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An iterative and parallel solver based on domain decomposition for the *h-p* version of the finite element method

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Abstract

In this paper, we propose a new interative and parallel solver, based on domain decomposition, for the *h-p* version of the finite element method in two dimensions. It improves our previous work in two aspects: (1) A subdomain may contain several super-elements of the coarse mesh, thus can be of arbitrary shape and size. This makes the solver more efficient and more flexible in computational practice. (2) The *p*-version components (i.e., the high order side and internal modes) in every element are treated separately, which results in better parallelism.

Keywords: Iterative and parallel solver; Preconditioning; The *h-p* version of the finite element method

AMS classification: 65F10; 65N30; 65N55

1. Introduction

For the solution of large-scale linear systems resulting from the finite element discretization of elliptic problems, iterative solvers usually offer distinct advantages over direct ones, e.g., no fill-in and thus requiring almost the same amount of memory as the original data. In recent years they have been gaining acceptance in the finite element community in spite of some limitations [19]. Many successful iterative solution techniques have been developed in the past decade. Among them are the preconditioned conjugate gradient method using preconditioning techniques based on the domain decomposition method [11, 19, 26].

Preconditioning techniques for the *h*-version and the *p*-version of the finite element method have been extensively investigated in the past decade [2, 6–8, 12, 18, 19, 22, 23, 25]. Preconditioning

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for the h-p version in $\mathbb{R}^2$ is recently discussed in [1, 15, 16, 20, 21], and techniques for the h-p version in $\mathbb{R}^3$ are addressed for the first time in [17]. By properly incorporating the preconditioning of the h-version and the p-version, we proposed in [15] a preconditioner for the h-p version of the finite element method using locally quasi-uniform meshes. It requires only that the partition of each super-element is quasi-uniform. The condition number for the preconditioned system is shown to be of order $\max_i(1+\ln[(H_i p_i)/h_i])^2$, where H_i is the diameter of the ith super-element, h_i is the characteristic diameter of the elements in the ith super-element, and p_i is the maximum polynomial degree used in it. This result covers both the h-version and the p-version as special cases.

In this paper, we extend the preconditioning technique proposed in our previous paper [15], to make it more flexible and more efficient in practical applications. There are two major improvements: First, we distinguish between the super-elements and the subdomains. We define here a subdomain as the union of several super-elements of the coarse mesh. Thus the subdomains are allowed to be of arbitrary sizes and shapes, e.g., a strip-shaped layer in a geometric mesh. This flexibility is important in practical computations. Second, the p-version components (i.e., the high order side and internal modes) are treated separately in the preconditioner. Such a treatment results in better parallelism, and is convenient for implementation in new finite element codes as well as in existing ones. In addition, we prove that the same bound of the condition number as in [15] for the preconditioned system is retained.

An outline of this paper is as follows. In Section 2, we introduce the partition of the domain into the coarse mesh and the fine mesh, and the approximation subspace for the h-p version of the finite element method. In Section 3, we construct a preconditioner based on a two-level orthogonalization, and describe an iterative and parallel solver for the h-p version of the finite element method. In Section 4, the condition numbers of the preconditioned linear systems are examined for two different types of finite element approximations. Finally, in Section 5, we discuss the application of the preconditioning to the h-p version with a geometric mesh.

2. Two-level mesh and finite element space

2.1. Model problem and two-level mesh

Let Ω be a polygonal domain in $\mathbb{R}^2$. $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces. Given $f \in L^2(\Omega)$, consider the following Poisson equation with homogeneous Dirichlet condition:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Define a bilinear form $a_\Omega(\cdot,\cdot)$: $H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}^1$ as

$$a_\Omega(u,v) = \int\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\Omega.$$

Then the weak formulation of (2.1) is to find $u \in H_0^1(\Omega)$, such that

$$a_\Omega(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega) \tag{2.2}$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(\Omega)$.

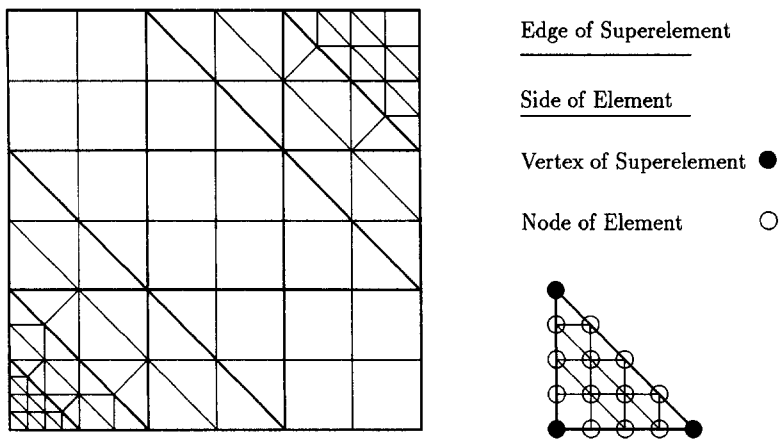

Fig. 1. Locally quasi-uniform mesh, superelement and elements.

We now consider the approximation of (2.2) by the h-p version of the finite element method. We first describe the partition of the domain Ω into a two-level mesh:

(1) *Coarse mesh.* We divide Ω into a family of non-overlapping triangles or quadrilaterals K_i, $1 \leqslant i \leqslant I$, i.e., $\bar{\Omega} = \bigcup_{i=1}^{I} \bar{K}_i$. We will hereafter refer to the K_i's as *super-elements*, and use H_i to denote the diameter of K_i.

(2) *Fine mesh.* Each super-element K_i is further partitioned into a number of quasi-uniform triangles or quadrilaterals, i.e., $\bar{K}_i = \bigcup_{j=1}^{N_i} \bar{\omega}_{i,j}$. The $\omega_{i,j}$, $1 \leqslant j \leqslant N_i$, are of approximately the same size. We will hereafter refer to the $\omega_{i,j}$'s as *elements*, and use h_i to denote the characteristic diameter of the elements in K_i.

We require that both of the above partitions are regular, i.e.,

(1) The intersection of any two super-elements (resp. elements) can only be a vertex, a whole edge, or an empty set;
(2) Any interior angle θ of a super-element (resp. element) satisfies $0 < \theta_0 \leqslant \theta \leqslant \theta_1 < \pi$;
(3) The ratio of the longest edge over the shortest edge of any quadrilateral super-element (resp. element) is bounded from above by a constant;

see Ciarlet [9] for details. Note that we do not require the coarse or fine mesh to be quasi-uniform in the above partitions. The quasi-uniformity is only required in each super-element, rather than in the whole domain. We call this kind of partition *locally quasi-uniform*, see Fig. 1 for a typical locally quasi-uniform mesh. This requirement is weaker than the quasi-uniformity of the coarse or fine mesh, but stronger than the shape regular requirement. It can be satisfied by most of the finite element meshes used in practices. In particular it is satisfied by the geometric mesh, which plays an essential role in the h-p version of the finite element method [5, 14].

To distinguish between the coarse and fine meshes, we will in this paper refer to the vertices and edges of an element ω as *nodes* and *sides*, while still use *vertices* and *edges* to refer to those of a super-element K_i. We also use L to denote the number of sides of an element (or the number of the edges of a super-element), i.e., $L=3$ if the element (or super-element) is triangular, or $L=4$ if it is quadrilateral. Furthermore, we will use Γ (with appropriate subscripts) to denote the edges of the super-elements, and use γ (with subscripts) to denote the sides of the elements.

2.2. Shape functions and finite element space

We next describe the approximation subspace for the h-p version discretization of (2.2). Let T be the equilateral triangle of side length 1, and let Q be the square with side length 1. They will be used below as the reference elements. For any positive integer p, let

$\mathscr{P}_p^1(T)$=the set of polynomials defined on T of total degree $\leqslant p$;

$\mathscr{P}_p^2(Q)$=the set of polynomials defined on Q of separate degree $\leqslant p$;

$\mathscr{P}_p^3(Q)$=the union of the set of polynomials defined on Q of total degree $\leqslant p$ with the set of polynomials of degree 1 in one variable and of degree $\leqslant p$ in the other.

Note that $P_p^3(Q)$ coincides with the space of the serendipity element in [27]. To simplify notations, we use $\hat{\omega}$ to denote either T or Q, and use $\mathscr{P}_p(\hat{\omega})$ to denote either $\mathscr{P}_p^1(T)$, $\mathscr{P}_p^2(Q)$ or $\mathscr{P}_p^3(Q)$, with the obvious understanding that $\hat{\omega}=T$ if the element is triangular or $\hat{\omega}=Q$ if the element is quadrilateral.

Now on the reference element $\hat{\omega}$, with $L+1$ positive integers $p^{(l)}$, $0\leqslant l\leqslant L$, we define the following three sets of shape functions (or modes):

(1) The set $\Psi^{[N]}(\hat{\omega})$ of *nodal* shape functions. It is composed of the linear or bilinear functions which have the value one at one node of $\hat{\omega}$, and zero at all others;

(2) The set $\Psi_{p^{(l)}}^{[S_l]}(\hat{\omega})$, $l=1,\ldots,L$, of *side* shape functions. If $\hat{\gamma}_l$ is a side of $\hat{\omega}$, then a side shape function associated with $\hat{\gamma}_l$ is zero on $\partial\hat{\omega}\backslash\hat{\gamma}_l$. $\Psi_{p^{(l)}}^{[S_l]}(\hat{\omega})$ is composed of all side shape functions in $\mathscr{P}_{p^{(l)}}(\hat{\omega})$ associated with $\hat{\gamma}_l$;

(3) The set $\Psi_{p^{(0)}}^{[B]}(\hat{\omega})$ of *internal* shape (or *bubble*) functions. These shape functions are in $\mathscr{P}_{p^{(0)}}(\hat{\omega})$ and vanish on $\partial\hat{\omega}$.

For brevity, we also use $\Psi^{[N]}(\hat{\omega})$, $\Psi_{p^{(l)}}^{[S_l]}(\hat{\omega})$ and $\Psi_{p^{(0)}}^{[B]}(\hat{\omega})$ to denote the spaces spanned by the corresponding set of shape functions. There are many different ways of creating shape functions satisfying the above conditions. We refer to [3, 24] for details.

We define the polynomial subspace on the reference element $\hat{\omega}$ as follows

$$\Psi(\hat{\omega})=\Psi^{[N]}(\hat{\omega})\bigcup_{l=1}^{L}\Psi_{p^{(l)}}^{[S_l]}(\hat{\omega})\cup\Psi_{p^{(0)}}^{[B]}(\hat{\omega}).$$

We now construct the finite element space for the h-p version approximation. Assume that $L+1$ positive integers $p_\omega^{(l)}$, $0\leqslant l\leqslant L$, are assigned to each element ω and its sides, which constitute the distribution of polynomial degrees used in the h-p version discretization. Let F_ω be the affine mapping from the reference element $\hat{\omega}$ onto ω. Then we define

$$\Psi(\omega)=\Psi^{[N]}(\omega)\cup\Psi^{[S]}(\omega)\cup\Psi^{[B]}(\omega),$$

where

$$\Psi^{[N]}(\omega)=\{\hat{v}\circ F_\omega^{-1}\mid\forall\hat{v}\in\Psi^{[N]}(\hat{\omega})\};$$

$$\Psi^{[S_l]}(\omega)=\{\hat{v}\circ F_\omega^{-1}\mid\forall\hat{v}\in\Psi_{p_\omega^{(l)}}^{[S_l]}(\hat{\omega})\};$$

$$\Psi^{[S]}(\omega)=\bigcup_{l=1}^{L}\Psi^{[S_l]}(\omega);$$
$$\Psi^{[B]}(\omega)=\{\hat{v}\circ F_\omega^{-1}\mid\forall\hat{v}\in\Psi^{[B]}_{p_\omega^{(0)}}(\hat{\omega})\}.$$

We define the approximation subspace in Ω by

$$\Psi(\Omega)=\{v\mid v|_\omega\in\Psi(\omega),\forall\omega\subset\Omega\}\cap H_0^1(\Omega).$$

The *h-p* version discretization of (2.2) is to find $u\in\Psi(\Omega)$ such that

$$a_\Omega(u,\phi)=(f,\phi),\quad\forall\phi\in\Psi(\Omega). \tag{2.3}$$

3. Preconditioner

3.1. Subdomains and two-level orthogonalization

To construct the preconditioner, we need a suitable decomposition of the approximation subspace $\Psi(\Omega)$. We first introduce the *subdomains* (or *substructures*) $\Omega_n, 1\leqslant n\leqslant N$, by combining a number of super-elements. Define $\bar{\Omega}_n$ as the union of several super-elements $\bar{K}_i$. We require that each super-element belongs to one and only one subdomain. Thus the Ω_n's are non-overlapping and $\bar{\Omega}=\bigcup_{n=1}^{N}\bar{\Omega}_n$. We denote by $\partial\Omega_n$ the boundary of Ω_n. Furthermore, we require that no vertex of a super-element is in the interior of a subdomain. Note that the Ω_n's may consist of an arbitrary number of subdomains, and thus they can be of different sizes and shapes.

Next, we describe a procedure of two-level orthogonalization.

(1) For every element ω, we make its side shape functions orthogonal to its internal shape functions, with respect to $a_\omega(\cdot,\cdot)$. In other words, we decompose $\Psi(\omega)$ into a direct sum as follows:

$$\Psi(\omega)=\Psi^{[N]}(\omega)\oplus\tilde{\Psi}^{[S]}(\omega)\oplus\Psi^{[B]}(\omega), \tag{3.1}$$

where

$$\tilde{\Psi}^{[S]}(\omega)=\bigoplus_{l=1}^{L}\tilde{\Psi}^{[S_l]}(\omega);$$
$$\tilde{\Psi}^{[S_l]}(\omega)=\{v\in\Psi^{[S_l]}(\omega)\mid a_\omega(v,\phi)=0,\forall\phi\in\Psi^{[B]}(\omega)\}.$$

The above process is usually called a *partial orthogonalization* (or the *elimination* of interior modes) in the *p*-version, see, e.g., [2]. It is easy to see that a function in $\tilde{\Psi}^{[S_l]}(\omega)$ is uniquely determined by its values on the side γ_l of ω.

(2) Let $\partial\Omega_n=\bigcup_{\ell=1}^{J_n}\Gamma_\ell$, and let $K_{j(\ell)}\subset\Omega_n$ be a super-element with Γ_ℓ as one of its edges. We define

$$\Psi^{[N]}(K_{j(\ell)})=\{v\mid v|_\omega\in\Psi^{[N]}(\omega),\forall\omega\subset K_{j(\ell)}\}\cap H^1(K_{j(\ell)}),$$
$$\Psi^{[N_l]}(K_{j(\ell)})=\{v\in\Psi^{[N]}(K_{j(\ell)})\mid v=0\text{ on }\partial K_{j(\ell)}\},$$

and

$$\tilde{\Psi}^{[N_\ell]}(K_{j(\ell)})=\{v\in\Psi^{[N]}(K_{j(\ell)})\,|\,v=0 \text{ on } \partial K_{j(\ell)}\backslash\Gamma_\ell \text{ and } a_{K_{j(\ell)}}(v,\phi)=0,\forall\phi\in\Psi^{[N_l]}(K_{j(\ell)})\}. \tag{3.2}$$

The above procedure actually makes the nodal modes associated with the nodes on $\Gamma_\ell\subset\partial\Omega_n$ orthogonal to those with nodes inside the super-element $K_{j(\ell)}$. It is equivalent to computing the Schur complement of the linear element approximations on $K_{j(\ell)}$, see, e.g., [6]. Clearly, a function in $\tilde{\Psi}^{[N_\ell]}(K_{j(\ell)})$ is uniquely determined by its values at the nodes on Γ_ℓ.

Once a set of basis function has been chosen, e.g., a Lagrange basis, then the above two-level orthogonalization is equivalent to a transformation of the basis functions. The spaces with tilde are those spanned by the transformed bases, while the spaces without tilde are spanned by the original bases. In computations these orthogonalizations can be accomplished by suitable block eliminations of the local stiffness matrices.

Remark 3.1. To define the preconditioner, a procedure of two-level orthogonalization is also used in [15, 16, 20, 21], in which the first step of the orthogonalization is the same as (3.1). However, the second step of the orthogonalization in [15] is carried out for all the nodal modes associated with nodes on the boundaries of the super-elements, while in this paper it is carried out only for the nodal modes associated with nodes on the boundaries of the subdomains. Thus the preconditioning technique of this paper requires much less work than those of [15, 16], if a subdomain contains a large number of super-elements.

3.2. Decomposition of the approximation space in Ω_n

Let $\Psi(\Omega_n)$ be the set of the restrictions of functions in $\Psi(\Omega)$ to Ω_n. To construct the preconditioner, we need only to consider in each subdomain Ω_n the decomposition of $\Psi(\Omega_n)$ see [18]. Clearly,

$$\Psi(\Omega_n)=\Psi^{[N]}(\Omega_n)\oplus\Psi^{[S]}(\Omega_n)\oplus\Psi^{[B]}(\Omega_n), \tag{3.3}$$

where

$$\Psi^{[N]}(\Omega_n)=\{v\,|\,v|_\omega\in\Psi^{[N]}(\omega),\forall\omega\subset\Omega_n\}\cap H^1(\Omega_n),$$

$$\tilde{\Psi}^{[S]}(\Omega_n)=\{v\,|\,v|_\omega\in\tilde{\Psi}^{[S]}(\omega),\forall\omega\subset\Omega_n\}\cap H^1(\Omega_n),$$

$$\Psi^{[B]}(\Omega_n)=\{v\,|\,v|_\omega\in\Psi^{[B]}(\omega),\forall\omega\subset\Omega_n\}.$$

$\Psi^{[N]}(\Omega_n)$ represents the h-version components of the h-p version approximation in Ω_n, while $\Psi^{[S]}(\Omega_n)$ and $\Psi^{[B]}(\Omega_n)$ represent the p-version components.

We now introduce a subspace associated with the linear/bilinear element on the coarse mesh. For any super-element $K\subset\Omega_n$, let F_K be the linear/bilinear mapping from the reference element $\hat{\omega}$ onto K. Define

$$\Psi^{[V]}(\Omega_n)=\{v|_K=p_1\circ F_K^{-1},\forall p_1\in\mathscr{P}_1(\hat{\omega}),\forall K\subset\Omega_n\}\cap H^1(\Omega_n).$$

In the preconditioner defined below, this space is used to define a problem on the coarse mesh. It will provide an overall communication of information, which is a necessary mechanism for preconditioning in problems involving a large number of super-elements, see, e.g., [25]. However, $\Psi^{[V]}(\Omega_n)$

is usually not a subspace of $\Psi(\Omega_n)$. We have to build another subspace which is included in $\Psi(\Omega_n)$ and which has the same degree of freedom as $\Psi^{[V]}(\Omega_n)$. For this purpose, we define Π_h as the piecewise linear or bilinear interpolation based on the fine mesh, i.e., for any $v\in C(\bar{\Omega}_n)$,

$$\Pi_h v|_\omega=(\hat{\Pi}\hat{v})\circ F_\omega^{-1},\quad \forall\omega\subset\Omega_n, \tag{3.4}$$

where $\hat{v}=v\circ F_\omega$ and $\hat{\Pi}\hat{v}$ is the linear or bilinear interpolation of $\hat{v}$ at the nodes of $\hat{\omega}$. We define

$$\Psi^{[V_h]}(\Omega_n)=\Pi_h\Psi^{[V]}(\Omega_n). \tag{3.5}$$

Further, for all $\Gamma_\ell\subset\partial\Omega_n$, let $\tilde{\Psi}^{[N_{\Gamma_\ell}]}(\Omega_n)$ be the extension by zero of the functions in $\tilde{\Psi}^{[N_{\Gamma_\ell}]}(K_{j(\ell)})$ (cf. (3.2)), and define

$$\Psi^{[N_I]}(\Omega_n)=\{v\in\Psi^{[N]}(\Omega_n)\mid v=0 \text{ on } \partial\Omega_n\}.$$

Then

$$\Psi^{[N]}(\Omega_n)=\Psi^{[V_h]}(\Omega_n)\oplus\Psi^{[N_I]}(\Omega_n)\bigoplus_{\ell=1}^{J_n}\tilde{\Psi}^{[N_{\Gamma_\ell}]}(\Omega_n). \tag{3.6}$$

Let γ_m, $1\leqslant m\leqslant M_n$, be the sides in $\bar{\Omega}_n$, and ω_{m_1} and ω_{m_2} be the elements sharing γ_m as one of their sides. Define

$$\tilde{\Psi}^{[S_{\gamma_m}]}(\Omega_n)=\{v\in\tilde{\Psi}^{[S]}(\Omega_n)\mid v=0 \text{ on } \Omega_n\backslash(\omega_{m_1}\cup\omega_{m_2})\}. \tag{3.7}$$

Then we can decompose $\tilde{\Psi}^{[S]}(\Omega_n)$ as follows:

$$\tilde{\Psi}^{[S]}(\Omega_n)=\bigoplus_{m=1}^{M_n}\tilde{\Psi}^{[S_{\gamma_m}]}(\Omega_n). \tag{3.8}$$

By (3.3), (3.5) and (3.8), we summarize the decomposition of $\Psi(\Omega_n)$ as follows:

$$\Psi(\Omega_n)=\Psi^{[V_h]}(\Omega_n)\oplus\Psi^{[N_I]}(\Omega_n)\bigoplus_{\ell=1}^{J_n}\tilde{\Psi}^{[N_{\Gamma_\ell}]}(\Omega_n)\bigoplus_{m=1}^{M_n}\tilde{\Psi}^{[S_{\gamma_m}]}(\Omega_n)\oplus\Psi^{[B]}(\Omega_n). \tag{3.9}$$

3.3. Preconditioner and its matrix representation

For any $v\in\Psi(\Omega_n)$, we have

$$v=v^{V_h}+v^{N_I}+\sum_{\ell=1}^{J_n}v^{N_{\Gamma_\ell}}+\sum_{m=1}^{M_n}v^{S_{\gamma_m}}+v^B \tag{3.10}$$

with each component belonging to the corresponding subspace in Ω_n listed above. We can define the preconditioner C as the stiffness matrix with respect to the following inner product

$$C_\Omega(u,v)=\sum_{n=1}^{N} C_{\Omega_n}(u,v), \tag{3.11}$$

where

$$C_{\Omega_n}(u,v)=a_{\Omega_n}(u^{V_h},v^{V_h})+a_{\Omega_n}(u^{N_I},v^{N_I})+\sum_{\ell=1}^{L} a_{\Omega_n}(u^{N_{\Gamma_\ell}},v^{N_{\Gamma_\ell}})$$

$$+\sum_{m=1}^{M_n} a_{\Omega_n}(u^{S_{\gamma_m}},v^{S_{\gamma_m}})+a_{\Omega_n}(u^{B},v^{B}). \tag{3.12}$$

The stiffness matrix in Ω_n associated with the preconditioner is then of the form

$$\begin{pmatrix} V_hV_h & & & & 0 \\ & N_IN_I & & & \\ & & \widehat{N_\Gamma N_\Gamma} & & \\ & & & \widehat{S_\gamma S_\gamma} & \\ 0 & & & & BB \end{pmatrix},$$

where

$$\widehat{N_\Gamma N_\Gamma}=\begin{pmatrix} N_{\Gamma_1}N_{\Gamma_1} & & & 0 \\ & N_{\Gamma_2}N_{\Gamma_2} & & \\ & & \cdots & \\ 0 & & & N_{\Gamma_L}N_{\Gamma_L} \end{pmatrix},$$

$$\widehat{S_\gamma S_\gamma}=\begin{pmatrix} S_{\gamma_1}S_{\gamma_1} & & & 0 \\ & S_{\gamma_2}S_{\gamma_2} & & \\ & & \cdots & \\ 0 & & & S_{\Gamma_{M_n}}S_{\Gamma_{M_n}} \end{pmatrix},$$

and BB is also a block diagonal matrix with each block corresponding to the internal modes in one element.

Remark 3.2. In practical computations, we usually compute only the local stiffness matrix for each element of the fine mesh, and then "assembly" them to form the global stiffness matrix. To compute the block V_hV_h used in the above preconditioner, we can make use of these local stiffness matrices as follows: Consider a super-element K. Let F_K be the linear or bilinear mapping from $\hat{\omega}$ to K, and let $\hat{\phi}_j(x)$, $1\leqslant j\leqslant L$, be the linear or bilinear basis functions in $\hat{\omega}$. Then $\phi_j(x)=\hat{\phi}_j\circ F_K^{-1}$, $1\leqslant j\leqslant L$, are the basis functions in the super-element K, and on an element ω

$$\Pi_h\phi_j(\boldsymbol{x})=\sum_{1\leqslant l\leqslant L}\phi_j(q_l)\phi_l^{(\omega)}(\boldsymbol{x}),$$

where $\phi_l^{(\omega)}$, $1 \leqslant l \leqslant L$, are the basis functions in the element ω, and q_l, $1 \leqslant l \leqslant L$, are the nodes of ω. Thus

$$a_K(\Pi_h\phi_j, \Pi_h\phi_{j'}) = \sum_{\omega \subset K} a_\omega(\Pi_h\phi_j, \Pi_h\phi_{j'})$$

$$= \sum_{\omega \subset K} \left(\sum_{1 \leqslant l, l' \leqslant L} \phi_j(q_l)\phi_{j'}(q_{l'}) a_\omega(\phi_l^{(\omega)}, \phi_{l'}^{(\omega)}) \right).$$

Since the $a_\omega(\phi_l^{(\omega)}, \phi_{l'}^{(\omega)})$, $1 \leqslant l, l' \leqslant L$, are entries of the local stiffness matrix in ω, we can obtain the entries of $V_h V_h$ by linear combinations of the entries of the local stiffness matrices in the elements.

3.4. Iterative and parallel algorithm

The whole iterative process for the *h-p* version of the finite element approximations of two-dimensional elliptic problems, using the above domain decomposition preconditioning technique, can be described as follows:

(1) Partition the domain Ω into super-elements and locally quasi-uniform elements, prescribe to every element and every side of the elements a positive integer, which constitutes the distribution of polynomial degrees used in the *h-p* version discretization, and define the subdomains by combining a number of super-elements;

(2) Select a specific kind of side and internal shape functions on the reference elements, and compute for each element the local stiffness matrix and load vector, which constitute the linear algebraic system of the *h-p* version approximation;

(3) Make the side modes in each side of the elements orthogonal to its internal modes, and make the nodal modes, associated with nodes on a super-element edge which lies on the boundary of subdomains orthogonal to those associated with the nodes inside the super-element;

(4) Solve the linear algebraic system by a preconditioned iterative method, e.g., a preconditioned conjugate gradient method, using the preconditioner defined in (3.11). This involves mainly the computation of the residual vectors r of the intermediate solutions and the matrix-vector multiplication $C^{-1}r$. The evaluation of r is straightforward. The computation of $w = C^{-1}r$ is equivalent to finding $w \in \Psi(\Omega)$ such that

$$C_\Omega(w, \phi) = (r, \phi), \quad \forall \phi \in \Psi(\Omega).$$

The solution $w = w^{V_h} + w^{N_I} + \sum_{\ell=1}^{J_n} w^{N_{\Gamma_\ell}} + \sum_{m=1}^{M_n} w^{S_{im}} + w^B$ can be obtained by the following steps:

(4a) Solution of w^{V_h}: This step is basically to solve a problem in Ω with the linear or bilinear approximation based on the coarse mesh;

(4b) Solution of w^{N_I} for every subdomain: This step is to solve a homogeneous Dirichlet problem on each subdomain using the linear or bilinear approximation based on the fine mesh;

(4c) Solution of $w^{N_{\Gamma_\ell}}$ for every super-element edge lying on the subdomain boundary: For each ℓ, this step involves only the two super-elements adjacent to the subdomain boundary. It is in fact the same as solving a homogeneous Dirichlet problem on the two super-elements using the linear or bilinear approximation based on the fine mesh;

(4d) Solution of $w^{S_{\gamma_m}}$ for every element side: For each m, this step is to solve a homogeneous Dirichlet problem on the two elements sharing γ_m by the high order approximations;

(4e) Solution of w^B: This step essentially involves the solution of a homogeneous Dirichlet problem for each element. This computation is completely independent for different elements.

Note that the computations in step (4b) can be in parallel on the subdomain level, those in (4c) can be in parallel on the super-element level, and those in (4d) and (4e) in parallel on the element level. Thus this algorithm is highly parallelizable. In addition, if H_i and h_i are small, (4b) and (4c) involve the major part of the computations. If p_i is large, then (4d) and (4e) also involve the major part of the computations.

4. Analysis of the condition number

In this section, we give two theorems regarding the condition number of the preconditioned system described in Section 3. In the proof we will emphasize the new techniques used in this paper, and skip those developed in our previous work. For more details refer to [15, 16].

Theorem 4.1. *Assume that the finite element space is defined by* $\mathscr{P}_p(\hat{\omega})=\mathscr{P}_p^1(T)$ *for triangular elements and* $\mathscr{P}_p(\hat{\omega})=\mathscr{P}_p^2(Q)$ *for quadrilateral elements, and that the distribution of polynomial degrees satisfies*

$$p_\omega^{(0)} \geqslant \max_{1\leqslant l\leqslant L} p_\omega^{(l)}, \quad \forall \omega \subset \Omega. \tag{4.1}$$

Let C *be the preconditioner defined in* (3.11). *Then there exist positive constants* c_1 *and* c_2, *independent of* H_i, h_i *and* p_i, *such that for all* $u\in\Psi(\Omega)$

$$c_1 a_\Omega(u,u) \leqslant C_\Omega(u,u) \leqslant c_2 \max_{1\leqslant i\leqslant I} \left(1+\ln\frac{H_i p_i}{h_i}\right)^2 a_\Omega(u,u). \tag{4.2}$$

Proof. Clearly we need only to prove (4.2) in every subdomain. Let Π_h be the piecewise linear/bilinear interpolation based on the fine mesh (cf. (3.4)), and let similarly Π_H be the piecewise linear/bilinear interpolation based on the coarse mesh. It is easy to see that on the subdomain Ω_n

$$u = u^{V_h} + u^{N_I} + \sum_{\ell=1}^{J_n} u^{N_{\Gamma_\ell}} + \sum_{m=1}^{M_n} u^{S_{\gamma_m}} + u^B, \tag{4.3}$$

where each of the components is defined by

$$u^{V_h} = \Pi_h \Pi_H u;$$

$$u^{N_{\Gamma_\ell}} = \begin{cases} \Pi_h(u - \Pi_H u), & \text{on } \Gamma_\ell, \\ 0, & \text{on the edges of the super-elements other than } \Gamma_\ell; \end{cases}$$

$$u^{N_I} = \Pi_h u - \left(u^{V_h} + \sum_{\ell=1}^{J_n} u^{N_{\Gamma_\ell}}\right), \text{ on } \Omega_n;$$

$$u^{S_{\gamma m}} = \begin{cases} u - \Pi_h u, & \text{on } \gamma_m, \\ 0, & \text{on the sides of the elements other than } \gamma_m; \end{cases}$$

$$u^B = u - \left(\Pi_h u + \sum_{m=1}^{M_n} u^{S_{\gamma m}} \right), \text{ on } \Omega_n.$$

Note that a function in the space $\tilde{\Psi}^{N_{\Gamma_\ell}}$ is uniquely determined by its values on Γ_ℓ and that a function in $\tilde{\Psi}^{S_{\gamma m}}$ is determined uniquely by its values on γ_m, see Section 3.1. Therefore the above formulas define the functions uniquely.

We now consider the left-hand side inequality of (4.2). First, by the Cauchy–Schwarz inequality

$$C_{\Omega_n}(u,u) \leqslant 5(C_{\Omega_n}(u^{V_h}, u^{V_h}) + C_{\Omega_n}(u^{N_I}, u^{N_I}) + C_{\Omega_n}\left(\sum_{\ell=1}^{J_n} u^{N_{\Gamma_\ell}}, \sum_{\ell=1}^{J_n} u^{N_{\Gamma_\ell}} \right)$$

$$+ C_{\Omega_n}\left(\sum_{m=1}^{M_n} u^{S_{\gamma m}}, \sum_{m=1}^{M_n} u^{S_{\gamma m}} \right) + C_{\Omega_n}(u^B, u^B)).$$

Since each super-element or element has at most four edges or sides, we find that

$$C_{\Omega_n}\left(\sum_{\ell=1}^{J_n} u^{N_{\Gamma_\ell}}, \sum_{\ell=1}^{J_n} u^{N_{\Gamma_\ell}} \right) \leqslant 4 \sum_{\ell=1}^{J_n} C_{\Omega_n}(u^{N_{\Gamma_\ell}}, u^{N_{\Gamma_\ell}}),$$

$$C_{\Omega_n}\left(\sum_{m=1}^{M_n} u^{S_{\gamma m}}, \sum_{m=1}^{M_n} u^{S_{\gamma m}} \right) \leqslant 4 \sum_{m=1}^{M_n} C_{\Omega_n}(u^{S_{\gamma m}}, u^{S_{\gamma m}}),$$

from which the left-hand side inequality of (4.2) follows readily.

For the right-hand side inequality of (4.2), we derive a bound for each term in (4.3) as follows:

(i) By [15, Lemma 4.2], we can easily show that for each $K_i \subset \Omega_n$,

$$a_{K_i}(u^{V_h}, u^{V_h}) = a_{K_i}(\Pi_h \Pi_H u, \Pi_h \Pi_H u)$$

$$\leqslant c \|u\|^2_{L^\infty(K_i)} \sum_{\ell=1}^{L} a_{K_i}(\Pi_h \phi_\ell, \Pi_h \phi_\ell)$$

$$\leqslant c \left(1 + \ln \frac{H_i p_i}{h_i} \right) \left(|u|^2_{H^1(K_i)} + \frac{1}{(H_i)^2} \|u\|^2_{L^2(K_i)} \right).$$

Note that the above inequality holds if we replace u by $u - \alpha$ with an arbitrary number α. Hence it implies, by an quotient space argument, that

$$a_{K_i}(u^{V_h}, u^{V_h}) \leqslant c \left(1 + \ln \frac{H_i p_i}{h_i} \right) |u|^2_{H^1(K_i)}.$$

Summing up the above inequalities for all $K_i \subset \Omega_n$, we find

$$a_{\Omega_n}(u^{V_h}, u^{V_h}) \leqslant c \max_{K_i \subset \Omega_n} \left(1 + \ln \frac{H_i p_i}{h_i} \right) |u|^2_{H^1(\Omega_n)}. \tag{4.4}$$

(ii) Let $K_{j(\ell)}$ be a super-element in Ω_n with Γ_ℓ as one of its edges. We obtain by [15, (4.16)]

$$a_{\Omega_n}(u^{N_{\Gamma_\ell}}, u^{N_{\Gamma_\ell}}) = a_{K_{j(\ell)}}(u^{N_{\Gamma_\ell}}, u^{N_{\Gamma_\ell}})$$
$$\leqslant c\left(1+\ln\frac{H_{j(\ell)}}{h_{j(\ell)}}\right)\left(1+\ln\frac{H_{j(\ell)}p_{j(\ell)}}{h_{j(\ell)}}\right)|u|^2_{H^1(K_{j(\ell)})}.$$

By the fact that every super-element has at most four edges, we have further that

$$\sum_{\ell=1}^{J_n} a_{\Omega_n}(u^{N_{\Gamma_\ell}}, u^{N_{\Gamma_\ell}}) \leqslant c \max_{K_i\subset\Omega_n}\left(1+\ln\frac{H_i}{h_i}\right)\left(1+\ln\frac{H_i p_i}{h_i}\right)|u|^2_{H^1(\Omega_n)}. \tag{4.5}$$

(iii) By the definition of u^{N_I}, (4.4) and (4.5), we find

$$a_{\Omega_n}(u^{N_I}, u^{N_I}) \leqslant c\left(a_{\Omega_n}(\Pi_h u, \Pi_h u) + a_{\Omega_n}(u^{V_h}, u^{V_h}) + \sum_{\ell=1}^{J_n} a_{\Omega_n}(u^{N_{\Gamma_\ell}}, u^{N_{\Gamma_\ell}})\right)$$
$$\leqslant c \max_{K_i\subset\Omega_n}\left(1+\ln\frac{H_i}{h_i}\right)\left(1+\ln\frac{H_i p_i}{h_i}\right)|u|^2_{H^1(\Omega_n)}. \tag{4.6}$$

(iv) We get from [15, (4.20)],

$$\sum_{m=1}^{M_i} a_{\Omega_n}(u^{S_{im}}, u^{S_{im}}) \leqslant c \max_{K_i\subset\Omega_n}(1+\ln p_i)^2|u|^2_{H^1(\Omega_n)}. \tag{4.7}$$

(v) Finally, by the definition of u^B and (4.7), we easily find that

$$a_{\Omega_n}(u^B, u^B) \leqslant c \max_{K_i\subset\Omega_n}\left(1+\ln\frac{H_i p_i}{h_i}\right)^2|u|^2_{H^1(\Omega_n)}. \tag{4.8}$$

The right-hand side inequality in (4.2) now follows immediately from (4.5)–(4.8). □

Theorem 4.2. *Assume that the finite element space is defined by* $\mathscr{P}_p(\hat{\omega})=\mathscr{P}^1_p(T)$ *for triangular elements and* $\mathscr{P}_p(\hat{\omega})=\mathscr{P}^3_p(Q)$ *for quadrilateral elements. Let* (4.1) *hold for all the elements. Then there exist positive constants* c_3 *and* c_4, *independent of* H_i, h_i *and* p_i, *such that for all* $u\in\Psi(\Omega)$

$$c_3 a_\Omega(u,u) \leqslant C_\Omega(u,u) \leqslant c_4 \max_{1\leqslant i\leqslant I}\left(1+\ln\frac{H_i}{h_i}+p_i\ln p_i\right)^2 a_\Omega(u,u). \tag{4.9}$$

Proof. The proof of this theorem is similar to that of Theorem 4.1. To handle the elements for the case $\mathscr{P}_p(\hat{\omega})=\mathscr{P}^3_p(Q)$, we shall use [2, Theorem 7.6] instead of [2, Theorem 7.5], and we obtain, in place of (4.7), the following inequality (see the derivation of [15, (4.20)]),

$$\sum_{m=1}^{M_i} a_{\Omega_n}(u^{S_{im}}, u^{S_{im}}) \leqslant c \max_{K_i\subset\Omega_n}(1+p_i\ln p_i)^2|u|^2_{H^1(\Omega_n)}. \tag{4.10}$$

(4.9) now follows readily from (4.4)–(4.6), (4.8) and (4.10). □

Remark 4.1. The condition number bound given by (4.2) and (4.9) cover both of the h-version and the p-version as special cases. For $p_i \equiv 1$, they reduce to the bound of the h-version discussed in [10]. In this case the orthogonalization is carried out only with respect to the nodal modes associated with the nodes on the super-element edges lying on the boundary of subdomains. If, on the other hand, each element is a super-element, then $H_i \equiv h_i$, and (4.2) and (4.9) reduce to the bounds of the p-version discussed in [2]. In these cases the orthogonalization is carried out only with respect to the side modes in every element.

Remark 4.2. The condition number bound given by (4.10) for the h-p version seems to be the best we can expect, because it fully reduces to that of the h-version and the p-version, as indicated in the previous remark.

5. Application to a geometric mesh

We now apply the preconditioning technique described in the previous sections to the h-p version with a geometric mesh. Since the h-p version with a geometric mesh can result in exponential convergence, a discussion of such an application is of practical importance.

For simplicity, we take as an example the case when Ω is an L-shaped domain consisting of three unit squares, see Fig. 2. We assume that the solution of (2.2) has a corner singularity only at the origin $O=(0,0)$, and that it belongs to the countably normed space $\mathbb{B}^2_\beta(\Omega)$, $0<\beta<1$. We refer to [4, 13] for regularity results in the space $\mathbb{B}^2_\beta(\Omega)$. Usually the geometric mesh Ω^n_σ contains layers Ω_i, $1 \leqslant i \leqslant n$, given according to the distance to the origin, and there are several elements (triangular or quadrilateral) $\Omega_{i,j}$, $1 \leqslant j \leqslant J(i)$, in the ith layer such that with a mesh factor $\sigma \in (0,1)$

$$h_{i,j} = \operatorname{diam}(\Omega_{i,j}) \approx \sigma^{n-i+1}, \quad 1 \leqslant j \leqslant J(i), \ 1 \leqslant i \leqslant n; \tag{5.1}$$

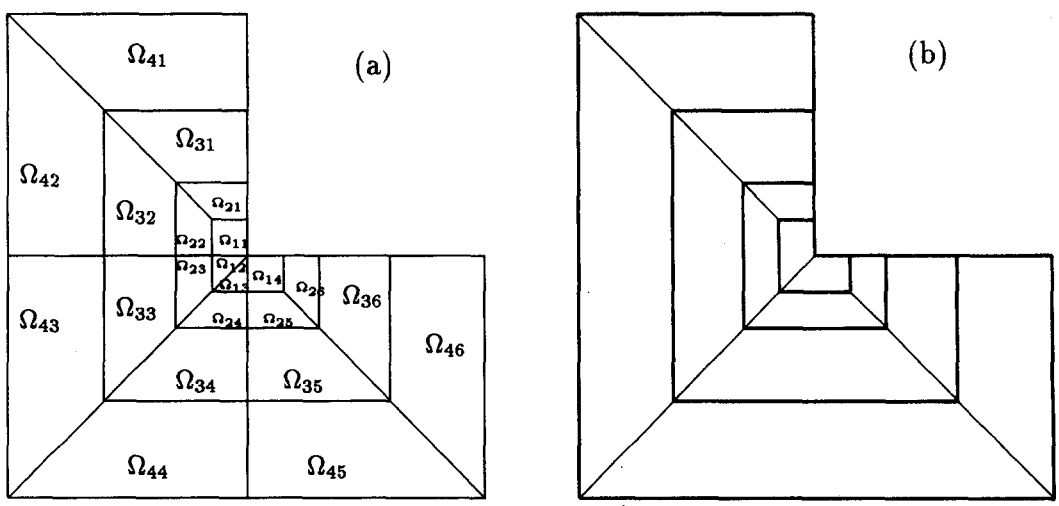

Fig. 2. (a) Fine mesh; (b) coarse mesh (——) and subdomain (—).

and

$$\begin{cases} d_{1,j}=0, & 1\leqslant j\leqslant J(1); \\ d_{i,j}=\operatorname{dist}(O,\Omega_{i,j})\approx\sigma^{n-i}, & 1\leqslant j\leqslant J(i),\ 1<i\leqslant n. \end{cases} \tag{5.2}$$

Let Ω_σ^n be the fine mesh, and let $\Theta_\sigma^n=\{\Theta_{i,k},\ 1\leqslant k\leqslant K(i),\ 1\leqslant i\leqslant n\}$ be the coarse mesh. In Fig. 2, $n=4$, $K(1)=2$, $K(2)=K(3)=K(4)=4$, and

$$\Theta_{1,1}=\Omega_{1,1}\cup\Omega_{1,2}, \qquad \Theta_{1,2}=\Omega_{1,3}\cup\Omega_{1,4},$$

$$\Theta_{i,1}=\Omega_{i,1}, \qquad \Theta_{i,2}=\Omega_{i,2}\cup\Omega_{i,3},$$

$$\Theta_{i,3}=\Omega_{i,4}\cup\Omega_{i,5}, \qquad \Theta_{i,4}=\Omega_{i,6}, \quad 2\leqslant i\leqslant 4.$$

The subdomains Ω_i, $1\leqslant i\leqslant n$, contain all super-elements $\Theta_{i,k}$, $1\leqslant k\leqslant K(i)$, i.e., $\Omega_i=\bigcup_{k=1}^{K(i)}\Theta_{i,k}=\bigcup_{j=1}^{I(j)}\Omega_{i,j}$. Therefore, $H_i/h_i\approx 2$. By $P=\{p_{i,j},\ 1\leqslant j\leqslant J(i), 1\leqslant i\leqslant n\}$, we denote a distribution of polynomial degrees, where $p_{i,j}$ is the separate degree of the element $\Omega_{i,j}$. The uniform distribution $p_{i,j}=[\mu n]$ and the linear distribution $p_{i,j}=[\mu i]$ are associated with the geometric mesh Ω_σ^n with a degree factor $\mu>0$. By applying Theorem 4.1, we can draw the following conclusion:

Theorem 5.1. *Let the fine (geometric) mesh Ω_σ^n, the coarse (geometric) mesh Θ_σ^n, and the uniform or linear distribution of polynomial degrees be given as above, and let each layer Ω_i be a subdomain, $1\leqslant i\leqslant n$. Then, for the preconditioner defined in* (3.11), *there exist positive constants c_5 and c_6 depending on μ but not on n, such that for all $u\in\Psi(\Omega)$*

$$c_5 a_\Omega(u,u)\leqslant C_\Omega(u,u)\leqslant c_6(1+\ln n)^2 a_\Omega(u,u). \qquad \square$$

Remark 5.1. If the linear distribution of polynomial degrees $p_{ij}=[\mu i]$ is used, then the term $(1+\ln(H_i p_i/h_i))\approx(1+\ln\mu+\ln i)$ increases as i increases, and $\max_i(1+\ln(H_i p_i/h_i))\approx(1+\ln\mu+\ln n)$. Hence we may combine the elements in different layers, e.g., the first and second layers, into one super-element, with a corresponding ratio $H_i/h_i\approx 1/\sigma$. If n is large, e.g., $n\geqslant 1/\sigma$, then this combination will not affect the condition number of the preconditioned system, and will reduce the number of subdomains and super-elements. It also makes the number of degree of freedoms in each subdomain more balanced.

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