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Comparison of performance of iterative methods for singular and nonsingular saddle point linear systems arising from Navier–Stokes equations ☆

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Abstract

We compare the convergence performance of iterative methods for solving singular and nonsingular linear systems arising from discretization and linearization of the steady-state Navier–Stokes equations. With a combination of analytic and empirical results, we study the effects of singularity and nonsingularity on convergence. We will show that the convergence performance for solving a singular linear system is better than that for solving a nonsingular linear system.

Keywords: Navier–Stokes equations; Iterative methods; GMRES; Singular or nonsingular linear systems

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1. Introduction

The aim of this paper is to compare the convergence performance of the Krylov subspace iterative methods for solving singular and nonsingular linear systems arising from discretization and linearization of the steady-state Navier–Stokes equations, i.e., the Oseen equations

$$\begin{aligned} &-\nu\Delta u + w\cdot \operatorname{grad} u + \operatorname{grad} p = f, \\ &-\operatorname{div} u = 0, \end{aligned} \quad \text{in } \Omega, \tag{1.1}$$

with suitable boundary conditions on $\partial\Omega$, where w is given such that $\operatorname{div} w = 0$.

Discretization of (1.1) leads to a saddle point linear system of the form

$$\begin{pmatrix} F & B^{\mathrm{T}} \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{1.2}$$

F is a discrete convection–diffusion operator, i.e., it has the form $F = \nu A + N$, where $A = A^{\mathrm{T}}$ is a discrete diffusion operator, $N = -N^{\mathrm{T}}$ is a discrete convection operator, B and B^{T} are discrete divergence and gradient operators, respectively.

Since each row sum of B^{T} equals zero, (1.2) is a (consistent) singular linear system. The zero eigenvalue of the coefficient matrix of (1.2) corresponding to the hydrostatic pressure. The solution $[u^{\mathrm{T}}, p^{\mathrm{T}}]^{\mathrm{T}}$ is not unique, each other solution has the form $[u^{\mathrm{T}}, p^{\mathrm{T}} + c^{\mathrm{T}}]^{\mathrm{T}}$, where c is an arbitrary constant vector. One usually eliminates the last row and the last column of the coefficient matrix of (1.2) to get a nonsingular linear system

$$\begin{pmatrix} F & \widehat{B}^{\mathrm{T}} \\ \widehat{B} & 0 \end{pmatrix}\begin{pmatrix} u \\ \hat{p} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{1.3}$$

It is easy to see that the unique solution $[u^{\mathrm{T}}, \hat{p}^{\mathrm{T}}]^{\mathrm{T}}$, of (1.3) is corresponding to the particular solution $[u^{\mathrm{T}}, \hat{p}^{\mathrm{T}}, 0]^{\mathrm{T}}$ of (1.2).

In this paper, we compare the convergence performance of a Krylov subspace iterative method (cf. [11]) such as GMRES for solving the singular linear system (1.2) and the nonsingular linear system (1.3).

Table 1
Iterative counts for the convergence of the preconditioned GMRES solver

ν	$h = 1/16$		$h = 1/32$		$h = 1/64$	
	Singular	Nonsingular	Singular	Nonsingular	Singular	Nonsingular
1	22	31	41	56	84	117
1/10	23	32	40	55	81	118
1/20	34	46	47	65	86	121
1/30	49	64	58	80	89	128
1/50	84	102	87	116	107	157
1/100	178	200	182	228	191	258

It is well known that GMRES is available for solving nonsingular linear systems. In fact we have the following [9].

Proposition 1.1. *If the GMRES algorithm breaks down at step j, then the iterative solution produced by GMRES at step j is the solution of the nonsingular system. Moreover, the degree j equals to the degree of the minimal polynomial of the initial residual vector.*

For a consistent singular linear system, if the index of the coefficient matrix is equal to 1, then GMRES is also an available solver. In fact we have the following (cf. [1,6]).

Proposition 1.2. *If the GMRES algorithm breaks down at step j, then the iterative solution produced by GMRES at step j is a solution of the consistent singular linear system. Moreover, the degree of the minimal polynomial of the initial residual vector is equal to* $j + 1$.

Remark 1.1. From Proposition 1.2 we can see that for a consistent singular linear system with index = 1 the setting reduces to a nonsingular case on the

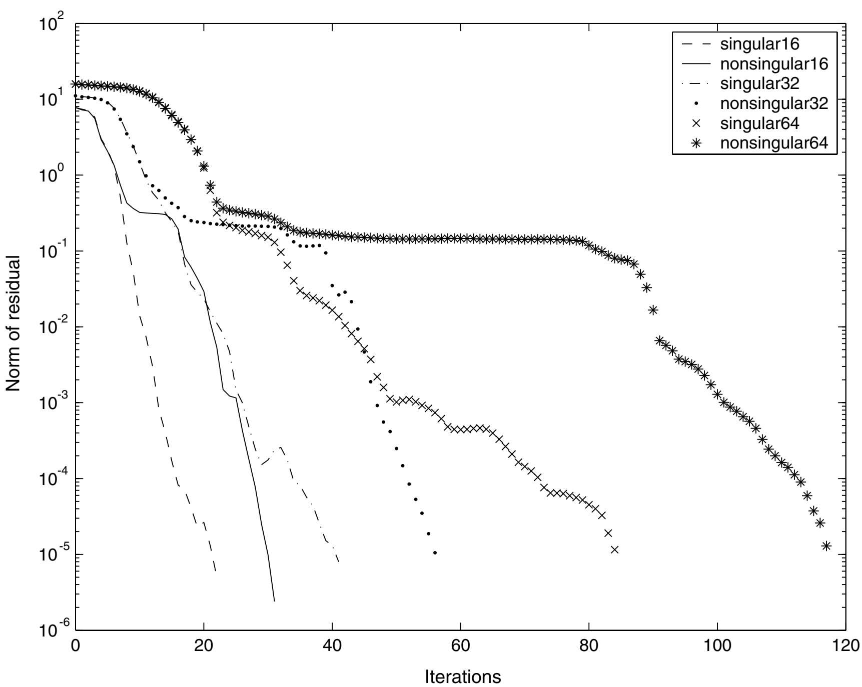

Fig. 1. Convergence history for $\nu = 1$.

range of the coefficient matrix. Therefore, we can exclude the zero eigenvalue when we analyze the convergence behavior of GMRES.

Remark 1.2. Since the coefficient matrix of the consistent singular linear system (1.2) has only the single zero eigenvalue, the index is equal to 1 and GMRES is available.

2. Performance of GMRES iteration

The test problem is a "leaky" two-dimensional lid-driven cavity problem in a square domain $(0 \leqslant x \leqslant 1 : 0 \leqslant y \leqslant 1)$. The boundary conditions are $u_1 = u_2 = 0$ on the three forced walls $(x = 0,\ y = 0,\ x = 1)$, and $u_1 = 1,\ u_2 = 0$ on the moving wall $(y = 1)$. We take a circulating flow field: $w_1 = 8x(x-1) \times (1-2y)$, $w_2 = 8(2x-1)y(y-1)$.

To discretize (1.1) we use the "marker and cell" (MAC) finite difference scheme (cf. [5,2]) based on $ne \times ne$ uniform grids of square meshes. Then we obtain the consistent singular linear system (1.2) and the corresponding nonsingular linear system (1.3).

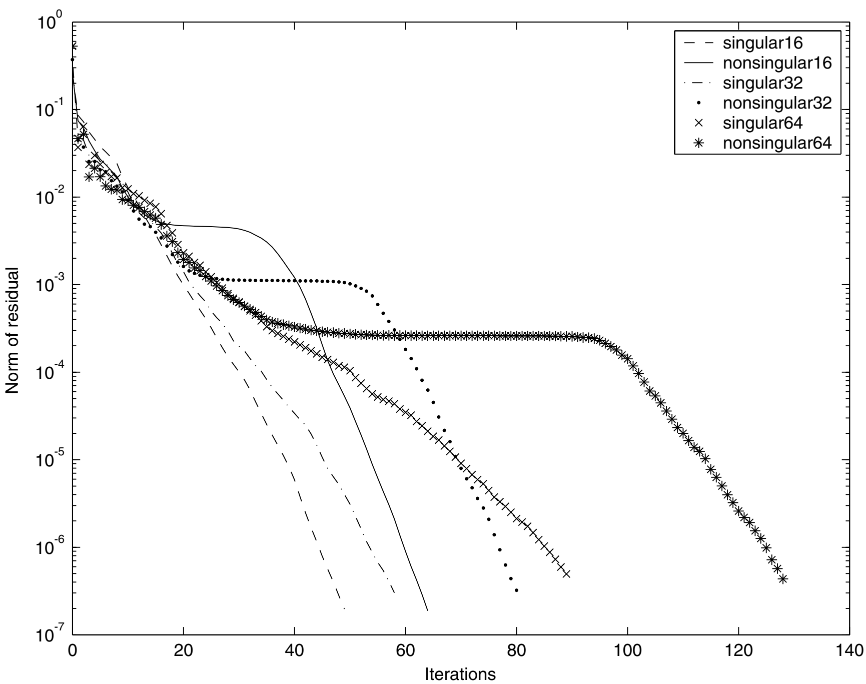

Fig. 2. Convergence history for $v = 1/30$.

We use preconditioning GMRES to solve these two linear systems and compare their convergence behaviors. Let (cf. [3])

$$M = \begin{pmatrix} F & B^{\mathrm{T}} \\ & -\frac{1}{v}I \end{pmatrix} \quad \text{and} \quad \widehat{M} = \begin{pmatrix} F & \widehat{B}^{\mathrm{T}} \\ & -\frac{1}{v}I \end{pmatrix}, \tag{2.1}$$

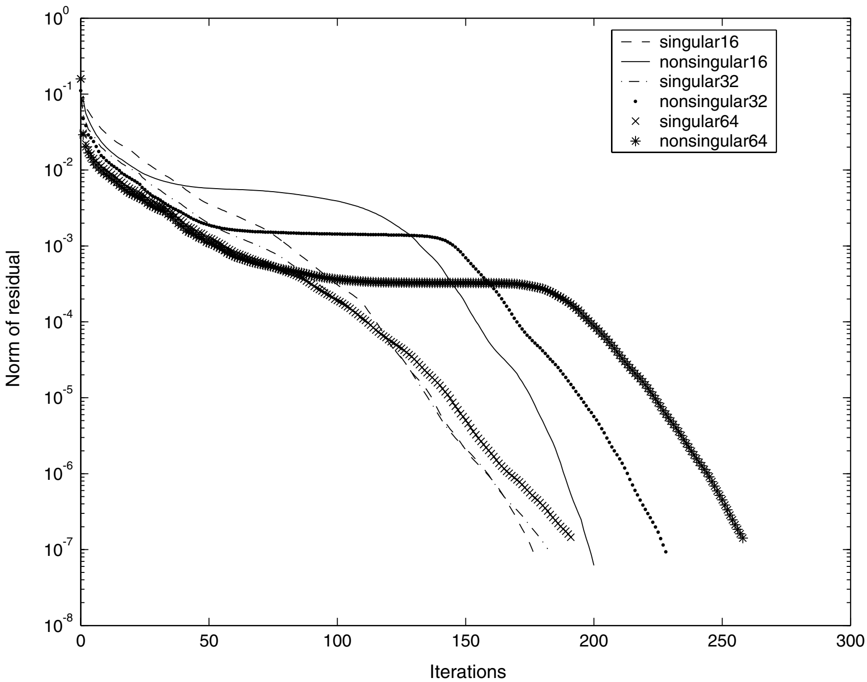

Fig. 3. Convergence history for $v = 1/100$.

Table 2
Iterative courts for the convergence of the preconditioned restarted GMRES(20) solver

v	$h = 1/16$		$h = 1/32$		$h = 1/64$	
	Singular	Nonsingular	Singular	Nonsingular	Singular	Nonsingular
1	2(2)	3(5)	3(7)	28(15)	6(5)	505(15)
1/10	2(4)	3(8)	3(1)	16(4)	5(14)	912(16)
1/20	2(16)	4(15)	3(8)	57(4)	5(20)	939(6)
1/30	3(16)	12(18)	4(7)	40(3)	6(8)	861(2)
1/50	7(2)	63(16)	7(6)	224(13)	9(2)	665(20)
1/100	22(2)	215(20)	20(12)	508(15)	22(17)	1934(7)

then the (left) preconditioners $L_\tau U_\tau$ and $\widehat{L}_\tau \widehat{U}_\tau$ are taken as incomplete LU factorizations with a drop tolerance τ (cf. [8]) of M and $\widehat{M}$, respectively:

$$M = L_\tau U_\tau + R_\tau, \qquad \widehat{M} = \widehat{L}_\tau \widehat{U}_\tau + \widehat{R}_\tau,$$

where L_τ and $\widehat{L}_\tau$ are lower triangular matrices, and U_τ and $\widehat{U}_\tau$ are upper triangular matrices.

Remark 2.1. For simplicity we consider the preconditioners deriving from (2.1), more efficient and complicated preconditioners are discussed in [2,10,4].

All computations described in the paper were performed using MATLAB on an Intel Pendum IV PC computer.

Table 1 shows the number of iterations required by the preconditioned GMRES solver, for both singular and nonsingular linear systems and a verity of values of the mesh size h and the viscosity v. The drop tolerance is equal to 0.01. The initial guess was identically zero, and the stopping criterion was

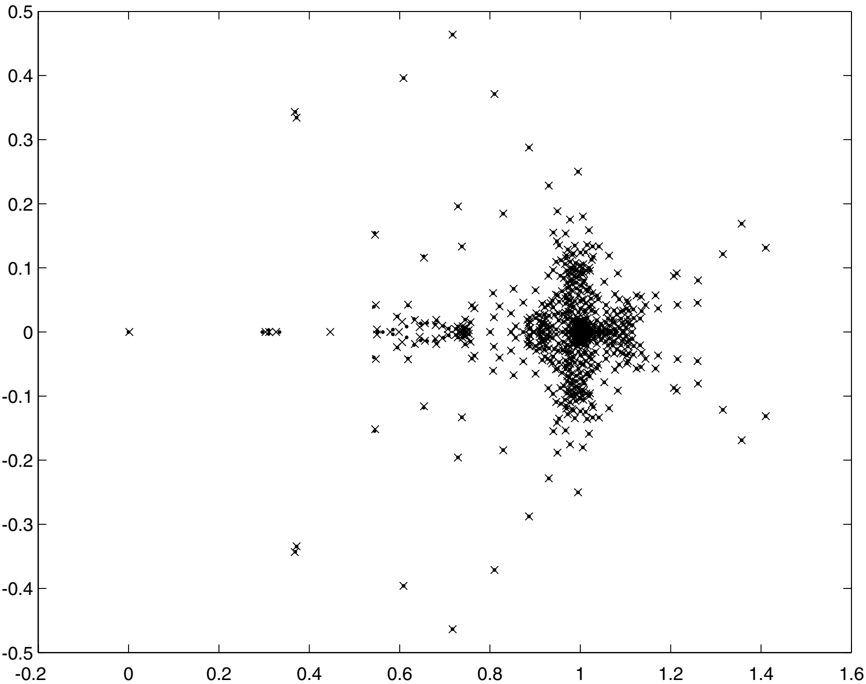

Fig. 4. Eigenvalues of preconditioned singular (●) and nonsingular (×) matrices, $v = 1$.

$$\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 10^{-6}.$$

Figs. 1–3 show the details of the convergence histories for the entries corresponding to $v = 1$, 1/30 and 1/100, respectively.

These experiments show that all the convergence behaviors of the singular cases are better than those of the corresponding nonsingular cases. The figures suggest that the asymptotic convergence behavior of the preconditioned GMRES is independent of being singular case or corresponding nonsingular case, but there is a period of slow convergence in the early stages of the iteration, and this latency period (cf. [4]) of the nonsingular case is larger than that of the corresponding singular case.

Table 2 shows the analogous results by using the preconditioned restated GMRES(20). In the table, $m(k)$ means the number of iterations is $(m-1)\times 20 + k$. From this table, we can see that all the convergence performances of the singular cases are significantly better than those of the corresponding nonsingular cases. Thus, in practical computation we should use restated GMRES to solve singular linear systems.

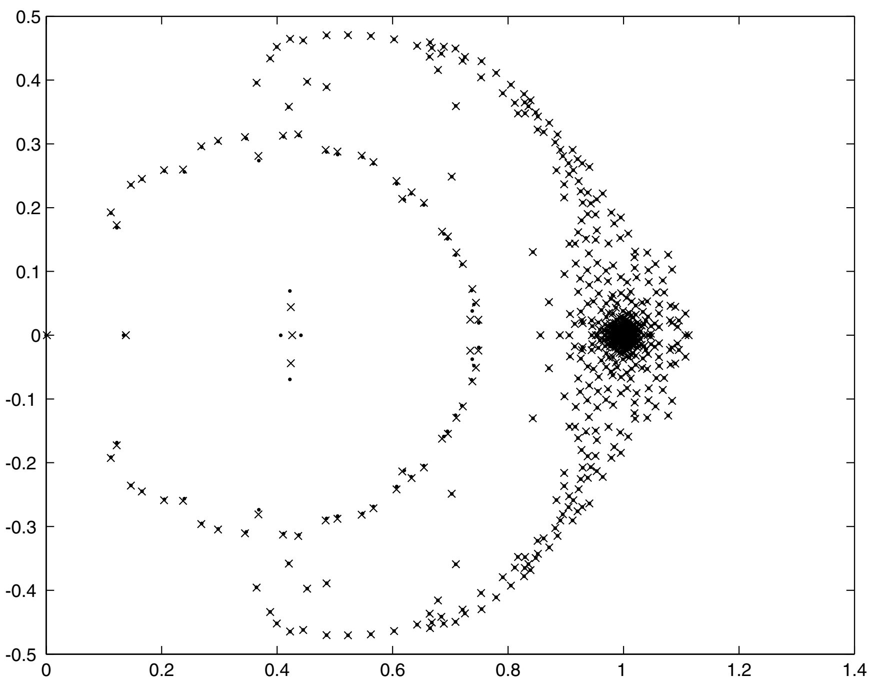

Fig. 5. Eigenvalues of preconditioned singular (●) and nonsingular (×) matrices, $v = 1/30$.

3. Behavior of eigenvalues and analysis of GMRES convergence

It is well known that eigenvalues play a central role for the convergence behavior of the Krylov subspace methods (cf. [6,7]). Let the preconditioned system under consideration be denoted now by

$$\mathscr{A}x = g, \tag{3.1}$$

and let the residual with an iterate $x^{(k)}$ is given by

$$r^{(k)} = g - \mathscr{A}x^{(k)}.$$

Assume $\mathscr{A} = V\Lambda V^{-1}$ is diagonalizable, and let $\sigma(\mathscr{A})$ denote the set of the eigenvalues of $\mathscr{A}$. Then for the bound on convergence of GMRES it holds [9]

$$\|r^{(k)}\|_2 \leqslant k(V) \min_{p_k(0)=1} \max_{\lambda\in\sigma(\mathscr{A})} |p_k(\lambda)| \|r^{(0)}\|_2, \tag{3.2}$$

where the minimum is over all polynomials of the degree k taken on the value 1 at the origin.

Figs. 4–6 plot the eigenvalues of the preconditioned singular and non-singular matrices for $h = 1/16$ corresponding to $v = 1$, 1/30 and 1/100, respectively. Figs. 7–9 plot the corresponding 51 eigenvalues with smallest magnitudes.

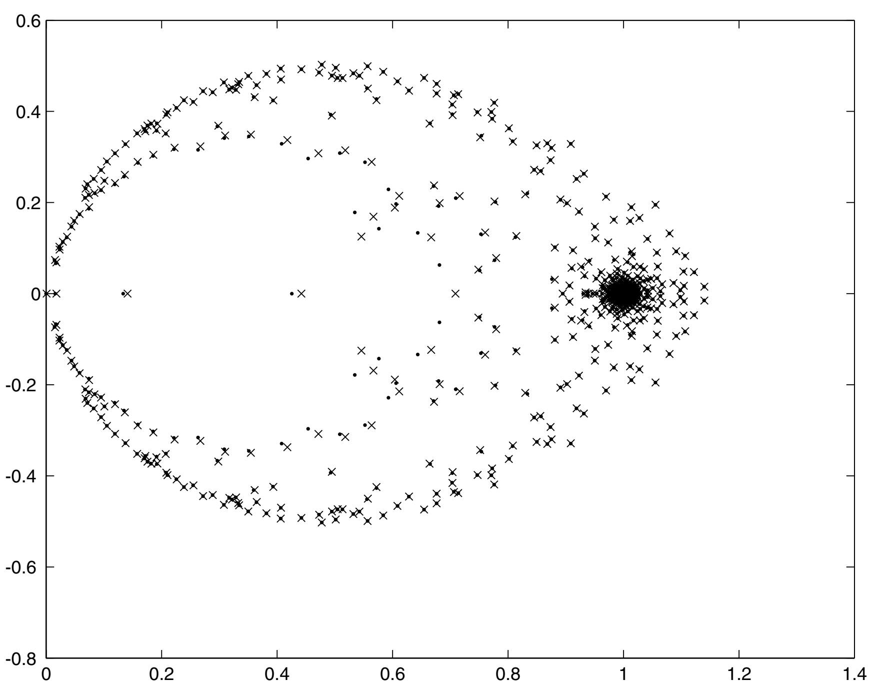

Fig. 6. Eigenvalues of preconditioned singular (●) and nonsingular (×) matrices, $v = 1/100$.

In Tables 3–5 we list the corresponding 10 eigenvalues with the smallest magnitudes.

From these figures and tables we can see

(i) All the eigenvalues in the singular case (except a zero eigenvalue) and in the nonsingular case are in the open right half-plane.
(ii) The difference between the smallest eigenvalue and the next smallest eigenvalue is decreasing with the decreasing of v.

In order to examine why the convergence behavior in singular case is better than that in nonsingular case as shown in Section 2. We need a convergence bound for the GMRES iteration that establishes a connection between the latencies displayed in the initial stage of the solution process and the outlying eigenvalues of the preconditioned systems. Let us split $\sigma(\mathscr{A})$ into two sets:

$$\sigma(\mathscr{A}) = \sigma_c(\mathscr{A}) \cup \sigma_{out}(\mathscr{A}),$$

where $\sigma_c(\mathscr{A})$ denotes a clustered set of eigenvalues of $\mathscr{A}$ which is bounded by a circle centered at c with the radius R, and $\sigma_{out}(\mathscr{A}) = \{\lambda_1, \ldots, \lambda_1\}$ denotes a set

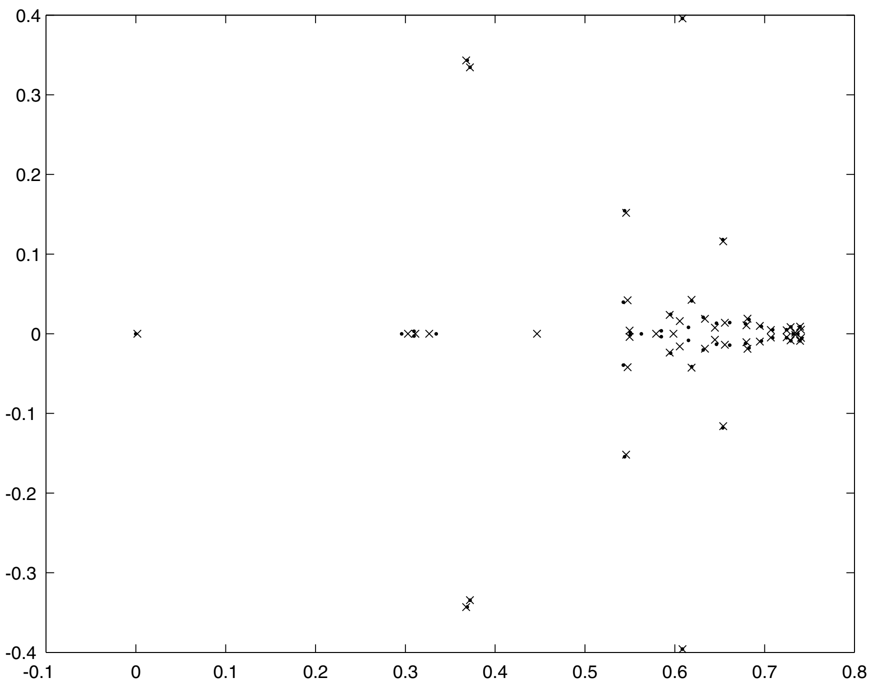

Fig. 7. Fifty-one smallest eigenvalues of preconditioned singular (●) and nonsingular (×) matrices, $v = 1$.

of l outliers. From (i) we can select $c > R > 0$. Then taking $k = l + q$ in (3.2) we have (cf. [9])

$$\min_{p_{l+q}(0)=1} \max_{\lambda \in \sigma(\mathscr{A})} |p_{l+q}(\lambda)| \leqslant \max_{\lambda \in \sigma_c(\mathscr{A})} \left| \left(1 - \frac{\lambda}{\lambda_1}\right) \cdots \left(1 - \frac{\lambda}{\lambda_l}\right) \right| \left(\frac{R}{c}\right)^{k-1}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\|r^{(k)}\|_2 \leqslant \kappa(V) \left(\frac{R}{c}\right)^{k-1} \max_{\lambda \in \sigma_c(\mathscr{A})} \left| \left(1 - \frac{\lambda}{\lambda_1}\right) \cdots \left(1 - \frac{\lambda}{\lambda_l}\right) \right|. \tag{3.4}$$

The bound (3.4) suggests that if some outlier λ_j is small, then the factor $(1 - \frac{\lambda}{\lambda_j})$ in (3.4) will be large which slows down the convergence or, equivalently, makes the length of the period of latency increasing.

We now compare the convergence bounds by using (3.4) to the singular and nonsingular cases. Since in the singular case the eigenvalue $\lambda = 0$ is not to do with the convergence behavior of GMRES and the set of all nonzero eigenvalues in the singular case and the set of all eigenvalues except the smallest one are well approximated each other, the term

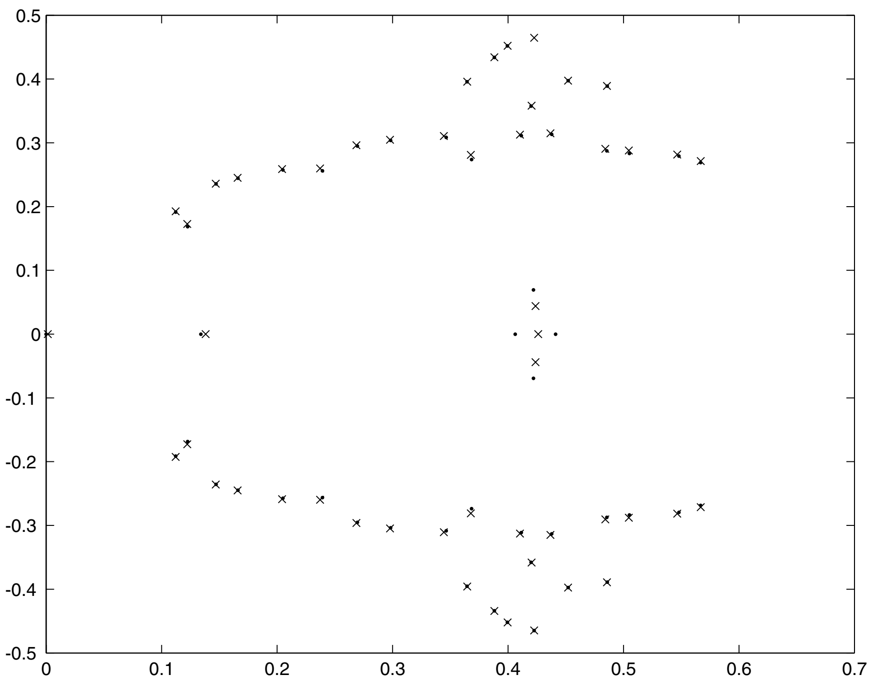

Fig. 8. Fifty-one smallest eigenvalues of preconditioned singular (●) and nonsingular (×) matrices, $v = 1/30$.

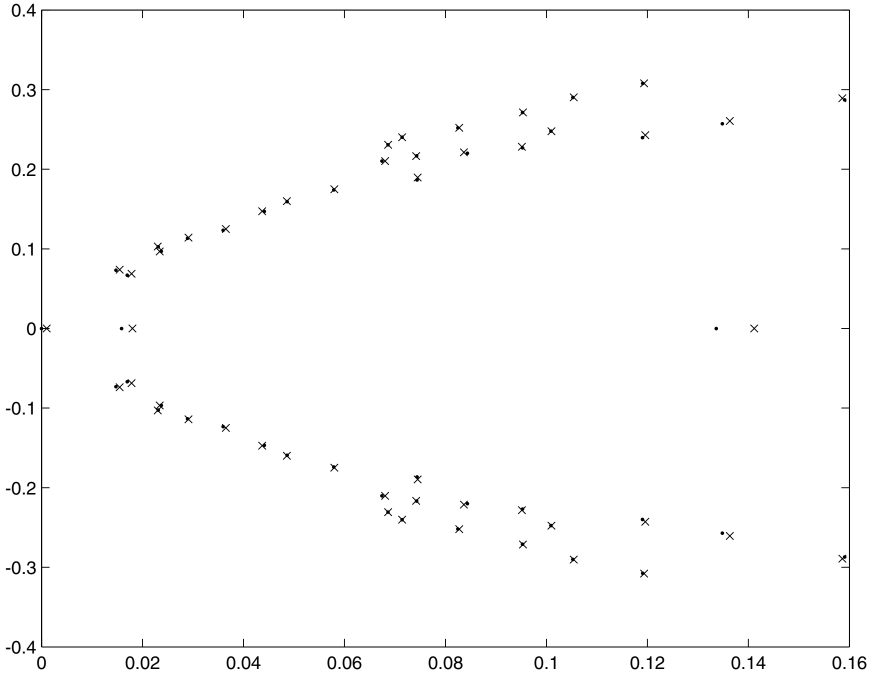

Fig. 9. Fifty-one smallest eigenvalues of preconditioned singular (●) and nonsingular (×) matrices, $v = 1/100$.

Table 3
Ten smallest eigenvalues, $v = 1$

k	Singular	Nonsingular
1	0	$1.5850e^{-3}$
2	$2.95916e^{-1}$	$3.0296e^{-1}$
3	$3.0962e^{-1} - 2.9370e^{-3}i$	$3.1118e^{-1}$
4	$3.0962e^{-1} + 2.9370e^{-3}i$	$3.265le^{-1}$
5	$3.3436e^{-1}$	$4.4653e^{-1}$
6	$3.7189e^{-1} - 3.3453e^{-1}i$	$3.7184e^{-1} - 3.3435e^{-1}i$
7	$3.7189e^{-1} + 3.3453e^{-1}i$	$3.7184e^{-1} + 3.3435e^{-1}i$
8	$3.6892e^{-1} - 3.4305e^{-1}i$	$3.6769e^{-1} - 3.4315e^{-1}i$
9	$3.6892e^{-1} + 3.4305e^{-1}i$	$3.6769e^{-1} + 3.4315e^{-1}i$
10	$5.4277e^{-1} - 3.9608e^{-2}i$	$5.4747e^{-1} - 4.2063e^{-2}i$

$$\max_{\lambda \in \sigma_c(\mathscr{A})} \left| \left(1 - \frac{\lambda}{\lambda_1}\right) \cdots \left(1 - \frac{\lambda}{\lambda_l}\right) \right|, \tag{3.5}$$

in (3.4) for the nonsingular case is significantly larger than that for the singular case.

Table 4
Ten smallest eigenvalues, $v = 1/30$

k	Singular	Nonsingular
1	0	$1.4469e^{-3}$
2	$1.3405e^{-1}$	$1.3819e^{-1}$
3	$1.2251e^{-1} - 1.6849e^{-1}i$	$1.2227e^{-1} - 1.7263e^{-1}i$
4	$1.2251e^{-1} + 1.6849e^{-1}i$	$1.2227e^{-1} + 1.7263e^{-1}i$
5	$1.1238e^{-1} - 1.9148e^{-1}i$	$1.2224e^{-1} - 1.9249e^{-1}i$
6	$1.1238e^{-1} + 1.9148e^{-1}i$	$1.2224e^{-1} + 1.9249e^{-1}i$
7	$1.4725e^{-1} - 2.3572e^{-1}i$	$1.4692e^{-1} - 2.3595e^{-1}i$
8	$1.4725e^{-1} + 2.3572e^{-1}i$	$1.4292e^{-1} + 2.3595e^{-1}i$
9	$1.6623e^{-1} - 2.4461e^{-1}i$	$1.6582e^{-1} - 2.4516e^{-1}i$
10	$1.6623e^{-1} + 2.4461e^{-1}i$	$1.6582e^{-1} + 2.4516e^{-1}i$

Table 5
Ten smallest eigenvalues, $v = 1/100$

k	Singular	Nonsingular
1	0	$1.0368e^{-3}$
2	$1.5831e^{-2}$	$1.7994e^{-2}$
3	$1.6953e^{-2} - 6.6978e^{-2}i$	$1.7817e^{-2} - 6.8643e^{-2}i$
4	$1.6953e^{-2} + 6.6978e^{-2}i$	$1.7817e^{-2} + 6.8643e^{-2}i$
5	$1.4716e^{-2} - 7.3280e^{-2}i$	$1.5468e^{-2} - 7.3782e^{-2}i$
6	$1.4716e^{-2} + 7.3280e^{-2}i$	$1.5468e^{-2} + 7.3782e^{-2}i$
7	$2.3710e^{-2} - 9.6703e^{-2}i$	$2.3394e^{-2} - 9.6648e^{-2}i$
8	$2.3710e^{-2} + 9.6703e^{-2}i$	$2.3394e^{-2} + 9.6648e^{-2}i$
9	$2.3122e^{-2} - 1.0265e^{-1}i$	$2.3029e^{-2} - 1.0296e^{-1}i$
10	$2.3122e^{-2} + 1.0265e^{-1}i$	$2.3029e^{-2} + 1.0296e^{-1}i$

Remark 3.1. How to analyze the convergence performance for the restarted GMRES algorithm and to examine why the convergence behavior in the singular case is significantly better than that in the nonsingular case as shown in Section 2 are open problems.

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