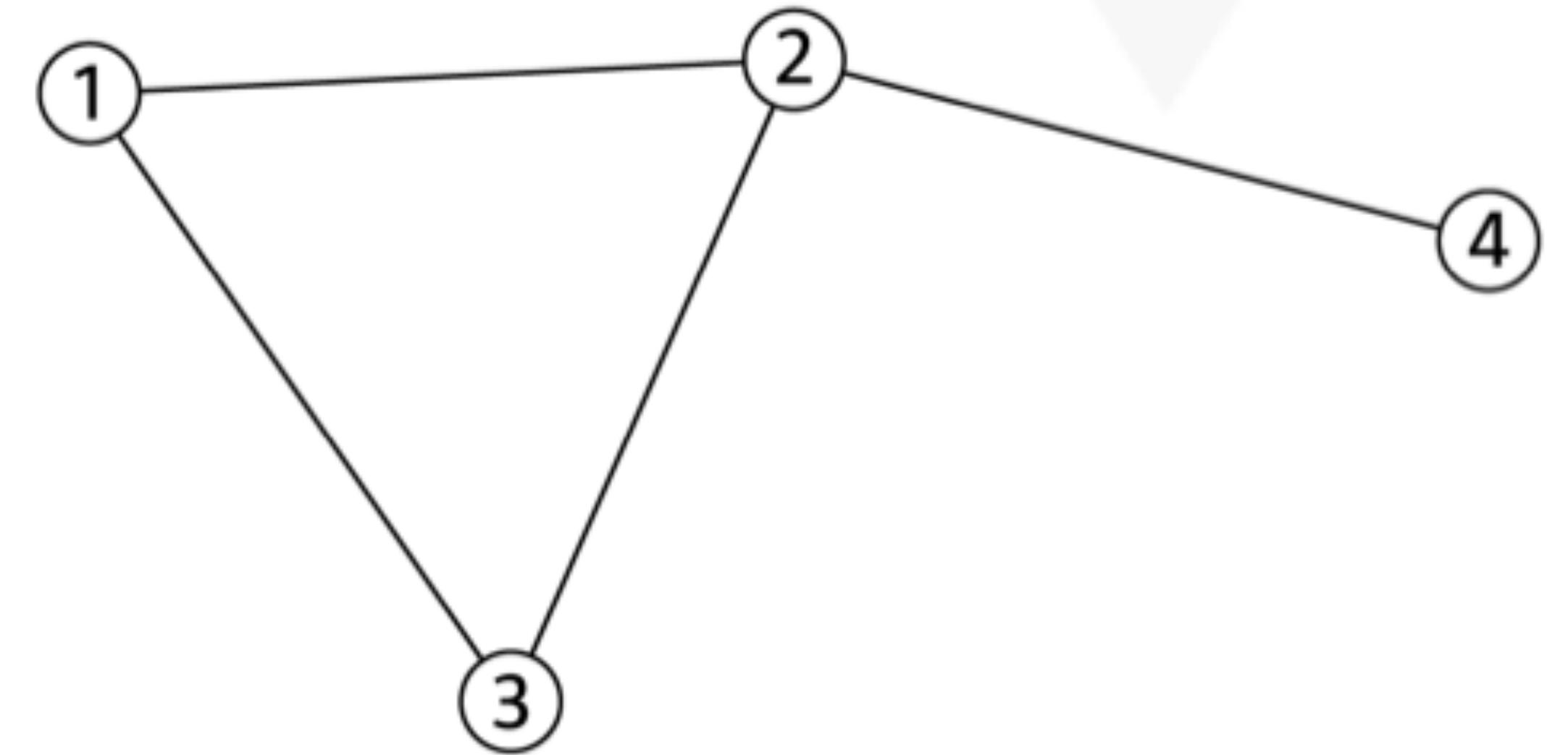
$$k_4 = 1$$



Every network has an **average degree** $\langle k \rangle$

$$\begin{aligned}k_1 &= 2 \\k_2 &= 3\end{aligned}$$

$$\begin{aligned}k_3 &= 2 \\k_4 &= 1\end{aligned}$$

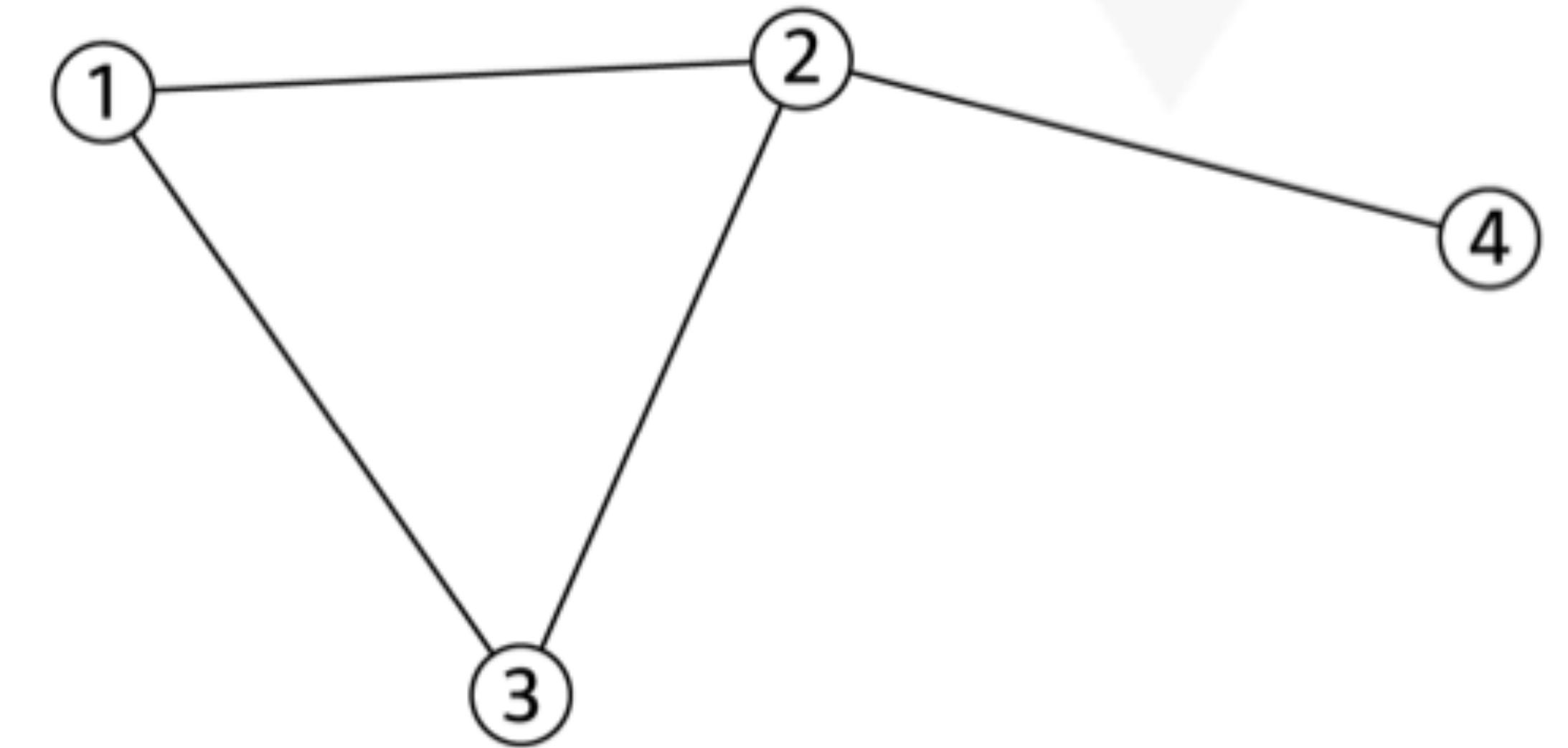
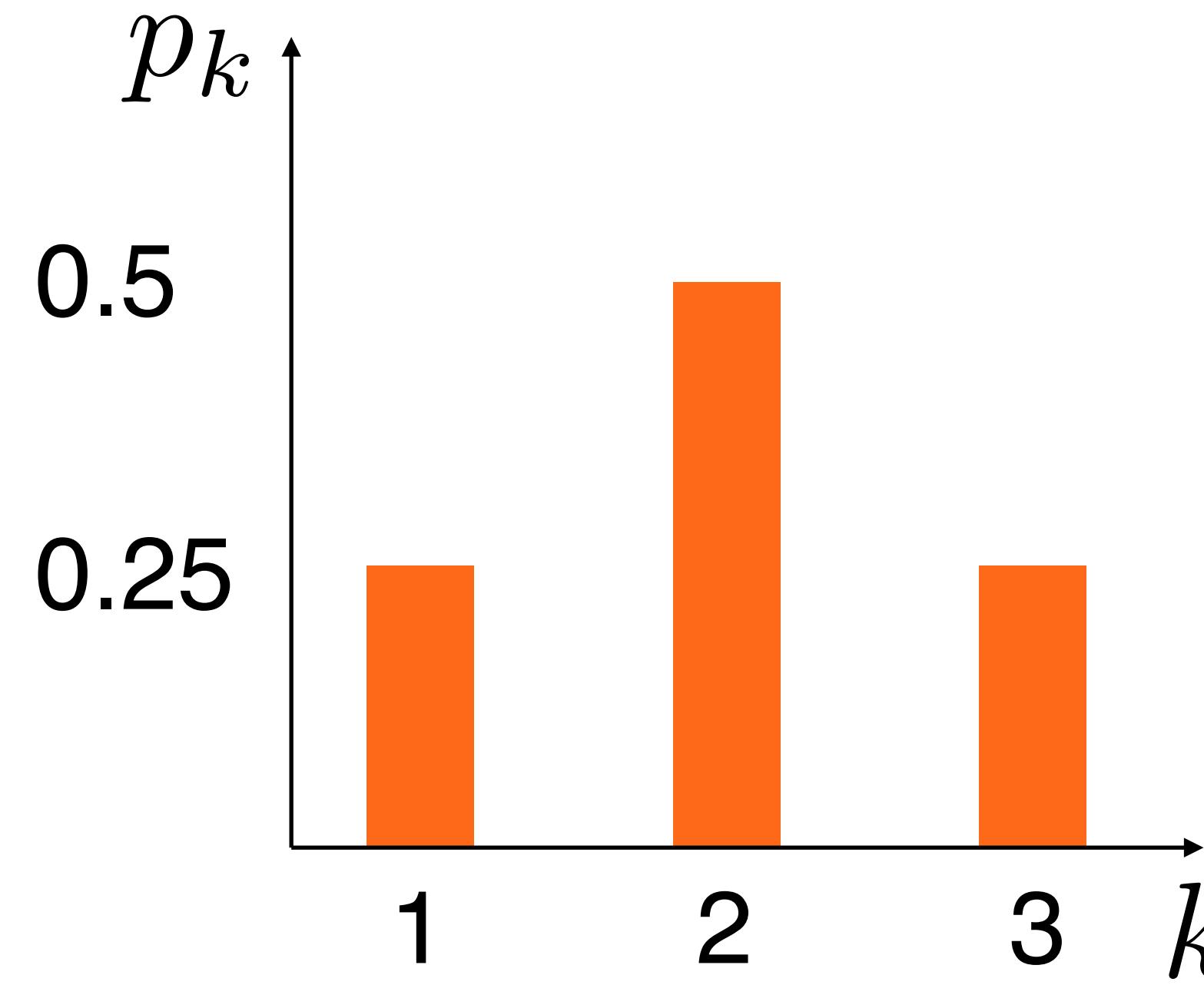


$$\langle k \rangle = \frac{1}{N} \sum_{i=1}^N k_i \quad \langle k \rangle = \frac{2 + 3 + 2 + 1}{4} = 2$$

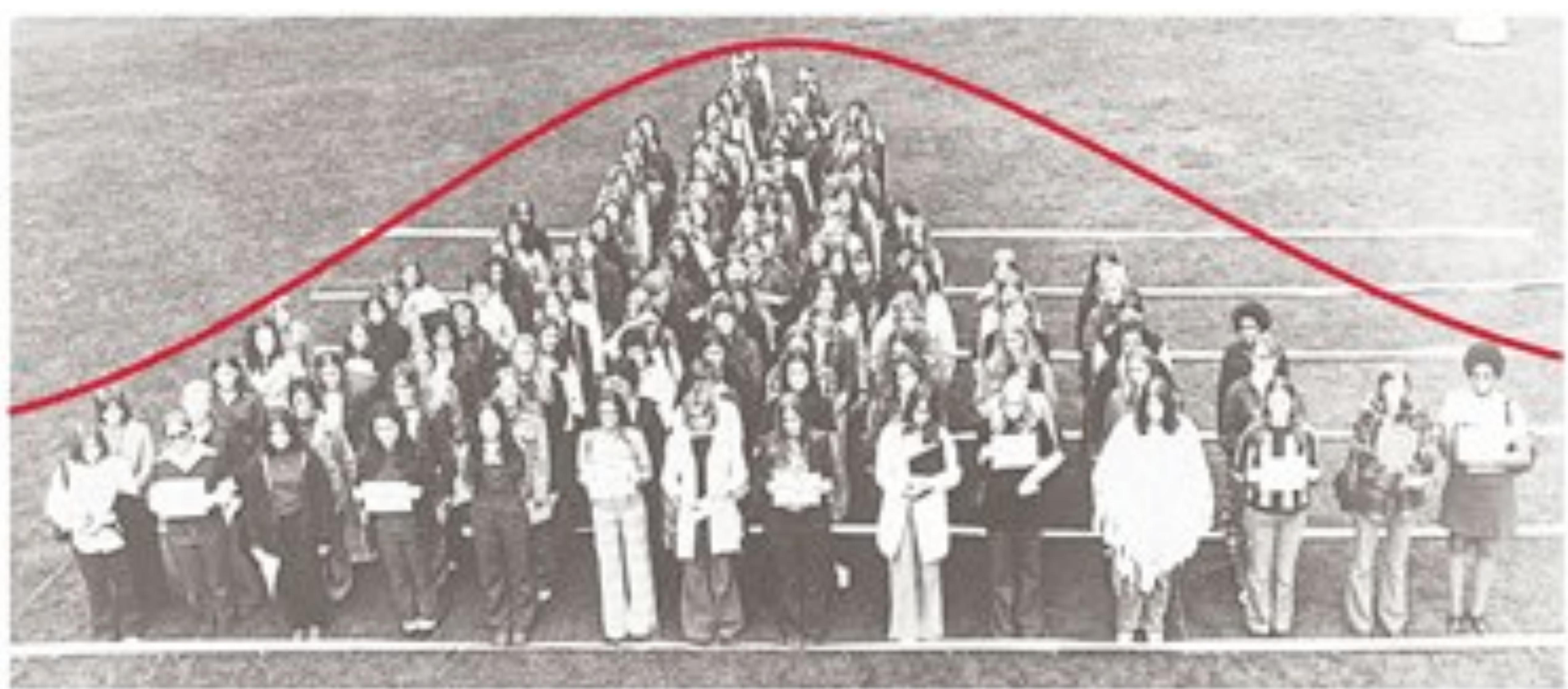
The degree distribution $p_k = N_k/N$ captures the probabilities that a node has a certain degree

$$\begin{aligned} k_1 &= 2 \\ k_2 &= 3 \end{aligned}$$

$$\begin{aligned} k_3 &= 2 \\ k_4 &= 1 \end{aligned}$$



A network's degree distribution tells us something fundamental about how individuals in the system connect



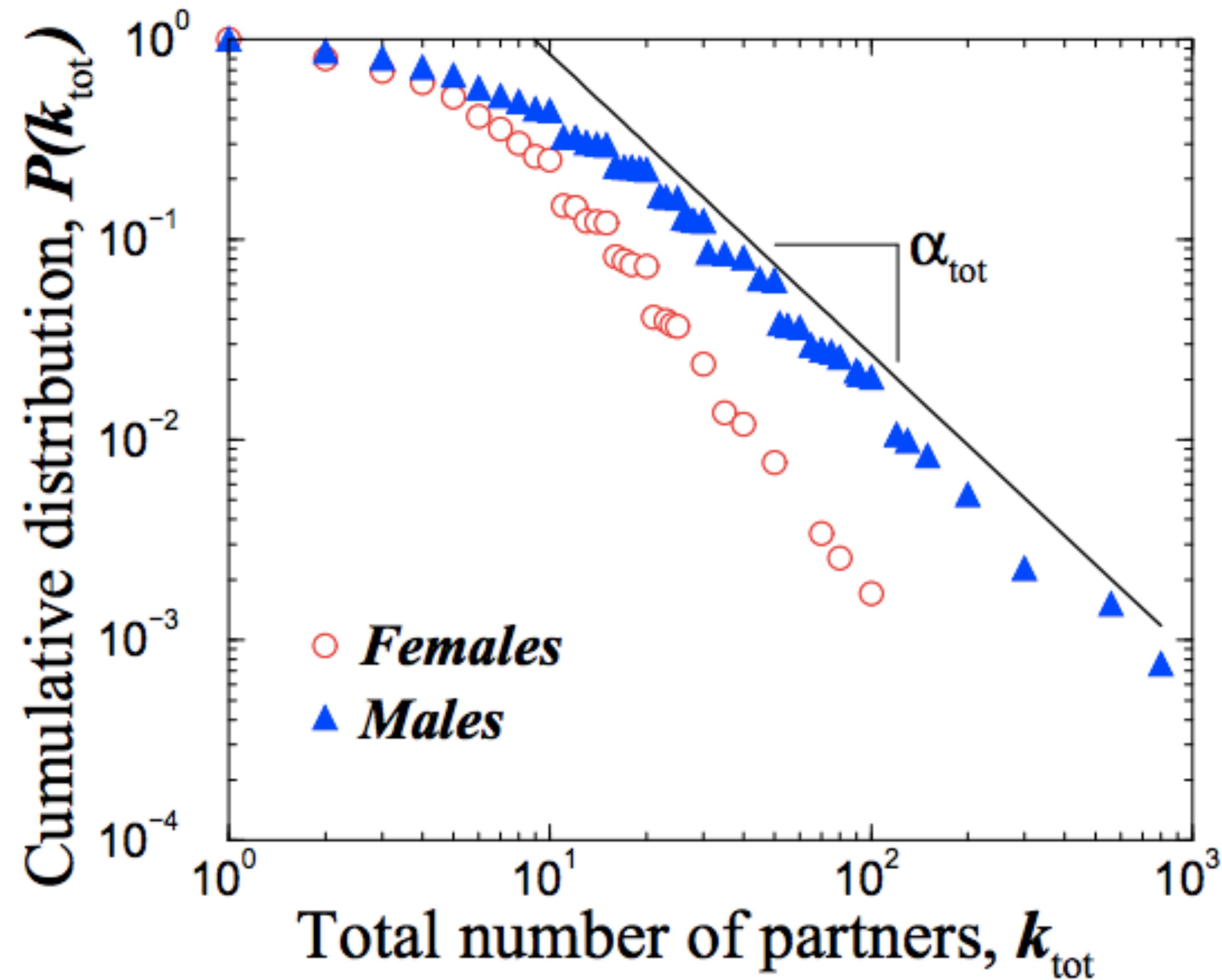


23/10 /2 * (62-15)*365 ~ 20000



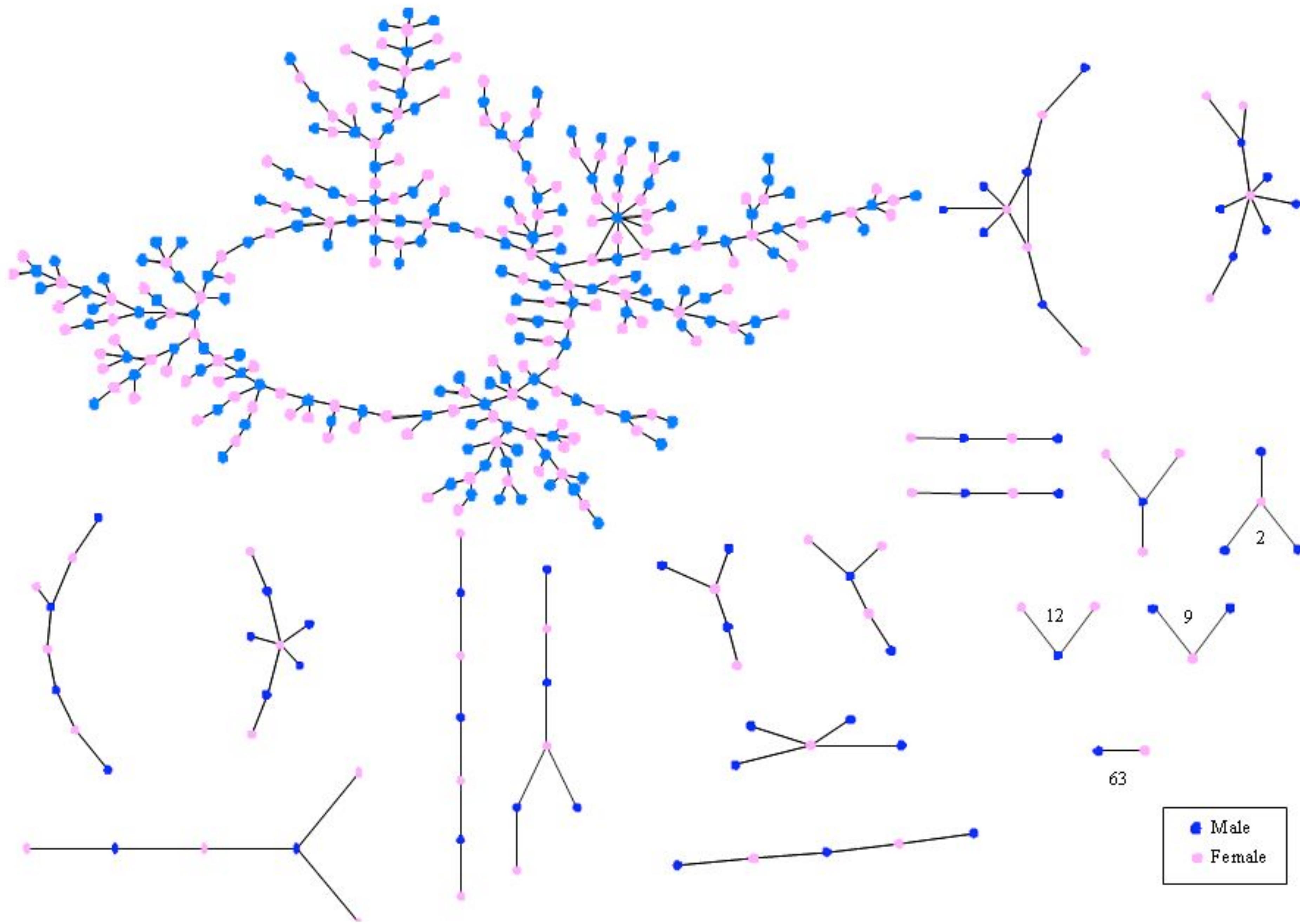
In a 1999 interview shortly before his death, he regretted not having explained the sexual climate at the time of his escapades, and warned other men who admired him for it, closing with the words: "With all of you men out there who think that having a thousand different ladies is pretty cool, I have learned in my life I've found out that having one woman a thousand different times is much more satisfying."^[100] Chamberlain also acknowledged that he never came close to marrying and had no intention of raising any children.^[37]

Sexual behavior is heavy-tailed

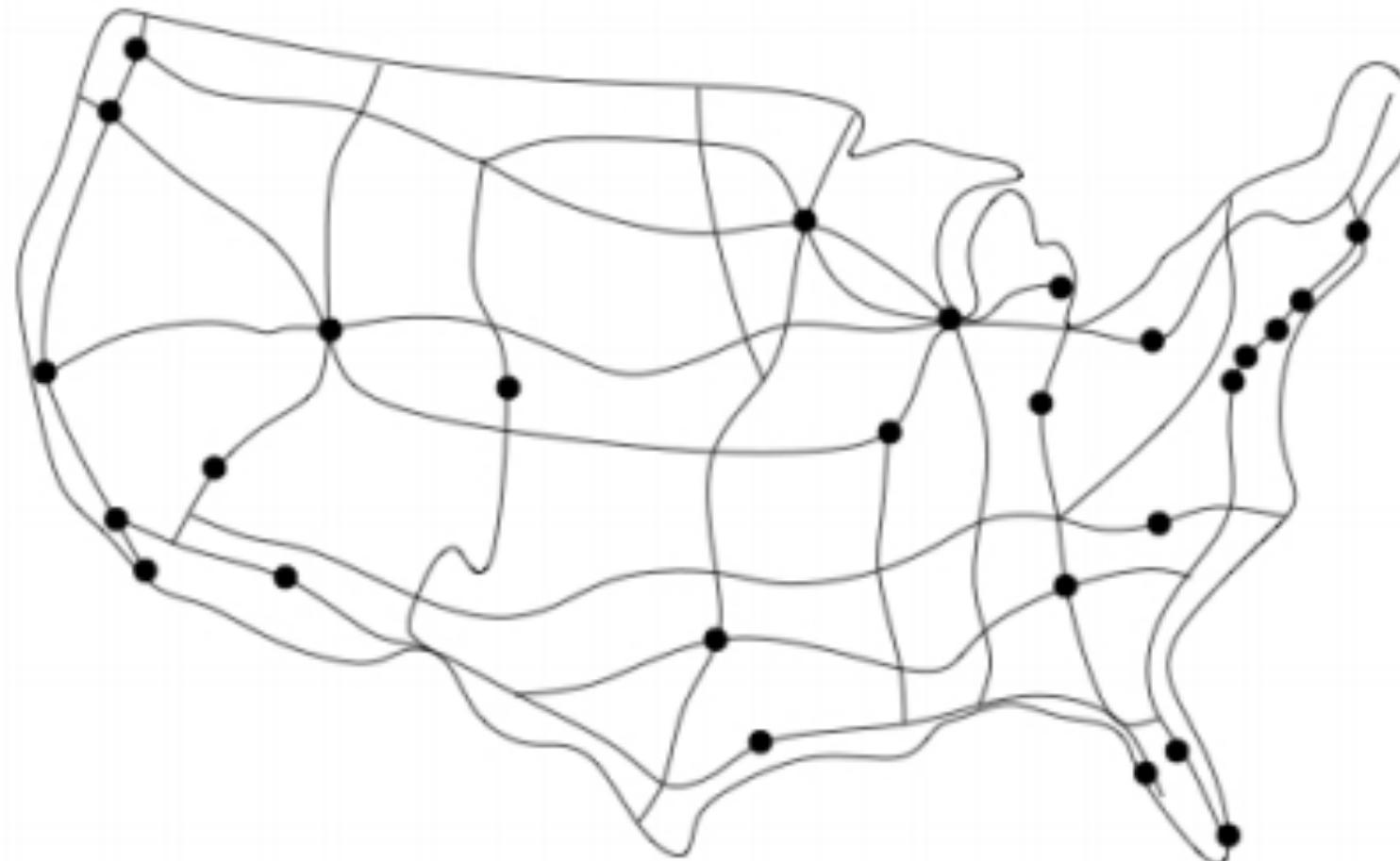
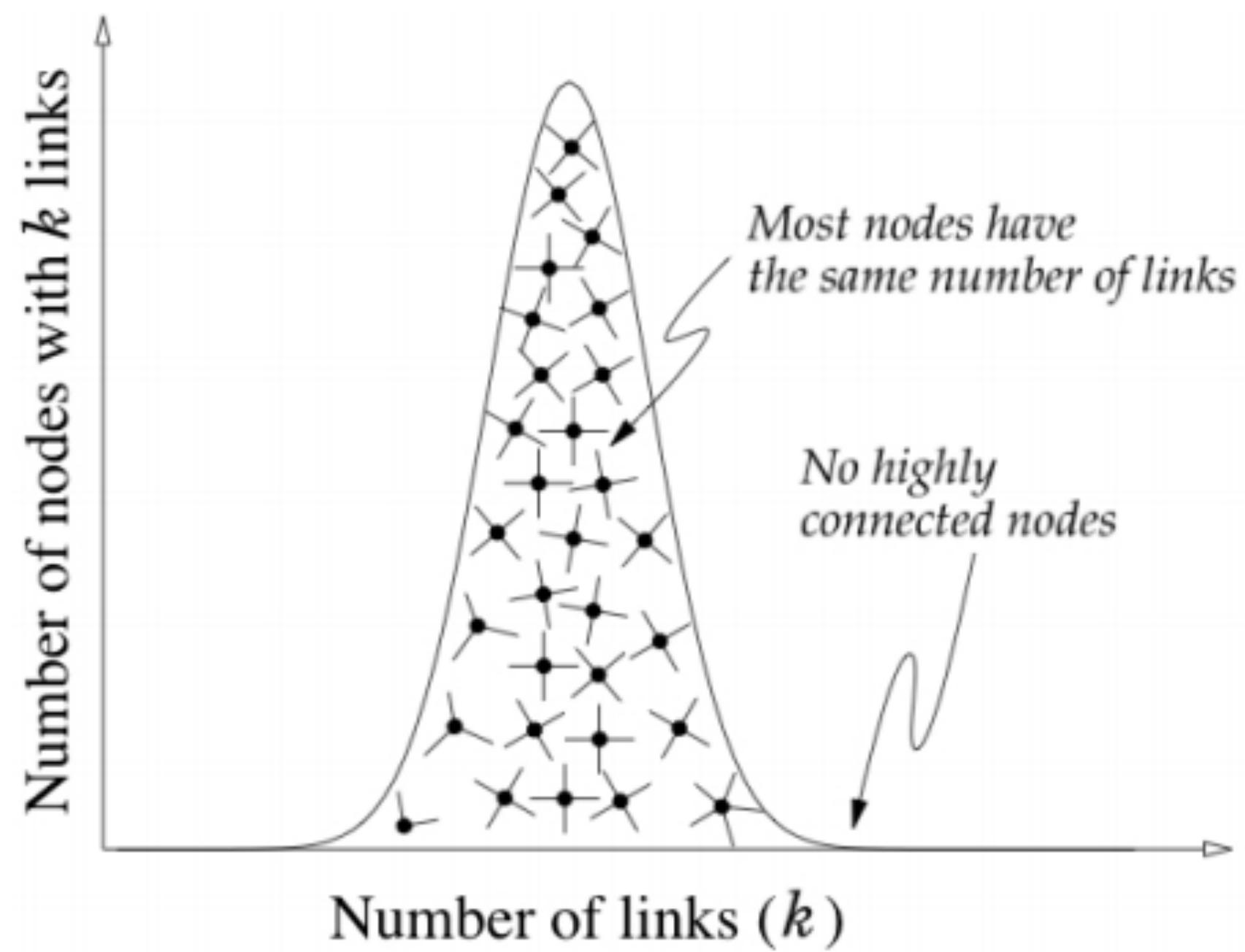


If our heights were like our sex lives



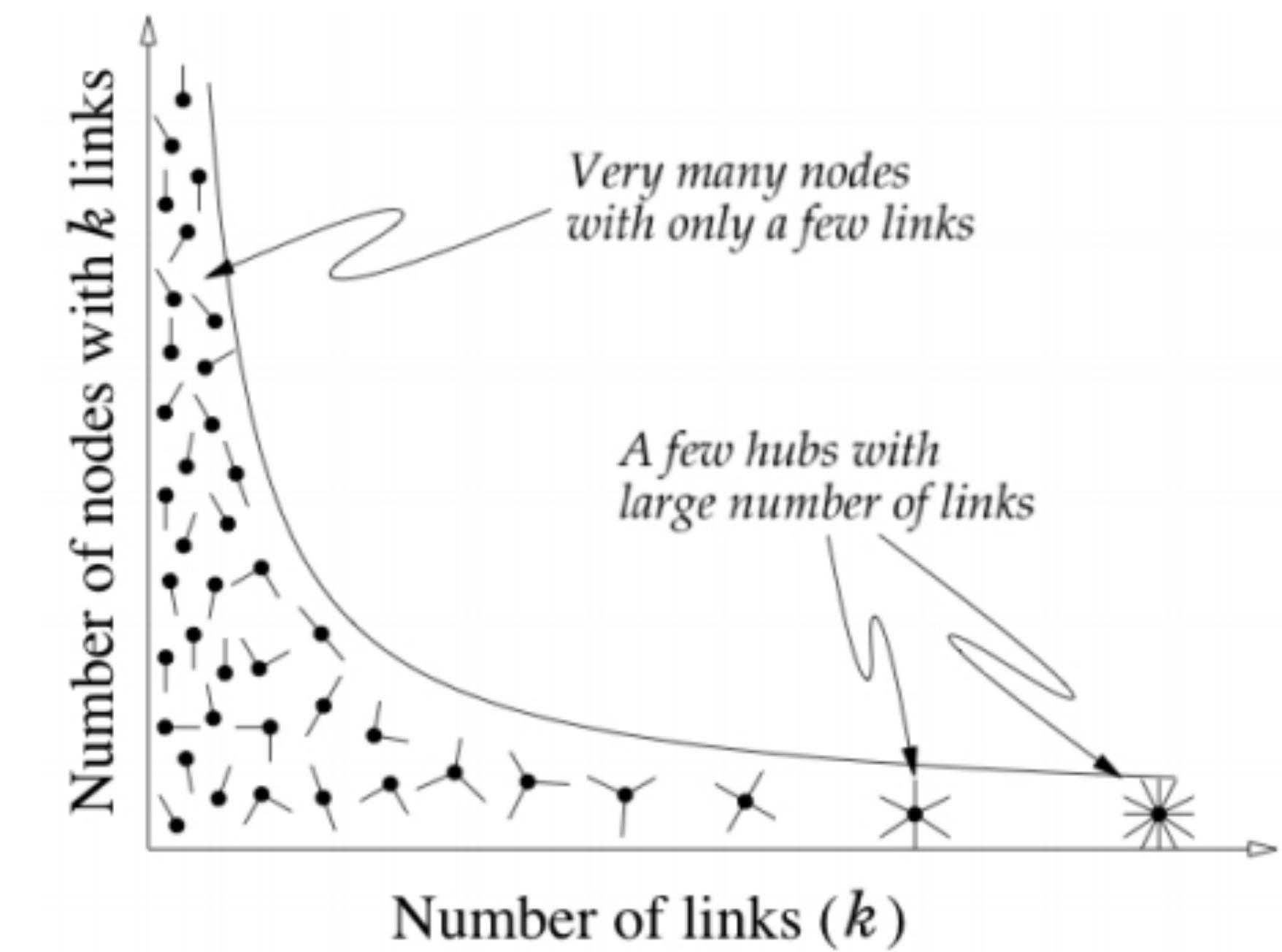


Thin-tailed network



Streets

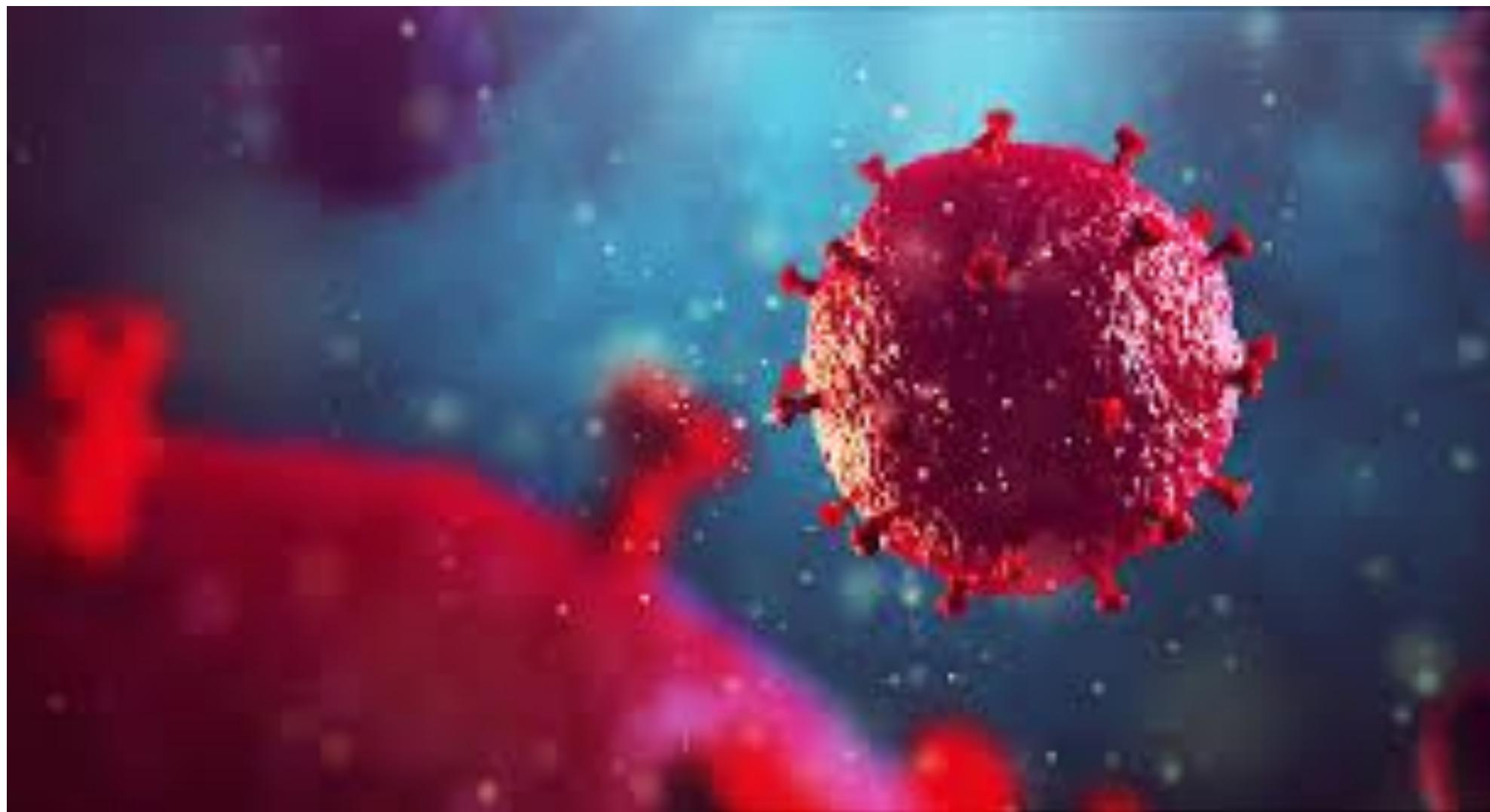
Heavy-tailed network



Airlines

A heavy-tailed degree distribution means we have hubs

Hubs completely change network processes
like epidemic spreading



Heavy-tailed distributions govern the world

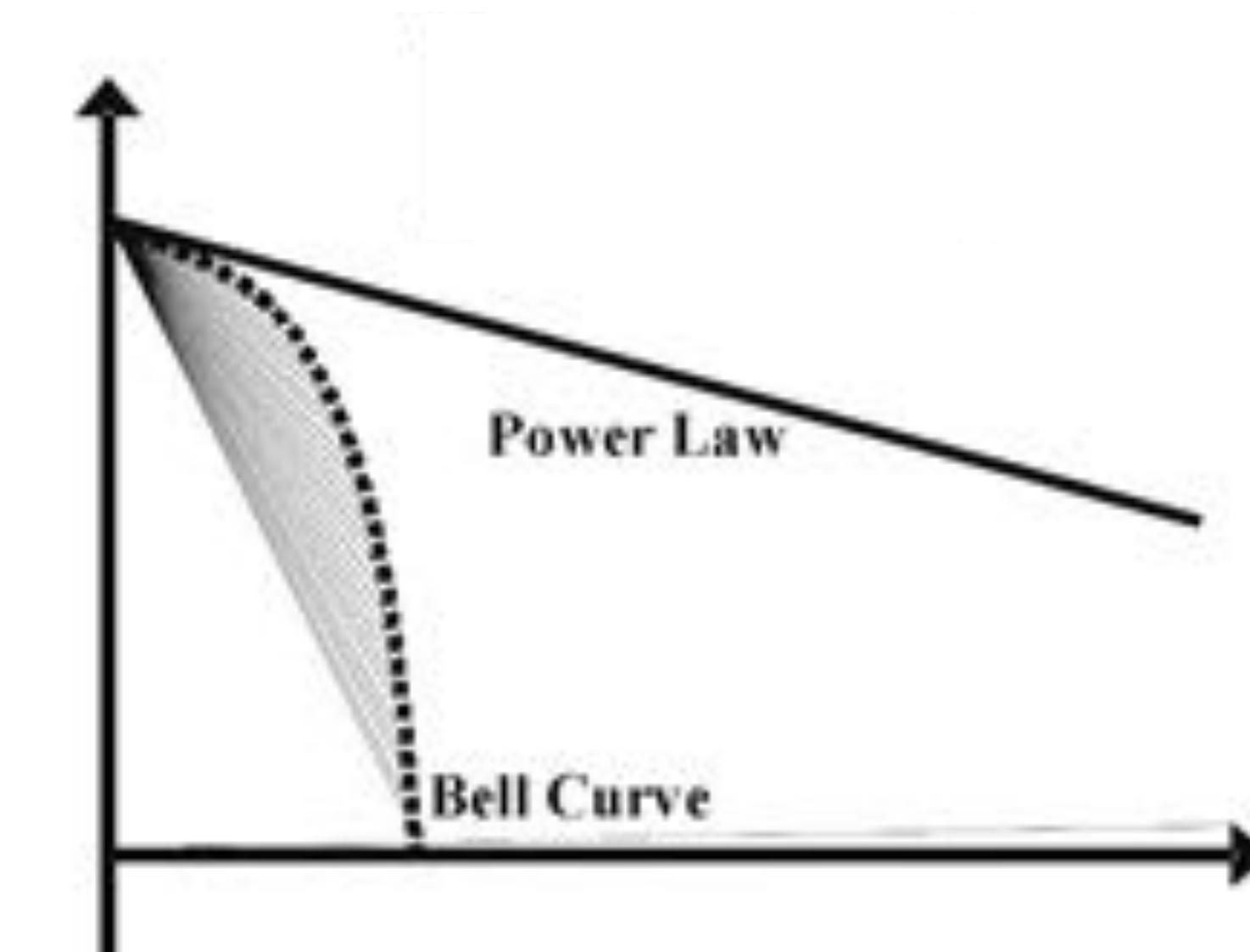
20th century statistics



Strange outliers

Focus on the
head/center

21th century statistics add:



Not outliers but
part of the system

Focus on the tail

Organizing principles of networks

Many networks are:

- 1) Heavy-tailed
- 2) Sparse
- 3) Small-world
- 4) Clustered

Organizing principles of networks

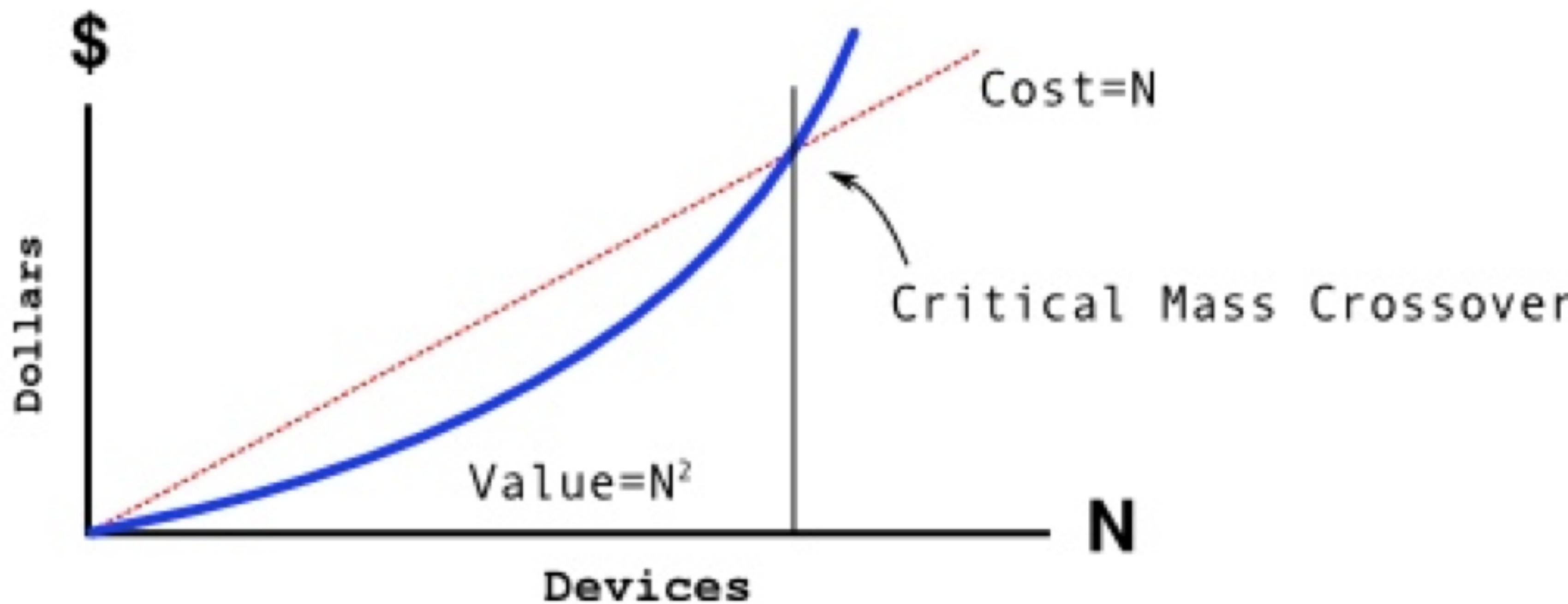
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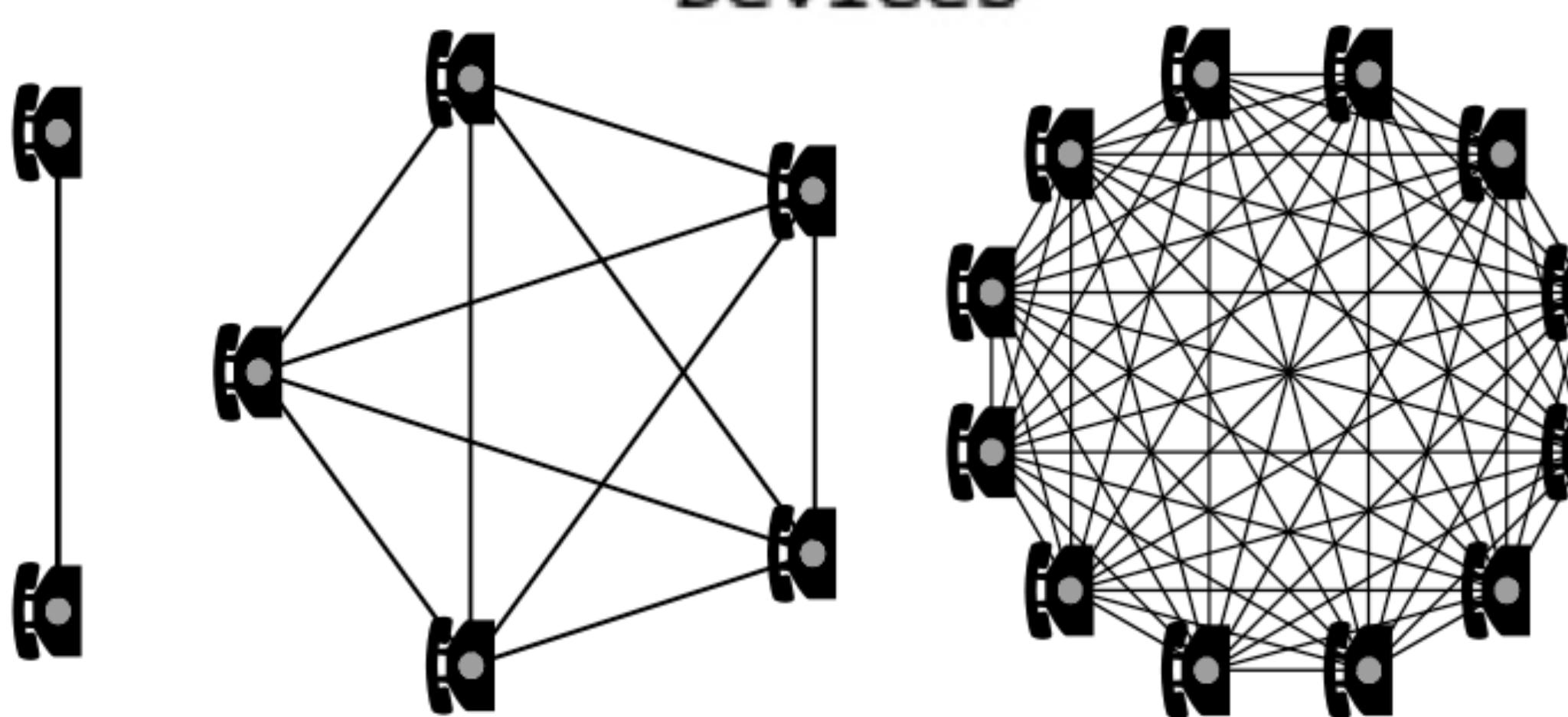
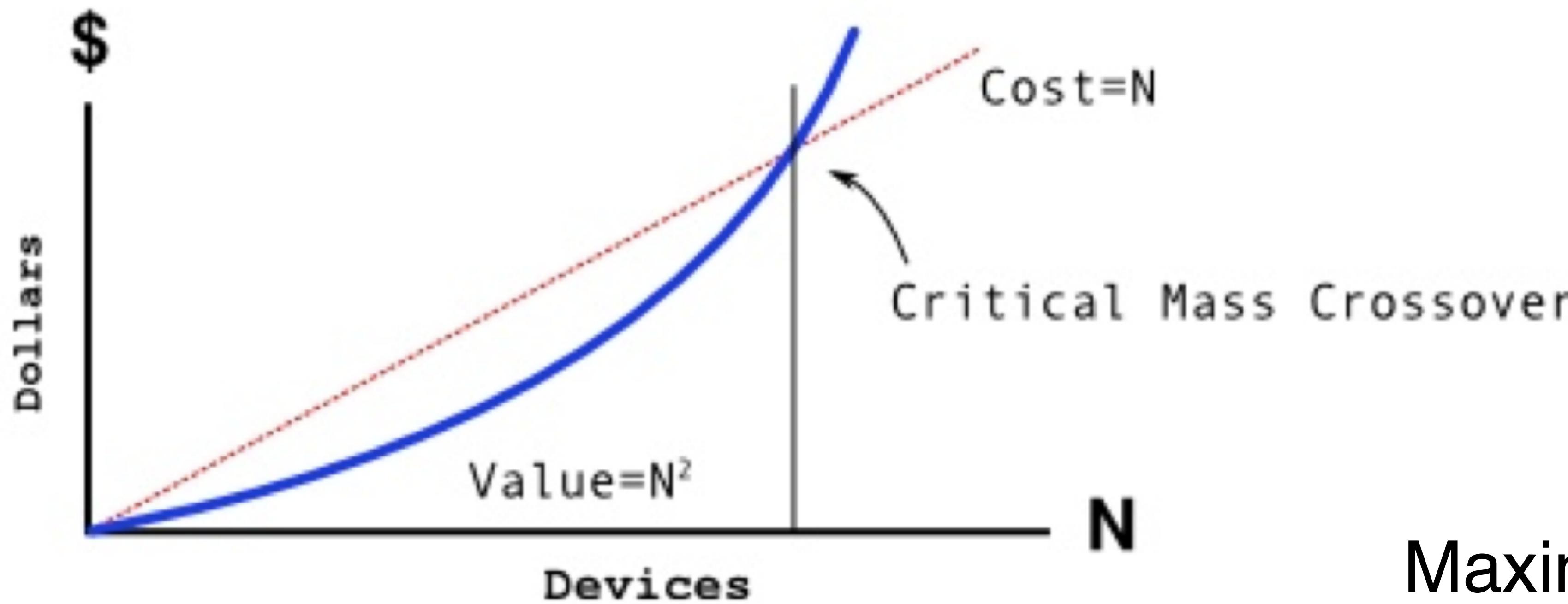
The person who bought the first fax,
what were they thinking???



Metcalf's law states that the value of a communication network increases with the square of users



Metcalf's law states that the value of a communication network increases with the square of users



Maximum possible number of links:

$$L_{\max} = \frac{N(N - 1)}{2} \sim N^2$$

Sparsity means: Although a lot of links are possible, only very few are actually there: $L \ll L_{\max}$

Contrasting the actual number of links L with the maximum possible number L_{\max} we define **density d** :

$$d = L/L_{\max}$$

Sparsity means: Although a lot of links are possible, only very few are actually there: $L \ll L_{\max}$

Contrasting the actual number of links L with the maximum possible number L_{\max} we define **density d** :

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Most networks are sparse

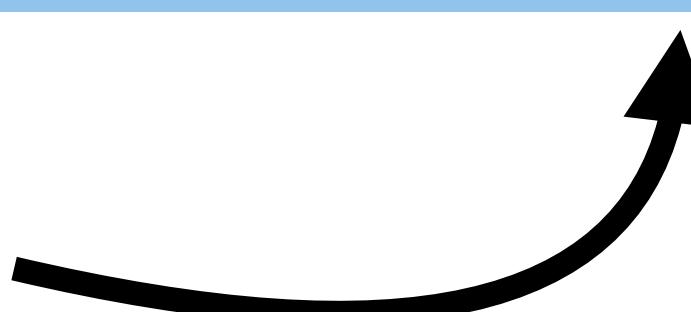
Most networks have low density: $d \ll 1$

Sparsity means: Although a lot of links are possible, only very few are actually there: $d \ll 1$

Network	Nodes (N)	Links (L)	Density (d)	Average degree ($\langle k \rangle$)
Facebook Northwestern Univ.	10,567	488,337	0.009	92.4
IMDB movies and stars	563,443	921,160	0.000006	3.3
IMDB co-stars	252,999	1,015,187	0.00003	8.0
Twitter US politics	18,470	48,365	0.0001	2.6
Enron email	87,273	321,918	0.00004	3.7
Wikipedia math	15,220	194,103	0.0008	12.8
Internet routers	190,914	607,610	0.00003	6.4
US air transportation	546	2,781	0.02	10.2
World air transportation	3,179	18,617	0.004	11.7
Yeast protein interactions	1,870	2,277	0.001	2.4
<i>C. elegans</i> brain	297	2,345	0.03	7.9

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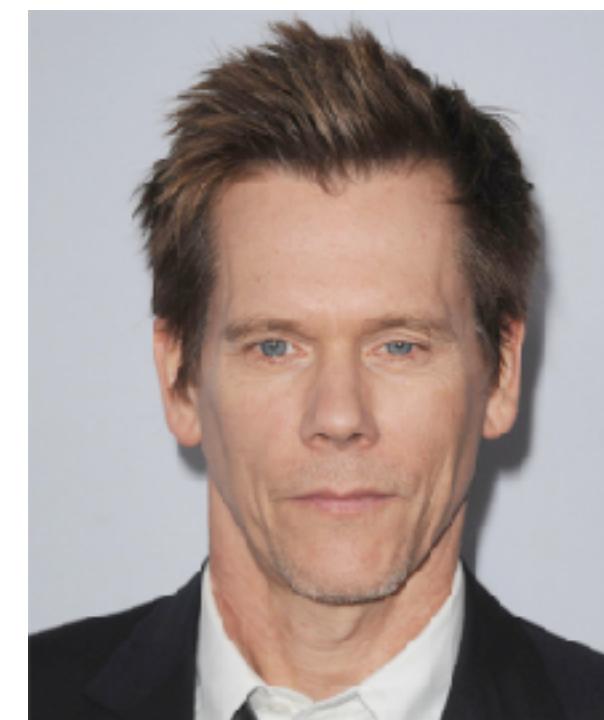
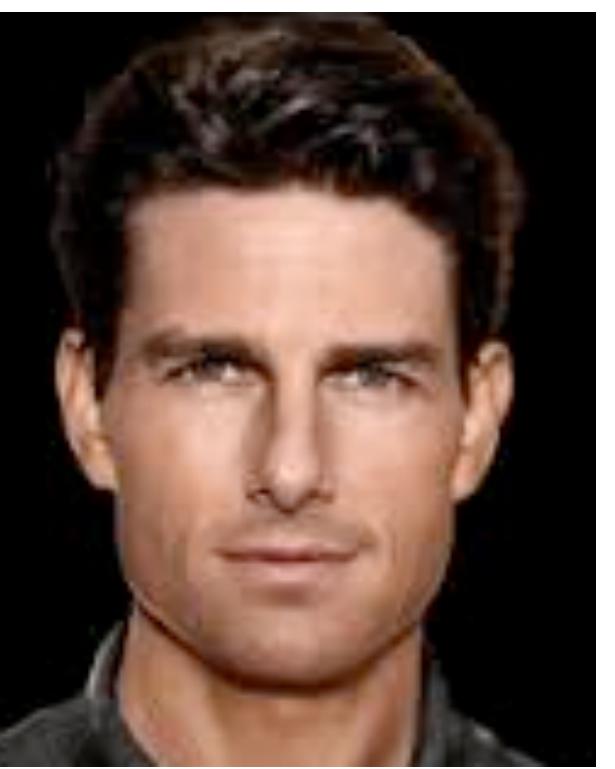
Also: $\langle k \rangle \ll \langle k \rangle_{\max} = N - 1$ 

Organizing principles of networks

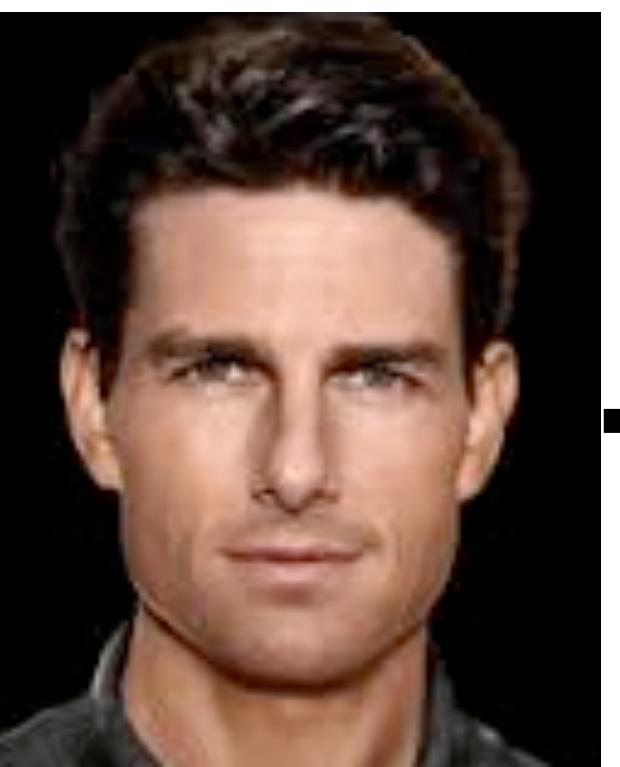
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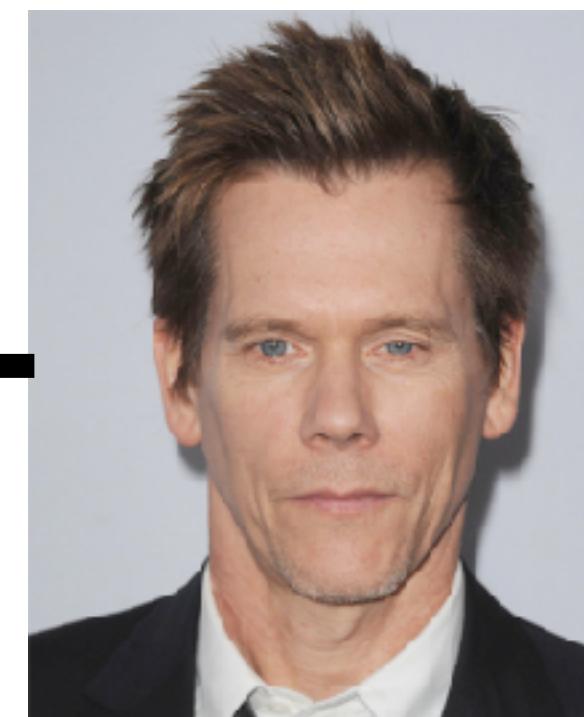
The Kevin Bacon game



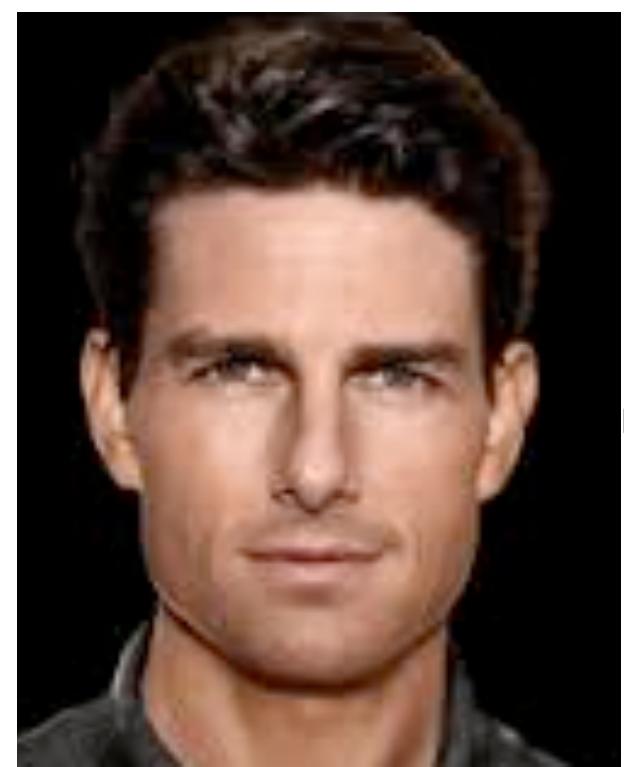
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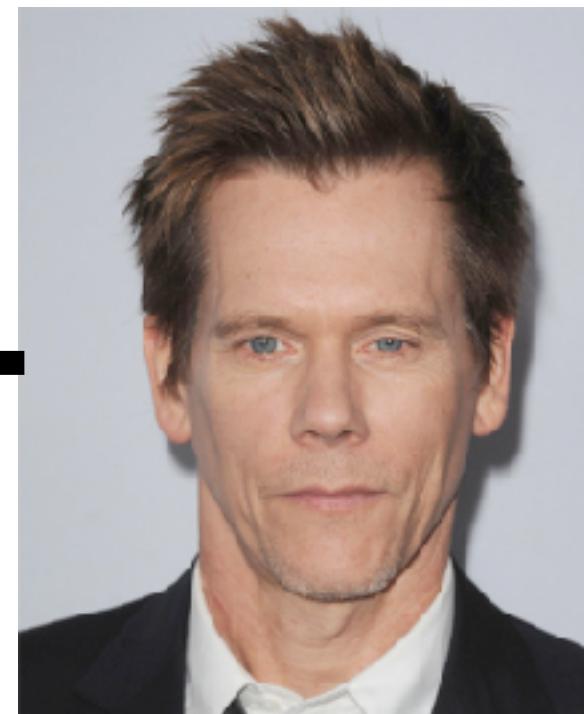
A few good
men



The Kevin Bacon game



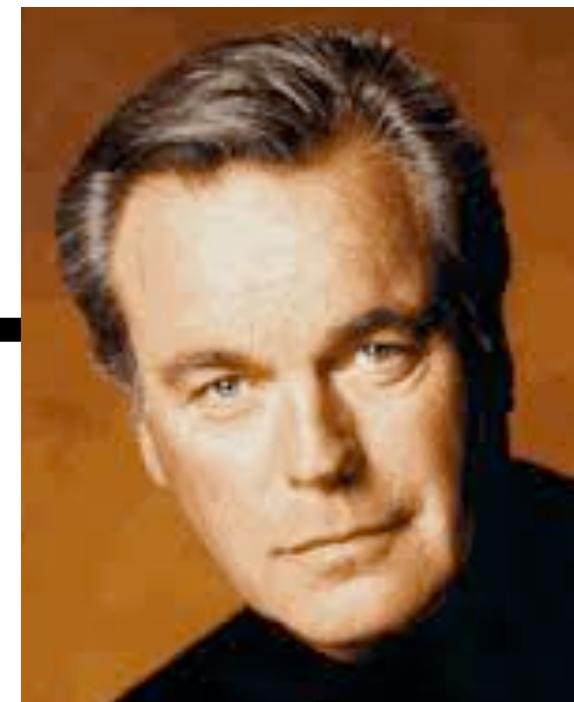
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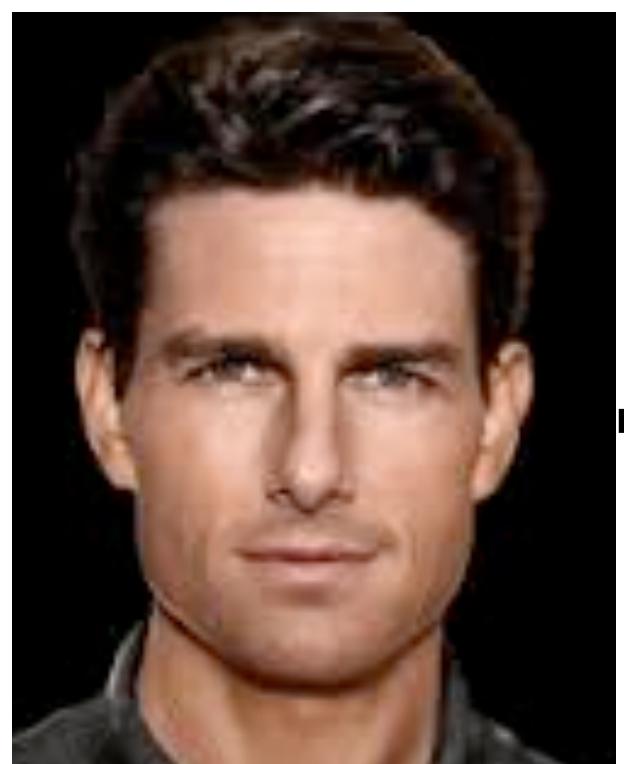
The Kevin Bacon game



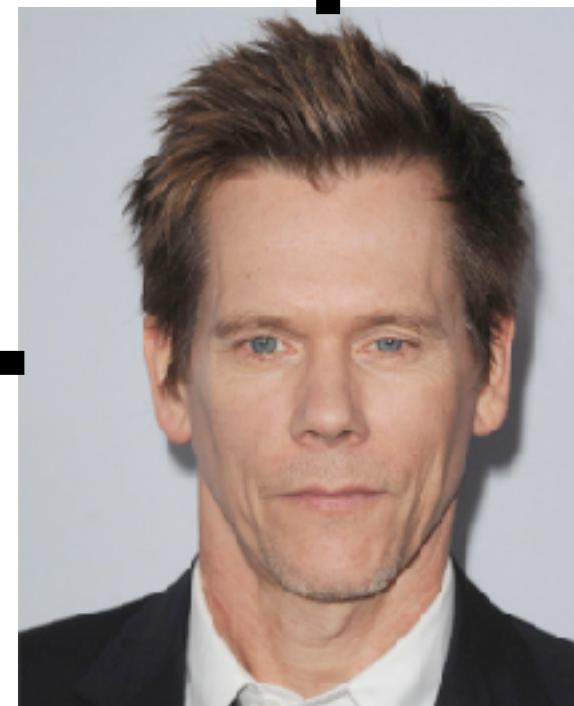
Austin
Powers



Wild things



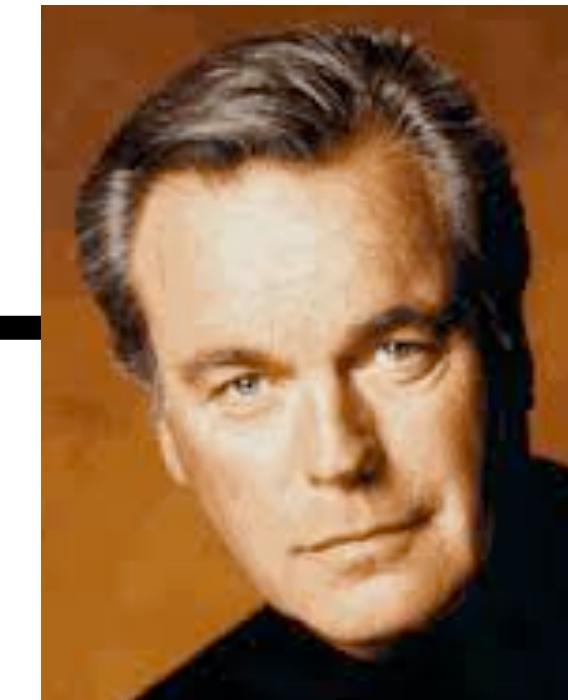
A few good
men



The Kevin Bacon game



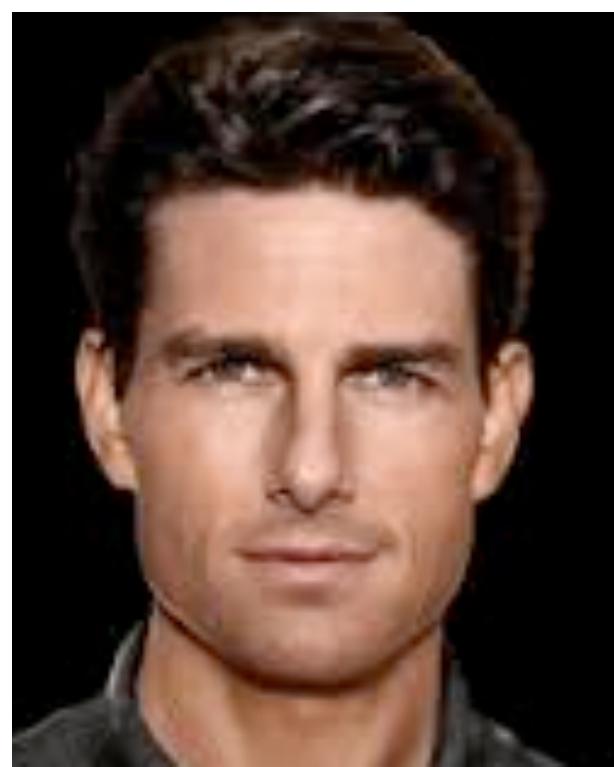
Austin
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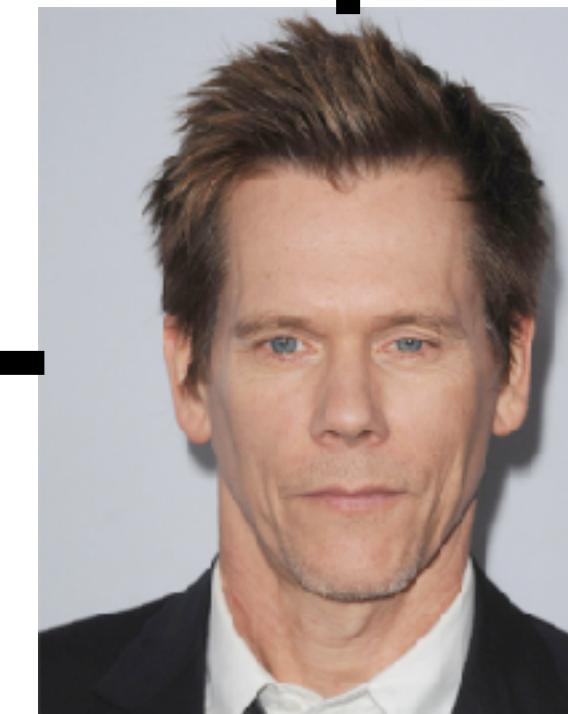
Let's make
it illegal



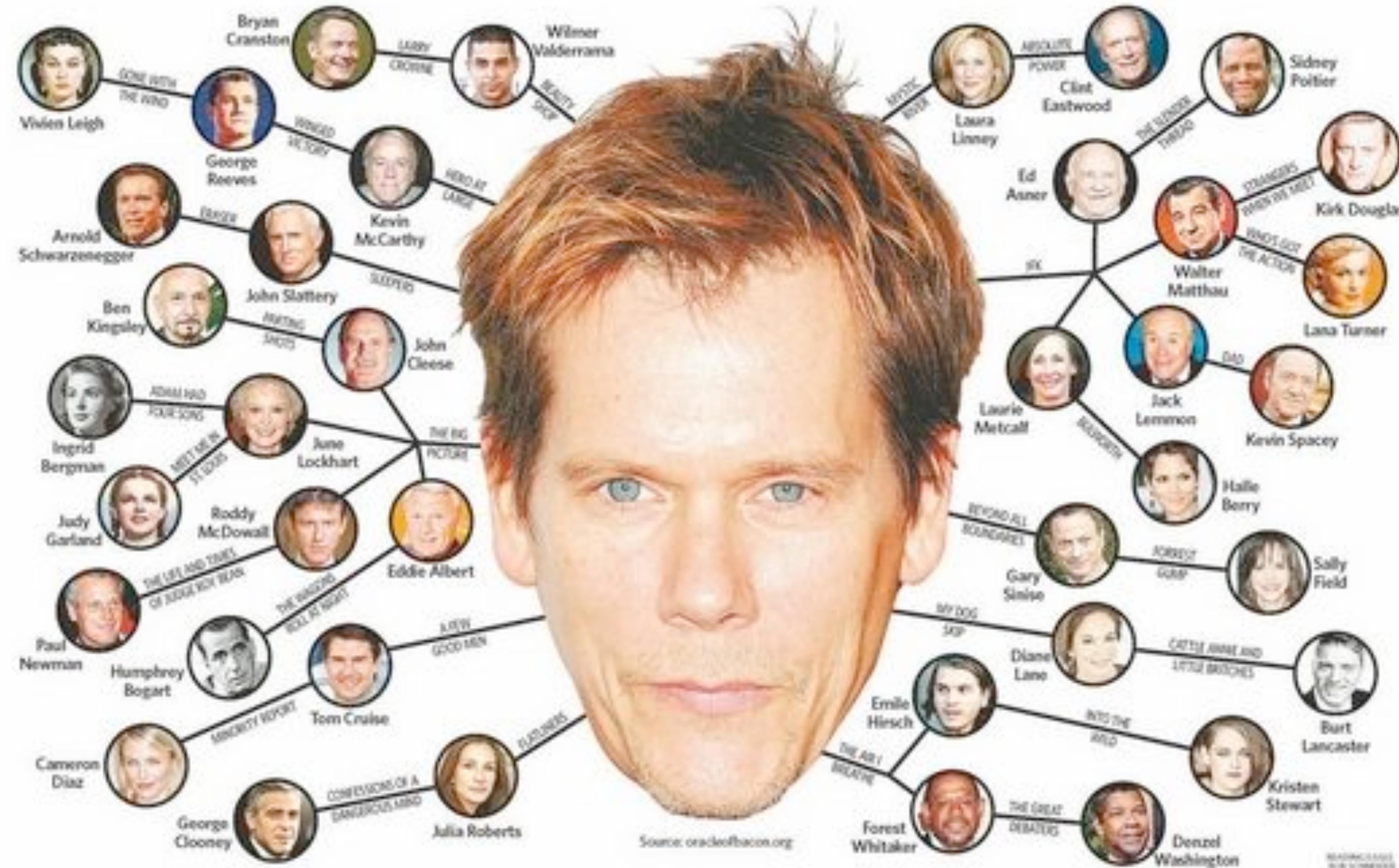
Wild things



A few good
men



Six degrees of Kevin Bacon: Everyone is connected with Kevin Bacon, although he is not the most famous actor



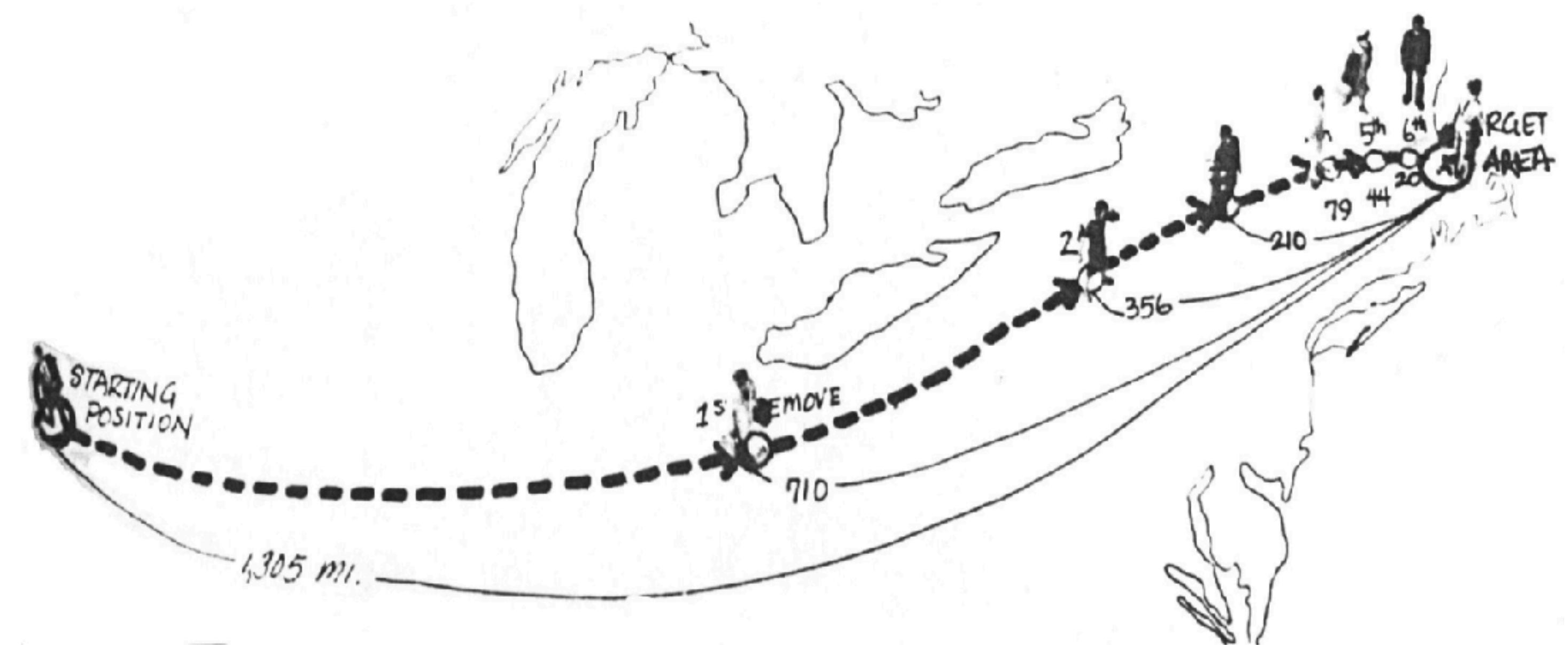
Six degrees of Erdős

MR Erdos Number = 5

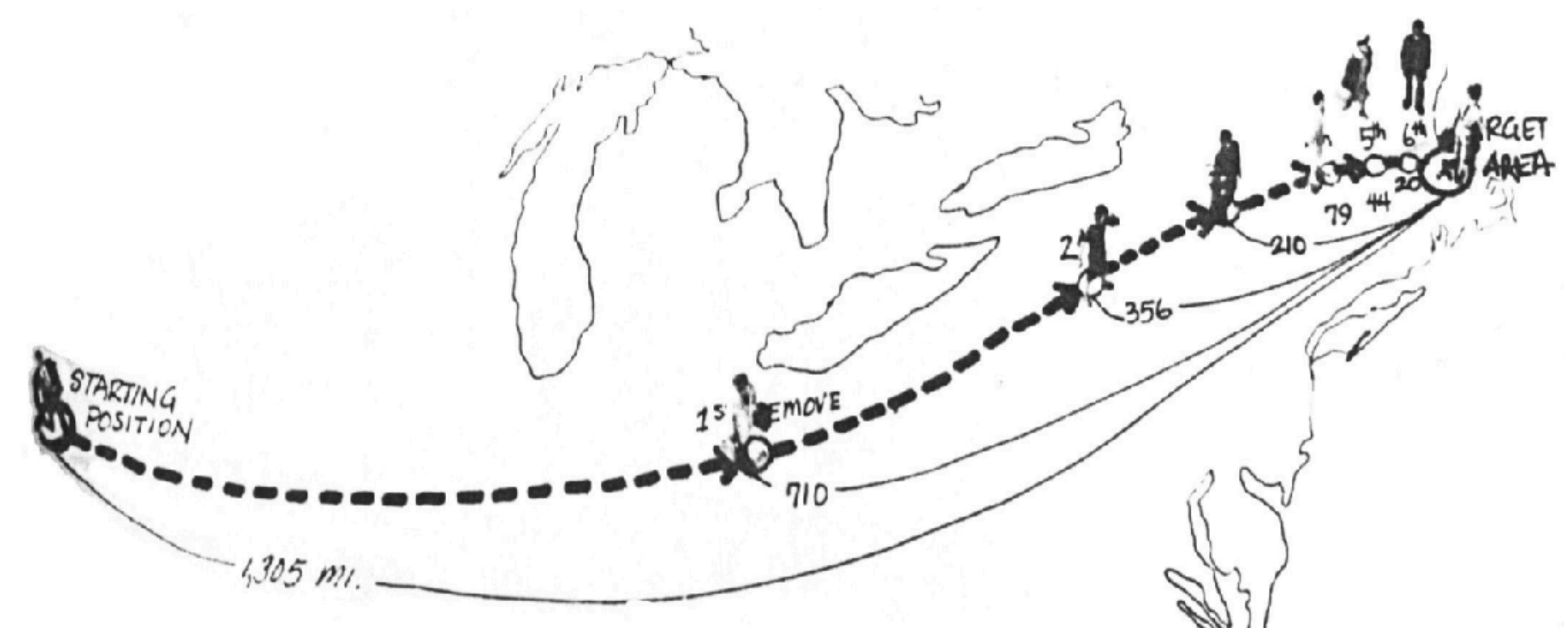
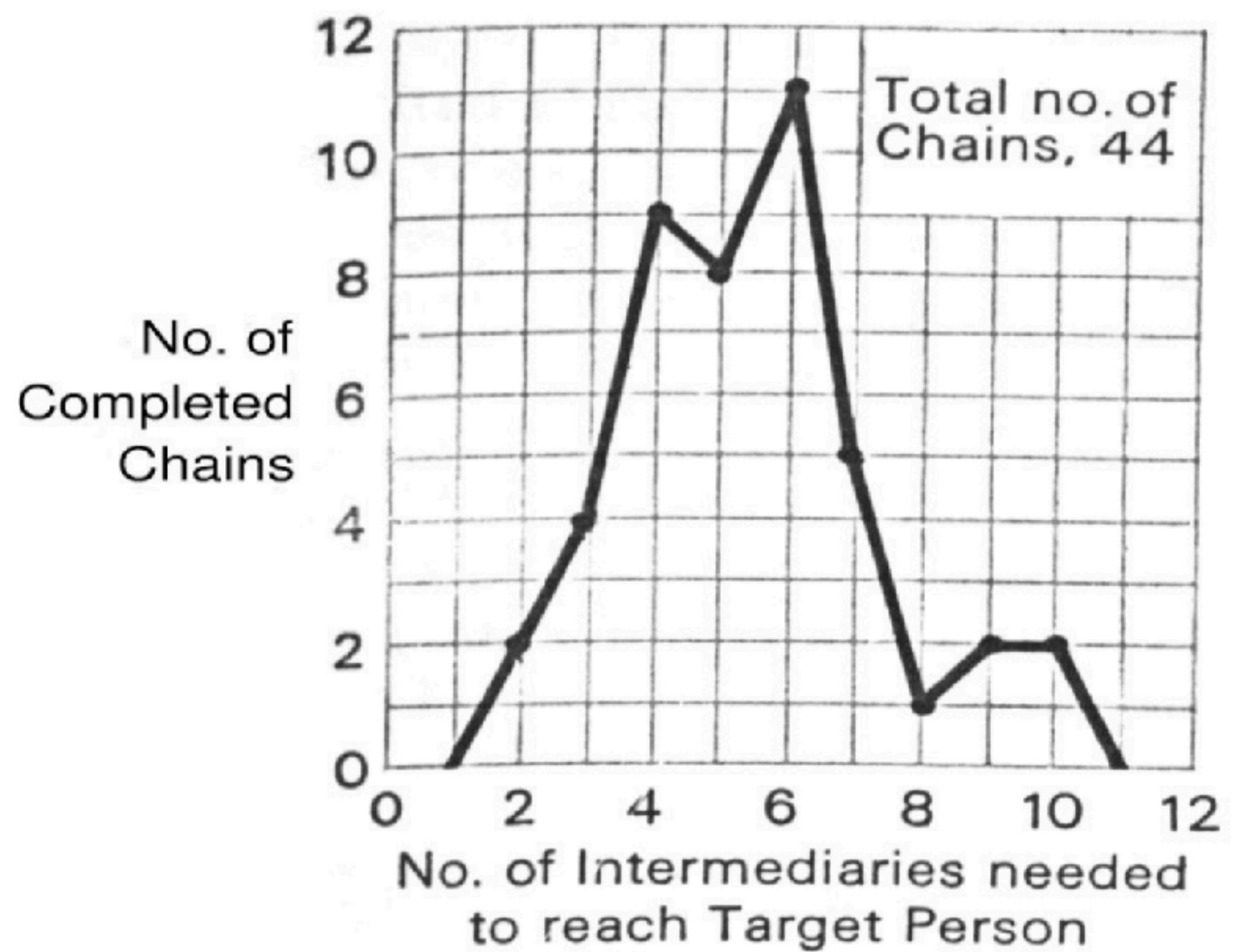
Michael Szell	coauthored with	Stefan Thurner ¹	MR2969349
Stefan Thurner ¹	coauthored with	Murray Gell-Mann	MR2793803
Murray Gell-Mann	coauthored with	Sheldon Lee Glashow	MR0134224
Sheldon Lee Glashow	coauthored with	Daniel J. Kleitman	MR0172630
Daniel J. Kleitman	coauthored with	Paul Erdős ¹	MR0228375



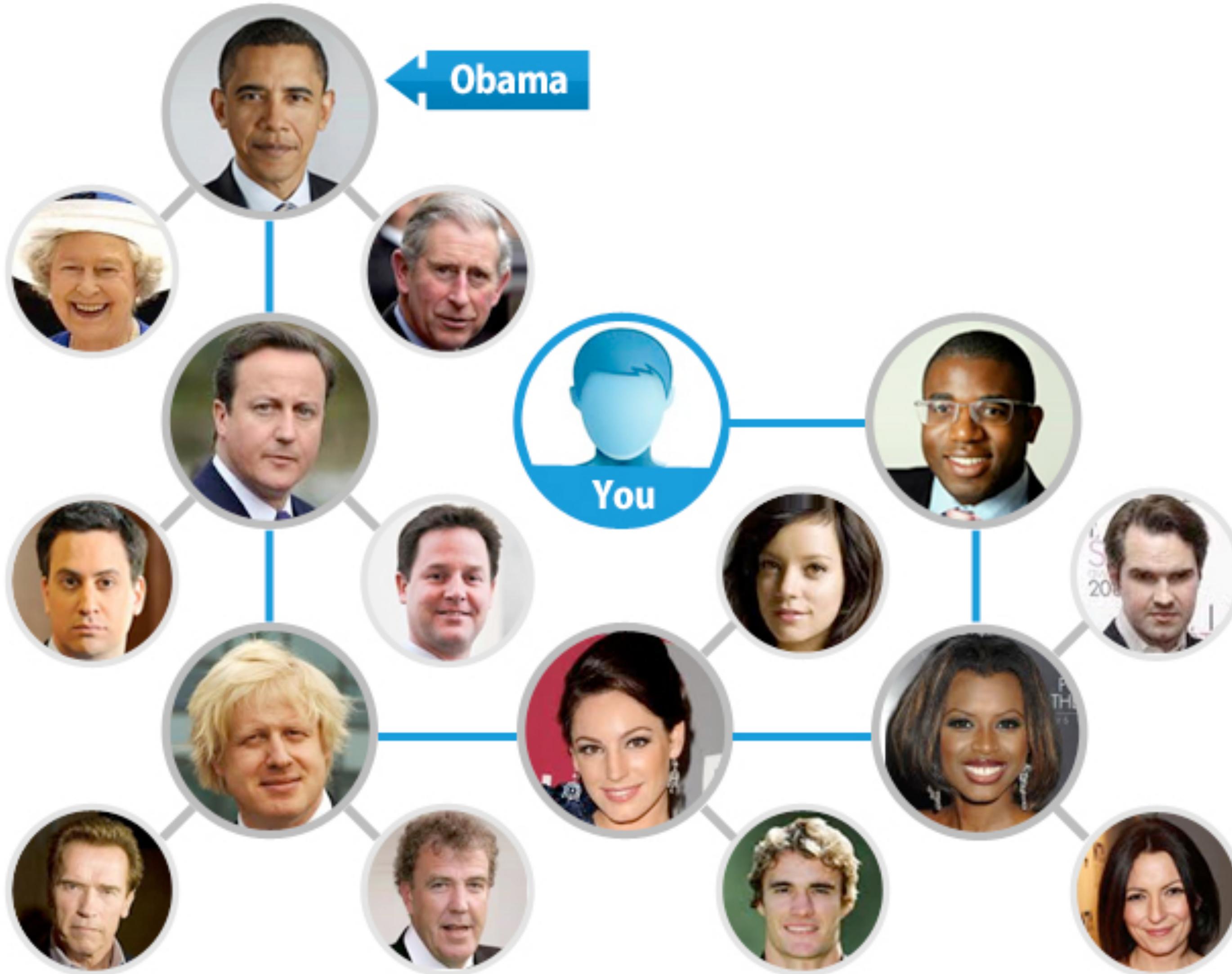
Stanley Milgram, 1967: Deliver a letter to somebody you don't know



The letters which reached their destination had a median degree of separation of 6: **The world is small**



Six degrees of separation means you are connected to almost any person in the world through 6 steps



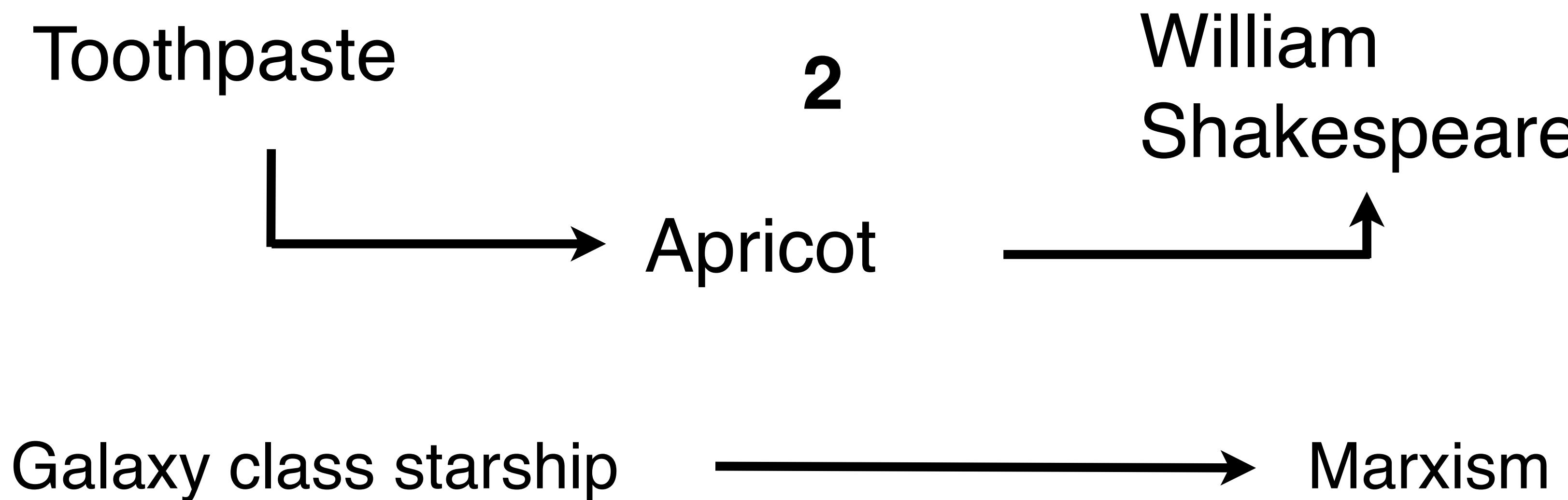
Six degrees also applies to knowledge (Wikipedia)

Toothpaste → William
Shakespeare

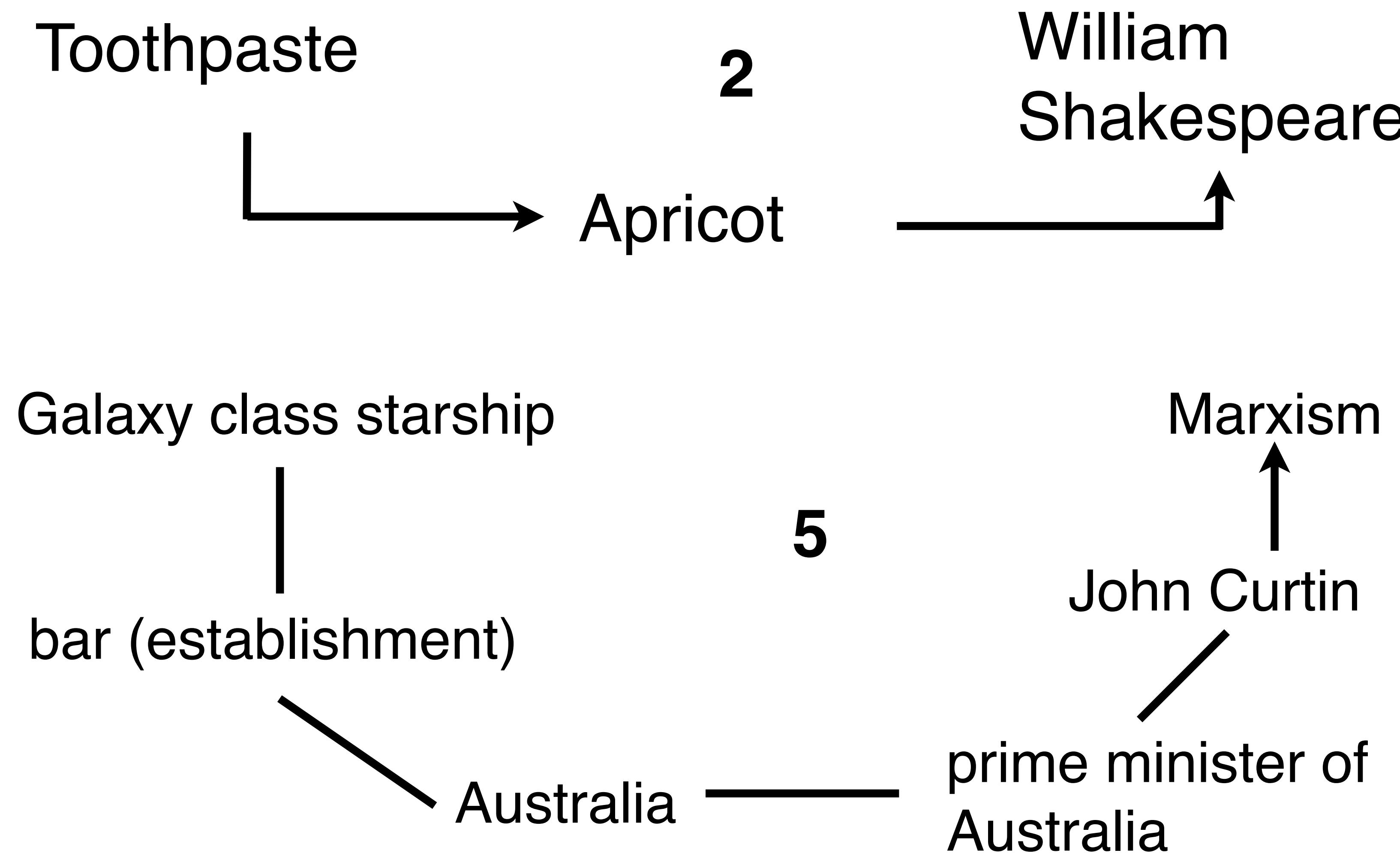
Six degrees also applies to knowledge (Wikipedia)



Six degrees also applies to knowledge (Wikipedia)



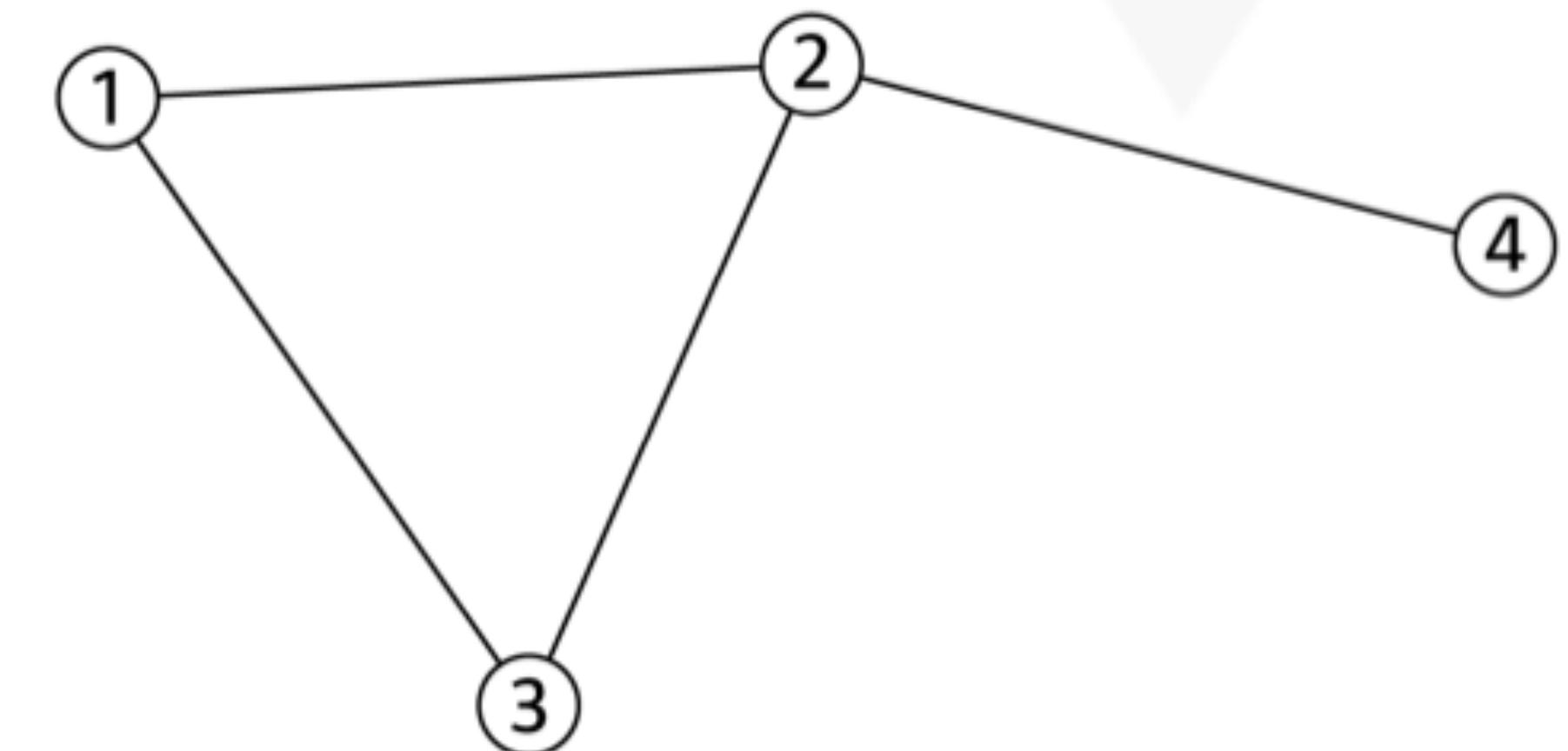
Six degrees also applies to knowledge (Wikipedia)



To formalize this idea, we need some definitions:

Walk: A sequence of neighboring nodes

$$\{n_1, n_2, n_1, n_3, n_2, n_4\}$$



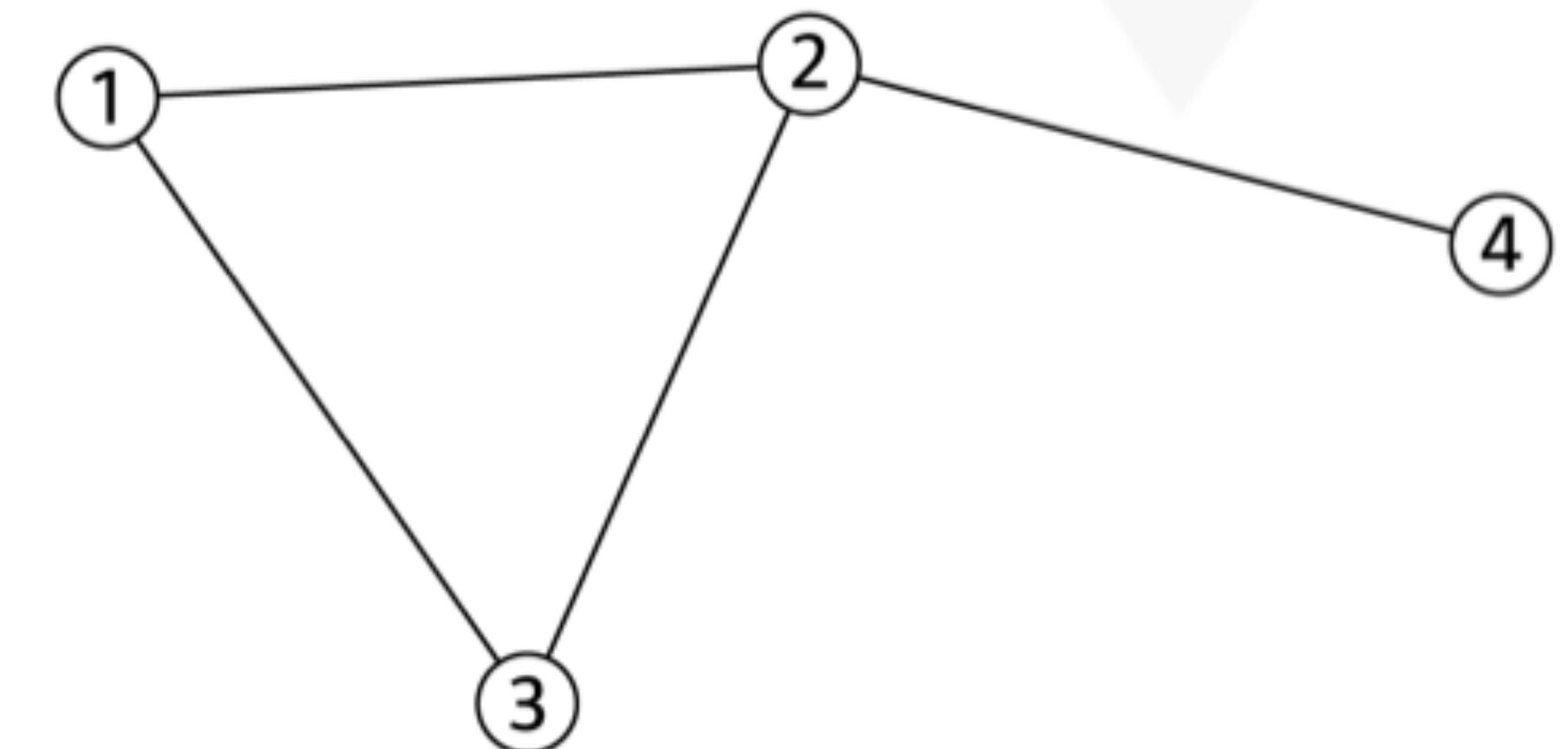
To formalize this idea, we need some definitions:

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Path: A walk where no node is repeated

$$\{n_1, n_3, n_2, n_4\}$$



To formalize this idea, we need some definitions:

Walk: A sequence of neighboring nodes

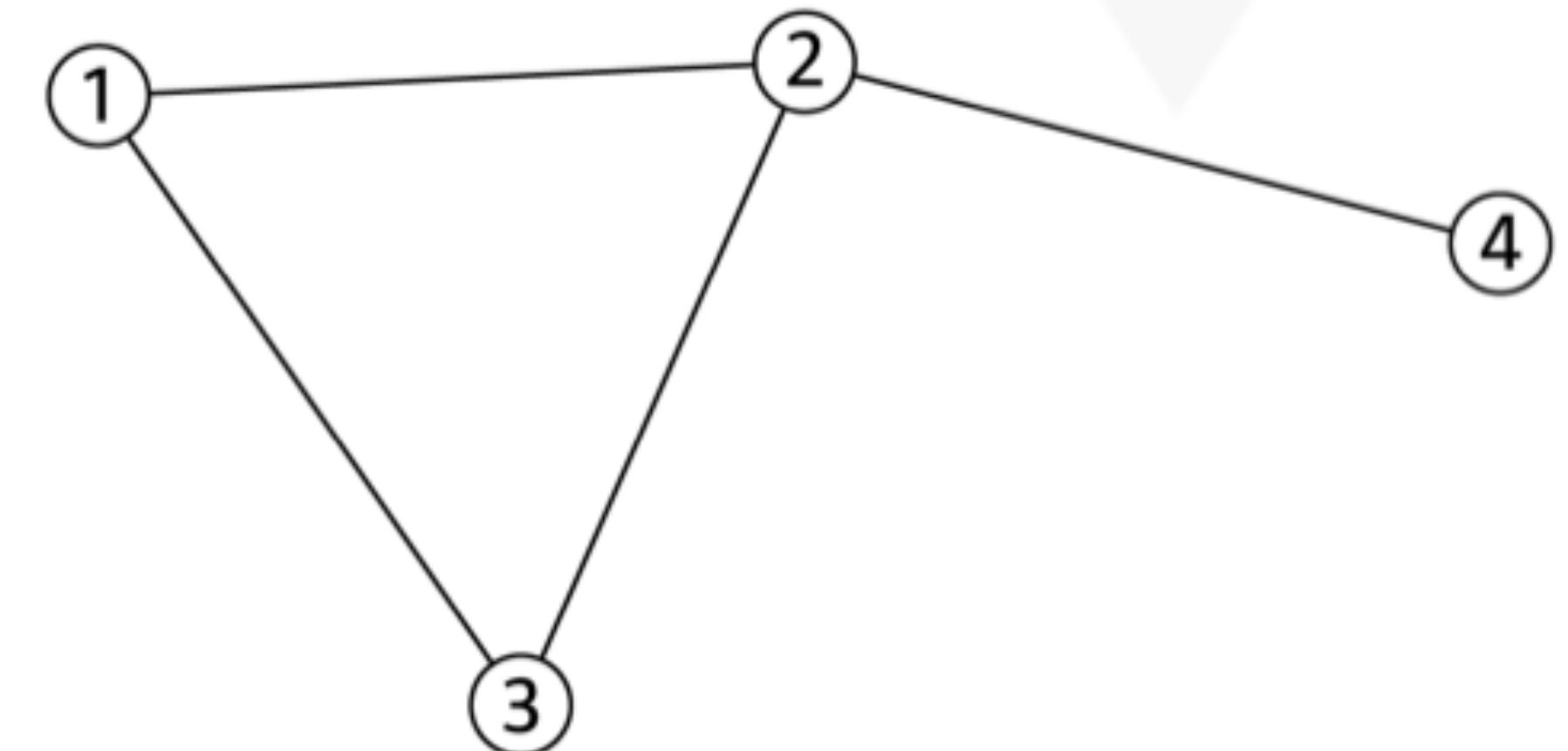
$$\{n_1, n_2, n_1, n_3, n_2, n_4\}$$

Path: A walk where no node is repeated

$$\{n_1, n_3, n_2, n_4\}$$

Shortest path: A path of minimal length

$$\{n_1, n_2, n_4\}$$



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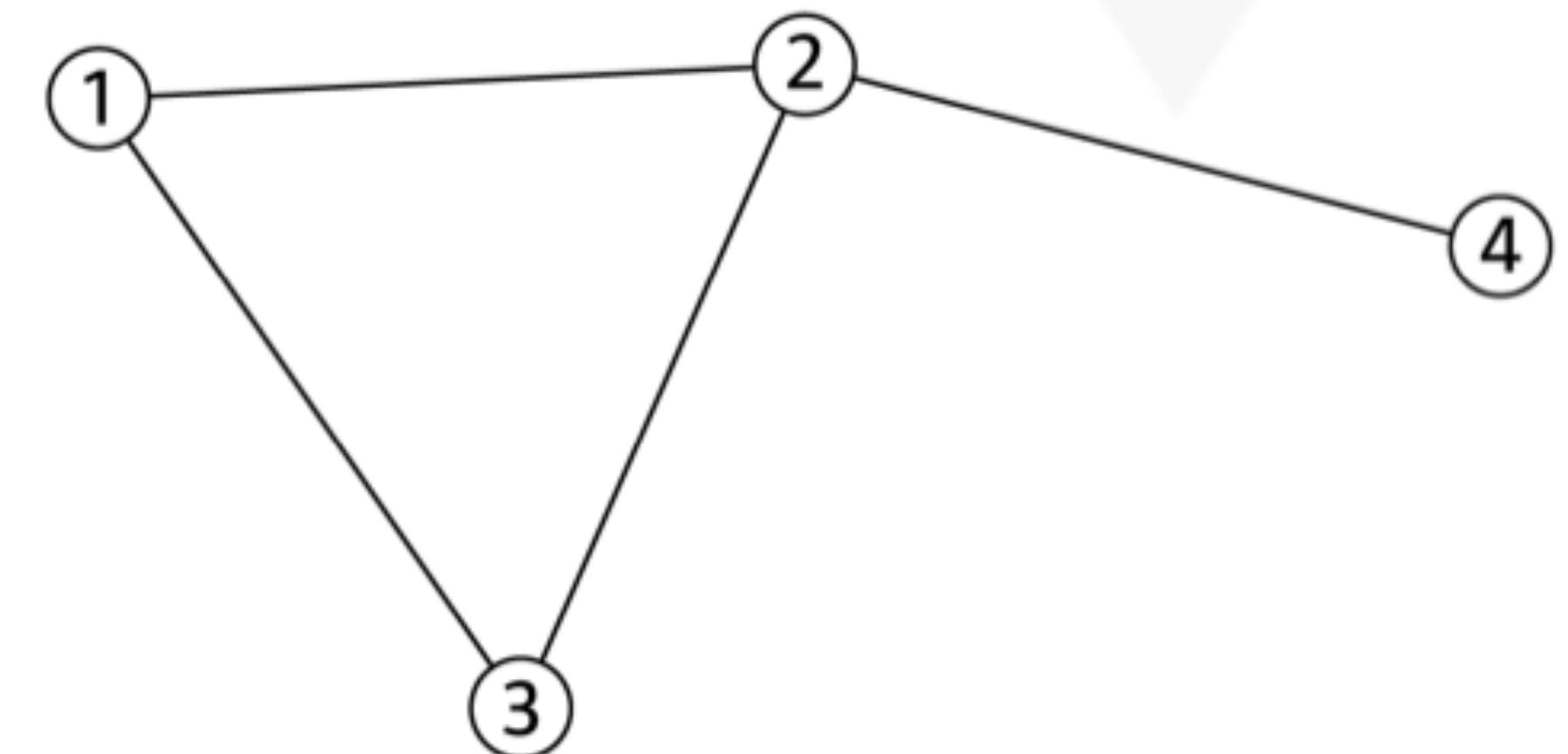
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Shortest path: A path of minimal length

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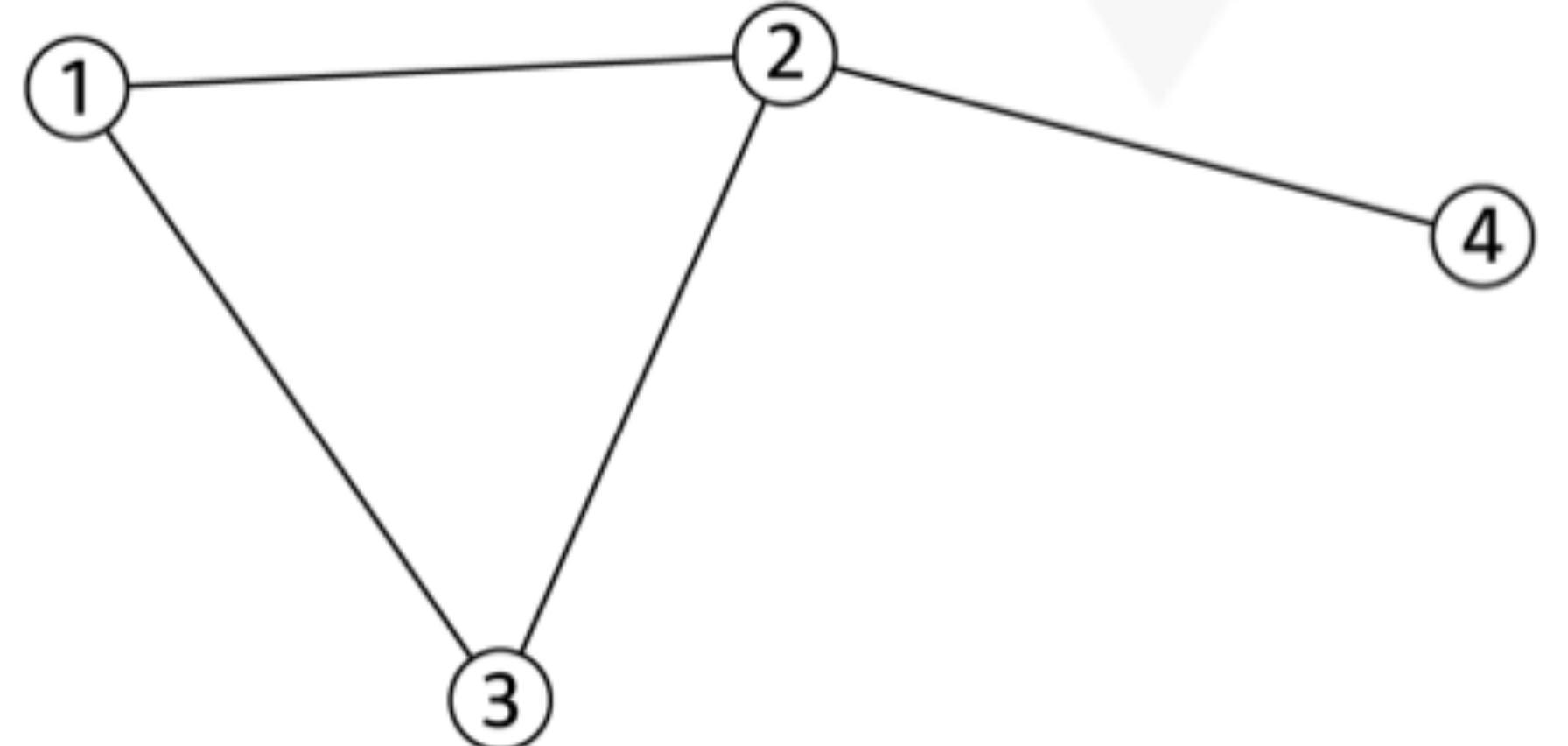
Graph distance: Length of shortest path

$$d_{1,4} = 2$$



The **average path length** ℓ is the mean graph distance over all pairs of nodes

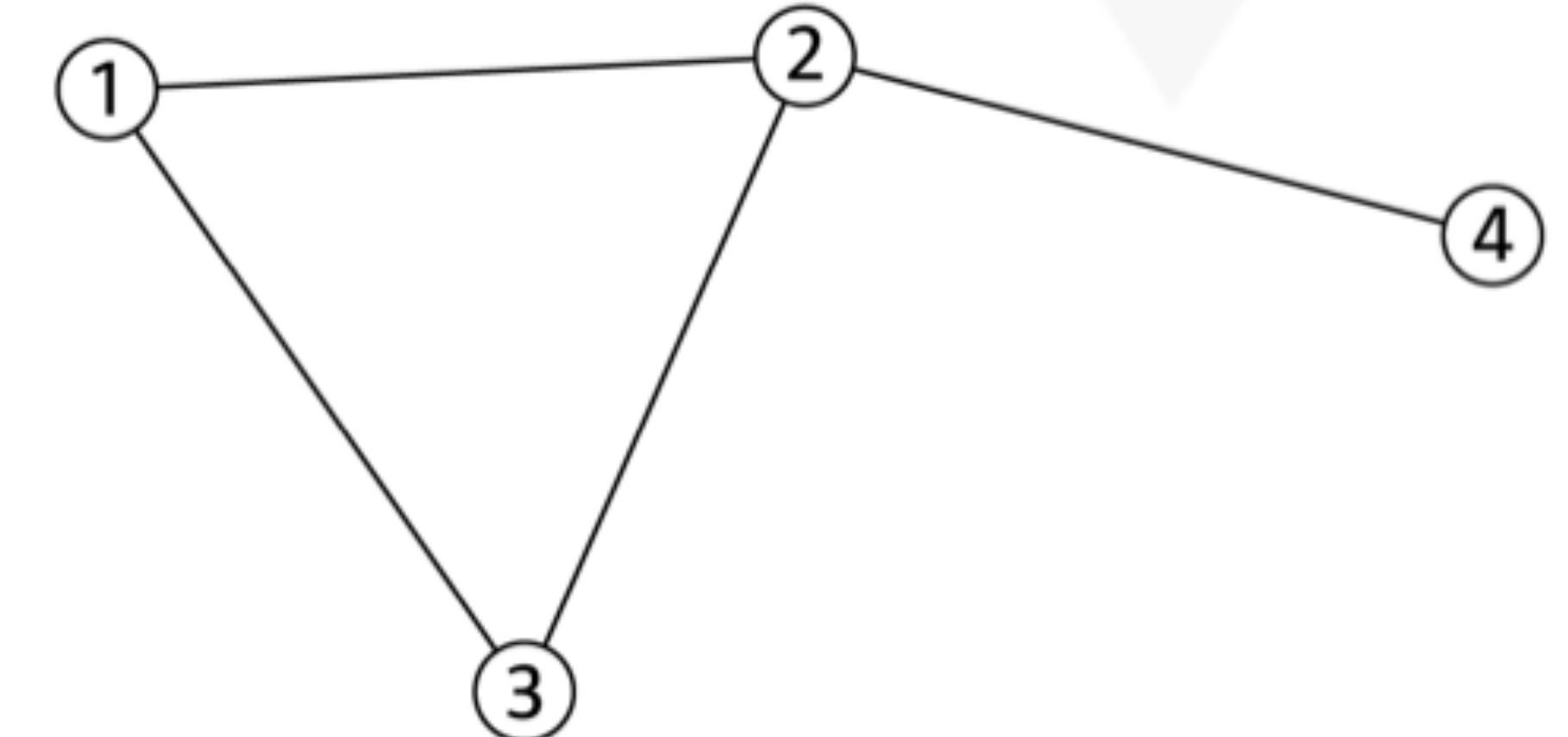
$$\ell = \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{j=1}^N d_{i,j}$$



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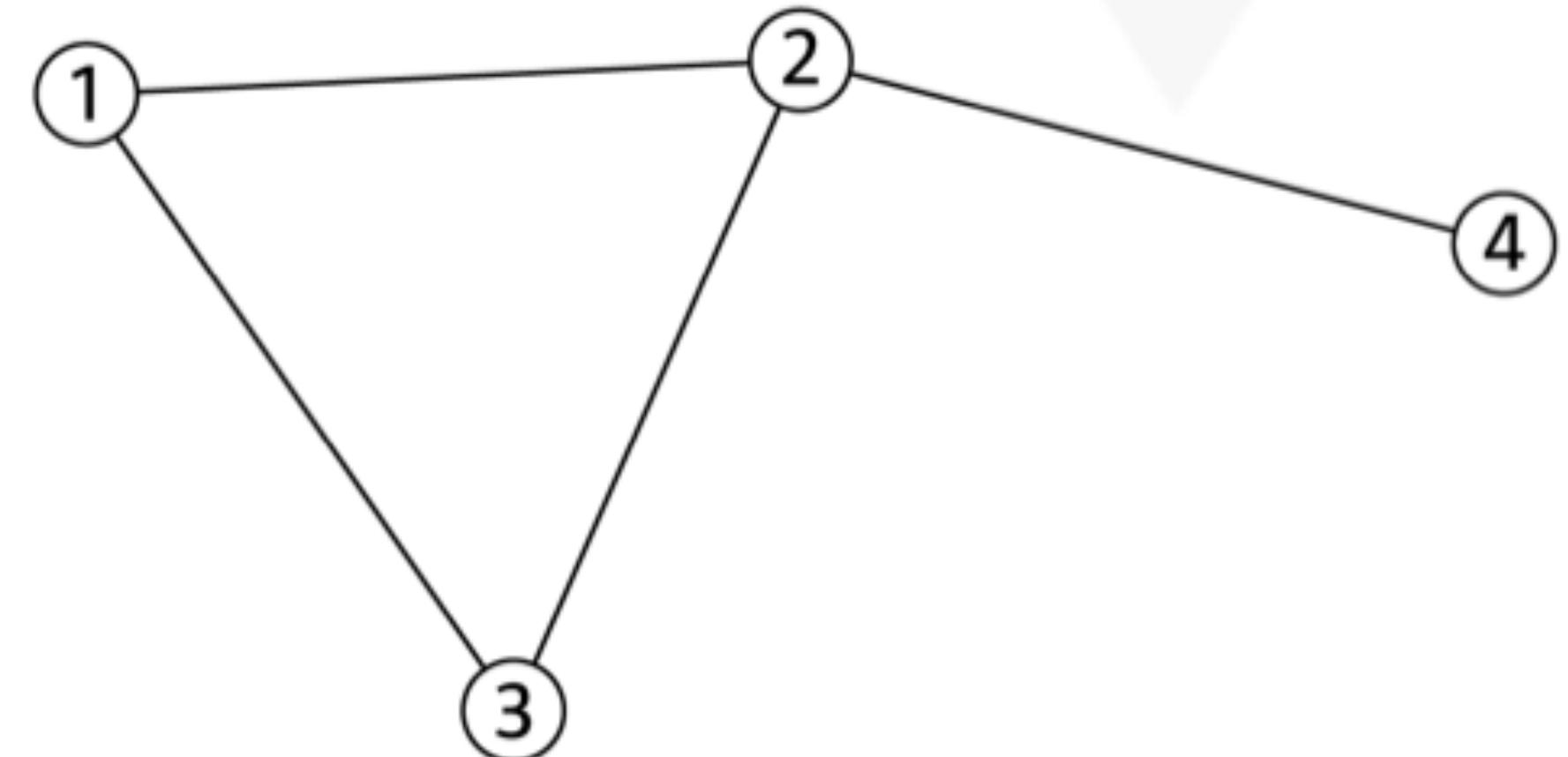
$$\ell = \frac{4 + 3 + 4 + 5}{4(4 - 1)} = \frac{4}{3}$$



The diameter D is the maximum length of shortest paths

$$\ell = \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{j=1}^N d_{i,j}$$

$$\ell = \frac{4 + 3 + 4 + 5}{4(4 - 1)} = \frac{4}{3}$$

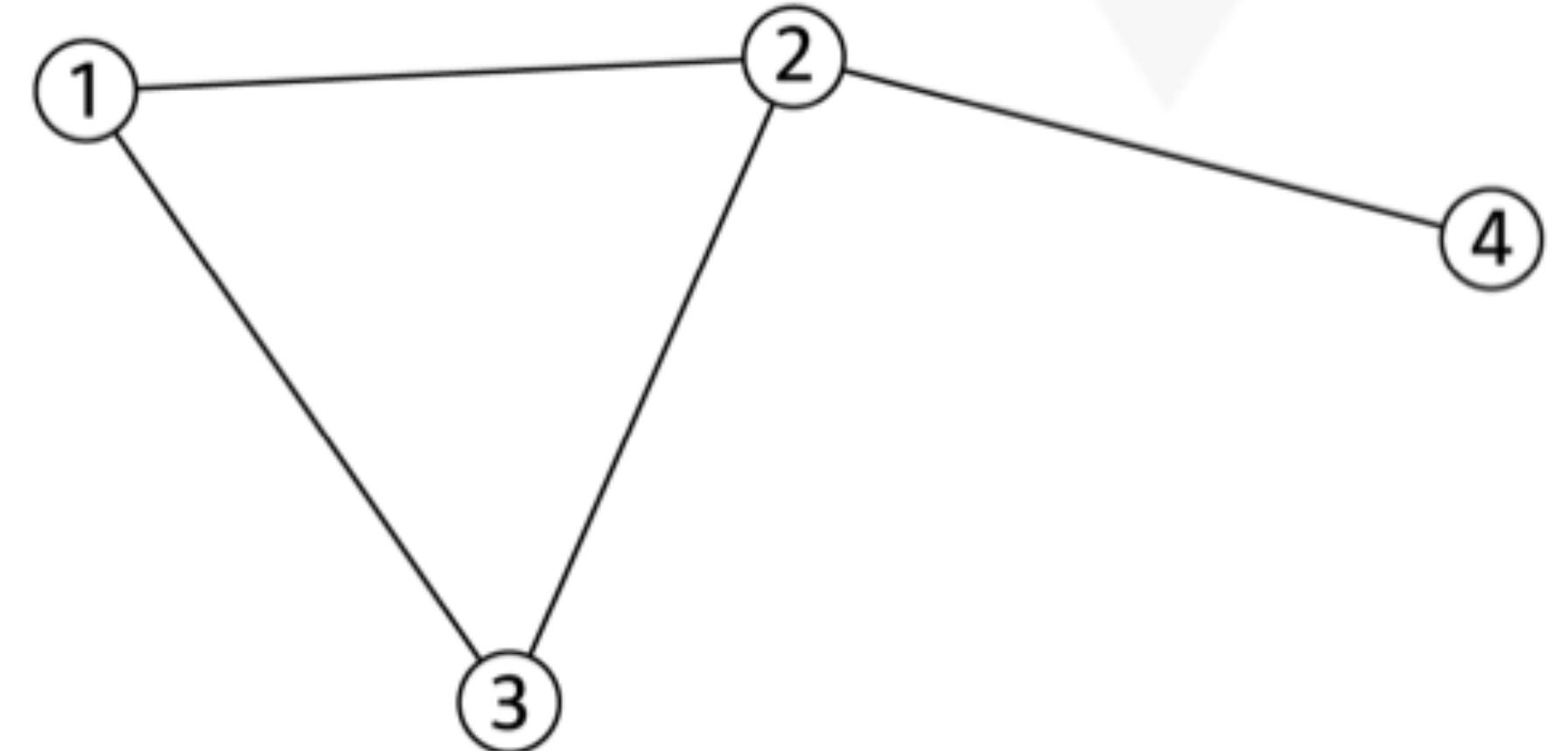


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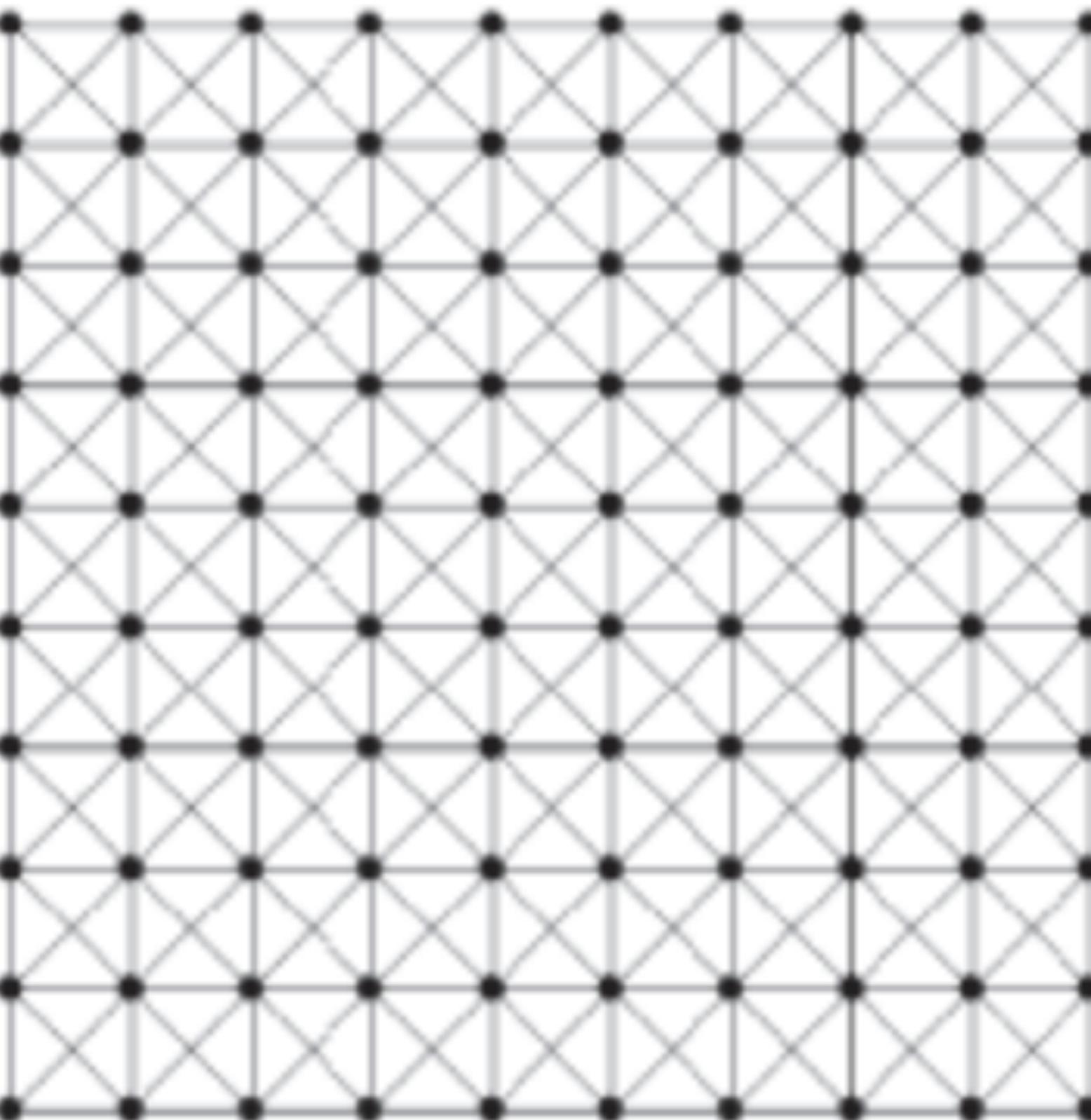
$$\ell = \frac{4 + 3 + 4 + 5}{4(4 - 1)} = \frac{4}{3}$$

$$D = 2$$



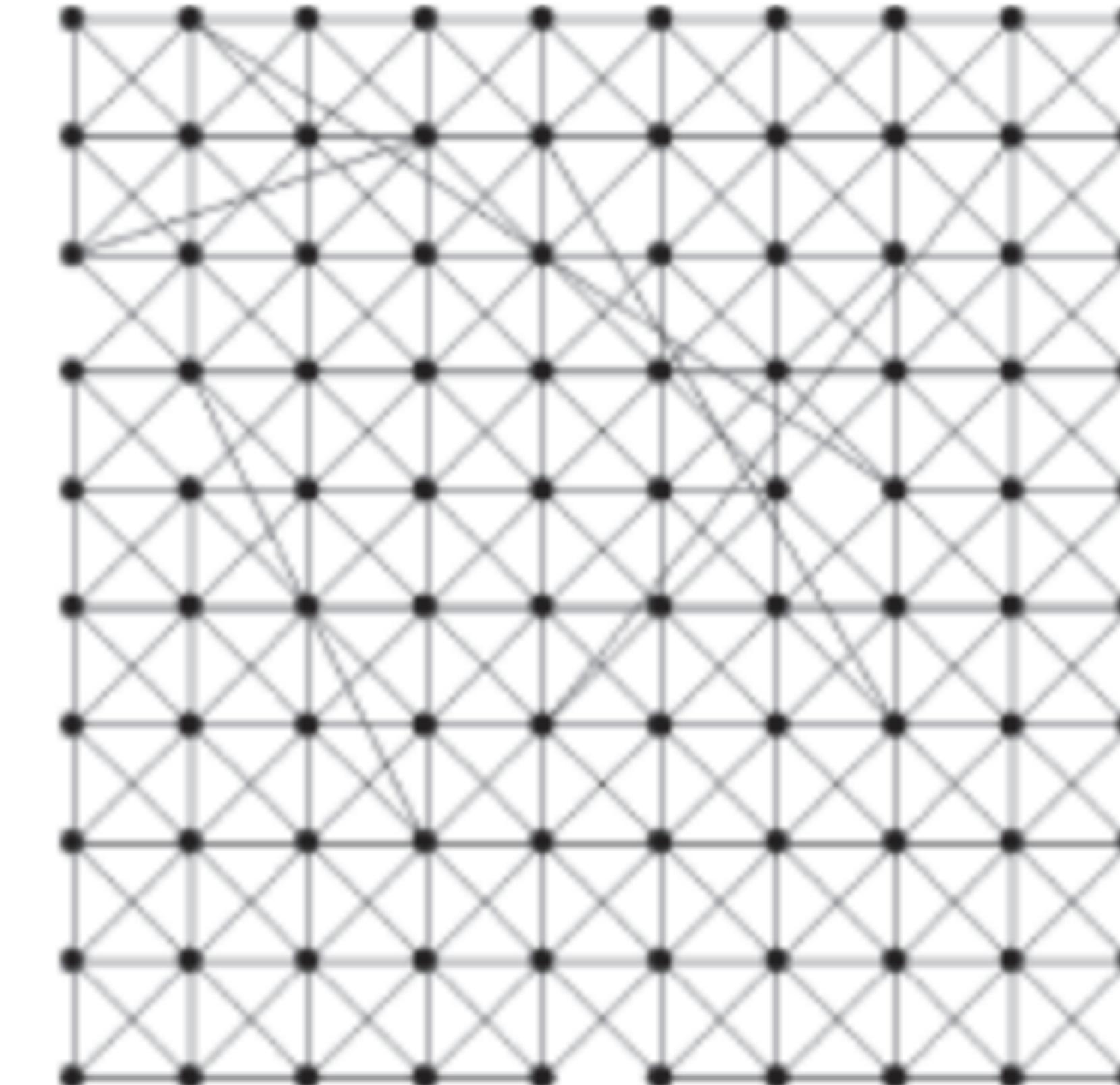
A small world network has short average path length

$$\ell \sim N$$



Regular network $\phi=0$

$$\ell \sim \log N$$



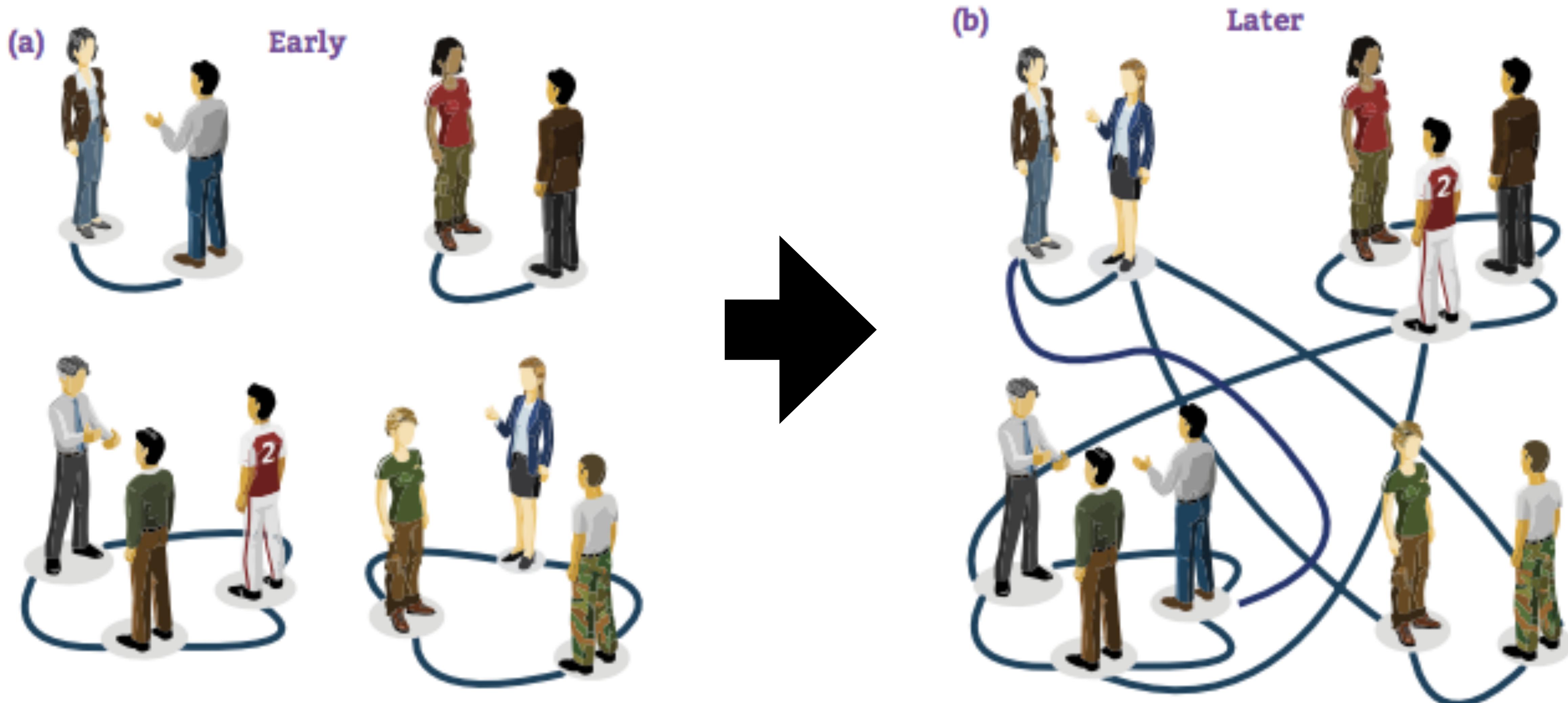
Small-world
network $\phi=0.01$

Organizing principles of networks

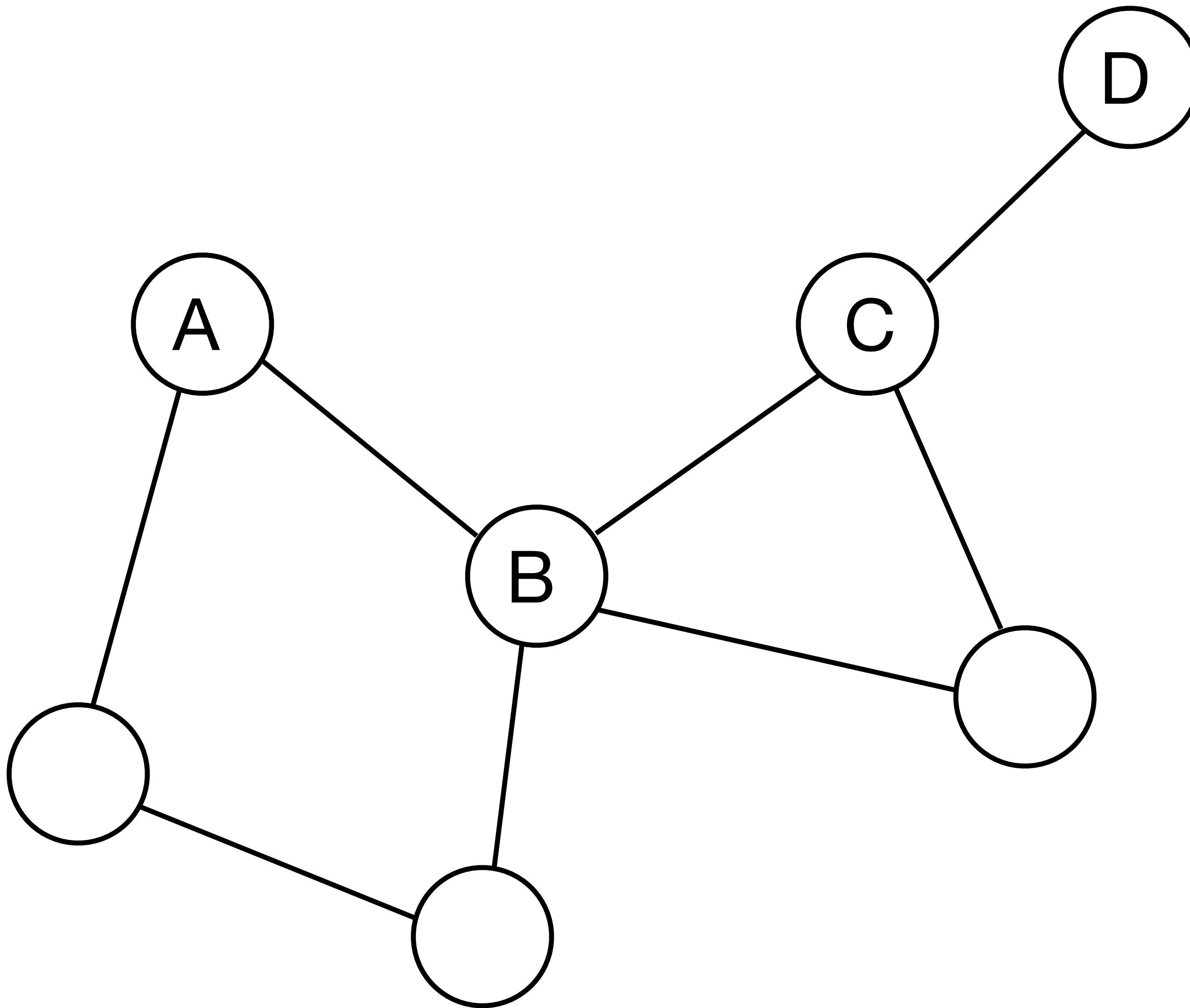
Many networks are:

- 1) Heavy-tailed
- 2) Sparse
- 3) Small-world
- 4) Clustered

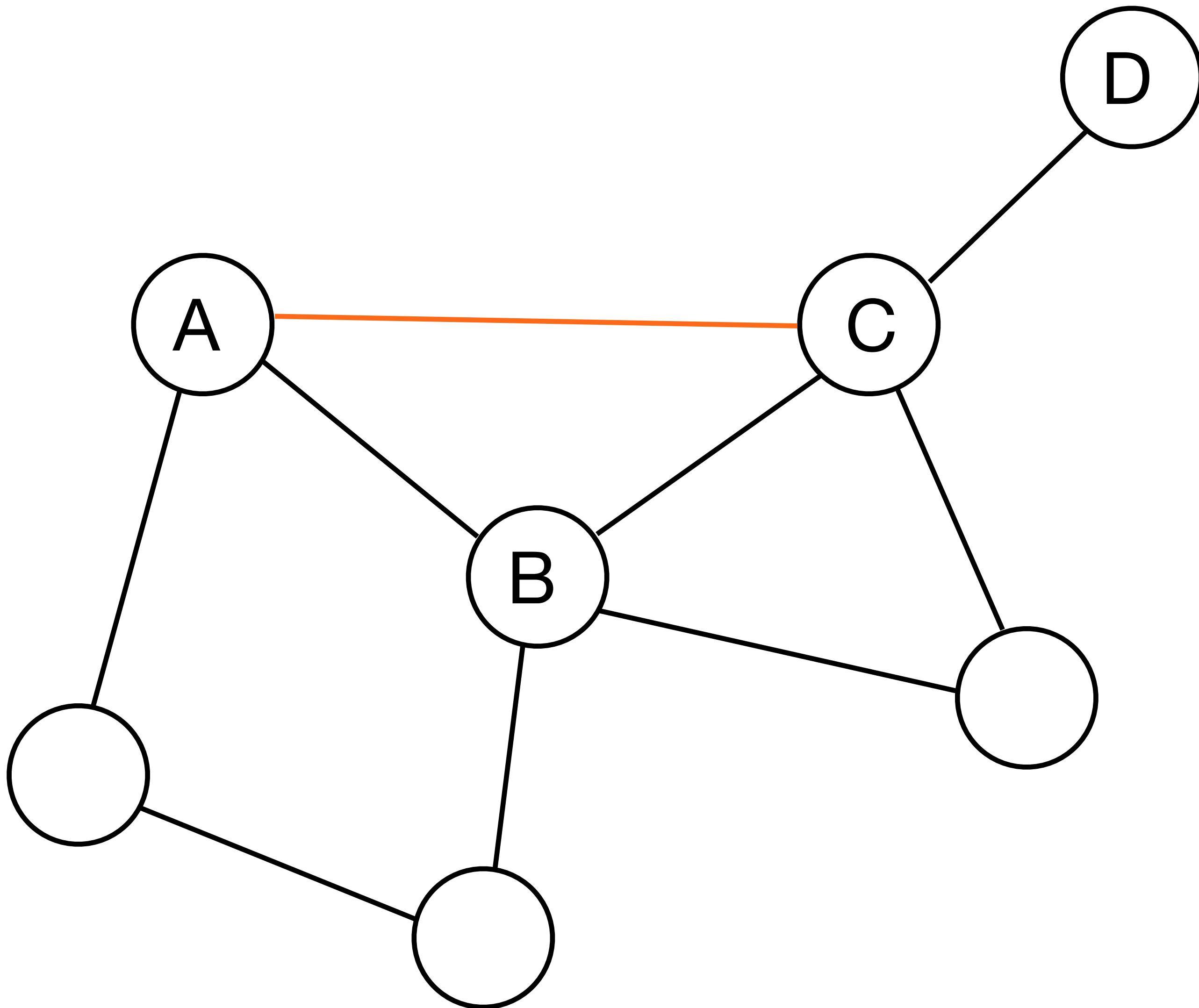
In a cocktail party you introduce each other to new people



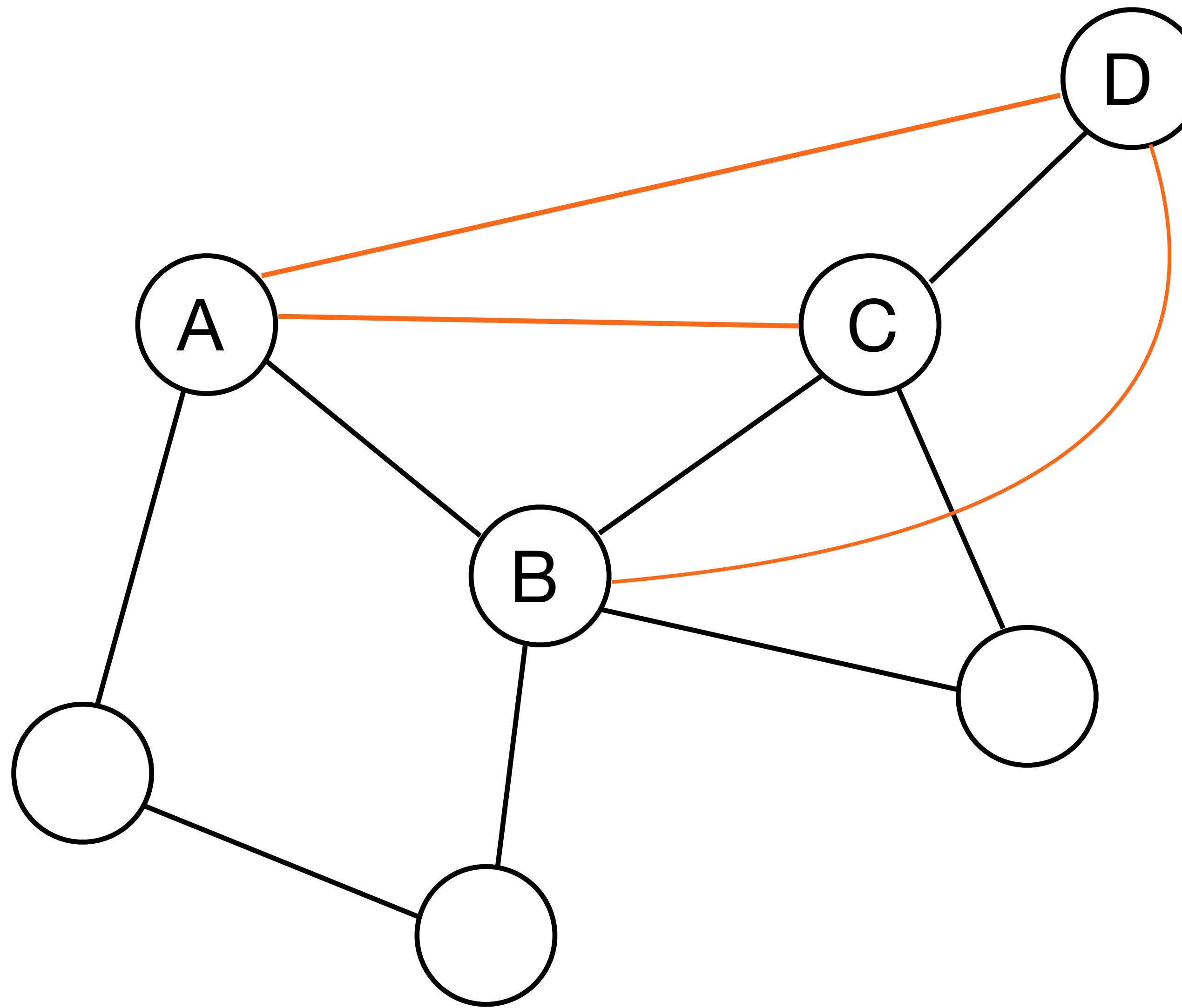
Who are more likely to connect: A–C or A–D?



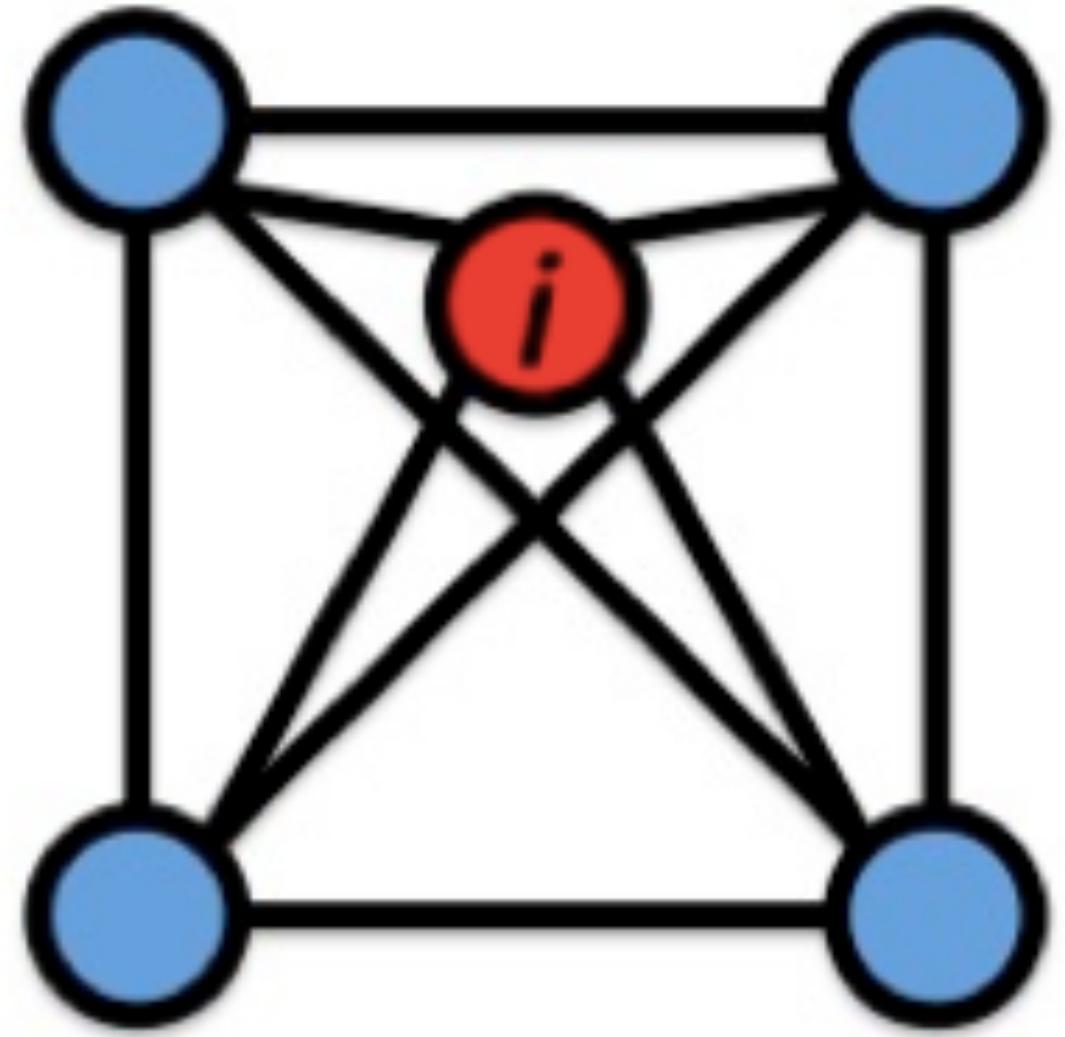
A and C already have a common friend, B



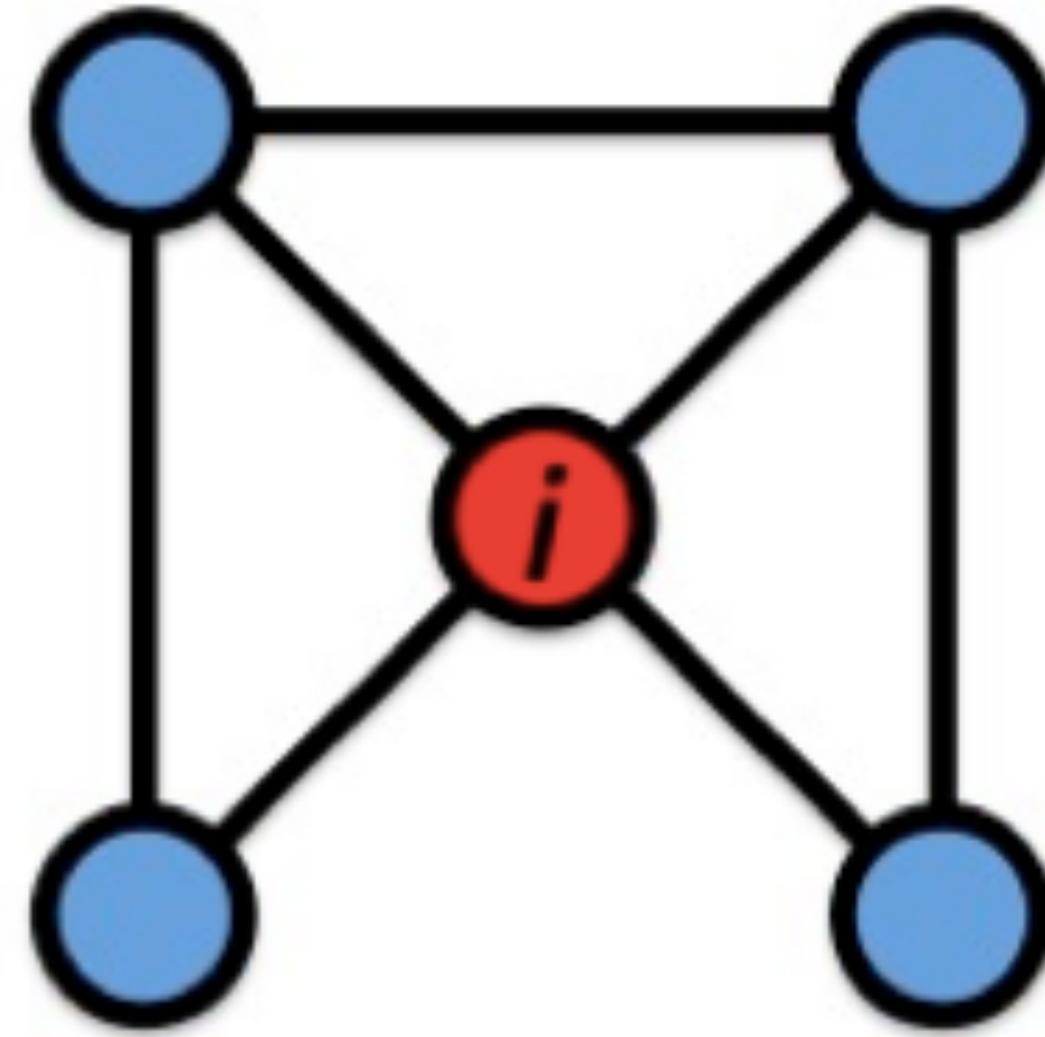
Closing open triangles is called **triadic closure**



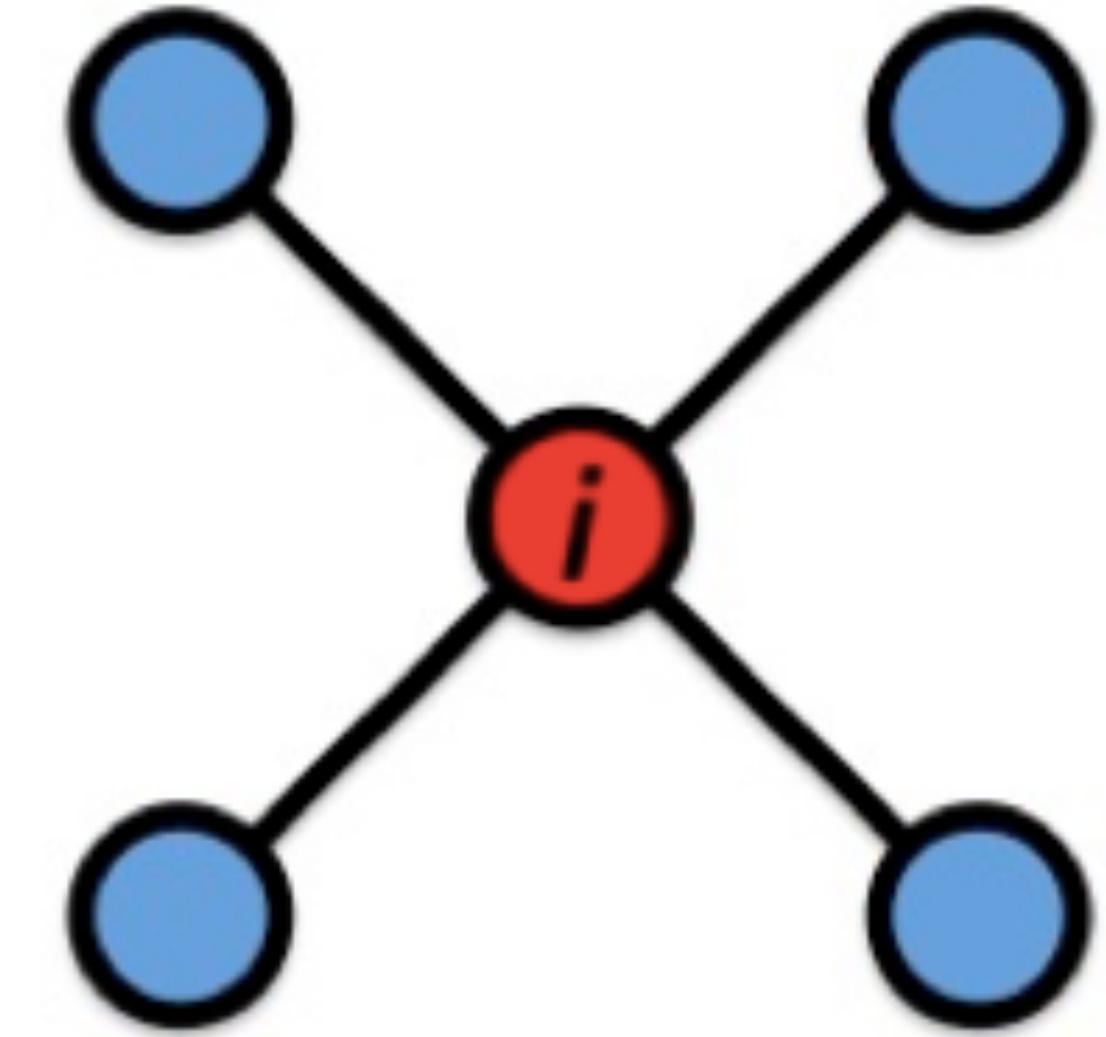
The clustering coefficient c_i of a node i measures the fraction of its neighbor pairs who are linked



$$c_i = 1$$



$$c_i = 1/2$$



$$c_i = 0$$

By definition, $0 \leq c_i \leq 1$

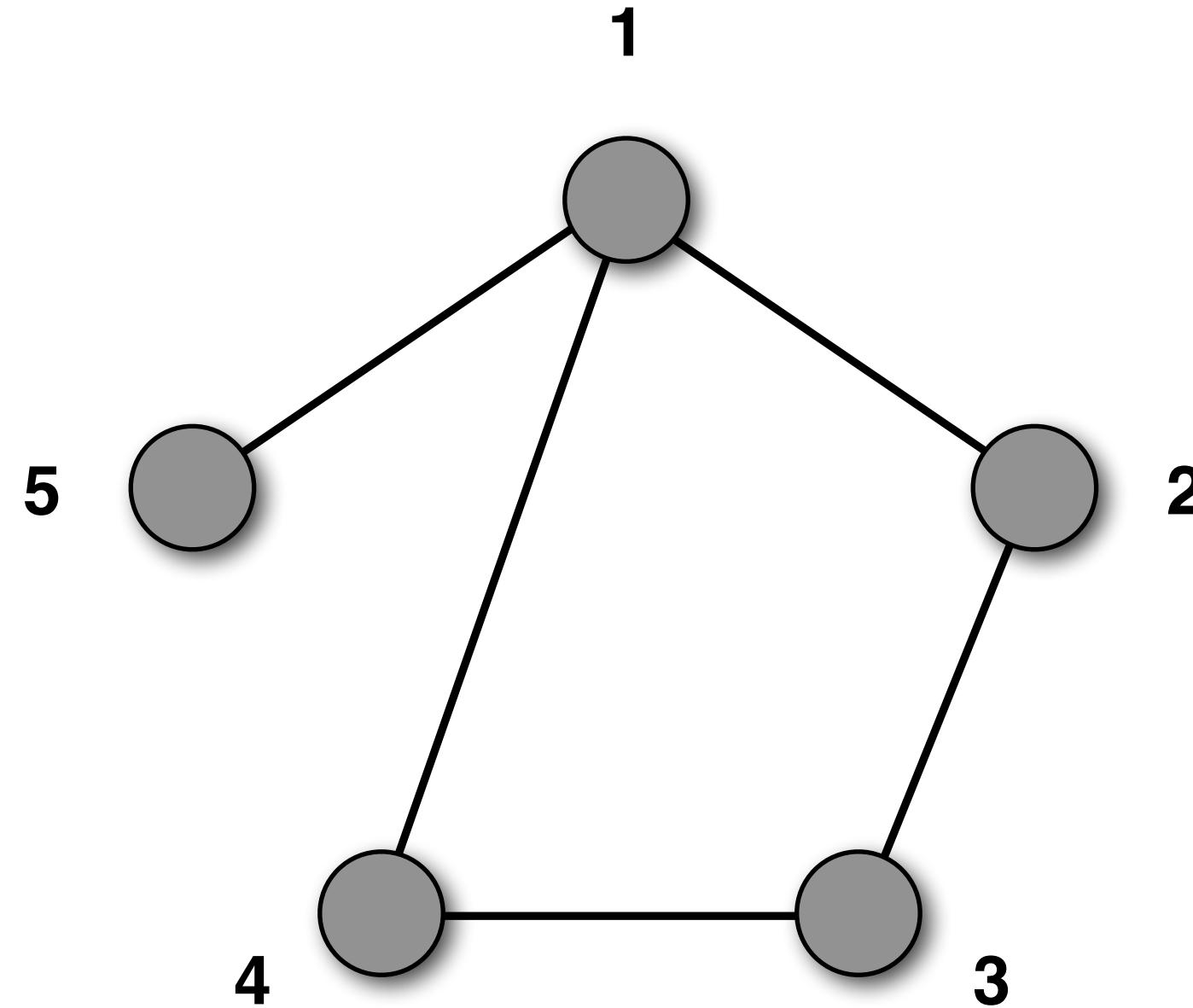
Organizing principles of networks

Many networks are:

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Network data structures

The **adjacency matrix** A stores all possible connections
Realized connections get a 1

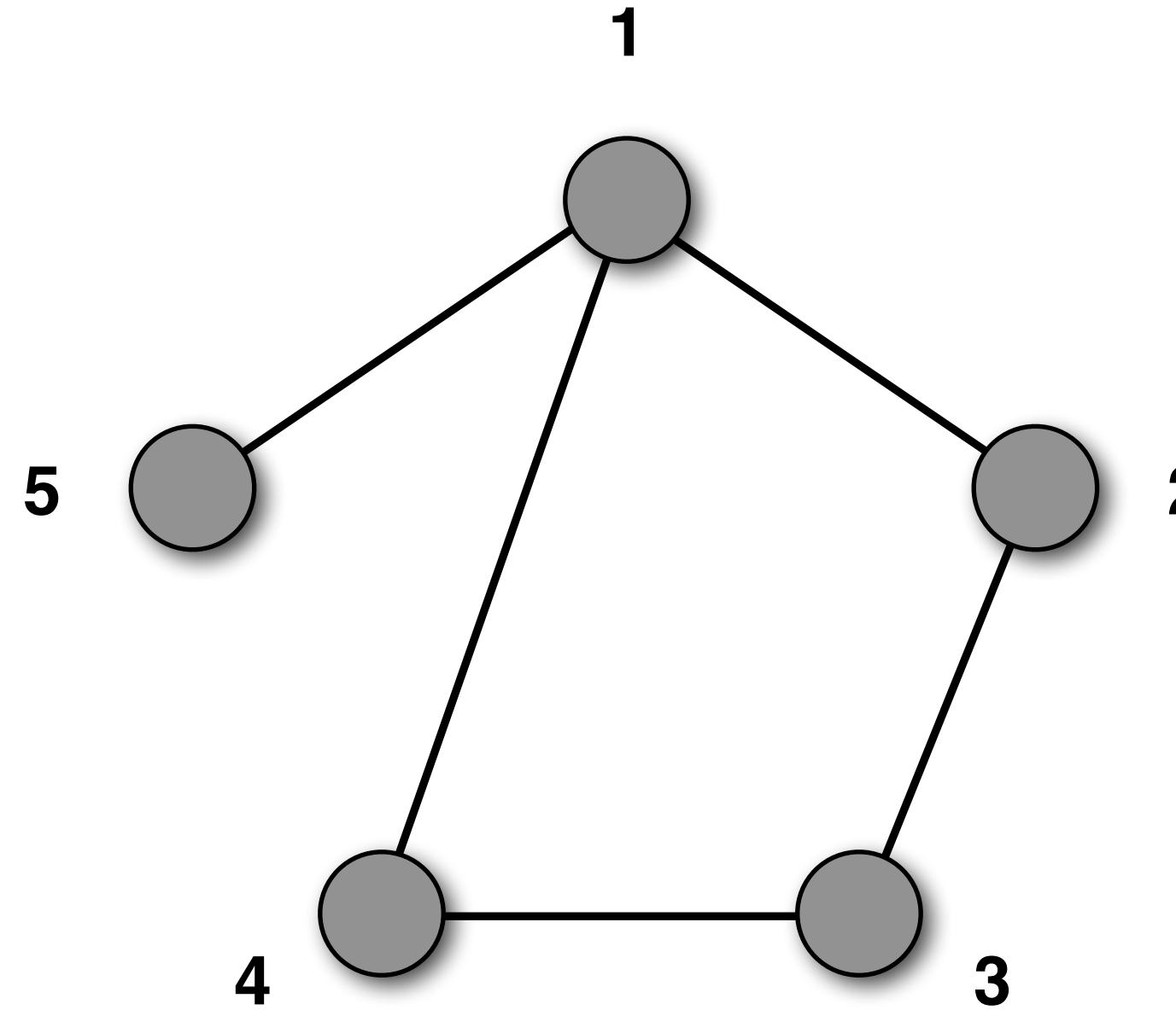


$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_{N1} & \dots & \dots & A_{NN} \end{pmatrix}$$

$A_{ij} = A_{ji} = 1$ If there is a link between
node i and node j

$A_{ij} = A_{ji} = 0$ If node i and node j are
not connected

The **adjacency matrix** A stores all possible connections
Realized connections get a 1



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$A_{ij} = A_{ji} = 1$ If there is a link between
node i and node j

$A_{ij} = A_{ji} = 0$ If node i and node j are
not connected

The **adjacency matrix** is good for dense networks, but not practical for real networks that are large and sparse



Easy to use analytical formulas

Easy to find/remove/add a link: $\mathcal{O}(1)$

Useful for dense networks



Needs a lot of memory: $\mathcal{O}(N^2)$

Inconvenient for many numerical calculations

The adjacency matrix is not practical for large networks



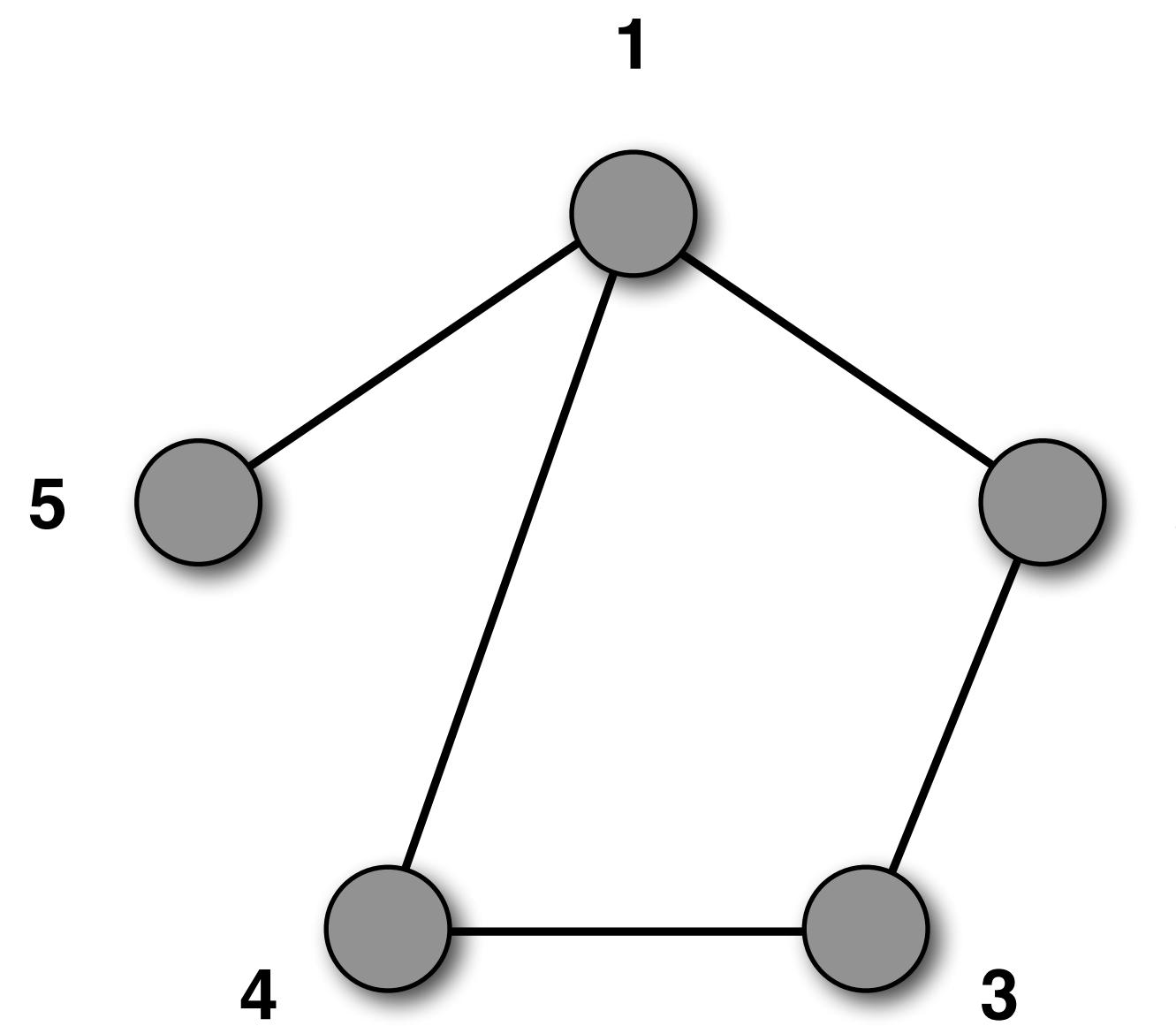
Needs a lot of memory: $\mathcal{O}(N^2)$

A matrix takes $4N^2$ bytes.

Assuming 10 GB = 10^{10} bytes of RAM, the largest possible network satisfies $4N^2 = 10^{10}$, which means $N = 50000$.

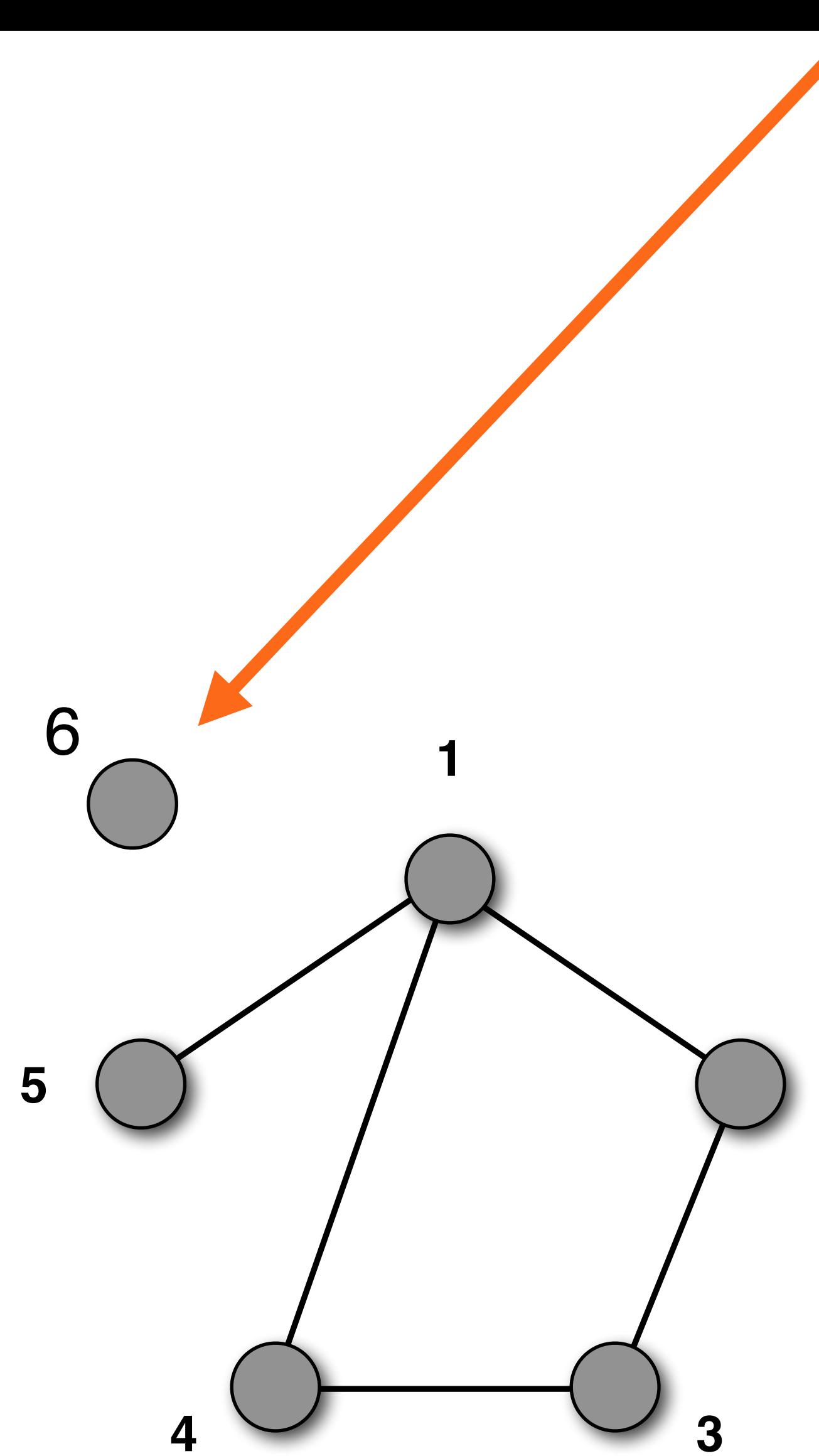
The edge list (I, J) stores the node IDs of connected links

$$I = \begin{pmatrix} \dots \\ i \\ \dots \end{pmatrix} \quad J = \begin{pmatrix} \dots \\ j \\ \dots \end{pmatrix} \quad \text{Only if } A_{ij} = 1$$



$$I = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad J = \begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 4 \end{pmatrix}$$

The edge list (I, J) is missing isolated nodes



$$I = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad J = \begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 4 \end{pmatrix}$$

The **edge list** needs less space, but has disadvantages for calculations



Needs less memory: $2L$

Convenient for data collection

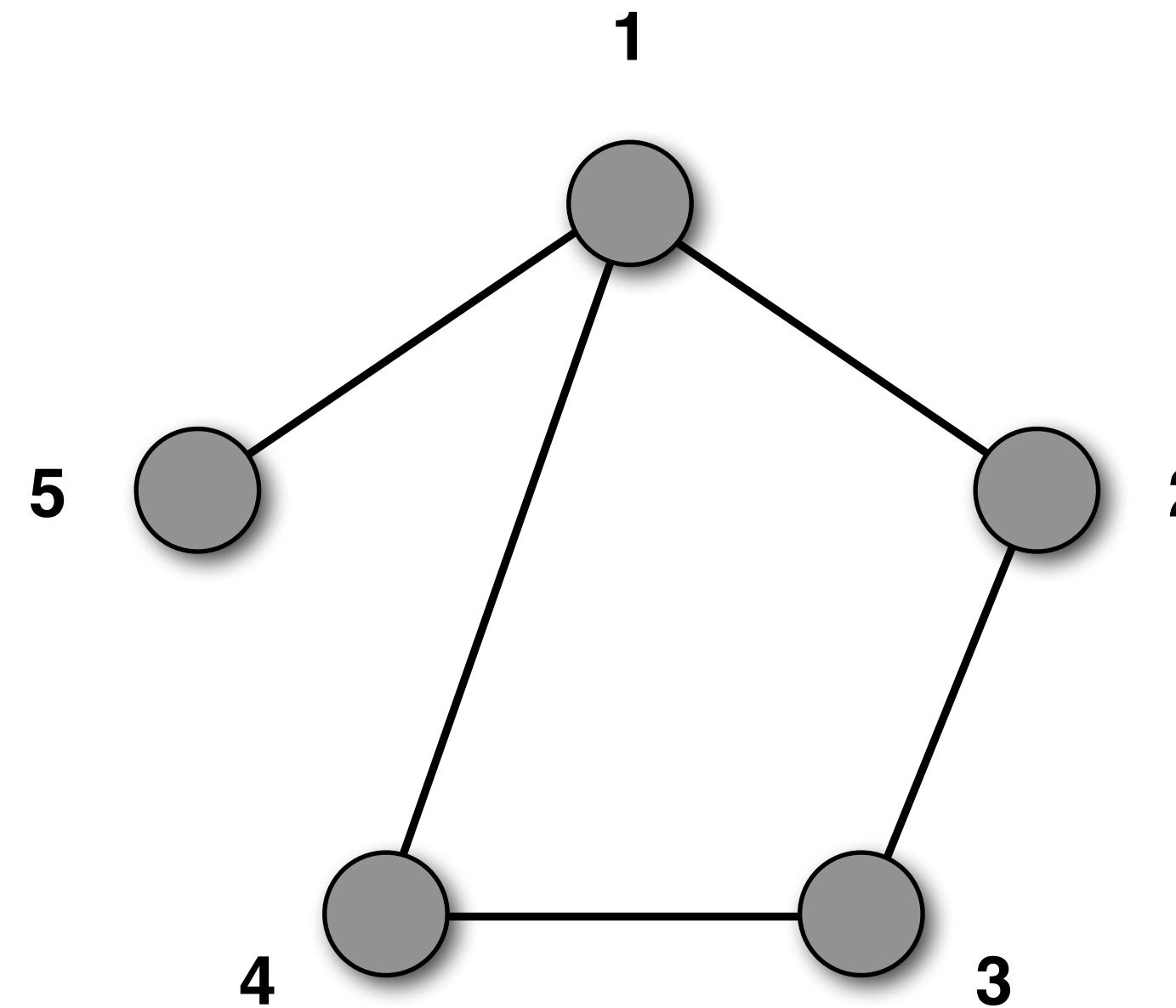
Convenient for data storage
(See: snap.stanford.edu/data/index.html)



Not fast to find edges: $\mathcal{O}(L)$

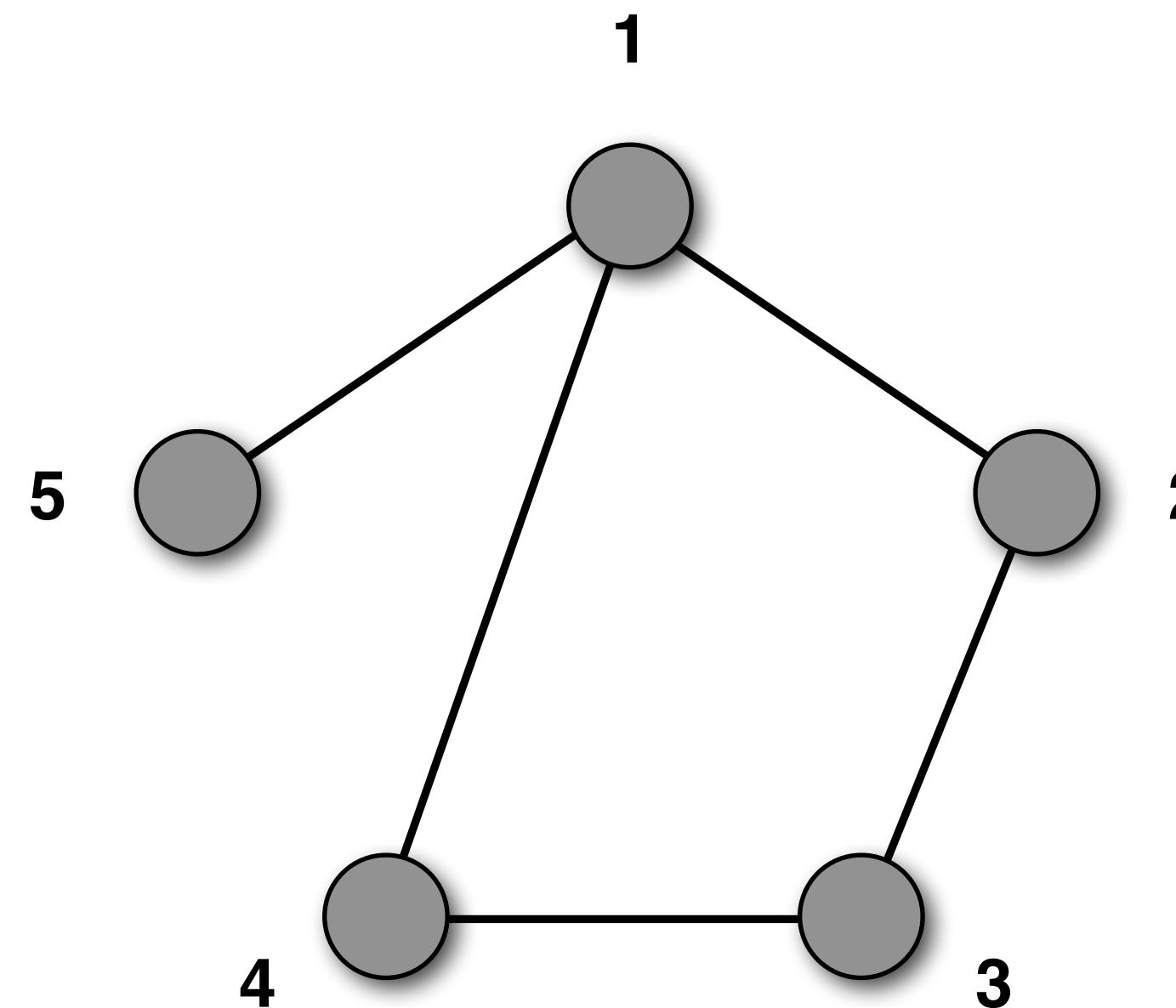
Inconvenient for calculations involving node neighbors

The **adjacency list** stores nodes with their lists of neighbors



Node	Neighbors
1	$\{k_1 \text{ neighbors}\}$
...	
i	$\{k_i \text{ neighbors}\}$
...	
N	$\{k_N \text{ neighbors}\}$

The **adjacency list** stores nodes with their lists of neighbors



Node	Neighbors
1	2,4,5
2	1,3
3	2,4
4	1,3
5	1

The **adjacency list** needs little space and is convenient for many calculations



Needs little memory: $2L$

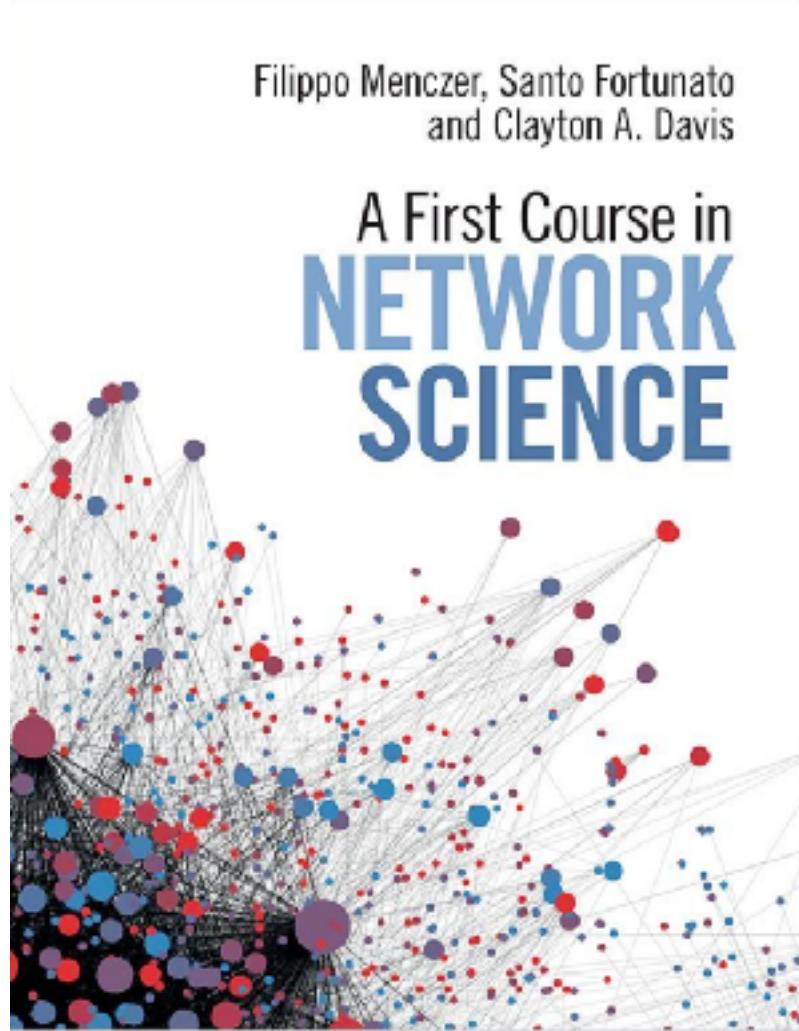
Convenient for many calculations involving neighbors (BFS, spreading processes,...)

Fast to add elements: $\mathcal{O}(1)$

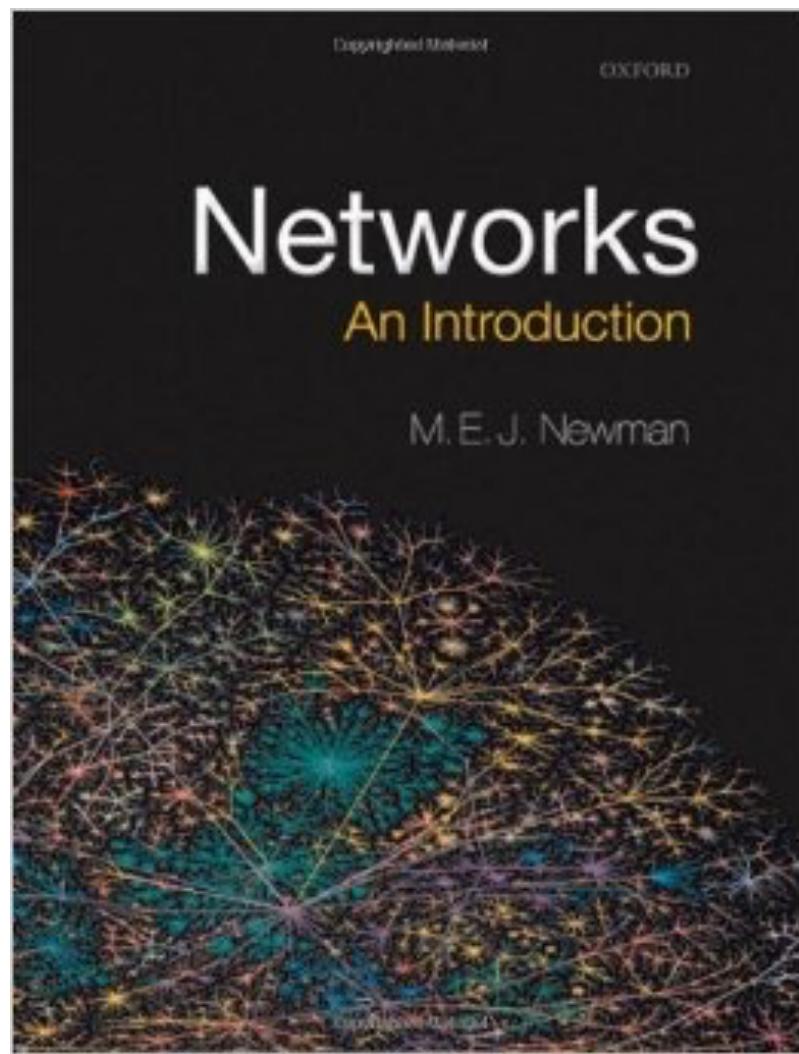
Not convenient to find/remove elements: $\mathcal{O}(L/N)$

Jupyter

Sources and further materials for today's class



F. Menczer, S. Fortunato, C.A. Davis
A first course in Network Science
Cambridge University Press (2020)



M.E.J. Newman.
Networks: An Introduction.
Oxford University Press (2010)

Organizing principles of networks

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